COMMUN. MATH. SCI.
Vol. 20, No. 3, pp. 829–853

TRAVELING WAVES IN A KELLER-SEGEL MODEL WITH LOGISTIC GROWTH*

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Abstract. Bacterial diffusion, proliferation and chemotactic aggregation play an important role in forming a traveling wave in a model for chemotaxis. In this paper, we investigate the existence and non-existence of traveling wave solutions of a Keller-Segel type model for chemotaxis, where logistic cell growth is considered and chemotactic sensitivity function is a general C^1 function that represents positive or negative chemotaxis. To show the existence of traveling waves, we use techniques from dynamical system theory. By applying the techniques, we determine the range of parameter values of the bacterial chemotaxis and the kinetics of cell and chemical for which traveling wave solutions exist. Furthermore, we examine the monotonicity of the traveling wave solutions. Finally, we conclude that the traveling waves are spectrally unstable.

Keywords. Traveling waves; chemotaxis; Keller-Segel model; cell growth; reaction diffusion system.

AMS subject classifications. Primary: 34C37; 35C07; 35K57; 35Q92; 92C17; Secondary: 34A34; 35B35.

1. Introduction

Some bacteria move toward favorable environments or away from noxious substances, which is called chemotaxis. To understand their chemotactic behavior at the population level, a mathematical model, formulated by Keller and Segel in the 1970s [22], has become widely used to model bacterial chemotaxis, and various types of the model have been developed. In particular, the interplay between bacterial diffusion and aggregation in the Keller-Segel model [22, 23] successfully described the formation of traveling bands observed in experiments [1], and, further, the model has been extensively studied for pattern formation from theoretical and numerical points of view. For example, related research on traveling waves in various contexts can be found in [3, 13, 14, 19, 21, 24, 26, 28–30, 33, 47] and references therein; see [17, 18, 42, 44] for the literature review on traveling waves of Keller-Segel models.

Recently, as continuation of work [31, 32, 40] on traveling wave solutions to a special case of Keller-Segel models (referred to as the minimal Keller-Segel model in [17]), there is a growing interest in Keller-Segel type models incorporated with cell growth and death. These models can describe more diverse biological scenarios including systems in which the time scale of bacterial migration is similar or slower than that of cell proliferation. For instance, the initiation of angiogenesis where the cell growth and chemotaxis play an important role in forming new blood vessels [9, 43], and some experimental results on non-equilibrium pattern formation were explained with an interplay between cell growth and chemotaxis in [8, 41, 46]. Also, from a mathematical point of view, it is intriguing to investigate the chemotactic effects on formulation and propagation of traveling wave solutions by comparing to the formulation of traveling waves in Fisher's equation where cells' chemotactic behavior is not considered.

The goal of this paper is to investigate the existence and stability of traveling wave

*Received: September 08, 2020; Accepted (in revised form): September 29, 2021. Communicated by Min Tang.

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solutions in a Keller-Segel type model with cell growth and death:

$$\begin{cases} u_t = Du_{xx} - (u\chi(v)v_x)_x + \mu u(1-u) \\ v_t = \alpha u + \beta v, \end{cases} \tag{1.1}$$

where $u(x,t) \geq 0$ and $v(x,t) \geq 0$ represent the density of a cell population and the chemical concentration, respectively, at spatial position $x \in \mathbb{R}$ and time $t \geq 0$. We assume $D > 0$, $\chi(v) \in C^1$, $\mu > 0$, and $\alpha, \beta \in \mathbb{R}$.

In the dynamics of cell density in system (1.1), $D > 0$ represents the cell diffusion coefficient and $\chi(v)$ describes the chemotactic sensitivity. Since $\chi(v)$ is determined by interactions between their intracellular signalling pathway and the bacterial movement in response to chemical stimuli [17, 34, 42], $\chi(v)$ depends upon bacterial species and biological scenarios, and in turn each chemotaxis model may have different $\chi(v)$. The prototype of $\chi(v)$ takes χ_0 (linear), $\frac{\chi_0}{v}$ (logarithmic), $\frac{\chi_0}{(k+v)^2}$, $k > 0$ (receptor) forms [17, 42]. Here, if the chemical is an attractant, then $\chi_0 \geq 0$, i.e., the cells move along the chemical gradient direction. If the chemical is a repellent, $\chi_0 \leq 0$, i.e., the cells move away from the chemical and it corresponds to negative chemotaxis. The logistic growth ($\mu > 0$) is a standard choice of the kinetics and has been used to govern some cells [4, 8, 17].

In the chemical kinetics in system (1.1), the choice of α and β depends on biological systems; $\alpha < 0$ (resp., $\alpha > 0$) corresponds the consumption (resp., secretion) rate of the chemical; $\beta > 0$ (resp., $\beta < 0$) represents the chemical growth (resp., degradation) rate. Particularly, our model (1.1) with $\alpha > 0$ and $\beta < 0$ describes bacterial chemotaxis in which bacteria consume chemicals produced by themselves such as *Escherichia coli* and *Salmonella typhimurium* bacteria [7, 46]; also, chemorepellent can be produced by cells themselves [20]. On the other hand, compared to the minimal model for chemotaxis in [17], our model (1.1) contains no chemical diffusion. In biological systems, zero diffusion may occur when chemical signals are confined to a rigid extracellular structure or substrate (e.g. angiogenesis), or chemicals are carried by large water molecules [17, 44]. Also, diffusion in the chemical signal can be negligible when chemical diffusion is much smaller than other biological process; see [10, 13, 25, 34, 42] for comparison of parameters.

Analysis of a traveling wave solution of a chemotxis model with the logistic source and a linear chemical reaction, in terms of existence and a minimal speed, has been established by the authors in [5,27,32,35–39]. Most of the studies considered a parabolic-parabolic model or a parabolic-elliptic model by introducing non-zero chemical diffusion and by taking a constant $\chi(v) \geq 0$. In [32], the authors showed the existence by using a homotopy argument; in [35–39] the authors used a sub-supersolution approach. The model of (1.1) with zero chemical diffusion and a general sensitivity function $\chi(v) \geq 0$ was studied by [5, 27] for the case of $\alpha < 0$ and $\beta > 0$, which describes biological systems where chemoattractant such as oxygen is consumed by cells to balance the exponential chemical growth. The authors in [27] established the existence, non-existence and a minimal speed by using a method in [2] where a Keller-Segel type model involves a logistic source term and $\chi(v) = 1/v$ whose singularity was removed by the Hopf-Cole transformation. The result of [27] was improved further by the author in [5] and was also extended to the case of non-zero chemical diffusion.

Our current contribution toward existence and non-existence of traveling wave solution is as follows. We consider a general case of C^1 sensitivity function $\chi(v)$ as the chemotactic term may vary depending on the biological system under consideration (see [42] for a brief list of $\chi(v)$). Since we consider both $\chi(v) > 0$ and $\chi(v) < 0$, our results can describe both chemoattractant and chemorepellent activities. In addition to the results of [5,27], by considering the case of $\alpha,\beta\in\mathbb{R}$ in (1.1), the model (1.1) enables to describe more biological scenarios.

To prove the existence of traveling wave solutions of (1.1), we employ dynamical systems theory to show the existence of a heteroclinic orbit connecting two steady states in a system of ordinary differential equations (ODEs) derived from (1.1). Different from the dynamical systems approach used in [2,27] where a trapping region containing a heteroclinic orbit was constructed; in general construction of a trapping region is not tractable [6,11,44]. Thus, in this paper, to obtain a heteroclinic orbit, we construct a certain region satisfying the conditions of *the principle of Ważeski* [11,12,15,45] in the three dimensional phase space; more specifically, we apply the shooting method, based on Ważeski's theorem, that was introduced in [12]. According to Ważeski's theorem, the region contains an invariant set, and the invariant set turns out to be our desired heteroclinic orbit. Here, the descriptive region helps to understand the interplay between the chemotaxis and the kinetics of cells and chemical in forming a traveling wave solution. Furthermore, it helps to deduce the monotonicity of the heteroclinic orbit.

Our main theorems of existence of traveling wave solutions of Keller-Segel model (1.1) are as follows:

Theorem 1.1 (Existence when $\chi(v)>0$). *Let $D,\mu,\alpha>0$ and $\beta<0$. Assume a C^1-function $\chi(v)$ satisfies*

$$0<\frac{D\mu}{\alpha}\le\chi(v)\le\frac{D|\beta|}{2\alpha}.$$

Then there is a minimal speed $s^*>0$ such that if $s>s^*$, the system (1.1) has a traveling wave solution $(u,v)(x,t)=(U,V)(\xi)$, where $\xi=x-st$, satisfying

(i) *$0\le U(\xi)\le 1$ and $0\le V(\xi)\le\frac{\alpha}{|\beta|}$ for any $\xi\in(-\infty,\infty)$*

(ii) *$(U,V)(\xi)$ converges to $(1,\frac{\alpha}{|\beta|})$ and $(0,0)$ exponentially as $\xi\to-\infty$ and $\xi\to\infty$, respectively.*

In particular, if $\chi(v)$ is a constant satisfying $\chi(v)\equiv\frac{D\mu}{\alpha}\le 1$, then U and V are monotonically decreasing. Namely, $U'\le 0$ and $V'\le 0$ for any $\xi\in(-\infty,\infty)$.

There is no traveling wave solution connecting $(1,\frac{\alpha}{|\beta|})$ and $(0,0)$ whose speed s satisfies $s<s^*$.

Theorem 1.2 (Existence when $\chi(v)<0$). *Let $D,\mu,\alpha>0$ and $\beta<0$. Assume $\chi(v)\in C^1$ satisfies*

$$\chi(v)<0,\qquad \chi'(v)\ge 0\qquad\text{and}\qquad 0<|\chi(v)|\le\frac{D\mu}{2\alpha}\le\frac{D|\beta|}{8\alpha}$$

for any $0<v\le\frac{\alpha}{|\beta|}$. Then, there is a minimal speed $s^*>0$ such that if $s>s^*$, the system (1.1) has a traveling wave solution $(u,v)(x,t)=(U,V)(\xi)$, where $\xi=x-st$, satisfying

(i) *$0\le U(\xi)\le 1$ and $0\le V(\xi)\le\frac{\alpha}{|\beta|}$ for any $\xi\in(-\infty,\infty)$*

(ii) *$(U,V)(\xi)$ converges to $(1,\frac{\alpha}{|\beta|})$ and $(0,0)$ exponentially as $\xi\to-\infty$ and $\xi\to\infty$, respectively.*

(iii) *V is monotonically decreasing; that is, $V'\le 0$ for any $\xi\in(-\infty,\infty)$.*

There is no traveling wave solution connecting $(1,\frac{\alpha}{|\beta|})$ and $(0,0)$ whose speed s satisfies $s<s^{**}$ for some $s^{**}\le s^*$.

As $\mu>0$, the system (1.1) admits infinitely many traveling wave solutions with different wave speeds. Here, we note that $\mu>0$ is sufficient to admit a traveling wave solution of (1.1) when the sensitive function $\chi(v)$ is C^1 [40]. In the biological sense, $\mu>0$ causes a wave to keep propagating [42]; otherwise, a small diffusion of cells can decrease the wave speed and broaden the wave profile. The exact values of s^* and s^{**} can be found in Theorem 2.1 and Theorem 2.2.

In the following, we present the non-existence of traveling wave solution of (1.1). To prove the theorem, we apply dynamical systems theory in conjunction with a Lyapunov function as in [27].

THEOREM 1.3 (Non-existence). *Let $D,\mu>0$ and $\alpha,s\geq 0$ and $\beta\in\mathbb{R}$. Assume that a C^1-function $\chi(v)$ does not change a sign.*

(1) *If either $\alpha=0$ and $\beta\neq 0$ or $\alpha>0$ and $\beta=0$, the system (1.1) has no non-trivial traveling wave solution of speed $s\geq 0$. For $\alpha,\beta>0$, a non-trivial traveling wave solution with speed $s>0$ does not exist.*

(2) *If $\alpha=\beta=0$ or $\alpha\beta\neq 0$, the system (1.1) has no non-trivial traveling wave solution of speed $s=0$ satisfying $0\leq U\leq 1$.*

In Theorems 1.1–1.3, only the case of $s\geq 0$ is addressed. The case of $s<0$ can be examined by using a change of variable, $\tilde{\xi}:=-\xi$, and the results are similar to those from Theorems 1.1–1.3.

Lastly, we investigate the stability of the traveling wave solutions (U,V) obtained from Theorems 1.1-1.2. Here, the stability of a traveling wave means that if an initial perturbation between a traveling wave solution and a solution to the Cauchy problem (1.1) with the initial data (u_0,v_0) satisfying

$$(u_0,v_0)(x)=(u,v)(x,0)\to\begin{cases}(U,V)(-\infty) & \text{as} \quad x\to-\infty\\ (U,V)(\infty) & \text{as} \quad x\to\infty\end{cases} \tag{1.2}$$

is small in some space, the solution (u,v) to the Cauchy problem converges to the traveling wave solution (U,V) as time evolves. We note that since our model (1.1) has infinitely many traveling wave solutions whose speeds are not isolated, it is easy to see the nonlinear instability of the family of traveling wave solutions in the unweighted L^2 space. In this work, we particularly verify the spectral instability of the family of traveling wave solutions (see Theorem 1.4) by examining the location of essential spectrum of the linearized operator of (1.1). Our spectral instability result is led by $\mu>0$ in the cell growth term, and is independent of the rates of chemical growth $\alpha>0$ and degradation $\beta<0$.

THEOREM 1.4. *Let $\chi(v)\in C^3$. The traveling wave solutions of the system (1.1), obtained in Theorem 1.1 and Theorem 1.2, are spectrally unstable in the L^2 norm.*

The remainder of this paper is organized as follows. In Section 2 we reformulate the system (1.1) and restate Theorem 1.1 and Theorem 1.2 in terms of the reformulated system. In Sections 3 and 4, the existence of traveling wave solutions are proved for the cases of $\chi(v)>0$ and $\chi(v)<0$, respectively. In Section 5, we prove Theorem 1.3. In Section 6, Theorem 1.4 is proved. We conclude with a brief summary in Section 7.

2. Reformulation of the problem

In the following, we reformulate the system (1.1) to show the existence of traveling wave solutions (U,V) of (1.1), which are desired in Theorem 1.1 and Theorem 1.2 for $\alpha>0$, $\beta<0$ and $s>0$. We state our main theorems of the existence of traveling wave solution for the transformed system.

In this paper, we are interested in a smooth traveling wave solution of the system of partial differential equations (PDEs) (1.1) with the speed s of the form

$$(u,v)(x,t) = (U,V)(\xi), \quad \xi := x - st$$

that satisfies

$$\begin{cases} U(\xi) \geq 0, \quad V(\xi) \geq 0, & \forall \xi \in (-\infty,\infty) \\ (U,V)(\pm\infty) = (u_\pm, v_\pm), & (U',V')(\pm\infty) = (0,0). \end{cases} \tag{2.1}$$

To enhance the understanding of the relations of parameters, we simplify the setting of parameters by using the following scaling:

$$\tilde{u}(x,t) := u(\sqrt{D}x,t), \ \ \tilde{v}(x,t) := v(\sqrt{D}x,t), \ \ \tilde{\chi}(\tilde{v}) := \frac{1}{D}\chi(\tilde{v}). \tag{2.2}$$

Then, (1.1) is transformed into

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} - (\tilde{u}\tilde{\chi}(\tilde{v})\tilde{v}_x)_x + \mu\tilde{u}(1-\tilde{u}) \\ \tilde{v}_t = \alpha\tilde{u} + \beta\tilde{v}, \end{cases} \tag{2.3}$$

and a traveling wave solution $(U,V)(\xi)$, $\xi = x - st$, of (1.1) also corresponds to $(\tilde{U},\tilde{V})(\tilde{\xi})$, where

$$\tilde{\xi} := x - ct, \qquad c := \frac{s}{\sqrt{D}}. \tag{2.4}$$

By substituting $(\tilde{U},\tilde{V})(\tilde{\xi})$ into the system (2.3) and using (2.4), we obtain the following system of ODEs:

$$\begin{cases} -c\tilde{U}' = \tilde{U}'' - (\tilde{U}\tilde{\chi}(\tilde{V})\tilde{V}')' + \mu\tilde{U}(1-\tilde{U}) \\ -c\tilde{V}' = \alpha\tilde{U} + \beta\tilde{V}, \end{cases} \tag{2.5}$$

where $' = \frac{d}{d\tilde{\xi}}$. It is noticed that the existence of a solution of the ODE system (2.5) connecting $(\tilde{u}_-,\tilde{v}_-)$ and $(\tilde{u}_+,\tilde{v}_+)$ implies the existence of a traveling wave solution (1.1) satisfying (2.1).

Note that $c \geq 0$ by the definition of c in (2.4) and $s \geq 0$. Since we are particularly interested in $\alpha > 0$ and $\beta < 0$ to address the existence of a traveling wave solution of the system (1.1), we fix $\alpha = 1$ and define a new constant $\gamma := -\beta > 0$ for simplicity in the remainder of this paper. Further, for the ease of analysis in finding a solution of (2.5), we convert the second order ODE to a system of first order ODEs by introducing a new variable:

$$\tilde{W}(\xi) := c\tilde{U} + \tilde{U}' - \tilde{U}\tilde{\chi}(\tilde{V})\tilde{V}' - c\tilde{u}_-. \tag{2.6}$$

For brevity of notation, we drop the tildes. Then, we derive the ODE system

$$\begin{cases} U' = -cU + \frac{1}{c}U\chi(V)(\gamma V - U) + W \\ V' = \frac{1}{c}(\gamma V - U) \\ W' = \mu U(U-1), \end{cases} \tag{2.7}$$

where

$$O := (0,0,0) \qquad \text{and} \qquad C := (1, \tfrac{1}{\gamma}, c) \tag{2.8}$$

are the equilibrium points of the system (2.7).

If there is a heteroclinic orbit of the ODE system (2.7) connecting the equilibria O and C, the orbit is a solution of ODE system (2.5). Using the transformations (2.2), (2.4) and (2.6), the orbit is also identified as a traveling wave solution of (1.1) satisfying (2.1). Therefore, we aim to prove the existence of a heteroclinic orbit of the system (2.7) connecting the two equilibria in (2.8) as follows:

THEOREM 2.1. *For $\mu > 0$ and $\gamma > 0$, we assume that a C^1 function $\chi(v)$ satisfies*

$$\mu \le \chi(v) \le \frac{\gamma}{2} \tag{2.9}$$

for $0 \le v \le \frac{1}{\gamma}$. Let

$$c^* := 2\sqrt{\mu} > 0. \tag{2.10}$$

Then, for any $c > c^*$ the system (2.7) has a solution $(U,V,W)(\xi)$, $\xi := x - ct$, satisfying

(i) *$0 \le U(\xi) \le 1$, $0 \le V(\xi) \le \frac{1}{\gamma}$, and $0 \le W(\xi) \le c$ for any $\xi \in (-\infty, \infty)$.*

(ii) *$(U,V,W)(\xi)$ converges to $(1, \frac{1}{\gamma}, c)$ and $(0,0,0)$ exponentially as $\xi \to -\infty$ and $\xi \to \infty$, respectively.*

In particular, if $\chi(v) \equiv \mu \le 1$, it is satisfied that $0 < W(\xi) \le cU(\xi) < c$ and $0 < \gamma V(\xi) \le U(\xi) < 1$. Furthermore, $U', V' \le 0$ and $W' \le 0$ for any $\xi \in (-\infty, \infty)$.

If $c < c^*$, there is no traveling wave solution connecting $(1, \frac{1}{\gamma}, c)$ and $(0,0,0)$ with speed c.

THEOREM 2.2. *Let $\chi(v) < 0$ be a C^1 function and satisfy*

$$\chi'(v) \ge 0 \tag{2.11}$$

for any $0 \le v \le \frac{1}{\gamma}$. Let $\mu > 0$ and $\gamma > 0$ satisfy

$$0 < |\chi(v)| \le \frac{\mu}{2} \le \frac{\gamma}{8} \tag{2.12}$$

for any $0 < v \le \frac{1}{\gamma}$. For any $c > c^*$, where

$$c^* := \sqrt{\gamma} > 0, \tag{2.13}$$

the system (2.7) has a heteroclinic orbit $(U,V,W)(\xi)$, $\xi := x - ct$, satisfying

(i) *$0 \le U(\xi) \le 1$, $0 \le V(\xi) \le \frac{1}{\gamma}$, and $0 \le W(\xi) \le c$ for any $\xi \in (-\infty, \infty)$.*

(ii) *$(U,V,W)(\xi)$ converges to $(1, \frac{1}{\gamma}, c)$ and $(0,0,0)$ exponentially as $\xi \to -\infty$ and $\xi \to \infty$, respectively.*

(iii) *$V'(\xi), W'(\xi) \le 0$ for all $\xi \in (-\infty, \infty)$.*

If $c < 2\sqrt{\mu}$, there is no traveling wave solution connecting $(1, \frac{1}{\gamma}, c)$ and $(0,0,0)$ with speed c.

REMARK 2.1. The condition $c > c^* \ge 2\sqrt{\mu} > 0$ in (2.10) and (2.13) is necessary to have non-negative U and V, and details can be found in Section 3.1. Noticing that there is no traveling wave solution connecting $(1,\frac{1}{\gamma},c)$ and $(0,0,0)$ for $c<2\sqrt{\mu}$, s^* in Theorem 1.1 and s^{**} in Theorem 1.2 can be explicitly obtained by (2.4). In Theorem 2.1 and Theorem 2.2, the assumptions on $\chi(v)$, μ and γ are sufficient to obtain our desired solution of the system by applying the principle of Ważeski.

Theorem 1.1 and Theorem 1.2, our main theorems of this work, are consequences of Theorem 2.1 and Theorem 2.2. The proofs of Theorem 2.1 and Theorem 2.2 are given in Section 3 and in Section 4, respectively. To prove the theorems, we mainly use a tool that was formulated by Ważeski [45]. Before giving the proofs, we introduce our notation for the flow of (2.7) and recall some definitions and *the principle of Ważeski.* We refer to [12, 15, 45] for more details.

NOTATION 2.1. *We denote the flow of the system (2.7) by* $\varphi_\xi(P)$*, where* $\varphi:\mathbb{R}\times\mathbb{R}^3\to\mathbb{R}^3$ *satisfies*

(i) $\varphi_0(P)=P$

(ii) *for all* ξ *and* $\eta\in\mathbb{R}$, $\varphi_\xi(\varphi_\eta(P))=\varphi_{\xi+\eta}(P)$

for all $P\in\mathbb{R}^3$.

DEFINITION 2.1 ([15, p. 278]). *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be sets of a topological space and* $\mathcal{A}$ *be a subset of* $\mathcal{B}$. *If a continuous map* $g:\mathcal{B}\to\mathcal{A}$ *satisfies* $g(A)=A$ *for any* $A\in\mathcal{A}$, *then* $\mathcal{A}$ *is said to be a retract of* $\mathcal{B}$.

DEFINITION 2.2 ([15, p. 280]). *Let* $\Omega=\mathbb{R}^3-\{O,C\}\subset\mathbb{R}^3$, *where* O *and* C *are given in (2.8), and* Ω^0 *an open subset in* Ω. *Let* $\varphi_\xi(P)$ *be a flow of the system (2.7). A point* $P\in\Omega\cap\partial\Omega^0$ *is an egress point of* Ω^0 *with respect to (2.7) if there is an* $\varepsilon>0$ *such that* $\varphi_\xi(P)\in\Omega^0$ *for* $-\varepsilon\le\xi<0$. *If an egress point* $P\in\Omega^0$ *satisfies* $\varphi_\xi(P)\notin\overline{\Omega^0}$ *for* $0<\xi\le\varepsilon$ *for small* $\varepsilon>0$, *the egress point* P *is called a strict egress point of* Ω^0. *We denote the set of egress points of* Ω^0 *by* Ω^0_{e} *and the set of strict egress points by* Ω^0_{se}.

The following theorem is one of the modified versions of the principle of Ważeski, and it was introduced by H. Fan and X.-B. Lin [12].

THEOREM 2.3 ([12, Lemma 1.2]). *Assume that all egress points of* Ω^0 *are strict egress points, i.e.,* $\Omega^0_{\mathrm{e}}=\Omega^0_{\mathrm{se}}$. *Let* $\mathcal{S}$ *be a nonempty subset of* $\Omega^0\cup\Omega^0_{\mathrm{e}}$ *such that* $\mathcal{S}\cap\Omega^0_{\mathrm{e}}$ *is not a retract of* $\mathcal{S}$ *but is a retract of* Ω^0_{e}. *Suppose that there are two mutually disjoint open subsets* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *of* $\mathcal{S}\cap\partial\Omega^0$ *such that there is a smooth curve segment* $\overline{P_1P_2}\subset\Omega^0$ *when* P_1 *and* P_2 *are strict egress points of* $\mathcal{S}_1$ *and* $\mathcal{S}_2$, *respectively. Then, there is a* $P_3\in\overline{P_1P_2}$ *satisfying that* $\varphi_\xi(P_3)$ *remains in* Ω^0 *for all positive* ξ.

3. Proof of the existence of traveling wave solutions for $\chi(v)>0$

In this section we prove Theorem 2.1. The outline of this section is as follows: We linearize the system (2.7) at the equilibria O and C given in (2.8), and analyze the behavior of a solution. Based on the analysis, we construct an open set Ω^0 and obtain strict egress points of Ω^0 to apply Theorem 2.3. Consequently we conclude that there is a heteroclincic orbit of (2.7) that connects O and C, and hence Theorem 2.1 is proved.

3.1. Linearized system. In the following we investigate a trajectory of the system (2.7) in a neighborhood of the equilibrium points O and C given in (2.8).

At $O=(0,0,0)$, the Jacobian matrix of (2.7) is

$$\begin{pmatrix} -c & 0 & 1 \\ -\frac{1}{c} & \frac{\gamma}{c} & 0 \\ -\mu & 0 & 0 \end{pmatrix} \tag{3.1}$$

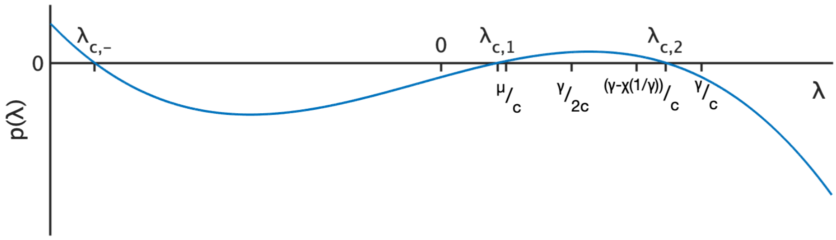

FIG. 3.1. *Graph of characteristic polynomial* $p(\lambda)$ *given in (3.4) for* $\chi(v)>0$, *where* $\lambda_{c,-}$, $\lambda_{c,1}$ *and* $\lambda_{c,2}$ *correspond to the roots of* $p(\lambda)$.

and its eigenvalues are

$$\lambda_{o,1}=\frac{\gamma}{c}>0,\ \ \lambda_{o,2}=\frac{-c+\sqrt{c^2-4\mu}}{2}<0,\ \ \lambda_{o,3}=\frac{-c-\sqrt{c^2-4\mu}}{2}<0.$$

The corresponding eigenvectors are

$$\vec{v}_{o,1}=(0,1,0)^\top,\qquad \vec{v}_{o,i}=(\gamma-c\lambda_{o,i},1,\gamma(c+\lambda_{o,i})+\mu c)^\top,\qquad i=1,2. \tag{3.2}$$

At $C=(1,\frac{1}{\gamma},c)$, the Jacobian matrix of the system (2.7) is

$$\begin{pmatrix} -c-\frac{1}{c}\chi(\frac{1}{\gamma}) & \frac{\gamma}{c}\chi(\frac{1}{\gamma}) & 1 \\ -\frac{1}{c} & \frac{\gamma}{c} & 0 \\ \mu & 0 & 0 \end{pmatrix}, \tag{3.3}$$

and its characteristic polynomial is

$$p(\lambda)=-\lambda^3+\Big(\frac{\gamma}{c}-\frac{\chi\big(\frac{1}{\gamma}\big)}{c}-c\Big)\lambda^2+(\gamma+\mu)\lambda-\frac{\gamma\mu}{c}. \tag{3.4}$$

By (2.9), it is straightforward to show that $p(\lambda)$ has one negative root $\lambda_{c,-}$ and two distinct positive roots $\lambda_{c,1}$ and $\lambda_{c,2}$; particularly, these two positive eigenvalues satisfy (see Figure 3.1)

$$0<\lambda_{c,1}<\frac{\mu}{c}<\frac{\gamma}{2c}<\frac{\gamma-\chi(1/\gamma)}{c}<\lambda_{c,2}<\frac{\gamma}{c} \tag{3.5}$$

and the corresponding eigenvectors are

$$\vec{v}_{c,i}=\Big(1,(\gamma-c\lambda_{c,i})^{-1},\mu\lambda_{c,i}^{-1}\Big)^\top,\qquad i=1,2. \tag{3.6}$$

Therefore we have the following lemma.

LEMMA 3.1. *Assume that conditions in Theorem 2.1 and* $c>c^*$, *where* c^* *is given in (2.10). At the equilibrium point* O, *the system (2.7) has a 2-dimensional local stable manifold* $W^{s}_{loc}(O)$. *At the equilibrium point* C, *(2.7) has a 2-dimensional local unstable manifold* $W^{u}_{loc}(C)$.

3.2. Construction of an open set containing an invariant set. To prove the existence of a solution of the system (2.7) that connects the equilibria O and C in (2.8), we construct an open set $\Omega^0 \subset \Omega = \mathbb{R}^3 - \{O,C\}$ and identify egress points and strict egress points of Ω^0. Then, the open set Ω^0 contains an invariant set by Theorem 2.3; it turns out that the invariant set is our desired heteroclinic orbit connecting O and C. The details are discussed below.

To begin with, we define an open set $\Omega^0 \subset \Omega = \mathbb{R}^3 - \{O,C\}$ whose boundary is surrounded by the following seven faces (Figure 3.2(A)):

$$\begin{aligned}
\mathcal{F}_1 &:= \{(U,V,W) \mid 0 \le U \le 1,\ V=0,\ kU \le W \le c\},\\
\mathcal{F}_2 &:= \{(U,V,W) \mid U=1,\ 0 \le V \le \tfrac{1}{\gamma},\ k \le W \le c\},\\
\mathcal{F}_3 &:= \{(U,V,W) \mid 0 \le U \le 1,\ V=\tfrac{1}{\gamma},\ k \le W \le c\},\\
\mathcal{F}_4 &:= \{(U,V,W) \mid U=0,\ 0 \le V \le \tfrac{1}{\gamma},\ k\gamma V \le W \le c\},\\
\mathcal{F}_5 &:= \{(U,V,W) \mid 0 \le U \le 1,\ 0 \le V \le \tfrac{1}{\gamma},\ W=c\},\\
\mathcal{F}_6 &:= \{(U,V,W) \mid 0 \le \gamma V \le U \le 1,\ W=kU\},\\
\mathcal{F}_7 &:= \{(U,V,W) \mid 0 \le U \le \gamma V \le 1, W=k\gamma V\},
\end{aligned} \tag{3.7}$$

where k is defined as

$$k := \frac{c-\sqrt{c^2-4\mu}}{2} > 0. \tag{3.8}$$

Now we investigate egress points and strict egress points of Ω^0 by Definition 2.2. More specifically, we verify that all egress points are strict egress points and the set $\Omega^0_{\rm se}$ is decomposed by two disjoint sets (Figure 3.2(A)):

$$\Omega^0_{\rm e} = \Omega^0_{\rm se} = \mathcal{S}_1 \cup \mathcal{S}_2, \tag{3.9}$$

where

$$\begin{aligned}
\mathcal{S}_1 &:= \mathcal{F}_1 - \Big(\mathcal{F}_2 \cup \mathcal{F}_4\Big),\\
\mathcal{S}_2 &:= \Big(\mathcal{F}_3 - (\mathcal{F}_2 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup \mathcal{F}_7)\Big) \cup \Big(\mathcal{F}_7 - \mathcal{F}_4\Big).
\end{aligned} \tag{3.10}$$

To show (3.9), we first rule out points $(U,V,W) \in \partial\Omega^0$ satisfying

$$\vec{n}\cdot(U',V',W') < 0,$$

where $\vec{n}$ is an outward vector at each point (U,V,W) of $\partial\Omega^0$, since these points are not egress points by the definition of egress points in Definition 2.2. It is noticed that a point $(U,V,W) \in \partial\Omega^0$ is classified as a strict egress point if $\vec{n}\cdot(U',V',W') > 0$ holds. Thus, for a point $(U,V,W) \in \partial\Omega^0$ satisfying $\vec{n}\cdot(U',V',W') = 0$, we further analyze to verify whether or not the point is an egress point.

In the following (F1)–(F7), we identify all possible strict egress points of the relative interiors of faces $\mathcal{F}_1, \cdots, \mathcal{F}_7$. We denote the relative interior of face $\mathcal{F}$ by $\mathrm{Int}(\mathcal{F})$:

(F1) The interior of $\mathcal{F}_1$, which is denoted by $\mathrm{Int}(\mathcal{F}_1)$, has an outward normal vector $\vec{n}_{\mathcal{F}_1} = (0,-1,0)^\top$. At any point on $\mathrm{Int}(\mathcal{F}_1)$ we have $\vec{n}_{\mathcal{F}_1}\cdot(U',V',W') = \frac{1}{c}U > 0$, which implies $\mathrm{Int}(\mathcal{F}_1) \subset \Omega^0_{\rm se}$.

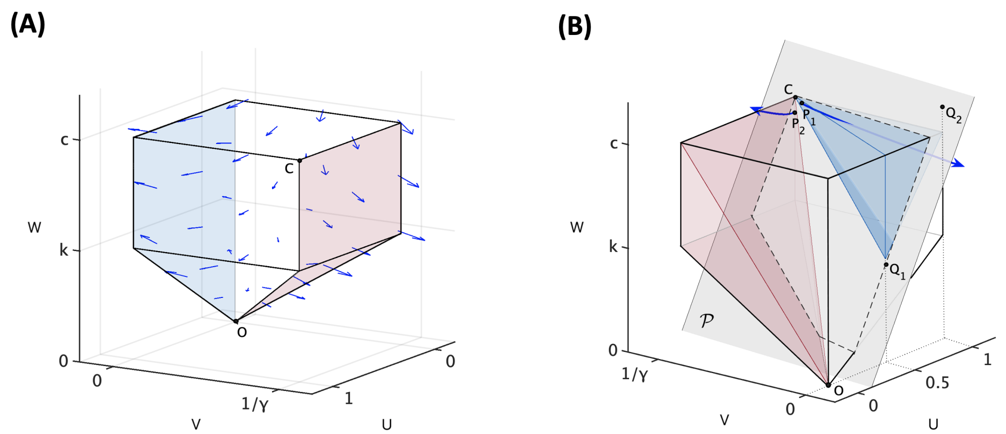

FIG. 3.2. *The case when* $\mu=\frac{1}{4},\gamma=1,c=1$ *and* $\chi(v)=\frac{4}{(3+v)^2}$. *(A) Sketch of* Ω^0 *with the set of egress points* $\Omega^0_{\rm e}=\mathcal{S}_1\cup\mathcal{S}_2$ *(*$\mathcal{S}_1:$ *blue face,* $\mathcal{S}_2:$ *red face). A collection of blue arrows represents a vector field on* Ω^0 *associated with the system (2.7) for* $\chi(v)>0$. *Two points* O *and* C *are the equilibrium points. (B) Sketch of* Ω^0 *with* Ω^1 *(blue tetrahedron) and* Ω^2 *(red pentahedron). The dashed polygon represents the intersection of* Ω^0 *and the plane* $\mathcal{P}$ *(gray plane). Two blue trajectories are flows of (2.7) initiated at* P_1 *and* P_2*, respectively.*

(**F2**) At $(U,V,W)\in\text{Int}(\mathcal{F}_2)$, the outward normal vector $\vec{n}_{\mathcal{F}_2}=(1,0,0)^\top$ satisfies $\vec{n}_{\mathcal{F}_2}\cdot(U',V',W')=-cU+W+\frac{1}{c}U\chi(V)(\gamma V-U)<0$. Thus $\text{Int}(\mathcal{F}_2)\not\subset\Omega^0_{\rm e}$.

(**F3**) At $(U,V,W)\in\text{Int}(\mathcal{F}_3)$, since the outward normal vector $\vec{n}_{\mathcal{F}_3}=(0,1,0)^\top$ satisfies $\vec{n}_{\mathcal{F}_3}\cdot(U',V',W')=\frac{1}{c}(1-U)>0$, $\text{Int}(\mathcal{F}_3)\subset\Omega^0_{\rm se}$.

(**F4**) At $(U,V,W)\in\text{Int}(\mathcal{F}_4)$, we have $\vec{n}_{\mathcal{F}_4}\cdot(U',V',W')=-W<0$, where $\vec{n}_{\mathcal{F}_4}=(-1,0,0)^\top$ is an outward normal vector. Thus, $\text{Int}(\mathcal{F}_4)\not\subset\Omega^0_{\rm e}$.

(**F5**) Using an outward normal vector $\vec{n}_{\mathcal{F}_5}=(0,0,1)$ at $(U,V,W)\in\text{Int}(\mathcal{F}_5)$, we obtain $\vec{n}_{\mathcal{F}_5}\cdot(U',V',W')=\mu U(U-1)<0$, and hence $\text{Int}(\mathcal{F}_5)\not\subset\Omega^0_{\rm e}$.

(**F6**) Using an outward normal vector $\vec{n}_{\mathcal{F}_6}=(k,0,-1)$, we obtain

$$\vec{n}_{\mathcal{F}_6}\cdot(U',V',W')=U\left(\frac{k}{c}\chi(V)(\gamma V-U)-\mu U\right)<0,$$

where it is used that k in (3.8) satisfies $k^2-ck+4\mu=0$. Thus, $\text{Int}(\mathcal{F}_6)\not\subset\Omega^0_{\rm e}$.

(**F7**) The face $\text{Int}(\mathcal{F}_7)$ has an outward normal vector $\vec{n}_{\mathcal{F}_7}=(0,\gamma k,-1)$. We then have $\vec{n}_{\mathcal{F}_7}\cdot(U',V',W')>0$ for $(U,V,W)\in\text{Int}(\mathcal{F}_7)$, and hence $\text{Int}(\mathcal{F}_7)\subset\Omega^0_{\rm se}$.

To this end, we examine that points on each edge of $\partial\Omega^0$ are either in $\Omega^0_{\rm se}$ or not in $\Omega^0_{\rm e}$:

(**E1**) If there is a $\xi_0\in\mathbb{R}$ satisfying $(U(\xi_0),V(\xi_0),W(\xi_0))\in\mathcal{F}_1\cap\mathcal{F}_4-\{O\}$, direct calculation yields $V(\xi)=\frac{1}{2}V''(\xi_0)(\xi-\xi_0)^2+O(|\xi-\xi_0|^3)$ for sufficiently small $|\xi-\xi_0|$, and hence $V''(\xi)<0$ for sufficiently small $|\xi-\xi_0|$. Also it is noticed that $\{O\}$ is an invariant set. Thus, by the definition of an egress point, $\mathcal{F}_1\cap\mathcal{F}_4\not\subset\Omega^0_{\rm e}$.

(**E2**) Any point $(U,V,W)\in\mathcal{F}_1\cap\mathcal{F}_2$, $U'<0$ and $V'<0$. We deduce that $\mathcal{F}_1\cap\mathcal{F}_2\not\subset\Omega^0_{\rm e}$.

(**E3**) Any point in $\mathcal{F}_2\cap\mathcal{F}_3-\{C\}$ satisfies $V'=0$ and $V''>0$. Using the approximation to the Taylor polynomials as in (E1) and noticing that $\{C\}$ is an invariant set, we have $\mathcal{F}_2\cap\mathcal{F}_3\not\subset\Omega^0_{\rm e}$.

(**E4**) Since $U'>0$ and $V'>0$ for any point in $\mathcal{F}_3\cap\mathcal{F}_4-\{(0,\frac{1}{\gamma},k),(0,\frac{1}{\gamma},c)\}$, $\mathcal{F}_3\cap\mathcal{F}_4-\{(0,\frac{1}{\gamma},k),(0,\frac{1}{\gamma},c)\}\not\subset\Omega^0_{\rm se}$. In fact, from (E8) and (E12), $(0,\frac{1}{\gamma},k),(0,\frac{1}{\gamma},c)\notin\Omega^0_{\rm e}$.

(**E5**) Since $V'<0$ and $W'<0$ at any point in $\mathcal{F}_1\cap\mathcal{F}_5-\{(0,0,c),(1,0,c)\}$, $\mathcal{F}_1\cap\mathcal{F}_5-\{(0,0,c),(1,0,c)\}\subset\Omega^0_{\mathrm{se}}$. From (E1) and (E2), $(0,0,c),(1,0,c)\notin\Omega^0_{\mathrm{e}}$.

(**E6**) Since $U'<0, W'=0$ and $W''<0$ at any point in $\mathcal{F}_2\cap\mathcal{F}_5-\{C\}$, $\mathcal{F}_2\cap\mathcal{F}_5\not\subset\Omega^0_{\mathrm{e}}$.

(**E7**) At any point in $\mathcal{F}_3\cap\mathcal{F}_5-\{(0,\frac{1}{\gamma},c),C\}$, $V'>0$ and $W'<0$. Thus, $\mathcal{F}_3\cap\mathcal{F}_5-\{(0,\frac{1}{\gamma},c),C\}\not\subset\Omega^0_{\mathrm{e}}$.

(**E8**) We have $U'>0$ and $W''<0$ on $\mathcal{F}_4\cap\mathcal{F}_5$, and hence $\mathcal{F}_4\cap\mathcal{F}_5\not\subset\Omega^0_{\mathrm{e}}$.

(**E9**) Since $V'<0$ and $W'<0$ on $\mathcal{F}_1\cap\mathcal{F}_6-\{O,(1,0,k)\}$, we have $\mathcal{F}_1\cap\mathcal{F}_6-\{O,(1,0,k)\}\subset\Omega^0_{\mathrm{se}}$. Here, $(1,0,k)\notin\Omega^0_{\mathrm{e}}$ by (E10).

(**E10**) Since $U'<0$ on $\mathcal{F}_2\cap\mathcal{F}_6$, $\mathcal{F}_2\cap\mathcal{F}_6\not\subset\Omega^0_{\mathrm{e}}$.

(**E11**) Noticing that $V'>0$ and $W'<0$ for any point in $\mathcal{F}_3\cap\mathcal{F}_7-\{(1,\frac{1}{\gamma},k),(0,\frac{1}{\gamma},k)\}$, we have $\mathcal{F}_3\cap\mathcal{F}_7-\{(1,\frac{1}{\gamma},k),(0,\frac{1}{\gamma},k)\}\subset\Omega^0_{se}$. In fact, $(1,\frac{1}{\gamma},k)\notin\Omega^0_{\mathrm{e}}$ from (E10).

(**E12**) Since $U'>0$ and $W''<0$ on $\mathcal{F}_4\cap\mathcal{F}_7-\{O\}$, we have $\mathcal{F}_4\cap\mathcal{F}_7\not\subset\Omega^0_{\mathrm{e}}$.

(**E13**) It is noticed that $U'<0$, $V'=0$, and $W'<0$ along $\mathcal{F}_6\cap\mathcal{F}_7-\{O,(1,\frac{1}{\gamma},k)\}$. We can derive that $\mathcal{F}_6\cap\mathcal{F}_7-\{O,(1,\frac{1}{\gamma},k)\}\subset\Omega^0_{\mathrm{se}}$.

The following lemma sums up our analysis in (F1)–(F7) and (E1)–(E13).

LEMMA 3.2. *Assume that conditions in Theorem 2.1 hold. Let $\Omega=\mathbb{R}^3-\{O,C\}$ and $\Omega^0\subset\Omega$ be an open set whose boundary is surrounded by seven faces $\mathcal{F}_1,\cdots,\mathcal{F}_7$ in (3.7). Then the set of strict egress points satisfies $\Omega^0_{\mathrm{e}}=\Omega^0_{\mathrm{se}}=\mathcal{S}_1\cup\mathcal{S}_2$, where disjoint sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are given in (3.10).*

3.3. Existence of a heteroclinic orbit. In the previous subsection, an open set $\Omega^0\subset\Omega=\mathbb{R}^3-\{O,C\}$, where O and C are given in (2.8), is constructed. Also, a set of strict egress points of Ω^0, denoted by Ω^0_{se}, is identified. In what follows, we aim to prove Theorem 2.1 by applying Theorem 2.3 to Lemma 3.2. To do this, we need two points P_1 and P_2 that satisfy the conditions in Theorem 2.3. The following two lemmas guarantee the existence of our desired P_1 and P_2.

LEMMA 3.3. *For $P\in\Omega$, let $\varphi_\xi(P)$ be a flow of the system (2.7). Let Ω^1 be an open subset of Ω^0 and be surrounded by the following faces:*

$$\begin{aligned}
\mathcal{F}_{1'}&:=\{(U,V,W)\ |\ \tfrac{1}{2}\le U\le 1,\ V=0,\ cU\le W\le c\}\subset\mathcal{F}_1,\\
\mathcal{F}_8&:=\{(U,V,W)\ |\ \tfrac{1}{2}\le U\le 1,\ 0\le V\le \tfrac{1}{\gamma}(2U-1),\ W=cU\},\\
\mathcal{F}_9&:=\{(U,V,W)\ |\ \tfrac{1}{2}\le U\le 1,\ V=\tfrac{1}{\gamma}(2U-1),\ cU\le W\le c\},\\
\mathcal{F}_{5'}&:=\{(U,V,W)\ |\ \tfrac{1}{2}\le U\le 1,\ 0\le V\le \tfrac{1}{\gamma}(2U-1),\ W=c\}\subset\mathcal{F}_5,
\end{aligned}$$

where $\mathcal{F}_1$ and $\mathcal{F}_5$ are given in (3.7).

(1) *For each $P\in\overline{\Omega^1}-\{O,C\}$, there is a $\xi_0>0$ such that $\varphi_{\xi_0}(P)\in\mathcal{F}_{1'}\cap\mathcal{S}_1$, where $\mathcal{S}_1$ is defined in (3.10).*

(2) *For any sufficiently small neighborhood $\mathcal{N}$ of C, there is a $P_1\in W^{\mathrm{u}}_{\mathrm{loc}}(C)\cap\Omega^1$ such that $\varphi_{\xi_1}(P_1)\in\mathcal{S}_1$ for some $\xi_1>0$, where $W^{\mathrm{u}}_{\mathrm{loc}}(C)$ is the local unstable manifold (relative to $\mathcal{N}$) at C.*

Proof. **(1)** It is noticed that there is no $P\in\Omega^1$ such that $\varphi_\xi(P)$ stays in Ω^1 for all $\xi>0$. Indeed, $V',W'<0$ in Ω^1 by (2.7) and any steady states of V and W are not contained in Ω^1, so $\varphi_\xi(P)$ must hit $\overline{\Omega^1}$ at $\xi=\xi_0$ for some $\xi_0>0$. Therefore, to complete the proof, it is sufficient to show that any trajectory initiating in Ω^1 transverses to only

$\mathcal{F}_1 \subset \mathcal{S}_1$. In other words, no trajectory starting in Ω^1 passes through the faces $\mathcal{F}_8$, $\mathcal{F}_9$ and $\mathcal{F}_{5'}$ forward in time.

Using $\vec{n}_{\mathcal{F}_8} = (c, 0, -1)^\top$ and (2.9), we derive

$$\vec{n}_{\mathcal{F}_8} \cdot (U', V', W') = U\Big(\chi(V)(\gamma V - U) + \mu(1-U)\Big) < 0$$

for any $(U,V,W) \in \mathcal{F}_8 - \{C\}$. It means that no trajectory starting in Ω^1 goes through $\mathcal{F}_8$ forward in time. With the outward normal vector $\vec{n}_{\mathcal{F}_9} = (-2, \gamma, 0)^\top$ of $\mathcal{F}_9$, we have

$$\vec{n}_{\mathcal{F}_9} \cdot (U', V', W') = -2(-cU + W) + \Big(-\frac{2}{c}U\chi(V) + \frac{\gamma}{c}\Big)(\gamma V - U) < 0$$

for any $(U,V,W) \in \mathcal{F}_9 - \{C\}$, where (2.9) is used. It concludes that no trajectory starting in Ω^1 transverses to $\mathcal{F}_9$ forward in time. Similarly, any trajectory initiating in Ω^1 cannot transverse to $\mathcal{F}_{5'}$ forward in time according to the analysis in (F5) in Section 3.2. Therefore, we conclude that if $P \in \Omega^1$, the flow $\varphi_\xi(P)$ must hit $\mathcal{F}_{1'} \cap \mathcal{S}_1$ at $\xi = \xi_0$ for some $\xi_0 > 0$, i.e., $\varphi_{\xi_0}(P) \in \mathcal{S}_1$.

(2) Since every trajectory starting in Ω^1 transverses to $\mathcal{S}_1$, it is enough to show that for any small neighborhood $\mathcal{N}$ of C, the local unstable manifold (relative to $\mathcal{N}$) at C satisfies $W^{\mathrm{u}}_{\mathrm{loc}}(C) \cap \Omega^1 \neq \emptyset$. That is, it suffices to show that the intersection of the tangent space of $W^{\mathrm{u}}_{\mathrm{loc}}(C)$ and Ω^1 is nonempty in $\mathcal{N}$.

In fact, $W^{\mathrm{u}}_{\mathrm{loc}}(C)$ is tangent to the following plane $\mathcal{P}$ at C (Figure 3.2(B)):

$$\mathcal{P} := \Big\{(U,V,W) \;\Big|\; W = c + \frac{\mu\gamma}{c\lambda_{c,1}\lambda_{c,2}}(U-1) - \frac{\mu(c\lambda_{c,1}-\gamma)(c\lambda_{c,2}-\gamma)}{c\lambda_{c,1}\lambda_{c,2}}\Big(V - \frac{1}{\gamma}\Big)\Big\}. \quad (3.11)$$

This plane $\mathcal{P}$ is the translated eigenspace with the eigenvectors $\vec{v}_{c,1}$ and $\vec{v}_{c,2}$ along the vector $\vec{C} = (1, \frac{1}{\gamma}, c)$, where $\vec{v}_{c,1}$ and $\vec{v}_{c,2}$ are given in (3.6). Moreover, using $\frac{\gamma}{2c} < \lambda_{c,2} < \frac{\gamma}{c}$ from (3.5), it is straightforward to show that in $\mathcal{P}$ there are two points

$$Q_1 = \Big(\frac{1}{2}, 0, q_1\Big) \qquad \text{and} \qquad Q_2 = (1, 0, q_2)$$

such that

$$q_1 < c \qquad \text{and} \qquad q_2 > c.$$

It follows that there is a point $Q = (q_3, 0, q_4) \in \mathcal{P}$, where

$$\frac{1}{2} \le q_3 \le 1 \qquad \text{and} \qquad cq_3 \le q_4 \le c,$$

which implies that $\overline{QC} \cap \mathcal{P} \cap \Omega^1 \neq \emptyset$ by the construction of Ω^1 (see Figure 3.2(B)). Therefore, we conclude that $W^{\mathrm{u}}_{\mathrm{loc}}(C) \cap \Omega^1 \neq \emptyset$ for any small neighborhood of C. In other words, for any sufficiently small neighborhood $\mathcal{N}$ at C we can choose a $P_1 \in W^{\mathrm{u}}_{\mathrm{loc}}(C) \cap \Omega^1$; moreover, there is a $\xi_1 > 0$ such that $\varphi_{\xi_1}(P_1) \in \mathcal{F}_{1'} \subset \mathcal{S}_1$ by (1). ∎

In a similar way to the proof of Lemma 3.3, we can also derive the following lemma, so we omit the proof.

LEMMA 3.4. *For $P \in \Omega$ let $\varphi_\xi(P)$ be a flow of the system (2.7). Let Ω^2 be an open subset of Ω^0 and be surrounded by $\mathcal{F}_{10}$, $\mathcal{F}_{11}$, $\mathcal{F}_7$, $\mathcal{F}_{4'}$ and $\mathcal{F}_3$, where $\mathcal{F}_{10}$, $\mathcal{F}_{11}$, and $\mathcal{F}_{4'}$ are defined as*

$$\mathcal{F}_{10} := \{(U,V,W) \mid 0 \le U \le 1,\ 0 \le W = c\gamma V \le c\},$$

$$\mathcal{F}_{11} := \{(U,V,W) \mid 0 \le U = V \le \tfrac{1}{\gamma},\ k\gamma V \le W \le c\gamma V\},$$
$$\mathcal{F}_{4'} := \{(U,V,W) \mid U = 0,\ 0 \le \gamma V \le 1,\ k\gamma V \le W \le c\gamma V\} \subset \mathcal{F}_4$$

and $\mathcal{F}_3, \mathcal{F}_4$ and $\mathcal{F}_7$, are given in (3.7). Here $0 < k < \frac{c}{2}$ is given in (3.8).

(1) For any $P \in \Omega^2 - \{O, C\}$, there is a $\xi_0 > 0$ such that $\varphi_{\xi_0}(P) \in \mathcal{F}_3 \cup \mathcal{F}_7 \cap \mathcal{S}_2$, where $\mathcal{S}_2$ is defined in (3.10).

(2) For any sufficiently small neighborhood $\mathcal{N}$ of C, there is a $P_2 \in W^{\mathrm{u}}_{\mathrm{loc}}(C) \cap \Omega^2$ such that $\varphi_{\xi_2}(P_2) \in \mathcal{S}_2$ for some $\xi_2 > 0$, where $W^{\mathrm{u}}_{\mathrm{loc}}(C)$ is the local unstable manifold (relative to $\mathcal{N}$) at C.

Now we choose a sufficiently small neighborhood $\mathcal{N}$ of C so that we can pick $P_1 \in W^{\mathrm{u}}_{\mathrm{loc}}(C) \cap \Omega^1$ and $P_2 \in W^{\mathrm{u}}_{\mathrm{loc}}(C) \cap \Omega^2$ satisfying Lemma 3.3 and Lemma 3.4, respectively. See Figure 3.2(B). In fact, the neighborhood $\mathcal{N}$ can be sufficiently small enough to have a smooth segment $\overline{P_1P_2} \subset W^{\mathrm{u}}_{\mathrm{loc}}(C) \cap \Omega^0$ in the neighborhood $\mathcal{N}$. Notice that $\mathcal{S}_1$ and $\mathcal{S}_2$ are disjoint subsets of the set of strict egress points, Ω^0_{se}. Then, by Theorem 2.3 there is a $P_3 \in \overline{P_1P_2} \subset W^{\mathrm{u}}_{\mathrm{loc}}(C) \cap \Omega^0$ such that $\varphi_\xi(P_3)$ remains in Ω^0 for $\xi > 0$. Furthermore, by the structure of Ω^0, the flow $\varphi_\xi(P_3)$ satisfies $W'(\xi) < 0$ for all $\xi \in (-\infty, \infty)$, and hence $\varphi_\xi(P_3)$ converges to O as $\xi \to \infty$. Therefore, $\varphi_\xi(P_3)$ is a heteroclinic orbit connecting O and C. In particular, by (3.2) and (3.5), O and C are hyperbolic for $c > c^*$, where c^* is given in (2.10). By the stable and unstable manifold theorems, we can conclude that the flow must converge O and C exponentially as $\xi \to \pm\infty$. On the other hand, O has a stable focus if $c < c^*$ by (3.2). It implies that there is no trajectory $(U,V,W)(\xi)$ such that $U \ge 0$ and $V \ge 0$ as $\xi \to \infty$. As a result, we have the following theorem.

THEOREM 3.1.

(1) *Assume that conditions in Theorem 2.1 hold and $c > c^*$, where c^* is given in (2.10). The system (2.7) has at least one solution $(U,V,W)(\xi)$, $\xi = x - ct$, such that $(U,V,W)(-\infty) = C$ and $(U,V,W)(\infty) = O$, where O and C are given in (2.8). Moreover, $(U,V,W)(\xi)$ converges to O and C exponentially as $\xi \to -\infty$ and $\xi \to \infty$.*

(2) *If $c < c^*$, there is no traveling wave solution with speed c that connects O and C.*

REMARK 3.1. Let (U,V,W) be a solution that is obtained from Theorem 3.1. Then, by the construction of Ω^1 in Lemma 3.3, there is no $\xi \in (-\infty, \infty)$ such that $cU(\xi) \le W(\xi)$ and $\gamma V(\xi) \le 2U(\xi) - 1$.

To complete the proof of Theorem 2.1, it remains to investigate the monotonicity of the solutions U and V.

COROLLARY 3.1. *Let $\chi(v)$ be a constant satisfying $0 < \chi(v) \equiv \mu \le 1$. Then the solution $(U,V,W)(\xi)$ that is obtained from Theorem 3.1 satisfies*

(i) $0 < W(\xi) \le cU(\xi) < c$

(ii) $0 < \gamma V(\xi) \le U(\xi) < 1$

for all $\xi \in (-\infty, \infty)$. Moreover, $U' \le 0$ and $V' \le 0$ for any $\xi \in (-\infty, \infty)$.

Proof. For a fixed c, let (U,V,W) be a solution obtained by Theorem 3.1. To prove (i), according to Remark 3.1, it suffices to prove that there is no $\xi \in (-\infty, \infty)$ satisfying both

$$cU(\xi) < W(\xi) \qquad \text{and} \qquad \gamma V(\xi) > 2U(\xi) - 1. \tag{3.12}$$

In what follows we prove (3.12) by contradiction. To derive a contradiction, we assume that there is $\xi_0 \in (-\infty,\infty)$ such that (3.12) holds. By Remark 3.1, we then have the following two possible cases:

(C1) the assumption (3.12) holds for all $\xi \in (-\infty,\xi_0]$,

(C2) there is a $\xi_1 < \xi_0$ such that $W(\xi_1) \le cU(\xi_1)$ and $\gamma V(\xi_1) > 2U(\xi_1) - 1$, and (3.12) holds for all $\xi \in (\xi_1,\xi_0]$.

If (C1) is true, using U' and W' in (2.7) and $\chi(v) \equiv \mu$ leads to

$$\begin{aligned} U' &= -cU + \frac{\mu}{c} U(\gamma V - U) + W \\ &> -cU + \frac{1}{c} W' + W \end{aligned} \tag{3.13}$$

for any $\xi \in (-\infty,\xi_0)$. Multiplying the above inequality by $e^{c\xi}$ yields

$$c(e^{c\xi}U(\xi))' > (e^{c\xi}W(\xi))'.$$

Furthermore, integrating the above estimate with respect to ξ over $(-\infty,\xi_0)$ and multiplying the result by $e^{-c\xi_0}$, we conclude

$$cU(\xi_0) - W(\xi_0) > 0.$$

This is a contradiction to our assumption $cU(\xi_0) < W(\xi_0)$. If (C2) is true, we derive from (3.13) for $\xi \in (\xi_1,\xi_0]$ that

$$cU(\xi_0) - W(\xi_0) > e^{c(\xi_1 - \xi_0)}(cU(\xi_1) - W(\xi_1)).$$

However, our assumption $cU(\xi_0) < W(\xi_0)$ implies $cU(\xi_1) < W(\xi_1)$, so this is a contradiction to the assumption of (C2). From the two cases (C1) and (C2), it is shown that (3.12) cannot be possible. By Remark 3.1 we conclude that $W(\xi) \le cU(\xi)$ for any $\xi \in (-\infty,\infty)$; the proof of (i) is completed.

Now we prove (ii) by contradiction. Assume to the contrary that there is a $\xi_1 \in (-\infty,\infty)$ such that

$$0 < U(\xi_1) < \gamma V(\xi_1) < 1. \tag{3.14}$$

It is noticed that for a sufficiently small $\varepsilon > 0$

$$(U - \gamma V)(\xi) = (U - \gamma V)(\xi_1) + \big((U' - \gamma V')(\xi_1)\big)(\xi - \xi_1) + O(|\xi - \xi_1|^2) \tag{3.15}$$

for $|\xi - \xi_1| < \varepsilon$. Applying the earlier result $W \le cU$, we can derive that

$$U' - \gamma V' = -cU + W + \frac{1}{c}\Big(U\chi(V) - \gamma\Big)(\gamma V - U) < 0$$

at $\xi = \xi_1$ by (2.9) and (3.14). It follows from (3.15) that $U(\xi) < \gamma V(\xi)$ for $\xi_1 \le \xi \le \xi_1 + \varepsilon$. In addition, the continuation argument leads to

$$U(\xi) < \gamma V(\xi), \qquad \xi \ge \xi_1.$$

Thus, by the equation of V' in (2.7)

$$V'(\xi) > 0, \qquad \xi \ge \xi_1. \tag{3.16}$$

However, since $V(\xi) \geq 0$ for any $\xi \in (-\infty, \infty)$ and $V \to 0$ as $\xi \to \infty$, (3.16) is not possible. Therefore, (3.14) is not true and the proof of (ii) is completed.

Applying (i) and (ii) to U' and V' in (2.7), we finally conclude that $U' \leq 0$ and $V' \leq 0$ for all $\xi \in (-\infty, \infty)$. ☐

As a consequence of Theorem 3.1 and Corollary 3.1, Theorem 2.1 is proved. Moreover, by using the transformations (2.2), (2.4) and (2.6), the proof of Theorem 1.1 is also completed only when $\alpha = 1$ (we fixed $\alpha = 1$ in Section 2). Now, define transforming variables

$$\bar{v}(x,t) := \frac{1}{\alpha} v(x,t), \qquad \bar{\chi}(\bar{v}) := \alpha \chi(\bar{v}). \tag{3.17}$$

Here, v and $\chi(v)$ are given in the system (1.1), where $\alpha = 1$ is assumed. Then, by (3.17), Theorem 1.1 can be proved for any $\alpha > 0$.

4. Proof of the existence of traveling waves for $\chi(v) < 0$

In this section we prove Theorem 2.2. In a similar way to the proof of Theorem 2.1 in Section 3, we first investigate the local behavior of solutions of the system (2.7) at the equilibrium points O and C given in (2.8). We then construct an open set Ω^0 and identify its strict egress points. In fact, since the open set Ω^0 built in Section 3 is sufficient to derive our desired result (Theorem 2.2), we use the same Ω^0 for this section. However, the behavior of trajectories in the Ω^0 differs from those of the case where $\chi(v) > 0$. In turn, with the same construction of Ω^0 as in Section 3, more restrictions on $\gamma, \chi(v)$ and c are required especially when we show that all egress points must be strict egress points (e.g., see Remark 4.2). Also, (strict) egress points are different from those in Section 3. We also employ Theorem 2.3 to show the existence of a heteroclinic orbit of (2.7) connecting O and C. Techniques and details in the following subsections are analogous to those from Sections 3.1 and 3.2.

4.1. Linearized system. At the equilibrium points O and C of the system (2.7), the Jacobian matrices are given in (3.1) and (3.3) respectively. In particular, at C, the Jacobian matrix (3.3) has the characteristic polynomial $p(\lambda)$ given in (3.4) and $p(\lambda)$ has the following roots: one negative eigenvalue $\lambda_{c,-}$ and two positive eigenvalues $\lambda_{c,1}$ and $\lambda_{c,2}$ satisfying

$$\lambda_{\mathrm{c},-} < -c < 0 < \frac{\mu}{2c} < \lambda_{\mathrm{c},1} < \frac{\mu}{c} < \frac{\gamma}{c} < \lambda_{\mathrm{c},2} < \frac{\gamma - \chi(1/\gamma)}{c} \tag{4.1}$$

(see Figure 4.1). Here, the first and fourth inequalities can be shown by (2.12) and (2.13). The eigenvectors associated with the positive eigenvalues satisfy (3.6). In conclusion, under the assumptions in Theorem 2.2, the system (2.7) has a 2-dimensional local stable manifold at O and a 2-dimensional local unstable manifold at C, denoted by $W^{\mathrm{s}}_{\mathrm{loc}}(O)$ and $W^{\mathrm{u}}_{\mathrm{loc}}(C)$, respectively.

4.2. Construction of an open set containing an invariant set. Open set $\Omega^0 \subset \Omega = \mathbb{R}^3 - \{O, C\}$ is given in Section 3.2. In a similar way to identifying the sets of egress points and strict egress points of Ω^0, denoted by Ω^0_{e} and Ω^0_{se}, in Section 3.2, we investigate Ω^0_{e} and Ω^0_{se} with respect to the system (2.7) for $\chi(v) < 0$ in the following lemma (see Figure 4.2(A) for the sketch of Ω^0 and $\Omega_{\mathrm{e}} = \Omega^0_{\mathrm{se}}$).

LEMMA 4.1. *Assume that the conditions in Theorem 2.2 hold. Let $\Omega = \mathbb{R} - \{O, C\}$ and $\Omega^0 \subset \Omega$ be an open set whose boundary is surrounded by faces $\mathcal{F}_1, \cdots, \mathcal{F}_7$ given in (3.7). Then the (strict) egress point set of Ω^0 satisfies*

$$\Omega^0_{\mathrm{e}} = \Omega^0_{\mathrm{se}} = \mathcal{S}_1 \cup \mathcal{S}_2,$$

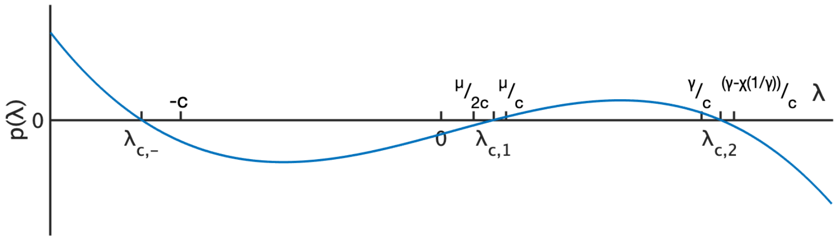

FIG. 4.1. *Graph of characteristic polynomial* $p(\lambda)$ *in (3.4) for* $\chi(v)<0$

where the disjoint sets $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *are defined as*

$$\begin{aligned}\mathcal{S}_1 &:= \mathcal{F}_1\cup\mathcal{F}_2-\Big((\mathcal{F}_1\cap\mathcal{F}_4)\cup\mathcal{F}_{2'}\Big)\\ \mathcal{S}_2 &:= (\mathcal{F}_3\cup\mathcal{F}_7)-\Big((\mathcal{F}_2\cap\mathcal{F}_3)\cup(\mathcal{F}_3\cap\mathcal{F}_4)\cup(\mathcal{F}_3\cap\mathcal{F}_5)\cup(\mathcal{F}_4\cap\mathcal{F}_7)\Big),\end{aligned} \tag{4.2}$$

with

$$\mathcal{F}_{2'} := \{(1,V,W)\ |\ 0\le\gamma V\le 1,\ W\le c-\frac{\chi(V)}{c}(\gamma V-1)\}\subset\mathcal{F}_2. \tag{4.3}$$

Proof. The construction of Ω^0 is given in Section 3, where $\chi(v)>0$ is assumed, so the following proof is similar to that of Theorem 3.2. In fact, under the assumption of $\chi(v)>0$, it is enough to examine the faces $\mathcal{F}_2$ and $\mathcal{F}_6$, edges $\mathcal{F}_1\cap\mathcal{F}_2$, $\mathcal{F}_2\cap\mathcal{F}_5$ and $\mathcal{F}_2\cap\mathcal{F}_6$. Hence, only the items (F2), (F6), (E2), (E6) and (E10) in the proof of Theorem 3.2 will be changed as follows:

(F2) For any $(U,V,W)\in\mathrm{Int}(\mathcal{F}_{2'})$, $\vec{n}_{\mathcal{F}_{2'}}\cdot(U',V',W')=-cU+W+\frac{1}{c}U\chi(V)(\gamma V-U)<0$. On the one hand $\mathrm{Int}(\mathcal{F}_{2'})\not\subset\Omega^0_{\mathrm{e}}$. On the other hand, any $(U,V,W)\in\mathrm{Int}(\mathcal{F}_2)$ satisfying $W>c-\frac{\chi(V)}{c}(\gamma V-1)$ is contained in Ω^0_{se}. In what follows, we examine points contained in

$$\{(1,V,W)\ |\ 0<\gamma V<1,\ W=c-\frac{\chi(V)}{c}(\gamma V-1)\}, \tag{4.4}$$

which is contained in the boundary of the face $\mathcal{F}_{2'}$ in (4.3). To show that any points in (4.4) are not included in Ω^0, it suffices to have $U''>0$. Indeed, by (2.11) we have $U''=\frac{1}{c^2}\chi'(V)(\gamma V-U)^2+\frac{\gamma}{c}U\chi(V)(\gamma V-1)>0$ for any point satisfying (4.4). Then, by the approximation to the Taylor polynomials, all points satisfying (4.4) are not in Ω^0_{e}.

(F6) Note that $\vec{n}_{\mathcal{F}_6}\cdot(U',V',W')=U(\frac{k}{c}\chi(V)(\gamma V-U)-\mu U)$. Here, by $|\chi(v)|<2\mu$ in (2.12), $k|\chi(V)|<\mu c$. Therefore, $\vec{n}_{\mathcal{F}_6}<0$ except for $U=0$, and $\mathrm{Int}(\mathcal{F}_6)\not\subset\Omega^0_{\mathrm{e}}$.

(E2) By (4.5) and $0<k<\frac{c}{2}$, for any point in $\mathcal{F}_1\cap\mathcal{F}_2$ satisfies $U'<0$. Thus, $\mathcal{F}_1\cap\mathcal{F}_2\not\subset\Omega^0_{\mathrm{e}}$.

(E6) For $\chi(v)<0$, we have $W'=0$ and $W''>0$ for any points in $\mathcal{F}_2\cap\mathcal{F}_5-\{C\}$. Hence, $\mathcal{F}_2\cap\mathcal{F}_5\not\subset\Omega^0_{\mathrm{e}}$.

(E10) Using (4.5) and $0<k<\frac{c}{2}$, $U'<0$ for any point in $\mathcal{F}_2\cap\mathcal{F}_6$, so $\mathcal{F}_2\cap\mathcal{F}_6\not\subset\Omega^0_{\mathrm{e}}$.

∎

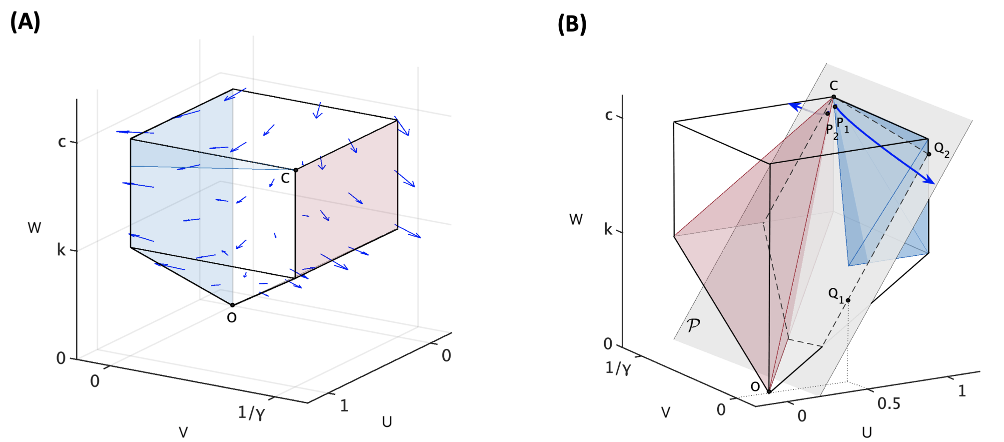

FIG. 4.2. *The case when* $\mu=\frac{1}{4},\gamma=1,c=1$ *and* $\chi(v)=-\frac{1}{8}$. *(A) Sketch of* Ω^0 *with* $\Omega^0_{\rm e}=\mathcal{S}_1\cup\mathcal{S}_2$ *(*$\mathcal{S}_1$*: blue face,* $\mathcal{S}_2$*: red face) and a vector field (a collection of blue arrows) on* Ω^0 *associated with the system (2.7) for* $\chi(v)<0$*. Two points* O *and* C *are the equilibrium points. (B) Sketch of* Ω^0 *with* Ω^1 *(blue tetrahedron) and* Ω^2 *(red tetrahedron). The dashed polygon is the intersection of* Ω^0 *and the plane* $\mathcal{P}$ *(gray plane). The flows of (2.7) initiated at* P_1 *and* P_2 *(blue trajectories) pass through* Ω^1 *and* Ω^2*, respectively.*

REMARK 4.1. For any $0\le v\le\frac{1}{\gamma}$, (2.12) and (2.13) lead to

$$c^2\ge 2|\chi(v)|, \tag{4.5}$$

and we have

$$k<c-\frac{1}{c}\chi(V)(\gamma V-1)<c, \tag{4.6}$$

where k is given in (3.8). The estimate (4.6) allows $\mathcal{S}_1$ to be contractible. The contractibility of $\mathcal{S}_1$ is required to apply Theorem 2.3 (the principle of Ważeski) with Ω^0.

REMARK 4.2. The condition $\chi'(v)\ge 0$ in (2.11) is sufficient to show $\Omega^0_{\rm e}=\Omega^0_{\rm se}$, which is not required in the case where $\chi(v)>0$. The details can be found in (F2) given in the proof of Lemma 4.1.

4.3. Existence of a heteroclinic orbit. Continuing the previous subsection, it remains to choose points P_1 and P_2 that satisfy the conditions in Theorem 2.3, and these points are selected in the following lemma. Applying Lemma 4.2 to Theorem 2.3 (the principle of Ważeski), Theorem 2.2 is proved.

LEMMA 4.2. *Assume that the conditions in Lemma 4.1 hold. Let* $\varphi_\xi(P)$ *be a flow of the system (2.7). Let* $\Omega^0=\mathbb{R}-\{O,C\}$*, where* O *and* C *are given in (2.8).*
(1) *Let* Ω^1 *be an open subset of* Ω^0 *surrounded by four faces* $\mathcal{F}_{1''}$*,* $\mathcal{F}_{2''}$*,* $\mathcal{F}_a$*, and* $\mathcal{F}_b$*, where*

$$\mathcal{F}_{1''}:=\left\{(U,0,W)\ \middle|\ \tfrac{1}{2}\le U\le 1,\ \tfrac{c}{2}\le W\le cU\right\}\subset\mathcal{F}_1,$$

$$\mathcal{F}_{2''}:=\left\{(1,V,W)\ \middle|\ 0\le\gamma V\le 1,\ \tfrac{c}{2}\big(\gamma V+1\big)\le W\le c\right\}\subset\mathcal{F}_{2'}\subset\mathcal{F}_2,$$

$$\mathcal{F}_a:=\left\{(U,V,W)\ \middle|\ \tfrac{1}{2}\le U\le 1,\ 0\le V\le\tfrac{1}{\gamma}(2U-1),\ W=\tfrac{c}{2}\big(\gamma V+1\big)\right\},$$

$$\mathcal{F}_b := \left\{(U,V,W) \mid \tfrac{1}{2} \le U \le 1,\ 0 \le V \le \tfrac{1}{\gamma}(2U-1),\ W = cU\right\},$$

where $\mathcal{F}_1, \mathcal{F}_{2'}$ and $\mathcal{F}_2$ are defined in (3.7) and (4.3), respectively. Then, for any $P \in \overline{\Omega^1} - \{C\}$ there exists a $\xi_0 > 0$ such that $\varphi_{\xi_0}(P) \in \mathcal{S}_1$, where $\mathcal{S}_1$ is given in (4.2). Moreover, there is a point $P_1 \in W^{\mathrm{u}}_{\mathrm{loc}}(C) \cap \Omega^1$ such that $\varphi_{\xi_1}(P_1) \in \mathcal{S}_1$ for some $\xi_1 > 0$.
(2) Let Ω^2 be an open subset of Ω^0 surrounded by four faces $\mathcal{F}_7$, $\mathcal{F}_c$, $\mathcal{F}_d$ and $\mathcal{F}_{3''}$ such as

$$\mathcal{F}_c := \{(U,V,W) \mid 0 \le U = \gamma V \le 1,\ kU \le W \le cU\},$$
$$\mathcal{F}_d := \{(U,V,W) \mid 0 \le cU = W \le 1,\ 0 \le V \le \tfrac{1}{\gamma}\},$$
$$\mathcal{F}_{3''} := \left\{\left(U,\tfrac{1}{\gamma},W\right) \mid 0 \le U \le 1,\ k \le W \le cU\right\} \subset \mathcal{F}_3,$$

where $\mathcal{F}_7$ and $\mathcal{F}_3$ are given in (3.7) and $0 < k < \frac{c}{2}$ is defined in (3.8). Then, for any $P \in \overline{\Omega^2} - \{O, C\}$ the flow $\varphi_\xi(P)$ hits $\mathcal{S}_2$ at some $\xi = \xi_0 > 0$, where $\mathcal{S}_2$ is given in (4.2). In addition, there exists a point $P_2 \in \Omega^2 \cap W^{\mathrm{u}}_{\mathrm{loc}}(C)$ satisfying $\varphi_{\xi_2}(P_2) \in \mathcal{S}_2$ for some $\xi_2 > 0$.

Proof. **(1)** In a similar way to deriving Lemma 3.3(a), we can show that any trajectory initiating in $\overline{\Omega^1} - \{C\}$ transverses to the boundary of $\mathcal{F}_{1''} \cup \mathcal{F}_{2''}$, where (4.5) and $2\mu \le \gamma$ from (2.12) are used. It remains to prove the existence of $P_1 \in W^{\mathrm{u}}_{\mathrm{loc}}(C)$ satisfying $\varphi_{\xi_1}(P_1) \in \mathcal{S}_1$ for some $\xi_1 > 0$. In an analogous manner for Lemma 3.3(b), we aim to show that $W^{\mathrm{u}}_{\mathrm{loc}}(C) \cap \Omega^1 \ne \emptyset$ by using the translated eigenspace $\mathcal{P}$ in (3.11) with eigenvectors $\vec{v}_{\mathrm{c},1}$ and $\vec{v}_{\mathrm{c},2}$ along $\vec{C} = (1, 1/\gamma, c)$. In fact, we can derive from $\lambda_{\mathrm{c},-} < -c$ in (4.1) and the characteristic polynomial $p(\lambda)$ in (3.4) for $\chi(v) < 0$ that

$$0 < \lambda_{\mathrm{c},1}\lambda_{\mathrm{c},2} < \frac{\gamma\mu}{c^2} \qquad \text{and} \qquad c(\lambda_{\mathrm{c},1} + \lambda_{\mathrm{c},2}) < \gamma + \mu + \lambda_{\mathrm{c},1}\lambda_{\mathrm{c},2}, \tag{4.7}$$

where $\lambda_{\mathrm{c},-}$ is a negative eigenvalue and $\lambda_{\mathrm{c},1}$ and $\lambda_{\mathrm{c},2}$ are positive eigenvalues of (3.3). By using $2\mu \le \gamma$ from (2.12) and $c^2 \ge 4\mu$ from (2.13), (4.1) and (4.7), we can show the existence of $(\frac{1}{2}, 0, q_1), (1, 0, q_2) \in \mathcal{P}$ satisfying

$$0 < q_1 < \frac{c}{2} \qquad \text{and} \qquad \frac{c}{2} < q_2 < c. \tag{4.8}$$

The details in finding $(\frac{1}{2}, 0, q_1)$ and $(1, 0, q_2)$ are as follows. Let $(\frac{1}{2}, 0, q_1) \in \mathcal{P}$, where

$$q_1 = c - \frac{\mu\gamma}{2c\lambda_{\mathrm{c},1}\lambda_{\mathrm{c},2}} + \frac{\mu(c\lambda_{\mathrm{c},1} - \gamma)(c\lambda_{\mathrm{c},2} - \gamma)}{c\gamma\lambda_{\mathrm{c},1}\lambda_{\mathrm{c},2}} \tag{4.9}$$

by $\mathcal{P}$ in (3.11). We shall show that $q_1 < \frac{c}{2}$. In fact, we have

$$q_1 < \frac{c}{2} \Longleftrightarrow \frac{\mu(c\lambda_{\mathrm{c},1} - \gamma)(c\lambda_{\mathrm{c},2} - \gamma)}{c\gamma\lambda_{\mathrm{c},1}\lambda_{\mathrm{c},2}} < -\frac{c}{2} + \frac{\mu\gamma}{2c\lambda_{\mathrm{c},1}\lambda_{\mathrm{c},2}},$$

and

$$\frac{\mu(c\lambda_{\mathrm{c},1} - \gamma)(c\lambda_{\mathrm{c},2} - \gamma)}{c\gamma\lambda_{\mathrm{c},1}\lambda_{\mathrm{c},2}} < 0 < -\frac{c}{2} + \frac{\mu\gamma}{2c\lambda_{\mathrm{c},1}\lambda_{\mathrm{c},2}}$$

holds by (4.1) and (4.7). Thus, $\mathcal{P}$ contains a point $(\frac{1}{2}, 0, q_1)$ such that $q_1 < \frac{c}{2}$. Let $(1, 0, q_2) \in \mathcal{P}$, where

$$q_2 = c + \frac{\mu(c\lambda_1 - \gamma)(c\lambda_2 - \gamma)}{c\gamma\lambda_1\lambda_2}.$$

By (4.1), $q_2 < c$. To show that $q_2 > \frac{c}{2}$, i.e.,

$$q_2 > \frac{c}{2} \Longleftrightarrow c\mu\gamma(\lambda_1+\lambda_2) < \mu c^2\lambda_1\lambda_2 + \mu\gamma^2 + \frac{c^2}{2}\gamma\lambda_1\lambda_2, \tag{4.10}$$

it is enough to prove the second inequality in (4.10). In fact, by (4.7)

$$c\mu\gamma(\lambda_1+\lambda_2) < \mu\gamma^2 + \mu^2\gamma + \mu\gamma\lambda_1\lambda_2,$$

by $4\mu \le \gamma$ from (2.12) and (4.1)

$$\mu^2\gamma < 2\mu c^2\lambda_1\lambda_2 \le \mu c^2\lambda_1\lambda_2 + \frac{c^2}{4}\gamma\lambda_1\lambda_2,$$

and by (2.13)

$$\mu\gamma\lambda_1\lambda_2 \le \frac{c^2}{4}\lambda_1\lambda_2.$$

Therefore, the second inequality in (4.10) holds.

Since $\mathcal{P}$ contains two points $(\frac{1}{2}, 0, q_1)$ and $(1, 0, q_2)$ satisfying (4.8), the tangent space $\mathcal{P}$ must intersect with $\mathcal{F}_{2''}$ and Ω^1. As a result, we can deduce that $W^{\mathrm{u}}_{\mathrm{loc}}(C) \cap \mathcal{F}_{2''} \neq \emptyset$ in a sufficiently small neighborhood of C. Therefore, $W^{\mathrm{u}}_{\mathrm{loc}}(C) \cap \Omega^1 \neq \emptyset$.

(2) Similarly, it is straightforward to show that any trajectory initiating in $\overline{\Omega^2} - \{C\}$ must hit the boundary of $\mathcal{F}_{3''} \cup \mathcal{F}_7$. Furthermore, since $W^{\mathrm{u}}_{\mathrm{loc}}(C) \cap \mathcal{F}_{3''} \neq \emptyset$, the desired proof is completed. ∎

Note that two points P_1 and P_2 found in Lemma 4.2 satisfy the conditions of Theorem 2.3 (see Figure 4.2(B)). It implies that there is a point $P_3 \in W^{\mathrm{u}}_{\mathrm{loc}}(C) \cap \Omega^0$ such that $\varphi_\xi(P_3) \in \Omega^0$ for all $\xi > 0$. According to the structure of Ω^0, the flow $\varphi_\xi(P_3)$ satisfies $W' < 0$, $\varphi_\xi(P_3)$ must converge to O as $\xi \to \infty$. Consequently, $\varphi_\xi(P_3)$ is a heteroclinic orbit connecting O and C. In particular, note that by (3.2) and (4.1), O and C are hyperbolic for $c > c^*$, where c^* is given in (2.13). By the stable and unstable manifold theorems, we can deduce the exponential decay of U and V as $\xi \to \pm\infty$. However, O has a stable focus if $c < 2\sqrt{\mu} \le c^*$. Hence, there is no traveling wave connecting O and C if $c < 2\sqrt{\mu}$.

THEOREM 4.1. (1) *Assume that conditions in Theorem 2.2 and $c > c^*$, where c^* is given in (2.13). The system (2.7) has a solution $(U,V,W)(\xi)$, $\xi = x - ct$, satisfying $(U,V,W)(-\infty) = C$ and $(U,V,W)(\infty) = O$, where O and C are defined in (2.8). Furthermore, $(U,V,W)(\xi)$ converges to O and C exponentially as $\xi \to -\infty$ and $\xi \to \infty$, respectively.*

(2) *There is no traveling wave solution connecting O and C with speed c if $c < 2\sqrt{\mu}$.*

REMARK 4.3. If (U,V,W) is a solution obtained from Theorem 4.1, there is no $\xi_0 \in (-\infty, \infty)$ satisfying $(U,V,W)(\xi_0) \in \Omega^1 \cup \Omega^2$ according to the construction of Ω^1 and Ω^2 and Lemma 4.2.

In the following corollary, we can further explore the monotonicity of V.

COROLLARY 4.1. *Let $(U,V,W)(\xi)$, $\xi = x - ct$, be a solution of the system (2.7) that is obtained from Theorem 4.1. For any $\xi \in (-\infty, \infty)$, $\gamma V(\xi) \le U(\xi)$ and $V'(\xi) \le 0$.*

Proof. To begin, we note that $\gamma V(\xi) - U(\xi) > 0$ is equivalent to $V'(\xi) > 0$ by V' in (2.7). In the following, we use proof by contradiction. Assume to the contrary that

$V'(\xi)>0$ for some ξ. Then, since $0\le V\le \frac{1}{\gamma}$ and $V(-\infty)=\frac{1}{\gamma}$ by Theorem 4.1, there is a $\xi_0\in(-\infty,\infty)$ such that $V'(\xi_0)=0$ and $V'(\xi)>0$ for $\xi_0\le\xi\le\xi_0+\varepsilon$ for some $\varepsilon>0$. Furthermore, by Remark 4.3 the solution (U,V,W) satisfies either $W(\xi_0)=cU(\xi_0)$ or $W(\xi_0)>cU(\xi_0)$.

In the case of $W(\xi_0)=cU(\xi_0)$, it follows from U' in (2.7) that $U'(\xi_0)=0$, and hence for sufficiently small $|\xi-\xi_0|$ we have

$$W(\xi)-cU(\xi)=W'(\xi_0)(\xi-\xi_0)+O(|\xi-\xi_0|^2). \tag{4.11}$$

Noticing that $W'(\xi_0)=\mu U(\xi_0)(U(\xi_0)-1)<0$ by Theorem 4.1, $W(\xi)<cU(\xi)$ for $\xi>\xi_0$, where $|\xi-\xi_0|$ is sufficiently small, by (4.11). This contradicts to $W(\xi)\ge cV(\xi)$ by Remark 4.3. In the case of $W(\xi_0)>cU(\xi_0)$, we obtain

$$V'(\xi)=V''(\xi_0)(\xi-\xi_0)+O(|\xi-\xi|^3)$$

for sufficiently small $|\xi-\xi_0|$. Here $V''(\xi_0)=-\frac{1}{c}U'(\xi_0)<0$, where $W(\xi_0)\ge cU(\xi_0)$, by Remark 4.3, is used. It implies that if $\varepsilon>0$ is sufficiently small, $V'(\xi)<0$ for all $\xi_0<\xi<\xi_0+\varepsilon$, which is a contradiction. Therefore, $V'(\xi)\le 0$ for any $\xi\in(-\infty,\infty)$. In other words, $\gamma V(\xi)\le U(\xi)$ for all $\xi\in(-\infty,\infty)$. □

Theorem 2.2 is finally proved by Theorem 4.1 and Corollary 4.1. Furthermore, by transforming variables (2.2), (2.4), (2.6) and (3.17), the proof of Theorem 1.2 is completed.

5. Non-existence of traveling wave solution

In this section, we prove Theorem 1.3 by showing that there is no traveling wave solution to the system (2.5), where transforms (2.2) and (2.4) are used. According to Theorem 1.3 (1) and (2), we present the proof by considering the following four cases: (i) $\alpha=0$ and $\beta\ne 0$, (ii) $\alpha\ne 0$ and $\beta=0$, (iii) $\alpha=\beta=0$, and (iv) $\alpha\beta\ne 0$.

Proof. (**Proof of Theorem 1.3.**)

(i) Assuming that $\alpha=0$ and $c\ge 0$, the system (2.5) has two steady states $(u_\pm,v_\pm)\in\{(0,0),(1,0)\}$. It implies that traveling wave V must be a pulse connecting $(0,0)$ and non-negative if a traveling wave exists. However, if $\alpha=0$ and $c\ge 0$, the second equation of (2.5) implies that V must be trivial, which is a contradiction.

(ii) Let $\alpha\ne 0$, $\beta=0$ and $c>0$. By setting a new variable W in (2.6) and using the second equation of (2.5) for V' with $\beta=0$, we have

$$\begin{cases} U'=-cU-\frac{\alpha}{c}U^2\chi(V)+W \\ V'=-\frac{\alpha}{c}U \\ W'=\mu U(U-1). \end{cases} \tag{5.1}$$

Here, (5.1) has a continuum of equilibria $(0,v_\pm,0)$ for $v_\pm\ge 0$.

In the following, by assuming that (5.1) has a solution connecting two steady states $(0,v_\pm,0)$, we derive a contradiction. If (5.1) admits a traveling wave solution connecting $(0,v_\pm,0)$, then

$$0\le V(\xi)\le m, \qquad \forall\xi\in(-\infty,\infty) \tag{5.2}$$

for some $m\ge 0$ due to continuity of V. Now define a C^1 function $L(\xi)$ as

$$L(\xi)=L(U(\xi),W(\xi))=\frac{1}{2c}U^2+\frac{1}{2c\mu}W^2\ge 0$$

for $\xi\in(-\infty,\infty)$, where $L(U,V,W)=0$ if and only if $U=W=0$. By setting

$$\mathcal{B}:=\Big\{(U,V,W)\,\Big|\,\big(\max_{0\le V\le m}|\chi(V)|\big)U\le\frac{c^2}{4|\alpha|},\quad 0\le V\le m,\quad |W|\le\frac{c}{4}\Big\},$$

where $m\ge 0$ is given in (5.2), it follows

$$\frac{dL}{d\xi}\le -\frac{1}{2}U^2+U^2\Big(-\frac{1}{2}+\frac{|\alpha|}{c^2}U|\chi(V)|+\frac{|W|}{c}\Big)\le -\frac{1}{2}U^2 \tag{5.3}$$

for any $(U,V,W)\in\mathcal{B}$. Furthermore, noticing that U and W are non-trivial pulses connecting 0, there is a ξ_0 satisfying

$$\begin{cases}0\le\big(\max_{0\le V\le m}|\chi(V)|\big)U(\xi)\le\big(\max_{0\le V\le m}|\chi(V)|\big)U(\xi)<\dfrac{c^2}{8|\alpha|}, & \forall\xi<\xi_0,\\ 0\le|W(\xi)|\le|W(\xi_0)|<\dfrac{c}{8}, \qquad \forall\xi<\xi_0, & \\ U(\xi_0)>0 \quad\text{or}\quad |W(\xi_0)|>0. & \end{cases} \tag{5.4}$$

Then, for any $(U,V,W)\in\mathcal{B}$ and for $\xi<\xi_0$, (5.3) and (5.4) imply that

$$0=L(-\infty)\ge L(\xi)\ge L(\xi_0)\ge\frac{1}{2c}U^2(\xi_0)+\frac{1}{2c\mu}W^2(\xi_0)>0,$$

which is a contradiction.

(iii) Assume to the contrary that there is a non-trivial traveling wave solution (U,V) whose speed is $c=0$ and that satisfies $0\le U\le 1$. By using a variable W in (2.6), the system (2.5) is reduced to

$$U'=W,\qquad V'=0,\qquad W'=\mu U(U-1)$$

and W must be a pulse connecting 0. However, by the assumption $0\le U\le 1$, $W'\le 0$ for any $\xi\in(-\infty,\infty)$. Therefore, U must be 0 or 1, which are trivial solutions. It contradicts our assumption.

(iv) In a similar way to showing (iii), the proof can be completed. ◻

6. Spectral instability of the traveling wave solutions

Suppose that $\{\vec{\phi}_s:=(U,V)(x-st)\}$ is the family of traveling wave solutions of the system (1.1) that is obtained from Theorem 1.1 and Theorem 1.2. In this section we examine the stability of the traveling wave solutions against perturbation in a certain space; we particularly verify the spectral instability of the family of traveling wave solutions in the unweighted L^2 space.

In fact, it is easy to see that the family of traveling wave solutions is nonlinearly unstable in L^2 norm since their speeds s are not isolated, i.e., there is a sequence $\{t_n\}$ such that $\|\vec{\phi}_s(\cdot-st_n)-\vec{\phi}_{s'}(\cdot-s't_n))\|\ge\varepsilon_0$ for a given $\varepsilon_0>0$. Here, we note that the nonlinear and spectral instability do not guarantee the linear instability, and the linear stability and instability are still unknown. Different from a parabolic-parabolic system, where the linearlized operator generates an analytic semigroup, our system (1.1) is partially parabolic as there is no chemical diffusion term; in turn, one cannot directly apply stability or instability theories based on analytic semigroup theories as in [16, 21, 26, 27, 33].

In what follows, we prove the spectral instability of the traveling wave solutions by investigating the essential spectrum of linear operator of the perturbation between a traveling wave solution and a solution to the Cauchy problem (1.1), (1.2). More details can be found in [16, p. 136]. By observing the loci of the essential spectrum, we confirm that the logistic cell growth rate $\mu>0$ determines the spectral instability regardless of the rates of chemical growth $\alpha>0$ and degradation $\beta<0$.

Consider the Cauchy problem (1.1), (1.2) in the moving coordinate $\xi=(x-st,t)$

$$\begin{cases} u_t = Du_{\xi\xi}+su_\xi-\left(u\chi(v)v_\xi\right)_\xi+\mu u(1-u)\\ v_t = sv_\xi+\alpha u+\beta v\end{cases} \tag{6.1}$$

with initial data

$$(u,v)(\xi,0)\to(u_\pm,v_\pm)=\begin{cases}(0,0) & \text{as} \quad \xi\to\infty\\ (1,-\frac{\beta}{\alpha}) & \text{as} \quad \xi\to-\infty.\end{cases}$$

Here $(u,v)(\xi,0)$ can be considered to form

$$(u,v)(\xi,0)=(U,V)(\xi)+(\psi,\eta)(\xi,0), \tag{6.2}$$

where $(U,V)(\xi)$ is a traveling wave solution for any $s>s^*$ obtained from Theorem 1.1 and Theorem 1.2. That is, $(\psi,\eta)(\xi,t)$ is the perturbation $(u-U,v-V)$. Then the solution to the Cauchy problem (1.1), (6.2) can be written as

$$(u,v)(\xi,t)=(U,V)(\xi)+(\psi,\eta)(\xi,t). \tag{6.3}$$

Assume that $\chi(v)\in C^3$ satisfies conditions in Theorem 1.1 and Theorem 1.2. By reformulating the problem (1.1), (6.2) in terms of the perturbation (ψ,η) by (6.1) and (2.1) and linearizing the resulting problem, we obtain

$$\begin{pmatrix}\varphi_t\\ \psi_t\end{pmatrix}=\mathcal{L}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}, \tag{6.4}$$

where $\mathcal{L}$ denotes

$$\begin{pmatrix} D\frac{\partial^2}{\partial\xi^2}+A_1(\xi)\frac{\partial}{\partial\xi}+A_2(\xi) & -U\chi(V)\frac{\partial^2}{\partial\xi^2}+A_3(\xi)\frac{\partial}{\partial\xi}+A_4(\xi)\\ \alpha & s\frac{\partial}{\partial\xi}+\beta\end{pmatrix} \tag{6.5}$$

with

$$\begin{aligned} A_1(\xi)&=s-\chi(V)V',\\ A_2(\xi)&=-\chi'(V)V'^2-\chi(V)V''+\mu(1-2U),\\ A_3(\xi)&=-U'\chi(V)-2UV'\chi'(V),\\ A_4(\xi)&=-U'V'\chi'(V)-UV'^2\chi''(V)-UV''\chi'(V).\end{aligned}$$

It is noticed that $\mathcal{L}:H^2(\mathbb{R})\times H^2(\mathbb{R})\to X$, where $\mathcal{L}$ is given in (6.5) and $X=L^2(\mathbb{R})\times L^2(\mathbb{R})$, is a closed operator in the space X. With

$$A_1(\pm\infty)=s,\qquad A_2(\pm\infty)=\mu(1-2u_\pm),\qquad A_3(\pm\infty)=A_4(\pm\infty)=0,$$

the asymptotic operator $\mathcal{L}_\pm$ at $\xi=\pm\infty$ becomes

$$\mathcal{L}_\pm=\begin{pmatrix} D\frac{\partial^2}{\partial\xi^2}+s\frac{\partial}{\partial\xi}+\mu(1-2u_\pm) & -u_\pm\chi(v_\pm)\frac{\partial^2}{\partial\xi^2}\\ \alpha & s\frac{\partial}{\partial\xi}+\beta\end{pmatrix}. \tag{6.6}$$

Letting

$$A^{\pm}(\tau) := \begin{pmatrix} -D\tau^2 + s\tau i + \mu(1-2u_{\pm}) & u_{\pm}\chi(v_{\pm})\tau^2 \\ \alpha & s\tau i + \beta \end{pmatrix}, \tag{6.7}$$

it follows from the spectral theory in [16] that the boundary of essential spectrum $\sigma_{ess}(\mathcal{L})$ is described by the curves $S^+ \cup S^-$, where

$$S^{\pm} = \{\lambda \in \mathbb{C} \mid \det(A_{\pm}(\tau) - \lambda I) = 0 \text{ for some } \tau \in \mathbb{R}\}. \tag{6.8}$$

If we particularly evaluate $(u_+, v_+) = (0,0)$ into $A^+(\tau)$ in (6.7), $\lambda \in S^+$ are given as

$$\lambda = \beta + s\tau i, \ -D\tau^2 + \mu + s\tau i. \tag{6.9}$$

Moreover, the second λ in (6.9) satisfies $Re(\lambda) = -D\tau^2 + \mu > 0$ for some $\tau \in \mathbb{R}$ as $\mu > 0$. It follows that all of the traveling wave solutions constructed from Theorems 1.1 and 1.2 have unstable essential spectrum on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$, and therefore, are spectrally unstable.

7. Discussion

In this paper, we have investigated a traveling wave solution in a mathematical model for chemotaxis in terms of existence, non-existence and spectral instability by employing dynamcial system theory and spectral analysis. As a Keller-Segel type model, our model contains logistic growth of cells, zero chemical diffusion, and a linear chemical reaction; particularly, the sensitivity function is unspecified and can represent positive or negative chemotaxis. Indeed, differences in species and experimental settings may take different mathematical forms or terms in a chemotaxis model, and hence our results can be applied to diverse biological scenarios.

In this work, it is assumed that there is no chemical diffusion. However, the chemical diffusion can be large compared to the other chemotactic and diffusion coefficients [17, 44]; furthermore, these values can vary up to two orders of magnitude [25, 42]. In addition to this paper, most of past studies considered zero diffusion and non-zero diffusion separately. In terms of the existence of traveling waves, for example, Salako and Shen in [38] established the existence of traveling wave solution of our model by adding a positive chemical diffusion, which has a lower bound, and by using a positive constant sensitivity function. Based on the results from this paper and [38], a sufficiently small chemical diffusion and the chemical diffusion limit have not been studied yet. Moreover, the non-zero chemical diffusion has been studied only for the case where the sensitivity function is a positive constant. Our future directions include consideration of non-zero chemical diffusion and a general class of sensitivity functions, and comparison of the results with those of experiments in [46], and further we plan to study the chemical diffusion limit.

When it comes to the stability of our traveling wave solutions, a question on the linear stability/instability is still remaining. Different from the diffusion-reaction systems with all non-zero diffusion terms, the spectral stability/instability and nonlinear stability/instability do not guarantee the linear stability/instability; moreover, our spectral instability has been verified only in the unweighted L^2 space. As recommended by one of the anonymous referees, it will be more interesting to investigate spectral and linear stability of the wave in some exponentially weighted spaces.

Acknowledgement. The authors would like to thank the editor and anonymous referees who kindly reviewed this manuscript and provided valuable suggestions and comments to improve this work.

REFERENCES

[1] J. Adler, *Chemotaxis in bacteria*, Science, 153:708–716, 1966. 1

[2] S. Ai, W. Huang, and Z.A. Wang, *Reaction, diffusion and chemotaxis in wave propagation*, Discrete Contin. Dyn. Syst. Ser. B, 20:1–21, 2015. 1

[3] S. Ai and Z.A. Wang, *Traveling bands for the Keller-Segel model with population growth*, Math. Biosci. Eng., 12:717–737, 2015. 1

[4] D. Balding and D.L.S. McElwain, *A mathematical model of tumour-induced capillary growth*, J. Theor. Biol., 114:53–73, 1985. 1

[5] J.J. Bramburger, *Exact minimum speed of traveling waves in a Keller–Segel model*, Appl. Math. Lett., 111:106594, 2020. 1

[6] J.J. Bramburger and D. Goluskin, *Minimum wave speeds in monostable reaction-diffusion equations: sharp bounds by polynomial optimization*, Proc. R. Soc. Lond. A, 476:20200450, 2020. 1

[7] E.O. Budrene and H.C. Berg, *Dynamics of formation of symmetrical patterns by chemotactic bacteria*, Nature, 376:49–53, 1995. 1

[8] M.A.J. Chaplain and A.M. Stuart, *A model mechanism for the chemotactic response of endothelial cells to tumour angiogenesis factor*, Math. Med. Biol., 10:149–168, 1993. 1, 1

[9] L. Corrias, B. Perthame, and H. Zaag, *Global solutions of some chemotaxis and angiogenesis systems in high space dimensions*, Milan J. Math., 72:1–28, 2004. 1

[10] F.W. Dahlquist, P. Lovely, and D.E. Koshland, *Quantitative analysis of bacterial migration in chemotaxis*, Nature New Biol., 236:120–123, 1972. 1

[11] S.R. Dunbar, *Traveling wave solutions of diffusive Lotka-Volterra equations: a heteroclinic connection in* $\mathbb{R}^4$, Trans. Amer. Math. Soc., 286:557–594, 1984. 1

[12] H. Fan and X.B. Lin, *A dynamical systems approach to traveling wave solutions for liquid/vapor phase transition*, in J. Mallet-Paret, J. Wu, Y. Yi, and H. Zhu (eds.), Infinite Dimensional Dynamical Systems, Springer, New York, 101–117, 2013. 1, 2, 2, 2.3

[13] R.M. Ford and D.A. Lauffenburger, *Analysis of chemotactic bacterial distributions in population migration assays using a mathematical model applicable to steep or shallow attractant gradients*, Bull. Math. Biol., 53:721–749, 1991. 1, 1

[14] M. Funaki, M. Mimura, and T. Tsujikawa, *Travelling front solutions arising in the chemotaxis-growth model*, Interfaces Free Bound., 8:223–245, 2006. 1

[15] P. Hartman, *Ordinary Differential Equations*, Second Edition, Birkhäuser, Basle, 1982. 1, 2, 2.1, 2.2

[16] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer, 3–40, 2006. 6, 6

[17] T. Hillen and K.J. Painter, *A user′s guide to PDE models for chemotaxis*, J. Math. Biol., 58:183–217, 2009. 1, 1, 7

[18] D. Horstmann, *From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I.*, Jahresber. Deutsch. Math.-Verein., 105:103–165, 2003. 1

[19] D. Horstmann and A. Stevens, *A constructive approach to traveling waves in chemotaxis*, J. Nonlinear Sci., 14:1–25, 2004. 1

[20] C. Jin, C. Krüger, and C.C. Maass, *Chemotaxis and autochemotaxis of self-propelling droplet swimmers*, Proc. Natl. Acad. Sci. U.S.A., 114:5089–5094, 2017. 1

[21] H.Y. Jin, J. Li, and Z.A. Wang, *Asymptotic stability of traveling waves of a chemotaxis model with singular sensitivity*, J. Differ. Equ., 255:193–219, 2013. 1, 6

[22] E.F. Keller and L.A. Segel, *Model for chemotaxis*, J. Theor. Biol., 30:225–234, 1971. 1

[23] E.F. Keller and L.A. Segel, *Traveling bands of chemotactic bacteria: a theoretical analysis*, J. Theor. Biol., 30:235–248, 1971. 1

[24] D. Lauffenburger, C.R. Kennedy, and R. Arix, *Traveling bands of chemotactic bacteria in the context of population growth*, Bull. Math. Biol., 46:19–40, 1984. 1

[25] P. Lewus and R.M. Ford, *Quantification of random motility and chemotaxis bacterial transport coefficients using individual-cell and population-scale assays*, Biotechnol. Bioeng., 75:292–304, 2001. 1, 7

[26] T. Li, H. Liu, and L. Wang, *Oscillatory traveling wave solutions to an attractive chemotaxis system*, J. Differ. Equ., 12:7080–7098, 2016. 1, 6

[27] T. Li and J. Park, *Traveling waves in a chemotaxis model with logistic growth*, Discrete Contin. Dyn. Syst. Ser. B, 24:6465–6480, 2019. 1, 1, 6

[28] T. Li and Z.A. Wang, *Nonlinear stability of traveling waves to a hyperbolic-parabolic system modeling chemotaxis*, SIAM J. Appl. Math., 70:1522–1541, 2010. 1

[29] J. Li and Z.A. Wang, *Travelling wave solutions of the density-suppressed motility model*, J. Differ. Equ., 301:1–36, 2021. 1

[30] R. Lui and Z.A. Wang, *Traveling wave solutions from microscopic to macroscopic chemotaxis*

models, J. Math. Biol., 61:739–761, 2010. 1

[31] M. Mimura and T. Tsujikawa, *Aggregating pattern dynamics in a chemotaxis model including growth*, Phys. A: Stat. Mech. Appl., 230:499–543, 1996. 1

[32] G. Nadin, B. Perthame, and L. Ryzhik, *Traveling waves for the Keller–Segel system with Fisher birth terms*, Interfaces Free Bound., 10:517–538, 2008. 1, 1

[33] T. Nagai and T. Ikeda, *Traveling waves in a chemotactic model*, J. Math. Biol., 30:169–184, 1991. 1, 6

[34] J. Park and Z. Aminzare, *A mathematical description of bacterial chemotaxis in response to two stimuli*, Bull. Math. Biol., 84:9, 2022. 1

[35] R.B. Salako and W. Shen, *Existence of traveling wave solutions to parabolic-elliptic-elliptic chemotaxis systems with logistic source*, Discrete Contin. Dyn. Syst. Ser. S, 13:293–319, 2020. 1

[36] R.B. Salako and W. Shen, *Spreading speeds and traveling waves of a parabolic-elliptic chemotaxis system with logistic source on* $\mathbb{R}^N$, Discrete Contin. Dyn. Syst. Ser. A, 37:6189–6225, 2017. 1

[37] R.B. Salako and W. Shen, *Existence of traveling wave solutions of parabolic–parabolic chemotaxis systems*, Nonlinear Anal. Real World Appl., 42:93–119, 2018. 1

[38] R.B. Salako and W. Shen, *Traveling wave solutions for fully parabolic Keller-Segel chemotaxis systems with a logistic source*, Electron. J. Differ. Equ., 2020(53):1–18, 2020. 1, 7

[39] R.B. Salako, W. Shen, and S. Xue, *Can chemotaxis speed up or slow down the spatial spreading in parabolic–elliptic Keller–Segel systems with logistic source?* J. Math. Biol., 79:1455–1490, 2019. 1

[40] H. Schwetlick, *Traveling waves for chemotaxis–systems*, Proc. Appl. Math. Mech., 3:476–478, 2003. 1, 1

[41] R. Tyson, S.R. Lubkinm, and J.D. Murray, *Model and analysis of chemotactic bacterial patterns in a liquid medium*, J. Math. Biol., 38:350–375, 1999. 1

[42] M.J. Tindall, P.K. Maini, L. Steven, and J.P. Armitage, *Overview of mathematical approaches used to model bacterial chemotaxis II: bacterial populations*, Bull. Math. Biol., 70:1570–1607, 2008. 1, 1, 1, 7

[43] Z.A. Wang, *Wavefront of an angiogenesis model*, Discrete Contin. Dyn. Syst. Ser. B, 17:2849–2860, 2012. 1

[44] Z.A. Wang, *Mathematics of traveling waves in chemotaxis –Review paper–*, Discrete Contin. Dyn. Syst. Ser. B, 18:601–641, 2013. 1, 1, 7

[45] T. Ważeski, *Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaries*, Ann. Soc. Polon. Math., 20:279–313, 1948. 1, 2

[46] D.E. Woodward, R. Tyson, M.R. Myerscough, J.D. Murray, E.O. Budrene, and H.C. Berg, *Spatio-temporal patterns generated by Salmonella typhimurium*, Biophys. J., 68:2181–2189, 1995. 1, 1, 7

[47] C. Xue, H.J. Hwang, K.J. Painter, and R. Erban, *Travelling waves in hyperbolic chemotaxis equations*, Bull. Math. Biol., 73:1695–1733, 2010. 1