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AN ENERGY PRESERVING DISCRETIZATION METHOD FOR THE THERMODYNAMIC KURAMOTO MODEL AND COLLECTIVE BEHAVIORS*

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Abstract. We provide an energy preserving discretization method for the thermodynamic Kuramoto (TK) model on a lattice and investigate its emergent dynamics, and show a smooth transition from the proposed discrete model to the corresponding continuous model. The thermodynamic Kuramoto model describes the temporal evolution of the phase and temperature at each lattice point in a domain. To integrate the continuous model numerically, one needs to discretize the continuous model in a suitable way so that the resulting discrete model exhibits the same emergent features as the corresponding continuous model. The naive forward Euler discretization for phase-temperature configuration does not conserve a total energy, which causes inconsistency with the continuous model. Thus, we instead propose an implicit scheme which preserves energy and satisfies entropy principle, and provide several sufficient frameworks leading to the emergent collective behaviors and uniform-in-time smooth transition from the discrete model to the continuous model.

Keywords. Emergence; entropy principle; Kuramoto model; thermodynamics.

AMS subject classifications. 39A30; 34E10; 65L05.

1. Introduction

In this paper, we propose and investigate an energy preserving discretization for thermodynamic Kuramoto (TK) lattice model [25]. The TK (lattice) model describes the synchronization of phases/frequencies and homogenization of temperatures of Kuramoto oscillators at each lattice point, and it consists of the Kuramoto-like phase dynamics and autonomous temperature equations. Here, *"synchronization"* is a collective phenomenon which represents an adjustment of rhythms of the coupled oscillators [4, 13, 15, 16]. Since Christiaan Huygens' first observation in 1665, synchronization behavior in nature, such as firing neurons and flashing fireflies, were observed by applied mathematicians, physicists and biologists [1–3, 5, 30, 31, 33, 39, 40]. After Vicsek's seminal work [38] on the modeling of collective dynamics, the velocity alignment of interacting individuals moving with a constant speed were also successfully described by a synchronization of their heading angles [10–12, 17]. Despite its omnipresence in nature and human society, a rigorous mathematical study had been initiated by two pioneers, Arthur Winfree and Yoshiki Kuramoto about a half-century ago. In [29], Y. Kuramoto suggested a mathematical model to describe synchronization phenomenon from the system of linearly coupled Stuart-Landau oscillators. Consider a lattice Λ consisting of N lattice points $x_\alpha \in \Lambda$ with $\alpha = 1, \cdots, N$, and assume that Kuramoto oscillators are located at each lattice point. Denote $\theta_\alpha = \theta_\alpha(t)$ the phase of the Kuramoto oscillator at each lattice point x_α. Then, in the absence of thermal interaction, the dynamics of

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phase θ_α is governed by the following ODEs:

$$\begin{cases} \dot{\theta}_\alpha = \nu_\alpha + \dfrac{\kappa}{N}\displaystyle\sum_{\beta=1}^{N} \sin(\theta_\beta - \theta_\alpha), \quad t>0, \\ \theta_\alpha(0) = \theta_\alpha^{in}, \quad \alpha = 1,\cdots,N, \end{cases} \tag{1.1}$$

where $\kappa > 0$ is a global scale parameter of coupling strengths and ν_α is a real-valued constant representing the natural frequency of α-th oscillator in the absence of coupling. Recently, complete synchronization of the Kuramoto model [29] has been extensively studied in various settings, e.g., phase-locking [7, 8, 16, 28, 36, 37] for a restricted initial configuration, a generic initial configuration [20], locally interacting topology [24], effects of inertia and frustration [22, 23], temperature effect [25], etc. For readers who are interested in the related works, we also refer to a survey article [15]. In particular, the authors in [25] introduced a generalization of (1.1) coupled with temperature equation from the thermodynamic Cucker-Smale model [9, 21, 26]. In what follows, we call such a coupled model as the thermodynamic Kuramoto (TK) model. More precisely, let $\theta_\alpha = \theta_\alpha(t)$ and ν_α be the phase and natural frequency of the α-th oscillator, respectively, and let $T_\alpha = T_\alpha(t)$ be a corresponding temperature of α-th oscillator. Then, the temporal evolution of $\{(\theta_\alpha, T_\alpha)\}_{\alpha=1}^N$ is given by the following first-order system

$$\begin{cases} \dot{\theta}_\alpha = \nu_\alpha + \dfrac{\kappa_1}{N}\displaystyle\sum_{\beta=1}^{N} \frac{\psi_{\alpha\beta}}{T_\alpha} \sin(\theta_\beta - \theta_\alpha), \quad t>0, \\ \dot{T}_\alpha = \dfrac{\kappa_2}{N}\displaystyle\sum_{\beta=1}^{N} \frac{\zeta_{\alpha\beta} T_*^2}{(T_*^2 + \eta^2 T_\alpha)} \left(\frac{1}{T_\alpha} - \frac{1}{T_\beta}\right), \quad t>0, \\ (\theta_\alpha, T_\alpha)\Big|_{t=0} = (\theta_\alpha^{in}, T_\alpha^{in}), \quad \alpha = 1,\cdots,N, \end{cases} \tag{1.2}$$

where κ_1 and κ_2 are positive coupling strengths, and coefficients $\psi_{\alpha\beta}, \zeta_{\alpha\beta}$ represent the positive, symmetric newtork topology on lattice Λ:

$$0 < \psi_{\alpha\beta} = \psi_{\beta\alpha} < \infty, \quad 0 < \zeta_{\alpha\beta} = \zeta_{\beta\alpha} < \infty, \quad \forall\ \alpha,\beta = 1,\cdots,N.$$

As a thermomechanically consistent model, (1.2) follows the energy conservation and entropy principle, where the energy $\mathcal{E}$ and entropy $\mathcal{S}$ are defined as

$$\mathcal{E} := \sum_{\alpha=1}^{N} \left(T_\alpha + \frac{\eta^2}{2T_*^2}|T_\alpha|^2\right), \quad \mathcal{S} := \sum_{\alpha=1}^{N} \ln T_\alpha.$$

The constants T_* and η are physical constants with dimensions $[T]$ and $[T^{-\frac{1}{2}}]$, which are assumed to be unity ($T_* = 1$ and $\eta = 1$) for simplicity.

Then, system (1.2) has also been extensively studied from diverse perspectives, e.g., emergence of complete synchronization [25], continuum limit [18] and a mean-field limit [19]. However, discrete-time approximation of (1.2) was not considered in follow-up research of [25]. For the Kuramoto model (1.1), the discrete-time approximation can be written as

$$\theta_i(n+1) = \theta_i(n) + \nu_i h + \frac{\kappa h}{N}\sum_{j=1}^{N} \sin(\theta_j(n) - \theta_i(n)), \quad i = 1,\cdots,N, \tag{1.3}$$

where $h := \Delta t > 0$ is a fixed time step.

In [6], Choi and Ha first proved the complete phase synchronization of (1.3) with identical oscillators ($\nu_i \equiv \nu$) when initial phases are confined in a half circle and κh is sufficiently small. More precisely, if

$$\nu_1 = \cdots = \nu_N = \nu, \quad 0 < \kappa h < 1, \quad \max_{1 \le i,j \le N} |\theta_i(0) - \theta_j(0)| < \pi,$$

(1.3) exhibits complete phase synchronization as n tends to infinity:

$$\lim_{n \to \infty} (\theta_i(n) - \theta_j(n)) = 0, \quad \forall\ i,j = 1, \cdots, N.$$

Then, Zhang and Zhu [41] provided a synchronization estimate for the first-order Euler-type discrete Kuramoto model (1.3) by utilizing the finite-in-time consistency of Euler method and showing the convergence of generic discrete gradient system under the uniform boundedness assumption of phases. Recently, Shim [32] presented a critical value of time-step to achieve phase/frequency synchronization for generic initial data without using any consistency result between discrete and continuous Kuramoto models. Therefore, it is natural to address the following two questions:

- (Q1): What is a structure-preserving discretization of (1.2)?
- (Q2): If such structure-preserving discretization exists, under what conditions on system parameters and initial configurations, does the proposed discretized model exhibit emergent collective behaviors and smooth transition toward the continuous model (1.2)?

The first question (Q1) deals with the discretization of (1.2), which enjoys the same qualitative and quantitative properties as in the continuous TK model. Since the first-order Euler discretization for (1.2) does not conserve the total energy (see Section 2.2), we propose the following discretization of (1.2) to satisfy energy conservation and entropy principle:

$$\begin{cases} \theta_\alpha(n+1) = \theta_\alpha(n) + \nu_\alpha h + \dfrac{\kappa_1 h}{N T_\alpha(n)} \displaystyle\sum_{\beta=1}^N \psi_{\alpha\beta} \sin(\theta_\beta(n) - \theta_\alpha(n)), \quad n \in \mathbb{Z}_{\ge 0}, \\ f(T_\alpha(n+1)) = f(T_\alpha(n)) + \dfrac{\kappa_2 h}{N} \displaystyle\sum_{\beta=1}^N \zeta_{\alpha\beta} \left(\dfrac{1}{T_\alpha(n)} - \dfrac{1}{T_\beta(n)} \right), \quad f(x) = x + \dfrac{x^2}{2}, \\ (\theta_\alpha(0), T_\alpha(0)) = (\theta_\alpha^{in}, T_\alpha^{in}), \quad \alpha = 1, \cdots, N, \end{cases} \tag{1.4}$$

where $h > 0$ is a fixed time step. Then, for the discretized entropy and total energy functional

$$\mathcal{E}(n) := \sum_{\alpha=1}^N \Big(T_\alpha(n) + \frac{1}{2} |T_\alpha(n)|^2 \Big), \quad \mathcal{S}(n) := \sum_{\alpha=1}^N \ln T_\alpha(n), \quad n \in \mathbb{Z}_{\ge 0}, \tag{1.5}$$

one can also verify that the energy and entropy functionals satisfy

$$\mathcal{E}(n) = \mathcal{E}(0), \quad \mathcal{S}(n) \ge \mathcal{S}(0), \quad n = 0, 1, 2, \cdots.$$

We refer to Lemma 2.1 for details.

The main results of this paper are devoted to settle (Q1)–(Q2). First, we provide a sufficient framework leading to the temperature homogenization and complete synchronization of the discrete TK model (1.4) in terms of system parameters and initial configurations: Let $\{(\theta_\alpha, T_\alpha)\}_{\alpha=1}^N$ be a solution to (1.4) subject to the initial data $\{(\theta_\alpha^{in}, T_\alpha^{in})\}_{\alpha=1}^N$, and suppose the initial data $\{(\theta_\alpha^{in}, T_\alpha^{in})\}_{\alpha=1}^N$, coupling strengths κ_1, κ_2 and communication weights $(\psi_{\alpha\beta})_{\alpha\beta}, (\zeta_{\alpha\beta})_{\alpha\beta}$ satisfy

$$\begin{aligned} &0 \le \mathcal{D}(\nu) := \max_{1\le\alpha,\beta\le N} |\nu_\alpha - \nu_\beta| \le \frac{\kappa_1 \psi_m}{T_M^{in}}, \quad \mathcal{D}(\Theta^{in}) := \max_{1\le\alpha,\beta\le N} |\theta_\alpha^{in} - \theta_\beta^{in}| < \pi - \gamma, \\ &\psi_m := \min_{1\le\alpha,\beta\le N} \psi_{\alpha\beta} > 0, \quad \gamma := \sin^{-1}\left(\frac{\mathcal{D}(\nu) T_M^{in}}{\kappa_1 \psi_m}\right) \le \frac{\pi}{2}, \quad T_M^{in} := \max_{1\le\alpha\le N} T_\alpha^{in} > 0. \end{aligned} \tag{1.6}$$

Then, if the time step h is sufficiently small to satisfy

$$\begin{aligned} &0 < \frac{\kappa_1 h \psi_M}{T_m^{in}} < 1, \quad 0 < \kappa_2 h \bar{\zeta}_M < f(T_m^{in}) T_m^{in}, \\ &\psi_M := \max_{\alpha,\beta} \psi_{\alpha\beta}, \quad \bar{\zeta}_M := \max_{1\le\alpha\le N} \frac{1}{N} \sum_{\beta=1}^N \zeta_{\alpha\beta}, \quad T_m^{in} := \min_\alpha T_\alpha^{in} > 0, \end{aligned} \tag{1.7}$$

there exists a constant vector $(\theta_1^\infty, \cdots, \theta_N^\infty, T^\infty)$ such that (see Theorem 3.1 and Theorem 4.1 for details),

$$\lim_{n\to\infty} \max_{1\le\alpha\le N} \Big(|\theta_\alpha(n) - (\theta_\alpha^\infty + n\nu_c h)| + |T_\alpha(n) - T^\infty| \Big) = 0, \quad \nu_c := \frac{1}{N}\sum_{\alpha=1}^N \nu_\alpha.$$

In particular, for a homogeneous ensemble with the same natural frequency ν, we show that the complete phase synchronization emerges exponentially fast, and there exists a constant vector $(\theta^\infty, T^\infty)$ such that

$$\lim_{n\to\infty} \max_{1\le\alpha\le N} \Big(|\theta_\alpha(n) - (\theta^\infty + n\nu h)| + |T_\alpha(n) - T^\infty| \Big) = 0.$$

Second, we show the smooth transition of our discrete system (1.4) to its continuous counterpart (1.2). Let $\{(\theta_\alpha, T_\alpha)\}_{\alpha=1}^N$ and $\{(\theta_\alpha^h, T_\alpha^h)\}_{\alpha=1}^N$ be the solution of system (1.2) and (1.4), respectively. Under the conditions (1.6)–(1.7), one can derive a uniform-in-time convergence (see Theorem 5.1 for details),

$$\lim_{h\to 0} \sup_{0\le n<\infty} \big(\|\Theta^h(n) - \Theta(nh)\|_\infty + \|\mathcal{T}^h(n) - \mathcal{T}(nh)\|_\infty \big) = 0,$$

where Θ and $\mathcal{T}$ represent N-vectors of phases and temperatures:

$$\begin{aligned} &\Theta^h(n) := (\theta_1^h(n), \cdots, \theta_N^h(n)), \quad \mathcal{T}^h(n) := (T_1^h(n), \cdots, T_N^h(n)), \quad n = 0, 1, 2, \cdots, \\ &\Theta(t) := (\theta_1(t), \cdots, \theta_N(t)), \quad \mathcal{T}(t) := (T_1(t), \cdots, T_N(t)), \quad t \ge 0. \end{aligned}$$

The rest of this paper is organized as follows. In Section 2, we briefly review the emergent dynamics of continuous TK model (1.2) and some elementary a priori estimates of the energy conserving discretization (1.4). In Section 3, we provide sufficient conditions to satisfy entropy principle and emergence of temperature homogenization. In Section 4, we provide estimates for the emergence of phase/frequency synchronization. In Section 5, we study a uniform-in-time convergence of (1.4) to (1.2) as $h \to 0$. Finally, Section 6 is devoted to a brief summary of our main results.

Notation: The relation $A(t) \lesssim B(t)$ represents an inequality $A(t) \le CB(t)$ for a positive constant C and all $t \ge 0$. For notational simplicity, we also use following handy notations from time to time:

$$\max_{\alpha} := \max_{1\le\alpha\le N}, \quad \max_{\alpha,\beta} := \max_{1\le\alpha,\beta\le N}, \quad \sum_{\alpha} := \sum_{\alpha=1}^{N}, \quad \sum_{\alpha,\beta} := \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}.$$

Throughout the paper, we denote Θ and $\mathcal{T}$ the N-vectors of phases and temperatures, respectively, i.e.,

$$\Theta := (\theta_1,\cdots,\theta_N), \quad \mathcal{T} := (T_1,\cdots,T_N).$$

For given index set $\mathcal{I}$ (e.g. $\mathbb{Z}_{\ge 0}$ or $\mathbb{R}_{\ge 0}$) and arbitrary N-vector valued function $X = (x_1,\cdots,x_N) : \mathcal{I} \to \mathbb{R}^N$, we denote $x_M(u), x_m(u)$ and $x_c(u)$ the maximum, minimum and average of $x_\alpha(u)$'s, respectively:

$$x_M(u) := \max_{\alpha} x_\alpha(u), \quad x_m(u) := \min_{\alpha} x_\alpha(u), \quad x_c(u) := \frac{1}{N}\sum_{\alpha} x_\alpha(u), \quad \forall\, u \in \mathcal{I}.$$

We also denote $\mathcal{D}(X)$ the maximal difference between x_α's:

$$\mathcal{D}(X) := \max_{\alpha,\beta} |x_\alpha - x_\beta| = x_M - x_m.$$

Moreover, if there is no confusion on X, we set $M_{(\cdot)} : \mathcal{I} \to \{1,\cdots,N\}$ and $m_{(\cdot)} : \mathcal{I} \to \{1,\cdots,N\}$ as the functions satisfying

$$x_{M_u}(u) = x_M(u), \quad x_{m_u}(u) = x_m(u), \quad \forall\, u \in \mathcal{I}.$$

Finally, following the notations in (1.6) and (1.7), we denote

$$\begin{aligned} &\psi_m := \min_{\alpha,\beta} \psi_{\alpha\beta}, \quad \psi_M := \max_{\alpha,\beta} \psi_{\alpha\beta}, \\ &\zeta_m := \min_{\alpha,\beta} \zeta_{\alpha\beta}, \quad \zeta_M := \max_{\alpha,\beta} \zeta_{\alpha\beta}, \quad \overline{\zeta}_M := \max_{\alpha} \frac{1}{N}\sum_{\beta} \zeta_{\alpha\beta}. \end{aligned}$$

2. Preliminaries

In this section, we first introduce the continuous TK lattice model and its emergent dynamics in [25], and then present several a priori estimates for the energy conserving discretization scheme.

2.1. Continuous TK lattice model.

Recall that the dynamics of temperature fields at each lattice point x_α is given by the following coupled system:

$$\begin{cases} \dot{T}_\alpha = \dfrac{\kappa_2}{N(1+T_\alpha)} \displaystyle\sum_{\beta} \zeta_{\alpha\beta}\left(\frac{1}{T_\alpha} - \frac{1}{T_\beta}\right), & t > 0, \\ T_\alpha(0) = T_\alpha^{in}, \quad \alpha = 1,\cdots,N. \end{cases} \tag{2.1}$$

Then, the temperature configuration $\{T_\alpha\}_{\alpha=1}^N$ exhibits an asymptotic homogenization.

PROPOSITION 2.1. [25] *Let $\{T_\alpha\}_{\alpha=1}^N$ be a global solution to (2.1) with the initial data $\{T_\alpha^{in}\}_{\alpha=1}^N$, and suppose the coupling strength κ_2, communication weights $(\zeta_{\alpha\beta})_{\alpha\beta}$ and initial temperatures $\{T_\alpha^{in}\}_{\alpha=1}^N$ satisfy*

$$\kappa_2 > 0, \quad T_M^{in} := \max_\alpha T_\alpha^{in}, \quad T_m^{in} = \min_\alpha T_\alpha^{in} > 0, \quad (\zeta_{\alpha\beta})_{\alpha\beta} = (\zeta_{\beta\alpha})_{\alpha\beta}, \quad \zeta_m > 0.$$

Then, the following assertions hold.

(1) *T_M and T_m are monotonically decreasing and increasing in t, respectively:*

$$0 < T_m(0) \le T_m(t) \le T_M(t) \le T_M(0), \quad \forall\ t \in \mathbb{R}_{\ge 0}.$$

(2) *The functionals $\mathcal{E}$ and $\mathcal{S}$ satisfy*

$$\mathcal{E}(t) = \mathcal{E}(0), \quad \mathcal{S}(t) \ge \mathcal{S}(0), \quad \forall\ t \in \mathbb{R}_{\ge 0}.$$

(3) *The temperature diameter $\mathcal{D}(\mathcal{T})$ converges to zero exponentially:*

$$\mathcal{D}(\mathcal{T}(t)) \le \mathcal{D}(\mathcal{T}(0)) \exp\left(-\frac{\kappa_2 \zeta_m t}{|T_M^{in}|^2 (1 + T_M^{in})} \right), \quad \forall\ t \in \mathbb{R}_{\ge 0},$$

and all temperatures $T_1, \cdots, T_N$ converge to a positive constant T^∞ determined by the energy conservation:

$$T^\infty + \frac{1}{2}|T^\infty|^2 = \frac{1}{N} \sum_\alpha \left(T_\alpha^{in} + \frac{1}{2}|T_\alpha^{in}|^2 \right).$$

On the other hand, the phase dynamics of system (1.2) is given by the following Kuramoto-like ordinary differential equation, which is not closed by itself:

$$\dot{\theta}_\alpha = \nu_\alpha + \frac{\kappa_1}{N} \sum_{\beta=1}^N \frac{\psi_{\alpha\beta}}{T_\alpha} \sin(\theta_\beta - \theta_\alpha), \quad \alpha = 1, \cdots, N, \quad t \in \mathbb{R}_{\ge 0}. \tag{2.2}$$

Note that as long as T_α's are all constants and $\psi \equiv 1$, system (2.2) reduces to the Kuramoto model (1.1) with coupling strength $\tilde{\kappa}_1 = \frac{\kappa_1}{T^\infty}$.

PROPOSITION 2.2. [25] (Homogeneous ensemble) *Suppose that the coupling strengths, natural frequencies and the initial data $\{(\theta_\alpha^{in}, T_\alpha^{in})\}_{\alpha=1}^N$ satisfy*

$$\kappa_1 > 0, \quad \kappa_2 > 0, \quad \nu_\alpha \equiv \nu_0, \quad \alpha = 1, \cdots, N, \quad 0 < T_1^{in} \le \cdots \le T_N^{in} < \infty,$$

and let $\{(\theta_\alpha, T_\alpha)\}_{\alpha=1}^N$ be a solution to (1.2) subject to the initial data $\{(\theta_\alpha^{in}, T_\alpha^{in})\}_{\alpha=1}^N$. Then, the following assertions hold.

(1) (Dichotomy in asymptotic dynamics)*: If $\psi_{\alpha\beta} = 1$ for all α, β and the initial Kuramoto order parameter R^{in} is strictly positive,*

$$R^{in} := \left| \frac{1}{N} \sum_\beta e^{i\theta_\beta^{in}} \right| > 0,$$

there exists a constant $\varphi^\infty \in \mathbb{R}$ such that $\Theta^\infty := (\theta_1^\infty, \cdots, \theta_N^\infty)$ is either one-point cluster or bi-polar configuration:

$$\lim_{t \to \infty} (\theta_\alpha(t) - \nu_0 t) =: \theta_\alpha^\infty \in \{\varphi^\infty, \varphi^\infty + \pi\}, \quad \forall\ \alpha = 1, \cdots, N.$$

(2) (Formation of one-point cluster): *If the initial phases are confined in a half circle,*

$$\mathcal{D}(\Theta^{in}) < \pi,$$

there exists a common constant phase $\theta^\infty \in \mathbb{R}$ *such that for any* $\Lambda \in \left(0, \frac{\kappa_1}{T^\infty}\right)$,

$$|\theta_\alpha(t) - \nu t - \theta^\infty| \lesssim e^{-\Lambda t} \text{ as } t \to \infty, \quad \alpha = 1, \cdots, N.$$

For the Kuramoto model (1.1) with distributed natural frequencies, the complete phase synchronization as in Proposition 2.2 (2) cannot be achieved, whereas asymptotic phase-locking will emerge, i.e., relative phases $(\theta_\alpha - \theta_\beta)_{1\le\alpha,\beta\le N}$ converge to constant values in a large coupling regime ($\kappa_1 \gg \mathcal{D}(\nu)$). A key ingredient of the complete synchronization for generic initial data in [20] is a reformulation of (1.1) as a gradient flow with analytical potential [13,20,35]. Unfortunately, the phase dynamics (2.2) cannot be rewritten as a gradient flow due to presence of T_α in the coefficient of $\sin(\theta_\beta - \theta_\alpha)$ as it is. Instead, it can be written as a gradient flow with exponentially decaying source term, which is still enough to derive an emergence of phase-locked state in a large coupling regime.

Proposition 2.3 ([25] (Heterogeneous ensemble)). *Suppose the coupling strengths, natural frequencies and the initial data* $\{(\theta_\alpha^{in}, T_\alpha^{in})\}_{\alpha=1}^N$ *satisfy*

$$\begin{aligned}
&\kappa_2 > 0, \quad 0 < T_1^{in} \le \cdots \le T_N^{in} < \infty, \quad \kappa_1\psi_m > \mathcal{D}(\nu)T_N^{in} > 0, \quad \nu := (\nu_1, \cdots, \nu_N),\\
&\mathcal{D}(\Theta^{in}) < \pi - \gamma, \quad \gamma := \arcsin\left(\frac{\mathcal{D}(\nu)T_N^{in}}{\kappa_1\psi_m}\right) < \frac{\pi}{2}, \quad \zeta_m > 0,
\end{aligned}$$

and let $\{(\theta_\alpha, T_\alpha)\}_{\alpha=1}^N$ *be a solution to (1.2) subject to the initial data* $\{(\theta_\alpha^{in}, T_\alpha^{in})\}_{\alpha=1}^N$. *Then, there exists a constant vector* $\Theta^\infty = (\theta_1^\infty, \cdots, \theta_N^\infty)$ *such that*

$$\lim_{t\to\infty}(\theta_\alpha(t) - \nu_c t) = \theta_\alpha^\infty, \quad \nu_\alpha + \frac{\kappa_1}{NT^\infty}\sum_\beta \psi_{\alpha\beta}\sin(\theta_\beta^\infty - \theta_\alpha^\infty) = 0, \quad \alpha = 1, \cdots, N,$$

where T^∞ *is the asymptotic temperature determined by the energy conservation in Proposition 2.1(3). Moreover, the set*

$$\{\Theta : \mathcal{D}(\Theta) < \pi - \gamma\}$$

is positively invariant.

Remark 2.1. Although it is not stated in [25] in detail, one can also show that the set

$$\{\Theta : \mathcal{D}(\Theta) \le D\}$$

is positively invariant, for every $D \in [\gamma, \pi - \gamma]$ (see Lemma 4.1 for the analogous result for discrete TK model).

2.2. An energy preserving discretization scheme. In this subsection, we briefly explain a motivation for our discretization scheme. As the most simple and naive discretization scheme based on the first-order Euler method, one might have

$$\begin{cases}
\theta_\alpha^h(n+1) - \theta_\alpha^h(n) = \nu_\alpha h + \dfrac{\kappa_1 h}{NT_\alpha^h(n)}\displaystyle\sum_\beta \psi_{\alpha\beta}\sin(\theta_\beta^h(n) - \theta_\alpha^h(n)), \quad n \in \mathbb{Z}_{\ge 0},\\
T_\alpha^h(n+1) = T_\alpha^h(n) + \dfrac{\kappa_2 h}{N}\displaystyle\sum_\beta \frac{\zeta_{\alpha\beta}}{1 + T_\alpha^h(n)}\left(\frac{1}{T_\alpha^h(n)} - \frac{1}{T_\beta^h(n)}\right), \quad n \in \mathbb{Z}_{\ge 0},
\end{cases} \tag{2.3}$$

and also consider a discrete total energy for (2.3), similar to (1.5):

$$\mathcal{E}^h(n) := \sum_\alpha \Big(T^h_\alpha(n) + \frac{1}{2}|T^h_\alpha(n)|^2 \Big), \quad n \in \mathbb{Z}_{\geq 0}.$$

Then, one can check the monotone increasing property of the energy functional $\mathcal{E}^h$:

$$\begin{aligned}
&\mathcal{E}^h(n+1) - \mathcal{E}^h(n) \\
&\quad = \sum_\alpha \Big(\big(T^h_\alpha(n+1) - T^h_\alpha(n)\big) + \frac{1}{2}\big(T^h_\alpha(n+1) - T^h_\alpha(n)\big)\big(T^h_\alpha(n+1) + T^h_\alpha(n)\big) \Big) \\
&\quad = \sum_\alpha \big(T^h_\alpha(n+1) - T^h_\alpha(n)\big) \Big(1 + \frac{1}{2}\big(T^h_\alpha(n+1) - T^h_\alpha(n)\big) + T^h_\alpha(n) \Big) \\
&\quad = \sum_\alpha \left(\frac{\kappa_2 h}{N} \sum_\beta \frac{\zeta_{\alpha\beta}}{1 + T^h_\alpha(n)} \left(\frac{1}{T^h_\alpha(n)} - \frac{1}{T^h_\beta(n)} \right) \right) \\
&\qquad\quad \times \left(1 + T^h_\alpha(n) + \frac{\kappa_2 h}{2N} \sum_\beta \frac{\zeta_{\alpha\beta}}{1 + T^h_\alpha(n)} \left(\frac{1}{T^h_\alpha(n)} - \frac{1}{T^h_\beta(n)} \right) \right) \\
&\quad = \frac{1}{2}\left(\frac{\kappa_2 h}{N}\right)^2 \sum_\alpha \left(\sum_\beta \frac{\zeta_{\alpha\beta}}{1 + T^h_\alpha(n)} \left(\frac{1}{T^h_\alpha(n)} - \frac{1}{T^h_\beta(n)} \right) \right)^2 \geq 0.
\end{aligned} \tag{2.4}$$

Therefore, as long as temperatures are away from the homogenized state, the R.H.S. of (2.4) is strictly positive and energy is strictly increasing:

$$\mathcal{E}^h(n+1) > \mathcal{E}^h(n), \quad \forall\, n \in \mathbb{Z}_{\geq 0},$$

which is *inconsistent* with the corresponding result for the continuous model (see Proposition 2.1 (2)). Since the limit temperature of (2.3) is larger than its counterpart on (1.2), the difference of equilibrium phases in (1.2) and (2.3) does not coincide asymptotically by the nonlinear relations

$$\begin{aligned}
&\nu_\alpha + \frac{\kappa_1}{N T^h(\infty)} \sum_\beta \psi_{\alpha\beta} \sin(\theta^h_\beta(\infty) - \theta^h_\alpha(\infty)) = 0, \\
&\nu_\alpha + \frac{\kappa_1}{N T(\infty)} \sum_\beta \psi_{\alpha\beta} \sin(\theta_\beta(\infty) - \theta_\alpha(\infty)) = 0,
\end{aligned}$$

where

$$\begin{aligned}
&\theta^h_\alpha(\infty) := \lim_{n\to\infty} \theta^h_\alpha(n), \quad T^h(\infty) := \lim_{n\to\infty} T^h_1(n) = \cdots = \lim_{n\to\infty} T^h_N(n), \\
&\theta_\alpha(\infty) := \lim_{t\to\infty} \theta_\alpha(t), \quad T(\infty) := \lim_{t\to\infty} T_1(t) = \cdots = \lim_{t\to\infty} T_N(t), \quad T(\infty) < T^h(\infty).
\end{aligned}$$

This motivates us to propose an energy preserving discretization model. Therefore, without any modification of phase difference equation $(2.3)_1$, we consider the following discretization for temperatures:

$$\begin{cases}
f(T_\alpha(n+1)) - f(T_\alpha(n)) = \dfrac{\kappa_2 h}{N} \displaystyle\sum_\beta \zeta_{\alpha\beta} \left(\frac{1}{T_\alpha(n)} - \frac{1}{T_\beta(n)} \right), \quad \forall\, n \in \mathbb{Z}_{\geq 0}, \\
T_\alpha(0) = T^{in}_\alpha, \quad \alpha = 1, \cdots, N,
\end{cases} \tag{2.5}$$

where $f(x)=x+\frac{x^2}{2}$. In this way, the temperature dynamics is given implicitly to achieve the conservation of total energy $\mathcal{E}$. This is critical for later analysis, since the total energy $\mathcal{E}$ indeed determines the asymptotic temperature T^∞, and the different choice of T^∞ would lead to a different asymptotic phase configuration even up to translation.

Now, we derive an analogous result to Proposition 2.1 for discrete system (2.5), except the monotone increasing property of discrete entropy $\mathcal{S}$.

Lemma 2.1 (Energy conservation). *Let $\{\mathcal{T}(n)\}_{n\geq 0}$ be a solution to (2.5) subject to the initial data $\{T_\alpha^{in}\}_{\alpha=1}^N$, and suppose the coupling strength and initial temperatures satisfy*

$$\kappa_2>0, \quad T_m^{in}:=\min_\alpha T_\alpha^{in}>0, \quad f(T_m^{in})T_m^{in}>\kappa_2 h\bar{\zeta}_M. \tag{2.6}$$

Then, we have the following results:

(1) *$\{T_{M_n}(n)\}_n$ and $\{T_{m_n}(n)\}_n$ are non-increasing and non-decreasing in n, respectively:*

$$0<T_{m_0}(0)\leq T_{m_n}(n)\leq T_{M_n}(n)\leq T_{M_0}(0), \quad \forall\ n\in\mathbb{Z}_{\geq 0}.$$

(2) *The energy functional $\mathcal{E}$ is conserved:*

$$\mathcal{E}(n)=\mathcal{E}(0), \quad \forall\ n\in\mathbb{Z}_{\geq 0}. \tag{2.7}$$

Proof. (1) For the desired monotonicity of T_m, we use an induction argument. Suppose that

$$0<T_m^{in}=T_{m_0}(0)\leq\cdots\leq T_{m_k}(k), \quad \forall\ k=0,\cdots,n. \tag{2.8}$$

Then, we have

$$\begin{aligned}
&f(T_{m_{n+1}}(n+1))-f(T_{m_n}(n))\\
&=\frac{\kappa_2 h}{N}\sum_{\beta=1}^N\zeta_{m_{n+1}\beta}\left(\frac{1}{T_{m_{n+1}}(n)}-\frac{1}{T_\beta(n)}\right)+f(T_{m_{n+1}}(n))-f(T_{m_n}(n))\\
&\geq\frac{\kappa_2 h}{N}\sum_{\beta=1}^N\zeta_{m_{n+1}\beta}\left(\frac{1}{T_{m_{n+1}}(n)}-\frac{1}{T_{m_n}(n)}\right)+f(T_{m_{n+1}}(n))-f(T_{m_n}(n))\\
&\geq\left(f(T_{m_{n+1}}(n))+\frac{\kappa_2 h\bar{\zeta}_M}{T_{m_{n+1}}(n)}\right)-\left(f(T_{m_n}(n))+\frac{\kappa_2 h\bar{\zeta}_M}{T_{m_n}(n)}\right)\geq 0,
\end{aligned}$$

where we used (2.8) and the monotone increasing property of the function

$$x\mapsto f(x)+\frac{\kappa_2 h\bar{\zeta}_M}{x}=x+\frac{x^2}{2}+\frac{\kappa_2 h\bar{\zeta}_M}{x},$$

for all x satisfying $x^2(1+x)\geq\kappa_2 h\bar{\zeta}_M$. Moreover, by using

$$x^2(1+x)\geq xf(x) \quad \text{for every } x>0,$$

we have

$$f(T_\alpha(n))+\frac{\kappa_2 h}{N}\sum_{\beta=1}^N\zeta_{\alpha\beta}\left(\frac{1}{T_\alpha(n)}-\frac{1}{T_\beta(n)}\right)\geq f(T_{m_n}(n))-\kappa_2 h\bar{\zeta}_M\frac{1}{T_{m_n}(n)}>0.$$

Then, since $f(x)=x+\frac{x^2}{2}$ is strictly increasing for $x>0$, this implies that the equation

$$f(x)=f(T_\alpha(n))+\frac{\kappa_2 h}{N}\sum_{\beta=1}^N \zeta_{\alpha\beta}\left(\frac{1}{T_\alpha(n)}-\frac{1}{T_\beta(n)}\right)(>0)$$

always has a unique positive solution $x=T_\alpha(n+1)$. Finally, by using similar argument, one can also show that $T_{M_n}(n)$ is nonincreasing.

(2) It suffices to check that $\sum_{\alpha=1}^N f(T_\alpha(n))$ is conserved. Thus, we add (2.5) over α and use $\zeta_{\alpha\beta}=\zeta_{\beta\alpha}$ to obtain

$$\begin{aligned}&\sum_\alpha f(T_\alpha(n+1))-\sum_\alpha f(T_\alpha(n))\\&=\frac{\kappa_2 h}{N}\sum_{\alpha,\beta}\zeta_{\alpha\beta}\left(\frac{1}{T_\alpha(n)}-\frac{1}{T_\beta(n)}\right)=\frac{\kappa_2 h}{N}\sum_{\alpha,\beta}\zeta_{\alpha\beta}\left(\frac{1}{T_\beta(n)}-\frac{1}{T_\alpha(n)}\right)=0,\end{aligned}$$

which is the desired result. □

REMARK 2.2. Compared to the result in Proposition 2.1, the smallness of h in (2.6) is the only condition to add to guarantee the same energy conservation law for the discrete TK model.

3. Emergence of temperature homogenization

In this section, we study emergent dynamics of system (1.4) following the methods in [25] for the continuous-time TK model. Although our presented results slightly differ from the original ones, all the results in what follows are still consistent with the continuous-time TK lattice model as $h\to 0$. First, we present an exponential decay of temperature diameter.

LEMMA 3.1. *Suppose the coupling strength and initial temperatures satisfy* (2.6)*, and let* $\{\mathcal{T}(n)\}_{n\geq 0}$ *be a solution to* (2.5) *with the initial data* $\{T_\alpha^{in}\}_{\alpha=1}^N$. *Then, the sequence of temperature diameter* $\{\mathcal{D}(\mathcal{T}(n))\}_{n\geq 0}$ *satisfies*

$$\mathcal{D}(\mathcal{T}(n))\leq \mathcal{D}(\mathcal{T}(0))\left(1-\frac{\kappa_2\zeta_m h}{|T_M^{in}|^2(1+T_M^{in})}\right)^n,\quad \forall\ n\in\mathbb{Z}_{\geq 0}.$$

Proof. First, observe that $(f^{-1})(x)=-1+\sqrt{1+2x}$ and

$$f(T_{m_{n+1}}(n+1))-f(T_{m_n}(n))\geq \frac{\kappa_2\zeta_m h}{N}\sum_{\beta=1}^N\left(\frac{1}{T_{m_n}(n)}-\frac{1}{T_\beta(n)}\right)=:a_n>0,$$

$$f(T_{M_{n+1}}(n+1))-f(T_{M_n}(n))\leq \frac{\kappa_2\zeta_m h}{N}\sum_{\beta=1}^N\left(\frac{1}{T_{M_n}(n)}-\frac{1}{T_\beta(n)}\right)=:b_n<0.$$

Then, the temperature diameter at $(n+1)$-th time step $\mathcal{D}(\mathcal{T}(n+1))$ satisfies the following inequality:

$$\begin{aligned}&\mathcal{D}(\mathcal{T}(n+1))-\mathcal{D}(\mathcal{T}(n))\\&\leq f^{-1}(f(T_{M_n}(n))+b_n)-f^{-1}(f(T_{m_n}(n))+a_n)-\mathcal{D}(\mathcal{T}(n))\\&=\sqrt{(1+T_{M_n}(n))^2+2b_n}-\sqrt{(1+T_{m_n}(n))^2+2a_n}-\mathcal{D}(\mathcal{T}(n))\end{aligned}$$

$$
\begin{aligned}
&= \frac{2b_n}{\sqrt{(1+T_{M_n}(n))^2+2b_n}+(1+T_{M_n}(n))} - \frac{2a_n}{\sqrt{(1+T_{m_n}(n))^2+2a_n}+(1+T_{m_n}(n))} \\
&\le \frac{b_n-a_n}{1+T_{M_n}(n)} = \frac{\kappa_2\zeta_m h}{1+T_{M_n}(n)}\left(\frac{1}{T_{M_n}(n)}-\frac{1}{T_{m_n}(n)}\right) \\
&= -\left(\frac{\kappa_2\zeta_m h}{(1+T_{M_n}(n))T_{M_n}(n)T_{m_n}(n)}\right)\mathcal{D}(\mathcal{T}(n)) \\
&\le -\left(\frac{\kappa_2\zeta_m h}{|T_M^{in}|^2(1+T_M^{in})}\right)\mathcal{D}(\mathcal{T}(n)),
\end{aligned} \tag{3.1}
$$

where the second inequality comes from

$$
\begin{aligned}
a_n &\le \kappa_2 h\bar{\zeta}_M\left(\frac{1}{T_{m_n}(n)}-\frac{1}{T_{M_n}(n)}\right) \le f(T_{m_n}(n))T_{m_n}(n)\left(\frac{1}{T_{m_n}(n)}-\frac{1}{T_{M_n}(n)}\right) \\
&\le (T_{M_n}(n)-T_{m_n}(n))\left(1+\frac{T_{m_n}(n)}{2}\right) \le \frac{1}{2}\left[(1+T_{M_n}(n))^2-(1+T_{m_n}(n))^2\right],
\end{aligned}
$$

and the monotone decreasing property $\{T_{M_n}(n)\}_{n\ge 0}$ is used in the last inequality. Finally, we apply (3.1) n times iteratively and conclude the desired result. ∎

REMARK 3.1. Fix a finite time $t<\infty$, and let $h_n := \frac{t}{n}$. Then, the result in Lemma 3.1 says that for the mesh size $h=h_n$, the temperature diameter $\mathcal{D}(\mathcal{T})$ at n-th step satisfies

$$
\mathcal{D}(\mathcal{T}(n)) \le \mathcal{D}(\mathcal{T}(0))\left(1-\frac{\kappa_2\zeta_m t}{|T_M^{in}|^2(1+T_M^{in})}\cdot\frac{1}{n}\right)^n,
$$

which converges to the decay rate in Proposition 2.1(3) as $n\to\infty$ (*i.e.*, $h_n\to 0$).

However, showing the entropy nondecreasing property is more complicated than others, since we have to compare the multiplicative terms $\Pi_\alpha T_\alpha(n)$ and $\Pi_\alpha T_\alpha(n+1)$.

LEMMA 3.2 (Monotonicity of entropy). *Suppose the coupling strength and initial temperatures satisfy*

$$
T_m^{in}>0, \quad 3f(T_m^{in})T_m^{in} > 16\kappa_2 h\bar{\zeta}_M > 0,
$$

and let $\{\mathcal{T}(n)\}_{n\ge 0}$ be a solution to $(1.4)_2$ with the initial data $\{T_\alpha^{in}\}_{\alpha=1}^N$. Then, we have

$$
\mathcal{S}(n+1) \ge \mathcal{S}(n), \quad \forall\, n\in\mathbb{Z}_{\ge 0}.
$$

Proof. We first recall the following inequality, which can be found from basic calculus:

$$
\ln(1+x) \ge x - x^2, \quad \forall\, x \ge -\frac{1}{2}. \tag{3.2}
$$

Note that the increment of entropy $\mathcal{S}$ at the n-th step can be written as

$$
\begin{aligned}
\mathcal{S}(n+1)-\mathcal{S}(n) &= \sum_\alpha \ln\left(1+\frac{T_\alpha(n+1)-T_\alpha(n)}{T_\alpha(n)}\right) \\
&= \sum_\alpha \ln\left(1+\frac{\sqrt{(1+T_\alpha(n))^2+2hc_\alpha(n)}-(1+T_\alpha(n))}{T_\alpha(n)}\right) \\
&=: \sum_\alpha \ln(1+d_\alpha(n)),
\end{aligned} \tag{3.3}
$$

where $c_\alpha(n)$ is given by the right-hand side of $(1.4)_2$ divided by h and satisfies:

$$|c_\alpha(n)| = \left| \frac{\kappa_2}{N} \sum_\beta \zeta_{\alpha\beta} \left(\frac{1}{T_\alpha(n)} - \frac{1}{T_\beta(n)} \right) \right| \le \frac{\kappa_2 \overline{\zeta}_M}{T_m^{in}} \le \frac{f(T_m^{in})}{2h} \le \frac{f(T_\alpha(n))}{2h}.$$

Therefore, we have

$$\frac{\sqrt{(1+T_\alpha(n))^2 + 2hc_\alpha(n)} - (1+T_\alpha(n))}{T_\alpha(n)} \ge \frac{\sqrt{1+T_\alpha(n) + \frac{1}{4}T_\alpha(n)^2} - (1+T_\alpha(n))}{T_\alpha(n)} = -\frac{1}{2},$$

and apply (3.2) to each summand of (3.3) to obtain

$$\mathcal{S}(n+1) - \mathcal{S}(n) \ge \sum_\alpha \left(d_\alpha(n) - d_\alpha(n)^2 \right).$$

On the other hand, $d_\alpha(n)$ also satisfies

$$\begin{aligned}
&- \frac{hc_\alpha(n)}{T_\alpha(n)(1+T_\alpha(n))} \\
&= \frac{2hc_\alpha(n)}{T_\alpha(n)\left(\sqrt{(1+T_\alpha(n))^2 + 2hc_\alpha(n)} + (1+T_\alpha(n)) \right)} - \frac{hc_\alpha(n)}{T_\alpha(n)(1+T_\alpha(n))} \\
&= \frac{2hc_\alpha(n)}{T_\alpha(n)} \left(\frac{1+T_\alpha(n) - \sqrt{(1+T_\alpha(n))^2 + 2hc_\alpha(n)}}{\left(\sqrt{(1+T_\alpha(n))^2 + 2hc_\alpha(n)} + (1+T_\alpha(n)) \right)(2+2T_\alpha(n))} \right) \\
&= -\frac{(2hc_\alpha(n))^2}{T_\alpha(n)} \left(\frac{1}{\left(\sqrt{(1+T_\alpha(n))^2 + 2hc_\alpha(n)} + (1+T_\alpha(n)) \right)^2 (2+2T_\alpha(n))} \right).
\end{aligned}$$

This implies the following inequalities for each $n \in \mathbb{Z}_{\ge 0}$:

$$\begin{aligned}
d_\alpha(n) &\ge \frac{hc_\alpha(n)}{T_\alpha(n)(1+T_\alpha(n))} - \frac{8h^2 c_\alpha(n)^2}{9T_\alpha(n)(1+T_\alpha(n))^3}, \\
d_\alpha(n)^2 &= \frac{(2hc_\alpha(n))^2}{T_\alpha(n)^2 \left(\sqrt{(1+T_\alpha(n))^2 + 2hc_\alpha(n)} + (1+T_\alpha(n)) \right)^2} \\
&\le \frac{16h^2 c_\alpha(n)^2}{9T_\alpha(n)^2(1+T_\alpha(n))^2}.
\end{aligned} \tag{3.4}$$

Therefore, we substitute (3.4) into (3.3) to obtain the lower bound of the entropy increment:

$$\begin{aligned}
&\mathcal{S}(n+1) - \mathcal{S}(n) \\
&\ge \sum_\alpha \left[\frac{hc_\alpha(n)}{T_\alpha(n)(1+T_\alpha(n))} - \frac{h^2 c_\alpha(n)^2 (24 + 16T_\alpha(n))}{9T_\alpha(n)^2(1+T_\alpha(n))^3} \right] \\
&= \frac{\kappa_2 h}{N} \sum_{\alpha,\beta} \frac{\zeta_{\alpha\beta}}{T_\alpha(n)(1+T_\alpha(n))} \left(\frac{1}{T_\alpha(n)} - \frac{1}{T_\beta(n)} \right) - \sum_\alpha \frac{h^2 c_\alpha(n)^2 (24 + 16T_\alpha(n))}{9T_\alpha(n)^2(1+T_\alpha(n))^3}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\kappa_2 h}{2N} \sum_{\alpha,\beta} \frac{\zeta_{\alpha\beta}(1+T_\alpha(n)+T_\beta(n))(T_\beta(n)-T_\alpha(n))^2}{T_\alpha(n)^2 T_\beta(n)^2(1+T_\alpha(n))(1+T_\beta(n))} - \sum_\alpha \frac{h^2 c_\alpha(n)^2(24+16T_\alpha(n))}{9T_\alpha(n)^2(1+T_\alpha(n))^3} \\
&= \frac{\kappa_2 h}{2N} \sum_{\alpha,\beta} \frac{\zeta_{\alpha\beta}(1+T_\alpha(n)+T_\beta(n))}{(1+T_\alpha(n))(1+T_\beta(n))} \left(\frac{1}{T_\alpha(n)} - \frac{1}{T_\beta(n)}\right)^2 - \sum_\alpha \frac{h^2 c_\alpha(n)^2(24+16T_\alpha(n))}{9T_\alpha(n)^2(1+T_\alpha(n))^3} \\
&= \frac{\kappa_2 h}{2N} \sum_{\alpha,\beta} \frac{\zeta_{\alpha\beta}(1+T_\alpha(n)+T_\beta(n))}{(1+T_\alpha(n))(1+T_\beta(n))} \left(\frac{1}{T_\alpha(n)} - \frac{1}{T_\beta(n)}\right)^2 \\
&\quad - \left(\frac{\kappa_2 h}{N}\right)^2 \sum_\alpha \frac{(24+16T_\alpha(n))}{9T_\alpha(n)^2(1+T_\alpha(n))^3} \left|\sum_\beta \zeta_{\alpha\beta}\left(\frac{1}{T_\alpha(n)} - \frac{1}{T_\beta(n)}\right)\right|^2 \\
&\geq \frac{\kappa_2 h}{N} \sum_\alpha \frac{1}{2(1+T_\alpha(n))} \sum_\beta \zeta_{\alpha\beta}\left(\frac{1}{T_\alpha(n)} - \frac{1}{T_\beta(n)}\right)^2 \\
&\quad - \left(\frac{\kappa_2 h}{N}\right)^2 \sum_\alpha \frac{8}{3T_\alpha(n)^2(1+T_\alpha(n))^2} \left|\sum_\beta \zeta_{\alpha\beta}\left(\frac{1}{T_\alpha(n)} - \frac{1}{T_\beta(n)}\right)\right|^2 \\
&\geq \frac{\kappa_2 h}{N} \sum_\alpha \frac{1}{2(1+T_\alpha(n))} \left(\sum_\beta \zeta_{\alpha\beta}\left(\frac{1}{T_\alpha(n)} - \frac{1}{T_\beta(n)}\right)^2\right) \\
&\quad - \left(\frac{\kappa_2 h}{N}\right)^2 \sum_\alpha \frac{8}{3T_\alpha(n)^2(1+T_\alpha(n))^2} \left(\sum_\beta \zeta_{\alpha\beta}\left(\frac{1}{T_\alpha(n)} - \frac{1}{T_\beta(n)}\right)^2\right)\left(\sum_\beta \zeta_{\alpha\beta}\right),
\end{aligned}
$$

where we used the Cauchy-Schwartz inequality in the last inequality. Finally, the condition

$$3f(T_m^{in})T_m^{in} > 16\kappa_2 h\bar{\zeta}_M$$

makes the above lower bound nonnegative, as a consequence of the following inequality:

$$\frac{\kappa_2 h}{N} \sum_\alpha \left(\frac{1}{2(1+T_\alpha(n))} - \frac{8\kappa_2 h\bar{\zeta}_M}{3T_\alpha(n)^2(1+T_\alpha(n))^2}\right)\left(\sum_\beta \zeta_{\alpha\beta}\left(\frac{1}{T_\alpha(n)} - \frac{1}{T_\beta(n)}\right)^2\right) \geq 0.$$

□

Remark 3.2. From (3.3) and (3.4), one can easily verify that the entropy increment satisfies

$$\mathcal{S}(n+1) - \mathcal{S}(n) = \frac{\kappa_2 h}{2N} \sum_{\alpha,\beta} \frac{\zeta_{\alpha\beta}(1+T_\alpha(n)+T_\beta(n))}{(1+T_\alpha(n))(1+T_\beta(n))}\left(\frac{1}{T_\alpha(n)} - \frac{1}{T_\beta(n)}\right)^2 + \mathcal{O}(h^2),$$

which is consistent with Proposition 2.1 as $h \to 0$.

Now, we show the temperature homogenization of (2.5) as n tends to infinity. In fact, this is an immediate consequence of Lemma 2.1 and Lemma 3.1. It also satisfies

Theorem 3.1. *Suppose the coupling strength and initial temperatures satisfy* (2.6), *and let* $\{\mathcal{T}(n)\}_{n\geq 0}$ *be a solution to* (2.5) *with the initial data* $\{T_\alpha^{in}\}_{\alpha=1}^N$. *Then, there exists a positive temperature* T^∞ *such that*

$$\lim_{n\to\infty} T_\alpha(n) = T^\infty, \quad \forall\ \alpha = 1, \cdots, N.$$

Proof. It follows from Lemma 2.1(1) that there exist positive constant temperatures $\overline{T}^\infty$ and $\underline{T}^\infty$ such that

$$\lim_{n\to\infty} T_m(n) = \underline{T}^\infty \quad \text{and} \quad \lim_{n\to\infty} T_M(n) = \overline{T}^\infty.$$

On the other hand, since $\mathcal{D}(\mathcal{T}(n)) = T_M(n) - T_m(n)$ tends to zero, one has

$$0 = \lim_{n\to\infty} (T_M(n) - T_m(n)) = \overline{T}^\infty - \underline{T}^\infty.$$

This yields the desired estimate:

$$\overline{T}^\infty = \underline{T}^\infty =: T^\infty.$$

□

4. Emergence of complete synchronization

Now, we show the emergence of synchronizations of (1.4), which correspond to the results for continuous TK model (1.2) in Section 2.1. In [25], the authors verified that the TK model can be viewd as a perturbed system of the Kuramoto model with exponentially decaying error, and they employed the dissipative nature of perturbed gradient system to deduce the desired synchronization results. However, we provide an alternative approach without showing any dissipative property of discrete perturbed gradient system. First, we define the complete phase/frequency synchronization and phase-locked state for the discrete TK model (1.4).

DEFINITION 4.1. *Let $\{(\Theta(n), \mathcal{T}(n))\}_{n\geq 0}$ be a solution to system (1.4).*

(1) $\{(\Theta(n), \mathcal{T}(n))\}_{n\geq 0}$ *exhibits complete phase synchronization if and only if*

$$\lim_{n\to\infty} \max_{\alpha,\beta} |\theta_\alpha(n) - \theta_\beta(n)| = 0.$$

(2) $\{(\Theta(n), \mathcal{T}(n))\}_{n\geq 0}$ *exhibits complete frequency synchronization if and only if*

$$\lim_{n\to\infty} \Big| \big(\theta_\alpha(n+1) - \theta_\beta(n+1)\big) - \big(\theta_\alpha(n) - \theta_\beta(n)\big) \Big| = 0, \quad \forall\ \alpha, \beta = 1, \cdots, N.$$

(3) *The state $(\Theta^\infty, T^\infty) = (\theta_1^\infty, \cdots, \theta_N^\infty, T^\infty)$ is a phase-locked state of (1.4) if and only if $(\Theta^\infty, T^\infty)$ satisfies*

$$T^\infty > 0, \quad \nu_\alpha + \frac{\kappa_1}{NT^\infty} \sum_\beta \psi_{\alpha\beta} \sin(\theta_\beta^\infty - \theta_\alpha^\infty) = 0, \quad \forall\ \alpha = 1, \cdots, N.$$

(4) $\{(\Theta(n), \mathcal{T}(n))\}_{n\geq 0}$ *exhibits the asymptotic phase-locking if and only if*

$$\lim_{n\to\infty} (\theta_\alpha(n) - \theta_\beta(n)) =: \theta_{\alpha\beta}^\infty \in \mathbb{R}, \quad \forall\ \alpha, \beta = 1 \cdot \cdots, N.$$

In fact, the phase-locked state represents the equilibria of (1.4), and the frequency synchronization means an asymptotic coherence of all discrete frequencies $\{\theta_\alpha(n+1) - \theta_\alpha(n)\}_{\alpha=1}^N$ as n tends to infinity. Moreover, the limit of $\{(\Theta(n), \mathcal{T}(n))\}_{n\geq 0}$ is always a phase-locked state if it exists, and the emergence of asymptotic phase-locking directly implies the emergence of complete frequency synchronization.

4.1. A positively invariant set. In this subsection, we present two elementary lemmas to be used in the proof of our main result, the emergence of asymptotic phase-locking. First, we set a framework $(\mathcal{A})$ as:

- $(\mathcal{A}1)$: system parameters κ_1, κ_2, ν and initial data $\{(\Theta^{in}, \mathcal{T}^{in})\}$ satisfy (2.6) and

$$\kappa_1, \kappa_2 > 0, \quad 0 \le \mathcal{D}(\nu) \le \frac{\kappa_1 \psi_m}{T_M^{in}}, \quad \psi_m > 0. \tag{4.1}$$

- $(\mathcal{A}2)$: time-step h satisfies

$$0 < \frac{\kappa_1 h \psi_M}{T_m^{in}} < 1. \tag{4.2}$$

Recall that for the continuous-time TK oscillator ensemble (1.2), the authors in [25] verified the positive invariance of the set

$$\mathcal{S}_{\pi-\gamma} := \{\Theta : \mathcal{D}(\Theta) \le \pi - \gamma\}, \quad \gamma := \sin^{-1}\left(\frac{D(\nu) T_M^{in}}{\kappa_1 \psi_m}\right),$$

and showed the convergence of associated TK flow $(\Theta, \mathcal{T})$ to a fixed point $(\Theta^\infty, \mathcal{T}^\infty)$, provided that the system parameters κ_1, κ_2, ν and initial data $\{(\Theta^{in}, \mathcal{T}^{in})\}$ satisfy (4.1) and

$$\nu_c = 0, \quad \Theta^{in} \in \mathcal{S}_{\pi-\gamma}. \tag{4.3}$$

According to [25], the t-derivative of the phase diameter $\frac{d\mathcal{D}(\Theta)}{dt}$ becomes non-positive whenever the Θ configuration of the TK flow touches the boundary of $\mathcal{S}_{\pi-\gamma}$. For the discrete TK model (1.4), however, the temporal evolution of the phase-diameter $\{\mathcal{D}(\Theta(n))\}_{n\ge 0}$ becomes more complicated. Since the arguments of the phase maxima and minima among $\{\theta_\alpha(n)\}_{\alpha=1}^N$ now change discretely in n, we cannot compute the exact value of the difference between the n-th and $(n+1)$-th phase diameters from (1.4), and the discrete TK flow $(\Theta(n), \mathcal{T}(n))_{n\ge 0}$ barely touches the boundary of $\mathcal{S}_{\pi-\gamma}$. Still, we can obtain analogous results of [25] by using the similar arguments in Lemma 3.1.

At the end of this section, we show the emergence of asymptotic phase-locking of (1.4) under the framework $\mathcal{A}$ and (4.3) for sufficiently small $h > 0$. First, we define an admissible set for initial phase configuration:

$$\mathcal{S}_D := \{\Theta : \mathcal{D}(\Theta) \le D\}, \quad D \in \mathbb{R}.$$

Then, the following preparatory lemma is necessary to proceed our argument further.

Lemma 4.1. *Suppose that the framework $(\mathcal{A}1)-(\mathcal{A}2)$ hold. Then, the set $\mathcal{S}_D$ is positively invariant set of* (1.4) *for every D in $[\gamma, \pi-\gamma]$.*

Proof. Let M_n and m_n be the argument indices of maximum and minimum phases among $\{\theta_\alpha(n)\}_{\alpha=1}^N$, i.e.,

$$\theta_{M_n}(n) = \max_\alpha \theta_\alpha(n), \quad \theta_{m_n}(n) = \min_\alpha \theta_\alpha(n), \quad n \in \mathbb{Z}_{\ge 0}.$$

Then, as long as $\mathcal{D}(\Theta(n))$ is smaller than π, we have

$$\mathcal{D}(\Theta(n+1)) = \theta_{M_{n+1}}(n+1) - \theta_{m_{n+1}}(n+1)$$

$$
\begin{aligned}
&= \theta_{M_{n+1}}(n) - \theta_{m_{n+1}}(n) + (\nu_{M_{n+1}} - \nu_{m_{n+1}})h \\
&\quad + \frac{\kappa_1 h}{N} \sum_\beta \left[\frac{\psi_{M_{n+1}\beta}}{T_{M_{n+1}}(n)} \sin(\theta_\beta(n) - \theta_{M_{n+1}}(n)) - \frac{\psi_{m_{n+1}\beta}}{T_{m_{n+1}}(n)} \sin(\theta_\beta(n) - \theta_{m_{n+1}}(n)) \right] \\
&\le \theta_{M_{n+1}}(n) - \theta_{m_{n+1}}(n) + \mathcal{D}(\nu)h + \frac{\kappa_1 h \psi_M}{T_m^{in}} \left(\theta_{M_n}(n) - \theta_{M_{n+1}}(n) - \theta_{m_n}(n) + \theta_{m_{n+1}}(n) \right) \\
&\quad + \frac{\kappa_1 h}{N} \sum_\beta \left[\frac{\psi_{M_{n+1}\beta}}{T_{M_{n+1}}(n)} \sin(\theta_\beta(n) - \theta_{M_n}(n)) - \frac{\psi_{m_{n+1}\beta}}{T_{m_{n+1}}(n)} \sin(\theta_\beta(n) - \theta_{m_n}(n)) \right] \\
&\le \theta_{M_{n+1}}(n) - \theta_{m_{n+1}}(n) + \mathcal{D}(\nu)h + \frac{\kappa_1 h \psi_M}{T_m^{in}} \left(\theta_{M_n}(n) - \theta_{M_{n+1}}(n) - \theta_{m_n}(n) + \theta_{m_{n+1}}(n) \right) \\
&\quad + \frac{\kappa_1 h \psi_m}{N T_M^{in}} \sum_\beta \left[\sin(\theta_\beta(n) - \theta_{M_n}(n)) - \sin(\theta_\beta(n) - \theta_{m_n}(n)) \right] \\
&\le \mathcal{D}(\Theta(n)) + \left(\mathcal{D}(\nu) - \frac{\kappa_1 \psi_m}{T_M^{in}} \sin \mathcal{D}(\Theta(n)) \right) h,
\end{aligned}
$$

where we used $\sin(x+y) \le \sin x + \sin y$ for $0 \le x, y \le x+y < \pi$ and

$$
\theta_{M_n}(n) - \theta_{M_{n+1}}(n) - \theta_{m_n}(n) + \theta_{m_{n+1}}(n) \ge 0, \quad \frac{\kappa_1 h \psi_M}{T_m^{in}} < 1
$$

in the last inequality. On the other hand, the condition (4.2) also implies that the function

$$
x \mapsto x + \left(\mathcal{D}(\nu) - \frac{\kappa_1 \psi_m}{T_M^{in}} \sin x \right) h
$$

is strictly increasing. Therefore, if we further assume $\mathcal{D}(\Theta(n)) \le D$, we have

$$
\begin{aligned}
\mathcal{D}(\Theta(n+1)) &\le \mathcal{D}(\Theta(n)) + \left(\mathcal{D}(\nu) - \frac{\kappa_1 \psi_m}{T_M^{in}} \sin \mathcal{D}(\Theta(n)) \right) h \\
&\le D + \left(\mathcal{D}(\nu) - \frac{\kappa_1 \psi_m}{T_M^{in}} \sin D \right) h \le D, \quad \forall\ D \in [\gamma, \pi - \gamma].
\end{aligned}
$$

□

REMARK 4.1. The assumptions (4.1) are also consistent with their corresponding assumptions in Proposition 2.3. In particular, if $\mathcal{D}(\nu) = 0$, the phase diameter satisfies

$$
\mathcal{D}(\Theta(n+1)) \le \left(1 - \frac{\kappa_1 \psi_m h}{T_M^{in}} \frac{\sin \mathcal{D}(\Theta(n))}{\mathcal{D}(\Theta(n))} \right) \mathcal{D}(\Theta(n)) \le \left(1 - \frac{\kappa_1 \psi_m h}{T_M^{in}} \frac{\sin \mathcal{D}(\Theta^{in})}{\mathcal{D}(\Theta^{in})} \right) \mathcal{D}(\Theta(n)),
$$

and the phase diameter $\mathcal{D}(\Theta(n))$ converges to zero exponentially:

$$
\mathcal{D}(\Theta(n)) \le \left(1 - \frac{\kappa_1 \psi_m h}{T_M^{in}} \frac{\sin \mathcal{D}(\Theta^{in})}{\mathcal{D}(\Theta^{in})} \right)^n \mathcal{D}(\Theta^{in}).
$$

In fact, one can also show that the basin of attraction of the set $\mathcal{S}_\gamma$ contains $\mathcal{S}_{\pi-\gamma-\varepsilon}$ for every $\varepsilon \in (0, \pi - 2\gamma)$.

COROLLARY 4.1. *Suppose that the framework* $(\mathcal{A}1)-(\mathcal{A}2)$ *hold. Then, for every* $\varepsilon \in (0, \frac{\pi}{2} - \gamma)$*, all discrete TK flows started from* $\mathcal{S}_{\pi-\gamma-\varepsilon}$ *arrive at* $\mathcal{S}_{\gamma+\varepsilon}$ *in finite time.*

Proof. Let $\{(\Theta(n),\mathcal{T}(n))\}_{n\geq 0}$ be a solution to (1.4) subject to the initial data $(\Theta^{in},\mathcal{T}^{in})$, and suppose that Θ^{in} satisfies

$$\mathcal{D}(\Theta^{in})\leq \pi-\gamma-\varepsilon.$$

Then, as long as $\mathcal{D}(\Theta(n))\in[\gamma+\varepsilon,\pi-\gamma-\varepsilon]$, we have

$$\begin{aligned}\mathcal{D}(\Theta(n+1))&\leq \mathcal{D}(\Theta(n))+\left(\mathcal{D}(\nu)-\frac{\kappa_1\psi_m}{T_M^{in}}\sin\mathcal{D}(\Theta(n))\right)h\\&\leq \mathcal{D}(\Theta(n))+\frac{\kappa_1\psi_m h}{T_M^{in}}(\sin\gamma-\sin(\gamma+\varepsilon)).\end{aligned}$$

Therefore, if $\Theta(n)\notin\mathcal{S}_{\gamma+\varepsilon}$ for all integers $n\leq n_\varepsilon-1$, where n_ε is a positive integer satisfying

$$n_\varepsilon h\geq \frac{T_M^{in}(\pi-2(\gamma+\varepsilon))}{\kappa_1\psi_m(\sin(\gamma+\varepsilon)-\sin\gamma)},$$

we immediately obtain $\Theta(n_\varepsilon)\in\mathcal{S}_{\gamma+\varepsilon}$:

$$\begin{aligned}\mathcal{D}(\Theta(n_\varepsilon))&\leq \mathcal{D}(\Theta(0))+\frac{\kappa_1\psi_m n_\varepsilon h}{T_M^{in}}(\sin\gamma-\sin(\gamma+\varepsilon))\\&\leq \pi-\gamma-\varepsilon+\frac{\kappa_1\psi_m}{T_M^{in}}(\sin\gamma-\sin(\gamma+\varepsilon))\cdot\frac{T_M^{in}(\pi-2(\gamma+\varepsilon))}{\kappa_1\psi_m(\sin(\gamma+\varepsilon)-\sin\gamma)}\\&=\gamma+\varepsilon.\end{aligned}$$

□

From now on, we rewrite the discrete TK model (1.4) as a difference equation in a rotating frame with angular velocity $\nu_c=\frac{1}{N}\sum_\alpha\nu_\alpha$: for $\hat{\theta}_\alpha(n):=\theta_\alpha(n)-n\nu_c h$ and $\hat{\nu}_\alpha:=\nu_\alpha-\nu_c$, we have

$$\begin{cases}\hat{\theta}_\alpha(n+1)-\hat{\theta}_\alpha(n)=\hat{\nu}_\alpha h+\dfrac{\kappa_1 h}{NT_\alpha(n)}\displaystyle\sum_\beta\psi_{\alpha\beta}\sin(\hat{\theta}_\beta(n)-\hat{\theta}_\alpha(n)),\quad\forall\, n\in\mathbb{Z}_{\geq 0},\\ f(T_\alpha(n+1))-f(T_\alpha(n))=\dfrac{\kappa_2 h}{N}\displaystyle\sum_\beta\zeta_{\alpha\beta}\left(\frac{1}{T_\alpha(n)}-\frac{1}{T_\beta(n)}\right),\quad f(x)=x+\frac{x^2}{2},\\ (\hat{\theta}_\alpha(0),T_\alpha(0))=(\hat{\theta}_\alpha^{in},T_\alpha^{in}):=(\theta_\alpha^{in},T_\alpha^{in}),\quad\alpha=1,\cdots,N,\quad\displaystyle\sum_\alpha\hat{\nu}_\alpha=0.\end{cases}\tag{4.4}$$

Then, one can prove that the phase configurations $\hat{\Theta}(n)$ are uniformly bounded in n under the framework $(\mathcal{A})$ and if initial phase Θ^{in} satisfy $\mathcal{D}(\Theta^{in})\leq\pi-\gamma$. To see this, we set the average phase $\hat{\theta}_c$:

$$\hat{\theta}_c(n):=\frac{1}{N}\sum_\alpha\hat{\theta}_\alpha(n),$$

and prove its uniform boundedness in n.

LEMMA 4.2. *Let $(\hat{\Theta},\mathcal{T})$ be a solution to (4.4) subject to the initial data $(\hat{\Theta}^{in},\mathcal{T}^{in})$, and suppose the initial data $(\hat{\Theta}^{in},\mathcal{T}^{in})$ satisfies*

$$\mathcal{D}(\hat{\Theta}^{in})\leq\pi-\gamma.$$

Then, under the framework $(\mathcal{A}1)$–$(\mathcal{A}2)$*, the phases* $\hat{\theta}_\alpha(n) = \theta_\alpha(n) - n\nu_c h$ *are uniformly bounded in* n *for each* α*.*

Proof. Since $\left\{\mathcal{D}(\hat{\Theta}(n))\right\}_{n\ge 0}$ is uniformly bounded, it suffices to show that

$$\sup_{0\le n<\infty} |\hat{\theta}_c(n)| < \infty.$$

On the other hand, the increment of the average phase $\hat{\theta}_c$ is bounded by

$$\begin{aligned}
\left|\hat{\theta}_c(n+1) - \hat{\theta}_c(n)\right| &= \left|\sum_{\alpha,\beta} \frac{\kappa_1 h \psi_{\alpha\beta}}{N^2 T_\alpha(n)} \sin(\hat{\theta}_\beta(n) - \hat{\theta}_\alpha(n))\right| \\
&= \left|\sum_{\alpha,\beta} \frac{\kappa_1 h \psi_{\alpha\beta}}{N^2}\left(\frac{1}{T^\infty} - \frac{1}{T_\alpha(n)}\right) \sin(\hat{\theta}_\beta(n) - \hat{\theta}_\alpha(n))\right| \\
&\le \frac{\kappa_1 h \psi_M}{(T_m^{in})^2}\mathcal{D}(\mathcal{T}(n)) \le \frac{\kappa_1 h \psi_M}{(T_m^{in})^2}\mathcal{D}(\mathcal{T}^{in})\left(1 - \frac{\kappa_2 \zeta_m h}{(T_M^{in})^2(1+T_M^{in})}\right)^n.
\end{aligned}$$

Therefore, we obtain a desired n-independent upper bound of $|\hat{\theta}_c(n)|$, which is also independent of h:

$$|\hat{\theta}_c(n)| \le |\hat{\theta}_c^{in}| + \frac{\kappa_1 \psi_M (T_M^{in})^2(1+T_M^{in})}{\kappa_2 \zeta_m (T_m^{in})^2}\mathcal{D}(\mathcal{T}^{in}) < \infty, \quad \forall\, n \in \mathbb{Z}_{\ge 0}, \quad h > 0.$$

∎

REMARK 4.2. From the proof of Lemma 4.2, one can also see that the average phase $\hat{\theta}_c(n)$ converges as $n \to \infty$.

4.2. Emergence of asymptotic phase-locking. Now, we show the emergence of asymptotic phase-locking for the discrete TK model (1.4).

LEMMA 4.3. *Let* $(\Theta, \mathcal{T})$ *and* $(\widetilde{\Theta}, \widetilde{\mathcal{T}})$ *be solutions to* (1.4) *subject to initial data* $(\Theta^{in}, \mathcal{T}^{in})$ *and* $(\widetilde{\Theta}^{in}, \widetilde{\mathcal{T}}^{in})$*, respectively. Moreover, suppose the system parameters* κ_1, κ_2, ν *and initial data* $(\Theta^{in}, \mathcal{T}^{in})$, $(\widetilde{\Theta}^{in}, \widetilde{\mathcal{T}}^{in})$ *satisfy* $(\mathcal{A}1)-(\mathcal{A}2)$ *and*

$$\varepsilon > 0, \quad 0 \le \mathcal{D}(\Theta^{in}), \mathcal{D}(\widetilde{\Theta}^{in}) \le \gamma + \varepsilon \le \frac{\pi}{2}, \quad T_\alpha^{in}, \widetilde{T}_\alpha^{in} \subset [L, U] \subset (0, \infty) \quad \forall\, \alpha = 1, \cdots, N. \tag{4.5}$$

Then, if the two solutions have the same total energies, we have

$$\mathcal{D}(\phi(n)) \le (1-bh)^n \mathcal{D}(\phi(0)) + a\frac{(1-bh)^n - (1-ch)^n}{c-b},$$

where $\phi_\alpha := \theta_\alpha - \tilde{\theta}_\alpha, \phi(n) = (\phi_1(n), \cdots, \phi_N(n))$ *and*

$$\begin{aligned}
&a := \frac{4\kappa_1 \psi_M (U-L)}{L^2}, \quad b := \frac{\sin(2(\gamma+\varepsilon))}{2(\gamma+\varepsilon)} \cdot \frac{\kappa_1 \psi_m}{T^\infty}, \quad c := \frac{\kappa_2 \zeta_m}{U^2(1+U)}, \\
&f(T^\infty) = \frac{1}{N}\sum_\alpha f(T_\alpha^{in}) = \frac{1}{N}\sum_\alpha f(\widetilde{T}_\alpha^{in}), \quad T^\infty > 0.
\end{aligned} \tag{4.6}$$

Proof. First of all, Theorem 3.1 implies that all temperatures $T_\alpha(n)$ and $\widetilde{T}_\alpha(n)$ converge to T^∞ as n tends to infinity. In addition, we also have $\{\Theta(n)\}_{n\ge 0}, \left\{\widetilde{\Theta}(n)\right\}_{n\ge 0} \subset$

$\mathcal{S}_{\gamma+\varepsilon}$ from the positive invariance of $\mathcal{S}_{\gamma+\varepsilon}$ in Lemma 4.1. Now, for each nonnegative integer n, we set

$$A_{\alpha,\beta}(n) = \frac{\theta_\beta(n)-\theta_\alpha(n)+\tilde{\theta}_\beta(n)-\tilde{\theta}_\alpha(n)}{2}, \quad \forall\ \alpha,\beta=1,\cdots,N,$$
$$\phi_{M_n}(n) = \max_\alpha \phi_\alpha(n), \quad \phi_{m_n}(n)=\min_\alpha \phi_\alpha(n), \quad M_n, m_n \in \{1,\cdots,N\}.$$

Then, the difference between $\phi_{M_{n+1}}(n+1)$ and $\phi_{M_n}(n)$ is given by

$$\begin{aligned}
&\frac{\phi_{M_{n+1}}(n+1)-\phi_{M_n}(n)}{h}\\
&= \frac{\kappa_1}{NT^\infty}\sum_{\beta=1}^N \psi_{M_{n+1}\beta}\Big(\sin(\theta_\beta(n)-\theta_{M_{n+1}}(n)) - \sin(\tilde{\theta}_\beta(n)-\tilde{\theta}_{M_{n+1}}(n))\Big)\\
&\quad + \frac{\kappa_1}{N}\sum_{\beta=1}^N \psi_{M_{n+1}\beta}\left(\frac{1}{T_{M_{n+1}}(n)} - \frac{1}{T^\infty}\right)\sin(\theta_\beta(n)-\theta_{M_{n+1}}(n))\\
&\quad - \frac{\kappa_1}{N}\sum_{\beta=1}^N \psi_{M_{n+1}\beta}\left(\frac{1}{\widetilde{T}_{M_{n+1}}(n)} - \frac{1}{T^\infty}\right)\sin(\tilde{\theta}_\beta(n)-\tilde{\theta}_{M_{n+1}}(n))\\
&\quad + \frac{\phi_{M_{n+1}}(n)-\phi_{M_n}(n)}{h} =: \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4.
\end{aligned}$$

In the sequel, we estimate each $\mathcal{I}_i$ one by one.

- (Estimate on $\mathcal{I}_1$): From the definition of $A_{\alpha,\beta}$, we have

$$|A_{\alpha,\beta}(n)| \le \frac{\mathcal{D}(\Theta(n))+\mathcal{D}(\widetilde{\Theta}(n))}{2} \le \gamma+\varepsilon, \quad \forall\ \alpha,\beta=1,\cdots,N, \quad n\in\mathbb{Z}_{\ge 0}.$$

Then, for each $\beta=1,2,,\cdots,N$, one can find an upper bound of the summand of $\mathcal{I}_1$.

$$\begin{aligned}
&\sin(\theta_\beta(n)-\theta_{M_{n+1}}(n)) - \sin(\tilde{\theta}_\beta(n)-\tilde{\theta}_{M_{n+1}}(n))\\
&= 2\cos\left(A_{M_{n+1},\beta}(n)\right)\sin\left(\frac{\phi_\beta-\phi_{M_{n+1}}}{2}(n)\right)\\
&\le 2\cos\left(A_{M_{n+1},\beta}(n)\right)\sin\left(\frac{\phi_\beta-\phi_{M_n}}{2}(n)\right) + 2\cos\left(A_{M_{n+1},\beta}(n)\right)\left(\frac{\phi_{M_n}-\phi_{M_{n+1}}}{2}(n)\right)\\
&\le -2(\cos(\gamma+\varepsilon))\sin\left(\frac{\phi_{M_n}(n)-\phi_\beta(n)}{2}\right) + \phi_{M_n}(n)-\phi_{M_{n+1}}(n),
\end{aligned}$$

where we used $|\sin x - \sin y| \le |x-y|$ in the first inequality. Therefore, we have

$$\begin{aligned}
\mathcal{I}_1 &\le \frac{\kappa_1\psi_m}{NT^\infty}\sum_{\beta=1}^N\left(-2(\cos(\gamma+\varepsilon))\sin\left(\frac{\phi_{M_n}(n)-\phi_\beta(n)}{2}\right) + \phi_{M_n}(n)-\phi_{M_{n+1}}(n)\right)\\
&\le -\frac{2(\cos(\gamma+\varepsilon))\kappa_1\psi_m}{NT^\infty}\sum_{\beta=1}^N\left(\sin\left(\frac{\phi_{M_n}(n)-\phi_\beta(n)}{2}\right)\right) + \frac{\phi_{M_n}(n)-\phi_{M_{n+1}}(n)}{h}\\
&= -\frac{2(\cos(\gamma+\varepsilon))\kappa_1\psi_m}{NT^\infty}\sum_{\beta=1}^N\left(\sin\left(\frac{\phi_{M_n}(n)-\phi_\beta(n)}{2}\right)\right) - \mathcal{I}_4,
\end{aligned}$$

where we used $\frac{\kappa_1 \psi_m h}{T^\infty} < 1$ obtained from ($\mathcal{A}2$) in the second inequality.

- (Estimate on $\mathcal{I}_2$ and $\mathcal{I}_3$): From direct calculation, we have

$$\begin{aligned}
|\mathcal{I}_2| &= \left| \frac{\kappa_1}{N} \sum_{\beta=1}^{N} \psi_{M_{n+1}\beta} \left(\frac{1}{T_{M_{n+1}}(n)} - \frac{1}{T^\infty} \right) \sin(\theta_\beta(n) - \theta_{M_{n+1}}(n)) \right| \\
&\le \frac{\kappa_1}{N} \sum_{\beta=1}^{N} \psi_{M_{n+1}\beta} \left| \frac{1}{T_{M_{n+1}}(n)} - \frac{1}{T^\infty} \right| \le \frac{\kappa_1 \psi_M \mathcal{D}(\mathcal{T}(n))}{L^2},
\end{aligned}$$

where we used

$$L \le T_m^{in} \le T_{M_{n+1}}(n), \quad L \le T^\infty, \quad |T^\infty - T_{M_{n+1}}(n)| \le \mathcal{D}(\mathcal{T}(n)),$$

in the last inequality. Similarly, the upper bound of $|\mathcal{I}_3|$ can be estimated by

$$\begin{aligned}
|\mathcal{I}_3| &= \left| \frac{\kappa_1}{N} \sum_{\beta=1}^{N} \psi_{M_{n+1}\beta} \left(\frac{1}{\widetilde{T}_{M_{n+1}}(n)} - \frac{1}{T^\infty} \right) \sin(\tilde{\theta}_\beta(n) - \tilde{\theta}_{M_{n+1}}(n)) \right| \\
&\le \frac{\kappa_1}{N} \sum_{\beta=1}^{N} \psi_{M_{n+1}\beta} \left| \frac{1}{\widetilde{T}_{M_{n+1}}(n)} - \frac{1}{T^\infty} \right| \le \frac{\kappa_1 \psi_M \mathcal{D}(\widetilde{\mathcal{T}}(n))}{L^2}.
\end{aligned}$$

Then, we combine the estimates of $\mathcal{I}_i,\ i = 1,2,3,4$, to obtain

$$\begin{aligned}
&\frac{\phi_{M_{n+1}}(n+1) - \phi_{M_n}(n)}{h} \\
&\le -\frac{2(\cos(\gamma+\varepsilon))\kappa_1 \psi_m}{N T^\infty} \sum_{\beta=1}^{N} \sin\left(\frac{\phi_{M_n}(n) - \phi_\beta(n)}{2} \right) + \frac{\kappa_1 \psi_M (\mathcal{D}(\mathcal{T}(n)) + \mathcal{D}(\widetilde{\mathcal{T}}(n)))}{L^2}.
\end{aligned}$$

Moreover, by taking $\tilde{\phi}_\alpha := \tilde{\theta}_\alpha - \theta_\alpha$ and applying the above argument to $\tilde{\phi}_\alpha$, we have

$$\begin{aligned}
&\frac{\phi_{m_{n+1}}(n+1) - \phi_{m_n}(n)}{h} \\
&\ge -\frac{2(\cos(\gamma+\varepsilon))\kappa_1 \psi_m}{N T^\infty} \sum_{\beta=1}^{N} \sin\left(\frac{\phi_{m_n}(n) - \phi_\beta(n)}{2} \right) - \frac{\kappa_1 \psi_M (\mathcal{D}(\mathcal{T}(n)) + \mathcal{D}(\widetilde{\mathcal{T}}(n)))}{L^2}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\frac{\mathcal{D}(\phi(n+1)) - \mathcal{D}(\phi(n))}{h} \\
&\le -\frac{2(\cos(\gamma+\varepsilon))\kappa_1 \psi_m}{T^\infty} \sin\frac{\mathcal{D}(\phi(n))}{2} + \frac{2\kappa_1 \psi_M (\mathcal{D}(\mathcal{T}(n)) + \mathcal{D}(\widetilde{\mathcal{T}}(n)))}{L^2} \\
&\le -\frac{2(\cos(\gamma+\varepsilon))\kappa_1 \psi_m}{T^\infty} \sin\frac{\mathcal{D}(\phi(n))}{2} + \frac{4\kappa_1 \psi_M (U-L)}{L^2} \left(1 - \frac{\kappa_2 \zeta_m h}{U^2(1+U)} \right)^n \\
&\le -\frac{\sin(2(\gamma+\varepsilon))\kappa_1 \psi_m}{2(\gamma+\varepsilon) T^\infty} \mathcal{D}(\phi(n)) + \frac{4\kappa_1 \psi_M (U-L)}{L^2} \left(1 - \frac{\kappa_2 \zeta_m h}{U^2(1+U)} \right)^n, \qquad (4.7)
\end{aligned}$$

where we used $\sin(x+y)\le \sin x+\sin y$ for $0\le x,y\le x+y<\pi$ and

$$0\le \frac{\phi_{M_n}(n)-\phi_\beta(n)}{2}=\frac{(\theta_{M_n}(n)-\theta_\beta(n))-(\tilde\theta_{M_n}(n)-\tilde\theta_\beta(n))}{2}\le \gamma+\varepsilon,$$
$$0\ge \frac{\phi_{m_n}(n)-\phi_\beta(n)}{2}=\frac{(\theta_{m_n}(n)-\theta_\beta(n))-(\tilde\theta_{m_n}(n)-\tilde\theta_\beta(n))}{2}\ge -(\gamma+\varepsilon).$$

Then, the inequality (4.7) is equivalent to

$$\begin{aligned}&\mathcal{D}(\phi(n+1))\\ &\le \left(1-\frac{\sin(2(\gamma+\varepsilon))}{2(\gamma+\varepsilon)}\cdot\frac{\kappa_1 h\psi_m}{T^\infty}\right)\mathcal{D}(\phi(n))+\frac{4\kappa_1 h\psi_M(U-L)}{L^2}\left(1-\frac{\kappa_2\zeta_m h}{U^2(1+U)}\right)^n,\end{aligned}$$

which can be simplified as

$$\mathcal{D}(\phi(n+1))+\frac{a}{c-b}(1-ch)^{n+1}\le (1-bh)\left(\mathcal{D}(\phi(n))+\frac{a}{c-b}(1-ch)^n\right),$$
$$a:=\frac{4\kappa_1\psi_M(U-L)}{L^2},\quad b:=\frac{\sin(2(\gamma+\varepsilon))}{2(\gamma+\varepsilon)}\cdot\frac{\kappa_1\psi_m}{T^\infty},\quad c:=\frac{\kappa_2\zeta_m}{U^2(1+U)}.$$

Therefore, we have

$$\mathcal{D}(\phi(n))\le (1-bh)^n\mathcal{D}(\phi(0))+a\frac{(1-bh)^n-(1-ch)^n}{c-b}.$$

□

REMARK 4.3. Under the framework $(\mathcal{A}1)-(\mathcal{A}2)$, the positiveness of $1-bh$ and $1-ch$ are always guaranteed. To see $1-bh>0$, we use the assumption $(\mathcal{A}2)$ and obtain

$$bh\le \frac{\kappa_1\psi_m h}{T^\infty}\le \frac{\kappa_1\psi_M h}{T_m^{in}}<1.$$

To see $1-ch>0$, we use (2.6) in $(\mathcal{A}1)$ and obtain

$$ch\le \frac{\kappa_2\bar\zeta_M h}{|T_m^{in}|^2(1+\frac{T_m^{in}}{2})}<1.$$

Moreover, if $(\Theta,\mathcal{T})$ and $(\widetilde\Theta,\widetilde{\mathcal{T}})$ are the solutions of continuous TK model (1.2), one can also obtain the following analogous estimate for $\phi(t)$:

$$\mathcal{D}(\phi(t))\le \mathcal{D}(\phi(0))e^{-bt}+\frac{a}{c-b}\left(e^{-bt}-e^{-ct}\right).$$

Now, we are ready to show the main result of this section.

THEOREM 4.1. *Let $\{(\hat\Theta,\mathcal{T})\}$ be a solution to (4.4) subject to the initial data $\{(\Theta^{in},\mathcal{T}^{in})\}$ satisfying $(\mathcal{A}1)-(\mathcal{A}2)$, and suppose that initial phase diameter $\mathcal{D}(\Theta^{in})$ is strictly less than $\pi-\gamma$. Then, the solution $\{(\hat\Theta,\mathcal{T})\}$ converges to a phase-locked state as $n\to\infty$.*

Proof. Fix $\varepsilon\in\left(0,\frac{\pi}{2}-\gamma\right)$. From Lemma 4.1 and Corollary 4.1, there exists a positive integer n'_ε such that

$$n\ge n'_\varepsilon \Longrightarrow \mathcal{D}(\hat\Theta(n))\le \gamma+\varepsilon.$$

Then, we apply Lemma 4.3 to

$$\left\{\left(\hat{\Theta}(n'_\varepsilon+n),\mathcal{T}(n'_\varepsilon+n)\right)\right\}_{n\geq 0} \quad \text{and} \quad \left\{\left(\hat{\Theta}(n'_\varepsilon+n+1),\mathcal{T}(n'_\varepsilon+n+1)\right)\right\}_{n\geq 0}$$

to obtain

$$\begin{aligned}
&\max_{\alpha,\beta}|(\hat{\theta}_\alpha(n'_\varepsilon+n+1)-\hat{\theta}_\alpha(n'_\varepsilon+n))-(\hat{\theta}_\beta(n'_\varepsilon+n+1)-\hat{\theta}_\beta(n'_\varepsilon+n))| \\
&\leq (1-b'h)^n\cdot\max_{\alpha,\beta}|(\hat{\theta}_\alpha(n'_\varepsilon+1)-\hat{\theta}_\alpha(n'_\varepsilon))-(\hat{\theta}_\beta(n'_\varepsilon+1)-\hat{\theta}_\beta(n'_\varepsilon))| \\
&\quad +a'\frac{(1-b'h)^n-(1-c'h)^n}{c'-b'},
\end{aligned}$$

where

$$a':=\frac{4\kappa_1\psi_M(T_M^{in}-T_m^{in})}{{T_m^{in}}^2},\quad b':=\frac{\sin(2(\gamma+\varepsilon))}{2(\gamma+\varepsilon)}\cdot\frac{\kappa_1\psi_m}{T^\infty},\quad c':=\frac{\kappa_2\zeta_m}{{T_M^{in}}^2(1+T_M^{in})}.$$

Therefore, for each pair (α,β), there exists a constant $\theta_{\alpha\beta}^\infty$ satisfying

$$\theta_{\alpha\beta}^\infty=\lim_{n\to\infty}\left(\hat{\theta}_\alpha(n)-\hat{\theta}_\beta(n)\right).$$

On the other hand, recall that Lemma 4.2 and Remark 4.2 imply the convergence of $\hat{\theta}_c(n)$ as $n\to\infty$. The convergence of $\hat{\theta}_\alpha(n)$ is then given by

$$\lim_{n\to\infty}\hat{\theta}_\alpha(n)=\lim_{n\to\infty}(\hat{\theta}_\alpha(n)-\hat{\theta}_c(n))+\lim_{n\to\infty}\hat{\theta}_c(n)=\frac{1}{N}\sum_\beta\theta_{\alpha\beta}+\lim_{n\to\infty}\hat{\theta}_c(n),\quad\forall\ \alpha=1,\cdots,N,$$

which is the desired result. □

5. From discrete model to continuous model

In this section, we study the uniform-in-time convergence from the discrete TK model to the continuous model as $h\to 0$. Since all norms in $\mathbb{R}^N$ are equivalent, we consider the ℓ^∞-norm throughout this section.

5.1. Finite-in-time transition.

Recall that the discrete TK model is

$$\begin{cases}
\theta_\alpha(n+1)-\theta_\alpha(n)=\nu_\alpha h+\dfrac{\kappa_1 h}{NT_\alpha(n)}\displaystyle\sum_{\beta=1}^N\psi_{\alpha\beta}\sin(\theta_\beta(n)-\theta_\alpha(n)),\quad\forall\ n\in\mathbb{Z}_{\geq 0},\\
f(T_\alpha(n+1))-f(T_\alpha(n))=\dfrac{\kappa_2 h}{N}\displaystyle\sum_{\beta=1}^N\zeta_{\alpha\beta}\left(\frac{1}{T_\alpha(n)}-\frac{1}{T_\beta(n)}\right),\quad f(x)=x+\dfrac{x^2}{2},\\
(\theta_\alpha(0),T_\alpha(0))=(\theta_\alpha^{in},T_\alpha^{in}),\quad\alpha=1,\cdots,N.
\end{cases}\tag{5.1}$$

This implicit difference equation is indeed a first-order Euler scheme for phase-energy states $\{(\theta_\alpha,\mathcal{E}_\alpha:=f(T_\alpha))\}_{\alpha=1}^N$. To see this, consider the restriction of function f to $(0,\infty)$:

$$f:(0,\infty)\to(0,\infty),\quad f(x)=x+\frac{x^2}{2},$$

and denote its inverse g. Since all T_α's have a positive lower bound T_m^{in} under the condition (2.6), one can obtain the following alternative form:

$$\begin{cases} \theta_\alpha(n+1)-\theta_\alpha(n)=\nu_\alpha h+\dfrac{\kappa_1 h}{Ng(\mathcal{E}_\alpha(n))}\displaystyle\sum_{\beta=1}^N \psi_{\alpha\beta}\sin(\theta_\beta(n)-\theta_\alpha(n)), \quad n\in\mathbb{Z}_{\geq 0}, \\ \mathcal{E}_\alpha(n+1)-\mathcal{E}_\alpha(n)=\dfrac{\kappa_2 h}{N}\displaystyle\sum_{\beta=1}^N \zeta_{\alpha\beta}\left(\dfrac{1}{g(\mathcal{E}_\alpha(n))}-\dfrac{1}{g(\mathcal{E}_\beta(n))}\right), \quad n\in\mathbb{Z}_{\geq 0}, \\ (\theta_\alpha(0),\mathcal{E}_\alpha(0))=\left(\theta_\alpha^{in},f(T_\alpha^{in})\right), \quad \alpha=1,\cdots,N, \\ T_\alpha(n):=g(\mathcal{E}_\alpha(n)), \quad n\in\mathbb{Z}_{\geq 0}, \quad \alpha=1,\cdots,N. \end{cases} \tag{5.2}$$

In addition, the right-hand sides of $(5.2)_1$ and $(5.2)_2$ are globally Lipschitz in

$$\left\{(\theta_1,\cdots,\theta_N,\mathcal{E}_1,\cdots,\mathcal{E}_N):\theta_\alpha\in\mathbb{R},\ 0<\mathcal{E}_m\leq\mathcal{E}_\alpha\leq\mathcal{E}_M<\infty\right\},$$

for every fixed $\mathcal{E}_m:=T_m^{in}+\frac{|T_m^{in}|^2}{2}$ and $\mathcal{E}_M:=T_M^{in}+\frac{|T_M^{in}|^2}{2}$. Therefore, the classical theory of the Euler discretization method yields the consistency and convergence of Euler discretization to its continuous counterpart.

Proposition 5.1 ([34]). *Suppose that the function $p:\mathbb{R}^d\to\mathbb{R}$ has a global Lipschitz constant L. Then, the Cauchy problem*

$$\frac{dx}{dt}=p(x), \quad x(0)=x^{in}, \quad t>0,$$

and the corresponding one-step Euler method

$$x_{n+1}^h=x_n^h+hp(x_n^h), \quad x_0^h=x^{in}, \quad n\in\mathbb{Z}_{\geq 0}$$

satisfy the following error estimates:

(1) *Consistency: As time step h tends to 0, the truncation error*

$$E_1^h(n):=\frac{1}{h}\|x((n+1)h)-x(nh)-hp(x(nh))\|$$

converges to zero uniformly in any finite time, i.e.,

$$\lim_{h\to 0}\max_{0\leq n\leq\frac{\tau}{h}} E_1^h(n)=0, \quad \forall\ \tau<\infty.$$

(2) *Convergence: As time step h tends to 0, the global error*

$$E_2^h(n):=\left\|x(nh)-x_n^h\right\|$$

can be controlled by the truncation error and Lipschitz constant. More precisely,

$$E_2^h(n)\leq\frac{\max_{0\leq n\leq\frac{\tau}{h}}E_1^h(n)}{L}\left(e^{Lnh}-1\right), \quad \forall\ (n,h,\tau) \text{ satisfying } 0\leq nh\leq\tau.$$

This result shows that the Euler discretization gives a consistent approximation to given ODE in any finite time, but the error grows exponentially in general. Therefore, the convergence result we want to verify beyond this general theory is to show the uniform-in-time convergence of (5.1)(or equivalently, (5.2)) to (1.4) as the time step tends to zero. Below, we define the uniform-in-time convergence of discrete TK model, inspired from the definitions in [27, 41].

DEFINITION 5.1. *Let $(\Theta,\mathcal{T})$ be a solution to the continuous system (1.2) and $(\Theta^h,\mathcal{T}^h)$ be a solution to the discrete system (1.4).*

(1) *The discrete system (1.4) converges to (1.2) in $[0,\tau]$ if and only if*

$$\limsup_{h\to 0}\sup_{0\le n\le \frac{\tau}{h}}\left(\|\Theta(nh)-\Theta^h(n)\|+\|\mathcal{T}(nh)-\mathcal{T}^h(n)\|\right)=0.$$

(2) *The discrete system (1.4) converges to (1.2) uniformly in time if and only if*

$$\limsup_{h\to 0}\sup_{0\le n<\infty}\left(\|\Theta(nh)-\Theta^h(n)\|+\|\mathcal{T}(nh)-\mathcal{T}^h(n)\|\right)=0.$$

On the other hand, note that the functions $f(t)=t+\frac{t^2}{2}$ and $\left(f^{-1}\right)(t)=-1+\sqrt{1+2t}$ satisfy

$$f(x)-f(y)=(x-y)\left(1+\frac{x+y}{2}\right),\quad \left(f^{-1}\right)(x)-\left(f^{-1}\right)(y)=\frac{2x-2y}{\sqrt{1+2x}+\sqrt{1+2y}}$$

for every positive real numbers x and y. Therefore, since the temperatures have uniform-in-time positive upper and lower bounds along the continuous and discrete TK flows, Proposition 5.1 yields the convergence of (1.4) to (1.2) in every finite time interval $[0,\tau]$. Moreover, the discrete system (1.4) converges to (1.2) uniformly in time if and only if:

$$\limsup_{h\to 0}\sup_{0\le n<\infty}\left(\|\Theta(nh)-\Theta^h(n)\|+\|\mathcal{E}(nh)-\mathcal{E}^h(n)\|\right)=0,$$

where $\mathcal{E}$ is an abbreviation of a vector $(\mathcal{E}_1,\cdots,\mathcal{E}_N)$.

5.2. Uniform-in-time transition. Now, we show the uniform-in-time transition of (1.4) to (1.2) by combining the results in Section 3 and Section 4.1. The following lemma gives a uniform-in-time convergence from finite-time convergence, which was used implicitly in [14].

LEMMA 5.1. *Let $\{a^h(n)\}_{h>0}$ be a one-parameter family of sequences in $\mathbb{R}^N$, and let $a(\cdot)$ be a curve on $\mathbb{R}^N$. Suppose that*

$$a^h(\infty):=\lim_{n\to\infty}a^h(n),\quad a(\infty):=\lim_{t\to\infty}a(t)$$

exist and there are two continuous functions p_1 and p_2 satisfying

$$\begin{aligned}&\|a^h(n)-a^h(\infty)\|\le p_1(nh),\quad \|a(nh)-a(\infty)\|\le p_2(nh),\ \forall\ n,h>0,\\&\lim_{t\to\infty}p_1(t)=\lim_{t\to\infty}p_2(t)=0.\end{aligned}$$

If we further assume

$$\limsup_{h\to 0}\sup_{0\le n\le\frac{\tau}{h}}\left(\|a(nh)-a^h(n)\|\right)=0,\quad \forall\ \tau>0,\tag{5.3}$$

then $\{a^h(n)\}_{h>0}$ converges to $a(\cdot)$ uniformly in time.

Proof. First, fix arbitrary $\varepsilon > 0$. Then, one can find a positive constant τ_0 such that

$$|p_1(t)| < \frac{\varepsilon}{5}, \quad |p_2(t)| < \frac{\varepsilon}{5} \quad \forall\, t > \tau_0.$$

On the other hand, from (5.3), one can also find a positive constant $h_0 < 1$ such that

$$\sup_{0 \le n \le \frac{\tau_0+1}{h}} \|a(nh) - a^h(n)\| < \frac{\varepsilon}{5} \quad \forall\, 0 < h < h_0.$$

Then, we have

$$\begin{aligned}
\|a^h(n) - a(nh)\| &\le \|a^h(n) - a^h(\infty)\| + \|a^h(\lfloor \frac{\tau_0+1}{h} \rfloor) - a^h(\infty)\| \\
&\quad + \|a^h(\lfloor \frac{\tau_0+1}{h} \rfloor) - a(h\lfloor \frac{\tau_0+1}{h} \rfloor)\| \\
&\quad + \|a(h\lfloor (\tau_0+1)/h \rfloor) - a(\infty)\| + \|a(\infty) - a(nh)\| \\
&\le p_1(nh) + p_1(h\lfloor \frac{\tau_0+1}{h} \rfloor) + \frac{\varepsilon}{5} + p_2(nh) + p_2(h\lfloor \frac{\tau_0+1}{h} \rfloor) < \varepsilon,
\end{aligned}$$

for every $0 < h < h_0$ and $n > \frac{\tau_0+1}{h}$. Hence, we obtain

$$\limsup_{h \to 0} \sup_{0 \le n < \infty} \|a(nh) - a^h(n)\| \le \varepsilon, \quad \forall\, \varepsilon > 0,$$

which implies the desired result. □

THEOREM 5.1. *Let $(\Theta, \mathcal{T})$ and $(\Theta^h, \mathcal{T}^h)$ be solutions to (1.2) and (1.4) subject to the same initial data $(\Theta^{in}, \mathcal{T}^{in})$. Moreover, suppose the system parameters κ_1, κ_2, ν and initial data $(\Theta^{in}, \mathcal{T}^{in})$ satisfy $(\mathcal{A}1)-(\mathcal{A}2)$, and the initial phase diameter satisfies*

$$\mathcal{D}(\Theta^{in}) \le \gamma.$$

Then, as h tends to zero, $(\Theta^h, \mathcal{T}^h)$ converges to $(\Theta, \mathcal{T})$ uniformly in time.

Proof. First, we show the uniform transition of discrete phases to its continuous counterpart. From Proposition 2.3 and Theorem 4.1, $\hat{\theta}_\alpha(nh) = \theta_\alpha(nh) - \nu_c nh$ and $\hat{\theta}^h_\alpha(n) = \theta_\alpha - \nu_c nh$ converge as $n \to \infty$, for each $\alpha = 1, \cdots, N$. Therefore, we apply Lemma 4.1 to $\phi_\alpha(n) := \hat{\theta}^h_\alpha(n) - \hat{\theta}^{h,\infty}_\alpha := \hat{\theta}^h_\alpha(n) - \lim_{n\to\infty} \hat{\theta}^h_\alpha(n)$ and obtain

$$\begin{aligned}
&\left|\hat{\theta}^h_\alpha(n+1) - \hat{\theta}^h_\alpha(n)\right| \\
&\le \frac{\kappa_1 h}{N T^h_\alpha(n)} \sum_{\beta=1}^N \psi_{\alpha\beta} \left| \sin(\hat{\theta}^h_\beta(n) - \hat{\theta}^h_\alpha(n)) - \sin(\hat{\theta}^{h,\infty}_\beta - \hat{\theta}^{h,\infty}_\alpha) \right| \\
&\le \frac{\kappa_1 h \psi_M}{T^{in}_m} \left((1-bh)^n \left(\max_{\alpha,\beta} \left\{ \hat{\theta}^{in}_\alpha - \hat{\theta}^{in}_\beta - \hat{\theta}^{h,\infty}_\alpha + \hat{\theta}^{h,\infty}_\beta \right\} \right) + a \frac{(1-bh)^n - (1-ch)^n}{c-b} \right),
\end{aligned}$$

$$\begin{aligned}
&\left|\hat{\theta}^{h,\infty}_\alpha - \hat{\theta}^h_\alpha(n)\right| \\
&\le \frac{\kappa_1 \psi_M}{T^{in}_m} \left(\frac{(1-bh)^n}{b} \left(\max_{\alpha,\beta} \left\{ \hat{\theta}^{in}_\alpha - \hat{\theta}^{in}_\beta - \hat{\theta}^{h,\infty}_\alpha + \hat{\theta}^{h,\infty}_\beta \right\} \right) + a \frac{\frac{(1-bh)^n}{b} - \frac{(1-ch)^n}{c}}{c-b} \right) \\
&\le \frac{\kappa_1 \psi_M}{T^{in}_m} \left(\frac{e^{-bnh}}{b} \left(\max_{\alpha,\beta} \left\{ \hat{\theta}^{in}_\alpha - \hat{\theta}^{in}_\beta - \hat{\theta}^{h,\infty}_\alpha + \hat{\theta}^{h,\infty}_\beta \right\} \right) + a \frac{\frac{e^{-bnh}}{b} + \frac{e^{-cnh}}{c}}{|b-c|} \right) =: p_1(nh),
\end{aligned}$$

where a,b,c are the constants independent of h and n as in (4.6). Similarly, we have

$$|\hat{\theta}_\alpha(nh)-\hat{\theta}_\alpha^\infty|\le p_1(nh), \quad \hat{\theta}_\alpha^\infty := \lim_{t\to\infty}\hat{\theta}_\alpha(t).$$

Therefore, we apply Lemma 5.1 to obtain the uniform-in-time transition of $\theta_\alpha^h(n)$ to $\theta_\alpha(nh)$.

For the transition of temperatures, we use Proposition 2.1, Lemma 3.1 and apply Lemma 5.1 to obtain the desired result. □

6. Conclusion

In this paper, we have provided an energy conserving discretization scheme for the thermodynamic Kuramoto model on a lattice. The thermodynamic Kuramoto model on a lattice was recently introduced in a series of works by the first and third authors and their collaborators in [25] as a correction of the Kuramoto model in an externally applied temperature field. The effect of temperature appears in the coupling strength part inversely. Thus, smaller temperatures can enhance the degree of collective behaviors which is consistent with our intuition. A naive and simple first-order Euler-type discretization does not conserve a total energy. In order to cure this apparent inconsistency with the continuous model, we proposed an energy conserving implicit discretization scheme for the thermodynamic Kuramoto lattice model so that the resulting discrete system exhibits the same dynamic features as the continuous one. For the proposed discrete model, we provided sufficient frameworks leading to the temperature homogenization and complete synchronization in terms of system parameters and initial configurations. Moreover, we also provided estimates for continuous passage from the discrete model to the corresponding continuous model in any finite interval for any generic initial data, whereas we can also extend finite-time lattice limit to the whole time interval for some admissible class of system parameters and initial data.

REFERENCES

[1] J.A. Acebron, L.L. Bonilla, C.J.P. Vicente, F. Ritort, and R. Spigler, *The Kuramoto model: A simple paradigm for synchronization phenomena*, Rev. Mod. Phys., 77:137–185, 2005. 1

[2] G. Albi, N. Bellomo, L. Fermo, S.-Y. Ha, L. Pareschi, D. Poyato, and J. Soler, *Vehicular traffic, crowds, and swarms. On the kinetic theory approach towards research perspective*, Math. Model. Meth. Appl. Sci., 29:1901–2005, 2019. 1

[3] N. Bellomo, S.-Y. Ha, and N. Outada, *Towards a mathematical theory of behavioral swarms*, ESAIM Control Optim. Calc. Var., 26:125, 2020. 1

[4] D. Benedetto, E. Caglioti, and U. Montemagno, *On the complete phase synchronization for the Kuramoto model in the mean-field limit*, Commun. Math. Sci., 13:1775–1786, 2015. 1

[5] J. Buck and E. Buck, *Biology of synchronous flashing of fireflies*, Nature, 211:562–564, 1966. 1

[6] Y.-P. Choi and S.-Y. Ha, *A simple proof of the complete consensus of discrete-time dynamical networks with time-varying couplings*, Int. J. Numer. Anal. Model. Ser. B, 1:58–69, 2010. 1

[7] Y.-P. Choi, S.-Y. Ha, S. Jung, and Y. Kim, *Asymptotic formation and orbital stability of phase-locked states for the Kuramoto model*, Phys. D, 241:735–754, 2012. 1

[8] N. Chopra and M.W. Spong, *On exponential synchronization of Kuramoto oscillators*, IEEE Trans. Automat. Contr., 54:353–357, 2009. 1

[9] F. Cucker and S. Smale, *Emergent behavior in flocks*, IEEE Trans. Automat. Contr., 52:852–862, 2007. 1

[10] P. Degong and S. Motsch, *Continuum limit of self-driven particles with orientation interaction*, Math. Model. Meth. Appl. Sci., 18:1193–1215, 2008. 1

[11] P. Degong and S. Motsch, *Large-scale dynamics of the persistent Turing Walker model of fish behavior*, J. Stat. Phys., 131:989–1021, 2008. 1

[12] P. Degong and S. Motsch, *Macroscopic limit of self-driven particles with orientation interaction*, C.R. Math. Acad. Sci. Paris, 345:555–560, 2007. 1

[13] J.-G. Dong and X. Xue, *Synchronization analysis of Kuramoto oscillators*, Commun. Math. Sci., 11:465–480, 2013. 1, 2.1

[14] J.-G. Dong, S.-Y. Ha, and D. Kim, *A nonlinear functional approach for mono-cluster flocking to the discrete Cucker-Smale model*, Commun. Math. Sci., 17:2239–2256, 2019. 5.2

[15] F. Dörfler and F. Bullo, *Synchronization in complex networks of phase oscillators: A survey*, Automatica, 50:1539–1564, 2014. 1, 1

[16] F. Dörfler and F. Bullo, *On the critical coupling for Kuramoto oscillators*, SIAM. J. Appl. Dyn. Syst., 10:1070–1099, 2011. 1, 1

[17] S.-Y. Ha, E. Jeong, and M.-J. Kang, *Emergent behavior of a generalized Viscek-type flocking model*, Nonlinearity, 23:3139–3156, 2010. 1

[18] S.-Y. Ha, M. Kang, H. Park, T. Ruggeri, and W. Shim, *Emergent behaviors of the continuum thermodynamic Kuramoto model in a large coupling regime*, J. Differ. Equ., 300(5):519-564, 2021. 1

[19] S.-Y. Ha, M. Kang, H. Park, T. Ruggeri, and W. Shim, *Uniform stability and uniform-in-time mean-field limit of the thermodynamic Kuramoto model*, Quart. Appl. Math., 79:445–478, 2021. 1

[20] S.-Y. Ha, H.W. Kim, and S.W. Ryoo, *Emergence of phase-locked states for the Kuramoto model in a large coupling regime*, Commun. Math. Sci., 14:1073–1091, 2016. 1, 2.1

[21] S.-Y. Ha, J. Kim, and T. Ruggeri, *Emergent behaviors of thermodynamic Cucker-Smale particles*, SIAM J. Math. Anal., 50:3092–3121, 2019. 1

[22] S.-Y. Ha, Y. Kim, and Z. Li, *Asymptotic synchronization behavior of Kuramoto type models with frustrations*, Netw. Heterog. Media, 9:33–64, 2014. 1

[23] S.-Y. Ha, Y. Kim, and Z. Li, *Large-time dynamics of Kuramoto oscillators under the effects of inertia and frustration*, SIAM J. Appl. Dyn. Syst., 13:466–492, 2014. 1

[24] S.-Y. Ha, Z. Li and X. Xue, *Formation of phase-locked states in a population of locally interacting Kuramoto oscillators*, J. Differ. Equ., 255:3053–3070, 2013. 1

[25] S.-Y. Ha, H. Park, T. Ruggeri, and W. Shim, *Emergent behaviors of thermodynamic Kuramoto ensemble on a regular ring lattice*, J. Stat. Phys., 181:917–943, 2020. 1, 1, 1, 2, 2.1, 2.2, 2.3, 2.1, 3, 4, 4.1, 4.1, 6

[26] S.-Y. Ha and T. Ruggeri, *Emergent dynamics of a thermodynamically consistent particle model*, Arch. Ration. Mech. Anal., 223:1397–1425, 2017. 1

[27] S.-Y. Ha and X. Zhang, *Uniform-in-time transition from discrete dynamics to continuous dynamics in the Cucker-Smale flocking*, Math. Model. Meth. Appl. Sci., 28:1699–1735, 2018. 5.1

[28] A. Jadbabaie, N. Motee, and M. Barahona, *On the stability of the Kuramoto model of coupled nonlinear oscillators*, Proc. Am. Control Conf., 4296–4301, 2004. 1

[29] Y. Kuramoto, *Self-entrainment of a population of coupled non-linear oscillators*, International Symposium on Mathematical Problems in Theoretical Physics, Lecture Notes in Phys., 39:420–422, 1975. 1, 1

[30] C.S. Peskin, *Mathematical Aspects of Heart Physiology*, Courant Institute of Mathematical Sciences, New York, 1975. 1

[31] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences*, Cambridge University Press, Cambridge, 2001. 1

[32] W. Shim, *On the generic complete synchronization of the discrete Kuramoto model*, Kinet. Relat. Models, 13:979–1005, 2020. 1

[33] S.H. Strogatz, *From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators*, Phys. D, 143:1–20, 2000. 1

[34] E. Süli and D. Mayers, *An Introduction to Numerical Analysis*, Cambridge University Press, Cambridge, 2003. 5.1

[35] J.L. van Hemmen and W.F. Wreszinski, *Lyapunov function for the Kuramoto model of nonlinearly coupled oscillators*, J. Stat. Phys., 72:145–166, 1993. 2.1

[36] M. Verwoerd and O. Mason, *On computing the critical coupling coefficient for the Kuramoto model on a complete bipartite graph*, SIAM J. Appl. Dyn. Syst., 8:417–453, 2009. 1

[37] M. Verwoerd and O. Mason, *Global phase-locking in finite populations of phase-coupled oscillators*, SIAM J. Appl. Dyn. Syst., 7:134–160, 2008. 1

[38] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Schochet, *Novel type of phase transition in a system of self-driven particles*, Phys. Rev. Lett., 75:1226–1229, 1995. 1

[39] A.T. Winfree, *The Geometry of Biological Time*, Springer, New York, 1980. 1

[40] A.T. Winfree, *Biological rhythms and the behavior of populations of coupled oscillators*, J. Theor. Biol., 16:15–42, 1967. 1

[41] X. Zhang and T. Zhu, *Emergent behaviors of the discrete-time Kuramoto model for generic initial configuration*, Commun. Math. Sci., 18:535–570, 2020. 1, 5.1