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ON THE ANALYTICITY AND GEVREY REGULARITY OF SOLUTIONS TO THE THREE-DIMENSIONAL INVISCID BOUSSINESQ EQUATIONS IN A HALF SPACE*

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Abstract. In this paper, we address the problem of analyticity up to the boundary to the 3D inviscid Boussinesq equations in a half space $\mathbb{R}^3_+$. Furthermore, we prove the persistence of Gevrey regularity and obtain lower bounds on the radius of Gevrey regularity.

Keywords. Boussinesq equations; analyticity radius; Gevrey class; half space.

AMS subject classifications. 35Q35; 35B65; 76B03.

1. Introduction

In this paper, we consider the inviscid Boussinesq equations in a half space $\mathbb{R}^3_+$

$$u_t + u\cdot\nabla u + \nabla p = \theta e_3, \tag{1.1}$$

$$\theta_t + u\cdot\nabla\theta = 0, \tag{1.2}$$

$$\nabla\cdot u = 0, \tag{1.3}$$

$$u\cdot n = 0, \tag{1.4}$$

where $x\in\Omega=\{x\in\mathbb{R}^3 : x_3>0\}$ and $t\geq 0$ and $n=(0,0,-1)$ is the outward unit normal vector. Here, $u(x,t)$ is the velocity vector field, $e_3=(0,0,1)$ is the unit vector, p is the scalar pressure and θ is the scalar density. The Boussinesq equations are an important model in the study of geophysical fluids and the Rayleigh–Bénard convection. There have been many efforts devoted to understanding the Boussinesq system. For the global existence and stability of solutions, recent progress has been made given fractional or full dissipation (cf. [1,12,14,17,18,20,24–27]).

The inviscid Boussinesq equations are however much harder to study since there is no dissipation in the system and the global regularity, even in 2D, still remains a challenging open problem. The 2D inviscid Boussinesq system shares some key features with the 3D incompressible Euler equations such as the vortex stretching effect and axisymmetry [22]. Thus, it has been studied by different authors. In [6], Chae and Nam addressed local existence of solutions and obtained a blow-up criterion for the inviscid Boussinesq equations in Sobolev spaces. Contributions on the local results of solutions have also been made in other spaces, such as Besov, Hölder, and Triebel-Lizorkin-Lorentz spaces (cf. [7,8,11,21,23,28]).

Recently, there were many authors studying the decay rates of radius of analyticity and the persistence of Gevrey regularity for fluid equations where the Gevrey regularity was orginally addressed by Foias and Temam for the Navier–Stokes equations [13]. The analyticity of solutions to the incompressible Euler equations with analytic initial data has been studied by many people [2–4, 15, 16, 19]. Bardos and Benachour [4] proved the persistence of analyticity for the Euler equations on a 3D bounded domain and obtained the decay rates of the radius of analyticity. As for the case of 3D periodic domain, by using the method of Gevrey regularity, Levermore and Oliver obtained the

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same rates [19]. Kukavica and Vicol improved previous decay rates depending only algebraically on $\|\omega(t)\|_{H^r}$ and $\exp(\int_0^t \|\nabla u(t')\|_{L^\infty}\,dt')$ in [15] for the Euler equations. Later, they extended the result to the 3D half space with delicate pressure estimates on the boundary [16] and obtained a better lower bound for decay rates of $\tau(t)$ replacing $(t+1)^{-2}$ with $(t+1)^{-1}$.

In a recent paper, by using a deductive method from [5], Cheng and Xu in [10] addressed the analyticity of smooth solutions for the inviscid Boussinesq equations on $\mathbb{T}^d$, where $d=2,3$. However, it is still unknown whether one can discover the decay rates for the radius of analyticity for the inviscid Boussinesq equations in a 3D half space with the presence of a boundary and whether the previous techniques can be adapted. Therefore, in this paper, we give positive answers to the above questions by addressing the analyticity of the solution and the persistence of Gevrey regularity for the inviscid Boussinesq equations in $\mathbb{R}^3_+$. The main difficulties come from the coupling term $u\cdot\nabla\theta$ which needs a careful treatment (cf. Lemma 2.4). Furthermore, due to the appearance of boundary conditions, the method in [5, 9, 10] no longer applies here and we will need to adapt the pressure estimates from [16]. To the best of our knowledge, this is the first result regarding the analyticity and Gevrey regularity of solutions to the 3D inviscid Boussinesq equations in a domain with boundary conditions.

The paper is organized as follows. In Section 2, we introduce notation and state our main results, Theorem 2.1 and Theorem 2.2 together with a key lemma addressing the coupling term $u\cdot\nabla\theta$. In Section 3, we give the proofs for the main theorems. In Section 4, we give the proofs of Lemmas 2.3 and 2.4.

2. Notation and main results

Recall that for $s\geq 1$ a smooth function f is uniformly of Gevrey-class s if there exist $M,\tau>0$ such that

$$\|\partial^\alpha f\|_{L^\infty}\leq M\frac{|\alpha|!^s}{\tau^{|\alpha|}},$$

for all $x\in\Omega$ and all multi-indices $\alpha\in\mathbb{N}_0^3$. Note that when $s=1$, we say that f is real-analytic and when $s>1$, f is C^∞ smooth but might not be analytic. For a multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)\in N_0^3$, we let $\alpha'=(\alpha_1,\alpha_2)$ and define the Sobolev and Lipschitz semi-norms $|\cdot|_m$ and $|\cdot|_{m,\infty}$ by

$$|v|_m=\sum_{|\alpha|=m}M_\alpha\|\partial^\alpha v\|_{L^2},$$

and

$$|v|_{m,\infty}=\sum_{|\alpha|=m}M_\alpha\|\partial^\alpha v\|_{L^\infty}, \tag{2.1}$$

where

$$M_\alpha=\frac{|\alpha'|!}{\alpha'!}=\binom{\alpha_1+\alpha_2}{\alpha_1}.$$

Furthermore, we define the norms X_τ and Y_τ by

$$\|v\|_{X_\tau}=\sum_{m=3}^\infty|v|_m\frac{\tau^{m-3}}{(m-3)!^s},$$

and

$$\|v\|_{Y_\tau} = \sum_{m=4}^{\infty} |v|_m \frac{(m-3)\tau^{m-4}}{(m-3)!^s},$$

where the spaces X_τ and Y_τ are defined by

$$X_\tau = \{v \in C^\infty : \|v\|_{X_\tau} < \infty\},$$

and

$$Y_\tau = \{v \in C^\infty : \|v\|_{Y_\tau} < \infty\}.$$

The following is our first main result addressing the analyticity of the solution in $\mathbb{R}^3_+$.

THEOREM 2.1. *Let $r > 9/2$, and assume that $\|u_0\|_{H^r} < \infty$ with $\nabla \cdot u_0 = 0$ and $\|\theta_0\|_{H^r} < \infty$. We further assume that u_0 and θ_0 are real-analytic in Ω. Then the unique solution $(u(t), \theta(t)) \in C(0, T; H^r(\Omega))$ of the initial value problem (1.1)–(1.4) is real-analytic for all time $t < T$, where $T \in [0, \infty)$. Moreover, the uniform radius of space analyticity $\tau(t)$ satisfies*

$$\tau(t) \geq \frac{1}{C_0(t+1)} \exp\big(-C \int_0^t 1 + \|\nabla u(s,\cdot)\|_{L^\infty} + \|\nabla\theta(s,\cdot)\|_{L^\infty}\, ds\big), \tag{2.2}$$

where C depends only on r.

The following is our second main result which shows the persistence of Gevrey regularity for the Boussinesq equations in $\mathbb{R}^3_+$.

THEOREM 2.2. *Let $r > 9/2$, and assume that $\|u_0\|_{H^r} < \infty$ with $\nabla \cdot u_0 = 0$ and $\|\theta_0\|_{H^r} < \infty$. We further assume that u_0 and θ_0 are uniformly of Gevrey–class s in Ω. Then there exists a unique solution $(u(t), \theta(t)) \in C(0, T; H^r(\Omega))$ of the initial value problem (1.1)–(1.4) uniformly of Gevrey–class s for all time $t < T$, where $T \in [0, \infty)$. Moreover, the uniform radius $\tau(t)$ of Gevrey–class regularity of the solution satisfies the lower bound (2.2).*

Next, we need the following pressure estimates.

LEMMA 2.1 ([16] Pressure estimates). *Assume that p is a smooth solution of the Neumann problem*

$$-\Delta p = v \text{ in } \Omega,$$
$$\frac{\partial p}{\partial n} = 0 \text{ on } \partial\Omega,$$

with $v \in C^\infty$. Then there is a universal constant $C > 0$ such that

$$\|\partial_3 \partial^\alpha p\|_{L^2} \leq C \sum_{\substack{s,t \in \mathbb{N}_0, |\beta| = m-1 \\ \beta' - \alpha' = (2s, 2t)}} \binom{s+t}{s} \|\partial^\beta v\|_{L^2}$$

for any $m \geq 1$ and any multi-index $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = m$ and $\alpha_3 \neq 0$. Additionally, if $\alpha_3 \geq 2$, then

$$\|\partial_1 \partial^\alpha p\|_{L^2} \leq C \sum_{\substack{s,t \in \mathbb{N}_0, |\beta| = m-1 \\ \beta' - \alpha' = (2s+1, 2t)}} \binom{s+t}{s} \|\partial^\beta v\|_{L^2},$$

$$\|\partial_2\partial^\alpha p\|_{L^2} \le C \sum_{\substack{s,t\in\mathbb{N}_0,|\beta|=m-1\\ \beta'-\alpha'=(2s,2t+1)}} \binom{s+t}{s} \|\partial^\beta v\|_{L^2},$$

where $C>0$ *is a universal constant.*

The following lemma is used to estimate the upper bound for the commutator term.

LEMMA 2.2 ([16]). *Denote that*

$$I_1 = \sum_{m=3}^{\infty} \sum_{|\alpha|=m} \sum_{\beta\le\alpha,\beta\neq 0} \frac{\tau(t)^{m-3}}{(m-3)!^s} M_\alpha \binom{\alpha}{\beta} \|\partial^\beta u\cdot\nabla\partial^{\alpha-\beta}u\|_{L^2}.$$

Then there exists a sufficiently large constant $C>0$ *such that*

$$I_1 \le C(\mathcal{C}_1 + \mathcal{C}_2\|u\|_{Y_\tau}),$$

where

$$\mathcal{C}_1 = |u|_{1,\infty}|u|_3 + |u|_{2,\infty}|u|_2 + \tau|u|_{2,\infty}|u|_3$$

and

$$\mathcal{C}_2 = \tau|u|_{1,\infty} + \tau^2|u|_{2,\infty} + \tau^3|u|_{3,\infty} + \tau^{3/2}\|u\|_{X_\tau}.$$

The following lemma shall be used to estimate the pressure term in terms of u and θ.

LEMMA 2.3. *Denote that*

$$I_2 = \sum_{m=3}^{\infty} \sum_{|\alpha|=m,\alpha_3\neq 0} \frac{\tau(t)^{m-3}}{(m-3)!^s} M_\alpha \|\nabla\partial^\alpha p\|_{L^2}.$$

Then there exists a sufficiently large constant $C>0$ *such that and*

$$I_2 \le C(\mathcal{L}_1 + \mathcal{L}_2\|u\|_{Y_\tau} + \tau\|\theta\|_{Y_\tau}),$$

where

$$\mathcal{L}_1 \le \tau|u|_{1,\infty}|u|_3 + |u|_{2,\infty}|u|_2 + \tau|u|_{2,\infty}|u|_3 + |\theta|_3,$$

$$\mathcal{L}_2 \le \tau|u|_{1,\infty} + \tau^2|u|_{2,\infty} + \tau^3|u|_{3,\infty} + \tau^{3/2}\|u\|_{X_\tau}.$$

In the following lemma, we give estimates for higher order derivatives of $u\cdot\nabla\theta$.

LEMMA 2.4. *Denote that*

$$I_4 = \sum_{m=3}^{\infty} \sum_{|\alpha|=m} \sum_{\beta\le\alpha,\beta\neq 0} \frac{\tau(t)^{m-3}}{(m-3)!^s} M_\alpha \binom{\alpha}{\beta} \|\partial^\beta u\cdot\nabla\partial^{\alpha-\beta}\theta\|_{L^2}.$$

There exists a sufficiently large constant $C>0$ *such that*

$$I_4 \le C(\mathcal{K}_1 + \mathcal{K}_2\|\theta\|_{Y_\tau} + \tau^{3/2}\|u\|_{X_\tau}\|\theta\|_{Y_\tau} + \tau^{3/2}\|\theta\|_{X_\tau}\|u\|_{Y_\tau}),$$

where

$$\mathcal{K}_1 = |u|_{1,\infty}|\theta|_3 + |u|_{2,\infty}|\theta|_2 + \tau|u|_{2,\infty}|\theta|_3 + \tau|u|_{1,\infty}|\theta|_3 + \tau|\theta|_{2,\infty}|u|_3 + \tau|\theta|_{1,\infty}|u|_3$$

and

$$\mathcal{K}_2 = \tau(t)|u|_{1,\infty} + \tau^2|u|_{2,\infty} + \tau^3|u|_{3,\infty} + \tau|\theta|_{1,\infty} + \tau^2|\theta|_{2,\infty}.$$

3. Proofs of main theorems

Next, we give the proofs of our theorems.

Proof. We first apply ∂^α to Equation (1.1) and take the L^2 inner product with $\partial^\alpha u$ obtaining

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha u\|_{L^2}^2 + \langle \partial^\alpha (u\cdot\nabla u), \partial^\alpha u\rangle + \langle \partial^\alpha \nabla p, \partial^\alpha u\rangle = \langle \partial^\alpha \theta e_3, \partial^\alpha u\rangle.$$

Notice that $\langle u\cdot\nabla\partial^\alpha u, \partial^\alpha u\rangle = 0$ since $\nabla\cdot u = 0$. Furthermore, since $n=(0,0,-1)$ and $u\cdot n = 0$ on $\partial\Omega$, we have that $\partial^\alpha u\cdot n = 0$ for all α such that $\alpha_3 = 0$ and due to the divergence-free condition of u in (1.3), we will have $\langle \nabla\partial^\alpha p, \partial^\alpha u\rangle = 0$ whenever $\alpha_3 = 0$. Therefore, we multiply Equation (1.1) by M_α and sum over $|\alpha| = m$ and apply the Cauchy-Schwarz inequality obtaining

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u|_m \le \sum_{|\alpha|=m}\sum_{\beta\le\alpha,\beta\neq 0} M_\alpha \binom{\alpha}{\beta}\|\partial^\beta u\cdot\nabla\partial^{\alpha-\beta}u\|_{L^2} + \sum_{|\alpha|=m,\alpha_3\neq 0} M_\alpha\binom{\alpha}{\beta}\|\nabla\partial^\alpha p\|_{L^2}\\ + \sum_{|\alpha|=m} M_\alpha\binom{\alpha}{\beta}\|\partial^\alpha\theta\|_{L^2}.\end{aligned}$$

Since we have a following a *priori* estimate

$$\frac{d}{dt}\|u\|_{X_{\tau(t)}} = \dot{\tau}(t)\|u\|_{Y_{\tau(t)}} + \sum_{m=3}^{\infty}\left(\frac{d}{dt}|u|_m\right)\frac{\tau(t)^{m-3}}{(m-3)!^s}.$$

Thus, combining with (2.1), we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_{X(\tau)} \le \dot{\tau}(t)\|u\|_{Y_{\tau(t)}} + I_1 + I_2 + I_3,$$

where

$$I_1 = \sum_{m=3}^{\infty}\sum_{|\alpha|=m}\sum_{\beta\le\alpha,\beta\neq 0}\frac{\tau(t)^{m-3}}{(m-3)!^s} M_\alpha\binom{\alpha}{\beta}\|\partial^\beta u\cdot\nabla\partial^{\alpha-\beta}u\|_{L^2},$$

$$I_2 = \sum_{m=3}^{\infty}\sum_{|\alpha|=m,\alpha_3\neq 0}\frac{\tau(t)^{m-3}}{(m-3)!^s} M_\alpha\|\nabla\partial^\alpha p\|_{L^2},$$

and

$$I_3 = \sum_{m=3}^{\infty}\sum_{|\alpha|=m}\frac{\tau(t)^{m-3}}{(m-3)!^s} M_\alpha\|\partial^\alpha\theta\|_{L^2} = \|\theta\|_{X_\tau}.$$

By Lemmas 2.1, 2.2, and 2.4, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{X_\tau} \le \dot{\tau}(t)\|u\|_{Y_{\tau(t)}} + C\|u\|_{H^r}^2(1+\tau(t)^2)\\ + C\|u\|_{Y_\tau}\left(\tau\|\nabla u\|_{L^\infty} + (\tau^2+\tau^3)\|u\|_{H^r}^2 + \tau^{3/2}\|u\|_{X_\tau}\right)\\ + \|\theta\|_{H^r} + \tau\|\theta\|_{Y_\tau} + \|\theta\|_{X_\tau}.\end{aligned}$$

Next, we apply ∂^α to Equation (1.1) and take the L^2 inner product with $\partial^\alpha\theta$ obtaining

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha\theta\|_{L^2}^2+\langle\partial^\alpha(u\cdot\nabla\theta),\partial^\alpha\theta\rangle=0.$$

Notice that $\langle u\cdot\nabla\partial^\alpha\theta,\partial^\alpha\theta\rangle=0$ since $\nabla\cdot u=0$. Thus, we multiply Equation (1.2) by M_α and sum over $|\alpha|=m$ and apply the Cauchy-Schwarz inequality and obtaining

$$\frac{d}{dt}|\theta|_m\leq\sum_{|\alpha|=m}\sum_{\beta\leq\alpha,\beta\neq0}M_\alpha\binom{\alpha}{\beta}\|\partial^\beta u\cdot\nabla\partial^{\alpha-\beta}\theta\|_{L^2}.$$

Therefore, we have an a *priori* estimate

$$\begin{aligned}\frac{d}{dt}\|\theta\|_{X_\tau}&\leq\dot\tau(t)\|\theta\|_{Y_{\tau(t)}}+\sum_{m=3}^\infty\sum_{|\alpha|=m}\sum_{\beta\leq\alpha,\beta\neq0}\frac{\tau(t)^{m-3}}{(m-3)!^s}M_\alpha\binom{\alpha}{\beta}\|\partial^\beta u\cdot\nabla\partial^{\alpha-\beta}\theta\|_{L^2}\\&=\dot\tau(t)\|\theta\|_{Y_{\tau(t)}}+I_4.\end{aligned}$$

Thus, by Lemma 2.4 we get

$$\begin{aligned}\frac{d}{dt}\|\theta\|_{X_\tau}\leq{}&\dot\tau(t)\|\theta\|_{Y_\tau}+(1+\tau)\|u\|_{H^r}\|\theta\|_{H^r}+\tau^{3/2}\|u\|_{X_\tau}\|\theta\|_{Y_\tau}+\tau^{3/2}\|\theta\|_{X_\tau}\|u\|_{Y_\tau}\\&+\left(\tau(\|\nabla u\|_{L^\infty}+\|\nabla\theta\|_{L^\infty})+(\tau^2+\tau^3)\|u\|_{H^r}+\tau^2\|\theta\|_{H^r}\right)\|\theta\|_{Y_\tau}.\end{aligned}$$

By adding the two a priori estimates, we get

$$\begin{aligned}\frac{d}{dt}(\|u\|_{X_\tau}+\|\theta\|_{X_\tau})\leq{}&\dot\tau(t)(\|u\|_{Y_\tau}+\|\theta\|_{Y_\tau})+C\|u\|_{H^r}^2(1+\tau(t)^2)+(1+\tau)\|u\|_{H^r}\|\theta\|_{H^r}\\&+\tau\|\theta\|_{Y_\tau}+C\|u\|_{Y_\tau}\left(\tau\|\nabla u\|_{L^\infty}+(\tau^2+\tau^3)\|u\|_{H^r}^2+\tau^{3/2}\|u\|_{X_\tau}\right)\\&+\|\theta\|_{X(\tau)}+\tau^{3/2}\|u\|_{X_\tau}\|\theta\|_{Y_\tau}+\tau^{3/2}\|\theta\|_{X_\tau}\|u\|_{Y_\tau}+\|\theta\|_{H^r}\\&+\left(\tau(\|\nabla u\|_{L^\infty}+\|\nabla\theta\|_{L^\infty})+(\tau^2+\tau^3)\|u\|_{H^r}+\tau^2\|\theta\|_{H^r}\right)\|\theta\|_{Y_\tau}.\end{aligned}$$

Let

$$F=\|u\|_{X_\tau}+\|\theta\|_{X_\tau}$$

and

$$N=\|u\|_{Y_\tau}+\|\theta\|_{Y_\tau}.$$

Then, we have

$$\begin{aligned}\frac{d}{dt}F\leq{}&\dot\tau(t)N+C\|u\|_{H^r}^2(1+\tau(t)^2)+(1+\tau)\|u\|_{H^r}\|\theta\|_{H^r}+\tau^{3/2}FN+\|\theta\|_{H^r}\\&+CN\left(\tau(\|\nabla u\|_{L^\infty}+\|\nabla\theta\|_{L^\infty})+(\tau^2+\tau^3)\|u\|_{H^r}+\tau^2\|\theta\|_{H^r}+\tau+1\right)\\={}&N\big(\dot\tau(t)+C\|u\|_{H^r}^2(1+\tau(t)^2)+(1+\tau)\|u\|_{H^r}\|\theta\|_{H^r}+\|\theta(t)\|_{H^r}\\&+C\left((\tau^2+\tau^3)\|u\|_{H^r}+\tau^2\|\theta\|_{H^r}+1\right)+\tau^{3/2}F\big)\\&+C\tau(\|\nabla u\|_{L^\infty}+\|\nabla\theta\|_{L^\infty}+1).\end{aligned}\tag{3.1}$$

Thus, if $\tau(t)$ decreases fast enough so that for all $0\leq t\leq T$, we have

$$\dot\tau(t)+C\tau(\|\nabla u\|_{L^\infty}+\|\nabla\theta\|_{L^\infty}+1)+C\left((\tau^2+\tau^3)\|u\|_{H^r}+\tau^2\|\theta\|_{H^r}\right)+\tau^{3/2}F\leq0.\tag{3.2}$$

Then, (3.1) implies

$$\frac{d}{dt}F \le C\|u(t)\|_{H^r}^2(1+\tau(0)^2)+(1+\tau(0))\|u(t)\|_{H^r}\|\theta(t)\|_{H^r}+\|\theta(t)\|_{H^r},$$

where we used the facts that $N \ge 0$ and $\tau(t) \le \tau(0)$ for $0 \le t \le T$. Thus, for $0 \le t \le T$, we have

$$\begin{aligned} F(t) &\le F(0)+C\int_0^t \|u(t')\|_{H^r}^2(1+\tau(0)^2)+(1+\tau(0))\|u(t')\|_{H^r}\|\theta(t')\|_{H^r}+\|\theta(t')\|_{H^r}\,dt' \\ &\le F(0)+C\int_0^t \|u(t')\|_{H^r}^2+\|u(t')\|_{H^r}\|\theta(t')\|_{H^r}+\|\theta(t')\|_{H^r}\,dt' = L(t). \end{aligned} \tag{3.3}$$

Since τ must be chosen to be a decreasing function, a sufficient condition for (3.2) to hold is that

$$\dot\tau(t)+C\tau(\|\nabla u\|_{L^\infty}+\|\nabla\theta\|_{L^\infty}+1)+C\tau^{3/2}\Big(C'_{\tau(0)}(\|u\|_{H^r}+\|\theta\|_{H^r})+F\Big)\le 0,$$

where $C'_{\tau(0)}=\tau^{1/2}(0)+\tau^{3/2}(0)$. Next, for simplicity, we denote

$$G(t)=\exp\left(C\int_0^t(1+\|\nabla u(t')\|_{L^\infty}+\|\nabla\theta(t')\|_{L^\infty})\,dt'\right),$$

where the constant $C>0$ is chosen to be large enough so that

$$\|u(t)\|_{H^r}^2+\|\theta(t)\|_{H^r}^2 \le (\|u_0\|_{H^r}^2+\|\theta_0\|_{H^r}^2)G(t).$$

Thus, we get

$$G^{-1}(t)(\|u(t)\|_{H^r}^2+\|\theta(t)\|_{H^r}^2)\le \|u_0\|_{H^r}^2+\|\theta_0\|_{H^r}^2. \tag{3.4}$$

If we let

$$\tau(t)=G(t)^{-1/2}\left(\tau(0)^{-1/2}+C\int_0^t\Big(C'_{\tau(0)}(\|u\|_{H^r}+\|\theta\|_{H^r})+L(t')\Big)G(t')^{-1}\,dt'\right)^{-1/2}. \tag{3.5}$$

Then, we see (3.2) is satisfied by taking the derivative above. By (3.3) and (3.4), we get

$$\begin{aligned} &\tau(0)^{-1/2}+C\int_0^t\Big(C'_{\tau(0)}(\|u\|_{H^r}+\|\theta\|_{H^r})+L(t')\Big)G(t')^{-1}\,dt' \\ \le&\tau(0)^{-1/2} \\ &\quad +C\int_0^t\Big(C'_{\tau(0)}(\|u_0\|_{H^r}+\|\theta_0\|_{H^r})+Ct'(\|u_0\|_{H^r}^2+\|u_0\|_{H^r}\|\theta_0\|_{H^r}+\|\theta_0\|_{H^r})\Big)\,dt' \\ \le&C_0(1+t)^2. \end{aligned}$$

Therefore, by (3.5), we obtain the desired lower bound on the radius

$$\begin{aligned} \tau(t) &\ge \frac{1}{C_0(1+t)}G(t)^{-1} \\ &=\frac{1}{C_0(1+t)}\exp\left(-C\int_0^t(1+\|\nabla u(t')\|_{L^\infty}+\|\nabla\theta(t')\|_{L^\infty})\,dt'\right). \end{aligned} \tag{3.6}$$

The last inequality in (3.6) above gives the explicit dependence on the initial data and thus we conclude the a priori estimates that are used to prove Theorem 2.2. The proof can be made rigorous by considering an approximating solution $(u^{(n)},\theta^{(n)})$, $n\in N$, proving the above estimates for $(u^{(n)},\theta^{(n)})$, and then taking the limit as $n\to\infty$. We omit further details. □

Now we give the proof of Lemmas 2.3 and 2.4.

4. Proofs of Lemmas 2.3 and 2.4

Proof. **(Proof of Lemma 2.4.)** For I_4, we split the sum depending on the values of m and j, where $|\beta|=j$. We split them into low j when $|\beta|=j$ in I_{41}, intermediate j in I_{42}, and high j in I_{43} so that we couple with the estimates in Lemma 2.2. We obtain

$$I_4=\sum_{m=3}^{\infty}\sum_{|\alpha|=m}\sum_{\beta\le\alpha,\beta\ne0}\frac{\tau(t)^{m-3}}{(m-3)!^s}M_\alpha\binom{\alpha}{\beta}\|\partial^\beta u\cdot\nabla\partial^{\alpha-\beta}\theta\|_{L^2}=I_{41}+I_{42}+I_{43},$$

where

$$\begin{aligned}I_{41}=&\sum_{m=3}^{\infty}\sum_{|\alpha|=m}\sum_{\beta\le\alpha,|\beta|=1}\frac{\tau(t)^{m-3}}{(m-3)!^s}M_\alpha\binom{\alpha}{\beta}\|\partial^\beta u\cdot\nabla\partial^{\alpha-\beta}\theta\|_{L^2}\\&+\sum_{m=3}^{\infty}\sum_{|\alpha|=m}\sum_{\beta\le\alpha,|\beta|=2}\frac{\tau(t)^{m-3}}{(m-3)!^s}M_\alpha\binom{\alpha}{\beta}\|\partial^\beta u\cdot\nabla\partial^{\alpha-\beta}\theta\|_{L^2}=I_{411}+I_{412}\end{aligned}$$

and

$$\begin{aligned}I_{42}=&\sum_{m=6}^{\infty}\sum_{j=3}^{[m/2]}\sum_{|\alpha|=m}\sum_{\beta\le\alpha,|\beta|=j}\frac{\tau(t)^{m-3}}{(m-3)!^s}M_\alpha\binom{\alpha}{\beta}\|\partial^\beta u\cdot\nabla\partial^{\alpha-\beta}\theta\|_{L^2}\\&+\sum_{m=7}^{\infty}\sum_{j=[m/2]+1}^{m-3}\sum_{|\alpha|=m}\sum_{\beta\le\alpha,|\beta|=j}\frac{\tau(t)^{m-3}}{(m-3)!^s}M_\alpha\binom{\alpha}{\beta}\|\partial^\beta u\|_{L^\infty}\|\nabla\partial^{\alpha-\beta}\theta\|_{L^2}\\=&I_{421}+I_{422}\end{aligned}$$

and

$$\begin{aligned}I_{43}=&\sum_{m=5}^{\infty}\sum_{|\alpha|=m}\sum_{\beta\le\alpha,|\beta|=m-2}\frac{\tau(t)^{m-3}}{(m-3)!^s}M_\alpha\binom{\alpha}{\beta}\|\partial^\beta u\cdot\nabla\partial^{\alpha-\beta}\theta\|_{L^2}\\&+\sum_{m=4}^{\infty}\sum_{|\alpha|=m}\sum_{\beta\le\alpha,|\beta|=m-1}\frac{\tau(t)^{m-3}}{(m-3)!^s}M_\alpha\binom{\alpha}{\beta}\|\partial^\beta u\cdot\nabla\partial^{\alpha-\beta}\theta\|_{L^2}\\&+\sum_{m=4}^{\infty}\sum_{|\alpha|=|\beta|=m}\frac{\tau(t)^{m-3}}{(m-3)!^s}M_\alpha\binom{\alpha}{\beta}\|\partial^\beta u\|_{L^\infty}\|\nabla\partial^{\alpha-\beta}\theta\|_{L^2}=I_{431}+I_{432}+I_{433}.\end{aligned}$$

For I_{41}, we apply Hölder's inequality and get

$$\begin{aligned}I_{41}=&\sum_{|\alpha|=3}\sum_{\beta\le\alpha,\beta\ne0}M_\alpha\binom{\alpha}{\beta}\|\partial^\beta u\cdot\nabla\partial^{\alpha-\beta}\theta\|_{L^2}\\&+\sum_{m=4}^{\infty}\sum_{|\alpha|=m}\sum_{\beta\le\alpha,\beta\ne0}\frac{\tau(t)^{m-3}}{(m-3)!^s}M_\alpha\binom{\alpha}{\beta}\|\partial^\beta u\cdot\nabla\partial^{\alpha-\beta}\theta\|_{L^2}\end{aligned}$$

$$\le \sum_{|\alpha|=3}\sum_{\beta\le\alpha,\beta\neq 0} M_\alpha \binom{\alpha}{\beta} \|\partial^\beta u\|_{L^\infty}\|\nabla\partial^{\alpha-\beta}\theta\|_{L^2}$$
$$+\sum_{m=4}^{\infty}\sum_{|\alpha|=m}\sum_{\beta\le\alpha,\beta\neq 0}\frac{\tau(t)^{m-3}}{(m-3)!^s} M_\alpha \binom{\alpha}{\beta} \|\partial^\beta u\|_{L^\infty}\|\nabla\partial^{\alpha-\beta}\theta\|_{L^2}.$$

It is easy to see that for $\alpha,\beta\in\mathbb{N}_0^3$ with $\beta\le\alpha$, we have

$$\binom{\alpha'}{\beta'} M_\alpha M_\beta^{-1} M_{\alpha-\beta}^{-1} \le \binom{|\alpha|}{|\beta|}.$$

See [16] for a proof. Therefore, we get

$$\begin{aligned}
I_{411} \le& \sum_{|\alpha|=3}\sum_{\beta\le\alpha,\beta\neq 0}\left(M_\beta\|\partial^\beta u\|_{L^\infty}\right)\left(M_{\alpha-\beta}\|\nabla\partial^{\alpha-\beta}\theta\|_{L^2}\right) M_\alpha M_\beta^{-1} M_{\alpha-\beta}^{-1}\binom{\alpha}{\beta}\\
&+\sum_{m=4}^{\infty}\sum_{|\alpha|=m}\sum_{\beta\le\alpha,\beta\neq 0}\left(M_\beta\|\partial^\beta u\|_{L^\infty}\right)\left(M_{\alpha-\beta}\|\nabla\partial^{\alpha-\beta}\theta\|_{L^2}\frac{(m-3)\tau(t)^{m-4}}{(m-3)!^s}\right)\\
&\times M_\alpha M_\beta^{-1} M_{\alpha-\beta}^{-1}\binom{\alpha}{\beta}\frac{\tau(t)}{m-3}\\
\le& C\sum_{|\alpha|=3}\sum_{\beta\le\alpha,\beta\neq 0}\left(M_\beta\|\partial^\beta u\|_{L^\infty}\right)\left(M_{\alpha-\beta}\|\nabla\partial^{\alpha-\beta}\theta\|_{L^2}\right)\\
&+C\tau(t)\sum_{m=4}^{\infty}\sum_{|\alpha|=m}\sum_{\beta\le\alpha,\beta\neq 0}\left(M_\beta\|\partial^\beta u\|_{L^\infty}\right)\left(M_{\alpha-\beta}\|\nabla\partial^{\alpha-\beta}\theta\|_{L^2}\frac{(m-3)\tau(t)^{m-4}}{(m-3)!^s}\right)\\
\le& C|u|_{1,\infty}|\nabla\theta|_2 + C\tau(t)|u|_{1,\infty}\|\theta\|_{Y_\tau} \le C|u|_{1,\infty}|\theta|_3 + C\tau(t)|u|_{1,\infty}\|\theta\|_{Y_\tau}.
\end{aligned}$$

For I_{412}, we separate it as

$$\begin{aligned}
I_{412} =& \sum_{|\alpha|=3,4}\sum_{\beta\le\alpha,|\beta|=2}\frac{\tau(t)^{m-3}}{(m-3)!^s} M_\alpha\binom{\alpha}{\beta}\|\partial^\beta u\cdot\nabla\partial^{\alpha-\beta}\theta\|_{L^2}\\
&+\sum_{m=5}^{\infty}\sum_{|\alpha|=m}\sum_{\beta\le\alpha,|\beta|=2}\frac{\tau(t)^{m-3}}{(m-3)!^s} M_\alpha\binom{\alpha}{\beta}\|\partial^\beta u\cdot\nabla\partial^{\alpha-\beta}\theta\|_{L^2}\\
\le& \sum_{|\alpha|=3,4}\sum_{\beta\le\alpha,|\beta|=2}\tau^{m-3}\left(M_\beta\|\partial^\beta u\|_{L^\infty}\right)\left(M_{\alpha-\beta}\|\nabla\partial^{\alpha-\beta}\theta\|_{L^2}\right) M_\alpha M_\beta^{-1} M_{\alpha-\beta}^{-1}\binom{\alpha}{\beta}\\
&+\sum_{m=5}^{\infty}\sum_{|\alpha|=m}\sum_{\beta\le\alpha,|\beta|=2}\left(M_\beta\|\partial^\beta u\|_{L^\infty}\right)\left(M_{\alpha-\beta}\|\nabla\partial^{\alpha-\beta}\theta\|_{L^2}\frac{(m-4)\tau(t)^{m-5}}{(m-4)!^s}\right)\\
&\times M_\alpha M_\beta^{-1} M_{\alpha-\beta}^{-1}\binom{\alpha}{\beta}\frac{\tau(t)^2}{(m-3)(m-4)}\\
\le& C|u|_{2,\infty}|\theta|_2 + C\tau|u|_{2,\infty}|\theta|_3 + C\tau^2|u|_{2,\infty}\|\theta\|_{Y_\tau}.
\end{aligned}$$

For I_{421}, we know by Hölder's and Sobolev inequalities

$$\|\partial^\beta u\cdot\nabla\partial^{\alpha-\beta}\theta\|_{L^2} \le \|\partial^\beta u\|_{L^\infty}\|\nabla\partial^{\alpha-\beta}\theta\|_{L^2} \le \|\partial^\beta u\|_{L^2}^{1/4}\|\Delta\partial^\beta u\|_{L^2}^{3/4}\|\nabla\partial^{\alpha-\beta}\theta\|_{L^2}.$$

Thus,

$$I_{421} \leq \tau^{3/2} \sum_{m=6}^{\infty} \sum_{j=3}^{[m/2]} \sum_{|\alpha|=m} \sum_{\beta\leq\alpha,|\beta|=j} \left(M_\beta \|\partial^\beta u\|_{L^\infty} \frac{\tau(t)^{j-3}}{(j-3)!^s} \right)^{1/4}$$
$$\times \left(M_\beta \|\Delta\partial^\beta u\|_{L^2} \frac{\tau(t)^{j-1}}{(j-1)!^s} \right)^{3/4} \left(M_{\alpha-\beta} \|\nabla\partial^{\alpha-\beta}\theta\|_{L^2} \frac{(m-j-2)\tau(t)^{m-j-2}}{(m-j-2)!^s} \right) \mathcal{N}_{\alpha,\beta,s},$$

where for $s\geq 1$ and by (2.2) we have

$$N_{\alpha,\beta,s} = M_\alpha M_\beta^{-1} M_{\alpha-\beta}^{-1} \binom{\alpha}{\beta} \frac{(j-3)!^{s/4}(j-3)!^{3s/4}(m-j-2)!^s}{(m-3)!^s(m-j-2)} \leq C.$$

Therefore, we see

$$I_{421} \leq C\tau^{3/2} \sum_{m=6}^{\infty} \sum_{j=3}^{[m/2]} \left(|u|_j \frac{\tau(t)^{j-3}}{(j-3)!^s} \right)^{1/4} \left(|\Delta u|_j \frac{\tau(t)^{j-1}}{(j-1)!^s} \right)^{3/4}$$
$$\times \left(|\nabla\theta|_{m-j} \frac{(m-j-2)\tau(t)^{m-j-2}}{(m-j-2)!^s} \right) \leq C\tau^{3/2} \|u\|_{X_\tau} \|\theta\|_{Y_\tau}.$$

For the estimates of I_{422}, I_{431}, I_{432}, and I_{433}, we reverse the indices j and $m-j$ and apply Hölder's inequality and get

$$\|\partial^\beta u\cdot\nabla\partial^{\alpha-\beta}\theta\|_{L^2} \leq \|\partial^\beta u\|_{L^2} \|\nabla\partial^{\alpha-\beta}\theta\|_{L^\infty}.$$

Therefore, we can similarly obtain $C\tau^{3/2}\|\theta\|_{X_\tau}\|u\|_{Y_\tau}$ as the upper bound for I_{422}, I_{431}, I_{432}, and I_{433}.

$$I_{422} \leq C\tau^{3/2}\|\theta\|_{X_\tau}\|u\|_{Y_\tau},\ I_{431} \leq C\tau^3|u|_{3,\infty}\|\theta\|_{Y_\tau},\ I_{432} \leq C\tau|\theta|_{2,\infty}|u|_3 + C\tau^2|\theta|_{2,\infty}\|u\|_{Y_\tau},$$

and

$$I_{433} \leq C\tau|\theta|_{1,\infty}|u|_3 + C\tau|\theta|_{1,\infty}\|u\|_{Y_\tau}.$$

Thus, by combining the above estimates we get

$$I_4 \leq C(\mathcal{K}_1 + \mathcal{K}_2\|\theta\|_{Y_\tau} + \tau^{3/2}\|u\|_{X_\tau}\|\theta\|_{Y_\tau} + \tau^{3/2}\|\theta\|_{X_\tau}\|u\|_{Y_\tau}),$$

where

$$\mathcal{K}_1 = |u|_{1,\infty}|\theta|_3 + |u|_{2,\infty}|\theta|_2 + \tau|u|_{2,\infty}|\theta|_3 + \tau|u|_{1,\infty}|\theta|_3 + \tau|\theta|_{2,\infty}|u|_3 + \tau|\theta|_{1,\infty}|u|_3,$$

and

$$\mathcal{K}_2 = \tau(t)|u|_{1,\infty} + \tau^2|u|_{2,\infty} + \tau^3|u|_{3,\infty} + \tau|\theta|_{1,\infty} + \tau^2|\theta|_{2,\infty}.$$

Therefore, Lemma 2.4 is proved. □

Next, we prove Lemma 2.3.

Proof. **(Proof of Lemma 2.3.)** For I_2, we apply Lemma 2.1 and get

$$\sum_{|\alpha|=m,\alpha_3\neq 0} M_\alpha \|\partial_3\partial^\alpha p\|_{L^2}$$

$$\leq C \sum_{|\alpha|=m,\alpha_3\neq 0} \sum_{\substack{s,t\in\mathbb{N}_0,|\beta|=m-1\\ \beta'-\alpha'=(2s,2t)}} \binom{s+t}{s} \|\partial^\beta(\partial_i u_k \partial_k u_i)\|_{L^2}$$
$$+ C \sum_{|\alpha|=m,\alpha_3\neq 0} \sum_{\substack{s,t\in\mathbb{N}_0,|\beta|=m-1\\ \beta'-\alpha'=(2s,2t)}} \binom{s+t}{s} \|\partial^\beta(\partial_3 \theta)\|_{L^2}$$
$$= C \sum_{|\beta|=m-1} \sum_{s=0}^{[\beta_1/2]} \sum_{t=0}^{[\beta_2/2]} \binom{\beta_1+\beta_2-2s-2t}{\beta_1-2s} \binom{s+t}{s} \|\partial^\beta(\partial_i u_k \partial_k u_i)\|_{L^2}$$
$$+ C \sum_{|\beta|=m-1} \sum_{s=0}^{[\beta_1/2]} \sum_{t=0}^{[\beta_2/2]} \binom{\beta_1+\beta_2-2s-2t}{\beta_1-2s} \binom{s+t}{s} \|\partial^\beta(\partial_3 \theta)\|_{L^2}. \tag{4.1}$$

By the pressure estimates from Lemma 2.1, we get

$$\sum_{|\alpha|=m,\alpha_3\neq 0} M_\alpha \|\partial_3\partial^\alpha p\|_{L^2} \leq Cm \sum_{|\beta|=m-1} M_\beta \|\partial^\beta(\partial_i u_k \partial_k u_i)\|_{L^2} + Cm \sum_{|\beta|=m-1} M_\beta \|\partial^\beta(\partial_3 \theta)\|_{L^2}.$$

Thus,

$$\sum_{m=3}^{\infty} \left(\sum_{|\alpha|=m,\alpha_3\neq 0} M_\alpha \|\partial_3\partial^\alpha p\|_{L^2} \right) \leq \sum_{m=3}^{\infty} \sum_{|\beta|=m-1} M_\beta \|\partial^\beta(\partial_i u_k \partial_k u_i)\|_{L^2} \frac{m\tau(t)^{m-3}}{(m-3)!^s} + \sum_{m=3}^{\infty} \sum_{|\beta|=m-1} M_\beta \|\partial^\beta(\partial_3 \theta)\|_{L^2} \frac{m\tau(t)^{m-3}}{(m-3)!^s}.$$

For higher derivatives of $\partial_1 p$, we decompose as follows

$$\sum_{|\alpha|=m,\alpha_3\neq 0} M_\alpha \|\partial_1\partial^\alpha p\|_{L^2} = \sum_{|\alpha|=m,\alpha_3=1} M_\alpha \|\partial_1\partial^\alpha p\|_{L^2} + \sum_{|\alpha|=m,\alpha_3\geq 2} M_\alpha \|\partial_1\partial^\alpha p\|_{L^2} = J_1 + J_2.$$

For J_1, we have

$$J_1 = \sum_{|\alpha|=m,\alpha_3=1} M_\alpha \|\partial_1\partial^\alpha p\|_{L^2}$$
$$\leq C \sum_{|\alpha|=m,\alpha_3=1} M_\alpha \|\partial^{\alpha'}(\partial_i u_k \partial_k u_i)\|_{L^2} + C \sum_{|\alpha|=m,\alpha_3=1} M_\alpha \|\partial^{\alpha'}(\partial_3 \theta)\|_{L^2}$$
$$= C \sum_{|\beta|=m-1,\beta_3=0} M_\beta \|\partial^\beta(\partial_i u_k \partial_k u_i)\|_{L^2} + C \sum_{|\beta|=m-1,\beta_3=0} M_\beta \|\partial^\beta(\partial_3 \theta)\|_{L^2}.$$

For J_2, we have

$$J_2 \leq \sum_{|\alpha|=m,\alpha_3\geq 2} M_\alpha \|\partial_1\partial^\alpha p\|_{L^2} \leq CJ_{21} + CJ_{22},$$

where

$$\begin{aligned}J_{21}=&\sum_{|\alpha|=m,\alpha_3\geq 2}\sum_{\substack{s,t\in\mathbb{N}_0,|\beta|=m-1\\ \beta'-\alpha'=(2s+1,2t)}}\binom{s+t}{s}\|\partial^\beta(\partial_i u_k\partial_k u_i)\|_{L^2}\\ =&\sum_{|\beta|=m-1,\beta_1\geq 1}\sum_{s=0}^{[(\beta_1-1)/2]}\sum_{s=0}^{[(\beta_2-1)/2]}\binom{\beta_1-1+\beta_2-2s-2t}{\beta_1-2s-1}\binom{s+t}{s}\|\partial^\beta(\partial_i u_k\partial_k u_i)\|_{L^2}\\ \leq& Cm\sum_{|\beta|=m-1,\beta_1\geq 1}M_\beta\|\partial^\beta(\partial_i u_k\partial_k u_i)\|_{L^2}\end{aligned}$$

and

$$\begin{aligned}J_{22}=&\sum_{|\alpha|=m,\alpha_3\geq 2}\sum_{\substack{s,t\in\mathbb{N}_0,|\beta|=m-1\\ \beta'-\alpha'=(2s+1,2t)}}\binom{s+t}{s}\|\partial^\beta(\partial_3\theta)\|_{L^2}\\ =&\sum_{|\beta|=m-1,\beta_1\geq 1}\sum_{s=0}^{[(\beta_1-1)/2]}\sum_{s=0}^{[(\beta_2-1)/2]}\binom{\beta_1-1+\beta_2-2s-2t}{\beta_1-2s-1}\binom{s+t}{s}\|\partial^\beta(\partial_3\theta)\|_{L^2}\\ \leq& Cm\sum_{|\beta|=m-1,\beta_1\geq 1}M_\beta\|\partial^\beta(\partial_3\theta)\|_{L^2}.\end{aligned}$$

By [16], we know that there exists a positive universal constant C such that

$$\sum_{s=0}^{[\beta_1/2]}\sum_{t=0}^{[\beta_2/2]}\binom{\beta_1+\beta_2-2s-2t}{\beta_1-2s}\binom{s+t}{s}\leq Cm\binom{\beta_1+\beta_2}{\beta_1}.$$

Thus,

$$\begin{aligned}J_{21}=&\sum_{|\beta|=m-1,\beta_1\geq 1}\sum_{s=0}^{[(\beta_1-1)/2]}\sum_{s=0}^{[(\beta_2-1)/2]}\binom{\beta_1-1+\beta_2-2s-2t}{\beta_1-2s-1}\binom{s+t}{s}\|\partial^\beta(\partial_i u_k\partial_k u_i)\|_{L^2}\\ \leq& Cm\sum_{|\beta|=m-1,\beta_1\geq 1}M_\beta\|\partial^\beta(\partial_i u_k\partial_k u_i)\|_{L^2}\end{aligned}$$

and

$$\begin{aligned}J_{22}=&\sum_{|\beta|=m-1,\beta_1\geq 1}\sum_{s=0}^{[(\beta_1-1)/2]}\sum_{s=0}^{[(\beta_2-1)/2]}\binom{\beta_1-1+\beta_2-2s-2t}{\beta_1-2s-1}\binom{s+t}{s}\|\partial^\beta(\partial_3\theta)\|_{L^2}\\ \leq& Cm\sum_{|\beta|=m-1,\beta_1\geq 1}M_\beta\|\partial^\beta(\partial_3\theta)\|_{L^2}.\end{aligned}$$

Therefore, combining the above estimates of J_{21} and J_{22} yielding

$$\begin{aligned}\sum_{m=3}^{\infty}\left(\sum_{|\alpha|=m,\alpha_3\neq 0}M_\alpha\|\partial_1\partial^\alpha p\|_{L^2}\right)\leq&\sum_{m=3}^{\infty}\sum_{|\beta|=m-1}M_\beta\|\partial^\beta(\partial_i u_k\partial_k u_i)\|_{L^2}\frac{m\tau(t)^{m-3}}{(m-3)!^s}\\ &+\sum_{m=3}^{\infty}\sum_{|\beta|=m-1}M_\beta\|\partial^\beta(\partial_3\theta)\|_{L^2}\frac{m\tau(t)^{m-3}}{(m-3)!^s}.\end{aligned}$$

By symmetry between ∂_1 and ∂_2, we similarly get

$$\sum_{m=3}^{\infty}\left(\sum_{|\alpha|=m,\alpha_3\neq 0} M_\alpha\|\partial_2\partial^\alpha p\|_{L^2}\right)\leq \sum_{m=3}^{\infty}\sum_{|\beta|=m-1} M_\beta\|\partial^\beta(\partial_i u_k\partial_k u_i)\|_{L^2}\frac{m\tau(t)^{m-3}}{(m-3)!^s} + \sum_{m=3}^{\infty}\sum_{|\beta|=m-1} M_\beta\|\partial^\beta(\partial_3\theta)\|_{L^2}\frac{m\tau(t)^{m-3}}{(m-3)!^s}.$$

Combine the above estimates and we have

$$I_2\leq \sum_{m=3}^{\infty}\sum_{|\beta|=m-1} M_\beta\|\partial^\beta(\partial_i u_k\partial_k u_i)\|_{L^2}\frac{m\tau(t)^{m-3}}{(m-3)!^s} + \sum_{m=3}^{\infty}\sum_{|\beta|=m-1} M_\beta\|\partial^\beta(\partial_3\theta)\|_{L^2}\frac{m\tau(t)^{m-3}}{(m-3)!^s}\leq \sum_{m=3}^{\infty}\sum_{j=0}^{m-1} I_{2,j}+I_{2,\theta}, \tag{4.2}$$

where

$$I_{2,j}=\frac{m\tau(t)^{m-3}}{(m-3)!^s}\sum_{|\beta|=m-1}\sum_{\gamma=j,\gamma\leq\beta} M_\beta\|\partial^\gamma\partial_i u_k\cdot\partial^{\beta-\gamma}\partial_k u_i\|_{L^2},$$

and

$$I_{2,\theta}=\sum_{m=3}^{\infty}\sum_{|\beta|=m-1} M_\beta\frac{m\tau(t)^{m-3}}{(m-3)!^s}\|\partial^\beta(\partial_3\theta)\|_{L^2}$$
$$\leq C|\partial_3\theta|_2+\sum_{m=4}^{\infty} M_\beta\frac{m\tau(t)^{m-4}}{(m-3)!^s}|\partial_3\theta|_{m-1}\leq C|\theta|_3+C\tau\|\theta\|_{Y_\tau}.$$

Similarly to [16], we address $I_{2,j}$ according to the values of m and j by considering the cases of low j, intermediate j, and high j, thus we obtain for low j

$$\sum_{m=3}^{\infty} I_{2,0}\leq C|u|_{1,\infty}|u|_3+C\tau|u|_{1,\infty}\|u\|_{Y_\tau},$$

$$\sum_{m=3}^{\infty} I_{2,1}\leq C|u|_{2,\infty}|u|_2+\tau|u|_{2,\infty}|u|_3+C\tau^2|u|_{2,\infty}\|u\|_{Y_\tau},$$

$$\sum_{m=5}^{\infty} I_{2,2}\leq C\tau^2|u|_{3,\infty}|u|_3+C\tau^3|u|_{3,\infty}\|u\|_{Y_\tau}.$$

For intermediate j,

$$\sum_{m=8}^{\infty}\sum_{j=3}^{[m/2]-1} I_{2,j}\leq C\tau^{3/2}\|u\|_{X_\tau}\|u\|_{Y_\tau},$$

$$\sum_{m=6}^{\infty}\sum_{j=[m/2]-1}^{m-3} I_{2,j}\leq C\tau^{3/2}\|u\|_{X_\tau}\|u\|_{Y_\tau},$$

$$\sum_{m=8}^{\infty}\sum_{j=3}^{[m/2]-1} I_{2,j} \leq |u|_{1,\infty}|u|_3 + |u|_{2,\infty}|u|_2 + \tau|u|_{2,\infty}|u|_3 + \tau^2|u|_{3,\infty}|u|_3.$$

For high j,

$$\sum_{m=4}^{\infty} I_{2,m-2} \leq C\tau|u|_{2,\infty}|u|_3 + C\tau^2|u|_{2,\infty}\|u\|_{Y_\tau},$$

$$\sum_{m=3}^{\infty} I_{2,m-1} \leq C\tau|u|_{1,\infty}|u|_3 + C\tau|u|_{1,\infty}\|u\|_{Y_\tau}.$$

Thus, by combining the above estimates we get

$$\sum_{m=4}^{\infty} I_{2,m-2} \leq C\tau|u|_{2,\infty}|u|_3 + C\tau^2|u|_{2,\infty}\|u\|_{Y_\tau}$$

and

$$I_2 \leq C(\mathcal{L}_1 + \mathcal{L}_2\|u\|_{Y_\tau} + \tau\|\theta\|_{Y_\tau}),$$

where

$$\mathcal{L}_1 \leq \tau|u|_{1,\infty}|u|_3 + |u|_{2,\infty}|u|_2 + \tau|u|_{2,\infty}|u|_3 + |\theta|_3,$$

$$\mathcal{L}_2 \leq \tau|u|_{1,\infty} + \tau^2|u|_{2,\infty} + \tau^3|u|_{3,\infty} + \tau^{3/2}\|u\|_{X_\tau}.$$

Thus, we have the estimates of I_2. □

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