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GLOBAL SOLUTION FOR EQUATIONS GOVERNING THE LOW-FREQUENCY ION MOTION IN PLASMA*

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Abstract. Equations describing the interactions between Langmuir waves and the low-frequency response of ions are considered in the present work. Existence of global smooth solution is established for suitably small initial data. The proof is based on the analysis of higher order energy estimate and lower order decay estimate.

Keywords. Global solution; decay estimate; weighted estimate.

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1. Introduction

In this paper, we mainly concentrate on the following set of equations:

$$\begin{cases} n_t+\nabla\cdot(nu)=0, \\ n(u_t+(u\cdot\nabla)u)+\nabla n=\mu_0\Delta u+\mu_1\nabla(\nabla\cdot u)-\nabla|E|^2, \\ 2\mathrm{i}\varepsilon E_t+3\Delta E=(n-1)E. \end{cases} \tag{1.1}$$

Here, $n:\mathbb{R}^+\times\mathbb{R}^3\to\mathbb{R}$ denotes the low-frequency density of ions, $u:\mathbb{R}^+\times\mathbb{R}^3\to\mathbb{R}^3$ denotes the velocity of ions, $E:\mathbb{R}^+\times\mathbb{R}^3\to\mathbb{C}^3$ is the slowly varying complex amplitude of the high-frequency electric field, $\varepsilon=\sqrt{m/M}$ with m, M the mass of the electron, ion respectively, $\mu_0>0$ and $\mu_1\geq 0$ are viscous constants.

The third equation in the above system describes the propagation of a finite amplitude Langmuir wave packet in plasma, and the first two equations in (1.1) describe the motion of the low-frequency ions. By decomposing the motion of a plasma into lower frequency part and higher frequency part, system (1.1) can be derived from two-fluid system under the charge neutrality condition $n_i\approx n_e$ and the action of viscosity, see [3] for more details. So we can also regard (1.1) as a simplified two-fluid model.

If one neglects the effects of electric field, system (1.1) is then reduced to the compressible Navier-Stokes equations. Using energy methods, Kawashima [9] proved global existence result of classical solution with initial data sufficiently small. Decay asymptotic results for such system have been studied systematically in L^p with $p>1$ by Hoff and Zumbrun [7]. In that case, the potential for electric field is governed by a self-consistent Poisson equation, Li, Matsumura and Zhang [10] observed the Poisson equation could make the decay rate for the momentum more slower.

When $\mu_0=\mu_1=0$, general theory of finite amplitude envelope solitons was presented by Schamel, Yu and Shukla [14], and Liapunov stability of the Langmuir solitons was shown by Laedke and Spatschek [11]. In the viscous case, Guo and Huang [4] obtained global existence of weak solutions to the 1D initial-boundary value problem of (1.1) in Sobolev-Orlicz space, and later in [5], they proved existence and uniqueness of global strong solutions to the initial-boundary value problem in 2D case for small initial data.

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However, as far as we know, there are few results of system (1.1) in three dimensional case. In particular, we want to know the influence of the coupled electric field in the existence theory and the time decay problem of the solution. This is the main motivation of the work.

Throughout the paper, L^p $(p\geq 1)$ is the usual Lebesgue space, and H^N denotes the inhomogeneous Sobolev spaces equipped with the norms

$$\|u\|_{H^N}:=\|(1+|\xi|^2)^{N/2}\widehat{u}\|_{L^2}.$$

Here, $\widehat{u}=\widehat{u}(\xi)$ is the Fourier transform of u, namely,

$$\widehat{u}(\xi)=\mathcal{F}u:=\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}e^{-\mathrm{i}x\cdot\xi}u(x)dx.$$

The default integral domain is always taken over $\mathbb{R}^3$. For $x,y\geq 0$, the notation $x\lesssim y$ means that there exists a constant $C>0$ such that $x\leq Cy$.

Now we state the main result of the paper. We endow (1.1) with the following initial data

$$(n,u,E)(0,x)=(n_0(x),u_0(x),E_0(x)). \tag{1.2}$$

Theorem 1.1. *Assume $N\geq 6$ and the initial data satisfies*

$$\begin{aligned}\|(n_0-1,u_0)\|_{H^N}+\|(n_0-1,u_0)\|_{L^1}\leq\epsilon_0,\\ \|E_0\|_{H^N}+\||x|^2E_0\|_{L^2}\leq\epsilon_0\end{aligned} \tag{1.3}$$

with $\epsilon_0>0$ being sufficiently small, then the Cauchy problem (1.1)–(1.2) admits a unique global solution $(n(t,x),u(t,x),E(t,x))$ satisfying

$$n(t,x)-1\in C(\mathbb{R}^+;H^N),\quad u(t,x)\in C(\mathbb{R}^+;H^N),\quad E(t,x)\in C(\mathbb{R}^+;H^N),$$

and for all $t\geq 0$,

$$\begin{aligned}\|(n(t,x)-1,u(t,x),E(t,x))\|_{H^N}\lesssim\epsilon_0,\\ \|(n(t,x)-1,u(t,x))\|_{L^\infty}\lesssim\frac{\epsilon_0}{(1+t)^{9/8-}},\\ \|E(t,x)\|_{L^\infty}\lesssim\frac{\epsilon_0}{(1+t)^{5/4}}.\end{aligned} \tag{1.4}$$

Theorem 1.1 states that global smooth solution exists for a small perturbation around the constant equilibrium state $(n^*,u^*,E^*)=(1,0,0)$. The proof is to establish energy estimates and decay estimates in suitable spaces (see Section 2.2 for the strategy of the proof). As we will see later, it is crucial to obtain the quadratic contribution of the electric field in decay estimates by using precisely the null structure. As a result, we see that although linearized equations for the motion of ions and the propagation of the Langmuir wave evolve separately, the coupled term of electric field may eventually reduce the asymptotic decay rate in the nonlinear solution. The proof of Theorem 1.1 will also imply some decay estimates concerning the first and second derivatives of the solution.

2. Preliminaries

2.1. Linear decay estimates. Let $\rho := n-1$, then linearized system of (1.1) around the constant equilibrium state $(n^*, u^*, E^*) = (1,0,0)$ reads

$$\begin{cases} \rho_t + \nabla\cdot u = 0, \\ u_t + \nabla\rho - \mu_0\Delta u - \mu_1\nabla(\nabla\cdot u) = 0, \\ 2\mathrm{i}\varepsilon E_t + 3\Delta E = 0. \end{cases} \tag{2.1}$$

The first two equations in (2.1) are of coupled hyperbolic-parabolic type, and we rewrite them in the following form

$$U_t = B(\nabla)U, \tag{2.2}$$

where $U := (\rho, u)^t$ and $B(\nabla)$ is a Fourier multiplier 4×4 matrix. Solution of (2.2) with initial data $U(0,x) = U_0(x)$ is then given by

$$U(t,x) = e^{tB(\nabla)}U_0 := G(t,x) * U_0(x), \tag{2.3}$$

where $*$ denotes the convolution, and $G(t,x)$ is the Green's function, which in Fourier space can be expressed by

$$\widehat{G}(t,\xi) = \begin{pmatrix} \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} & -\mathrm{i}\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\xi^t \\ -\mathrm{i}\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\xi & e^{-\mu_0|\xi|^2 t}(I_{3\times 3} - \frac{\xi^t\xi}{|\xi|^2}) + \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-}\frac{\xi^t\xi}{|\xi|^2} \end{pmatrix}. \tag{2.4}$$

In (2.4), λ_+ and λ_- are the roots of the equation

$$x^2 + \nu|\xi|^2 x + |\xi|^2 = 0$$

with

$$\nu = \mu_0 + \mu_1 > 0,$$

namely,

$$\lambda_\pm = -\frac{1}{2}\nu|\xi|^2 \pm \frac{1}{2}\sqrt{\nu^2|\xi|^4 - 4|\xi|^2}. \tag{2.5}$$

The representation of $\widehat{G}$ was derived in [7]. For $\widehat{G}(t,\xi)$, we have the following two lemmas, which present pointwise estimates for the low-frequency case and the high-frequency case respectively.

LEMMA 2.1. *Assume $\widehat{G}(t,\xi)$ is given by (2.4). Then for any $R>0$ and $|\xi|\le R$, there exists $\alpha>0$ such that*

$$|\widehat{G}(t,\xi)| \lesssim e^{-\alpha|\xi|^2 t} \tag{2.6}$$

with the implicit constant depending on R.

LEMMA 2.2. *Let $\widehat{G}(t,\xi)$ be defined by (2.4). Then there exists $R_0>0$ sufficiently large such that*

$$|\widehat{G}(t,\xi)| \lesssim e^{-\beta t}, \quad \beta := \min\{\frac{1}{\nu},\ \mu_0 R_0^2\} \tag{2.7}$$

$$|\nabla\widehat{G}(t,\xi)| \lesssim e^{-\frac{1}{2\nu}t}|\xi|^{-2} + e^{-\frac{\mu_0}{2}|\xi|^2 t}|\xi|^{-1}, \tag{2.8}$$

$$|\Delta\widehat{G}(t,\xi)| \lesssim e^{-\frac{1}{2\nu}t}|\xi|^{-3} + e^{-\frac{\mu_0}{2}|\xi|^2 t}|\xi|^{-2} \tag{2.9}$$

for $|\xi| > R_0$.

Estimates in Lemma 2.1 and Lemma 2.2 are actually obtained in [7]. To make the paper self-contained, we give the proof in the appendix. Here, Lemma 2.2 is proved in a more straightforward way. In particular, we remark that applying the weighted inequality (2.13) below, L^∞ decay bound for the linear solution (2.3) can be established only using up to second order derivative estimates for $\widehat{G}$.

THEOREM 2.1. *For any* $t > 0$ *and multi-index* γ, *we have*

$$\|D^\gamma e^{tB(\nabla)}U_0\|_{L^2(\mathbb{R}^3)} \lesssim t^{-3/4-|\gamma|/2}(\|U_0\|_{L^1(\mathbb{R}^3)} + \|D^\gamma U_0\|_{L^2(\mathbb{R}^3)}), \tag{2.10}$$

$$\|D^\gamma e^{tB(\nabla)}U_0\|_{L^\infty(\mathbb{R}^3)} \lesssim t^{-3/2-|\gamma|/2}(\|U_0\|_{L^1(\mathbb{R}^3)} + \|D^\gamma U_0\|_{L^\infty(\mathbb{R}^3)}). \tag{2.11}$$

Proof. Let R_0 be determined by Lemma 2.2. According to (2.3), we use Plancherel's identity and (2.6)–(2.7) to obtain

$$\begin{aligned}
\|D^\gamma e^{tB(\nabla)}U_0\|_{L^2}^2 &= \int_{\mathbb{R}^3} |\widehat{G}(t,\xi)|^2 |\widehat{D^\gamma U_0}(\xi)|^2 d\xi \\
&= \int_{|\xi|\le R_0} |\widehat{G}(t,\xi)|^2 |\widehat{D^\gamma U_0}(\xi)|^2 d\xi + \int_{|\xi|>R_0} |\widehat{G}(t,\xi)|^2 |\widehat{D^\gamma U_0}(\xi)|^2 d\xi \\
&\lesssim \int_{|\xi|\le R_0} |\xi|^{2|\gamma|} e^{-2\alpha|\xi|^2 t} |\widehat{U_0}(\xi)|^2 d\xi + \int_{|\xi|>R_0} e^{-2\beta t} |\widehat{D^\gamma U_0}(\xi)|^2 d\xi \\
&\lesssim \|\widehat{U_0}(\xi)\|_{L^\infty}^2 \int_{\mathbb{R}^3} |\xi|^{2|\gamma|} e^{-2\alpha|\xi|^2 t} d\xi + e^{-2\beta t}\|D^\gamma U_0\|_{L^2}^2 \\
&\lesssim t^{-3/2-|\gamma|}\|U_0\|_{L^1}^2 + e^{-2\beta t}\|D^\gamma U_0\|_{L^2}^2.
\end{aligned}$$

Hence, the desired estimate (2.10) follows.

To prove (2.11), we introduce a smooth cut-off function $\psi(r)$ defined by

$$\psi(r) = \begin{cases} 1, & |r| \le R_0, \\ 0, & |r| > R_0 + 1, \end{cases}$$

then we decompose G into

$$G = G_L + G_H$$

with

$$\widehat{G_L} = \widehat{G}(t,\xi)\psi(|\xi|), \quad \widehat{G_H} = \widehat{G}(t,\xi)(1-\psi(|\xi|)).$$

For the low frequency part, we use Hausdorff-Young inequality and (2.6) to obtain

$$\begin{aligned}
\|D^\gamma(G_L * U_0)\|_{L^\infty} &\lesssim \||\xi|^{|\gamma|}\widehat{G_L}\cdot\widehat{U_0}\|_{L^1} \\
&\lesssim \||\xi|^{|\gamma|}\widehat{G_L}\|_{L^1}\|\widehat{U_0}\|_{L^\infty} \\
&\lesssim \||\xi|^{|\gamma|}e^{-\alpha|\xi|^2 t}\|_{L^1}\|U_0\|_{L^1} \\
&\lesssim t^{-3/2-|\gamma|/2}\|U_0\|_{L^1}.
\end{aligned} \tag{2.12}$$

For the high frequency part, applying the weighted inequality (see [1, Lemma 2.1])

$$\|f\|_{L^1(\mathbb{R}^3)} \lesssim \|xf\|_{L^2(\mathbb{R}^3)}^{1/2} \||x|^2 f\|_{L^2(\mathbb{R}^3)}^{1/2}, \tag{2.13}$$

which together with the estimates (2.8) and (2.9) yield that

$$\begin{aligned}\|D^\gamma(G_H * U_0)\|_{L^\infty} &\lesssim \|G_H\|_{L^1}\|D^\gamma U_0\|_{L^\infty}\\ &\lesssim \|xG_H\|_{L^2}^{1/2}\||x|^2 G_H\|_{L^2}^{1/2}\|D^\gamma U_0\|_{L^\infty}\\ &\lesssim t^{-\frac{1}{8}} e^{-\theta t}\|D^\gamma U_0\|_{L^\infty}\end{aligned} \tag{2.14}$$

for some $\theta > 0$. Hence, the desired estimate (2.11) follows from (2.12) and (2.14). □

The last equation in (2.1) is a Schrödinger equation, and it is known that (see, for example, [2]) for $p \in [2, +\infty]$,

$$\|e^{\mathrm{i}\kappa t\Delta} f\|_{L^p(\mathbb{R}^3)} \lesssim \frac{1}{t^{3(1/2-1/p)}}\|f\|_{L^{p'}(\mathbb{R}^3)}, \quad \frac{1}{p}+\frac{1}{p'}=1, \quad \kappa := \frac{3}{2\varepsilon}. \tag{2.15}$$

2.2. Strategy of the proof. In order to prove the main theorem, we define the following norms associated to the work space (recall that $U = (\rho, u)^t$)

$$\begin{aligned}\|U\|_{X_T} := \sup_{t\in[0,T)} \big(&\|U\|_{H^N} + (1+t)^{5/4}\|\nabla U\|_{W^{1,\infty}}\\ &+ (1+t)^{3/4-\delta}\|U\|_{L^2} + (1+t)^{5/4}\|\Delta U\|_{L^2}\big),\\ \|E\|_{Y_T} := \sup_{t\in[0,T)} \big(&\|E\|_{H^N} + \|xf\|_{L^2} + (1+t)^{-1/2}\||x|^2 f\|_{L^2}\big),\end{aligned}$$

where $\delta > 0$ is sufficiently small, and set

$$A_T := \|U\|_{X_T} + \|E\|_{Y_T}. \tag{2.16}$$

Note that in the norm $\|E\|_{Y_T}$, f denotes the the profile of E, namely,

$$f := e^{-\mathrm{i}\kappa t\Delta} E,$$

which yields $\|f\|_{H^s} = \|E\|_{H^s}$ for all $s \in \mathbb{R}$, and

$$f_t = e^{-\mathrm{i}\kappa t\Delta}(E_t - \mathrm{i}\kappa\Delta E) = -\frac{\mathrm{i}\kappa}{3} e^{-\mathrm{i}\kappa t\Delta}((n-1)E). \tag{2.17}$$

The introduction of weighted norms for the profile f in space Y_T is crucial, which in turn controls the decay estimates for E. Indeed, using (2.13) and (2.15), one sees that

$$\|E\|_{L^\infty} = \|e^{\mathrm{i}\kappa t\Delta} f\|_{L^\infty} \lesssim \min\{1, t^{-3/2}\}\|f\|_{L^1\cap H^2} \lesssim (1+t)^{-5/4}\|E\|_{Y_T}. \tag{2.18}$$

Meanwhile, by Gagliardo-Nirenberg inequality, the above work space also implies

$$\|U\|_{L^\infty} \lesssim \|U\|_{L^2}^{1/4}\|\Delta U\|_{L^2}^{3/4} \lesssim (1+t)^{-9/8+\delta/4} A_T, \tag{2.19}$$

$$\|\nabla U\|_{L^2} \lesssim \|U\|_{L^2}^{1/2}\|\Delta U\|_{L^2}^{1/2} \lesssim (1+t)^{-1+\delta/2} A_T. \tag{2.20}$$

In the following sections, our main attention is to prove the key *a priori* estimate

$$A_T \lesssim \epsilon_0 + A_T^{3/2}, \tag{2.21}$$

where ϵ_0 is related to the size of initial data, and the implicit constant in (2.21) is independent of T. Then applying the standard continuation argument, we finally obtain the existence of global solution as stated in Theorem 1.1 which satisfies $A_\infty \lesssim \epsilon_0$. As shown in the definition of the work space, the proof will consist of higher order energy estimate and lower order decay estimate. Using the dissipative structure of the fluid equation, loss of derivatives can be avoided by means of integration by parts, and closed energy estimate is obtained in the next section.

To derive decay estimates, we use Duhamel's principle to obtain

$$U(t,x) = e^{tB(\nabla)}U_0(x) + \int_0^t e^{(t-s)B(\nabla)}F(U(s,x))ds, \tag{2.22}$$

$$E(t,x) = e^{\mathrm{i}\kappa t\Delta}E_0(x) - \frac{\mathrm{i}\kappa}{3}\int_0^t e^{\mathrm{i}\kappa(t-s)\Delta}\rho(s,x)E(s,x)ds, \tag{2.23}$$

where the nonlinear term $F(U) := (F_1(U), F_2(U))^t$ with

$$F_1(U) := -\nabla\cdot(\rho u), \tag{2.24}$$

$$\begin{aligned} F_2(U) :=& \mu_0(\frac{1}{1+\rho}-1)\Delta u + \mu_1(\frac{1}{1+\rho}-1)\nabla(\nabla\cdot u) \\ &- (\frac{1}{1+\rho}-1)\nabla\rho - (u\cdot\nabla)u - \frac{1}{1+\rho}\nabla|E|^2. \end{aligned} \tag{2.25}$$

When applying (2.10)–(2.11) to estimate decay norms for U, one may see that the problematic term comes from the coupled electric term in (2.25), i.e., $\frac{1}{1+\rho}\nabla|E|^2$. Indeed, using (2.10)–(2.11) forces us to estimate L^1 norm of the term $\frac{1}{1+\rho}\nabla|E|^2$, which is nearly equal to $\nabla|E|^2$ if one ignores cubic and higher order terms. However, Hölder's estimate such as $L^2\times L^2$ type does not work here since there is no decay estimate for E or ∇E in L^2. To overcome this difficulty, we use the idea of the works [6, 8, 13] (see also [1]) and take advantage of the weighted norm for E as introduced in (2.16). Taking Fourier transform for this term yields that

$$\begin{aligned} \widehat{\nabla|E|^2}(s,\xi) &= \mathrm{i}\xi\int_{\mathbb{R}^3}\hat{E}(s,\xi-\eta)\hat{\bar{E}}(s,\eta)d\eta \\ &= \mathrm{i}\xi\int_{\mathbb{R}^3} e^{\mathrm{i}\kappa s\varphi(\xi,\eta)}\hat{f}(s,\xi-\eta)\hat{\bar{f}}(s,\eta)d\eta, \end{aligned} \tag{2.26}$$

where the phase $\varphi(\xi,\eta)$ is given by

$$\varphi(\xi,\eta) = -|\xi-\eta|^2 + |\eta|^2. \tag{2.27}$$

A key observation is that the derivative nonlinear structure in (2.26) provides a null resonance form for φ, namely,

$$\nabla_\eta\varphi = 2\xi, \tag{2.28}$$

we then use this identity to integrate (2.26) by parts in η and obtain extra decay factors which are sufficient to derive desired bound for the term $\|\nabla|E|^2\|_{L^1}$. See Section 4 for the detailed proof of decay estimates.

Now it remains to establish weighted estimates for the electric filed E, which are given in Section 5. To this end, we deduce from (2.23) or (2.17) that

$$f(t,x) = f(0,x) - \frac{\mathrm{i}\kappa}{3}\int_0^t e^{-\mathrm{i}\kappa s\Delta}\rho(s,x)(e^{\mathrm{i}\kappa s\Delta}f(s,x))ds,$$

and in frequency space,

$$\hat{f}(t,\xi)=\hat{f}(0,\xi)-\frac{i\kappa}{3}\int_0^t\int_{\mathbb{R}^3}e^{i\kappa s\tilde{\varphi}(\xi,\eta)}\hat{\rho}(s,\eta)\hat{f}(s,\xi-\eta)d\eta ds, \tag{2.29}$$

where

$$\tilde{\varphi}(\xi,\eta):=|\xi|^2-|\xi-\eta|^2. \tag{2.30}$$

Although applying operator ∇_ξ (or Δ_ξ) to (2.29) will produce extra growth factor s (or even s^2), we note that the L^p decay property for fluid equation would eliminate nicely some of these growth factors and eventually lead to the desired estimates for xf and $|x|^2f$.

3. Energy estimate of the solution

We show the energy estimate for system (1.1) in this section. During the proof, we need the following calculus inequalities [12, Lemma 3.4].

LEMMA 3.1. *Assume m is a nonnegative integer, then for all $u,v\in L^\infty(\mathbb{R}^3)\cap H^m(\mathbb{R}^3)$, we have*

$$\|uv\|_{H^m}\lesssim\|u\|_{H^m}\|v\|_{L^\infty}+\|u\|_{L^\infty}\|v\|_{H^m}, \tag{3.1}$$

$$\sum_{0\le|\gamma|\le m}\|D^\gamma(uv)-u(D^\gamma v)\|_{L^2}\lesssim\|\nabla u\|_{L^\infty}\|v\|_{H^{m-1}}+\|u\|_{H^m}\|v\|_{L^\infty}. \tag{3.2}$$

PROPOSITION 3.1. *Assume that $(\rho=n-1,u,E)\in C([0,T);H^N\times H^N\times H^N)$ is a smooth solution of system (1.1) satisfying $A_T\ll1$, where A_T is defined by (2.16). Then there holds*

$$\sup_{t\in[0,T)}\|(\rho(t,x),u(t,x),E(t,x))\|_{H^N}\lesssim\|(\rho_0(x),u_0(x),E_0(x))\|_{H^N}+A_T^{3/2}, \tag{3.3}$$

where the implicit constant C is independent of T.

Proof. System (1.1) admits the following energy identities:

$$\frac{d}{dt}\int_{\mathbb{R}^3}\rho dx=0, \tag{3.4}$$

$$\frac{d}{dt}\int_{\mathbb{R}^3}|E|^2dx=0, \tag{3.5}$$

$$\frac{d}{dt}\mathcal{M}+\mu_0\int_{\mathbb{R}^3}|\nabla u|^2dx+\mu_1\int_{\mathbb{R}^3}|\nabla\cdot u|^2dx=0 \tag{3.6}$$

with

$$\mathcal{M}:=\frac{1}{2}\int_{\mathbb{R}^3}n|u|^2dx+\int_{\mathbb{R}^3}|\nabla E|^2dx+\int_{\mathbb{R}^3}(n-1)|E|^2dx+\int_{\mathbb{R}^3}n\ln n dx.$$

Since

$$n\ln n=(1+\rho)\ln(1+\rho)=\rho+\frac{1}{2}\rho^2+O(\rho^3),$$

we then deduce from (3.4)–(3.6) that

$$\|(\rho,u)\|_{L^2}+\|E\|_{H^1}+\int_0^t\int_{\mathbb{R}^3}|\nabla u(s,x)|^2dxds\lesssim\|(\rho_0,u_0)\|_{L^2}+\|E_0\|_{H^1}. \tag{3.7}$$

Therefore, it remains to prove (3.3) for the highest order of derivative.

Let γ be a multi-index satisfying $|\gamma|=N$. The density equation of (1.1) implies that

$$\rho_t+\nabla\cdot u+\nabla\cdot(\rho u)=0, \tag{3.8}$$

and performing energy estimate for this equation gives

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|D^\gamma\rho|^2dx+\int_{\mathbb{R}^3}D^\gamma(\nabla\cdot u)D^\gamma\rho dx+\int_{\mathbb{R}^3}D^\gamma\nabla\cdot(\rho u)D^\gamma\rho dx=0. \tag{3.9}$$

Meanwhile, taking energy estimate for the momentum equation of (1.1) yields that

$$\begin{aligned}&\int_{\mathbb{R}^3}D^\gamma(nu_t)D^\gamma udx+\int_{\mathbb{R}^3}D^\gamma(n(u\cdot\nabla)u)D^\gamma udx\\ =&-\mu_0\int_{\mathbb{R}^3}|D^\gamma\nabla u|^2dx-\mu_1\int_{\mathbb{R}^3}|D^\gamma(\nabla\cdot u)|^2dx\\ &-\int_{\mathbb{R}^3}D^\gamma\nabla|E|^2D^\gamma udx-\int_{\mathbb{R}^3}D^\gamma\nabla nD^\gamma udx.\end{aligned} \tag{3.10}$$

Note that the second term in (3.9) cancels the last term in (3.10) since $\nabla n=\nabla\rho$. The third term in (3.9) can be decomposed as

$$\begin{aligned}\int_{\mathbb{R}^3}D^\gamma\nabla\cdot(\rho u)D^\gamma\rho dx&=\int_{\mathbb{R}^3}D^\gamma(\rho(\nabla\cdot u))D^\gamma\rho dx+\int_{\mathbb{R}^3}D^\gamma(\nabla\rho\cdot u)D^\gamma\rho dx\\ &=:I_1+I_2.\end{aligned} \tag{3.11}$$

By (3.1), we have

$$\begin{aligned}|I_1|&\le\|D^\gamma(\rho(\nabla\cdot u))\|_{L^2}\|D^\gamma\rho\|_{L^2}\\ &\lesssim(\|\rho\|_{H^N}\|\nabla u\|_{L^\infty}+\|\rho\|_{L^\infty}\|\nabla u\|_{H^N})\|D^\gamma\rho\|_{L^2}\\ &\lesssim(1+t)^{-5/4}A_T^3+(1+t)^{-9/8+\delta/4}A_T^2\|\nabla u\|_{H^N}\\ &\lesssim(1+t)^{-5/4}A_T^3+\tilde\epsilon\|\nabla u\|_{H^N}^2,\end{aligned} \tag{3.12}$$

and by (3.2),

$$\begin{aligned}|I_2|&\le\left|\int_{\mathbb{R}^3}[D^\gamma(\nabla\rho\cdot u)-((D^\gamma\nabla\rho)\cdot u)]D^\gamma\rho dx\right|+\left|\int_{\mathbb{R}^3}((D^\gamma\nabla\rho)\cdot u)D^\gamma\rho dx\right|\\ &\lesssim(\|\rho\|_{H^N}\|\nabla u\|_{L^\infty}+\|\nabla\rho\|_{L^\infty}\|u\|_{H^N})\|D^\gamma\rho\|_{L^2}+\|\nabla u\|_{L^\infty}\|D^\gamma\rho\|_{L^2}^2\\ &\lesssim(1+t)^{-5/4}A_T^3.\end{aligned} \tag{3.13}$$

Hence, one sees from (3.11)–(3.13) that

$$\left|\int_{\mathbb{R}^3}D^\gamma\nabla\cdot(\rho u)D^\gamma\rho dx\right|\lesssim(1+t)^{-5/4}A_T^3+\tilde\epsilon\|\nabla u\|_{H^N}^2. \tag{3.14}$$

Similarly, we can obtain

$$\begin{aligned}\left|-\int_{\mathbb{R}^3} D^\gamma\nabla|E|^2 D^\gamma u dx\right| &= \left|\int_{\mathbb{R}^3} D^\gamma |E|^2 D^\gamma \nabla\cdot u dx\right| \\ &\lesssim \|D^\gamma |E|^2\|_{L^2}\|D^\gamma\nabla\cdot u\|_{L^2} \\ &\lesssim \|E\|_{L^\infty}\|E\|_{H^N}\|\nabla u\|_{H^N} \\ &\lesssim (1+t)^{-5/2}A_T^4+\tilde{\epsilon}\|\nabla u\|_{H^N}^2. \end{aligned} \tag{3.15}$$

For the first term in (3.10), we treat it as

$$\begin{aligned}\int_{\mathbb{R}^3} D^\gamma(nu_t)D^\gamma u dx &= \int_{\mathbb{R}^3}[D^\gamma(nu_t)-nD^\gamma u_t]D^\gamma u dx+\int_{\mathbb{R}^3} nD^\gamma u_t D^\gamma u dx \\ &= \int_{\mathbb{R}^3}[D^\gamma(nu_t)-nD^\gamma u_t]D^\gamma u dx+\frac{1}{2}\int_{\mathbb{R}^3} n|D^\gamma u|_t^2 dx \end{aligned} \tag{3.16}$$

with

$$\begin{aligned}\left|\int_{\mathbb{R}^3}[D^\gamma(nu_t)-nD^\gamma u_t]D^\gamma u dx\right| &\lesssim (\|n\|_{H^N}\|u_t\|_{L^\infty}+\|\nabla n\|_{L^\infty}\|u_t\|_{H^{N-1}})\|D^\gamma u\|_{L^2} \\ &\lesssim (1+t)^{-5/4}A_T^3+\tilde{\epsilon}\|\nabla u\|_{H^N}^2, \end{aligned} \tag{3.17}$$

where we have used the following estimates in the last step

$$\begin{aligned}\|u_t\|_{L^\infty} &= \|-(u\cdot\nabla)u+n^{-1}(\mu_0\Delta u+\mu_1\nabla(\nabla\cdot u)-\nabla|E|^2-\nabla n)\|_{L^\infty} \\ &\lesssim (1+t)^{-5/4}A_T, \\ \|u_t\|_{H^{N-1}} &\lesssim \|\nabla u\|_{H^N}+A_T. \end{aligned}$$

In a similar way, we have

$$\begin{aligned}&\int_{\mathbb{R}^3} D^\gamma(n(u\cdot\nabla)u)D^\gamma u dx \\ &= \int_{\mathbb{R}^3}[D^\gamma((nu\cdot\nabla)u)-(nu\cdot\nabla)D^\gamma u]D^\gamma u dx+\int_{\mathbb{R}^3}(nu\cdot\nabla)D^\gamma u D^\gamma u dx \\ &= \int_{\mathbb{R}^3}[D^\gamma((nu\cdot\nabla)u)-(nu\cdot\nabla)D^\gamma u]D^\gamma u dx+\frac{1}{2}\int_{\mathbb{R}^3} n_t|D^\gamma u|^2 dx \end{aligned} \tag{3.18}$$

with

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}[D^\gamma((nu\cdot\nabla)u)-(nu\cdot\nabla)D^\gamma u]D^\gamma u dx\right| \\ &\lesssim (\|nu\|_{H^N}\|\nabla u\|_{L^\infty}+\|\nabla(nu)\|_{L^\infty}\|u\|_{H^N})\|D^\gamma u\|_{L^2} \\ &\lesssim (1+t)^{-19/8+\delta/4}A_T^4. \end{aligned} \tag{3.19}$$

Combining (3.9), (3.10) and (3.14)–(3.19), one sees

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{R}^3}(|D^\gamma\rho|^2+n|D^\gamma u|^2)dx\right)+\mu_0\|D^\gamma\nabla u\|_{L^2}^2+\mu_1\|D^\gamma\nabla\cdot u\|_{L^2}^2 \\ &\lesssim (1+t)^{-5/4}A_T^3+\tilde{\epsilon}\|\nabla u\|_{H^N}^2. \end{aligned} \tag{3.20}$$

Finally, we take energy estimate for the Schrödinger equation in (1.1) and obtain

$$\begin{aligned}\varepsilon\frac{d}{dt}\int_{\mathbb{R}^3}|D^\gamma E|^2dx &= \mathrm{Im}\int_{\mathbb{R}^3}D^\gamma(\rho E)D^\gamma\overline{E}dx\\ &\lesssim(\|\rho\|_{L^\infty}\|D^\gamma E\|_{L^2}+\|D^\gamma\rho\|_{L^2}\|E\|_{L^\infty})\|D^\gamma\overline{E}\|_{L^2}\\ &\lesssim(1+t)^{-9/8+\delta/4}A_T^3. \end{aligned}\tag{3.21}$$

Integrating (3.20)-(3.21) in time and summing over $|\gamma|=N$, then the desired bound (3.3) follows from (3.7) and the above two estimates by choosing $\tilde{\epsilon}$ sufficiently small. □

4. Decay estimates for U

This section deals with the L^∞ and L^2 decay estimates for U. Recall the representation (2.22) for U, that is,

$$U(t,x)=e^{tB(\nabla)}U_0(x)+\int_0^t e^{(t-s)B(\nabla)}F(U(s,x))ds,\tag{4.1}$$

where $F(U)$ denotes the nonlinear term defined in (2.24)–(2.25). Note that $F(U)$ has at least quadratic nonlinearity with at most two derivatives on U. As we will see later, the coupled term $\nabla|E|^2$ in the momentum equation will affect nonlinear decay rate. Decay estimates for this term are very important in our analysis. In particular, we remark that the derivative nonlinear structure in the argument is crucial. We first give the following key lemma.

Lemma 4.1. *Assume $(U,E)\in C([0,T);H^N\times H^N)$ satisfies system (1.1) with $T>0$. If $A_T\ll 1$, then there hold*

$$\|F(U(t,x))\|_{L^1}\lesssim(1+t)^{-1}A_T^2,\tag{4.2}$$

$$\|\nabla F(U(t,x))\|_{L^1}\lesssim(1+t)^{-11/8+\delta}A_T^2,\tag{4.3}$$

$$\|F(U(t,x))\|_{W^{1,\infty}}\lesssim(1+t)^{-5/4}A_T^2,\tag{4.4}$$

$$\|F(U(t,x))\|_{H^2}\lesssim(1+t)^{-5/4}A_T^2.\tag{4.5}$$

Proof. For small t such as $0<t\le 1$, we have $1+t\sim 1$, so the estimates (4.2)–(4.5) follows directly from the definition of $F(U)$ by Hölder inequality and Sobolev inequality only using the following energy bounds

$$\|U\|_{H^N}\lesssim A_T,\quad \|E\|_{H^N}\lesssim A_T.\tag{4.6}$$

Hence without loss of generality, we may assume $t>1$ and thus $1+t\sim t$.

Since

$$\frac{1}{1+\rho}-1=-\rho+\rho^2-\rho^3+\cdots=\rho+O(\rho),\tag{4.7}$$

then using the decay bounds coming from the definition A_T,

$$\|U\|_{L^2}\lesssim t^{-3/4+\delta}A_T,\quad \|\nabla U\|_{L^2}\lesssim t^{-1+\delta/2}A_T,\quad \|\Delta U\|_{L^2}\lesssim t^{-5/4}A_T,\tag{4.8}$$

we see that all the terms (see (2.24) and (2.25)) in $F(U)$ except the term $\nabla|E|^2$ can be treated by Cauchy-Schwarz inequality,

$$\|\nabla\cdot(\rho u)\|_{L^1}\lesssim\|\nabla\rho\|_{L^2}\|u\|_{L^2}+\|\rho\|_{L^2}\|\nabla u\|_{L^2}\lesssim t^{-7/4+3\delta/2}A_T^2,$$

$$\|(u\cdot\nabla)u\|_{L^1}\lesssim\|u\|_{L^2}\|\nabla u\|_{L^2}\lesssim t^{-7/4+3\delta/2}A_T^2,$$
$$\|(\frac{1}{1+\rho}-1)\nabla\rho\|_{L^1}\lesssim\|\rho\|_{L^2}\|\nabla\rho\|_{L^2}\lesssim t^{-7/4+3\delta/2}A_T^2,$$
$$\|(\frac{1}{1+\rho}-1)\Delta u\|_{L^1}\lesssim\|\rho\|_{L^2}\|\Delta u\|_{L^2}\lesssim t^{-2+\delta}A_T^2,$$
$$\|\frac{1}{1+\rho}\nabla|E|^2\|_{L^1}\lesssim\|\nabla|E|^2\|_{L^1}.$$

So in order to show (4.2), it suffices to prove

$$\|\nabla|E|^2(t,x)\|_{L^1}\lesssim t^{-1}A_T^2. \tag{4.9}$$

From (2.26), we have

$$\begin{aligned}\widehat{\nabla|E|^2}(t,\xi)&=\mathrm{i}\int_{\mathbb{R}^3}\xi e^{\mathrm{i}\kappa t\varphi(\xi,\eta)}\hat{f}(t,\xi-\eta)\hat{\bar{f}}(t,\eta)d\eta\\&=\frac{1}{2\kappa t}\int_{\mathbb{R}^3}\nabla_\eta e^{\mathrm{i}\kappa t\varphi(\xi,\eta)}\hat{f}(t,\xi-\eta)\hat{\bar{f}}(t,\eta)d\eta,\end{aligned}\tag{4.10}$$

where we have used the null resonance condition of φ ($\nabla_\eta\varphi=2\xi$) in the second equality. We integrate (4.10) by part in η to obtain

$$\begin{aligned}\widehat{\nabla|E|^2}(t,\xi)=&-\frac{1}{2\kappa t}\int_{\mathbb{R}^3}e^{\mathrm{i}\kappa t\varphi(\xi,\eta)}\nabla_\eta\hat{f}(t,\xi-\eta)\hat{\bar{f}}(t,\eta)d\eta\\&-\frac{1}{2\kappa t}\int_{\mathbb{R}^3}e^{\mathrm{i}\kappa t\varphi(\xi,\eta)}\hat{f}(t,\xi-\eta)\nabla_\eta\hat{\bar{f}}(t,\eta)d\eta.\end{aligned}$$

Returning back to the physical space yields

$$\nabla|E|^2=\frac{\mathrm{i}}{2\kappa t}[e^{\mathrm{i}\kappa t\Delta}(xf)\cdot\bar{E}+E\cdot e^{-\mathrm{i}\kappa t\Delta}(x\bar{f})]$$

from which we can get

$$\|\nabla|E|^2\|_{L^1}\lesssim t^{-1}\|E\|_{L^2}\|xf\|_{L^2}\lesssim t^{-1}A_T^2.$$

Hence, the bound (4.9) is proved.

The proof for (4.3) is similar. By (4.8), (2.18) and the bound

$$\|\nabla\Delta u\|_{L^2}\lesssim\|\Delta u\|_{L^2}^{1/2}\|D^4u\|_{L^2}^{1/2}\lesssim t^{-5/8}A_T^2,$$

it is easy to see that for $j=1,2,3$,

$$\|\partial_{x_j}\nabla\cdot(\rho u)\|_{L^1}\lesssim\|\Delta\rho\|_{L^2}\|u\|_{L^2}+\|\rho\|_{L^2}\|\Delta u\|_{L^2}+\|\nabla\rho\|_{L^2}\|\nabla u\|_{L^2}\lesssim t^{-2+\delta}A_T^2,$$
$$\|\partial_{x_j}(u\cdot\nabla)u\|_{L^1}\lesssim\|u\|_{L^2}\|\Delta u\|_{L^2}+\|\nabla u\|_{L^2}^2\lesssim t^{-2+\delta}A_T^2,$$
$$\|\partial_{x_j}[(\frac{1}{1+\rho}-1)\nabla\rho]\|_{L^2}\lesssim\|\rho\|_{L^2}\|\Delta\rho\|_{L^2}+\|\nabla\rho\|_{L^2}^2\lesssim t^{-2+\delta}A_T^2,$$
$$\|\partial_{x_j}[(\frac{1}{1+\rho}-1)\Delta u]\|_{L^1}\lesssim\|\nabla\rho\|_{L^2}\|\Delta u\|_{L^2}+\|\rho\|_{L^2}\|\nabla\Delta u\|_{L^2}\lesssim t^{-11/8+\delta}A_T^2,$$
$$\|\partial_{x_j}(\frac{1}{1+\rho}\nabla|E|^2)\|_{L^1}\lesssim\|\partial_{x_j}\nabla|E|^2\|_{L^1}+\|\partial_{x_j}\rho\nabla|E|^2\|_{L^1}$$

$$\lesssim \|\partial_{x_j}\nabla|E|^2\|_{L^1}+\|\nabla\rho\|_{L^2}\|E\|_{L^\infty}\|\nabla E\|_{L^2}$$
$$\lesssim \|\partial_{x_j}\nabla|E|^2\|_{L^1}+t^{-9/4+\delta/2}A_T^2.$$

For the term including the electric field, we obtain from (2.26)

$$\mathcal{F}[\partial_{x_j}\nabla|E|^2](t,\xi)=-\int_{\mathbb{R}^3}\xi\xi_j e^{\mathrm{i}\kappa t\varphi(\xi,\eta)}\hat{f}(t,\xi-\eta)\hat{\bar{f}}(t,\eta)d\eta.$$

Using the derivative structure and (2.27)–(2.28), we integrate by part twice in η_j and η, then there holds

$$\begin{aligned}\mathcal{F}[\partial_{x_j}\nabla|E|^2](t,\xi)=&\frac{1}{4\kappa^2t^2}\int_{\mathbb{R}^3}e^{\mathrm{i}\kappa t\varphi(\xi,\eta)}\nabla_\eta\nabla_{\eta_j}\hat{f}(t,\xi-\eta)\hat{\bar{f}}(t,\eta)d\eta\\&+\frac{1}{4\kappa^2t^2}\int_{\mathbb{R}^3}e^{\mathrm{i}\kappa t\varphi(\xi,\eta)}\nabla_{\eta_j}\hat{f}(t,\xi-\eta)\nabla_\eta\hat{\bar{f}}(t,\eta)d\eta\\&+\frac{1}{4\kappa^2t^2}\int_{\mathbb{R}^3}e^{\mathrm{i}\kappa t\varphi(\xi,\eta)}\nabla_\eta\hat{f}(t,\xi-\eta)\nabla_{\eta_j}\hat{\bar{f}}(t,\eta)d\eta\\&+\frac{1}{4\kappa^2t^2}\int_{\mathbb{R}^3}e^{\mathrm{i}\kappa t\varphi(\xi,\eta)}\hat{f}(t,\xi-\eta)\nabla_\eta\nabla_{\eta_j}\hat{\bar{f}}(t,\eta)d\eta,\end{aligned}$$

so it can be seen that

$$\begin{aligned}\|\partial_{x_j}\nabla|E|^2\|_{L^1}\lesssim&\, t^{-2}\|e^{\mathrm{i}\kappa t\Delta}(xx_jf)\|_{L^2}\|\bar{E}\|_{L^2}\\&+t^{-2}\|e^{\mathrm{i}\kappa t\Delta}(x_jf)\|_{L^2}\|e^{-\mathrm{i}\kappa t\Delta}(x\bar{f})\|_{L^2}\\&+t^{-2}\|e^{\mathrm{i}\kappa t\Delta}(xf)\|_{L^2}\|e^{-\mathrm{i}\kappa t\Delta}(x_j\bar{f})\|_{L^2}\\&+t^{-2}\|E\|_{L^2}\|e^{-\mathrm{i}\kappa t\Delta}(xx_j\bar{f})\|_{L^2}\\\lesssim&\, t^{-2}(\||x|^2f\|_{L^2}\|E\|_{L^2}+\|xf\|_{L^2}^2)\\\lesssim&\, t^{-3/2}A_T^2.\end{aligned}$$

Therefore, the bound (4.3) follows from the above estimates. Note that we need one extra derivative to produce better decay rate.

From (4.8) and the L^∞ decay bounds

$$\|U\|_{L^\infty}\lesssim t^{-9/8+\delta/4}A_T,\quad \|\nabla U\|_{W^{1,\infty}}\lesssim t^{-5/4}A_T, \tag{4.11}$$

it is easily to obtain the bounds (4.4) and (4.5) by using Hölder inequality and Sobolev inequality (note that the worst bound comes from the coupled term). We skip the details here for simplicity. ∎

Proposition 4.1. *Assume $(U,E)\in C([0,T);H^N\times H^N)$ satisfies system (1.1) with $T>0$. If $A_T\ll 1$, then we have*

$$(1+t)^{-5/4}\|\nabla U\|_{W^{1,\infty}}\lesssim \|U_0(x)\|_{L^1\cap H^4}+A_T^2, \tag{4.12}$$

for all $t\in[0,T)$, where the implicit constant is independent of T.

Proof. From (2.6) and (2.7), one sees $e^{tB(\nabla)}$ is a bounded linear operator from H^k to H^k,

$$\|e^{tB(\nabla)}\|_{\mathcal{L}(H^k\to H^k)}\lesssim 1. \tag{4.13}$$

So for small t such as $0<t\le 1$, we use (4.6), (4.13) and Sobolev embedding $H^2(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$ to obtain

$$\begin{aligned}\|\nabla U\|_{W^{1,\infty}} &\lesssim \|e^{tB(\nabla)}\nabla U_0\|_{W^{1,\infty}}+\int_0^t \|e^{(t-s)B(\nabla)}\nabla F(U)\|_{W^{1,\infty}}ds\\ &\lesssim \|U_0\|_{H^4}+\int_0^1 \|F(U)\|_{H^4}ds\\ &\lesssim \|U_0\|_{H^4}+A_T^2.\end{aligned}$$

Since in this case $1+t\sim 1$, we thus obtain the bound (4.12).

From now on we assume $t>1$. Decompose the expression for ∇U as

$$\begin{aligned}\nabla U(t,x)=&e^{tB(\nabla)}\nabla U_0(x)+\int_0^1 e^{(t-s)B(\nabla)}\nabla F(U(s,x)ds\\ &+\int_1^t e^{(t-s)B(\nabla)}\nabla F(U(s,x)ds.\end{aligned} \tag{4.14}$$

From the linear estimate (2.11), we see

$$\|e^{tB(\nabla)}\nabla U_0\|_{L^\infty}\lesssim t^{-2}\|U_0\|_{L^1\cap L^\infty}. \tag{4.15}$$

Also, from (2.11), there holds

$$\begin{aligned}\left\|\int_0^1 e^{(t-s)B(\nabla)}\nabla F(U)ds\right\|_{W^{1,\infty}} &\lesssim \int_0^1 \frac{1}{(t-s)^2}\|F(U)\|_{L^1\cap W^{1,\infty}}ds\\ &\lesssim \int_0^1 \frac{1}{(t-s)^2}A_T^2 ds\\ &\lesssim \frac{1}{(t-1)^2}A_T^2\\ &\sim \frac{1}{t^2}A_T^2,\end{aligned} \tag{4.16}$$

where we have used the following rough bounds

$$\|F(U)\|_{L^1}\lesssim A_T^2,\quad \|F(U)\|_{W^{1,\infty}}\lesssim A_T^2.$$

To estimate the third term in (4.14), we use (2.11) and Lemma 4.1 to get

$$\begin{aligned}\left\|\int_1^t e^{(t-s)B(\nabla)}\nabla F(U)ds\right\|_{L^\infty} &\lesssim \int_1^t \frac{1}{(t-s)^{3/2}}\|\nabla F(U)\|_{L^1\cap L^\infty}ds\\ &\lesssim \int_1^t \frac{1}{(t-s)^{3/2}}\cdot\frac{1}{s^{5/4}}A_T^2 ds\\ &\lesssim A_T^2\int_1^{t/2} \frac{1}{(t-s)^{3/2}}\cdot\frac{1}{s^{5/4}}ds\\ &\quad +A_T^2\int_{t/2}^t \frac{1}{(t-s)^{3/2}}\cdot\frac{1}{s^{5/4}}ds\\ &\lesssim \frac{1}{t^{5/4}}A_T^2,\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}\left\|\int_1^t e^{(t-s)B(\nabla)}\partial_{x_j}\nabla F(U)ds\right\|_{L^\infty} &\lesssim \int_1^t \frac{1}{(t-s)^2}\|\nabla F(U)\|_{L^1\cap L^\infty}ds\\ &\lesssim \int_1^t \frac{1}{(t-s)^2}\cdot\frac{1}{s^{5/4}}A_T^2 ds\\ &\lesssim \frac{1}{t^{5/4}}A_T^2. \end{aligned} \tag{4.18}$$

Therefore, the desired bound (4.12) follows from the estimates (4.14)–(4.18). ☐

With a similar argument as above, we can obtain L^2-type decay estimates.

PROPOSITION 4.2. *Under the same assumptions as Proposition 4.1, there hold that*

$$\sup_{t\in[0,T)}(1+t)^{3/4-\delta}\|U(t,x)\|_{L^2}\lesssim \|U_0\|_{L^1\cap L^2}+A_T^2, \tag{4.19}$$

$$\sup_{t\in[0,T)}(1+t)^{5/4}\|\Delta U(t,x)\|_{L^2}\lesssim \|U_0\|_{L^1\cap \dot{H}^2}+A_T^2, \tag{4.20}$$

where the implicit constant is independent of T.

Proof. As shown in Proposition 4.1, without loss of generality, we may assume $t>1$. Also, we only consider the time integral form 1 to t. One sees from (2.10) with $|\gamma|=0$, (4.2) and (4.5) that

$$\begin{aligned}\left\|\int_1^t e^{(t-s)B(\nabla)}F(U)ds\right\|_{L^2} &\lesssim \int_1^t \frac{1}{(t-s)^{3/4}}\|F(U)\|_{L^1\cap L^2}ds\\ &\lesssim \int_1^t \frac{1}{(t-s)^{3/4}}\cdot\frac{1}{s}A_T^2 ds\\ &\lesssim \frac{1}{t^{3/4-\delta}}A_T^2, \end{aligned}$$

which yields (4.19) as desired.

From (4.1), we have

$$\Delta U(t,x)=e^{tB(\nabla)}\Delta U_0(x)+\int_0^t e^{(t-s)B(\nabla)}\Delta F(U(s,x))ds, \tag{4.21}$$

so using (2.10) with $|\gamma|=2$, there holds

$$\|e^{tB(\nabla)}\Delta U_0(x)\|_{L^2}\lesssim \frac{1}{t^{7/4}}\|U_0\|_{L^1\cap\dot{H}^2}. \tag{4.22}$$

For the Duhamel's term, we use (2.10) with $|\gamma|=1$, (4.3) and (4.5) to obtain

$$\begin{aligned}\left\|\int_1^t e^{(t-s)B(\nabla)}\Delta F(U)ds\right\|_{L^2} &\lesssim \int_1^t \frac{1}{(t-s)^{5/4}}\|\nabla F(U)\|_{L^1\cap\dot{H}^1}ds\\ &\lesssim \int_1^t \frac{1}{(t-s)^{5/4}}\cdot\frac{1}{s^{5/4}}A_T^2 ds\\ &\lesssim \frac{1}{t^{5/4}}A_T^2. \end{aligned} \tag{4.23}$$

Thus, the bound (4.20) follows from (4.21)–(4.23). ☐

5. Weighted estimates for E

Now we prove the weighted estimates for the profile of E, which can control the L^∞ decay estimate as shown in (2.18). Recall the expression (2.29), i.e.,

$$\begin{aligned}\hat{f}(t,\xi) &= \hat{f}(0,\xi) - \frac{\mathrm{i}\kappa}{3}\int_0^t \int_{\mathbb{R}^3} e^{\mathrm{i}\kappa s\tilde{\varphi}(\xi,\eta)}\hat{\rho}(s,\eta)\hat{f}(s,\xi-\eta)d\eta ds \\ &=: \hat{f}(0,\xi) + \widehat{\mathcal{N}}(t,\xi) \end{aligned} \tag{5.1}$$

with

$$\tilde{\varphi}(\xi,\eta) := |\xi|^2 - |\xi-\eta|^2 = 2\xi\cdot\eta - |\eta|^2. \tag{5.2}$$

The main result of this section is stated as follows.

Proposition 5.1. *Let (ρ,u,E) be the solution of system (1.1) on $[0,T)\times\mathbb{R}^3$ with $T>0$. If $A_T \ll 1$, then we have*

$$\sup_{t\in[0,T)} (\|xf\|_{L^2} + (1+t)^{-1/2}\||x|^2 f\|_{L^2}) \lesssim \|xE_0\|_{L^2} + \||x|^2E_0\|_{L^2} + A_T^2. \tag{5.3}$$

Proof. We first note that if xE_0, $|x|^2E_0 \in L^2$, then as the argument shown in [15], we can obtain $xf, |x|^2 f \in C([0,T);H^N)$. Since $f := e^{-\mathrm{i}\kappa t\Delta}E$, one has

$$\|xf(0,x)\|_{L^2} = \|xE_0(x)\|_{L^2}, \quad \||x|^2 f(0,x)\|_{L^2} = \||x|^2E_0(x)\|_{L^2}.$$

Hence, in order to prove (5.3), we only need to show

$$\sup_{t\in[0,T)} (\|x\mathcal{N}(t,x)\|_{L^2} + (1+t)^{-1/2}\||x|^2\mathcal{N}(t,x)\|_{L^2}) \lesssim A_T^2. \tag{5.4}$$

As shown in Section 4, we can easily treat with the case $0\le t\le 1$ or the contribution of the time integral from 0 to 1. So without loss of generality, we may assume $t>1$ and only consider the contribution coming from the time integral in the interval $[1,t]$. We now denote

$$\widehat{\mathcal{N}'}(t,\xi) := -\frac{\mathrm{i}\kappa}{3}\int_1^t \int_{\mathbb{R}^3} e^{\mathrm{i}\kappa s\tilde{\varphi}(\xi,\eta)}\hat{\rho}(s,\eta)\hat{f}(s,\xi-\eta)d\eta ds. \tag{5.5}$$

Applying ∇_ξ to $\widehat{\mathcal{N}'}$ gives

$$\begin{aligned}\nabla_\xi\widehat{\mathcal{N}'} &= -\frac{\mathrm{i}\kappa}{3}\int_1^t\int_{\mathbb{R}^3}\nabla_\xi e^{\mathrm{i}\kappa s\tilde{\psi}}\hat{\rho}(\eta)\hat{f}(\xi-\eta)d\eta ds - \frac{\mathrm{i}\kappa}{3}\int_1^t\int_{\mathbb{R}^3} e^{\mathrm{i}s\kappa\tilde{\psi}}\hat{\rho}(\eta)\nabla_\xi\hat{f}(\xi-\eta)d\eta ds \\ &= \frac{2\kappa^2}{3}\int_1^t\int_{\mathbb{R}^3} se^{\mathrm{i}\kappa s\tilde{\psi}}\eta\hat{\rho}(\eta)\hat{f}(\xi-\eta)d\eta ds - \frac{\mathrm{i}\kappa}{3}\int_1^t\int_{\mathbb{R}^3} e^{\mathrm{i}s\kappa\tilde{\psi}}\hat{\rho}(\eta)\nabla_\xi\hat{f}(\xi-\eta)d\eta ds,\end{aligned}$$

then we use the Hölder inequality to obtain

$$\begin{aligned}\|x\mathcal{N}'\|_{L^2} &= \|\nabla_\xi\widehat{\mathcal{N}'}\|_{L^2} \\ &\lesssim \int_1^t s\|\nabla\rho\|_{L^2}\|E\|_{L^\infty}ds + \int_1^t \|\rho\|_{L^\infty}\|xf\|_{L^2}ds \\ &\lesssim A_T^2\int_1^t \frac{s}{(1+s)^{9/4-\delta/2}}ds + A_T^2\int_1^t\frac{1}{(1+s)^{9/8-\delta/4}}ds \lesssim A_T^2.\end{aligned} \tag{5.6}$$

Furthermore, taking Δ_ξ to (5.5), we get

$$\Delta_\xi \widehat{\mathcal{N}'} = K_1 + K_2 + K_3, \tag{5.7}$$

where

$$\begin{aligned} K_1 &= \frac{4\mathrm{i}\kappa^3}{3} \int_1^t \int_{\mathbb{R}^3} s^2 e^{\mathrm{i}\kappa s \tilde{\psi}} |\eta|^2 \hat{\rho}(\eta) \hat{f}(\xi - \eta) d\eta ds, \\ K_2 &= \frac{4\kappa^2}{3} \int_1^t \int_{\mathbb{R}^3} s e^{\mathrm{i}\kappa s \tilde{\psi}} \eta \hat{\rho}(\eta) \cdot \nabla_\xi \widehat{f}(\xi - \eta) d\eta ds, \\ K_3 &= -\frac{\mathrm{i}\kappa}{3} \int_1^t \int_{\mathbb{R}^3} e^{\mathrm{i}\kappa s \tilde{\psi}} \hat{\rho}(\eta) \Delta_\xi \widehat{f}(\xi - \eta) d\eta ds. \end{aligned}$$

The term K_1 is estimated by a direct $L^2 \times L^\infty$ estimate

$$\begin{aligned} \|K_1\|_{L^2} &\lesssim \int_1^t s^2 \|\Delta\rho\|_{L^2} \|E\|_{L^\infty} ds \\ &\lesssim A_T^2 \int_1^t \frac{s^2}{(1+s)^{5/2}} ds \\ &\lesssim (1+t)^{1/2} A_T^2, \end{aligned} \tag{5.8}$$

and term K_3 is estimated by $L^\infty \times L^2$ type estimate

$$\begin{aligned} \|K_3\|_{L^2} &\lesssim \int_1^t \|\rho\|_{L^\infty} \||x|^2 f\|_{L^2} ds \\ &\lesssim A_T^2 \int_1^t \frac{(1+s)^{1/2}}{(1+s)^{9/8-\delta/4}} ds \\ &\lesssim (1+t)^{3/8+\delta/4} A_T^2. \end{aligned} \tag{5.9}$$

To estimate the middle term, we first note that, by interpolation,

$$\|\nabla\rho\|_{L^4} \lesssim \|\nabla\rho\|_{L^2}^{1/4} \|\Delta\rho\|_{L^2}^{3/4} \lesssim \frac{1}{(1+s)^{19/16-\delta/8}} A_T^2$$

and by (2.15),

$$\begin{aligned} \|e^{\mathrm{i}\kappa s\Delta}(xf)\|_{L^4} &\lesssim \frac{1}{(1+s)^{3/4}} \|xf\|_{L^{4/3}} \\ &\lesssim \frac{1}{(1+s)^{3/4}} \|xf\|_{L^2}^{1/4} \||x|^2 f\|_{L^2}^{3/4} \\ &\lesssim \frac{1}{(1+s)^{3/8}} A_T, \end{aligned}$$

then using $L^4 \times L^4$ type estimate gives

$$\begin{aligned} \|K_2\|_{L^2} &\lesssim \int_1^t s \|\nabla\rho\|_{L^4} \|e^{\mathrm{i}\kappa s\Delta}(xf)\|_{L^4} ds \\ &\lesssim A_T^2 \int_1^t \frac{s}{(1+s)^{25/16-\delta/8}} ds \end{aligned}$$

$$\lesssim (1+t)^{7/16+\delta/8} A_T^2. \tag{5.10}$$

Inserting estimates (5.8)–(5.10) into (5.7), we obtain

$$\||x|^2\mathcal{N}'\|_{L^2} = \|\Delta_\xi \widehat{\mathcal{N}'}\|_{L^2} \lesssim (1+t)^{1/2} A_T^2,$$

which together with (5.6) yield the bound (5.4) as desired. This ends the proof of Proposition 5.1.

Finally, combining Proposition 3.1, Proposition 4.1, Proposition 4.2 and Proposition 5.1, we thus obtain the desired bound (2.21). □

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Appendix.

In this appendix, we prove Lemma 2.1 and Lemma 2.2.

Proof. **(Proof of Lemma 2.1.)** Recall (2.4) for the representation of $\widehat{G}(t,\xi)$. For sake of simplicity, we denote

$$\begin{aligned}
\widehat{G_1}(t,\xi) &:= \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-},\\
\widehat{G_2}(t,\xi) &:= -\mathrm{i}\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\xi^t,\\
\widehat{G_3}(t,\xi) &:= -\mathrm{i}\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\xi,\\
\widehat{G_4}(t,\xi) &:= e^{-\mu_0|\xi|^2 t}\Big(I_{3\times 3} - \frac{\xi^t\xi}{|\xi|^2}\Big) + \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-}\frac{\xi^t\xi}{|\xi|^2}.
\end{aligned}$$

Our aim is to show all the above four multipliers satisfy (2.6). Since

$$\nu^2|\xi|^4 - 4|\xi|^2 = 0 \Leftrightarrow |\xi| = 0 \text{ or } |\xi| = r_0 := \frac{2}{\nu},$$

there holds

$$\lambda_\pm = \begin{cases} -\frac{1}{2}\nu|\xi|^2 \pm \mathrm{i}\frac{1}{2}|\xi|\sqrt{4-\nu^2|\xi|^2}, & |\xi| < r_0,\\ -\frac{1}{2}\nu|\xi|^2 \pm \frac{1}{2}\nu|\xi|^2\sqrt{1-\frac{4}{\nu^2|\xi|^2}}, & |\xi| \geq r_0. \end{cases} \tag{A.1}$$

We begin to prove (2.6) for the multiplier G_2. Notice first that

$$\begin{aligned}
\lim_{|\xi|\to 0}\widehat{G_2} &= -\mathrm{i}\lim_{|\xi|\to 0}\frac{1+\lambda_+ t + o(\lambda_+ t) - (1+\lambda_- t + o(\lambda_- t))}{\mathrm{i}|\xi|\sqrt{4-\nu^2|\xi|^2}}\xi^t\\
&= -\mathrm{i}\lim_{|\xi|\to 0}\frac{\mathrm{i}|\xi|t\sqrt{4-\nu^2|\xi|^2} + |\xi|^2 t^2 O(1)}{\mathrm{i}|\xi|\sqrt{4-\nu^2|\xi|^2}}\xi^t\\
&= 0,
\end{aligned}$$

then we split this proof into the following three cases.

Case 1: $0 < |\xi| < \frac{9}{10}r_0$ and $|\xi| < R$. From (A.1), this case implies

$$|e^{\lambda_+ t}| = |e^{\lambda_- t}| = e^{-\frac{1}{2}\nu|\xi|^2 t},$$

thus we see

$$|\widehat{G_2}| \leq \frac{|e^{\lambda_+ t}| + |e^{\lambda_- t}|}{|\xi|\sqrt{4-\nu^2|\xi|^2}} |\xi^t| \lesssim e^{-\frac{1}{2}\nu|\xi|^2 t}.$$

Case 2: $\frac{9}{10}r_0 \leq |\xi| \leq \frac{11}{10}r_0$ and $|\xi| < R$ (we may assume $\frac{9}{10}r_0 < R$, otherwise this case does not exist). We use mean-valued theorem to get

$$e^{\lambda_+ t} - e^{\lambda_- t} = e^{(\theta\lambda_+ + (1-\theta)\lambda_-)t}(\lambda_+ - \lambda_-)t, \quad \theta \in (0,1).$$

So if $|\xi| < r_0$, we have

$$|\widehat{G_2}| = e^{-\frac{1}{2}\nu|\xi|^2 t} t|\xi| = e^{-\frac{1}{4}\nu|\xi|^2 t}(e^{-\frac{1}{4}\nu|\xi|^2 t} t|\xi|^2|)|\xi|^{-1} \lesssim e^{-\frac{1}{4}\nu|\xi|^2 t},$$

and if $|\xi| \geq r_0$, we have

$$\begin{aligned} |\widehat{G_2}| &= e^{-\frac{1}{2}\nu|\xi|^2 t} e^{\frac{1}{2}\nu|\xi|^2\sqrt{1-\frac{4}{\nu^2|\xi|^2}}(2\theta-1)t}|\xi|t \\ &\leq e^{-\frac{1}{2}\nu|\xi|^2 t} e^{\frac{\sqrt{21}}{22}\nu|\xi|^2 t}|\xi|t \\ &\lesssim e^{-\frac{1}{4}\nu|\xi|^2 t}. \end{aligned}$$

Case 3: $r > \frac{11}{10}r_0$ and $|\xi| < R$ (one can again assume $\frac{11}{10}r_0 < R$, or this case is empty). In this case, it is easy to see that there exist two constants c_1, c_2 satisfying

$$0 < c_1 < 1 - \sqrt{1 - \frac{4}{\nu^2|\xi|^2}} < c_2 < 1, \quad \frac{11}{10}r_0 < |\xi| < R,$$

then

$$e^{\lambda_+ t} = e^{-\frac{1}{2}\nu|\xi|^2 t\cdot(1-\sqrt{1-\frac{4}{\nu^2|\xi|^2}})} \leq e^{-\frac{c_1}{2}\nu|\xi|^2 t}.$$

This bound together with the trivial bound

$$|e^{\lambda_- t}| \leq e^{-\frac{1}{2}\nu|\xi|^2 t}$$

yield

$$|\widehat{G_2}| \leq \frac{|e^{\lambda_+ t}| + |e^{\lambda_- t}|}{\nu|\xi|^2\sqrt{1-\frac{4}{\nu^2|\xi|^2}}} |\xi^t| \lesssim e^{-\frac{c_1}{2}\nu|\xi|^2 t}.$$

Combining the estimates in Case 1–Case 3, we thus obtain (2.6) for G_2. The proof for G_3 is the same since $G_3 = G_2^t$. Note that $|\lambda_\pm| \lesssim 1$ if $|\xi| < R$ and

$$\frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} = e^{\lambda_+ t} - \lambda_+ \cdot \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}, \tag{A.2}$$

$$\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} = e^{\lambda_- t} + \lambda_+ \cdot \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}, \tag{A.3}$$

applying similar treatment as G_2 for the second term of (A.2) and (A.3), the desired bound for G_1 and G_4 thus follows. ∎

Proof. **(Proof of Lemma 2.2.)** We first show the estimates (2.7)–(2.9) hold for $\widehat{G_2}$. For $|\xi|\geq R_0$ with R_0 large enough, Taylor's formula implies

$$\sqrt{1-\frac{4}{\nu^2|\xi|^2}}=1-\frac{2}{\nu^2|\xi|^2}-\frac{2}{\nu^4|\xi|^4}-\frac{4}{\nu^6|\xi|^6}+\cdots,$$

so we have

$$\lambda_+=-\frac{1}{\nu}-\frac{1}{\nu^3|\xi|^2}-\frac{2}{\nu^5|\xi|^4}+\cdots, \tag{A.4}$$

$$\lambda_-=-\nu|\xi|^2+\frac{1}{\nu}+\frac{1}{\nu^3|\xi|^2}-\frac{2}{\nu^5|\xi|^4}+\cdots, \tag{A.5}$$

$$\lambda_+-\lambda_-=\nu|\xi|^2-\frac{2}{\nu}-\frac{2}{\nu^3|\xi|^2}-\frac{4}{\nu^5|\xi|^4}+\cdots. \tag{A.6}$$

Let $r=|\xi|$, from (A.4)–(A.6), it can be seen that

$$\begin{aligned} &\lambda_+\sim-\nu^{-1}, && \lambda_-\sim-\nu r^2, && \lambda_+-\lambda_-\sim\nu r^2,\\ &\lambda_+'\sim 2\nu^{-3}r^{-3}, && \lambda_-'\sim-2\nu r, && \lambda_+'-\lambda_-'\sim 2\nu r,\\ &\lambda_+''\sim-6\nu^{-3}r^{-4}, && \lambda_-''\sim-2\nu, && \lambda_+''-\lambda_-''\sim 2\nu. \end{aligned} \tag{A.7}$$

Define

$$g(r)=\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-},$$

using the bounds in (A.7), a direct computation gives

$$\begin{aligned} &|g(\xi)|=|g(r)|\lesssim\frac{e^{-\frac{1}{\nu}t}+e^{-\nu r^2t}}{r^2},\\ &|\nabla g(\xi)|\leq|g'(r)|\lesssim\frac{e^{-\frac{1}{2\nu}t}+e^{-\frac{\nu}{2}r^2t}}{r^3},\\ &|\Delta g(\xi)|=|g''(r)+\frac{2}{r}g'(r)|\lesssim\frac{e^{-\frac{1}{2\nu}t}+e^{-\frac{\nu}{2}r^2t}}{r^4}. \end{aligned} \tag{A.8}$$

Since $\widehat{G_2}=-ig(\xi)\xi^t$, one can easily see

$$\begin{aligned} &|\widehat{G_2}|\lesssim(e^{-\frac{1}{\nu}t}+e^{-\nu|\xi|^2t})|\xi|^{-1},\\ &|\nabla\widehat{G_2}|\lesssim(e^{-\frac{1}{2\nu}t}+e^{-\frac{\nu}{2}|\xi|^2t})|\xi|^{-2},\\ &|\Delta\widehat{G_2}|\lesssim(e^{-\frac{1}{2\nu}t}+e^{-\frac{\nu}{2}|\xi|^2t})|\xi|^{-3}. \end{aligned} \tag{A.9}$$

Clearly, the estimates in (A.9) still hold if we replace $\widehat{G_2}$ by $\widehat{G_3}$. In virtue of (A.2) and (A.3), we use the bounds (A.7), (A.8) and

$$\begin{aligned} &|e^{\lambda_+t}|\lesssim e^{-\frac{1}{\nu}t},\quad |\nabla e^{\lambda_+t}|\lesssim|\xi|^{-3}e^{-\frac{1}{2\nu}t},\quad |\Delta e^{\lambda_-t}|\lesssim|\xi|^{-4}e^{-\frac{1}{2\nu}t},\\ &|e^{\lambda_-t}|\lesssim e^{-\nu|\xi|^2t},\quad |\nabla e^{\lambda_-t}|\lesssim|\xi|^{-1}e^{-\frac{\nu}{2}|\xi|^2t},\quad |\Delta e^{\lambda_-t}|\lesssim|\xi|^{-2}e^{-\frac{\nu}{2}|\xi|^2t}, \end{aligned}$$

then it is not hard to find that

$$\begin{aligned} |\widehat{G_1}| &\lesssim |\xi|^{-2}e^{-\nu|\xi|^2 t} + e^{-\frac{1}{\nu}t}, \\ |\nabla\widehat{G_1}| &\lesssim |\xi|^{-3}e^{-\frac{\nu}{2}|\xi|^2 t} + |\xi|^{-3}e^{-\frac{1}{2\nu}t}, \\ |\Delta\widehat{G_1}| &\lesssim |\xi|^{-4}e^{-\frac{\nu}{2}|\xi|^2 t} + |\xi|^{-4}e^{-\frac{1}{2\nu}t} \end{aligned} \tag{A.10}$$

and

$$\begin{aligned} |\widehat{G_4}| &\lesssim e^{-\mu_0|\xi|^2 t} + |\xi|^{-2}e^{-\frac{1}{\nu}t}, \\ |\nabla\widehat{G_4}| &\lesssim |\xi|^{-1}e^{-\frac{\mu_0}{2}|\xi|^2 t} + |\xi|^{-3}e^{-\frac{1}{2\nu}t}, \\ |\Delta\widehat{G_4}| &\lesssim |\xi|^{-2}e^{-\frac{\mu_0}{2}|\xi|^2 t} + |\xi|^{-4}e^{-\frac{1}{2\nu}t}. \end{aligned} \tag{A.11}$$

Therefore, the desired bounds (2.7)–(2.9) follows from (A.9)–(A.11) clearly. ∎

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