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SUB-EXPONENTIAL CONVERGENCE TO STEADY-STATES FOR 1-D EULER-POISSON EQUATIONS WITH TIME-DEPENDENT DAMPING*

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Abstract. This paper is concerned with the Cauchy problem to Euler-Poisson equations for 1-D unipolar hydrodynamic model of semiconductors with time-dependent damping effect $-\frac{nu}{(1+t)^\lambda}$ for $\lambda\in(-1,0)\cup(0,1)$, where the damping is strong for $\lambda<0$ and weak for $\lambda>0$. For the strong damping case with $\lambda\in(-1,0)$, the system is proved to possess a unique global smooth solution time-asymptotically converging to the steady-state in the sub-exponential form $O((1+t)^{|\lambda|}e^{-\alpha(1+t)^{1-|\lambda|}})$ for some constant $\alpha>0$. For the weak damping case with $\lambda\in(0,1)$, when the doping profile is completely flat, the system is further proved to admit a unique global smooth solution converging to the constant steady-state in the sub-exponential form $O((1+t)^{-\frac{|\theta+\lambda|}{2}}e^{-\beta(1+t)^{1-|\lambda|}})$ for some number $\beta>0$. Specially, the index $\theta\in[\lambda,\infty)$ relies on the initial perturbation and could be large enough once the initial perturbation is sufficiently close to zero, such that the convergence rate involving the part of algebraic decay can be arbitrarily large. A new observation is that the time-dependent damping essentially affects the asymptotic behavior of solutions to Euler-Poisson system, and both the weak and strong damping effects cause the decay rates to be sub-exponential, which are slower than the regular exponential decay in the case of $\lambda=0$. The adopted approach for the proof in this paper is based on the elementary L^2-energy estimates but with some technical development.

Keywords. Euler-Poisson equations; unipolar hydrodynamic model; semiconductor; weak damping; strong damping; sub-exponential convergence; steady-states.

AMS subject classifications. 35B40; 35L50; 35L60; 35L65.

1. Introduction

In this paper, we consider the damped Euler-Poisson equations for one-dimensional unipolar hydrodynamic model of semiconductors

$$\begin{cases} n_t+(nu)_x=0, \\ (nu)_t+\big(nu^2+p(n)\big)_x=n\phi_x-\dfrac{nu}{(1+t)^\lambda}, \quad (x,t)\in\mathbb{R}\times(0,\infty). \\ \phi_{xx}=n-D(x), \end{cases} \tag{1.1}$$

Here the unknown functions $n(x,t)$, $u(x,t)$ and $\phi(x,t)$ represent the electronic density, the electronic velocity and the electrostatic potential, respectively. The given function $D(x)$ is the doping profile standing for the density of impurities (positive background ions) in semiconductor devices and $p=p(n)$ is the pressure-density relation. The term $-\frac{nu}{(1+t)^\lambda}$ represents the external fractional damping effect with a parameter $\lambda\in(-\infty,\infty)$. Mathematically, it is called the weak damping when $\lambda>0$, as the damping coefficient $(1+t)^{-\lambda}$ becomes vanishing; while, it is called the strong damping when $\lambda<0$, as the damping coefficient becomes enhancing. obviously, it is the regular damping when $\lambda=0$.

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Let $J:=nu$ be the current density and $E:=\phi_x$ be the electric field. Thus, the system (1.1) can be rewritten as

$$\begin{cases} n_t+J_x=0, \\ J_t+\left(\dfrac{J^2}{n}+p(n)\right)_x=nE-\dfrac{J}{(1+t)^\lambda}, \quad (x,t)\in\mathbb{R}\times(0,\infty). \\ E_x=n-D(x), \end{cases} \tag{1.2}$$

Our target is to study the large-time behavior of smooth solutions to (1.2) in the strong damping case with $\lambda\in(-1,0)$ and weak damping case with $\lambda\in(0,1)$, supplemented with the initial value

$$n(x,0)=n_0(x), \quad J(x,0)=J_0(x),\ x\in\mathbb{R}. \tag{1.3}$$

Throughout this paper, our assumption on the pressure function is

$$p(\cdot)\in C^4(0,\infty) \quad \text{and} \quad p'(s)>0 \text{ for } s>0. \tag{1.4}$$

A physical example is the isentropic flow with $p(n)=kn^\gamma$ for some constants $k>0$ and $\gamma\geq 1$.

Firstly introduced by Bløtekjær [2], the hydrodynamic model (1.1) then has been usually used in describing the charged particles such as electrons and holes in semiconductor devices [27, 40]. When $\lambda=0$, the structure of the subsonic/supersonic/transonic solutions to the unipolar hydrodynamic model for semiconductors (1.1) with the regular damping has been extensively investigated in [1, 3, 4, 8, 11, 12, 14, 15, 34, 36, 37, 41, 45, 52, 53, 55, 60], and the large-time behavior of subsonic solutions was studied in [18, 19, 21, 22, 33, 38, 42, 47, 54]. See [13, 16, 20, 23, 26, 43, 44, 56] for the bipolar system case. However, when $\lambda\neq 0$, the structure of the solutions to (1.1) becomes more complicated and challenging.

Regarding how the time-dependent damping effect makes the changes significantly to the structure of solutions, Wirth [57–59] first investigated the Cauchy problem to the following linear damped wave equation

$$u_{tt}+\frac{u_t}{(1+t)^\lambda}-\Delta u=0. \tag{1.5}$$

The global/blow-up solutions to (1.5) with the nonlinear source $\pm|u|^q$ or $\pm u|u|^{q-1}$ $(q>1)$ were studied by Nishihara and his collaborators in [39, 46, 48, 49].

For the compressible Euler equations with time-dependent damping

$$\begin{cases} n_t+\operatorname{div}(n\boldsymbol{u})=0, \\ (n\boldsymbol{u})_t+\operatorname{div}(n\boldsymbol{u}\otimes\boldsymbol{u})+\nabla p(n)=-\frac{\mu}{(1+t)^\lambda}n\boldsymbol{u}, \end{cases} \tag{1.6}$$

and the corresponding p-equations ($v:=n^{-1}$ is the specific volume)

$$\begin{cases} v_t-\operatorname{div}\boldsymbol{u}=0, \\ \boldsymbol{u}_t+\nabla p(v)=-\frac{\mu}{(1+t)^\lambda}\boldsymbol{u}, \end{cases} \tag{1.7}$$

this issue has received widespread attention recently. For the weak damping with $\lambda\in(0,1)$, the global existence of solutions were proved in [7, 9, 10, 30] for (1.7) and in [51] for (1.6) in the 1-D case, while in [24, 25, 28] for multi-dimensional (1.6). Among them, it was shown in [9, 30] that the solutions time-algebraically converge to the corresponding self-similar solutions (diffusion waves) when the far-field states are different; lately, the optimal convergence rates of the solutions for multi-dimensional (1.6) around the constant steady-states were obtained by Ji and Mei in [28] through the Fourier analysis technique and Green function method, where they recognized that the weaker damping effect for $\lambda\in(0,1)$ makes the solutions to decay at a faster time-algebraical rate. For the weak damping with critical value $\lambda=1$, we can refer to [7, 24, 25, 50, 51] that the solutions are proved to be blow-up once $0<\mu\le 3-d$, and to time-globally exist when $\mu>3-d$, where d is the spatial dimension. But, these results did not involve the asymptotic behavior of the global solution. For this, Geng-Lin-Mei [17] observed that the effects of hyperbolicity and parabolicity are equal in the 1-D (1.7) when $\lambda=1$ with $\mu>2$, and both cannot be ignored. Based on this point, they found the appropriate profile is the solution to the linear wave equation with damping, and further showed the algebraic convergence rates relate to μ. When $\lambda>1$, the damping effect is too weak, which causes that the damped Euler system is pretty similar to the pure Euler system, such that the weak damping cannot prevent the shock formation, and the solutions are proved to blow up at finite time for their gradients [7, 50]. When $\lambda\in[-1,0)$, the damping effect becomes stronger, the solutions to the multi-dimensional (1.6) are proved to globally exist and the optimal convergence rates for the solutions around the constant states are obtained in [29], where Ji and Mei realized that the stronger damping effect for $\lambda\in[-1,0)$ causes the convergence rates to slow down. The optimal convergence to the diffusion wave for $\lambda=-1$ was recently obtained by Li-Li-Mei-Zhang in [32].

For Euler-Poisson system with time-dependent damping, different from the damped Euler system mentioned above, the relevant studies are quite limited. For the doping profile $D(x)\equiv 0$, Li-Li-Mei-Zhang [31] first studied the 1-D bipolar hydrodynamic model (5×5 Euler-Poisson system) with the damping $\frac{\mu J_i}{(1+t)^\lambda}$ $(i=1,2)$, and showed that the solutions for the Cauchy problem time-algebraically converge to the self-similar solutions (diffusion waves) of the corresponding nonlinear porous media equations when $\lambda\in(-1,1)$ and $\mu=1$. For the critical case $\lambda=1$ with $\mu>2$, Luan-Mei-Rubino-Zhu [35] proved that the solutions of the above Cauchy problem algebraically converge to the constant steady-states, but the convergent rates are not sufficient compared to [17]. We notice that in [31], self-similar solutions can be regarded as the appropriate asymptotic profiles for the assumption $D(x)\equiv 0$, which makes that the energy estimates for the solutions can be smoothly established in some sense. However, when $D(x)>0$ (the physical case), the reasonable asymptotic profiles maybe the steady-states for the existence of the Poisson equation. Meanwhile, we usually expect the effect of Poisson equation to bring out some fast exponential decays for the solutions, rather than the algebraic decays in [17, 31]. But it is totally unknown for us how the decay rates in the exponential form are affected by the time-dependent weak/strong damping.

Therefore, for the nonzero doping profile, to investigate the solutions of the system (1.2) in the weak/strong damping cases converging to the steady-states will be the main issue for us in this paper. In what follows, we realize that both the weak damping with $\lambda\in(0,1)$ and the strong damping with $\lambda\in(-1,0)$ lead to the slower sub-exponential decay of the solutions, rather than the exponential decay for the regular damping case of $\lambda=0$ in [21, 22, 33, 38]. Here are some technical issues in the proof we need to point out. For the weak damping case with $\lambda\in(0,1)$, different from the previous studies for $\lambda=0$, the non-trivial doping profile will cause us some essential difficulty in establishing the energy estimates, and we have to assume the doping profile to be completely flat, namely $D(x)\equiv \mathit{constant}>0$, which also matches the studies on the weak-damped Klein-

Gordon equation by Burq-Raugel-Schlag [5,6] (see Remark 1.3 and Remark 3.2 below for details). For the strong damping case with $\lambda\in(-1,0)$, we find that the strong damping can eliminate the obstacle caused by the non-trivial doping profile, so we can allow $D(x)$ to be a non-trivial function. But if $\inf_{x\in\mathbb{R}} D(x)<\frac{|\lambda|}{2}$, different from the regular case with $\lambda=0$, we can not directly get all decay rates of the solutions by the energy method. To overcome this difficulty, we propose a new treating procedure: We first adopt the technical time-weighted energy method to derive the global existence of solutions with algebraic convergence rates, then we further enhance the algebraic decay rates to the sub-exponential rates sequentially when the time is sufficiently large.

In summary, we precisely state our main results as follows:

(i) When $\lambda\in(-1,0)$, for the flat non-constant doping profile $D(x)$, we expect that the asymptotic profile of (n,E,J) is the steady-state solution $(\bar{n},\bar{E},\bar{J})$ to the well-known unipolar drift-diffusion model for semiconductors, where the current density $\bar{J}(x)\equiv 0$. Then, we show that there exists a unique global solution to (1.2)-(1.3) with strong damping which sub-exponentially decays to the steady-state in the form of

$$\|n(t)-\bar{n}\|_{L^\infty(\mathbb{R})}+\|J(t)\|_{L^\infty(\mathbb{R})}+\|E(t)-\bar{E}\|_{L^\infty(\mathbb{R})}\lesssim(1+t)^{|\lambda|}e^{-\alpha(1+t)^{1-|\lambda|}}, \tag{1.8}$$

for some constant $\alpha>0$, provided the initial perturbation is sufficiently small.

(ii) When $\lambda\in(0,1)$, we have to restrict the doping profile $D(x)\equiv\widehat{\mathcal{D}}$ for some positive constant $\widehat{\mathcal{D}}$, and the expected steady-state is reduced to the trivial state $(\widehat{\mathcal{D}},0,0)$. We then prove that there exists a unique global solution to (1.2)-(1.3) with weak damping which sub-exponentially converges to the constant steady-state in the form of

$$\|n(t)-\widehat{\mathcal{D}}\|_{L^\infty(\mathbb{R})}+\|J(t)\|_{L^\infty(\mathbb{R})}+\|E(t)\|_{L^\infty(\mathbb{R})}\lesssim(1+t)^{-\frac{|\theta+\lambda|}{2}}e^{-\beta(1+t)^{1-|\lambda|}}, \tag{1.9}$$

for some constant $\beta>0$, provided the initial perturbation is sufficiently small. Here, the index $\theta\in[\lambda,\infty)$ is closely related to the initial perturbation and could be large enough as the initial perturbation reduces to zero. Moreover, for the regularly small initial perturbation, the slowest decay is $(1+t)^{-|\lambda|}e^{-\beta(1+t)^{1-|\lambda|}}$ for $\theta\geq\lambda$.

REMARK 1.1. It is worth comparing our results with the existing studies on the unipolar hydrodynamic model of semiconductors. For the regular damping case with $\lambda=0$, it was shown in [21,38] that the convergence rates of the solutions to the steady-states are time-exponential like $O(e^{-\nu t})$ for some positive number ν. Here, we show that, the time-dependent damping essentially affects the asymptotic behavior of the Euler-Poisson system. In fact, the weak damping effect with $\lambda\in(0,1)$ can lead to a slow convergence rate in the sub-exponential form of $(1+t)^{-\frac{|\theta+\lambda|}{2}}e^{-\beta(1+t)^{1-|\lambda|}}$ $(\lambda\leq\theta<\infty)$, while the strong damping with $\lambda\in(-1,0)$ causes the convergence rate to be slower in the sub-exponential form of $(1+t)^{|\lambda|}e^{-\alpha(1+t)^{1-|\lambda|}}$.

Particularly, we see the properties of the convergence rates depending on the parameter λ:

$$\text{weak damping case: } (1+t)^{-|\lambda|}e^{-\beta(1+t)^{1-|\lambda|}}\to e^{-\nu t},\ \text{as } \lambda\to 0^+,$$

for $\nu=\beta(0^+)$;

$$\text{strong damping case: } (1+t)^{|\lambda|}e^{-\alpha(1+t)^{1-|\lambda|}}\to e^{-\nu t},\ \text{as } \lambda\to 0^-,$$

for $\nu=\alpha(0^-)$.

Remark 1.2. It is also interesting to compare our results with the time-dependent compressible Euler system. As we know, the structure of the damped Euler equations essentially is parabolic-hyperbolic, like the damped wave equation

$$u_{tt}+\frac{\mu}{(1+t)^{\lambda}}u_t-u_{xx}=F,$$

which implies that the solutions time-asymptotically decay in the algebraic form (see [58,59]). For the compressible Euler system (1.6) with time-dependent damping, recently it was recognized that the weak damping case with $\lambda\in(0,1)$ makes the algebraic decay rate of the solutions to be faster than the regular damping case with $\lambda=0$, and much faster than the strong damping case with $\lambda\in[-1.0)$ (see [9,28–32]).

However, for the Euler-Poisson equations with time-dependent damping, the Poisson effect for the electric field causes the system to possess the hyperbolicity and strong dispersion, which makes the working equation resemble the Klein-Gordon equation. So, we may expect the solutions decay time-exponentially. Here, we observe that both the weak damping with $\lambda\in(0,1)$ and the strong damping with $\lambda\in(-1,0)$ make the decay of the solutions to be sub-exponential, which is slower than the exponential decay in the regular damping case. This is a new observation which is different from the damped Euler Equations (1.6).

Remark 1.3. Note that the sub-exponential decay estimates of the solution for the following Klein-Gordon equation

$$u_{tt}+\frac{\mu}{(1+t)^{\lambda}}u_t+\omega(x)u-u_{xx}=F \tag{1.10}$$

were first obtained by Burq-Raugel-Schlag in [5,6], in which they considered the case of $\lambda\in(0,\frac{1}{2})$ and proved that the solution converges to an equilibrium point in the sub-exponential rate of $e^{-\nu(1+t)^{1-\lambda}}$ for some positive constant ν. For their study, they assumed $\omega(x)\equiv constant>0$. It seems that this is a technical but crucial assumption in the proofs for deriving the sub-exponential decay.

For our study, a similar restriction $D(x)\equiv constant>0$ is also needed on the Euler-Poisson system (1.2) with $\lambda\in(0,1)$, and our decay result in the form of $(1+t)^{-\frac{\theta+\lambda}{2}}e^{-\nu(1+t)^{1-\lambda}}$ $(\lambda\leq\theta<\infty)$ for the case of $\lambda\in(0,1)$ is faster than the rate obtained in [5]. For the technical and detailed explanation, we refer to Remark 3.2 below.

Remark 1.4. For the cases of $\lambda\geq1$ and $\lambda\leq-1$, we may expect the solutions to be blow-up at finite time for $\lambda>1$, and to globally/non-globally exist for the critical case of $\lambda=1$, and to globally exist for $\lambda=-1$ with logarithmic decay, and to globally exist for $\lambda<-1$ with non-decay. These will be our targets in future.

Notations. Throughout this paper, $c_i,\tilde{c}_i,C_i,\tilde{C}_i$, etc. always represent some specific positive constants. C denotes the generic positive constant which maybe different in different lines. The derivatives of a real-valued function f on $\mathbb{R}$ are denoted by $\partial_x^k f$ $(k=1,2,\cdots)$ or $f_x,f_{xx}\cdots$. $H^k(\mathbb{R})$ $(k\geq0)$ is the usual Sobolev space whose norm is defined by

$$\|f\|_k^2:=\|f\|_{H^k(\mathbb{R})}^2=\sum_{i=0}^{k}\|\partial_x^i f\|^2$$

with $\|f\|^2:=\|f\|_{L^2(\mathbb{R})}^2$. For simplicity, we also denote $\|(f_1,f_2,\cdots,f_n)\|^2:=\sum_{i=0}^{n}\|f_i\|^2$. $A\lesssim$

B or $B \gtrsim A$ means that $A \le CB$ for some constant $C>0$, and $A \simeq B$ denotes that $A \lesssim B$ and $A \gtrsim B$.

2. Strong damping case with $-1<\lambda<0$

In this section, we study the global existence and large-time behavior of smooth solutions to the Cauchy problem (1.2)-(1.3) with time-gradually enhanced damping in the case of $\lambda \in (-1,0)$. To begin with, throughout this section, we assume that the doping profile satisfies

$$D(x)>0, \ \lim_{x\to\pm\infty} D(x)=D_\pm, \ D(\cdot)\in C(\mathbb{R}) \text{ and } D'(\cdot)\in H^2(\mathbb{R}). \tag{2.1}$$

Since the doping profile is nonzero, the expected asymptotic profiles of the solutions will be the non-trivial steady-states $(\bar{n},\bar{J},\bar{E})$ satisfying the following stationary equations corresponding to the well-known unipolar drift-diffusion model for semiconductors (see also the previous studies [21,38]):

$$\begin{cases} (\bar{n}\bar{E} - p(\bar{n})_x)_x = 0, \\ \bar{E}_x = \bar{n} - D(x), \end{cases} \tag{2.2}$$

with the boundary conditions

$$\lim_{x\to+\infty} \bar{n}(x) = D_+, \quad \lim_{x\to-\infty} \bar{n}(x) = D_-, \quad \lim_{x\to-\infty} \bar{E}(x) = 0. \tag{2.3}$$

Obviously, from $(2.2)_1$ and the boundary condition (2.3), it holds that

$$\bar{J}(x) := \bar{n}\bar{E} - p(\bar{n})_x \equiv const. = \bar{J}(-\infty) = D_-\bar{E}(-\infty) = 0 \quad \text{for } x\in\mathbb{R}. \tag{2.4}$$

It is well-known that the existence and uniqueness of the stationary solution have been obtained in [21,33,38]. Hence, we omit the proof and state the results as follows.

LEMMA 2.1 (Asymptotic profiles). *Suppose that $p(\bar{n})$ satisfies (1.4) and $D(x)$ satisfies (2.1). We define*

$$D_* := \inf_{x\in\mathbb{R}} D(x) \quad \textit{and} \quad D^* := \sup_{x\in\mathbb{R}} D(x). \tag{2.5}$$

Then, the steady-state system (2.2)-(2.3) possesses a unique smooth solution $(\bar{n},\bar{E})$ satisfying

$$D_* \le \bar{n}(x) \le D^* \ (x\in\mathbb{R}), \tag{2.6}$$

and

$$\begin{aligned} &\|\bar{n}-D\|_3 + \sup_{x\in\mathbb{R}} \big(|\bar{n}_x(x)| + |\bar{n}_{xx}(x)| + |\bar{n}_{xxx}(x)| + |\bar{E}(x)| + |\bar{E}_x(x)| + |\bar{E}_{xx}(x)|\big) \\ &\lesssim \|D'\|_2. \end{aligned} \tag{2.7}$$

Based on Lemma 2.1, we obtain the global existence and large-time behavior of the solutions to (1.2)-(1.3), which is summarized in the following theorem.

THEOREM 2.1 (Convergence). *For the case of $-1<\lambda<0$, suppose that (1.4) and (2.1) hold. Let $(\bar{n},\bar{E})$ be the solution to the stationary problem (2.2)-(2.3) and*

$$\rho_0(x) := \int_{-\infty}^{x} (n_0 - \bar{n})(y)dy. \tag{2.8}$$

Assume that $\rho_0(\cdot)\in H^3(\mathbb{R})$, $J_0(\cdot)\in H^2(\mathbb{R})$ and $\|\rho_0\|_3+\|J_0\|_2+\|D'\|_2$ is sufficiently small. Then, the Cauchy problem (1.2)-(1.3) admits a unique global-in-time smooth solution (n,J,E) satisfying for $t>0$,

$$\|n(t)-\bar{n}\|_2+(1+t)^{\frac{|\lambda|}{2}}\|J(t)\|+(1+t)^{|\lambda|}\|J_x(t)\|_1+\|E(t)-\bar{E}\|_3\lesssim\|\rho_0\|_3+\|J_0\|_2. \quad (2.9)$$

Furthermore, the global solution decays to the steady-state in the form of

$$\|n(t)-\bar{n}\|_2+\|J(t)\|_2+\|E(t)-\bar{E}\|_3\lesssim(1+t)^{|\lambda|}e^{-\alpha(1+t)^{1-|\lambda|}}(t\to\infty), \quad (2.10)$$

for some positive constant α.

Let us set

$$\mathcal{N}(x,t):=n(x,t)-\bar{n}(x), \quad \mathcal{E}(x,t):=E(x,t)-\bar{E}(x). \quad (2.11)$$

Thus, the unknown functions $(\mathcal{N},J,\mathcal{E})$ satisfy the system

$$\begin{cases}\mathcal{N}_t+J_x=0,\\ J_t+\left(\dfrac{J^2}{n}\right)_x+(p(n)-p(\bar{n}))_x=\mathcal{N}\bar{E}+\mathcal{N}\mathcal{E}+\bar{n}\mathcal{E}-\dfrac{J}{(1+t)^\lambda},\\ \mathcal{E}_x=\mathcal{N}.\end{cases} \quad (2.12)$$

This implies

$$\mathcal{E}_x=\mathcal{N}, \quad \mathcal{E}_t=-J. \quad (2.13)$$

Substituting (2.13) into $(2.12)_2$, we get the following nonlinear Klein-Gordon equation with time-dependent damping

$$\mathcal{E}_{tt}+(1+t)^{-\lambda}\mathcal{E}_t+\bar{n}\mathcal{E}-(p(\bar{n}+\mathcal{E}_x)-p(\bar{n}))_x=-\mathcal{E}\mathcal{E}_x-\bar{E}\mathcal{E}_x+\left(\frac{\mathcal{E}_t^2}{\bar{n}+\mathcal{E}_x}\right)_x. \quad (2.14)$$

Here, we choose

$$\begin{aligned}E(x,0)&=\int_{-\infty}^x(n_0-D)(y)dy\\ &=\int_{-\infty}^x(n_0-\bar{n})(y)dy+\int_{-\infty}^x(\bar{n}-D)(y)dy\\ &=\rho_0(x)+\bar{E}(x).\end{aligned}$$

Thus, from (2.13) again, we get the initial data for the Equation (2.14) as

$$\mathcal{E}(x,0)=\rho_0(x), \quad \mathcal{E}_t(x,0)=-J_0(x). \quad (2.15)$$

Now, we state the corresponding *a priori* estimates of the solution to (2.14)-(2.15) and the sub-exponential convergence as follows.

PROPOSITION 2.1 (*A priori* estimates in the case of $-1<\lambda<0$). *Under the conditions of Theorem 2.1, there exists a positive constant ε_1 sufficiently small such that, for any given $T>0$, if the solution to (2.14)-(2.15) on $[0,T]$ satisfying*

$$\sup_{0\le t\le T}\left[\|\mathcal{E}(t)\|_3^2+(1+t)^{-\lambda}\|(\mathcal{E}_t,\mathcal{E}_{tt})(t)\|^2+(1+t)^{-2\lambda}\|(\mathcal{E}_{xt},\mathcal{E}_{xxt},\mathcal{E}_{xtt})(t)\|^2\right]\le\varepsilon_1, \quad (2.16)$$

then, for any $t\in[0,T]$,

$$\begin{aligned}&\|\mathcal{E}(t)\|_3^2+(1+t)^{-\lambda}\|(\mathcal{E}_t,\mathcal{E}_{tt})(t)\|^2+(1+t)^{-2\lambda}\|(\mathcal{E}_{xt},\mathcal{E}_{xxt},\mathcal{E}_{xtt})(t)\|^2\\&\quad+\int_0^t\left[(1+s)^{\lambda}\|\mathcal{E}(s)\|_3^2+(1+s)^{-\lambda}\|\mathcal{E}_t(s)\|_2^2\right]ds\\&\quad+\int_0^t\left[(1+s)^{-2\lambda}\|\mathcal{E}_{tt}(s)\|^2+(1+s)^{-3\lambda}\|\mathcal{E}_{xtt}(s)\|^2\right]ds\\&\lesssim\|\rho_0\|_3^2+\|J_0\|_2^2.\end{aligned}\tag{2.17}$$

Based on the estimate (2.17), we can obtain the following sub-exponential decay estimates.

PROPOSITION 2.2 (Decay rates in the case of $-1<\lambda<0$). *Under the conditions of Theorem 2.1, there exists a positive constant* α *such that, if the solution to* (2.14)-(2.15) *globally exists and the estimate* (2.17) *holds for all* $t\ge 0$*, then,*

$$\|\mathcal{E}(t)\|_3+\|\mathcal{E}_t(t)\|_2\lesssim(1+t)^{|\lambda|}e^{-\alpha(1+t)^{1-|\lambda|}},\quad \text{as } t\to\infty.\tag{2.18}$$

Proof of Theorem 2.1. In fact, the local existence of solution to the initial problem (2.14)-(2.15) can be obtained by linearizing the system with standard linear iteration method, and we omit its detail. Then, from the *a priori* estimate (2.17) in Proposition 2.1, we can extend the local solution globally by using the usual continuity arguments and show that estimate (2.17) holds for all $t\ge 0$, provided the initial perturbation $\|\rho_0\|_3+\|J_0\|_2$ is sufficiently small. Equivalently, we have proved the global existence of solutions to the Cauchy problem (1.2)-(1.3). Finally, we can get the decay estimate (2.10) based on the estimate (2.18) in Proposition 2.2.

In what follows, the main target is to prove Proposition 2.1 and Proposition 2.2.

REMARK 2.1. As we show later, once $D_*>\frac{|\lambda|}{2}$, we can prove the *a priori* estimate with the sub-exponential decay directly. However, in order to remove such a restriction, we need to divide our proof into two steps. The first step is only to get the *a priori* estimates (2.17) with the algebraic decay, due to the effects by the nonlinear terms. Then, we improve the decay rates to be sub-exponential in (2.18).

2.1. A priori estimates.

From the assumption (2.16) and using the standard Sobolev inequality, we have

$$\begin{aligned}&\sup_{0\le t\le T}\left[\sum_{k=0}^{2}\|\partial_x^k\mathcal{E}(t)\|_{L^\infty(\mathbb{R})}+(1+t)^{-\frac{3}{4}\lambda}\|(\mathcal{E}_t,\mathcal{E}_{tt})(t)\|_{L^\infty(\mathbb{R})}+(1+t)^{-\lambda}\|\mathcal{E}_{xt}(t)\|_{L^\infty(\mathbb{R})}\right]\\&\lesssim\varepsilon_1.\end{aligned}\tag{2.19}$$

Thus, from (2.6) and the smallness of ε_1, it is easy to verify that

$$\frac{D_*}{2}\le n=\bar{n}+\mathcal{E}_x\le 2D^*.\tag{2.20}$$

To prove Proposition 2.1, we have to establish the following *a priori* estimates based on (2.19) and (2.20).

LEMMA 2.2. *Under the conditions of Proposition 2.1, it holds that for any* $t\in[0,T]$,

$$\|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t)(t)\|^2+\int_0^t\left[(1+s)^{-\lambda}\|\mathcal{E}_t(s)\|^2+(1+s)^{\lambda}\|(\mathcal{E},\mathcal{E}_x)(s)\|^2\right]ds$$

$$\lesssim \|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t)(0)\|^2. \tag{2.21}$$

Proof. Multiplying (2.14) by $(1+t)^\lambda\mathcal{E}+2\mathcal{E}_t$ and integrating the resulting equality with respect to x over $\mathbb{R}$ by parts, we obtain that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\Big[(1+t)^\lambda\mathcal{E}\mathcal{E}_t-\frac{\lambda}{2}(1+t)^{\lambda-1}\mathcal{E}^2+\frac{1}{2}\mathcal{E}^2+\bar{n}\mathcal{E}^2+\mathcal{E}_t^2\Big]dx+\big[2(1+t)^{-\lambda}-(1+t)^\lambda\big]\|\mathcal{E}_t(t)\|^2\\
&\quad+\int_{\mathbb{R}}\Big[(1+t)^\lambda\bar{n}+\frac{\lambda(\lambda-1)}{2}(1+t)^{\lambda-2}\Big]\mathcal{E}^2dx+\int_{\mathbb{R}}(p(\bar{n}+\mathcal{E}_x)-p(\bar{n}))\big[(1+t)^\lambda\mathcal{E}_x+2\mathcal{E}_{xt}\big]dx\\
&=-\int\frac{\mathcal{E}_t^2}{n}\big[(1+t)^\lambda\mathcal{E}_x+2\mathcal{E}_{xt}\big]dx-\int_{\mathbb{R}}(\mathcal{E}+\bar{E})\big[(1+t)^\lambda\mathcal{E}\mathcal{E}_x+2\mathcal{E}_x\mathcal{E}_t\big]dx\\
&=:K_{11}+K_{12}.
\end{aligned}\tag{2.22}$$

First, we have

$$\int_{\mathbb{R}}2(p(\bar{n}+\mathcal{E}_x)-p(\bar{n}))\mathcal{E}_{xt}dx=\frac{d}{dt}\int_{\mathbb{R}}\int_0^{\mathcal{E}_x}2(p(\bar{n}+\theta)-p(\bar{n}))d\theta dx, \tag{2.23}$$

and

$$(1+t)^\lambda\int_{\mathbb{R}}(p(\bar{n}+\mathcal{E}_x)-p(\bar{n}))\mathcal{E}_xdx\geq(1+t)^\lambda\int_{\mathbb{R}}p'(\bar{n})\mathcal{E}_x^2dx-C\varepsilon_1(1+t)^\lambda\|\mathcal{E}_x(t)\|^2 \tag{2.24}$$

after using Tayler's formula and (2.19). For the terms in the right-hand side of (2.22), it is easy to see that

$$K_{11}\lesssim\varepsilon_1(1+t)^{-\lambda}\|\mathcal{E}_t(t)\|^2, \tag{2.25}$$

and

$$K_{12}\lesssim(\|D'\|_2+\varepsilon_1)\big[(1+t)^{-\lambda}\|\mathcal{E}_t(t)\|^2+(1+t)^\lambda\|(\mathcal{E},\mathcal{E}_x)(t)\|^2\big] \tag{2.26}$$

from (2.19) and the estimate of $\bar{E}$ in (2.7). Since $-1<\lambda<0$ and $\bar{n}\geq D_*$, for all $t\geq 0$, one has

$$2(1+t)^{-\lambda}-(1+t)^\lambda\geq(1+t)^{-\lambda}, \tag{2.27}$$

and

$$(1+t)^\lambda\bar{n}+\frac{\lambda(\lambda-1)}{2}(1+t)^{\lambda-2}\geq(1+t)^\lambda D_*. \tag{2.28}$$

Now, putting (2.23)-(2.26) into (2.22) and combining (2.27)-(2.28), we get

$$\begin{aligned}
&\frac{d}{dt}Q_1(t)+(1+t)^{-\lambda}\|\mathcal{E}_t(t)\|^2+(1+t)^\lambda D_*\|\mathcal{E}(t)\|^2+\int_{\mathbb{R}}(1+t)^\lambda p'(\bar{n})\mathcal{E}_x^2dx\\
&\lesssim(\|D'\|_2+\varepsilon_1)\big[(1+t)^{-\lambda}\|\mathcal{E}_t(t)\|^2+(1+t)^\lambda\|(\mathcal{E},\mathcal{E}_x)(t)\|^2\big],
\end{aligned}$$

where

$$Q_1(t)=\int_{\mathbb{R}}\Big[(1+t)^\lambda\mathcal{E}\mathcal{E}_t-\frac{\lambda}{2}(1+t)^{\lambda-1}\mathcal{E}^2+\frac{1}{2}\mathcal{E}^2+\bar{n}\mathcal{E}^2+\mathcal{E}_t^2+2\int_0^{\mathcal{E}_x}(p(\bar{n}+\theta)-p(\bar{n}))d\theta\Big]dx.$$

It holds for $p'>0$ that

$$\int_0^{\mathcal{E}_x}(p(\bar{n}+\theta)-p(\bar{n}))d\theta\simeq\mathcal{E}_x^2. \tag{2.29}$$

By combining Cauchy-Schwarz inequality and (2.29), one gets

$$Q_1(t)\simeq\|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t)(t)\|^2. \tag{2.30}$$

Therefore, by using (2.6), $p'(\bar{n})>0$ and $\|D'\|_2+\varepsilon_1\ll 1$, there exits a constant $C_1>0$ such that

$$\frac{d}{dt}Q_1(t)+C_1\left[(1+t)^{-\lambda}\|\mathcal{E}_t(t)\|^2+(1+t)^{\lambda}\|(\mathcal{E},\mathcal{E}_x)(t)\|^2\right]\le 0. \tag{2.31}$$

Then, for any $t\in[0,T]$, integrating (2.31) over $(0,t)$ and applying (2.30), we obtain the desired estimates (2.21). This completes the proof of Lemma 2.2. ◻

LEMMA 2.3. *Under the conditions of Proposition 2.1, it holds that for any $t\in[0,T]$,*

$$\begin{aligned}&\|(\mathcal{E}_x,\mathcal{E}_{xx},\mathcal{E}_{xt})(t)\|^2+\int_0^t\left[(1+s)^{\lambda}\|\mathcal{E}_{xx}(s)\|^2+(1+s)^{-\lambda}\|\mathcal{E}_{xt}(s)\|^2\right]ds\\&\lesssim\|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t,\mathcal{E}_{xx},\mathcal{E}_{xt})(0)\|^2.\end{aligned} \tag{2.32}$$

Proof. Differentiating (2.14) with respect to x yields

$$\begin{aligned}&\mathcal{E}_{ttx}+(1+t)^{-\lambda}\mathcal{E}_{tx}+\bar{n}\mathcal{E}_x-(p(n)-p(\bar{n}))_{xx}\\&=-\bar{n}_x\mathcal{E}-(\mathcal{E}_x+\bar{E}_x)\mathcal{E}_x-(\mathcal{E}+\bar{E})\mathcal{E}_{xx}+\left(\frac{\mathcal{E}_t^2}{n}\right)_{xx}.\end{aligned} \tag{2.33}$$

Multiplying (2.33) by $(1+t)^{\lambda}\mathcal{E}_x+2\mathcal{E}_{xt}$ and integrating the resultant equation with respect to x over $\mathbb{R}$ by parts, we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\left[(1+t)^{\lambda}\mathcal{E}_x\mathcal{E}_{xt}-\frac{\lambda}{2}(1+t)^{\lambda-1}\mathcal{E}_x^2+\bar{n}\mathcal{E}_x^2+\frac{1}{2}\mathcal{E}_x^2+\mathcal{E}_{xt}^2\right]dx\\&\quad+\left[2(1+t)^{-\lambda}-(1+t)^{\lambda}\right]\|\mathcal{E}_{xt}(t)\|^2+\int_{\mathbb{R}}(p(n)-p(\bar{n}))_x\left[(1+t)^{\lambda}\mathcal{E}_{xx}+2\mathcal{E}_{xxt}\right]dx\\&=\int_{\mathbb{R}}\left[\frac{\lambda(1-\lambda)}{2}(1+t)^{\lambda-2}+(1+t)^{\lambda}\bar{n}\right]\mathcal{E}_x^2dx-\int_{\mathbb{R}}\left(\frac{\mathcal{E}_t^2}{n}\right)_x\left[(1+t)^{\lambda}\mathcal{E}_{xx}+2\mathcal{E}_{xxt}\right]dx\\&\quad-\int_{\mathbb{R}}(\bar{n}_x\mathcal{E}+\mathcal{E}\mathcal{E}_{xx}+\mathcal{E}_x^2+\bar{E}\mathcal{E}_{xx}+\bar{E}_x\mathcal{E}_x)\left[2\mathcal{E}_{xt}+(1+t)^{\lambda}\mathcal{E}_x\right]dx\\&=:K_{21}+K_{22}+K_{23}.\end{aligned} \tag{2.34}$$

Firstly, by (2.19) and the property of $\bar{n}$ in the estimate (2.7), we can estimate

$$\begin{aligned}&\int_{\mathbb{R}}2(p(n)-p(\bar{n}))_x\mathcal{E}_{xxt}dx+(1+t)^{\lambda}\int_{\mathbb{R}}(p(n)-p(\bar{n}))_x\mathcal{E}_{xx}dx\\&=\frac{d}{dt}\int_{\mathbb{R}}p'(n)\mathcal{E}_{xx}^2dx+(1+t)^{\lambda}\int_{\mathbb{R}}\left[p'(n)\mathcal{E}_{xx}^2+(p'(n)-p'(\bar{n}))\bar{n}_x\mathcal{E}_{xx}\right]dx-\int_{\mathbb{R}}p''(n)\mathcal{E}_{xx}^2\mathcal{E}_{xt}dx\\&\quad-\int_{\mathbb{R}}2\left[p''(n)\bar{n}_x\mathcal{E}_{xx}+(p'(n)-p'(\bar{n}))\bar{n}_{xx}+(p''(n)-p''(\bar{n}))\bar{n}_x^2\right]\mathcal{E}_{xt}dx\end{aligned}$$

$$\geq \frac{d}{dt}\int_{\mathbb{R}} p'(n)\mathcal{E}_{xx}^2 dx + (1+t)^\lambda \int_{\mathbb{R}} p'(n)\mathcal{E}_{xx}^2 dx \\ - C(\|D'\|_2 + \varepsilon_1)\big[(1+t)^\lambda \|\mathcal{E}_x(t)\|_1^2 + (1+t)^{-\lambda}\|\mathcal{E}_{xt}(t)\|^2\big], \tag{2.35}$$

where we handled the term $\int_{\mathbb{R}} p''(n)\mathcal{E}_{xx}^2\mathcal{E}_{xt}dx$ as

$$\int_{\mathbb{R}} p''(n)\mathcal{E}_{xx}^2\mathcal{E}_{xt}dx \lesssim (1+t)^{-\lambda}\|\mathcal{E}_{xt}(t)\|_{L^\infty(\mathbb{R})}(1+t)^\lambda\|\mathcal{E}_{xx}(t)\|^2 \\ \lesssim \varepsilon(1+t)^\lambda\|\mathcal{E}_{xx}(t)\|^2.$$

Secondly, by (2.7) and (2.19), we can estimate the terms appearing in the right-hand side of (2.34) as follows:

$$K_{21} \lesssim (1+t)^\lambda \|\mathcal{E}_x(t)\|^2. \tag{2.36}$$

$$K_{23} \lesssim (\|D'\|_2 + \varepsilon_1)\big[(1+t)^\lambda \|(\mathcal{E}, \mathcal{E}_x, \mathcal{E}_{xx})(t)\|^2 + (1+t)^{-\lambda}\|\mathcal{E}_{xt}(t)\|^2\big]. \tag{2.37}$$

$$K_{22} = \int_{\mathbb{R}} \Big[\frac{\mathcal{E}_t}{n^2}(\bar{n}_x + \mathcal{E}_{xx}) - \frac{2}{n}\mathcal{E}_{xt}\Big][(1+t)^\lambda \mathcal{E}_t\mathcal{E}_{xx} + 2\mathcal{E}_t\mathcal{E}_{xxt}]dx \\ \leq \frac{d}{dt}\int_{\mathbb{R}} \frac{\mathcal{E}_t^2}{n^2}\mathcal{E}_{xx}^2 dx + C(\|D'\|_2 + \varepsilon_1)\big[(1+t)^\lambda\|\mathcal{E}_{xx}(t)\|^2 + \|(\mathcal{E}_t, \mathcal{E}_{xt})(t)\|^2\big]. \tag{2.38}$$

Here, we have used that

$$\int_{\mathbb{R}} (1+t)^\lambda \Big[\frac{\mathcal{E}_t}{n^2}(\bar{n}_x + \mathcal{E}_{xx}) - \frac{2}{n}\mathcal{E}_{xt}\Big]\mathcal{E}_t\mathcal{E}_{xx}dx \lesssim (\|D'\|_2 + \varepsilon_1)(1+t)^\lambda\|(\mathcal{E}_t, \mathcal{E}_{xx})(t)\|^2,$$

and

$$\int_{\mathbb{R}} \Big[\frac{2}{n^2}\mathcal{E}_t^2(\bar{n}_x + \mathcal{E}_{xx}) - \frac{4}{n}\mathcal{E}_t\mathcal{E}_{xt}\Big]\mathcal{E}_{xxt}dx \\ = \frac{d}{dt}\int_{\mathbb{R}} \frac{\mathcal{E}_t^2}{n^2}\mathcal{E}_{xx}^2 dx - \int_{\mathbb{R}} \frac{2}{n^2}\mathcal{E}_t\mathcal{E}_{tt}\mathcal{E}_{xx}^2 dx + \int_{\mathbb{R}} \frac{2}{n^2}(n\mathcal{E}_{xt} - 3n\bar{n}_x\mathcal{E}_t - \mathcal{E}_t\mathcal{E}_{xx})\mathcal{E}_{xt}^2 dx \\ + \int_{\mathbb{R}} \frac{2}{n^3}(2\bar{n}_x^2\mathcal{E}_t + 2\bar{n}_x\mathcal{E}_t\mathcal{E}_{xx} + \mathcal{E}_t\mathcal{E}_{xx}^2 - n\bar{n}_{xx}\mathcal{E}_t)\mathcal{E}_t\mathcal{E}_{xt}dx \\ \leq \frac{d}{dt}\int_{\mathbb{R}} \frac{\mathcal{E}_t^2}{n^2}\mathcal{E}_{xx}^2 dx + C(\|D'\|_2 + \varepsilon_1)\big[(1+t)^\lambda\|\mathcal{E}_{xx}(t)\|^2 + \|(\mathcal{E}_t, \mathcal{E}_{xt})(t)\|^2\big],$$

where

$$-\int_{\mathbb{R}} \frac{2}{n^2}\mathcal{E}_t\mathcal{E}_{tt}\mathcal{E}_{xx}^2 dx \lesssim (1+t)^{-\frac{3}{2}\lambda}\|\mathcal{E}_t(t)\|_{L^\infty(\mathbb{R})}\|\mathcal{E}_{tt}(t)\|_{L^\infty(\mathbb{R})}(1+t)^{\frac{3}{2}\lambda}\|\mathcal{E}_{xx}(t)\|^2 \\ \lesssim \varepsilon_1(1+t)^\lambda\|\mathcal{E}_{xx}(t)\|^2.$$

Substituting (2.35)-(2.38) into (2.34) and employing (2.20), (2.27), $p'(n) > 0$ and $\|D'\|_2 + \varepsilon_1 \ll 1$, we have

$$\frac{d}{dt}Q_2(t) + C_2\big[(1+t)^\lambda\|\mathcal{E}_{xx}(t)\|^2 + (1+t)^{-\lambda}\|\mathcal{E}_{xt}(t)\|^2\big] \lesssim (1+t)^\lambda\|(\mathcal{E}, \mathcal{E}_x)(t)\|^2 + (1+t)^{-\lambda}\|\mathcal{E}_t(t)\|^2, \tag{2.39}$$

for some positive constant C_2, where

$$Q_2(t)=\int_{\mathbb{R}}\Big[(1+t)^{\lambda}\mathcal{E}_x\mathcal{E}_{xt}-\frac{\lambda}{2}(1+t)^{\lambda-1}\mathcal{E}_x^2+\bar{n}\mathcal{E}_x^2+\frac{1}{2}\mathcal{E}_x^2+\mathcal{E}_{xt}^2+p'(n)\mathcal{E}_{xx}^2-\frac{1}{n^2}\mathcal{E}_t^2\mathcal{E}_{xx}^2\Big]dx.$$

Using Cauchy-Schwarz inequality, we have

$$Q_2(t)\simeq\|(\mathcal{E}_x,\mathcal{E}_{xx},\mathcal{E}_{xt})(t)\|^2. \tag{2.40}$$

Then, for any $t\in[0,T]$, integrating (2.39) over $(0,t)$ and applying (2.40), we can get the desired estimates (2.32) by combining (2.21). This completes the proof of Lemma 2.3. □

Lemma 2.4. *Under the conditions of Proposition 2.1, it holds that for any $t\in[0,T]$,*

$$\begin{aligned}&(1+t)^{-\lambda}\|(\mathcal{E}_t,\mathcal{E}_{xt},\mathcal{E}_{tt})(t)\|^2+\int_0^t(1+s)^{-2\lambda}\|\mathcal{E}_{tt}(s)\|^2ds\\ &\lesssim\|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t,\mathcal{E}_{xx},\mathcal{E}_{xt})(0)\|^2.\end{aligned} \tag{2.41}$$

Proof. Differentiating (2.14) with respect to t gives

$$\begin{aligned}&\mathcal{E}_{ttt}+(1+t)^{-\lambda}\mathcal{E}_{tt}-\lambda(1+t)^{-\lambda-1}\mathcal{E}_t+\bar{n}\mathcal{E}_t-(p'(n)\mathcal{E}_{xt})_x\\ &=-\mathcal{E}\mathcal{E}_{xt}-\mathcal{E}_x\mathcal{E}_t-\bar{E}\mathcal{E}_{xt}+\left(\frac{\mathcal{E}_t^2}{n}\right)_{xt}.\end{aligned} \tag{2.42}$$

Multiplying (2.42) by $\mathcal{E}_t+2(1+t)^{-\lambda}\mathcal{E}_{tt}$ and integrating resulting equation with respect to x over $\mathbb{R}$ by parts, one gets

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\Big[\mathcal{E}_t\mathcal{E}_{tt}-\lambda(1+t)^{-2\lambda-1}\mathcal{E}_t^2+(1+t)^{-\lambda}\Big(\frac{1}{2}\mathcal{E}_t^2+\bar{n}\mathcal{E}_t^2+p'(n)\mathcal{E}_{xt}^2+\mathcal{E}_{tt}^2\Big)\Big]dx\\ &\quad+\big[2(1+t)^{-2\lambda}+\lambda(1+t)^{-\lambda-1}-1\big]\|\mathcal{E}_{tt}(t)\|^2+\big[1+\lambda(1+t)^{-\lambda-1}\big]\int_{\mathbb{R}}p'(n)\mathcal{E}_{xt}^2dx\\ &=-\int\Big[\lambda(1+t)^{-\lambda-1}\bar{n}+\bar{n}-\frac{\lambda}{2}(1+t)^{-\lambda-1}-\lambda(2\lambda+1)(1+t)^{-2\lambda-2}\Big]\mathcal{E}_t^2dx\\ &\quad-(1+t)^{-\lambda}\int_{\mathbb{R}}(2\mathcal{E}\mathcal{E}_{xt}\mathcal{E}_{tt}+2\mathcal{E}_x\mathcal{E}_t\mathcal{E}_{tt}+2\bar{E}\mathcal{E}_{xt}\mathcal{E}_{tt}+p''(n)\mathcal{E}_{xt}^3)dx\\ &\quad-\int_{\mathbb{R}}(\mathcal{E}\mathcal{E}_t\mathcal{E}_{xt}+\mathcal{E}_x\mathcal{E}_t^2+\bar{E}\mathcal{E}_t\mathcal{E}_{xt})dx-\int_{\mathbb{R}}\left(\frac{\mathcal{E}_t^2}{n}\right)_t\big[\mathcal{E}_{xt}+2(1+t)^{-\lambda}\mathcal{E}_{xtt}\big]dx\\ &=:K_{31}+K_{32}+K_{33}+K_{34}.\end{aligned} \tag{2.43}$$

It is easy to verify

$$K_{31}\lesssim\|\mathcal{E}_t(t)\|^2. \tag{2.44}$$

By (2.7) and (2.19), we can estimate the other terms in the right-hand side of (2.43) as follows:

$$K_{32}\lesssim(\|D'\|_2+\varepsilon_1)\big[\|(\mathcal{E}_t,\mathcal{E}_{xt})(t)\|^2+(1+t)^{-2\lambda}\|\mathcal{E}_{tt}(t)\|^2\big]. \tag{2.45}$$

$$K_{33}\lesssim(\|D'\|_2+\varepsilon_1)\|(\mathcal{E}_t,\mathcal{E}_{xt})(t)\|^2. \tag{2.46}$$

$$
\begin{aligned}
K_{34} &= \int_{\mathbb{R}} \Big(\frac{1}{n^2}\mathcal{E}_t^2\mathcal{E}_{xt}^2 - \frac{2}{n}\mathcal{E}_t\mathcal{E}_{xt}\mathcal{E}_{tt}\Big)dx + \int_{\mathbb{R}}(1+t)^{-\lambda}\Big(\frac{2}{n^2}\mathcal{E}_t^2\mathcal{E}_{xt}\mathcal{E}_{xtt} - \frac{4}{n}\mathcal{E}_t\mathcal{E}_{tt}\mathcal{E}_{xtt}\Big)dx \\
&= \frac{d}{dt}\int_{\mathbb{R}}(1+t)^{-\lambda}\frac{\mathcal{E}_t^2}{n^2}\mathcal{E}_{xt}^2dx + \lambda(1+t)^{-\lambda-1}\int_{\mathbb{R}}\frac{\mathcal{E}_t^2}{n^2}\mathcal{E}_{xt}^2dx + \int_{\mathbb{R}}\Big(\frac{\mathcal{E}_t^2}{n^2}\mathcal{E}_{xt}^2 - \frac{2}{n}\mathcal{E}_t\mathcal{E}_{xt}\mathcal{E}_{tt}\Big)dx \\
&\quad + \int_{\mathbb{R}}(1+t)^{-\lambda}\Big[\frac{2}{n^3}(\mathcal{E}_t^2\mathcal{E}_{xt} - n\mathcal{E}_t\mathcal{E}_{tt})\mathcal{E}_{xt}^2 - \frac{2}{n^2}(\mathcal{E}_t\mathcal{E}_{xx} + \bar{n}_x\mathcal{E}_t - n\mathcal{E}_{xt})\mathcal{E}_{tt}^2\Big]dx \\
&\le \frac{d}{dt}\int_{\mathbb{R}}(1+t)^{-\lambda}\frac{\mathcal{E}_t^2}{n^2}\mathcal{E}_{xt}^2dx + C(\|D'\|_2+\varepsilon_1)\big[\|\mathcal{E}_{xt}(t)\|^2 + (1+t)^{-\lambda}\|\mathcal{E}_{tt}(t)\|^2\big]. \qquad (2.47)
\end{aligned}
$$

From $-1<\lambda<0$, for any $t\ge 0$, one has the following two inequalities:

$$
1+\lambda(1+t)^{-\lambda-1} \ge 1+\lambda > 0. \qquad (2.48)
$$

$$
2(1+t)^{-2\lambda} + \lambda(1+t)^{-\lambda-1} - 1 \ge (1+\lambda)(1+t)^{-2\lambda}. \qquad (2.49)
$$

Analogous to (2.39), substituting (2.44)-(2.47) into (2.43) and combining (2.48)-(2.49), we have

$$
\frac{d}{dt}Q_3(t) + C_3\big[\|\mathcal{E}_{xt}(t)\|^2 + (1+t)^{-2\lambda}\|\mathcal{E}_{tt}(t)\|^2\big] \lesssim (1+t)^{-\lambda}\|\mathcal{E}_t(t)\|^2, \qquad (2.50)
$$

for some positive constant C_3, where

$$
\begin{aligned}
Q_3(t) &= \int_{\mathbb{R}}\Big[\mathcal{E}_t\mathcal{E}_{tt} - \lambda(1+t)^{-2\lambda-1}\mathcal{E}_t^2\Big]dx \\
&\quad + \int_{\mathbb{R}}(1+t)^{-\lambda}\Big(\frac{1}{2}\mathcal{E}_t^2 + \bar{n}\mathcal{E}_t^2 + p'(n)\mathcal{E}_{xt}^2 + \mathcal{E}_{tt}^2 - \frac{1}{n^2}\mathcal{E}_t^2\mathcal{E}_{xt}^2\Big)dx.
\end{aligned}
$$

Using Cauchy-Schwarz inequality, we have

$$
Q_3(t) \simeq (1+t)^{-\lambda}\|(\mathcal{E}_t,\mathcal{E}_{xt},\mathcal{E}_{tt})(t)\|^2. \qquad (2.51)
$$

Then, for any $t\in[0,T]$, integrating (2.50) over $(0,t)$ and applying (2.51), we have

$$
\begin{aligned}
&(1+t)^{-\lambda}\|(\mathcal{E}_t,\mathcal{E}_{xt},\mathcal{E}_{tt})(t)\|^2 + \int_0^t \big[\|\mathcal{E}_{xt}(s)\|^2 + (1+s)^{-2\lambda}\|\mathcal{E}_{tt}(s)\|^2\big]ds \\
&\lesssim \int_0^t (1+s)^{-\lambda}\|\mathcal{E}_t(s)\|^2ds + \|(\mathcal{E}_t,\mathcal{E}_{xt},\mathcal{E}_{tt})(0)\|^2 \\
&\lesssim \|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t,\mathcal{E}_{xx},\mathcal{E}_{xt})(0)\|^2. \qquad (2.52)
\end{aligned}
$$

Here, in order to get the last inequality in (2.52), we used (2.21) and the estimate

$$
\|\mathcal{E}_{tt}(0)\|^2 \lesssim \|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t,\mathcal{E}_{xt},\mathcal{E}_{xx})(0)\|^2,
$$

which comes from the Equation (2.14). Then, we get the estimate (2.41), which completes the proof of Lemma 2.4. ∎

Lemma 2.5. *Under the conditions of Proposition 2.1, it holds that for any $t\in[0,T]$,*

$$
\begin{aligned}
&\|(\mathcal{E}_{xx},\mathcal{E}_{xxx})(t)\|^2 + (1+t)^{-2\lambda}\|(\mathcal{E}_{xt},\mathcal{E}_{xxt},\mathcal{E}_{xtt})(t)\|^2 \\
&\quad + \int_0^t \big[(1+s)^{\lambda}\|\mathcal{E}_{xxx}(s)\|^2 + (1+s)^{-\lambda}\|\mathcal{E}_{xxt}(s)\|^2 + (1+s)^{-3\lambda}\|\mathcal{E}_{xtt}(s)\|^2\big]ds
\end{aligned}
$$

$$\lesssim \|\mathcal{E}(0)\|_3^2 + \|\mathcal{E}_t(0)\|_2^2. \tag{2.53}$$

Proof. Differentiating (2.14) in x twice yields

$$\begin{aligned}&\mathcal{E}_{ttxx} + (1+t)^{-\lambda}\mathcal{E}_{txx} + \bar{n}\mathcal{E}_{xx} - (p(n)-p(\bar{n}))_{xxx} + (\mathcal{E}+\bar{E})\mathcal{E}_{xxx}\\ &= -\bar{n}_{xx}\mathcal{E} - (2\bar{n}_x + \bar{E}_{xx})\mathcal{E}_x - (3\mathcal{E}_x + 2\bar{E}_x)\mathcal{E}_{xx} + \left(\frac{\mathcal{E}_t^2}{n}\right)_{xxx}.\end{aligned} \tag{2.54}$$

Multiplying (2.54) by $(1+t)^{\lambda}\mathcal{E}_{xx} + \frac{2}{1+\lambda}\mathcal{E}_{xxt}$ and integrating the resulting equality with respect to x over $\mathbb{R}$ by parts, we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\Big[(1+t)^{\lambda}\mathcal{E}_{xx}\mathcal{E}_{xxt} - \frac{\lambda}{2}(1+t)^{\lambda-1}\mathcal{E}_{xx}^2 + \frac{1}{2}\mathcal{E}_{xx}^2 + \frac{1}{1+\lambda}\mathcal{E}_{xxt}^2 + \frac{\bar{n}}{1+\lambda}\mathcal{E}_{xx}^2\Big]dx\\ &\quad + \Big[\frac{2}{1+\lambda}(1+t)^{-\lambda} - (1+t)^{\lambda}\Big]\|\mathcal{E}_{xxt}(t)\|^2 + \int_{\mathbb{R}}(p(n)-p(\bar{n}))_{xx}\Big[(1+t)^{\lambda}\mathcal{E}_{xxx} + \frac{2\mathcal{E}_{xxxt}}{1+\lambda}\Big]dx\\ &= -\int_{\mathbb{R}}\big[(\mathcal{E}+\bar{E})\mathcal{E}_{xxx} + (2\bar{n}_x + \bar{E}_{xx})\mathcal{E}_x + \bar{n}_{xx}\mathcal{E} + (3\mathcal{E}_x + 2\bar{E}_x)\mathcal{E}_{xx}\big]\Big[(1+t)^{\lambda}\mathcal{E}_{xx} + \frac{2\mathcal{E}_{xxt}}{1+\lambda}\Big]dx\\ &\quad - \int_{\mathbb{R}}\Big[\frac{\lambda(\lambda-1)}{2}(1+t)^{\lambda-2} + (1+t)^{\lambda}\bar{n}\Big]\mathcal{E}_{xx}^2 dx + \int_{\mathbb{R}}\left(\frac{\mathcal{E}_t^2}{n}\right)_{xxx}\Big[(1+t)^{\lambda}\mathcal{E}_{xx} + \frac{2\mathcal{E}_{xxt}}{1+\lambda}\Big]dx\\ &=: K_{41} + K_{42} + K_{43}.\end{aligned} \tag{2.55}$$

A straightforward calculation yields

$$(p(n)-p(\bar{n}))_{xx} = p'(n)\mathcal{E}_{xxx} + p''(n)\mathcal{E}_{xx}^2 + 2p''(n)\bar{n}_x\mathcal{E}_{xx} + (p'(n)-p'(\bar{n}))\bar{n}_{xx} + (p''(n)-p''(\bar{n}))\bar{n}_x^2.$$

Then, as similar with (2.35), we can estimate

$$\begin{aligned}&\int_{\mathbb{R}}\frac{2}{1+\lambda}(p(n)-p(\bar{n}))_{xx}\mathcal{E}_{xxxt}dx\\ &= \frac{d}{dt}\int_{\mathbb{R}}\frac{p'(n)}{1+\lambda}\mathcal{E}_{xxx}^2 dx + (1+t)^{\lambda}\int_{\mathbb{R}}p'(n)\mathcal{E}_{xxx}^2 dx - \int_{\mathbb{R}}\frac{p''(n)}{1+\lambda}\mathcal{E}_{xt}\mathcal{E}_{xxx}^2 dx - \int_{\mathbb{R}}\frac{4p''(n)}{1+\lambda}n_x\mathcal{E}_{xxx}\mathcal{E}_{xxt}dx\\ &\quad - \frac{2}{1+\lambda}\int_{\mathbb{R}}\big[p'''(n)(3\bar{n}_x\mathcal{E}_{xx} + 3\bar{n}_x^2 + \mathcal{E}_{xx}^2) + 3p''(n)\bar{n}_{xx}\big]\mathcal{E}_{xx}\mathcal{E}_{xxt}dx\\ &\quad - \frac{2}{1+\lambda}\int_{\mathbb{R}}\big[(p'(n)-p'(\bar{n}))\bar{n}_{xxx} + 3(p''(n)-p''(\bar{n}))\bar{n}_x\bar{n}_{xx} + (p'''(n)-p'''(\bar{n}))\bar{n}_x^3\big]\mathcal{E}_{xxt}dx\\ &\quad + (1+t)^{\lambda}\int_{\mathbb{R}}\big[p''(n)\mathcal{E}_{xx}^2 + 2p''(n)\bar{n}_x\mathcal{E}_{xx} + (p'(n)-p'(\bar{n}))\bar{n}_{xx} + (p''(n)-p''(\bar{n}))\bar{n}_x^2\big]\mathcal{E}_{xxx}dx\\ &\ge \frac{d}{dt}\int_{\mathbb{R}}\frac{p'(n)}{1+\lambda}\mathcal{E}_{xxx}^2 dx + (1+t)^{\lambda}\int_{\mathbb{R}}p'(n)\mathcal{E}_{xxx}^2 dx\\ &\quad - C(\|D'\|_2 + \varepsilon_1)\big[(1+t)^{\lambda}\|\mathcal{E}_x(t)\|_2^2 + (1+t)^{-\lambda}\|\mathcal{E}_{xxt}(t)\|^2\big].\end{aligned} \tag{2.56}$$

From (2.7) and (2.19), the terms in the right-hand side of (2.55) can be dealt as follows:

$$K_{41} \lesssim (\|D'\|_2 + \varepsilon_1)\big[(1+t)^{\lambda}\|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_{xx},\mathcal{E}_{xxx})(t)\|^2 + (1+t)^{-\lambda}\|\mathcal{E}_{xxt}(t)\|^2\big]. \tag{2.57}$$

$$K_{42} \lesssim (1+t)^{\lambda}\|\mathcal{E}_{xx}(t)\|^2. \tag{2.58}$$

$$\begin{aligned}K_{43} &= -\int_{\mathbb{R}}(1+t)^{\lambda}\left(\frac{\mathcal{E}_t^2}{n}\right)_{xx}\mathcal{E}_{xxx}dx - \frac{2}{1+\lambda}\int_{\mathbb{R}}\left(\frac{\mathcal{E}_t^2}{n}\right)_{xx}\mathcal{E}_{xxxt}dx\\ &\le \frac{d}{dt}\int_{\mathbb{R}}\frac{\mathcal{E}_t^2\mathcal{E}_{xxx}^2}{(1+\lambda)n^2}dx + C(\|D'\|_2 + \varepsilon_1)\big[(1+t)^{\lambda}\|\mathcal{E}_{xxx}(t)\|^2 + (1+t)^{-\lambda}\|\mathcal{E}_t(t)\|_2^2\big].\end{aligned} \tag{2.59}$$

Here, by noticing that

$$\left(\frac{\mathcal{E}_t^2}{n}\right)_{xx}=\frac{2}{n}(\mathcal{E}_t\mathcal{E}_{xxt}+\mathcal{E}_{xt}^2)+\frac{2}{n^3}\mathcal{E}_t^2(\bar{n}_x+\mathcal{E}_{xx})^2 \\ -\frac{1}{n^2}(4\mathcal{E}_t\mathcal{E}_{xt}\mathcal{E}_{xx}+4\bar{n}_x\mathcal{E}_t\mathcal{E}_{xt}+\bar{n}_{xx}\mathcal{E}_t^2+\mathcal{E}_t^2\mathcal{E}_{xxx}),$$

we have

$$\begin{aligned}&-2\int_{\mathbb{R}}\left(\frac{\mathcal{E}_t^2}{n}\right)_{xx}\mathcal{E}_{xxxt}dx\\ =&\frac{d}{dt}\int_{\mathbb{R}}\frac{\mathcal{E}_t^2}{n^2}\mathcal{E}_{xxx}^2dx+\int_{\mathbb{R}}\frac{2}{n^3}(\mathcal{E}_t^2\mathcal{E}_{xt}-n\mathcal{E}_t\mathcal{E}_{tt})\mathcal{E}_{xxx}^2dx-\int_{\mathbb{R}}\frac{2}{n^2}(\mathcal{E}_t\mathcal{E}_{xx}+\bar{n}_x\mathcal{E}_t-n\mathcal{E}_{xt})\mathcal{E}_{xxt}^2dx\\ &+2\int_{\mathbb{R}}\left[\frac{2\mathcal{E}_{xt}^2}{n}+\frac{2\mathcal{E}_t^2}{n^3}(\bar{n}_x+\mathcal{E}_{xx})^2-\frac{1}{n^2}(4\mathcal{E}_t\mathcal{E}_{xt}\mathcal{E}_{xx}+4\bar{n}_x\mathcal{E}_t\mathcal{E}_{xt}+\bar{n}_{xx}\mathcal{E}_t^2)\right]_x\mathcal{E}_{xxt}dx\\ \le&\frac{d}{dt}\int_{\mathbb{R}}\frac{\mathcal{E}_t^2}{n^2}\mathcal{E}_{xxx}^2dx+C(\|D'\|_2+\varepsilon_1)\left[(1+t)^{\lambda}\|\mathcal{E}_{xxx}(t)\|^2+(1+t)^{-\lambda}\|(\mathcal{E}_t,\mathcal{E}_{xt},\mathcal{E}_{xxt})(t)\|^2\right],\end{aligned}$$

where we have used that

$$\int_{\mathbb{R}}\frac{2}{n^3}(\mathcal{E}_t^2\mathcal{E}_{xt}-n\mathcal{E}_t\mathcal{E}_{tt})\mathcal{E}_{xxx}^2dx\lesssim\varepsilon_1(1+t)^{\lambda}\|\mathcal{E}_{xxx}(t)\|^2,$$

and

$$-\int_{\mathbb{R}}\frac{2}{n^2}(\mathcal{E}_t\mathcal{E}_{xx}+\bar{n}_x\mathcal{E}_t-n\mathcal{E}_{xt})\mathcal{E}_{xxt}^2dx\lesssim(\|D'\|_2+\varepsilon_1)\|\mathcal{E}_{xxt}(t)\|^2,$$

and

$$\begin{aligned}&\int_{\mathbb{R}}\left[\frac{2\mathcal{E}_{xt}^2}{n}+\frac{2\mathcal{E}_t^2}{n^3}(\bar{n}_x+\mathcal{E}_{xx})^2-\frac{1}{n^2}(4\mathcal{E}_t\mathcal{E}_{xt}\mathcal{E}_{xx}+4\bar{n}_x\mathcal{E}_t\mathcal{E}_{xt}+\bar{n}_{xx}\mathcal{E}_t^2)\right]_x\mathcal{E}_{xxt}dx\\ =&\int_{\mathbb{R}}\left[\frac{2}{n^3}(4\bar{n}_x^2+\bar{n}_x\mathcal{E}_{xx}+2\bar{n}_x\bar{n}_{xx}+2\bar{n}_{xx}\mathcal{E}_{xx})-\frac{1}{n^2}\bar{n}_{xxx}-\frac{6}{n^4}(\bar{n}_x+\mathcal{E}_{xx})^3\right]\mathcal{E}_t^2\mathcal{E}_{xxt}dx\\ &+\int_{\mathbb{R}}\left[\frac{12}{n^3}(2\bar{n}_x\mathcal{E}_t\mathcal{E}_{xx}+\bar{n}_x^2\mathcal{E}_t+\mathcal{E}_t\mathcal{E}_{xx}^2)-\frac{6}{n^2}(\mathcal{E}_{xx}\mathcal{E}_{xt}+\bar{n}_x\mathcal{E}_{xt}+\bar{n}_{xx}\mathcal{E}_t)\right]\mathcal{E}_{xt}\mathcal{E}_{xxt}dx\\ &+\int_{\mathbb{R}}\frac{2}{n^3}(2\bar{n}_x\mathcal{E}_t^2+2\mathcal{E}_t^2\mathcal{E}_{xx}-2n\mathcal{E}_t\mathcal{E}_{xt})\mathcal{E}_{xxx}\mathcal{E}_{xxt}dx-\int_{\mathbb{R}}\frac{4}{n^2}(\mathcal{E}_t\mathcal{E}_{xx}+\bar{n}_x\mathcal{E}_t-n\mathcal{E}_{xt})\mathcal{E}_{xxt}^2dx\\ \lesssim&(\|D'\|_2+\varepsilon_1)\left[(1+t)^{\lambda}\|\mathcal{E}_{xxx}(t)\|^2+(1+t)^{-\lambda}\|(\mathcal{E}_t,\mathcal{E}_{xt},\mathcal{E}_{xxt})(t)\|^2\right].\end{aligned}$$

Substituting (2.56)-(2.59) into (2.55), we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\left\{(1+t)^{\lambda}\mathcal{E}_{xx}\mathcal{E}_{xxt}+\frac{1}{2}\left[1-\lambda(1+t)^{\lambda-1}\right]\mathcal{E}_{xx}^2\right\}dx\\ &\quad+\frac{d}{dt}\int_{\mathbb{R}}\frac{1}{1+\lambda}\left[\mathcal{E}_{xxt}^2+\bar{n}\mathcal{E}_{xx}^2+\left(p'(n)-\frac{\mathcal{E}_t^2}{n^2}\right)\mathcal{E}_{xxx}^2\right]dx\\ &\quad+\left[\frac{2}{1+\lambda}(1+t)^{-\lambda}-(1+t)^{\lambda}\right]\|\mathcal{E}_{xxt}(t)\|^2+(1+t)^{\lambda}\int_{\mathbb{R}}p'(n)\mathcal{E}_{xxx}^2dx\\ \lesssim&(\|D'\|_2+\varepsilon_1)\left[(1+t)^{\lambda}\|\mathcal{E}_{xxx}(t)\|^2+(1+t)^{-\lambda}\|\mathcal{E}_{xxt}(t)\|^2\right]+(1+t)^{\lambda}\|\mathcal{E}(t)\|_2^2+\frac{\|\mathcal{E}_t(t)\|_1^2}{(1+t)^{\lambda}}.\end{aligned}\tag{2.60}$$

Differentiating (2.42) in x yields

$$\begin{aligned}&\mathcal{E}_{tttx}+(1+t)^{-\lambda}\mathcal{E}_{ttx}-\lambda(1+t)^{-\lambda-1}\mathcal{E}_{tx}+\bar{n}\mathcal{E}_{xt}-(p'(n)\mathcal{E}_{xxt})_x\\ =&-(\bar{n}_x\mathcal{E}_t+\mathcal{E}\mathcal{E}_{xxt}+\mathcal{E}_t\mathcal{E}_{xx}+2\mathcal{E}_x\mathcal{E}_{xt}+\bar{E}\mathcal{E}_{xxt}+\bar{E}_x\mathcal{E}_{xt})\\ &+(p''(n)\bar{n}_x\mathcal{E}_{xt}+p''(n)\mathcal{E}_{xx}\mathcal{E}_{xt})_x+\left(\frac{\mathcal{E}_t^2}{n}\right)_{xxt}.\end{aligned}\tag{2.61}$$

Multiplying (2.61) by $(1+t)^{-\lambda}\mathcal{E}_{xt}+\frac{2}{1+\lambda}(1+t)^{-2\lambda}\mathcal{E}_{xtt}$ and integrating it by parts with respect to x over $\mathbb{R}$, we obtain that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\Big[(1+t)^{-\lambda}\mathcal{E}_{xt}\mathcal{E}_{xtt}+\frac{\lambda}{2}(1+t)^{-\lambda-1}\mathcal{E}_{xt}^2-\frac{\lambda}{1+\lambda}(1+t)^{-3\lambda-1}\mathcal{E}_{xt}^2\\ &\quad+\frac{1}{1+\lambda}(1+t)^{-2\lambda}\Big(\mathcal{E}_{xtt}^2+p'(n)\mathcal{E}_{xxt}^2+\bar{n}\mathcal{E}_{xt}^2+\frac{1+\lambda}{2}\mathcal{E}_{xt}^2\Big)\Big]dx\\ &\quad+\Big[(1+t)^{-\lambda}+\frac{2\lambda}{1+\lambda}(1+t)^{-2\lambda-1}\Big]\int_{\mathbb{R}}p'(n)\mathcal{E}_{xxt}^2dx\\ &\quad+\Big[\frac{2}{1+\lambda}(1+t)^{-3\lambda}-(1+t)^{-\lambda}+\frac{2\lambda}{1+\lambda}(1+t)^{-2\lambda-1}\Big]\|\mathcal{E}_{xtt}(t)\|^2\\ =:&\sum_{m=1}^{5}K_{5m},\end{aligned}\tag{2.62}$$

where

$$\begin{aligned}K_{51}=&\int_{\mathbb{R}}\Big[\frac{(1+\lambda-2\bar{n})\lambda}{(1+\lambda)(1+t)^{\lambda+1}}+\frac{\lambda(3\lambda+1)}{(1+\lambda)(1+t)^{2(\lambda+1)}}-\frac{\lambda(\lambda+1)}{2(1+t)^2}-\bar{n}\Big](1+t)^{-\lambda}\mathcal{E}_{xt}^2dx\\ \lesssim&(1+t)^{-\lambda}\|\mathcal{E}_{xt}(t)\|^2,\end{aligned}\tag{2.63}$$

$$\begin{aligned}K_{52}=&-\int_{\mathbb{R}}(1+t)^{-\lambda}\big[(\bar{n}_x+\mathcal{E}_{xx})\mathcal{E}_t+(2\mathcal{E}_x+\bar{E}_x)\mathcal{E}_{xt}\big]\mathcal{E}_{xt}dx\\ &-\int_{\mathbb{R}}(1+t)^{-\lambda}\big[(\mathcal{E}+\bar{E}+p''(n)\bar{n}_x+p''(n)\mathcal{E}_{xx})\mathcal{E}_{xxt}\big]\mathcal{E}_{xt}dx\\ \lesssim&(\|D'\|_2+\varepsilon_1)(1+t)^{-\lambda}\|(\mathcal{E}_t,\mathcal{E}_{xt},\mathcal{E}_{xxt})(t)\|^2,\end{aligned}\tag{2.64}$$

$$\begin{aligned}K_{53}=&-\int_{\mathbb{R}}\frac{1}{1+\lambda}(1+t)^{-2\lambda}\big\{2[p''(n)(\bar{n}_x+\mathcal{E}_{xx})+\bar{E}+\mathcal{E}]\mathcal{E}_{xxt}\mathcal{E}_{xtt}+2(\bar{n}_x+\mathcal{E}_{xx})\mathcal{E}_t\mathcal{E}_{xtt}\big\}dx\\ &-\int_{\mathbb{R}}\frac{1}{1+\lambda}(1+t)^{-2\lambda}\big[2p''(n)\bar{n}_{xx}+4\mathcal{E}_x+2\bar{E}_x+p'''(n)(\bar{n}_x+\mathcal{E}_{xx})^2\big]\mathcal{E}_{xt}\mathcal{E}_{xtt}dx\\ \lesssim&(\|D'\|_2+\varepsilon_1)\big[(1+t)^{-3\lambda}\|\mathcal{E}_{xtt}(t)\|^2+(1+t)^{-\lambda}\|(\mathcal{E}_t,\mathcal{E}_{xt},\mathcal{E}_{xxt})(t)\|^2\big],\end{aligned}\tag{2.65}$$

$$\begin{aligned}K_{54}=&-\int_{\mathbb{R}}\frac{1}{1+\lambda}(1+t)^{-2\lambda}\big(p''(n)\mathcal{E}_{xt}\mathcal{E}_{xxt}^2+2p''(n)\mathcal{E}_{xt}\mathcal{E}_{xxx}\mathcal{E}_{xtt}\big)dx\\ \lesssim&(1+t)^{-\lambda}\|\mathcal{E}_{xt}(t)\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}\big[(1+t)^{-\lambda}\mathcal{E}_{xxt}^2+(1+t)^{-\lambda}\mathcal{E}_{xxx}\mathcal{E}_{xtt}\big]dx\\ \lesssim&\varepsilon_1\big[(1+t)^{-3\lambda}\|\mathcal{E}_{xtt}(t)\|^2+(1+t)^{-\lambda}\|\mathcal{E}_{xxt}(t)\|^2+(1+t)^{\lambda}\|\mathcal{E}_{xxx}(t)\|^2\big],\end{aligned}\tag{2.66}$$

and

$$\begin{aligned}K_{55}=&\int_{\mathbb{R}}\left(\frac{\mathcal{E}_t^2}{n}\right)_{xxt}\Big[(1+t)^{-\lambda}\mathcal{E}_{xt}+\frac{2}{1+\lambda}(1+t)^{-2\lambda}\mathcal{E}_{xtt}\Big]dx\\=&-\int_{\mathbb{R}}(1+t)^{-\lambda}\left(\frac{\mathcal{E}_t^2}{n}\right)_{xt}\mathcal{E}_{xxt}dx-\int_{\mathbb{R}}\frac{2}{1+\lambda}(1+t)^{-2\lambda}\left(\frac{\mathcal{E}_t^2}{n}\right)_{xt}\mathcal{E}_{xxtt}dx\\\le&\frac{d}{dt}\int_{\mathbb{R}}\frac{1}{1+\lambda}(1+t)^{-2\lambda}\frac{\mathcal{E}_t^2}{n^2}\mathcal{E}_{xxt}^2dx+C(\|D'\|_2+\varepsilon_1)\big[(1+t)^{-\lambda}\|(\mathcal{E}_{xt},\mathcal{E}_{tt},\mathcal{E}_{xxt})(t)\|^2\\&+(1+t)^{-3\lambda}\|\mathcal{E}_{xtt}(t)\|^2+(1+t)^{\lambda}\|\mathcal{E}_{xxx}(t)\|^2\big].\end{aligned}\tag{2.67}$$

Substituting (2.63)-(2.67) into (2.62), we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\Big\{(1+t)^{-\lambda}\mathcal{E}_{xt}\mathcal{E}_{xtt}+\Big[\frac{\lambda}{2}(1+t)^{-\lambda-1}-\frac{\lambda}{1+\lambda}(1+t)^{-3\lambda-1}\Big]\mathcal{E}_{xt}^2\\&\quad+\frac{1}{1+\lambda}(1+t)^{-2\lambda}\Big[\mathcal{E}_{xtt}^2+\Big(\bar{n}+\frac{1+\lambda}{2}\Big)\mathcal{E}_{xt}^2+\Big(p'(n)-\frac{\mathcal{E}_t^2}{n^2}\Big)\mathcal{E}_{xxt}^2\Big]\Big\}dx\\&\quad+\Big[(1+t)^{-\lambda}+\frac{2\lambda}{1+\lambda}(1+t)^{-2\lambda-1}\Big]\int_{\mathbb{R}}p'(n)\mathcal{E}_{xxt}^2dx\\&\quad+\Big[\frac{2}{1+\lambda}(1+t)^{-3\lambda}-(1+t)^{-\lambda}+\frac{2\lambda}{1+\lambda}(1+t)^{-2\lambda-1}\Big]\|\mathcal{E}_{xtt}(t)\|^2\\&\lesssim(\|D'\|_2+\varepsilon_1)\big[(1+t)^{-3\lambda}\|\mathcal{E}_{xtt}(t)\|^2+(1+t)^{-\lambda}\|\mathcal{E}_{xxt}(t)\|^2+(1+t)^{\lambda}\|\mathcal{E}_{xxx}(t)\|^2\big]\\&\quad+(1+t)^{-\lambda}\|(\mathcal{E}_t,\mathcal{E}_{xt},\mathcal{E}_{tt})(t)\|^2.\end{aligned}\tag{2.68}$$

Taking (2.60)+(2.68), we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}Q_4(x,t)dx+\Big[\frac{2}{1+\lambda}(1+t)^{-3\lambda}-(1+t)^{-\lambda}+\frac{2\lambda}{1+\lambda}(1+t)^{-2\lambda-1}\Big]\|\mathcal{E}_{xtt}(t)\|^2\\&\quad+\Big[\frac{3+\lambda}{1+\lambda}(1+t)^{-\lambda}-(1+t)^{\lambda}+\frac{2\lambda}{1+\lambda}(1+t)^{-2\lambda-1}\Big]\|\mathcal{E}_{xxt}(t)\|^2+\int_{\mathbb{R}}(1+t)^{\lambda}p'(n)\mathcal{E}_{xxx}^2dx\\&\lesssim(\|D'\|_2+\varepsilon_1)\big[(1+t)^{-3\lambda}\|\mathcal{E}_{xtt}(t)\|^2+(1+t)^{-\lambda}\|\mathcal{E}_{xxt}(t)\|^2+(1+t)^{\lambda}\|\mathcal{E}_{xxx}(t)\|^2\big]\\&\quad+(1+t)^{-\lambda}\|(\mathcal{E}_t,\mathcal{E}_{xt},\mathcal{E}_{tt})(t)\|^2+(1+t)^{\lambda}\|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_{xx})(t)\|^2,\end{aligned}\tag{2.69}$$

where

$$\begin{aligned}Q_4(x,t)=&(1+t)^{-\lambda}\mathcal{E}_{xt}\mathcal{E}_{xtt}+\frac{1}{1+\lambda}(1+t)^{-2\lambda}\Big[\mathcal{E}_{xtt}^2+p'(n)\mathcal{E}_{xxt}^2+\Big(\bar{n}+\frac{1+\lambda}{2}\Big)\mathcal{E}_{xt}^2-\frac{\mathcal{E}_t^2}{n^2}\mathcal{E}_{xxt}^2\Big]\\&+\Big[\frac{\lambda}{2}(1+t)^{-\lambda-1}-\frac{\lambda}{1+\lambda}(1+t)^{-3\lambda-1}\Big]\mathcal{E}_{xt}^2+(1+t)^{\lambda}\mathcal{E}_{xx}\mathcal{E}_{xxt}\\&+\frac{1}{2}\big[1-\lambda(1+t)^{\lambda-1}\big]\mathcal{E}_{xx}^2+\frac{1}{1+\lambda}\Big(\bar{n}\mathcal{E}_{xx}^2+\mathcal{E}_{xxt}^2+p'(n)\mathcal{E}_{xxx}^2-\frac{1}{n^2}\mathcal{E}_t^2\mathcal{E}_{xxx}^2\Big),\end{aligned}$$

and

$$\int_{\mathbb{R}}Q_4(x,t)dx\simeq\|(\mathcal{E}_{xx},\mathcal{E}_{xxx})(t)\|^2+(1+t)^{-2\lambda}\|(\mathcal{E}_{xt},\mathcal{E}_{xxt},\mathcal{E}_{xtt})(t)\|^2.\tag{2.70}$$

By $-1<\lambda<0$, for any $t\ge0$, we have

$$\frac{3+\lambda}{1+\lambda}(1+t)^{-\lambda}-(1+t)^{\lambda}+\frac{2\lambda}{1+\lambda}(1+t)^{-2\lambda-1}\ge2(1+t)^{-\lambda},\tag{2.71}$$

and

$$\frac{2}{1+\lambda}(1+t)^{-3\lambda}-(1+t)^{-\lambda}+\frac{2\lambda}{1+\lambda}(1+t)^{-2\lambda-1}\geq(1+t)^{-3\lambda}. \tag{2.72}$$

Then, putting (2.71)-(2.72) into (2.69) and using the smallness of $\|D'\|_2+\varepsilon_1$, for some positive constant C_4, we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}Q_4(x,t)dx+C_4\left[(1+t)^{-3\lambda}\|\mathcal{E}_{xtt}(t)\|^2+(1+t)^{-\lambda}\|\mathcal{E}_{xxt}(t)\|^2+(1+t)^{\lambda}\|\mathcal{E}_{xxx}(t)\|^2\right]\\&\lesssim(1+t)^{\lambda}\|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_{xx})(t)\|^2+(1+t)^{-\lambda}\|(\mathcal{E}_t,\mathcal{E}_{xt},\mathcal{E}_{tt})(t)\|^2. \end{aligned}\tag{2.73}$$

From the Equation (2.33), we can get

$$\|\mathcal{E}_{xtt}(0)\|^2\lesssim\|\mathcal{E}(0)\|_3^2+\|\mathcal{E}_t(0)\|_2^2.$$

Then, for any $t\in[0,T]$, integrating (2.73) over $(0,t)$ and applying (2.70), we finally obtain the desired high-order estimate (2.53) by combining the estimates (2.21), (2.32) and (2.41). This completes the proof of Lemma 2.5. ∎

Proof of Proposition 2.1. Combining Lemmas 2.2-2.5 implies Proposition 2.1.

2.2. Decay rates. When the estimate (2.17) in Proposition 2.1 holds for all $t\geq0$, from Sobolev inequality, it holds that

$$\begin{aligned}&\sup_{t\geq0}\left[\sum_{k=0}^{2}\|\partial_x^k\mathcal{E}(t)\|_{L^\infty(\mathbb{R})}+(1+t)^{-\frac{3}{4}\lambda}\|(\mathcal{E}_t,\mathcal{E}_{tt})(t)\|_{L^\infty(\mathbb{R})}+(1+t)^{-\lambda}\|\mathcal{E}_{xt}(t)\|_{L^\infty(\mathbb{R})}\right]\\&\lesssim\Phi_0,\end{aligned}\tag{2.74}$$

where $\Phi_0:=\|\rho_0\|_3+\|J_0\|_2$.

Note: *There exist positive constants T_0 and c_0 only dependent on λ and D_*, such that*

$$\begin{cases}2(1+t)^{-\lambda}-1\geq(1+t)^{-\lambda}\geq1,\\ \bar{n}+\dfrac{\lambda}{2}(1+t)^{-\lambda-1}\geq c_0,\end{cases}\qquad\text{for } t>T_0. \tag{2.75}$$

Obviously, $(1+t)^{-\lambda}\geq1$ and $(2.75)_1$ holds for all $t\geq0$ since $\lambda\in(-1,0)$. If $D_*>\frac{|\lambda|}{2}=-\frac{\lambda}{2}$, then, for all $t\geq0$,

$$\bar{n}+\frac{\lambda}{2}(1+t)^{-\lambda-1}\geq D_*+\frac{\lambda}{2}(1+t)^{-\lambda-1}\geq D_*+\frac{\lambda}{2}>0.$$

If $D_*<\frac{|\lambda|}{2}=-\frac{\lambda}{2}$, let $T_0=(-\frac{\lambda}{2D_*})^{\frac{1}{\lambda+1}}-1$, then for any $t>T_0$,

$$\bar{n}+\frac{\lambda}{2}(1+t)^{-\lambda-1}>D_*+\frac{\lambda}{2}(1+T_0)^{-\lambda-1}>0.$$

In conclusion, we can get $(2.75)_2$.

Proof. **(Proof of Proposition 2.2.)** Next, we further derive the sub-exponential decay estimates based on (2.74) and (2.75).

Step 1: Multiply (2.14) by $\mathcal{E}+2\mathcal{E}_t$ and integrate it over $\mathbb{R}$ by parts to obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\Big[\mathcal{E}\mathcal{E}_t+\frac{1}{2}(1+t)^{-\lambda}\mathcal{E}^2+\mathcal{E}_t^2+\bar{n}\mathcal{E}^2\Big]dx+\big[2(1+t)^{-\lambda}-1\big]\|\mathcal{E}_t(t)\|^2\\&\quad+\int_{\mathbb{R}}\Big[\bar{n}+\frac{\lambda}{2}(1+t)^{-\lambda-1}\Big]\mathcal{E}^2dx+\int_{\mathbb{R}}(p(n)-p(\bar{n}))(\mathcal{E}_x+2\mathcal{E}_{xt})dx\\&=-\int_{\mathbb{R}}\Big(\mathcal{E}_x\mathcal{E}^2+\bar{E}\mathcal{E}\mathcal{E}_x+2\bar{E}\mathcal{E}_x\mathcal{E}_t+2\mathcal{E}\mathcal{E}_x\mathcal{E}_t+\frac{1}{n}\mathcal{E}_t^2\mathcal{E}_x+\frac{2}{n}\mathcal{E}_t^2\mathcal{E}_{xt}\Big)dx=:I_0.\end{aligned}\tag{2.76}$$

By Taylor's formula and (2.74) we can estimate

$$\int_{\mathbb{R}}(p(n)-p(\bar{n}))(\mathcal{E}_x+2\mathcal{E}_{xt})dx\geq\frac{d}{dt}\int_{\mathbb{R}}p'(\bar{n})\mathcal{E}_x^2dx+\int_{\mathbb{R}}p'(\bar{n})\mathcal{E}_x^2dx-C\Phi_0\|\mathcal{E}_x(t)\|^2.\tag{2.77}$$

From (2.7) and (2.74), we have

$$I_0\lesssim(\Phi_0+\|D'\|_2)\|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t)(t)\|^2.\tag{2.78}$$

Hence, substituting (2.77)-(2.78) into (2.76), and combining (2.75) and the smallness of $\Phi_0+\|D'\|_2$ shows that

$$\frac{d}{dt}F_1(t)+C_5\|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t)(t)\|^2\leq 0,\tag{2.79}$$

for some constant $C_5>0$, where

$$F_1(t)=\int_{\mathbb{R}}\Big[\mathcal{E}\mathcal{E}_t+\frac{1}{2}(1+t)^{-\lambda}\mathcal{E}^2+\bar{n}\mathcal{E}^2+\mathcal{E}_t^2+p'(\bar{n})\mathcal{E}_x^2\Big]dx.$$

Now, by Cauchy-Schwarz inequality, there holds

$$\|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t)(t)\|^2\lesssim F_1(t)\lesssim(1+t)^{-\lambda}\|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t)(t)\|^2.\tag{2.80}$$

Thus, (2.79) and (2.80) imply that

$$\frac{d}{dt}F_1(t)+c_1(1+t)^{\lambda}F_1(t)\leq 0,$$

for some constant $c_1>0$. Moreover,

$$\frac{d}{dt}\Big[e^{\frac{c_1}{\lambda+1}(1+t)^{\lambda+1}}F_1(t)\Big]\leq 0.\tag{2.81}$$

Let $\alpha_1:=\frac{c_1}{\lambda+1}$, integrating (2.81) over (T_0,t) and using (2.80) again, we have

$$\|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t)(t)\|^2\lesssim\|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t)(T_0)\|^2e^{-\alpha_1(1+t)^{\lambda+1}}.\tag{2.82}$$

Step 2: Multiply (2.33) by $\mathcal{E}_x+2\mathcal{E}_{xt}$ and integrate it over $\mathbb{R}$ by parts to get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\Big[\mathcal{E}_x\mathcal{E}_{xt}+\frac{1}{2}(1+t)^{-\lambda}\mathcal{E}_x^2+\bar{n}\mathcal{E}_x^2+\mathcal{E}_{xt}^2\Big]dx+\big[2(1+t)^{-\lambda}-1\big]\|\mathcal{E}_{xt}(t)\|^2\\&\quad+\int_{\mathbb{R}}\Big[\frac{\lambda}{2}(1+t)^{-\lambda-1}+\bar{n}\Big]\mathcal{E}_x^2dx+\int_{\mathbb{R}}(p(n)-p(\bar{n}))_x(\mathcal{E}_{xx}+2\mathcal{E}_{xxt})dx\\&=-\int_{\mathbb{R}}\big[\bar{n}_x\mathcal{E}\mathcal{E}_x+2\bar{n}_x\mathcal{E}\mathcal{E}_{xt}+(\mathcal{E}+\bar{E})(2\mathcal{E}_{xx}\mathcal{E}_{xt}+\mathcal{E}_x\mathcal{E}_{xx})+(\mathcal{E}_x+\bar{E}_x)(2\mathcal{E}_x\mathcal{E}_{xt}+\mathcal{E}_x^2)\big]dx\end{aligned}$$

$$-\int_{\mathbb{R}}\left(\frac{\mathcal{E}_t^2}{n}\right)_x(\mathcal{E}_{xx}+2\mathcal{E}_{xxt})dx=:I_{21}+I_{22}. \tag{2.83}$$

By (2.7) and (2.74), we have

$$\begin{aligned}&\int_{\mathbb{R}}(p(n)-p(\bar{n}))_x(\mathcal{E}_{xx}+2\mathcal{E}_{xxt})dx\\ \geq&\frac{d}{dt}\int_{\mathbb{R}}p'(n)\mathcal{E}_{xx}^2dx+\int_{\mathbb{R}}p'(n)\mathcal{E}_{xx}^2dx-C(\Phi_0+\|D'\|_2)\|(\mathcal{E}_x,\mathcal{E}_{xt},\mathcal{E}_{xx})(t)\|^2.\end{aligned} \tag{2.84}$$

Analogously, we can estimate that

$$I_{21}\lesssim(\Phi_0+\|D'\|_2)\|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_{xt},\mathcal{E}_{xx})(t)\|^2, \tag{2.85}$$

and

$$I_{22}\leq\frac{d}{dt}\int_{\mathbb{R}}\frac{\mathcal{E}_t^2}{n^2}\mathcal{E}_{xx}^2dx+C(\Phi_0+\|D'\|_2)\|(\mathcal{E}_t,\mathcal{E}_{xt},\mathcal{E}_{xx})(t)\|^2. \tag{2.86}$$

Putting (2.84)-(2.86) into (2.83), and using (2.75) and the smallness of $\Phi_0+\|D'\|_2$, we have

$$\frac{d}{dt}F_2(t)+C_6\|(\mathcal{E}_x,\mathcal{E}_{xt},\mathcal{E}_{xx})(t)\|^2\lesssim\|(\mathcal{E},\mathcal{E}_t)(t)\|^2, \tag{2.87}$$

for some positive constant C_6, where

$$F_2(t)=\int_{\mathbb{R}}\left[\mathcal{E}_x\mathcal{E}_{xt}+\frac{1}{2}(1+t)^{-\lambda}\mathcal{E}_x^2+\bar{n}\mathcal{E}_x^2+\mathcal{E}_{xt}^2+p'(n)\mathcal{E}_{xx}^2-\frac{1}{n^2}\mathcal{E}_t^2\mathcal{E}_{xx}^2\right]dx.$$

Analogous to (2.80), it holds that

$$\|(\mathcal{E}_x,\mathcal{E}_{xt},\mathcal{E}_{xx})(t)\|^2\lesssim F_2(t)\lesssim(1+t)^{-\lambda}\|(\mathcal{E}_x,\mathcal{E}_{xt},\mathcal{E}_{xx})(t)\|^2. \tag{2.88}$$

By (2.87)-(2.88) and applying (2.82), there exits a positive constant $c_2>0$ such that

$$\frac{d}{dt}F_2(t)+c_2(1+t)^{\lambda}F_2(t)\lesssim\|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t)(T_0)\|^2e^{-\alpha_1(1+t)^{\lambda+1}}.$$

Furthermore, it holds that

$$\frac{d}{dt}\left[e^{\frac{c_2}{\lambda+1}(1+t)^{\lambda+1}}F_2(t)\right]\lesssim\|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t)(T_0)\|^2e^{\left(\frac{c_2}{\lambda+1}-\alpha_1\right)(1+t)^{\lambda+1}}. \tag{2.89}$$

Then, by setting $\tilde{\alpha}_2:=\frac{c_2}{\lambda+1}$ and integrating (2.89) over (T_0,t), we can get

$$\begin{aligned}e^{\tilde{\alpha}_2(1+t)^{\lambda+1}}F_2(t)&\lesssim\|(\mathcal{E}_x,\mathcal{E}_{xt},\mathcal{E}_{xx})(T_0)\|^2+\|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t)(T_0)\|^2\int_{T_0}^t e^{(\tilde{\alpha}_2-\alpha_1)(1+s)^{\lambda+1}}ds\\&\lesssim\|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t,\mathcal{E}_{xt},\mathcal{E}_{xx})(T_0)\|^2\left[1+\int_0^t e^{(\tilde{\alpha}_2-\alpha_1)(1+s)^{\lambda+1}}ds\right]\end{aligned} \tag{2.90}$$

by (2.88). Notice that $0<\lambda+1<1$ and $\frac{-\lambda}{\lambda+1}>0$ from $\lambda\in(-1,0)$, we can get

$$\int_0^t e^{(\tilde{\alpha}_2-\alpha_1)(1+s)^{\lambda+1}}ds=\frac{1}{\lambda+1}\int_1^{(1+t)^{\lambda+1}}e^{(\tilde{\alpha}_2-\alpha_1)s}s^{\frac{-\lambda}{\lambda+1}}ds$$

$$\leq \frac{1}{\lambda+1}\int_1^{(1+t)^{\lambda+1}} e^{(\tilde{\alpha}_2-\alpha_1)s}(1+t)^{-\lambda}ds$$
$$= \frac{(1+t)^{-\lambda}}{(\lambda+1)(\tilde{\alpha}_2-\alpha_1)}\left[e^{(\tilde{\alpha}_2-\alpha_1)(1+t)^{\lambda+1}} - e^{\tilde{\alpha}_2-\alpha_1}\right]. \tag{2.91}$$

In addition,

$$e^{-\tilde{\alpha}_2(1+t)^{\lambda+1}}\int_0^t e^{(\tilde{\alpha}_2-\alpha_1)(1+s)^{\lambda+1}}ds \lesssim (1+t)^{-\lambda}e^{-\alpha_2(1+t)^{\lambda+1}}, \tag{2.92}$$

where $\alpha_2=\min\{\alpha_1,\tilde{\alpha}_2\}$. Finally, from (2.88) we obtain

$$\|(\mathcal{E}_x,\mathcal{E}_{xt},\mathcal{E}_{xx})(t)\|^2 \lesssim \|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t,\mathcal{E}_{xt},\mathcal{E}_{xx})(T_0)\|^2(1+t)^{-\lambda}e^{-\alpha_2(1+t)^{\lambda+1}}. \tag{2.93}$$

Step 3: We multiply (2.54) by $\mathcal{E}_{xx}+2\mathcal{E}_{xxt}$ and integrate it over $\mathbb{R}$ by parts to give

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left[\mathcal{E}_{xx}\mathcal{E}_{xxt}+\frac{1}{2}(1+t)^{-\lambda}\mathcal{E}_{xx}^2+\mathcal{E}_{xxt}^2+\bar{n}\mathcal{E}_{xx}^2\right]dx+\left[2(1+t)^{-\lambda}-1\right]\|\mathcal{E}_{xxt}(t)\|^2\\
&\quad+\int_{\mathbb{R}}\left[\frac{\lambda}{2}(1+t)^{-\lambda-1}+\bar{n}\right]\mathcal{E}_{xx}^2dx+\int_{\mathbb{R}}(p(n)-p(\bar{n}))_{xx}(\mathcal{E}_{xxx}+2\mathcal{E}_{xxxt})dx\\
&=-\int_{\mathbb{R}}(\bar{E}\mathcal{E}_{xxx}+2\bar{n}_x\mathcal{E}_x+\bar{n}_{xx}\mathcal{E}+3\mathcal{E}_x\mathcal{E}_{xx}+2\bar{E}_x\mathcal{E}_{xx}+\bar{E}_{xx}\mathcal{E}_x)(\mathcal{E}_{xx}+2\mathcal{E}_{xxt})dx\\
&\quad-\int_{\mathbb{R}}\left(\frac{\mathcal{E}_t^2}{n}\right)_{xx}(\mathcal{E}_{xxx}+2\mathcal{E}_{xxxt})dx=:I_{31}+I_{32}.
\end{aligned} \tag{2.94}$$

From (2.7) and (2.74), we get that

$$\begin{aligned}
&\int_{\mathbb{R}}(p(n)-p(\bar{n}))_{xx}(\mathcal{E}_{xxx}+2\mathcal{E}_{xxxt})dx\\
\geq&\frac{d}{dt}\int_{\mathbb{R}}p'(n)\mathcal{E}_{xxx}^2dx+\int_{\mathbb{R}}p'(n)\mathcal{E}_{xxx}^2dx-C(\Phi_0+\|D'\|_2)\|(\mathcal{E}_x,\mathcal{E}_{xx},\mathcal{E}_{xxx},\mathcal{E}_{xxt})(t)\|^2.
\end{aligned} \tag{2.95}$$

Similarly, we can estimate that

$$I_{31}\lesssim(\Phi_0+\|D'\|_2)\|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_{xx},\mathcal{E}_{xxx},\mathcal{E}_{xxt})(t)\|^2, \tag{2.96}$$

and

$$I_{32}\leq\frac{d}{dt}\int_{\mathbb{R}}\frac{\mathcal{E}_t^2}{n^2}\mathcal{E}_{xxx}^2dx+C(\Phi_0+\|D'\|_2)\|(\mathcal{E}_x,\mathcal{E}_t)(t)\|_2^2. \tag{2.97}$$

Similarly as (2.87), substituting (2.95)-(2.97) into (2.94), we have

$$\frac{d}{dt}F_3(t)+C_7\|(\mathcal{E}_{xx},\mathcal{E}_{xxx},\mathcal{E}_{xxt})(t)\|^2\lesssim\|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t,\mathcal{E}_{xt})(t)\|^2 \tag{2.98}$$

for some positive constant C_7, where

$$F_3(t)=\int_{\mathbb{R}}\left[\mathcal{E}_{xx}\mathcal{E}_{xxt}+\frac{1}{2}(1+t)^{-\lambda}\mathcal{E}_{xx}^2+\bar{n}\mathcal{E}_{xx}^2+\mathcal{E}_{xxt}^2+p'(n)\mathcal{E}_{xxx}^2-\frac{\mathcal{E}_t^2}{n^2}\mathcal{E}_{xxx}^2\right]dx,$$

and

$$\|(\mathcal{E}_{xx},\mathcal{E}_{xxx},\mathcal{E}_{xxt})(t)\|^2\lesssim F_3(t)\lesssim(1+t)^{-\lambda}\|(\mathcal{E}_{xx},\mathcal{E}_{xxx},\mathcal{E}_{xxt})(t)\|^2. \tag{2.99}$$

From (2.98) and (2.99), by using (2.82) and (2.93), there exits a constant $c_3>0$ such that

$$\frac{d}{dt}F_3(t)+c_3(1+t)^\lambda F_3(t)\lesssim \|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t,\mathcal{E}_{xt},\mathcal{E}_{xx})(T_0)\|^2(1+t)^{-\lambda}e^{-\alpha_2(1+t)^{\lambda+1}}.$$

Then, by setting $\tilde{\alpha}_3:=\frac{c_3}{\lambda+1}$ we have

$$\frac{d}{dt}\left[e^{\tilde{\alpha}_3(1+t)^{\lambda+1}}F_3(t)\right]\lesssim \|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t,\mathcal{E}_{xt},\mathcal{E}_{xx})(T_0)\|^2(1+t)^{-\lambda}e^{(\tilde{\alpha}_3-\alpha_2)(1+t)^{\lambda+1}}. \tag{2.100}$$

Integrating (2.100) over (T_0,t) and employing (2.99) again, we can get

$$F_3(t)\lesssim \|(\mathcal{E},\mathcal{E}_x,\mathcal{E}_t,\mathcal{E}_{xt},\mathcal{E}_{xx},\mathcal{E}_{xxx},\mathcal{E}_{xxt})(T_0)\|^2e^{-\tilde{\alpha}_3(1+t)^{\lambda+1}}\left[1+\int_0^t\frac{e^{(\tilde{\alpha}_3-\alpha_2)(1+s)^{\lambda+1}}}{(1+s)^\lambda}ds\right].$$

Analogous to (2.91)-(2.92), we have

$$\begin{aligned} e^{-\tilde{\alpha}_3(1+t)^{\lambda+1}}\int_0^t(1+s)^{-\lambda}e^{(\tilde{\alpha}_3-\alpha_2)(1+s)^{\lambda+1}}ds &= \frac{e^{-\tilde{\alpha}_3(1+t)^{\lambda+1}}}{\lambda+1}\int_1^{(1+t)^{\lambda+1}}e^{(\tilde{\alpha}_3-\alpha_2)s}s^{\frac{-2\lambda}{\lambda+1}}ds\\ &\lesssim (1+t)^{-2\lambda}e^{-\alpha_0(1+t)^{\lambda+1}}, \end{aligned} \tag{2.101}$$

where $\alpha_0=\min\{\tilde{\alpha}_3,\alpha_2\}$. Thus, by (2.99) again, we get

$$\|(\mathcal{E}_{xx},\mathcal{E}_{xxx},\mathcal{E}_{xxt})(t)\|^2\lesssim \big(\|\mathcal{E}(T_0)\|_3^2+\|\mathcal{E}_t(T_0)\|_2^2\big)(1+t)^{-2\lambda}e^{-\alpha_0(1+t)^{1+\lambda}}. \tag{2.102}$$

In the end, combining (2.82), (2.93), (2.102) and employing (2.17), we obtain (2.18). This completes the proof of Proposition 2.2. □

3. Weak damping case with $0<\lambda<1$

In this section, we consider the Cauchy problem (1.2)-(1.3) with time-gradually vanished damping in the case of $\lambda\in(0,1)$, and obtain the global existence and large-time behavior of the smooth solution (n,J,E) nearby the constant steady-state as follows.

THEOREM 3.1 (Convergence). *For the case of $0<\lambda<1$, suppose that $D(x)\equiv\widehat{\mathcal{D}}$ for a constant $\widehat{\mathcal{D}}>0$ and the pressure function satisfies (1.4). Let*

$$\psi_0(x):=\int_{-\infty}^x(n_0(y)-\widehat{\mathcal{D}})dy. \tag{3.1}$$

Assume that $\psi_0(\cdot)\in H^3(\mathbb{R})$, $J_0(\cdot)\in H^2(\mathbb{R})$ and $\|\psi_0\|_3+\|J_0\|_2$ is sufficiently small, then, the Cauchy problem (1.2)-(1.3) admits a unique global-in-time smooth solution (n,J,E) and it holds that for some constant $\beta>0$,

$$\|n(t)-\widehat{\mathcal{D}}\|_2+\|J(t)\|_2+\|E(t)\|_3\le C_\theta(1+t)^{-\frac{\theta+\lambda}{2}}e^{-\beta(1+t)^{1-\lambda}}, \tag{3.2}$$

where $\theta\in[\lambda,\infty)$ and

$$0<C_\theta:=K^\theta(1+\theta)^{\frac{\theta}{1-\lambda}}(\|\psi_0\|_3+\|J_0\|_2)<\infty$$

for some constant $K>1$.

REMARK 3.1. Actually, the index θ is closely related to the initial perturbation. In order to obtain the global existence of the solution by the continuation argument, we need to guarantee that

$$K_*^{\theta}(\|\psi_0\|_3+\|J_0\|_2)\ll 1$$

where the constant $K_*>1$, see proposition below. In view of this, we find that the level of the smallness of initial perturbation needs to reach as $K_*^{-\theta}$. On the other hand, when the initial perturbation $\|\psi_0\|_3+\|J_0\|_2$ reduces to zero, the index θ could reach infinity, which leads to the fact that the algebraic decay in (3.2) could be fast enough.

In order to prove Theorem 3.1, we just need to obtain the *a priori* estimates of the solution. If we define

$$N(x,t):=n(x,t)-\widehat{\mathcal{D}}, \tag{3.3}$$

then the unknown functions (N,J,E) satisfy the system

$$\begin{cases} N_t+J_x=0,\\ J_t+\left(\dfrac{J^2}{n}+p(n)\right)_x=(\widehat{\mathcal{D}}+N)E-\dfrac{J}{(1+t)^{\lambda}},\\ E_x=N, \end{cases} \tag{3.4}$$

which implies

$$N=E_x,\quad J=-E_t. \tag{3.5}$$

Then, putting (3.5) into $(3.4)_2$, we can get that $E(x,t)$ satisfies the following nonlinear Klein-Gordon equation with weak damping

$$E_{tt}+(1+t)^{-\lambda}E_t+\widehat{\mathcal{D}}E-p(\widehat{\mathcal{D}}+E_x)_x=-EE_x+\left(\frac{E_t^2}{\widehat{\mathcal{D}}+E_x}\right)_x, \tag{3.6}$$

with the initial data

$$E(x,0)=\psi_0(x),\quad E_t(x,0)=-J_0(x). \tag{3.7}$$

REMARK 3.2. We now explain why we have to restrict $D(x)\equiv\widehat{\mathcal{D}}$ for the case of $\lambda\in(0,1)$: On one hand, the main working Equation (3.6) for the electric field $E(x,t)$ with weak damping is actually the damped Klein-Gordon Equation (1.10) studied in [5], in which Burq-Raugel-Schlag assumed $\omega(x)\equiv constant>0$ and obtained the sub-exponential rate. This assumption is actually equivalent to $D(x)\equiv constant>0$; on the other hand, when the doping profile $D(x)$ is non-constant, the expected asymptotic profile $(\bar{n},\bar{J},\bar{E})$ is the same as that for the case of $\lambda\in(-1,0)$. Setting $\tilde{\mathcal{E}}(x,t):=E(x,t)-\bar{E}(x)$, then, the main working equation for $\tilde{\mathcal{E}}(x,t)$ is reduced to Klein-Gordon equation as follows

$$\tilde{\mathcal{E}}_{tt}+(1+t)^{-\lambda}\tilde{\mathcal{E}}_t+\bar{n}\tilde{\mathcal{E}}-(p(\bar{n}+\tilde{\mathcal{E}}_x)-p(\bar{n}))_x=-\tilde{\mathcal{E}}\tilde{\mathcal{E}}_x-\bar{E}\tilde{\mathcal{E}}_x+\left(\frac{\tilde{\mathcal{E}}_t^2}{n}\right)_x. \tag{3.8}$$

We multiply (3.8) by $(1+t)^{\theta_1}\tilde{\mathcal{E}}+2(1+t)^{\theta_2}\tilde{\mathcal{E}}_t$ for two positive constants θ_1,θ_2, and integrate the resulting equality with respect to x over $\mathbb{R}$ by parts to get

$$\frac{d}{dt}\int_{\mathbb{R}}\Big[(1+t)^{\theta_1}\tilde{\mathcal{E}}\tilde{\mathcal{E}}_t-\frac{\theta_1}{2}(1+t)^{\theta_1-1}\tilde{\mathcal{E}}^2+\frac{1}{2}(1+t)^{\theta_1-\lambda}\tilde{\mathcal{E}}^2+(1+t)^{\theta_2}(\bar{n}\tilde{\mathcal{E}}^2+\tilde{\mathcal{E}}_t^2+p'(\bar{n})\tilde{\mathcal{E}}_x^2)\Big]dx$$

$$
\begin{aligned}
&+\int_{\mathbb{R}}\Big\{\big[(1+t)^{\theta_1}-\theta_2(1+t)^{\theta_2-1}\big]\bar{n}+\frac{\theta_1(\theta_1-1)}{2}(1+t)^{\theta_1-2}-\frac{\theta_1-\lambda}{2}(1+t)^{\theta_1-\lambda-1}\Big\}\tilde{\mathcal{E}}^2dx\\
&+\int_{\mathbb{R}}\big[2(1+t)^{\theta_2-\lambda}-(1+t)^{\theta_1}-\theta_2(1+t)^{\theta_2-1}\big]\tilde{\mathcal{E}}_t^2dx\\
&+\int_{\mathbb{R}}\big[(1+t)^{\theta_1}-\theta_2(1+t)^{\theta_2-1}\big]p'(\bar{n})\tilde{\mathcal{E}}_x^2dx\\
\lesssim&\int_{\mathbb{R}}\big[(1+t)^{\theta_1}|\bar{E}\tilde{\mathcal{E}}\tilde{\mathcal{E}}_x+\tilde{\mathcal{E}}^2\tilde{\mathcal{E}}_x+\tilde{\mathcal{E}}_x^3+\tilde{\mathcal{E}}_x\tilde{\mathcal{E}}_t^2|+(1+t)^{\theta_2}|\bar{E}\tilde{\mathcal{E}}_x\tilde{\mathcal{E}}_t+\tilde{\mathcal{E}}\tilde{\mathcal{E}}_x\mathcal{E}_t+\tilde{\mathcal{E}}_x^2\tilde{\mathcal{E}}_{xt}+\tilde{\mathcal{E}}_t^2\tilde{\mathcal{E}}_{xt}|\big]dx.
\end{aligned}
$$

In order to guarantee that for all $t\geq 0$, there exists some positive constant $\tilde{C}_0$ such that

$$
\begin{cases}
2(1+t)^{\theta_2-\lambda}-(1+t)^{\theta_1}-\theta_2(1+t)^{\theta_2-1}\geq \tilde{C}_0(1+t)^{\theta_2-\lambda},\\
(1+t)^{\theta_1}-\theta_2(1+t)^{\theta_2-1}\geq \tilde{C}_0(1+t)^{\theta_1},
\end{cases}
$$

we have to choose that θ_1,θ_2 satisfy

$$
\theta_1+\lambda<\theta_2<\theta_1+1. \tag{3.9}
$$

However, to deal with the term $(1+t)^{\theta_2}\bar{E}\tilde{\mathcal{E}}_x\tilde{\mathcal{E}}_t$, we need to choose

$$
2\theta_2\leq\theta_1+\theta_2-\lambda\Leftrightarrow\theta_2\leq\theta_1-\lambda, \tag{3.10}
$$

which contradicts with the left inequality of (3.9) since $\lambda\in(0,1)$. Hence, we choose a constant steady-state as the background solution such that $\bar{E}=0$, which can be derived from $D(x)\equiv const$.

PROPOSITION 3.1 (*A priori* estimate in the case of $0<\lambda<1$). *Under the conditions of Theorem 3.1, for the index $\theta\in[\lambda,\infty)$, let the constant M satisfy that*

$$
M^{1-\lambda}=K_0(\theta+1), \tag{3.11}
$$

where $K_0=\max\{\lambda^{-1},[\widehat{\mathcal{D}}(1-\lambda)]^{-1}\}>1$. There exists a positive constant ε_2 sufficiently small such that, for any given $\widetilde{T}>0$, if the solution to (3.6)-(3.7) on $[0,\widetilde{T}]$ satisfies

$$
\sup_{0\leq t\leq\widetilde{T}}(1+t)^{\frac{\theta+\lambda}{2}}\big(\|E(t)\|_3+\|E_t(t)\|_2\big)\leq\frac{\varepsilon_2}{M^{\frac{\theta+\lambda}{2}}}, \tag{3.12}
$$

then, for any $t\in[0,\widetilde{T}]$,

$$
\|E(t)\|_3+\|E_t(t)\|_2\lesssim K_*^{\theta}M^{\frac{\theta+\lambda}{2}}(\|\psi_0\|_3+\|J_0\|_2)(1+t)^{-\frac{\theta+\lambda}{2}}e^{-\beta(1+t)^{1-\lambda}}, \tag{3.13}
$$

for some positive constant $K_*>1$ only dependent on λ and $\widehat{\mathcal{D}}$.

Proof. On the one hand, by the *a priori* assumption (3.12) and the Sobolev inequality, one has

$$
\sup_{0\leq t\leq\widetilde{T}}(M+t)^{\frac{\theta+\lambda}{2}}\left(\sum_{k=0}^{2}\|\partial_x^kE(t)\|_{L^\infty(\mathbb{R})}+\sum_{k=0}^{1}\|\partial_x^kE_t(t)\|_{L^\infty(\mathbb{R})}\right)\lesssim\varepsilon_2. \tag{3.14}
$$

Thus, it follows from $\widehat{\mathcal{D}}>0$ and $\varepsilon_2\ll 1$ that

$$
\frac{\widehat{\mathcal{D}}}{2}\leq n=\widehat{\mathcal{D}}+E_x\leq 2\widehat{\mathcal{D}}. \tag{3.15}
$$

On the other hand, since M satisfies (3.11) and $0<\lambda<1$, one has the following inequalities:

$$(M+t)^{\lambda}(1+t)^{-\lambda}\geq 1; \tag{3.16}$$

$$(\theta+\lambda)(M+t)^{\lambda-1}\leq \lambda; \tag{3.17}$$

$$(1-\lambda)\widehat{\mathcal{D}}+\frac{\theta(\theta-1)}{2(M+t)^2}-\frac{\theta(M+t)^{-1}}{2(1+t)^{\lambda}}>0. \tag{3.18}$$

Next, we focus on establishing the *a priori* estimates of the solution based on (3.14)-(3.18).

Step 1: Multiplying (3.6) by $(M+t)^{\theta}E+2(M+t)^{\theta+\lambda}E_t$ and integrating it by parts with respect to x over $\mathbb{R}$ leads to

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left[(M+t)^{\theta}EE_t+(M+t)^{\theta+\lambda}(E_t^2+\widehat{\mathcal{D}}E^2)+\frac{(M+t)^{\theta}}{2(1+t)^{\lambda}}E^2-\frac{\theta(M+t)^{\theta}}{2(M+t)}E^2\right]dx\\
&\quad+\big[2(M+t)^{\lambda}(1+t)^{-\lambda}-1-(\theta+\lambda)(M+t)^{\lambda-1}\big](M+t)^{\theta}\|E_t(t)\|^2\\
&\quad+\left\{\big[1-(\theta+\lambda)(M+t)^{\lambda-1}\big]\widehat{\mathcal{D}}+\frac{\theta(\theta-1)}{2(M+t)^2}-\frac{\theta(M+t)^{-1}}{2(1+t)^{\lambda}}\right\}(M+t)^{\theta}\|E(t)\|^2\\
&\quad+\frac{\lambda(M+t)^{\theta}}{2(1+t)^{\lambda+1}}\|E(t)\|^2-\int_{\mathbb{R}}p(\widehat{\mathcal{D}}+E_x)_x\big[(M+t)^{\theta}E+2(M+t)^{\theta+\lambda}E_t\big]dx\\
&=-(M+t)^{\theta}\int_{\mathbb{R}}\left[\Big(E^2+\frac{E_t^2}{n}\Big)E_x+2(M+t)^{\lambda}\Big(EE_x+\frac{E_t}{n}E_{xt}\Big)E_t\right]dx=:L_1.
\end{aligned} \tag{3.19}$$

Here, from (3.14), we have

$$\begin{aligned}
L_1&\lesssim(M+t)^{\lambda}\|(E,E_x,E_{xt})\|_{L^{\infty}(\mathbb{R})}(M+t)^{\theta}\|(E,E_x,E_t)(t)\|^2\\
&\lesssim\varepsilon_2(M+t)^{\theta}\|(E,E_x,E_t)(t)\|^2,
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
&-\int_{\mathbb{R}}p(\widehat{\mathcal{D}}+E_x)_x\big[(M+t)^{\theta}E+2(M+t)^{\theta+\lambda}E_t\big]dx\\
&=-\int_{\mathbb{R}}(p(\widehat{\mathcal{D}}+E_x)-p(\widehat{\mathcal{D}}))_x\big[(M+t)^{\theta}E+2(M+t)^{\theta+\lambda}E_t\big]dx\\
&\geq\frac{d}{dt}\int_{\mathbb{R}}(M+t)^{\theta+\lambda}p'(\widehat{\mathcal{D}})E_x^2dx-C\varepsilon_2(M+t)^{\theta}\|E_x(t)\|^2\\
&\quad+\big[1-(\theta+\lambda)(M+t)^{\lambda-1}\big](M+t)^{\theta}\int_{\mathbb{R}}p'(\widehat{\mathcal{D}})E_x^2dx,
\end{aligned} \tag{3.21}$$

by using $p(\widehat{\mathcal{D}})_x=p'(\widehat{\mathcal{D}})\widehat{\mathcal{D}}_x=0$. Hence, substituting (3.20)-(3.21) into (3.19) and applying the inequalities (3.16)-(3.18), $p'(\widehat{\mathcal{D}})>0$ and $\varepsilon_2\ll 1$ gives

$$\frac{d}{dt}\int_{\mathbb{R}}G_1(x,t)dx+\tilde{C}_1(M+t)^{\theta}\|(E,E_x,E_t)(t)\|^2\leq 0, \tag{3.22}$$

where $\tilde{C}_1$ is a positive constant and

$$G_1(x,t)=(M+t)^{\theta+\lambda}(E_t^2+\widehat{\mathcal{D}}E^2+p'(\widehat{\mathcal{D}})E_x^2)+(M+t)^{\theta}EE_t+\left[\frac{(M+t)^{\theta}}{2(1+t)^{\lambda}}-\frac{\theta(M+t)^{\theta}}{2(M+t)}\right]E^2.$$

Then, by Cauchy-Schwarz inequality, we have

$$\int_{\mathbb{R}} G_1(x,t)dx \simeq (M+t)^{\theta+\lambda}\|(E,E_x,E_t)(t)\|^2. \tag{3.23}$$

Combining (3.22) and (3.23), we have

$$\frac{d}{dt}\int_{\mathbb{R}} G_1(x,t)dx + \tilde{c}_1(M+t)^{-\lambda}\int_{\mathbb{R}} G_1(x,t)dx \le 0,$$

for some positive constant $\tilde{c}_1$. Then,

$$\frac{d}{dt}\left[e^{\frac{\tilde{c}_1}{1-\lambda}(M+t)^{1-\lambda}}\int_{\mathbb{R}} G_1(x,t)dx\right] \le 0. \tag{3.24}$$

Let $\beta_1 := \frac{\tilde{c}_1}{1-\lambda} > 0$, for any $t \in [0,\widetilde{T}]$, integrating (3.24) over $(0,t)$ we can get

$$\int_{\mathbb{R}} G_1(x,t)dx \lesssim e^{\beta_1 M^{1-\lambda}}\int_{\mathbb{R}} G_1(x,0)dx e^{-\beta_1(1+t)^{1-\lambda}}.$$

Then, setting $K_1 := e^{\beta_1 K_0}$ and applying (3.23) again yields

$$\|(E,E_x,E_t)(t)\|^2 \lesssim K_1^\theta M^{\theta+\lambda}\|(E,E_x,E_t)(0)\|^2(M+t)^{-(\theta+\lambda)}e^{-\beta_1(M+t)^{1-\lambda}}. \tag{3.25}$$

Step 2: Differentiating (3.6) in x yields

$$E_{ttx} + (1+t)^{-\lambda}E_{tx} + \widehat{\mathcal{D}}E_x - (p'(n)E_{xx})_x = -EE_{xx} - E_x^2 + \left(\frac{E_t^2}{n}\right)_{xx}. \tag{3.26}$$

Multiplying (3.26) by $(M+t)^\theta E_x + 2(M+t)^{\theta+\lambda}E_{xt}$ and integrating it with respect to x over $\mathbb{R}$ yields

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\Big[(M+t)^\theta E_xE_{xt} + (M+t)^{\theta+\lambda}(E_{xt}^2 + \widehat{\mathcal{D}}E_x^2 + p'(n)E_{xx}^2)\Big]dx\\
&\quad + \frac{d}{dt}\int_{\mathbb{R}}\Big[\frac{(M+t)^\theta}{2(1+t)^\lambda}E_x^2 - \frac{\theta(M+t)^\theta}{2(M+t)}E_x^2\Big]dx\\
&\quad + \big[2(M+t)^\lambda(1+t)^{-\lambda} - 1 - (\theta+\lambda)(M+t)^{\lambda-1}\big](M+t)^\theta\|E_{xt}(t)\|^2\\
&\quad + \left\{\big[1-(\theta+\lambda)(M+t)^{\lambda-1}\big]\widehat{\mathcal{D}} + \frac{\theta(\theta-1)}{2(M+t)^2} - \frac{\theta(M+t)^{-1}}{2(1+t)^\lambda}\right\}(M+t)^\theta\|E_x(t)\|^2\\
&\quad + \frac{\lambda(M+t)^\theta}{2(1+t)^{\lambda+1}}\|E_x(t)\|^2 + \big[1-(\theta+\lambda)(M+t)^{\lambda-1}\big](M+t)^\theta\int_{\mathbb{R}} p'(n)E_{xx}^2dx\\
&= -\int_{\mathbb{R}}(M+t)^\theta\big[(M+t)^\lambda(p''(n)E_{xt}E_{xx}^2 - 2EE_{xx}E_{xt} - 2E_x^2E_{xt}) + (EE_xE_{xx} + E_x^3)\big]dx\\
&\quad + \int_{\mathbb{R}}\left(\frac{E_t^2}{n}\right)_{xx}\big[2(M+t)^{\theta+\lambda}E_{xt} + (M+t)^\theta E_x\big]dx =: L_{21} + L_{22}.
\end{aligned} \tag{3.27}$$

From (3.14), we have

$$L_{21} \lesssim \varepsilon_2(M+t)^\theta\|(E_x,E_{xt},E_{xx})(t)\|^2, \tag{3.28}$$

and

$$\begin{aligned}
L_{22} = & -\int_{\mathbb{R}} \left(\frac{E_t^2}{n}\right)_x \left[2(M+t)^{\theta+\lambda} E_{xxt} + (M+t)^{\theta} E_{xx}\right] dx \\
= & \frac{d}{dt} \int_{\mathbb{R}} (M+t)^{\theta+\lambda} \frac{E_t^2}{n^2} E_{xx}^2 dx + \int_{\mathbb{R}} [1-(\theta+\lambda)(M+t)^{\lambda-1}](M+t)^{\theta} \frac{E_t^2}{n^2} E_{xx}^2 dx \\
& + 2(M+t)^{\theta} \int_{\mathbb{R}} (M+t)^{\lambda} \left[\frac{E_{xt}^3}{n} - \frac{E_t}{n^2}(E_{xx}E_{xt}^2 + E_{tt}E_{xx}^2) + \frac{E_t^2}{n^3} E_{xt} E_{xx}^2\right] dx \\
& - 2(M+t)^{\theta} \int_{\mathbb{R}} \frac{E_t}{n} E_{tx} E_{xx} dx \\
\leq & \frac{d}{dt} \int_{\mathbb{R}} (M+t)^{\theta+\lambda} \frac{E_t^2}{n^2} E_{xx}^2 dx + C\varepsilon_2 (M+t)^{\theta} \|(E_{xx}, E_{xt})(t)\|^2. \qquad (3.29)
\end{aligned}$$

Here, in the last inequality of (3.29), we have used the fact that

$$\|E_{tt}(t)\|_{L^\infty(\mathbb{R})} \leq \sum_{k=0}^{2} \|\partial_x^k E(t)\|_{L^\infty(\mathbb{R})} + \sum_{k=0}^{1} \|\partial_x^k E_t(t)\|_{L^\infty(\mathbb{R})} \lesssim \varepsilon_2 (M+t)^{-\frac{\theta+\lambda}{2}},$$

which comes from the Equation (3.6) and the (3.14). Then, substituting (3.28)-(3.29) into (3.27), and applying (3.16)-(3.18), $p'(n) > 0$ and $\varepsilon_2 \ll 1$, there exits a constant $\tilde{C}_2 > 0$ such that

$$\frac{d}{dt} \int_{\mathbb{R}} G_2(x,t) dx + \tilde{C}_2 (M+t)^{\theta} \|(E_x, E_{xt}, E_{xx})(t)\|^2 \leq 0, \qquad (3.30)$$

where

$$\begin{aligned}
G_2(x,t) = & (M+t)^{\theta+\lambda} \left[\widehat{\mathcal{D}} E_x^2 + E_{xt}^2 + \left(p'(n) - \frac{E_t^2}{n^2}\right) E_{xx}^2\right] + (M+t)^{\theta} E_x E_{xt} \\
& + \frac{1}{2} (M+t)^{\theta} [(1+t)^{-\lambda} - \theta(M+t)^{-1}] E_x^2,
\end{aligned}$$

and

$$\int_{\mathbb{R}} G_2(x,t) dx \simeq (M+t)^{\theta+\lambda} \|(E_x, E_{xt}, E_{xx})(t)\|^2. \qquad (3.31)$$

Thus, (3.30) and (3.31) imply that for some positive constant $\tilde{c}_2$,

$$\frac{d}{dt} \int_{\mathbb{R}} G_2(x,t) dx + \tilde{c}_2 (M+t)^{-\lambda} \int_{\mathbb{R}} G_2(x,t) dx \leq 0.$$

Analogous to (3.25), letting $\beta_2 := \frac{\tilde{c}_2}{1-\lambda} > 0$ and $K_2 := e^{\beta_2 K_0}$, we have

$$\|(E_x, E_{xt}, E_{xx})(t)\|^2 \lesssim K_2^{\theta} M^{\theta+\lambda} \|(E_x, E_{xt}, E_{xx})(0)\|^2 (M+t)^{-(\theta+\lambda)} e^{-\beta_2 (M+t)^{1-\lambda}}. \qquad (3.32)$$

Step 3: Differentiating (3.6) in x twice yields

$$\begin{aligned}
& E_{ttxx} + (1+t)^{-\lambda} E_{txx} + \widehat{\mathcal{D}} E_{xx} - \left(p'(n) E_{xxx}\right)_x \\
& = -E E_{xxx} - 3 E_x E_{xx} + \left(p''(n) E_{xx}^2\right)_x + \left(\frac{E_t^2}{n}\right)_{xxx}. \qquad (3.33)
\end{aligned}$$

Multiplying (3.33) by $(M+t)^\theta E_{xx}+2(M+t)^{\theta+\lambda}E_{xxt}$ and integrating the resulting equations with respect to x over $\mathbb{R}$ by parts lead to

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\Big[(M+t)^\theta E_{xx}E_{xxt}+\frac{(M+t)^\theta}{2(1+t)^\lambda}E_{xx}^2-\frac{\theta}{2}(M+t)^{\theta-1}E_{xx}^2\\
&\quad+(M+t)^{\theta+\lambda}(E_{xxt}^2+\widehat{\mathcal{D}}E_{xx}^2+p''(n)E_{xt}E_{xxx}^2)\Big]dx\\
&\quad+\big[2(M+t)^\lambda(1+t)^{-\lambda}-1-(\theta+\lambda)(M+t)^{\lambda-1}\big](M+t)^\theta\|E_{xxt}(t)\|^2\\
&\quad+\Big\{\big[1-(\theta+\lambda)(M+t)^{\lambda-1}\big]\widehat{\mathcal{D}}+\frac{\theta(\theta-1)}{2(M+t)^2}-\frac{\theta(M+t)^{-1}}{2(1+t)^\lambda}\Big\}(M+t)^\theta\|E_{xx}(t)\|^2\\
&\quad+\frac{\lambda(M+t)^\theta}{2(1+t)^{\lambda+1}}\|E_{xx}(t)\|^2+\int_{\mathbb{R}}\big[1-(\theta+\lambda)(M+t)^{\lambda-1}\big](M+t)^\theta p'(n)E_{xxx}^2dx\\
&=:L_{31}+L_{32},
\end{aligned}\tag{3.34}$$

where, from (3.14), we have

$$\begin{aligned}
L_{31}=&(M+t)^\theta\int_{\mathbb{R}}(2p''(n)E_{xx}-E)\big[E_{xx}E_{xxx}+2(M+t)^\lambda E_{xxx}E_{xxt}\big]dx\\
&+(M+t)^\theta\int_{\mathbb{R}}(p'''(n)E_{xx}^2-3E_x)\big[E_{xx}^2+2(M+t)^\lambda E_{xx}E_{xxt}\big]dx\\
&+(M+t)^{\theta+\lambda}\int_{\mathbb{R}}p''(n)E_{xt}E_{xxx}^2dx\\
\lesssim&\,\varepsilon_2(M+t)^\theta\|(E_{xx},E_{xxt},E_{xxx})(t)\|^2,
\end{aligned}\tag{3.35}$$

and

$$\begin{aligned}
L_{32}=&-\int_{\mathbb{R}}\left(\frac{E_t^2}{n}\right)_{xx}\big[2(M+t)^{\theta+\lambda}E_{xxxt}+(M+t)^\theta E_{xxx}\big]dx\\
=&\frac{d}{dt}\int_{\mathbb{R}}(M+t)^{\theta+\lambda}\frac{E_t^2}{n^2}E_{xxx}^2dx-\int_{\mathbb{R}}(\theta+\lambda)(M+t)^{\theta+\lambda-1}\frac{E_t^2}{n^2}E_{xxx}^2dx\\
&-(M+t)^{\theta+\lambda}\int_{\mathbb{R}}\Big[\Big(\frac{E_t^2}{n^2}\Big)_tE_{xxx}^2-2\Big(\frac{E_t}{n}\Big)_xE_{xxt}^2\Big]dx\\
&+\int_{\mathbb{R}}4(M+t)^{\theta+\lambda}\Big(\frac{1}{n^3}E_t^2E_{xx}^2+\frac{1}{n}E_{xt}^2-\frac{2}{n^2}E_tE_{xt}E_{xx}\Big)_xE_{xxt}dx\\
&+(M+t)^\theta\int_{\mathbb{R}}\Big(\frac{4}{n^2}E_tE_{xt}E_{xx}+\frac{1}{n^2}E_t^2E_{xxx}-\frac{2}{n^3}E_t^2E_{xx}^2-\frac{2}{n}E_tE_{xxt}-\frac{2}{n}E_{xt}^2\Big)E_{xxx}dx\\
\le&\frac{d}{dt}\int_{\mathbb{R}}(M+t)^{\theta+\lambda}\frac{E_t^2}{n^2}E_{xxx}^2dx+C\varepsilon_2(M+t)^\theta\|(E_{xx},E_{xt})(t)\|_1^2.
\end{aligned}\tag{3.36}$$

Substituting (3.35)-(3.36) into (3.34), applying (3.16)-(3.18) and using $p'(n)>0$ and $\varepsilon_2\ll 1$ again, we have

$$\frac{d}{dt}\int_{\mathbb{R}}G_3(x,t)dx+\tilde{C}_3(M+t)^\theta\|(E_{xx},E_{xxt},E_{xxx})(t)\|^2\lesssim(M+t)^\theta\|E_{xt}(t)\|^2,\tag{3.37}$$

for some constant $\tilde{C}_3>0$, where

$$G_3(x,t)=(M+t)^{\theta+\lambda}\Big[E_{xxt}^2+\widehat{\mathcal{D}}E_{xx}^2+\Big(p'(n)-\frac{E_t^2}{n^2}\Big)E_{xxx}^2\Big]+(M+t)^\theta E_{xx}E_{xxt}$$

$$+\frac{1}{2}(M+t)^{\theta}[(1+t)^{-\lambda}-\theta(M+t)^{-1}]E_{xx}^2,$$

and

$$\int_{\mathbb{R}} G_3(x,t)dx \simeq (M+t)^{\theta+\lambda}\|(E_{xx},E_{xxt},E_{xxx})(t)\|^2. \tag{3.38}$$

Combining (3.37) and (3.38) and applying the estimate (3.32), we can get

$$\frac{d}{dt}\int_{\mathbb{R}} G_3(x,t)dx+\tilde{c}_3(M+t)^{-\lambda}\int_{\mathbb{R}} G_3(x,t)dx$$
$$\lesssim K_2^{\theta}M^{\theta+\lambda}\|(E_x,E_{xt},E_{xx})(0)\|^2(M+t)^{-\lambda}e^{-\beta_2(M+t)^{1-\lambda}},$$

for a constant $\tilde{c}_3>0$. Then

$$\frac{d}{dt}\left[e^{\tilde{\beta}_3(M+t)^{1-\lambda}}\int_{\mathbb{R}} G_3(x,t)dx\right]$$
$$\lesssim K_2^{\theta}M^{\theta+\lambda}\|(E_x,E_{xt},E_{xx})(0)\|^2\frac{e^{(\tilde{\beta}_3-\beta_2)(M+t)^{1-\lambda}}}{(M+t)^{\lambda}}, \tag{3.39}$$

where $\tilde{\beta}_3:=\frac{\tilde{c}_3}{1-\lambda}>0$. Integrating (3.39) over $(0,t)$ for any $t\in[0,\widetilde{T}]$, we can obtain

$$\int_{\mathbb{R}} G_3(x,t)dx \lesssim M^{\theta+\lambda}K_2^{\theta}\|(E_x,E_{xt},E_{xx})(0)\|^2e^{-\tilde{\beta}_3(M+t)^{1-\lambda}}\int_0^t\frac{e^{(\tilde{\beta}_3-\beta_2)(M+s)^{1-\lambda}}}{(M+s)^{\lambda}}ds$$
$$+M^{\theta+\lambda}K_3^{\theta}\|(E_{xx},E_{xxt},E_{xxx})(0)\|^2e^{-\tilde{\beta}_3(M+t)^{1-\lambda}}$$
$$\lesssim M^{\theta+\lambda}(K_2+K_3)^{\theta}\big(\|E_x(0)\|_2^2+\|E_{xt}(0)\|_1^2\big)e^{-\beta_3(M+t)^{1-\lambda}},$$

where $K_3=e^{\tilde{\beta}_3K_0}$ and $\beta_3:=\min\{\tilde{\beta}_3,\beta_2\}$. Here, we have used the calculation

$$\int_0^t\frac{e^{(\tilde{\beta}_3-\beta_2)(M+s)^{1-\lambda}}}{(M+s)^{\lambda}}ds=\frac{1}{1-\lambda}\int_1^{(M+t)^{1-\lambda}}e^{(\tilde{\beta}_3-\beta_2)s}ds$$
$$=\frac{1}{(1-\lambda)(\tilde{\beta}_3-\beta_2)}\big[e^{(\tilde{\beta}_3-\beta_2)(M+t)^{1-\lambda}}-e^{\tilde{\beta}_3-\beta_2}\big].$$

Therefore, by using (3.38) again and setting $K_*:=K_2+K_3$, we have

$$\|(E_{xx},E_{xxt},E_{xxx})(t)\|^2$$
$$\lesssim M^{\theta+\lambda}K_*^{\theta}\big(\|E_x(0)\|_2^2+\|E_{xt}(0)\|_1^2\big)(M+t)^{-\theta-\lambda}e^{-\beta_3(M+t)^{1-\lambda}}. \tag{3.40}$$

Finally, let $2\beta:=\min\{\beta_1,\beta_3\}$, then (3.25), (3.32) and (3.40) imply the *a priori* estimate (3.13). This completes the proof of Proposition 3.1. ∎

Proof of Theorem 3.1. In the end, similar to the proof of Theorem 2.1, we can prove Theorem 3.1 by the local existence of solution and the *a priori* estimates of the solution in Proposition 3.1.

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