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STABILITY OF MEASURE SOLUTIONS TO A GENERALIZED BOLTZMANN EQUATION WITH COLLISIONS OF A RANDOM NUMBER OF PARTICLES*

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Abstract. In the paper we study a measure version of the evolutionary nonlinear Boltzmann-type equation in which we admit a random number of collisions of particles. We consider first a stationary model and use two methods to find its fixed points: the first based on Zolotarev seminorm and the second on Kantorovich-Rubinstein maximum principle. Then a dynamic version of Boltzmann-type equation is considered and its asymptotic stability is shown.

Keywords. Generalized Boltzmann equation; stability; collisions of particles; Zolotariev seminorm; Kantorovich-Rubinstein maximum principle.

AMS subject classifications. 82B31; 82B21; 35Q20.

1. Introduction

In the paper we consider a nonlinear evolutionary measure-valued Boltzmann type equation of the form

$$\frac{d\psi}{dt}+\psi=\mathbb{P}\psi \qquad \text{for} \qquad t\geq 0 \tag{1.1}$$

where the operator $\mathbb{P}$ maps $\mathcal{M}_1(\mathbb{R}_+)$ the space of probability measures on $\mathbb{R}_+=[0,\infty)$ into itself. We are looking for $\psi:\mathbb{R}_+\to\mathcal{M}_{sig}(\mathbb{R}_+)$ with $\psi_0\in\mathcal{M}_1(\mathbb{R}_+)$, where $\mathcal{M}_{sig}(\mathbb{R}_+)$ (or shortly $\mathcal{M}_{sig}$) is the space of all signed measures on $\mathbb{R}_+$.

Equation (1.1) is a generalized version of the equation considered in [27] (see also Section 8.9 in [17], or [6] for the motivation),

$$\frac{\partial u(t,x)}{\partial t}+u(t,x)=\int_x^\infty \frac{dy}{y}\int_0^y u(t,y-z)u(t,z)dz:=Pu(x) \qquad t\geq 0, \qquad x\geq 0, \tag{1.2}$$

which describes energy changes subject to the collision operator $Pu(x)$ and which was obtained from Boltzmann equation corresponding to a spatially homogeneous gas with no external forces, using Abel transformation. To be more precise, in the theory of dilute gases Boltzmann equation in the general form $\frac{dF(t,x,v)}{dt}=C(F(t,x,v))$ gives us an information about time, position and velocity of particles of the dilute gas. This equation is a base for many mathematical models of colliding particles. In particular, for a spatially homogeneous gas we come to the Equation (1.2) with additional conditions saying that its solution u, for fixed t, is a density with first moment equal to 1, which in turn corresponds to the conservation law of mass and energy. The operator Pu is a density function of the random variable $\eta(\xi_1+\xi_2)$, where random variables η, ξ_1 and ξ_2 are independent and η is uniformly distributed, while ξ_1, ξ_2 have the same density function u. The assumption that η has uniform distribution on $[0,1]$ is quite restrictive

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and there are no physical reasons to assume that the distribution of energy of particles can be described only by its density (is absolutely continuous). Moreover collision of two particles maybe replaced by collision of a random number of particles. This is a reason that in what follows we shall consider a measure-valued version (1.1) of the Equation (1.2).

Let

$$D := \{\mu \in \mathcal{M}_1(\mathbb{R}_+) : m_1(\mu) = 1\}, \quad \text{with} \quad m_1(\mu) = \int_0^\infty x\mu(dx), \tag{1.3}$$

and denote by $\bar{D}$ a weak closure of D, which is of the form

$$\bar{D} := \{\mu \in \mathcal{M}_1(\mathbb{R}_+) : m_1(\mu) \le 1\}. \tag{1.4}$$

The operator P defined in (1.2) describes collision of two particles. In what follows we shall consider a general situation of collision of a random number of particles. To describe the collision operator in this case we start from recalling the convolution operator of order n and the linear operator P_φ, which is related to multiplication of random variables.

For every $n \in \mathbb{N}$ let $P_{*n} : \mathcal{M}_{sig} \to \mathcal{M}_{sig}$, be given by the formula

$$P_{*1}\mu := \mu, \quad P_{*(n+1)}\mu := \mu * P_{*n}\mu \quad \text{for} \quad \mu \in \mathcal{M}_{sig}. \tag{1.5}$$

It is easy to verify that $P_{*n}(\mathcal{M}_1(\mathbb{R}_+)) \subset \mathcal{M}_1(\mathbb{R}_+)$ for every $n \in \mathbb{N}$. Moreover, $P_{*n}|_{\mathcal{M}_1(\mathbb{R}_+)}$ has a simple probabilistic interpretation: If $\xi_1,\dots,\xi_n$ are independent random variables with the same probability distribution μ, then $P_{*n}\mu$ is the probability distribution of $\xi_1 + \dots + \xi_n$.

The second class of operators we are going to study is related to multiplication of random variables. Formal definition is as follows: Given $\mu, v \in \mathcal{M}_{sig}$, we define product $u \circ v$ by

$$(\mu \circ v)(A) := \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \mathbf{1}_A(xy)\mu(dx)v(dy) \quad \text{for} \quad A \in \mathcal{B}_{\mathbb{R}_+}. \tag{1.6}$$

and

$$\langle f, \mu \circ v \rangle = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} f(xy)\mu(dx)v(dy) \tag{1.7}$$

for every Borel measurable $f : \mathbb{R}_+ \to \mathbb{R}$ such that $(x,y) \mapsto f(xy)$ is integrable with respect to the product of $|\mu|$ and $|v|$. For fixed $\varphi \in \mathcal{M}_1$ define

$$P_\varphi \mu := \phi \circ \mu \quad \text{for} \quad \mu \in \mathcal{M}_{sig}. \tag{1.8}$$

Similarly as in the case of convolution it follows that $P_\varphi(\mathcal{M}_1) \subset \mathcal{M}_1$. For $\mu \in \mathcal{M}_1$ the measure $P_\varphi \mu$ has an immediate probabilistic interpretation: If φ and μ are probability distributions of random variables ξ and η respectively, then $P_\varphi \mu$ is the probability distribution of the product $\xi\eta$.

We introduce now definition of more general version of P allowing infinite number of collisions:

$$\mathbb{P} := \sum_{i=1}^\infty \alpha_i P_{\varphi_i} P_{*_i} = \sum_{i=1}^\infty \alpha_i \mathbb{P}_i, \tag{1.9}$$

where we have $\mathbb{P}_i := P_{\varphi_i} P_{*_i}$, $\sum_{i=1}^{\infty} \alpha_i = 1$, $\alpha_i \geq 0$, $\varphi_i \in \mathcal{M}_1$ and $m_1(\varphi_i) = 1/i$ and the limit of the series is considered in the weak topology sense, that is, $\sum_{i=1}^{n} \alpha_i P_{\varphi_i} P_{*_i} \Rightarrow \sum_{i=1}^{\infty} \alpha_i P_{\varphi_i} P_{*_i}$, which means $\sum_{i=1}^{n} \alpha_i P_{\varphi_i} P_{*_i}(f) \to \sum_{i=1}^{\infty} \alpha_i \mathbb{P}_i(f)$ as $n \to \infty$, for any continuous bounded function f defined on $\mathbb{R}$. From (1.9) it follows that $\mathbb{P}\mathcal{M}_1 \subset \mathcal{M}_1$. Using (1.5) and (1.8) it is easy to verify that for $\mu \in D$,

$$m_1(P_{*i}\mu) = i \quad \text{and} \quad m_1(P_{\varphi_i}\mu) = 1/i. \tag{1.10}$$

Given $\mu \in \mathcal{M}_1$ the value of $\mathbb{P}\mu$ can be considered as the probability distribution of a random variable ζ defined as

$$\zeta := \eta_\tau \Big(\sum_{j=1}^{\tau} \xi_{\tau j}\Big), \tag{1.11}$$

where we have sequences of independent random variables η_i, ξ_{ij}, $j = 1, 2, \ldots$, and τ, such that ξ_{ij} have the same probability distribution μ for $j = 1, 2, \ldots$, random variables η_i have the probability distribution φ_i and random variable τ takes values in the set $\{1, 2, \ldots\}$ with $P\{\tau = j\} = \alpha_j$. Physically this means that the number of colliding particles is random and energies of particles before a collision are independent quantities and that a particle after collision of i-th particles takes the η_i part of the sum of the energies of the colliding particles.

We can also define $\tilde{\eta}_i := i\eta_i$ and consider

$$\zeta := \tilde{\eta}_\tau \Big(\frac{1}{\tau} \sum_{j=1}^{\tau} \xi_{\tau j}\Big) \tag{1.12}$$

and write

$$\mathbb{P} = \sum_{i=1}^{\infty} \alpha_i \tilde{P}_{\varphi_i} \tilde{P}_{*_i} \tag{1.13}$$

where $\tilde{P}_{\varphi_i} = P_{\tilde{\varphi}_i}$ with $\tilde{\varphi}_i$ being the probability distribution of $i\eta_i$, and $\tilde{P}_{*_i}\mu$ is the probability distribution of $\frac{\xi_{i1} + \ldots + \xi_{ii}}{i}$.

Measure-valued solutions to the Boltzmann-type equations with different collision operators and even in the multidimensional case were studied in a number of papers, see e.g. [1,21] and [23]. In the paper [21], existence and stability of measure solutions to the spatially homogeneous Boltzmann equations that have polynomial and exponential moment production properties, is shown. In the paper [1], existence and uniqueness of measure solutions to one dimensional Boltzmann dissipative equation and then their asymptotics is considered. In this paper, first stationary (steady state) equation for a specific collision operator is studied and then dynamic fixed-point theorem is used. Asymptotics of solutions to the Boltzmann equation with infinite energy to so-called self-similar solutions was studied in [7]. Asymptotic property of self-similar solutions to the Boltzmann Equation for Maxwell molecules was then shown in [9]. Long-time behaviour of the solutions to the nonlinear Boltzmann equation for spatially uniform freely cooling inelastic Maxwell molecules was studied in [5]. Stability of Boltzmann equation with external potential was also considered in [28] and in the case of exterior problem in [16]. Solutions of the Boltzmann equation with collision of N particles and their limit behaviour when $N\to\infty$ over finite time horizon were studied in [2]. In [8], the N-particle model, which includes multi-particle interactions was considered. It is shown that under certain natural assumptions we obtain a class of equations which can be considered as the most general Maxwell-type model.

In [26] an individual based model describing phenotypic evolution in hermaphroditic populations which includes random and assortative mating of individuals is introduced. By increasing the number of individuals to infinity a nonlinear transport equation is obtained, which describes the evolution of phenotypic probability distribution. The main result of the paper is a theorem on asymptotic stability of trait (which concerns the model with more general operator P) with respect to Fortet-Mourier metric.

Stability problems of the Boltzmann-type Equation (1.1) with operator P corresponding to collision of two particles was studied in the paper [18]. The case with infinite number of particles was considered in [19] using Zolotariev seminorm approach. Properties of stationary solutions corresponding to collisions of two particles were studied in [20].

In this paper we study stability of solutions to one dimensional Boltzmann-type Equation (1.1) with operator P of the form $\mathbb{P}$ defined in (1.9). We show that if this equation has a stationary solution μ^*, such that its support covers $\mathbb{R}_+$, then taking into account positivity of solutions to (1.1)-(1.9) we have its asymptotical stability in Kantorovich - Wasserstein metric to μ^*. We consider first stationary equation and look for fixed points of the operator $\mathbb{P}$. For this purpose we adopt two methods to show the existence of fixed point of $\mathbb{P}$ with the first moment equal to 1. The first method is based on Zolotarev seminorm and in some sense simplifies the method used in [19]. The second method is based on Kantorovich-Rubinstein maximum principle and generalizes the results of [14] and then also of [18] to the case of a random number of colliding particles. We also show several characteristics of fixed points of $\mathbb{P}$. Results on the stationary equation are then used to study stability of the dynamic Boltzmann equation. The novelty of the paper is that we consider the Boltzmann-type Equation (1.1) with random unbounded number of colliding particles and show its stability in Kantorovich - Wasserstein metric using probabilistic methods, generalizing former results of [18, 19] and [14]. To improve readability of the paper an appendix is added to the paper where some important results, which are used in the paper, are formulated and their proofs are sketched.

2. Properties of the operator $\mathbb{P}$

We study first several properties of $\mathbb{P}$.

Proposition 2.1. *Operator $\mathbb{P}$ transforms the set D or $\bar{D}$ into itself. It is continuous with respect to weak topology in $\mathcal{M}_1(\mathbb{R}_+)$ i.e. whenever $\mathcal{M}_1(\mathbb{R}_+)\ni\mu_n\Rightarrow\mu$ we have $\mathbb{P}\mu_n\Rightarrow\mathbb{P}\mu$ as $n\to\infty$. Furthermore whenever $m_r(\varphi_i)=\int_{\mathbb{R}_+} x^r\varphi_i(dx)<\infty$ and $m_r(\mu)=\int_{\mathbb{R}_+} x^r\mu(dx)<\infty$ for $r\geq 1$, $i=1,2,\ldots$ then*

$$m_r(\mathbb{P}\mu)\leq\sum_{i=1}^{\infty}\alpha_i m_r(\varphi_i)i^r m_r(\mu). \tag{2.1}$$

Proof. Note first that for each $\mu\in\mathcal{M}_1(\mathbb{R}_+)$ we have that $P_{\varphi_i}P_{*_i}\mu\in\mathcal{M}_1(\mathbb{R}_+)$ for $i=1,2,\ldots$ and therefore $\mathbb{P}\mu\in\mathcal{M}_1(\mathbb{R}_+)$. Moreover for $\mu\in D$

$$m_1(P_{\varphi_i}P_{*_i}\mu)=\int_0^\infty\ldots\int_0^\infty x(y_1+y_2+\ldots+y_i)\varphi_i(dx)\mu(dy_1)\ldots\mu(dy_i)=\frac{1}{i}i=1 \tag{2.2}$$

so that $P_{\varphi_i} P_{*_i}\mu \in D$ and consequently $\mathbb{P}: D \mapsto D$. Similarly $\mathbb{P}: \bar{D} \mapsto \bar{D}$. For a given continuous bounded function $f: \mathbb{R}_+ \to \mathbb{R}$ and $\mathcal{M}_1(\mathbb{R}_+) \ni \mu_n \Rightarrow \mu$ we have

$$P_{\varphi_i} P_{*_i}\mu_n(f) = \int_0^\infty \dots \int_0^\infty f(x(y_1+y_2+\ldots+y_i))\varphi_i(dx)\mu_n(dy_1)\ldots\mu_n(dy_i) \to$$
$$\int_0^\infty \dots \int_0^\infty f(x(y_1+y_2+\ldots+y_i))\varphi_i(dx)\mu(dy_1)\ldots\mu(dy_i) = P_{\varphi_i} P_{*_i}\mu(f) \tag{2.3}$$

as $n \to \infty$. In fact, from continuity of

$$(y_1, y_2, \ldots, y_i) \to \int_0^\infty f(x(y_1+y_2+\ldots+y_i))\varphi_i(dx),$$

and weak convergence of the measures $\mu_n(dy_1)\ldots\mu_n(dy_i) \Rightarrow \mu(dy_1)\ldots\mu(dy_i)$, using (2.3) we immediately obtain that $\mathbb{P}\mu_n \Rightarrow \mathbb{P}\mu$ which is the desired continuity property. Now using ζ defined in (1.12), independence of random variables, as well as convexity, we obtain

$$m_r(\mathbb{P}\mu) = \int_{\mathbb{R}_+} x^r \mathbb{P}\mu(dx) = E[\zeta^r] = \sum_{i=1}^\infty \alpha_i E[\eta_i^r] i^r E\left[\left(\frac{1}{i}\sum_{j=1}^i \xi_{\tau j}\right)^r\right]$$
$$\le \sum_{i=1}^\infty \alpha_i m_r(\eta_i) i^r m_r(\mu), \tag{2.4}$$

which completes the proof. □

We comment below on the formula (2.1).

Remark 2.1. Note that since $m_1(\varphi_i) = \frac{1}{i}$ we have $\int_{\mathbb{R}_+} x\varphi_i(dx) \le \left(\int_{\mathbb{R}_+} x^r \varphi_i(dx)\right)^{\frac{1}{r}}$ and consequently $m_r(\varphi_i) \ge \frac{1}{i^r}$, so that in general we may not have that $\sum_{i=1}^\infty \alpha_i m_r(\varphi_i) i^r < \infty$.

This sum is finite however when for example we know that for a sufficiently large i we have that $\alpha_i m_r(\varphi_i) \le \frac{1}{i^\beta}$ with $\beta > 1 + r$. Finiteness of the sum above shall play an important role in the approach to study fixed points of $\mathbb{P}$ with the use of Zolotariev seminorm (see Section 3).

In what follows we are interested to find a fixed point of the operator $\mathbb{P}$ in the set D. It is clear that $\mu = \delta_0$ is a fixed point of $\mathbb{P}$ in $\bar{D}$. Typical way to find a fixed point is to consider iterations $\mathbb{P}\mu$ for $\mu \in D$ and since the measures $\{\mathbb{P}\mu, \mathbb{P}^2\mu, \ldots\}$ are tight, expect a limit to be a fixed point. However even when such iteration converges in a weak topology, its limit will be in $\bar{D}$ and it is not clear that such limit will be a fixed point. Note furthermore that when we have more than one particle, the operator $\mathbb{P}$ is nonlinear and we can not use several techniques typical for linear operators.

Remark 2.2. Notice that fixed point of $\mathbb{P}$ is not necessary in D since the set D is not closed in the weak topology. In fact, when $\mu_n\{\frac{1}{n}\} = \frac{n}{n+1}$ and $\mu_n\{n\} = \frac{1}{n+1}$, then $\mu_n \in D$ and $\mu_n \Rightarrow \mu := \delta_0$ so that total mass of μ is concentrated at 0, and $\int_0^\infty x\mu(dx) = 0$. Similary for $0 < \alpha < 1$, letting $\mu_n(1-\alpha) = 1 - \frac{1}{n}$, $\mu_n(1) = \frac{1-\alpha}{n}$ and $\mu_n(n) = \frac{\alpha}{n}$ we clearly have that $\mu_n \in D$ and $\mu_n \Rightarrow \delta_{1-\alpha}$, as $n \to \infty$. One can show that the closure $\bar{D}$ of the set D in the weak topology consists of all probability measures $\nu \in \mathcal{M}_1(\mathbb{R}_+)$ such that

$m_1(\nu)\le 1$. Consider the following example: $P\{\varphi_i=0\}=\frac{n}{n+1}$, $P\{\varphi_i=(n+1)\frac{1}{i}\}=\frac{1}{n+1}$ and $\mu([\Delta,\infty))=1$ with $m_1(\mu)=1$, $\Delta>0$ and fixed integer $n\ge 1$. Then the support of $\mathbb{P}\mu$ consists of 0 and the second part contained in $[\Delta(\frac{n+1}{i}),\infty)$ and inductively the support of $\mathbb{P}^k\mu$ contains 0 and its second part is contained in $[\Delta(\frac{(n+1)}{i})^k,\infty)$, for positive integer k. Since $\mathbb{P}^k\mu(0)=\frac{n}{n+1}$ it is clear that $\mathbb{P}^k\mu\Rightarrow\delta_0$, as $k\to\infty$, so that limit of $\mathbb{P}^k\mu$, which is also a fixed point of $\mathbb{P}$, is not in D.

On the other hand, when $P\{\varphi_i=\delta\}=\frac{n}{(n+1)-i\delta}$ and $P\{\varphi_i=(n+1)\frac{1}{i}\}=\frac{1-i\delta}{(n+1)-i\delta}$ with $i\delta<1$ and $\mu([\Delta,\infty))=1$ with $m_1(\mu)=1$, $\Delta>0$ we have that $\cup_k supp(\mathbb{P}^k\mu)$ is dense in $[0,\infty)$.

When support of φ_i contains a sequence of positive real numbers converging to 0, then $\cup_k supp(\mathbb{P}^k\mu)$ is dense in $[0,\infty)$ no matter what $\mu\in D$ is chosen. Assume additionally that $\frac{1}{i}\in supp\varphi_i$ for each $i=1,2,\ldots$. Then $supp(\mathbb{P}^k\mu)\subset supp(\mathbb{P}^{k+1}\mu)$, so that we have an increasing sequence of closed sets $supp(\mathbb{P}^k\mu)$ which cover the interval $(0,\infty)$.

Next two propositions show properties of the fixed point of $\mathbb{P}$.

Let $\Lambda:=\{i:\alpha_i>0\}$

Proposition 2.2. *Let $\mu\in D$ be a fixed point of $\mathbb{P}$. When there are at least two elements of Λ and $\tilde\varphi_i=\delta_1$ for $i\in\Lambda$ then either $\mu=\delta_1$ or $m_\beta(\mu)=\infty$ for any $\beta>1$. When $\mu=\delta_1$ then $\tilde\varphi_i=\delta_1$ for $i\in\Lambda$.*

Proof. Using (1.13) we have that the law of $\tilde\eta_\tau(\frac{1}{\tau}\sum_{j=1}^{\tau}\xi_{\tau j})$, when the law of ξ_{ij} is μ, is also μ. Then for $\beta>1$ we have

$$E\left\{\tilde\eta_\tau^\beta(\frac{1}{\tau}\sum_{j=1}^{\tau}\xi_{\tau j})^\beta\right\}=\sum_{i=1}^{\infty}\alpha_i m_\beta(\tilde\varphi_i)E\left\{(\frac{1}{i}\sum_{j=1}^{i}\xi_{ij})^\beta\right\}=m_\beta(\mu). \tag{2.5}$$

Notice now that by strict convexity we have for $\beta>1$ that $\left(\frac{1}{i}\sum\limits_{j=1}^{i}\xi_{ij}\right)^\beta<\frac{1}{i}\sum\limits_{j=1}^{i}\xi_{ij}^\beta$ with equality only when $\xi_{ij}=\xi_{ij'}$ for $j'\in\{1,2,\ldots,i\}$.

Therefore when there is at least one element in Λ before $i\in\Lambda$ and $\tilde\varphi_j=\delta_1$ for $j\in\Lambda$ we have

$$E\left\{(\frac{1}{i}\sum_{j=1}^{i}\tilde\eta_i\xi_{ij})^\beta\right\}<\sum_{j=1}^{i}\frac{1}{i}E\left\{\tilde\eta_i^\beta\xi_{ij}^\beta\right\}=m_\beta(\tilde\eta_i)m_\beta(\mu)=m_\beta(\mu) \tag{2.6}$$

with strict inequality whenever $m_\beta(\mu)<\infty$. Since ξ_{ij} and $\xi_{ij'}$ are independent for $j'\ne j$, they should be deterministic and because $m_1(\mu)=1$ we have that $\mu=\delta_1$. When $\mu=\delta_1$ by (2.5) and (2.6) we have

$$E\left\{\tilde\eta_\tau^\beta(\frac{1}{\tau}\sum_{j=1}^{\tau}\xi_{\tau j})^\beta\right\}=m_\beta(\tilde\varphi_i)m_\beta(\mu)=m_\beta(\mu), \tag{2.7}$$

which implies that $m_\beta(\tilde\varphi_i)=1=m_1(\tilde\varphi_i)$, which is again true only when $\tilde\varphi_i=\delta_1$. ∎

In the next proposition we adopt some arguments of Proposition 2.1 of [20].

Proposition 2.3. *Assume that for $1\ne k\in\Lambda$ there are $q_1,q_2\in supp\varphi_k$ such that $q_1>\frac{1}{k}$, $q_2<\frac{1}{k}$. Under the above assumptions, when μ^* is a fixed point of $\mathbb{P}$, its support is either $\{0\}$ or $[0,\infty)$.*

Proof. Notice that the support of each φ_i contains an element not greater that $\frac{1}{i}$. Therefore taking into account that $q_2 < \frac{1}{k}$ we have that when $0 \neq r \in supp\mu^*$ then the support of $\mathbb{P}\mu^*$ contains an element not greater than

$$\sum_{i=1, i\neq k}^{\infty} \alpha_i r + \alpha_k q_2 k r = \sum_{i=1}^{\infty} \alpha_i r + \alpha_k (q_2 - \frac{1}{k}) k r = r(1-(1-kq_2)\alpha_k).$$

Since $(1-(1-kq_2)\alpha_k) < 1$ iterating the operator $\mathbb{P}$ we obtain that $0 \in supp\mu^*$. Therefore to show that $supp\mu^*$, in the case when $supp\mu^* \neq \{0\}$, covers the whole interval $[0,\infty)$ it suffices to show that for any $r \in (0,\infty) \cap supp\mu^*$ we have that $A := supp\{\mathbb{P}_k^i \mu^*,\ for\ i = 1,2,\dots\}$ is dense in $[0,r]$. For a given $\varepsilon > 0$ we can find a positive integer m such that $\left(\frac{1}{k}\right)^m \leq \frac{\varepsilon}{rkq_1}$ and $(kq_2)^m \leq 1$. Then we can find a positive integer n such that $(kq_1)^{n-1}(kq_2)^m \leq (\frac{1}{k})^m kq_1 \leq 1$ and $(kq_1)^n (kq_2)^m > 1$. Consequently we have that $q_2^m (kq_1)^n \leq \frac{\varepsilon}{r}$ and $k^m q_2^m (kq_1)^n \geq 1$. Therefore $iq_2^m (kq_1)^n r \in A$ for $i = 1,2,\dots,k^m$, and $q_2^m (kq_1)^n r \leq \varepsilon$ while $k^m q_2^m (kq_1)^n r \geq r$. Since ε can be chosen arbitrarily small and A is closed we have that $[0,\infty) \subset A$, which we obtain taking into account that together with r the values kq_1^i are in $supp\mu^*$. □

REMARK 2.3. *In some cases there is only one fixed point of $\mathbb{P}$ which is $\delta_0 \notin D$. Given probability measure μ on $\mathbb{R}_+$ consider its characteristic function $\psi(t) = \int e^{itx}\mu(dx)$. If μ is a fixed point of $\mathbb{P}$ then in the case when $\Lambda = \{2\}$, ψ satisfies the following function equation*

$$\psi(t) = \int_{\mathbb{R}_+} [\psi(tz)]^2 \varphi_2(dz). \tag{2.8}$$

Assume $\varphi_2 = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$. Then (2.8) takes the form

$$\psi(t) = \frac{1}{2} + \frac{1}{2}[\psi(t)]^2. \tag{2.9}$$

The solutions to this quadratic equation are constant functions $\psi(t)$ and since $\psi(0) = 1$ the only solution which is a characteristic function is $\psi(t) \equiv 1$, which corresponds to $\mu = \delta_0$. Consequently we don't have fixed point of $\mathbb{P}$ in the set D.

In the space $\mathcal{M}_1(\mathbb{R}_+)$ consider *Kantorovich - Wasserstein metric* (see [15], Definition 4.3.1 or [10]) given by the formula

$$\|\mu_1 - \mu_2\|_{\mathcal{K}} = \sup\{|\mu_1(f) - \mu_2(f)| : f \in \mathcal{K}\} \qquad \text{for} \qquad \mu_1, \mu_2 \in \mathcal{M}_1(\mathbb{R}_+), \tag{2.10}$$

where $\mathcal{K}$ is the set of functions $f : \mathbb{R}_+ \to R$ which satisfy the condition

$$|f(x) - f(y)| \leq |x - y| \quad \text{for} \quad x, y \in \mathbb{R}_+.$$

We recall now another, Forter-Mourier metric $\|\cdot\|_{\mathcal{F}}$ in $\mathcal{M}_1(\mathbb{R}_+)$

$$\|\mu_1 - \mu_2\|_{\mathcal{F}} = \sup\{|\mu_1(f) - \mu_2(f)| : f \in \mathcal{F}\} \qquad \text{for} \qquad \mu_1, \mu_2 \in \mathcal{M}_1(\mathbb{R}_+), \tag{2.11}$$

where $\mathcal{F}$ consists of functions f from $\mathcal{K}$ such that $\|f\|_{sup} = \sup_{x \in \mathbb{R}_+} |f(x)| \leq 1$.

Almost immediately we have that $\|f\|_{\mathcal{F}} \leq \|f\|_{\mathcal{K}}$ for $\mu \in \mathcal{M}_1(\mathbb{R}_+)$, which means that Kantorovich - Wasserstein metric is stronger than Fortet-Mourier metric. Notice however that the convergence of probability measures in Fortet-Mourier metric is equivalent to weak convergence (see [13] or [10]). We have:

LEMMA 2.1. *The operator $\mathbb{P}$ is nonexpansive in $\bar{D}$ in Kantorovich - Wasserstein metric.*

Proof. We are going to show that for each $i=1,2,\ldots$ the operators $\mathbb{P}_i$ are nonexpansive in Kantorovich - Wasserstein metric. In fact, for $f\in\mathcal{K}$ and $\mu\in\bar{D}$ we have

$$\mathbb{P}_i\mu(f)=\int\limits_{\mathbb{R}_+}\ldots\int\limits_{\mathbb{R}_+} f(r(x_1+x_2+\ldots+x_i))\varphi_i(dr)\mu(dx_1)\mu(dx_2)\ldots\mu(dx_i). \tag{2.12}$$

Define for $\nu\in\bar{D}$

$$\begin{aligned}\tilde{f}(x):=&\int\limits_{\mathbb{R}_+}\ldots\int\limits_{\mathbb{R}_+} f(r(x+x_2+\ldots+x_i))\varphi_i(dr)(\mu(dx_2)\ldots\mu(dx_i)+\\ &\nu(dx_2)\mu(dx_3)\ldots\mu(dx_i)+\nu(dx_2)\nu(dx_3)\mu(dx_4)\ldots\mu(dx_i)+\ldots+\\ &\nu(dx_2)\nu(dx_3)\ldots\nu(dx_i)).\end{aligned} \tag{2.13}$$

Then $\tilde{f}\in\mathcal{K}$ (since $m_1(\varphi_i)=\frac{1}{i}$). Furthermore

$$\mathbb{P}_i\mu(f)-\mathbb{P}_i\nu(f)=\mu(\tilde{f})-\nu(\tilde{f}). \tag{2.14}$$

Hence $\|\mathbb{P}_i\mu-\mathbb{P}_i\nu\|_{\mathcal{K}}\le\|\mu-\nu\|_{\mathcal{K}}$ and consequently

$$\|\mathbb{P}\mu-\mathbb{P}\nu\|_{\mathcal{K}}\le\sum_{i=1}^{\infty}\alpha^i\|\mathbb{P}_i\mu-\mathbb{P}_i\nu\|_{\mathcal{K}}\le\|\mu-\nu\|_{\mathcal{K}}, \tag{2.15}$$

which completes the proof. □

REMARK 2.4. One can notice that operator $\mathbb{P}$ is not nonexpansive in Fortet-Mourier metric.

We also have:

COROLLARY 2.1. *Operator $\mathbb{P}$ transforms D into itself and is defined also as a limit of $\sum_{i=1}^{n}\alpha_i\mathbb{P}_i$ in Kantorovich - Wasserstein metric. Furthermore D is convex and closed in Kantorovich - Wasserstein metric.*

Proof. Notice that for $\mu\in D$ we have $\left(\sum_{i=1}^{n}\alpha_i\right)^{-1}\sum_{i=1}^{n}\alpha_i\mathbb{P}_i\mu\Rightarrow\mathbb{P}\mu$ as $n\to\infty$ and

$$m_1(\left(\sum_{i=1}^{n}\alpha_i\right)^{-1}\sum_{i=1}^{n}\alpha_i\mathbb{P}_i\mu)=1=m_1(\mathbb{P}\mu).$$

Therefore by the second part of Theorem A.2 we have that $\mathbb{P}\mu$ is a limit of $\sum_{i=1}^{n}\alpha_i\mathbb{P}_i\mu$ in Kantorovich - Wasserstein metric. Convexity of D is obvious. Closedness of D in Kantorovich - Wasserstein metric follows almost immediately from the following arguments: Convergence in Kantorovich - Wasserstein metric implies weak convergence and therefore the limit is a probability measure. Furthermore convergence in Kantorovich - Wasserstein metric implies convergence of the first moments. □

3. Fixed point of $\mathbb{P}$ using Zolotariev seminorm

In this section we shall introduce Zolotariev seminorm. Namely for $\mu\in\mathcal{M}_{sig}(\mathbb{R}_+)$ and $r\in(1,2)$ define

$$\|\mu\|_r := \sup\{\mu(f) : f\in\mathcal{F}_r\} \tag{3.1}$$

where $\mathcal{F}_r$ consists of differentiable functions $f:\mathbb{R}_+\to R$, which satisfy the condition

$$|f'(x)-f'(y)|\le |x-y|^{r-1} \quad \text{for} \quad x,y\in\mathbb{R}_+.$$

One can notice that when $f\in\mathcal{F}_r$ then also function $f(x)+\alpha+\beta x$ is in $\mathcal{F}_r$. Therefore $\|\mu\|_r=\infty$ whenever $\mu(\mathbb{R}_+)\ne 0$ or $m_1(\mu)\ne 0$. The following properties of Zolotariev seminorm will be used later on.

Lemma 3.1. *Assume that*

$$\sum_{i=1}^{\infty}\alpha_i m_r(\phi_i) i^r < \infty \tag{3.2}$$

then for $\mu,\nu\in D$, whenever $m_r(\mu)<\infty$ and $m_r(\nu)<\infty$ for $i=1,2,\ldots$, we have

$$\frac{1}{r}|m_r(\mu)-m_r(\nu)|\le\|\mu-\nu\|_r\le\frac{1}{r}|m_r|(\mu-\nu):=\frac{1}{r}\int_{\mathbb{R}_+} x^r|\mu-\nu|(dx), \tag{3.3}$$

$$\|P_{*_i}\mu-P_{*_i}\nu\|_r\le i\|\mu-\nu\|_r, \tag{3.4}$$

$$\|P_{\varphi_i}P_{*_i}\mu-P_{\varphi_i}P_{*_i}\nu\|_r\le m_r(\varphi_i) i\|\mu-\nu\|_r, \tag{3.5}$$

$$\|\mathbb{P}\mu-\mathbb{P}\nu\|_r\le\sum_{i=1}^{\infty}\alpha_i m_r(\phi_i) i\|\mu-\nu\|_r. \tag{3.6}$$

Proof. When $f\in\mathcal{F}_r$ then $f(x)=f(0)+\int_0^1 xf'(tx)dt$. Therefore

$$\begin{aligned}
\mu(f)-\nu(f) &= \int_{\mathbb{R}_+}\int_0^1 xf'(tx)\mu(dx)-\int_{\mathbb{R}_+}\int_0^1 xf'(tx)dt\nu(dx)\\
&=\int_{\mathbb{R}_+}\int_0^1 x(f'(tx)-f'(0))dt\mu(dx)-\int_{\mathbb{R}_+}\int_0^1 x(f'(tx)-f'(0))dt\nu(dx)\\
&\le\int_{\mathbb{R}_+}\int_0^1 x|f'(tx)-f'(0)|dt|\mu(dx)-\nu(dx)|\\
&=\int_{\mathbb{R}_+}\int_0^1 x|tx|^{r-1}dt|\mu(dx)-\nu(dx)|=\frac{1}{r}|m_r|(\mu-\nu),
\end{aligned} \tag{3.7}$$

which completes the proof of the second part of (3.3). For $f(x)=\frac{1}{r}x^r$ we have that $f\in\mathcal{F}_r$ and then

$$\|\mu-\nu\|_r \geq \frac{1}{r}\Big|\int_{\mathbb{R}_+} x^r\mu(dx) - \int_{\mathbb{R}_+} x^r\nu(dx)\Big| \geq \frac{1}{r}|m_r(\mu)-m_r(\nu)|. \tag{3.8}$$

Therefore we have (3.3).

To prove (3.4) notice that $P_{*_i}\mu - P_{*_i}\nu = \sum_{j=1}^{i}\mu_j*\mu - \sum_{j=1}^{i}\mu_j*\nu$ where $\mu_j=(P_{*_{i-j}}\mu)*(P_{*_{j-1}}\nu)$ with $P_{*_0}\mu=P_{*_0}\nu=\delta_0$. When $f\in\mathcal{F}_r$ then $\bar{f}(y)=\int_{\mathbb{R}_+} f(x+y)\mu_j(dx)$ is in $\mathcal{F}_r$. Therefore

$$\begin{aligned}\|\mu_j*(\mu-\nu)\|_r &= \sup_{f\in\mathcal{F}_r}\Big|\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f((x+y)\mu_j(dx)(\mu-\nu)(dy)\Big| \\ &= \sup_{f\in\mathcal{F}_r}\Big|\int_{\mathbb{R}_+}\bar{f}(y)(\mu-\nu)(dy)\Big| \leq \|\mu-\nu\|_r,\end{aligned} \tag{3.9}$$

from which (3.4) easily follows. Now

$$\begin{aligned}\|P_{\varphi_i}P_{*_i}\mu - P_{\varphi_i}P_{*_i}\nu\|_r &= \sup_{f\in\mathcal{F}_r}\left(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f(zx)\varphi_i(dz)P_{*_i}\mu(dx) - \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f(zx)\varphi_i(dz)P_{*_i}\nu(dx)\right) \\ &\leq m_r(\varphi_i)\sup_{f\in\mathcal{F}_r}\left(\int_{\mathbb{R}_+}\tilde{f}(x)P_{*_i}\mu(dx) - \int_{\mathbb{R}_+}\tilde{f}(x)P_{*_i}\nu(dx)\right) \\ &\leq m_r(\varphi_i)\|P_{*_i}\mu - P_{*_i}\nu\|_r,\end{aligned} \tag{3.10}$$

where $\tilde{f}(x)=\frac{1}{m_r(\varphi_i)}\int_{\mathbb{R}_+} f(zx)\varphi_i(dx)$ is an element of $\mathcal{F}_r$, provided that $f\in\mathcal{F}_r$. The estimation (3.5) follows now directly from (3.4). From (3.5) we immediately obtain (3.6). ∎

The main result of this section is the existence of a fixed point of $\mathbb{P}$. We use the same assumptions as in the paper [19], where similar result was obtained. Our proof is different and shows other properties of the fixed point of $\mathbb{P}$. It is formulated as follows.

THEOREM 3.1. *Assume that for some $r\in(1,2)$ we have $m_r(\varphi_i)<\frac{1}{i}$ for all $i\in\Lambda$, (3.2) is satisfied and there is $\mu\in D$ such that $m_r(\mu)<\infty$. Then $\mathbb{P}^n\mu$ converges in Kantorovich - Wasserstein metric to a unique $\mu^*\in D$, which is a fixed point of $\mathbb{P}$ such that $m_r(\mu^*)<\infty$.*

Proof. Under (3.2) using Proposition 2.1 we have that $m_r(\mathbb{P}^n\mu)<\infty$ for $n=1,2,\ldots$. By (3.6) we have that $\|\mathbb{P}^{j+1}\mu-\mathbb{P}^j\mu\|_r\leq\lambda^j\|\mathbb{P}\mu-\mu\|_r$, where $\lambda=\sum_{i=1}^{\infty}\alpha_i m_r(\phi_i)i$. Therefore

$$\|\mathbb{P}^n\mu-\mu\|_r = \Big\|\sum_{j=1}^{n}\mathbb{P}^j\mu-\mathbb{P}^{j-1}\mu\Big\|_r \leq \sum_{j=1}^{n}\lambda^j\|\mathbb{P}\mu-\mu\|_r, \tag{3.11}$$

where $\mathbb{P}^0\mu=\mu$. Since by (3.3)

$$\|\mathbb{P}^n\mu-\mu\|_r \geq \frac{1}{r}|m_r(\mathbb{P}^n\mu)-m_r(\mu)|, \tag{3.12}$$

we therefore have that for $n=1,2,\dots$

$$m_r(\mathbb{P}^n\mu)\le\frac{r}{1-\lambda}\|\mathbb{P}\mu-\mu\|_r+m_r(\mu):=\kappa<\infty. \tag{3.13}$$

The sequence of probability measures $\{\mu,\mathbb{P}\mu,\mathbb{P}^2\mu,\dots\}$ is tight and therefore there is μ^* and subsequence (n_k) such that $\mathbb{P}^{n_k}\mu\Rightarrow\mu^*$ as $k\to\infty$. By (2.1) and Corollary A.1 we have that $\|\mathbb{P}^{n_k}\mu-\mu^*\|_{\mathcal{K}}\to 0$ as $k\to\infty$. Consequently $\mu^*\in D$. Since for $K>0$

$$\int_{\mathbb{R}_+}(x^r\wedge K)\mathbb{P}^{n_k}\mu(dx)\le m_r(\mathbb{P}^{n_k}\mu)\le\kappa \tag{3.14}$$

and therefore letting $k\to\infty$ we have that $\int_{\mathbb{R}_+}(x^r\wedge K)\mu^*(dx)\le\kappa$, which by Fatou lemma gives that $m_r(\mu^*)\le\kappa$. Now

$$\|\mathbb{P}^{n_k}\mathbb{P}\mu-\mathbb{P}^{n_k}\mu\|_{\mathcal{K}}=\|\mathbb{P}\mathbb{P}^{n_k}\mu-\mathbb{P}^{n_k}\mu\|_{\mathcal{K}}\to\|\mathbb{P}\mu^*-\mu^*\|_{\mathcal{K}}:=\beta \tag{3.15}$$

and

$$\begin{aligned}&\|\mathbb{P}\mathbb{P}\mu^*-\mathbb{P}\mu^*\|_{\mathcal{K}}=\lim_{k\to\infty}\|\mathbb{P}\mathbb{P}\mathbb{P}^{n_k}\mu-\mathbb{P}\mathbb{P}^{n_k}\mu\|_{\mathcal{K}}\ge\\ &\lim_{k\to\infty}\|\mathbb{P}\mathbb{P}^{n_k+1}\mu-\mathbb{P}^{n_k+1}\mu\|_{\mathcal{K}}=\|\mathbb{P}\mu^*-\mu^*\|_{\mathcal{K}}=\beta,\end{aligned} \tag{3.16}$$

so that taking into account that $\|\mathbb{P}\mathbb{P}\mu^*-\mathbb{P}\mu^*\|_{\mathcal{K}}\le\|\mathbb{P}\mu^*-\mu^*\|_{\mathcal{K}}$ we obtain that $\|\mathbb{P}\mathbb{P}\mu^*-\mathbb{P}\mu^*\|_{\mathcal{K}}=\|\mathbb{P}\mu^*-\mu^*\|_{\mathcal{K}}$. By small modification of (3.15) and (3.16) we obtain that for any $n=0,1,\dots$

$$\|\mathbb{P}^{n+1}\mu^*-\mathbb{P}^n\mu^*\|_{\mathcal{K}}=\|\mathbb{P}\mu^*-\mu^*\|_{\mathcal{K}}. \tag{3.17}$$

On the other hand by (3.6) we have that $\|\mathbb{P}^{n+1}\mu^*-\mathbb{P}^n\mu^*\|_r\le\lambda^n\|\mathbb{P}\mu^*-\mu^*\|_r$. Therefore $\lim_{n\to\infty}\|\mathbb{P}^{n+1}\mu^*-\mathbb{P}^n\mu^*\|_r=0$. By Theorem A.3 we also have that $\lim_{n\to\infty}\|\mathbb{P}^{n+1}\mu^*-\mathbb{P}^n\mu^*\|_{\mathcal{K}}=0$. Therefore $\|\mathbb{P}\mu^*-\mu^*\|_{\mathcal{K}}=0$ and μ^* is a fixed point of $\mathbb{P}$. If there is another weak limit ν^* of subsequence of $\mathbb{P}^n\mu$, then $\|\mathbb{P}^n\mu^*-\mathbb{P}^n\nu^*\|_r\to 0$, as $n\to\infty$ and by Theorem A.3 again we have that $\|\mu^*-\nu^*\|_{\mathcal{K}}=\lim_{n\to\infty}\|\mathbb{P}^{n+1}\mu^*-\mathbb{P}^n\mu^*\|_{\mathcal{K}}=0$. Consequently any weak limit of a subsequence of $\mathbb{P}^n\mu$ is equal to μ^*, which means that $\mathbb{P}^n\mu\Rightarrow\mu^*$ and the convergence also holds in Kantorovich - Wasserstein metric. □

Remark 3.1. We may not exclude the case in which we have another fixed point $\nu^*\in D$ of $\mathbb{P}$ such that $m_r(\nu^*)=\infty$ for each $r>1$. Notice furthermore that in the case when φ_i is uniformly distributed over the interval $[0,\frac{1}{i}]$ we have $m_r(\varphi_i)=\frac{1}{r+1}\left(\frac{2}{i}\right)^r<\frac{1}{i}$ for all $r\in(1,2)$ and $i\in\Lambda\setminus\{1\}$. Therefore if (3.2) is satisfied and $1\notin\Lambda$, by the above theorem we have existence of a unique fixed point of $\mathbb{P}$ in D with finite r-th moment.

4. Contraction property of the operator $\mathbb{P}$

We have the following generalization of Theorem 5.2.2 of [14]

Theorem 4.1. *Assume for some $i\in\Lambda$ we have that 0 is an accumulation point of φ_i, where $m_1(\varphi_i)=\frac{1}{i}$. Then for $\mu,\nu\in D$ such that $\mu\neq\nu$ and*

$$supp(P_{*(i-1)}(\mu+\nu))=\mathbb{R}_+ \tag{4.1}$$

we have

$$\|\mathbb{P}_i\mu-\mathbb{P}_i\nu\|_{\mathcal{K}}<\|\mu-\nu\|_{\mathcal{K}} \tag{4.2}$$

and consequently

$$\|\mathbb{P}\mu-\mathbb{P}\nu\|_{\mathcal{K}}<\|\mu-\nu\|_{\mathcal{K}}. \tag{4.3}$$

Proof. Recall that $\mathbb{P}_i=P_{\varphi_i}P_{*_i}$ and assume that $\|\mathbb{P}_i\mu-\mathbb{P}_i\nu\|_{\mathcal{K}}=\|\mu-\nu\|_{\mathcal{K}}$. By Theorem A.4 there is $f_0\in\mathcal{K}$ such that

$$\|\mathbb{P}_i\mu-\mathbb{P}_i\nu\|_{\mathcal{K}}=\langle f_0,\mathbb{P}_i\mu-\mathbb{P}_i\nu\rangle. \tag{4.4}$$

Then

$$\|\mu-\nu\|_{\mathcal{K}}=\int_{\mathbb{R}_+{}^{i+1}} f_0((x_1+x_2+\ldots+x_i)r)\varphi_i(dr)$$
$$[\mu(dx_1)\mu(dx_2)\ldots\mu(dx_i)-\nu(dx_1)\nu(dx_2)\ldots\nu(dx_i)]=\langle f_1,\mu-\nu\rangle, \tag{4.5}$$

where

$$f_1(x)=\int_{\mathbb{R}_+{}^{i}} f_0((x+x_2+\ldots+x_i)r)\varphi_i(dr)\,[\mu(dx_2)\ldots\mu(dx_i)+$$
$$\nu(dx_2)\mu(dx_3)\ldots\mu(dx_i)+\nu(dx_2)\nu(dx_3)\mu(dx_4)\ldots\mu(dx_i)+\ldots+$$
$$\nu(dx_2)\nu(dx_3)\nu(dx_4)\ldots\mu(dx_i)+\nu(dx_2)\nu(dx_3)\nu(dx_4)\ldots\nu(dx_i)]. \tag{4.6}$$

Clearly, using again the fact that $m_1(\varphi_i)=\frac{1}{i}$ we have that $f_1\in\mathcal{K}$.

By Theorem A.4 there are two points $x_1,x_2\in\mathbb{R}_+$ such that $x_1<x_2$ and $|f_1(x_2)-f_1(x_1)|=x_2-x_1$. Since f_1 is nonexpansive (Lipschitz with constant less or equal to 1) we have that $f_1(x)=\theta x+\sigma$, for $x\in(x_1,x_2)$ with $|\theta|=1$. Therefore $|f_1(x_1+\varepsilon)-f_1(x_1)|=\varepsilon$ for $\varepsilon\in(0,x_2-x_1)$. Replacing f_0 by $-f_0$ we may assume that $f_1(x_1+\varepsilon)-f_1(x_1)=\varepsilon$ for $\varepsilon\in(0,x_2-x_1)$. We are going now to show that for $x\in\mathbb{R}_+$

$$f_0(x)=x+c \tag{4.7}$$

with a constant $c\in\mathbb{R}$. Consider now $u_1,u_2\in\mathbb{R}_+$ such that $u_1<u_2$. We want to show that then

$$f_0(u_2)-f_0(u_1)\geq u_2-u_1, \tag{4.8}$$

which by nonexpansiveness of f_0 implies that $f_0(u_2)-f_0(u_1)=u_2-u_1$ and therefore f_0 is of the form (4.7). Assume conversely that $f_0(u_2)-f_0(u_1)<u_2-u_1$. Since f_0 as a Lipschitzian mapping is almost everywhere differentiable there is $\bar{u}\in(u_1,u_2)$ such that $f_0'(\bar{u})<1$ and for $\delta\in(0,\delta_0)$ we have

$$\frac{f_0(\bar{u}+\delta)-f_0(\bar{u})}{\delta}<1. \tag{4.9}$$

Define

$$h(y_2,\ldots,y_i,r,\varepsilon)=\frac{f_0((x_1+\varepsilon+y_2+\ldots+y_i)r)-f_0((x_1+y_2+\ldots+y_i)r)}{\varepsilon r}. \tag{4.10}$$

By definition of f_1 for $\varepsilon\in(0,x_2-x_1)$ we have

$$1=\frac{f_1(x_1+\varepsilon)-f_1(x_1)}{\varepsilon}=\int_{(\mathbb{R}_+)^{i-1}}\int_{\mathbb{R}_+} h(y_2,\ldots,y_i,r,\varepsilon)r\phi_i(dr)\,[\mu(dy_2)$$
$$\ldots\mu(dy_i)+\nu(dy_2)\mu(dy_3)\ldots\mu(dy_i)+\nu(dy_2)\nu(dy_3)\mu(dy_4)\ldots\mu(dy_i)+$$

$$\ldots + \nu(dy_2)\nu(dy_3)\nu(dy_4)\ldots\mu(dy_i) + \nu(dy_2)\nu(dy_3)\nu(dy_4)\ldots\nu(dy_i)] :=$$
$$\int_{(\mathbb{R}_+)^i} h(y_2,\ldots,y_i,r,\varepsilon)q(dy_2,\ldots,dy_i,dr), \tag{4.11}$$

where we define implicitly, probability measure q. Since 0 is an accumulation point of $supp\phi$ there is $\bar{r} \in supp\phi$ such that $x_1\bar{r} < \bar{u}$. Then there is $(\bar{y}_2,\bar{y}_3,\ldots,\bar{y}_i) \in supp(P_{*(i-1)}(\mu+\nu))$ such that

$$\bar{u} - x_1\bar{r} = (\bar{y}_2 + \bar{y}_3 + \ldots, \bar{y}_i)\bar{r}. \tag{4.12}$$

Consequently for every $\bar{\varepsilon} \in (0, x_2 - x_1)$ such that $\bar{\varepsilon}\bar{r} < \delta_0$ we have

$$h(\bar{y}_2 + \bar{y}_3 + \ldots, \bar{y}_i, \bar{r}, \bar{\varepsilon}) = \frac{f_0(\bar{u} + \bar{\varepsilon}\bar{r}) - f_0(\bar{u})}{\bar{\varepsilon}\bar{r}} < 1. \tag{4.13}$$

Since $h \le 1$ by continuity of h and full support of $P_{*(i-1)}(\mu+\nu)$ we have that

$$\int_{(\mathbb{R}_+)^i} h(y_2,\ldots,y_i,r,\varepsilon)q(dy_2,\ldots,dy_i,dr) < 1, \tag{4.14}$$

a contradiction to (4.11). Therefore we have equality in (4.8). Consequently $f_0(x) = x + c$ for a constant c. Since $\mathbb{P}_i\mu$ and $\mathbb{P}_i\nu \in D$ we therefore have $\langle f_0, \mathbb{P}_i\mu - \mathbb{P}_i\nu\rangle = m_1(\mathbb{P}_i\mu) - m_1(\mathbb{P}_i\nu) = 0$ and by (4.4)

$$\|\mathbb{P}_i\mu - \mathbb{P}_i\nu\|_{\mathcal{K}} = \|\mu - \nu\|_{\mathcal{K}} = 0,$$

which contradicts the fact that $\mu \neq \nu$. □

Remark 4.1. Condition $supp(P_{*(i-1)}(\mu+\nu)) = \mathbb{R}_+$ is not very restrictive. It holds in particular when $supp(\mu+\nu) = \mathbb{R}_+$ or when $supp\mu = \mathbb{R}_+$.

We have the following consequences of Theorem 4.1.

Corollary 4.1. *If for some $i \in \Lambda$ we have that 0 is an accumulation point of ϕ_i and μ^* is a weak accumulation point of $\mathbb{P}^n\mu$ for $\mu \in D$ i.e. there is a sequence (n_k) such that $\mathbb{P}^{n_k}\mu \Rightarrow \mu^*$, as $k \to \infty$ and $supp\mu^* = \mathbb{R}_+$, $\mu^* \in D$, then μ^* is a fixed point of $\mathbb{P}$.*

Proof. Notice that $m_1(\mathbb{P}^{n_k}\mu) = m_1(\mu^*)$ so that by Theorem A.2 we have that $\|\mathbb{P}^{n_k}\mu - \mu^*\|_{\mathcal{K}} \to 0$ as $k \to \infty$. Therefore

$$\|\mathbb{P}^{n_{k+1}}\mathbb{P}\mu - \mathbb{P}^{n_{k+1}}\mu\|_{\mathcal{K}} \le \|\mathbb{P}^{n_k+1}\mathbb{P}\mu - \mathbb{P}^{n_k+1}\mu\|_{\mathcal{K}} \le \|\mathbb{P}^{n_k}\mathbb{P}\mu - \mathbb{P}^{n_k}\mu\|_{\mathcal{K}} \le \|\mathbb{P}\mu - \mu\|_{\mathcal{K}}, \tag{4.15}$$

so that there is a limit $\lim_{k\to\infty} \|\mathbb{P}^{n_k}\mathbb{P}\mu - \mathbb{P}^{n_k}\mu\|_{\mathcal{F}_0} := \beta$ and by continuity

$$\|\mathbb{P}^{n_k}\mathbb{P}\mu - \mathbb{P}^{n_k}\mu\|_{\mathcal{K}} = \|\mathbb{P}\mathbb{P}^{n_k}\mu - \mathbb{P}^{n_k}\mu\|_{\mathcal{K}} \to \|\mathbb{P}\mu^* - \mu^*\|_{\mathcal{K}} = \beta. \tag{4.16}$$

If $\mathbb{P}\mu^* \neq \mu^*$ and assumptions of Corollary 4.1 are satisfied, then by Theorem 4.1 we have that $\|\mathbb{P}\mathbb{P}\mu^* - \mathbb{P}\mu^*\|_{\mathcal{K}} < \|\mathbb{P}\mu^* - \mu^*\|_{\mathcal{K}} = \beta$, while

$$\|\mathbb{P}\mathbb{P}\mu^* - \mathbb{P}\mu^*\|_{\mathcal{K}} = \lim_{k\to\infty} \|\mathbb{P}\mathbb{P}\mathbb{P}^{n_k}\mu - \mathbb{P}\mathbb{P}^{n_k}\mu\|_{\mathcal{K}} \ge$$
$$\lim_{k\to\infty} \|\mathbb{P}\mathbb{P}^{n_{k+1}}\mu - \mathbb{P}^{n_{k+1}}\mu\|_{\mathcal{K}} = \|\mathbb{P}\mu^* - \mu^*\|_{\mathcal{K}} = \beta \tag{4.17}$$

and we have a contradiction. Therefore $\mathbb{P}\mu^* = \mu^*$. □

Corollary 4.2. *If for some $i\in\Lambda$ we have that 0 is an accumulation point of ϕ_i and $\mu^*\in D$ is a fixed point of $\mathbb{P}$, then there is no other fixed point $\nu^*\in D$. Consequently for $\nu\in D$ we have that any weakly convergent subsequence $\mathbb{P}^{n_k}\nu$ either converges to μ^*, as $n\to\infty$, or to a measure $\tilde{\mu}\in\bar{D}\setminus D$.*

Proof. By Proposition 2.3 since $\mu^*\in D$ we have that $supp\mu^*=\mathbb{R}_+$. Then we use again Theorem 4.1. Namely when $\nu^*\in D$ is another fixed point of $\mathbb{P}$ we have

$$\|\mu^*-\nu^*\|_{\mathcal{K}}=\|\mathbb{P}\mu^*-\mathbb{P}\nu^*\|_{\mathcal{K}}<\|\mu^*-\nu^*\|_{\mathcal{K}}, \tag{4.18}$$

which is a contradiction. Any sequence $(\mathbb{P}^n\nu)$ is compact in Fortet-Mourier metric and its subsequence converges to a measure $\tilde{\mu}$, and the convergence is also in Kantorovich-Wasserstein metric whenever $m_1(\mu)=1$. In the last case we have $\tilde{\mu}=\mu^*$ by uniqueness of fixed point of $\mathbb{P}$ in D. □

Using Theorem A.1 we can now strengthen the last corollary.

Corollary 4.3. *Assume there is $i\in\Lambda$ such that 0 be an accumulation point of φ_i. Assume that $\mu^*\in D$ is a fixed point of $\mathbb{P}$. Then for $\nu\in D$ for any limit of weakly convergent subsequence $\mathbb{P}^{n_k}\nu$ which belongs to D is also in Kantorovich-Wasserstein metric. Furthermore if the sequence $n\to\mathbb{P}^n\nu$ is uniformly integrable we have*

$$\lim_{n\to\infty}\|\mathbb{P}^n\nu-\mu^*\|_{\mathcal{K}}=0. \tag{4.19}$$

Proof. As above using Proposition 2.3, since $\mu^*\in D$, we have that $supp\mu^*=\mathbb{R}_+$. The first part easily follows from Corollary 4.2. It remains to notice only that uniform integrability of $n\to\mathbb{P}^n\nu$ together with weak compactness implies compactness of $n\to\mathbb{P}^n\nu$ in Kantorovich-Wasserstein metric. Since any subsequence converges to the same limit in Kantorovich-Wasserstein metric, we have (4.19). □

5. Asymptotic stability of the nonlinear Boltzmann-type equation

Consider now the following equation in the space of signed measures

$$\frac{d\psi}{dt}+\psi=\mathbb{P}\psi \qquad \text{for} \qquad t\geq 0 \tag{5.1}$$

with initial condition

$$\psi(0)=\psi_0, \tag{5.2}$$

where $\psi_0\in\mathcal{M}_1(\mathbb{R}_+)$ and $\psi:\mathbb{R}_+\to\mathcal{M}_{sig}(\mathbb{R}_+)$. In this section we show that the equation (1.1) may by considered in a convex closed subset D of a vector space of signed measures $\mathcal{M}_{sig}(\mathbb{R}_+)$. This approach seems to be quite natural and it is related to the classical results concerning the semigroups and differential equations on convex subsets of Banach spaces (see [11,19]). For details see Appendix.

We finish the paper with sufficient conditions for the asymptotic stability of solutions of the Equation (5.1) with respect to Kantorovich–Wasserstein metric.

Equation (5.1) together with the initial condition (5.2) may be considered in a convex subset D of the vector space of finite signed measures $\mathcal{M}_{sig}$. We have

Corollary 5.1. *Assume $\varphi_i\in\mathcal{M}_1, i\in\{1,2,...,\}$ is such that $m_1(\varphi_i)=\frac{1}{i}$, $\mathbb{P}$ is given by (1.9) with $\sum_{i=1}^{\infty}\alpha_i=1$, $\alpha_i\geq 0$. Then for every $\psi_0\in D$ there exists a unique solution ψ of problem (5.1), (5.2) taking values in D.*

The solutions of (5.1), (5.2) generate a semigroup of Markov operators $(P^t)_{t\geq 0}$ on D given by

$$P^t u_0 = u(t) \qquad \text{for} \qquad t\in\mathbb{R}_+, \qquad u_0\in D. \tag{5.3}$$

Now using Theorem 4.1 we can easily derive the following result.

THEOREM 5.1. *Let $\mathbb{P}$ be the operator given by (1.9). Moreover, let $(\varphi_1,\varphi_2,\ldots)$ be a sequence of probability measures such that $m_1(\varphi)=\frac{1}{i}$ and $\alpha_i\geq 0$ be a sequence of nonnegative numbers such that $\sum_{i=1}^{\infty}\alpha_i=1$. Assume that 0 is an accumulation point of $supp\varphi_i$ for some $i\in\Lambda$. If $\mathbb{P}$ has a fixed point $\mu^*\in D$ then*

$$\lim_{t\to\infty}||\psi(t)-\mu^*||_{\mathcal{K}}=0 \tag{5.4}$$

for every sequentially compact (in Kantorovich-Wasserstein metric) solution ψ of (5.1), (5.2).

Proof. First we have to prove that $(P^t)_{t\geq 0}$ is nonexpansive on D with respect to Kantorovich-Wasserstein metric. For this purpose let $\eta_0,\vartheta_0\in D$ be given. For $t\in\mathbb{R}_+$ define

$$v(t)=P^t\eta_0-P^t\vartheta_0. \tag{5.5}$$

Using (A.13), Corollary 2.1 and (5.3) it is easy to see that

$$v(t)=e^{-t}v_0+\int_0^t e^{-(t-s)}(\mathbb{P}(P^s\eta_0)-\mathbb{P}(P^s\vartheta_0))ds \quad \text{for} \quad t\in\mathbb{R}_+. \tag{5.6}$$

From Corollary 5.1 it follows immediately that

$$||v(t)||_{\mathcal{K}}\leq e^{-t}||v(0)||_{\mathcal{K}}+\int_0^t e^{-(t-s)}||v(s)||_{\mathcal{K}}ds \quad \text{for} \quad t\in\mathbb{R}_+. \tag{5.7}$$

Consequently

$$f(t)\leq ||v(0)||_{\mathcal{K}}+\int_0^t f(s)ds \quad \text{for} \quad t\in\mathbb{R}_+, \tag{5.8}$$

where $f(t)=e^t||v(t)||_{\mathcal{K}}$. From the Gronwall inequality it follows that

$$f(t)\leq e^t||v(0)||_{\mathcal{K}}. \tag{5.9}$$

This is equivalent to the fact that $(P^t)_{\geq t}$ is nonexpansive on D with respect to Kantorovich - Wasserstein metric. Notice that μ^*, as a fixed point of $\mathbb{P}$, is a stationary solution to the Equation (5.1) i.e. $P^t\mu^*=\mu^*$. To complete the proof it is sufficient to verify condition (A.2) of Theorem A.1.

From (A.13) and Proposition 2.3 and Theorem 4.1 it follows immediately that for $\psi_0\in D$ and $t>0$

$$\|P^t\psi_0-\mu^*\|_{\mathcal{K}}\leq e^{-t}\,\|\psi_0-\mu^*\|_{\mathcal{K}}+\int_0^t e^{-(t-s)}\,\|\mathbb{P}P^s\psi_0-\mathbb{P}\mu^*\|_{\mathcal{K}}\,ds$$

$$< e^{-t} \| \psi_0 - \mu^* \|_{\mathcal{K}} + (1-e^{-t}) \| P^s \psi_0 - \mu^* \|_{\mathcal{K}} \leq \| \psi_0 - \mu^* \|_{\mathcal{K}}. \tag{5.10}$$

By Theorem A.1 we immediately obtain (5.4). □

We shall now study nonlinear Boltzmann Equation (5.1) using Zolotariev seminorm following the results of [19]. Consider time discretized version of (5.1) with discretization step $h \in (0,1)$

$$\frac{d\psi_h}{dt}(d_h(t)) + \psi_h(d_h(t)) = \mathbb{P}\psi_h(d_h(t)) \qquad \text{for} \qquad t \geq 0 \tag{5.11}$$

with initial condition

$$\psi_h(0) = \psi_0, \tag{5.12}$$

where $\psi_0 \in \mathcal{M}_1(\mathbb{R}_+)$ and $d_h(t) = nh$ for $t \in [nh,(n+1)h)$. Then

$$\psi_h((n+1)h) = (1-h)\psi_h(nh) + h\mathbb{P}\psi_h(nh) := \mathbb{P}_h\psi_h(nh). \tag{5.13}$$

Notice that fixed point of the operator $\mathbb{P}$ is also a fixed point of $\mathbb{P}_h$ and vice versa. We have:

Lemma 5.1. *Under assumptions of Theorem 3.1 we have that*

(i) *when $m_r(\mu) < \infty$ then $m_r(\mathbb{P}^n_h\mu) < \infty$ for any positive integer n and $\mu \in \mathcal{M}_1(\mathbb{R}_+)$,*

(ii) *$\|\mathbb{P}_h\mu - \mathbb{P}_h\nu\|_r \leq \lambda_h \|mu - \nu\|_r$ for $\mu,\nu \in \mathcal{M}_1(\mathbb{R}_+)$ such that $m_r(\mu) < \infty$, $m_r(\nu) < \infty$ with $\lambda_h = 1 - h(1-\lambda)$, where $\lambda = \sum_{i=1}^{\infty} \alpha_i m_r(\varphi_i) i$,*

(iii) *$\|\mathbb{P}^n_h\mu - \mu^*\|_{\mathcal{K}} \leq 2^{1+\frac{1}{r}} \lambda_h^{\frac{n}{r}} (\|\mu - \mu^*\|_r)^{\frac{1}{r}} \leq K$ with $\mu^* \in D$ being the unique fixed point of $\mathbb{P}$ and $K = \frac{1}{r} 2^{1+\frac{1}{r}} (m_r(\mu) + m_r(\mu^*))$.*

Proof. Under (3.2) using Proposition 2.1 we have that $m_r(\mathbb{P}\mu) < \infty$ and therefore also $m_r(\mathbb{P}_h\mu) < \infty$. Then (i) follows by induction. (ii) follows directly from the definition of $\mathbb{P}$ and (3.6). For fixed point μ^* of $\mathbb{P}$, which is also a fixed point of $\mathbb{P}_h$ and $\mu \in \mathcal{M}_1(\mathbb{R}_+)$ such that $m_r(\mu) < \infty$ we have

$$\|\mathbb{P}^n_h\mu\|_r \leq \lambda^n_h \|\mu - \nu\|_r. \tag{5.14}$$

Therefore using Theorem A.3 and then (3.3) we obtain

$$\|\mathbb{P}^n_h\mu - \mu^*\|_{\mathcal{K}} \leq 2^{1+\frac{1}{r}} \lambda_h^{\frac{n}{r}} (\|\mu - \mu^*\|_{\mathcal{K}})^{\frac{1}{r}} \leq K \tag{5.15}$$

with $K = \frac{1}{r} 2^{1+\frac{1}{r}} (m_r(\mu) + m_r(\mu^*))$. □

We recall now Lemma 3 of [19].

Lemma 5.2. *Under the assumptions of Theorem 3.1, when $\mu = \psi_0$ is such that $m_r(\mu) < \infty$, we have*

$$\|\psi_h(t) - \psi(t)\|_{\mathcal{K}} \leq 4Kh(e^{2t} - 1). \tag{5.16}$$

We can now formulate:

Theorem 5.2. *Under assumptions of Theorem 3.1 when $\mu = \psi_0$ is such that $m_r(\mu) < \infty$ we have that*

$$\|\psi(t) - \mu^*\|_{\mathcal{K}} \leq K e^{-\frac{t}{r}(1-\lambda)}. \tag{5.17}$$

Proof. We follow the arguments of the proof of Theorem 1 of [19]. Fix $t>0$ and for a given $\varepsilon>0$ find positive integer n such that $\frac{t}{n}4K(e^{2t}-1)\leq\varepsilon$. Then by Lemma 5.2

$$\|\psi(t)-\mu^*\|_{\mathcal{K}}\leq\|\psi(t)-\psi_h(t)\|_{\mathcal{K}}+\|\psi_h(t)-\mu^*\|_{\mathcal{K}}\leq\varepsilon+K(1-\frac{t}{n}(1-\lambda))^{\frac{n}{r}}. \tag{5.18}$$

Since $1-x\leq e^{-x}$ for $x\geq 0$ we have that $(1-\frac{t}{n}(1-\lambda))^{\frac{n}{r}}\leq e^{-\frac{t}{r}(1-\lambda)}$ and the claim follows from (5.18) taking into account that ε could be chosen arbitrarily small. ∎

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Appendix. On a given complete metric space (E,ρ) consider a continuous operator T or continuous semigroup (T_t), for $t\geq 0$ transforming (E,ρ) into itself. Denote by $\omega(x)$ the set of all limiting points of the trajectory $n\to T^n x$ or $t\to T_t x$ respectively. We say that $n\to T^n x$ or $t\to T_t x$ is sequentially compact if from every sequence $T^{n_k}x$, $T_{t_{n_k}}x$ respectively, one could choose a convergent subsequence. Let $\mathcal{Z}$ be the set of all x such that the trajectory $t\to T_t x$ ($n\to T^n x$) is sequentially compact. We shall assume that $\mathcal{Z}$ is a nonempty set and let $\Omega=\bigcup_{\mu\in\mathcal{Z}}\omega(\mu)$. We have the following result formulated for semigroup T_t, which naturally holds for continuous operator T.

THEOREM A.1 (see Theorem 5.1.2 of [14]). *Assume that T_t is nonexpansive i.e.*

$$\rho(T_t x,T_t y)\leq\rho(x,y) \tag{A.1}$$

for $t\geq 0$ and there is $x^*\in\Omega$ such that for every $x\in\Omega$, $x\neq x^*$ there is $t(x)$ such that

$$\rho(T_{t(x)}x,T_{t(x)}x^*)<\rho(x,x^*). \tag{A.2}$$

Then for $z\in\mathcal{Z}$ we have

$$\lim_{t\to\infty}\rho(T_t z,x^*)=0. \tag{A.3}$$

DEFINITION A.1. *Sequence of probability measures μ_n defined on $\mathbb{R}_+$ is uniformly integrable when*

$$\sup_n\int_M^\infty x\mu_n(dx)\to 0, \tag{A.4}$$

whenever $M\to\infty$.

THEOREM A.2. *Assume that for sequence of probability measures μ_n defined on $\mathbb{R}_+$ we have that $m_1(\mu_n)<\infty$ and $\mu_n\Rightarrow\mu$, as $n\to\infty$. Then $m_1(\mu_n)\to m_1(\mu)$ if and only if measures μ_n are uniformly integrable. Furthermore for sequence of probability measures μ_n defined on $\mathbb{R}_+$ such that $m_1(\mu_n)<\infty$ and we have that convergence $\mu_n\Rightarrow\mu$, as $n\to\infty$, together with convergence of $m_1(\mu_n)\to m_1(\mu)$ is equivalent to convergence $\|\mu_n-\mu\|_{\mathcal{K}}\to 0$.*

Proof. By Skorokhod theorem (25.6 of [3]) there is a probability space (Ω,F,P) and nonnegative random variables X_n, X with laws μ_n and μ respectively such that $X_n(\omega)\to X(\omega)$ for each $\omega\in\Omega$. Uniform integrability of μ_n is equivalent to uniform integrability of X_n. By Theorem II T21 of [22] uniform integrability of X_n is equivalent to the convergence $m_1(\mu_n)\to m_1(\mu)$. To prove the last statement of the theorem, notice that when $\|\mu_n-\mu\|_{\mathcal{K}}\to 0$ we have also $\|\mu_n-\mu\|_{\mathcal{F}}\to 0$, so that $\mu_n\Rightarrow\mu$ and $m_1(\mu_n)\to m_1(\mu)$. Assume now that $\mu_n\Rightarrow\mu$, as $n\to\infty$ and $m_1(\mu_n)\to m_1(\mu)$. Then X_n defined above converges to X in $L^1(P)$ norm. In particular for any function f with Lipschitz constant not greater than 1 we have

$$|\mu_n(f)-\mu(f)|=|E[f(X_n)]-E[f(X)]|\le E[|f(X_n)-f(X)|]\le E|X_n-X|\to 0 \tag{A.5}$$

as $n\to\infty$, which means that we have also convergence in $\|\cdot\|_{\mathcal{K}}$ norm, which completes the proof. □

Remark A.1. The result above in not unexpected. For given measure $\mu\in D$ define $\bar{\mu}(A):=\int_A x\mu(dx)$ for Borel measurable set A. Then compactness of the closure of the sequence $\{\bar{\mu}_n\in D\}$ is by Theorem 6.2 of [4] equivalent to the tightness of measures $\{\bar{\mu}_n\}$, which is equivalent to (A.4).

Corollary A.1. *Whenever $D\ni\mu_n\Rightarrow\mu$ and $\sup_n m_\beta(\mu_n)<\infty$ for some $\beta>1$ we have $\|\mu_n-\mu\|_{\mathcal{K}}\to 0$ as $n\to\infty$.*

Proof. It is clear that

$$\sup_n\int_M^\infty x\mu_n(dx)\le\sup_n\Big(\int_0^\infty x^\beta\mu_n(dx)\Big)^{\frac{1}{\beta}}\Big(\int_0^\infty 1_{x\ge M}\mu_n(dx)\Big)^{\frac{\beta-1}{\beta}}. \tag{A.6}$$

Now $\int_0^\infty 1_{x\ge M}\mu_n(dx)\le\frac{1}{M}$, so that μ_n is uniformly integrable and it remains to use Theorem A.2. □

Before we formulate the next theorem we define metric d_r in the space of probability measures defined on $\mathbb{R}_+$ with finite r-th moments, where $r\in[1,2)$. Namely for probability measures μ and ν such that $m_r(\mu)<\infty$ and $m_r(\nu)<\infty$ let

$$d_r(\mu,\nu):=\inf\Big\{(E(|X-Y|^r))^{\frac{1}{r}}\Big\}, \tag{A.7}$$

where infimum is taken over probability measures P on $\mathbb{R}_+^2$ such that their marginals are μ and ν respectively. We have

Theorem A.3. *For $\mu,\nu\in D$ such that $m_r(\mu)<\infty$ and $m_r(\nu)<\infty$ with $r\in(1,2)$ we have*

$$\|\mu-\nu\|_{\mathcal{K}}\le 2(2\|\mu-\nu\|_r)^{\frac{1}{r}}. \tag{A.8}$$

Proof. By [25] we have that $d_r(\mu,\nu)\le 2(2\|\mu-\nu\|_r)^{\frac{1}{r}}$. Clearly $d_1(\mu,\nu)\le d_r(\mu,\nu)$. Since by Theorem 20.1 of [12] $d_1(\mu,\nu)=\|\mu-\nu\|_{\mathcal{K}}$ we obtain (A.8). □

We recall now Kantorovich-Rubinstein maximum principle for our metric $\|\cdot\|_{\mathcal{K}}$, see Corollary 6.2 of [24].

Theorem A.4. *For probability measures μ,ν defined on $\mathbb{R}_+$ there exists $f_0\in\mathcal{K}$ such that*

$$\|\mu-\nu\|_{\mathcal{K}}=\langle f_0,\mu-\nu\rangle. \tag{A.9}$$

Moreover when $f_0\in\mathcal{K}$ satisfies (A.9) for measures $\mu\ne\nu$ defined on $\mathbb{R}_+$ then there are two different points $x_1,x_2\in\mathbb{R}_+$ such that $|f_0(x_1)-f_0(x_2)|=|x_1-x_2|$.

Finally we recall now some known results related with ordinary differential equations in Banach spaces. For details see [11].

Let $(E,\|\cdot\|)$ be a Banach space and let $\tilde{D}$ be a closed, convex, nonempty subset of E. In the space E we consider *an evolutionary differential equation*

$$\frac{du}{dt} = -u + \tilde{P}u \qquad \text{for} \qquad t \in \mathbb{R}_+ \tag{A.10}$$

with the initial condition

$$u(0) = u_0, \qquad u_0 \in \tilde{D}, \tag{A.11}$$

where $\tilde{P} : \tilde{D} \to \tilde{D}$ is a given operator.

Function $u : \mathbb{R}_+ \to E$ is called *a solution to the problem* (A.10), (A.11) if it is strongly differentiable on $\mathbb{R}_+$, $u(t) \in \tilde{D}$ for all $t \in \mathbb{R}_+$ and u satisfies relations (A.10), (A.11).

We have:

Theorem A.5. *Assume that the operator $\tilde{P} : \tilde{D} \to \tilde{D}$ satisfies Lipschitz condition*

$$\|\tilde{P}v - \tilde{P}w\| \le l\,\|v - w\| \qquad \text{for} \qquad u, w \in \tilde{D}, \tag{A.12}$$

where l is a nonnegative constant. Then for every $u_0 \in \tilde{D}$ there exists a unique solution u to the problem (A.10), (A.11).

The standard proof of Theorem A.5 is based on the fact, that function $u : \mathbb{R}_+ \to \tilde{D}$ is a solution to (A.10), (A.11) if and only if it is continuous and satisfies the integral equation

$$u(t) = e^{-t} u_0 + \int_0^t e^{-(t-s)}\, \tilde{P}u(s)\, ds \qquad \text{for} \qquad t \in \mathbb{R}_+. \tag{A.13}$$

Due to completeness of $\tilde{D}$, the integral on the right-hand side is well defined and equation (A.13) may be solved by the method of successive approximations. Observe that, thanks to the properties of $\tilde{D}$, for every $u_0 \in \tilde{D}$ and for every continuous function $u : \mathbb{R}_+ \to \tilde{D}$ the right-hand side of (A.13) is also a function with values in $\tilde{D}$. The solutions of (A.13) generate a semigroup of operators $(\tilde{P}^{\,t})_{t \ge 0}$ on $\tilde{D}$ given by the formula

$$\tilde{P}^{\,t} u_0 = u(t) \qquad \text{for} \qquad t \in \mathbb{R}_+, \qquad u_0 \in \tilde{D}. \tag{A.14}$$

REFERENCES

[1] R. Alonso, V. Bagland, Y. Cheng, and B. Lods, *One-dimensional dissipative Boltzmann equation: measure solutions, cooling rate, and self-similar profile*, SIAM J. Math. Anal., 50:1278–1321, 2018. 1

[2] I. Ampatzoglou and N. Pavlovic, *A rigorous derivation of a ternary Boltzmann equation for a classical system of particles*, Commun. Math. Phys., 387:793–863, 2021. 1

[3] H.P. Billingsley, *Probability and Measure*, John Wiley, New York, 1986. 5

[4] H.P. Billingsley, *Convergence of Probability Measures*, John Wiley, New York, 1968. A.1

[5] M. Bisi, J.A. Carillo, and G. Toscani, *Decay rates in probability metrics towards homogeneous cooling states for the inelastic Maxwell model*, J. Stat. Phys., 124:625–653, 2006. 1

[6] A.V. Bobylev, *Exact solutions of the Boltzmann equations*, Sov. Phys. Dokl., 20:822–824, 1976. 1

[7] A.V. Bobylev and C. Cercignani, *Self-similar solutions of the Boltzman equation and their applications*, J. Stat. Phys., 106:1039–1071, 2002. 1

[8] A.V. Bobylev, C. Cercignani, and I.M. Gamba, *On the self-similar asymptotics for generalized non-linear kinetic Maxwell models*, Commun. Math. Phys., 291:599–644, 2009. 1

[9] A.V. Bobylev, C. Cercignani, and G. Toscani, *Proof of an asymptotic property of self-similar solutions of the Boltzmann equation for granular materials*, J. Stat. Phys., 111:403–417, 2003. 1

[10] V.I. Bogachev and A.V. Koleshnikov, *The Monge-Kantorovich problem: achievements, connections, and perspectives*, Russ. Math. Surv., 67:3–110, 2012. 2, 2

[11] M.G. Crandall, *Differential equations on convex sets*, J. Math. Soc. Japan, 22:443–455, 1970. 5, 5

[12] R.M. Dudley, *Probabilities and Metrics*, Aarhaus Universitet, Aarhaus, 1976. 5

[13] S. Ethier and T. Kurtz, *Markov Processes, Characterization and Convergence*, John Wiley and Sons, New York, 1986. 2

[14] H. Gacki, *Applications of the Kantorovich–Rubinstein maximum principle in the theory of Markov semigroups*, Diss. Math., 448:1–59, 2007. 1, 4, A.1

[15] J. Hutchinson, *Fractals and self–similarity*, Indiana Univ. J., 30:713–747, 1981. 2

[16] X. Jia, *Exterior problem for the Boltzmann equation with temperature difference: asymptotic stability of steady solutions*, J. Differ. Equ., 262:3642–3688, 2017. 1

[17] A. Lasota and M.C. Mackey, *Chaos, Fractals and Noise. Stochastic Aspects of Dynamics*, Springer, 1994. 1

[18] A. Lasota and J. Traple, *Kantorovich–Rubinstein maximum principle in the theory of the Tjon–Wu equation*, J. Differ. Equ., 159:578–596, 1999. 1

[19] A. Lasota and J. Traple, *Asymptotic stability of differential equations on convex sets*, J. Dyn. Differ. Equ., 15:335–355, 2003. 1, 3, 5, 5, 5, 5

[20] A. Lasota and J. Traple, *Properties of stationary solutions of a generalized Tjon–Wu equation*, J. Math. Anal. Appl., 335:669–682, 2007. 1, 2

[21] X. Lu and C. Mouhot, *On measure solutions of the Boltzmann equation, part I: Moment production and stability estimates*, J. Differ. Equ., 252:3305–3363, 2012. 1

[22] P.A. Meyer, *Probability and Potentials*, Blaisdell Pub. Co., 1966. 5

[23] Y. Morimoto, S. Wang, and T. Yang, *Measure valued solutions to the spatially homogeneous Boltzmann equation without angular cutoff*, J Stat. Phys., 165:866–906, 2016. 1

[24] S.T. Rachev, *Probability Metrics and the Stability of Stochastic Models*, Wiley, New York, 1991. 5

[25] E. Rio, *Distances minimales et distances idéales*, C.R. Acad. Sci. Paris, Sér. I, 326:1127–1130, 1998. 5

[26] R. Rudnicki and P. Zwolenski, *Model of phenotypic evolution in hermaphroditic populations*, J. Math. Biol., 70:1295–1321, 2015. 1

[27] J.A. Tjon and T.T. Wu, *Numerical aspects of the approach to a Maxwellian equation*, Phys. Rev. A, 19:883–888, 1979. 1

[28] G. Wang and Y. Wang, *Global stability of Boltzmann equation with large external potential for a class of large oscillation data*, J. Differ. Equ., 267:3610–3645, 2019. 1