COMMUN. MATH. SCI.
Vol. 20, No. 4, pp. 897–946

THE CAHN-HILLIARD-BOUSSINESQ SYSTEM WITH SINGULAR POTENTIAL*

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Abstract. We consider a Cahn-Hilliard-Boussinesq system with positive heat diffusivity and singular potential on a two-dimensional bounded domain with suitable boundary conditions. For the corresponding initial and boundary value problem we prove the existence of strong solutions and the well-posedness for weak solutions. Then we set the diffusivity equal to zero. In this case, the model can be viewed as an approximation of the two-dimensional compressible Navier-Stokes-Cahn-Hilliard system proposed in [J. Lowengrub, L. Truskinovsky, Proc. R. Soc. Lond. A., 454:2617–2654, 1998]. In particular, the heat equation turns out to be the continuity equation for the fluid density. In the case of zero diffusivity, existence and uniqueness of weak and strong solutions are established. In addition, we show that the solution to the diffusive problem does converge to the solution to the diffusionless when the diffusion coefficient goes to zero. In particular, we provide an error estimate for strong solutions. The validity of the uniform separation property from the pure states is finally proven for both the cases.

Keywords. Boussinesq equations; Cahn-Hilliard equation; logarithmic potential; weak and strong solutions; uniqueness; strict separation property.

AMS subject classifications. 35Q35; 76T06.

1. Introduction

One of the oldest approaches to multi-phase problems is the phase-field method characterized by the notion of diffuse interface. This means that the transition layer between the phases has a finite size. Thus there is no tracking mechanism for the interface, but the phase state is incorporated into the governing equations. The (diffuse) interface is associated with a smooth, but highly localized variation of the so-called phase-field variable. In the diffuse interface theory, the motion of a mixture of two incompressible viscous fluids and the evolution of the interface that separates them are described by the Model H (see, e.g., [4,30,31]). Although it is assumed that the fluids are macroscopically immiscible, the model accounts for a partial mixing on a small length scale measured by a parameter $\alpha > 0$, called capillary coefficient. Therefore the classical sharp interface between both fluids is replaced by an interfacial region and an order parameter related to the concentration difference of both fluids is introduced, leading to the coupling with the Cahn-Hilliard equation.

A compressible version of the model H is obtained in [40] with a rigorous physical derivation. Consider a mixture of two immiscible substances A and B, which is homogeneously distributed and isothermal. Let Ω be a bounded domain in $\mathbb{R}^N$, $N=2,3$, filled with a binary solution consisting of A and B atoms. We define their relative mass fraction (assumed to be non-uniform) as $\varphi_A(x)$ and $\varphi_B(x)$, with $\varphi_k:\Omega\to[0,1]$, $k=A$, B and $\varphi_A(x)+\varphi_B(x)\equiv 1$. The order parameter is then defined as $\varphi(x):=\varphi_A(x)-\varphi_B(x)$ so that $\varphi:\Omega\to[-1,1]$. The model H is thus expressed through the following compressible

*Received: April 21, 2021; Accepted (in revised form): October 12, 2021. Communicated by Chun Liu.

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Navier-Stokes-Cahn-Hilliard (NSCH) system

$$\begin{cases} \rho\partial_t\mathbf{u}+\rho(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla\pi-\mathrm{div}(\nu(\varphi)D\mathbf{u})-\nabla(\mathrm{div}\ \mathbf{u})=-\alpha\ \mathrm{div}(\rho\nabla\varphi\otimes\nabla\varphi)+\rho\mathbf{g} \\ \rho\partial_t\varphi+\rho\mathbf{u}\cdot\nabla\varphi=\Delta\left(-\frac{\alpha}{\rho}\mathrm{div}(\rho\nabla\varphi)+\Psi'(\varphi)\right) \\ \partial_t\rho+\mathbf{u}\cdot\nabla\rho+\rho\mathrm{div}(\mathbf{u})=0, \end{cases} \tag{1.1}$$

in $\Omega\times(0,T)$, for some given $T>0$, with suitable boundary and initial conditions. Here $\mathbf{u}$ represents the (volume averaged) velocity, $D\mathbf{u}$ is the strain rate tensor, π denotes the pressure, $\nu(\varphi)>0$ is the viscosity of the mixture (possibly depending on φ), $\mathbf{g}=-\gamma\mathbf{e}_n$ is the gravitational force and Ψ is the double-well potential defined by

$$\Psi(s)=\frac{\overline{\alpha}}{2}((1+s)\ln(1+s)+(1-s)\ln(1-s))-\frac{\alpha_0}{2}s^2 \quad \forall s\in[-1,1], \tag{1.2}$$

with $\overline{\alpha}$ such that $0<\overline{\alpha}<\alpha_0$, constants related to the temperature of the mixture. The potential defined in this way is called *singular*. However, many authors (see, e.g., [18,42] and references therein) considered a proper approximation, which avoids the fact that Ψ' is unbounded at the pure phases -1 and 1. The most common choice is a polynomial of fourth degree, typically $\Psi(s)=\frac{1}{4}(s^2-1)^2$ and usually called regular potential. However, the polynomial approximation does not ensure the existence of physical solutions, that is, solutions whose values are in $[-1,1]$, due to the lack of comparison principles for the Cahn-Hilliard equation.

As shown in the Appendix, performing a (formal) perturbation argument [26], we can obtain the following approximating system, which is the main subject of our analysis

$$\begin{cases} \partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\frac{1}{\rho^*}\nabla\pi-\mathrm{div}\left(\frac{\nu(\varphi)}{\rho^*}\nabla\mathbf{u}\right)=-\alpha\ \mathrm{div}(\nabla\varphi\otimes\nabla\varphi)-\frac{\rho}{\rho^*}\gamma\mathbf{e}_n \\ \mathrm{div}\ \mathbf{u}=0 \\ \partial_t\varphi+\mathbf{u}\cdot\nabla\varphi=\Delta\left(-\frac{\alpha}{\rho^*}\Delta\varphi+\frac{1}{\rho^*}\Psi'(\varphi)\right) \\ \partial_t\rho+\mathbf{u}\cdot\nabla\rho=0. \end{cases} \tag{1.3}$$

Assuming now $\rho^*=1$ and $\gamma=-1$, setting $\theta=\rho$, and imposing suitable boundary and initial conditions, we thus have the following problem for $(\mathbf{u},\varphi,\theta)$

$$\begin{cases} \partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla\pi-\mathrm{div}\,(\nu(\varphi)\nabla\mathbf{u})=\mu\nabla\varphi+\theta\mathbf{e}_2 \\ \mathrm{div}\ \mathbf{u}=0 \\ \partial_t\varphi+\mathbf{u}\cdot\nabla\varphi=\Delta\mu \\ \partial_t\theta+\mathbf{u}\cdot\nabla\theta=0 \\ \mu=\ -\alpha\Delta\varphi+\Psi'(\varphi) \\ \mathbf{u}(0)=\mathbf{u}_0, \quad \varphi(0)=\varphi_0, \quad \theta(0)=\theta_0 & \text{in } \Omega \\ \partial_{\mathbf{n}}\varphi=0, \quad \partial_{\mathbf{n}}\mu=0, \quad \mathbf{u}=\mathbf{0} & \text{on } \partial\Omega,\, t\in(0,T), \end{cases} \tag{1.4}$$

where μ is the so-called chemical potential. Observe that the so-called Korteweg force $-\alpha\ \mathrm{div}(\nabla\varphi\otimes\nabla\varphi)$ can be equivalently rewritten as $\mu\nabla\varphi$. Indeed we have

$$\mu\nabla\varphi=\nabla\left(\frac{\alpha}{2}|\nabla\varphi|^2+\Psi(\varphi)\right)-\alpha\ \mathrm{div}(\nabla\varphi\otimes\nabla\varphi),$$

so that the first term on the right-hand side can be viewed as an extra-pressure. Note that the fluid density ρ is here denoted by θ, and it is interpreted as the temperature of the mixture. Indeed, in the Boussinesq approximation the two quantities are linearly dependent: $\rho=\rho_0-\tilde{\alpha}\rho_0(\theta-\theta_0)$, where $\rho_0>0$ and $\tilde{\alpha}>0$ are a given density and the coefficient of thermal expansion, respectively. System (1.4) is also known as Cahn-Hilliard-Boussinesq system without diffusivity. We recall that the mere coupling of Navier-Stokes system with the convection-diffusion equation for θ, named Boussinesq equations, has been widely studied in the literature. We refer the reader, for instance, to [6,11,33,34,36–38,54] for the case without diffusivity and to [7,10,12,33,51] for the diffusive case.

Problem (1.4) with diffusion reads

$$\begin{cases}\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla\pi-\operatorname{div}(\nu(\varphi)\nabla\mathbf{u})=\mu\nabla\varphi+\theta\mathbf{e}_2\\ \operatorname{div}\,\mathbf{u}=0\\ \partial_t\varphi+\mathbf{u}\cdot\nabla\varphi=\Delta\mu\\ \partial_t\theta+\mathbf{u}\cdot\nabla\theta-\kappa\Delta\theta=0\\ \mu=-\alpha\Delta\varphi+\Psi'(\varphi)\\ \mathbf{u}(0)=\mathbf{u}_0,\quad \varphi(0)=\varphi_0,\quad \theta(0)=\theta_0 & \text{in } \Omega\\ \partial_{\mathbf{n}}\varphi=0,\quad \partial_{\mathbf{n}}\mu=0,\quad \mathbf{u}=\mathbf{0},\quad \theta=g & \text{on } \partial\Omega,\, t\in(0,T).\end{cases} \tag{1.5}$$

Here $\kappa>0$ and g is a given boundary datum.

Problems like and (1.4) and (1.5) have been considered in the literature, as far as we know, for regular potentials only. Problem (1.5) in the inviscid case has been analyzed in [52,53] in a two-dimensional bounded domain. The first contribution contains global existence and uniqueness of smooth solutions with smooth initial data. The latter is concerned with large-time behavior of solutions. The inviscid case is also considered in [15]. Here, some blow-up criteria for smooth solutions in three dimensional bounded domains are shown. In [41] the viscous case with no diffusivity and the inviscid case with diffusivity are analyzed. Global well-posedness results in a two-dimensional bounded domain are obtained. The viscous case with no diffusivity is also considered in [14]. The existence of a strong solution is proven, passing through the strong solutions to the system with diffusion, letting the thermal conductivity vanish.

On the other hand, it is worth recalling that the theoretical literature on NSCH systems is much richer. We just mention some contributions. The incompressible NSCH system was firstly considered in [46] and [8] (see also [5, 21–23] and the references therein). The case of singular potential and constant mobility was analyzed in [1] and, more recently, in [28]. Concerning the non-homogeneous case (i.e. non-constant density), in [29] the reader can find a detailed analysis of the literature on this subject. Model (1.1) has been studied in [3] where the existence of a global weak solution was obtained (see also [13]). Then the existence of local (in time) strong solutions have been proven in [2].

The main goal of this work is the analysis of problems (1.4) and (1.5) in a two-dimensional bounded domain with the physically relevant potential Ψ. Some basic ideas originated from the techniques devised in [28]. From now on, for the sake of simplicity, problems (1.4) and (1.5) will be called CHB_0 and CHB_κ, respectively. Our results are the following:

(1) existence (and uniqueness) of a strong solution to CHB_κ;
(2) existence of a weak solution to CHB_κ;
(3) continuous dependence on data and uniqueness of weak solutions to CHB_κ;
(4) existence and uniqueness of weak and strong solutions to CHB_0;

(5) error estimate for the difference of strong solutions to CHB_κ and CHB_0;
(6) regularization in finite time for any weak solution to CHB_κ;
(7) validity of the instantaneous separation property for any weak solution to CHB_κ, namely,

$$\forall \tau > 0 \ \ \exists\, \delta = \delta(\tau) > 0 : \|\varphi(t)\|_{C(\overline{\Omega})} \leq 1 - \delta \qquad \forall t \geq \tau;$$

(8) any strong solution to CHB_0 is strictly separated in $[0,T]$, $T \in (0,\infty)$.

It would be challenging to extend our analysis of (1.5) to the case of non-constant viscosity (see [28] for NSCH). Indeed, even though the existence of weak solutions can be shown using a suitable Galerkin approximation scheme (see Remark 4.3), the existence of more regular solutions is not at all straightforward. A further issue could be the long-time behavior of the solutions, namely, the existence of global and exponential attractors as well as the convergence of a given (weak) solution to a single equilibrium. It would also be very interesting to study the behavior of solutions to (1.5) when the viscosity ν vanishes (see, e.g., [12] for the Boussinesq equations and [16] for the incompressible NSCH).

Plan of the paper. In Section 2 we introduce the main assumptions and the functional framework as well as we report some basic tools from functional analysis. In Section 3 we give the notions of weak and strong solutions to problems CHB_κ, CHB_0, together with the main results of the paper. The other sections are devoted to the proofs of our main results. More precisely, Section 4 contains the proofs related to problem CHB_κ (see (1)-(3)). In Section 5, the proofs of (4)-(5) are given, while regularization and strict separation property (see (6)-(8)) are demonstrated in Section 6. The Appendix contains a derivation of (1.4) from (1.1) through a formal perturbation argument.

2. Preliminaries

2.1. Assumptions on the potential and its approximation. We take a slight generalization of the logarithmic potential Ψ, namely a quadratic perturbation of a singular (strictly) convex function in the closed interval [-1, 1]. More precisely, we consider

$$\Psi(s) = F(s) - \frac{\alpha_0}{2} s^2 \tag{2.1}$$

where $F \in C([-1,1]) \cap C^3(-1,1)$ is convex and fulfills

$$\lim_{s \to -1} F'(s) = -\infty \qquad \lim_{s \to 1} F'(s) = +\infty \qquad F''(s) \geq \overline{\alpha} \quad \forall\, s \in (-1,1),$$

namely we consider a double well potential, assuming $\tilde{\alpha} = \alpha_0 - \overline{\alpha} > 0$.

This means that

$$\Psi''(s) \geq -\tilde{\alpha} \qquad \forall s \in (-1,1). \tag{2.2}$$

We also extend $F(s) = +\infty$ for any $s \notin [-1,1]$. Notice that the above assumptions imply that there exists $s_0 \in (-1,1)$ such that $F'(s_0) = 0$. Without loss of generality, we assume that $s_0 = 0$ and that $F(s_0) = 0$ as well. In particular, this entails that $F(s) \geq 0$ for any $s \in [-1,1]$. Moreover we require that F'' is convex and

$$F''(s) \leq C e^{C|F'(s)|} \qquad \forall s \in (-1,1) \tag{2.3}$$

for some positive constant C. Also, we assume that there exists $\gamma \in (0,1)$ such that F'' is nondecreasing in $[1-\gamma,1)$ and nonincreasing in $(-1,-1+\gamma]$. These hypotheses are fulfilled by the logarithmic potential defined by (1.2) and extended by continuity at -1 and 1.

In the first existence result we need to introduce a suitable approximation Ψ_λ of Ψ. Let us recall the existence of a sequence of regular functions F_λ which approximate the singular function F. For any $\lambda > 0$ we introduce $\Psi_\lambda(s) = F_\lambda(s) - \frac{\alpha_0}{2}s^2$, where F_λ (see [19, (3.7)] and [28, Thm.4.1]) and we recall that F_λ enjoys the following properties: There exist $0 < \overline{\lambda} < \gamma \leq 1$ and $\hat{C} > 0$ such that

(1) $\Psi_\lambda(s) \geq \frac{\alpha_0}{2}s^2 - \hat{C} \geq -\hat{C}, \quad \forall \lambda \in (0,\overline{\lambda}], \forall s \in \mathbb{R}$;
(2) $\Psi_\lambda \in \mathcal{C}^2(\mathbb{R})$ for every $0 < \lambda \leq \overline{\lambda}$;
(3) as $\lambda \to 0$, $F_\lambda(s) \to F(s)$ for all $s \in \mathbb{R}$, $|F'_\lambda(s)| \to |F'(s)|$ for $s \in (-1,1)$ and F'_λ converges uniformly to F' on any compact subset of $(-1,1)$. Furthermore, $|F'_\lambda(s)| \to +\infty$ for every $|s| \geq 1$. Moreover, we have, for $\lambda \in (0,\overline{\lambda}]$, $F_\lambda(s) \leq F(s)$, for every $s \in [-1,1]$ and $|F'_\lambda(s)| \leq |F'(s)|$, for every $s \in (-1,1)$;
(4) $F''_\lambda(s) \geq 0$ $\forall s \in \mathbb{R}$ for $0 < \lambda \leq \overline{\lambda}$, entailing
$$\Psi''_\lambda(s) \geq -\alpha_0 \qquad \forall s \in \mathbb{R}; \tag{2.4}$$
(5) there exists a positive constant $C = C(\overline{\varphi}_0)$ (see [19] and [24]) such that, for $0 < \lambda \leq \overline{\lambda}$,
$$\int_\Omega |F'_\lambda(\varphi)| dx \leq C \left| \int_\Omega F'_\lambda(\varphi)(\varphi - \overline{\varphi}) dx \right| + C, \tag{2.5}$$
provided that $\overline{\varphi}_0 \in (-1,1)$, where we denote by $\overline{f}$ the integral mean of f over Ω.

2.2. Notation and function spaces. Let Ω be a smooth bounded domain of $\mathbb{R}^2$. For the velocity field we set

$$\mathbf{H}_\sigma = \overline{\{\mathbf{u} \in [C_0^\infty(\Omega)]^2 : \text{ div } \mathbf{u} = 0\}}^{[L^2(\Omega)]^2} \qquad \mathbf{V}_\sigma = \overline{\{\mathbf{u} \in [C_0^\infty(\Omega)]^2 : \text{ div } \mathbf{u} = 0\}}^{[H^1(\Omega)]^2}$$

$$\mathbf{W}_\sigma = [H^2(\Omega)]^2 \cap \mathbf{V}_\sigma.$$

For the order parameter we define
$$V = H^1(\Omega), \qquad V_2 = \{v \in H^2(\Omega) : \partial_{\mathbf{n}} v = 0 \text{ on } \partial\Omega\}.$$

For the temperature field we set
$$H = L^2(\Omega), \qquad V_\theta = H_0^1(\Omega) \qquad V_\theta^2 = V_\theta \cap H^2(\Omega).$$

We denote by $(\cdot,\cdot)$ the standard inner product in $\mathbf{H}_\sigma$ (or in H) and by $\|\cdot\|$ the induced norm. In $\mathbf{V}_\sigma$, owing to Poincaré's inequality, we can define the inner product $(\mathbf{u},\mathbf{v})_{\mathbf{V}_\sigma} = (\nabla \mathbf{u}, \nabla \mathbf{v})$ and the induced norm $\|\mathbf{v}\|_{\mathbf{V}_\sigma} = \|\nabla \mathbf{v}\|$. We then indicate by $(\cdot,\cdot)_1$ the canonical inner product in V, while $\|\cdot\|_1$ stands for its induced norm (we define as $\|\cdot\|_1$ the canonical norm of $[H^1(\Omega)]^2$ as well). Moreover, we consider the Stokes operator $\mathbf{A} = -P\Delta$, with domain $D(\mathbf{A}) = \mathbf{W}_\sigma$, where P is the Leray orthogonal projector onto $\mathbf{H}_\sigma$ and we define $\|\mathbf{v}\|^2_{\mathbf{W}_\sigma} := (\mathbf{A}\mathbf{v}, \mathbf{A}\mathbf{v})$.

We also recall that the trilinear form defined by

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{i,j=1}^{2} \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i \, dx,$$

for any $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in [H^1(\Omega)]^2$, satisfies

$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| \le C\|\nabla \mathbf{u}\|^{\frac{1}{2}} \|\mathbf{u}\|^{\frac{1}{2}} \|\nabla \mathbf{v}\| \; \|\mathbf{w}\|^{\frac{1}{2}} \|\nabla \mathbf{w}\|^{\frac{1}{2}} \tag{2.6}$$

for any $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathbf{V}_\sigma$. Moreover, due to its antisymmetry, for any $\mathbf{v}$, $\mathbf{w} \in H^1(\Omega)$ and $\mathbf{u} \in \mathbf{V}_\sigma$, we have that

$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| \le \|\mathbf{u}\|^{\frac{1}{2}} \|\mathbf{u}\|_1^{\frac{1}{2}} \|\mathbf{v}\|^{\frac{1}{2}} \|\mathbf{v}\|_1^{\frac{1}{2}} \|\mathbf{w}\|_1. \tag{2.7}$$

2.3. Basic inequalities and tools. We now recall some classical inequalities used in the proofs, valid for $\Omega \subset \mathbb{R}^2$ being any bounded domain with smooth boundary $\partial\Omega$.

- Trudinger-Moser inequality (see, e.g., [43])

$$\int_\Omega e^{|u|} dx \le C e^{C\|u\|_V^2}, \qquad \forall u \in V. \tag{2.8}$$

We also recall the following density result. Let X and Y be two Hilbert spaces such that Y is continuously and densely embedded into X. Let $T>0$ and p,q satisfy $1 \le p,q < \infty$. Applying [17, Thm.5.0.27] to $L^q(0,T;X)$, together with the representation formula of [9, Prop.II.5.11], we infer

Lemma 2.1. *$C^\infty([0,T];Y)$ is dense in*

$$E_{p,q} := \left\{ u \in L^p(0,T;X) : \frac{d}{dt} u \in L^q(0,T;X) \right\}.$$

Referring, for instance, to [28, Appendix B], we denote by $\mathbf{A}^{-1}$ the inverse map of the Stokes operator so that $\|\mathbf{f}\|_\flat := \|\nabla \mathbf{A}^{-1}\mathbf{f}\|$ is an equivalent norm on $\mathbf{V}_\sigma'$. We then have, due to regularity results, that

$$\exists C>0 \;\; s.t. \;\; \|\mathbf{u}\|_{H^2(\Omega)} \le C\|\mathbf{u}\|_{\mathbf{W}_\sigma} \;\; \forall \mathbf{u} \in \mathbf{W}_\sigma. \tag{2.9}$$

Furthermore, we introduce the Riesz isomorphism $A_0 : V_\theta \to V_\theta'$ by setting $< A_0 u, v >= (\nabla u, \nabla v)$ for every $v \in V_\theta$. Then, denoting by A_0^{-1} its inverse map, we have that $\|f\|_\sharp := \|\nabla A_0^{-1} f\|$ is a norm on V_θ' equivalent to the natural one and (see, e.g., [48, Ch.2])

$$\|A_0^{-1} f\|_{H^2(\Omega)} \le C\|f\| \;\; \forall f \in H. \tag{2.10}$$

Finally, recalling [28, Appendix A], we set $V_0 = \{v \in V : \overline{v} = 0\}$ and its dual V_0'. The restriction $\bar{A}_0$ to V_0 of $B_0 : V \to V'$ defined by $< B_0 u, v >= (\nabla u, \nabla v)$ for all $v \in V$, is an isomorphism from V_0 onto V_0'. Thus we denote by $\bar{A}_0{}^{-1}$ its inverse map and we set $\|f\|_* := \|\nabla \bar{A}_0{}^{-1} f\|$, which is a norm on V_0' equivalent to the canonical one.

2.4. The lift operator. We analyze the case of nonhomogeneous Dirichlet boundary conditions for θ and we introduce the lift operator θ_g presented, e.g., in [7, Sec.2]: It is the harmonic extension of the boundary datum g in Ω for any $t \in [0,T]$. Since Ω is smooth, from [39], if g belongs to $L^p(0,T;H^{m-1/2}(\partial\Omega))$ for some $m \ge -1$ and some $p \in [1,\infty]$, and $\partial_t g \in L^q(0,T;H^{k-1/2}(\partial\Omega))$ for some $k \ge -1$ and some $q \in [1,\infty]$, then $\theta_g \in L^p(0,T;H^m(\Omega))$ and $\partial_t \theta_g \in L^q(0,T;H^k(\Omega))$.

3. Main results

In this section, the generic constants $C>0$ (and in some cases $\tilde{C}>0$) appearing in the estimates, unless otherwise indicated, depend on T, the norms of the initial data, the domain Ω, the potential F, the parameters and the coefficients of the problem, but are independent of t.

3.1. Weak and strong solutions.

CHB$_\kappa$ system. The basic assumptions are

(1) $\kappa, \nu > 0$;

(2) $g \in H^1(0,T;H^{1/2}(\partial\Omega))$;

(3) $\varphi_0 \in V \cap L^\infty(\Omega)$ with $\|\varphi_0\|_{L^\infty(\Omega)} \le 1$, $|\overline{\varphi}_0| < 1$;

(4) $\mathbf{u}_0 \in \mathbf{H}_\sigma$;

(5) $\theta_0 \in H$.

Here $\overline{f}$ stands for the spatial average of f.

Let us introduce the notion of weak solution to CHB$_\kappa$.

Definition 3.1. *Let hypotheses (1)-(5) be satisfied. Given $T>0$, a triple ($\boldsymbol{u}$, φ, θ) is a weak solution to CHB$_\kappa$ on $[0,T]$ if*

- $\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{H}_\sigma) \cap L^2(0,T;\boldsymbol{V}_\sigma) \cap H^1(0,T;\boldsymbol{V}_\sigma')$;
- $\pi \in W^{-1,\infty}(0,T;H)$;
- $\varphi \in L^\infty(0,T;V) \cap L^4(0,T;V_2) \cap L^2(0,T;W^{2,p}(\Omega)) \cap H^1(0,T;V')$, $2 \le p < \infty$;
- $\varphi \in L^\infty(\Omega \times (0,T))$ *and* $|\varphi(x,t)| < 1$ *for a.a.* $(x,t) \in \Omega \times (0,T)$;
- $\theta \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap H^1(0,T;V_\theta')$ *and* $\theta = g$ *a.e. on* $\partial\Omega \times (0,T)$ *in the sense of traces;*

$$< \partial_t \boldsymbol{u}, \boldsymbol{w} > + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{w}) + (\nu \nabla \boldsymbol{u}, \nabla \boldsymbol{w}) = -(\varphi \nabla \mu, \boldsymbol{w}) + (\theta, \boldsymbol{e}_2 \cdot \boldsymbol{w}) \quad \forall \boldsymbol{w} \in \boldsymbol{V}_\sigma \tag{3.1}$$

$$< \partial_t \varphi, v > + (\nabla \mu, \nabla v) + (\boldsymbol{u} \cdot \nabla \varphi, v) = 0 \qquad \forall v \in V \tag{3.2}$$

$$< \partial_t \theta, \xi > + (\kappa \nabla \theta, \nabla \xi) + (\boldsymbol{u} \cdot \nabla \theta, \xi) = 0 \qquad \forall \xi \in V_\theta \tag{3.3}$$

for almost every $t \in (0,T)$;

- $\mu = -\alpha \Delta \varphi + \Psi'(\varphi)$ *a.e. in* $\Omega \times (0,T)$ *with* $\mu \in L^2(0,T;V)$;
- $\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \varphi(0) = \varphi_0, \qquad \theta(0) = \theta_0.$

Remark 3.1. Notice that any φ_0 in the class of admissible initial conditions has finite energy $\mathcal{E}(\varphi_0) < \infty$, where

$$\mathcal{E}(\varphi) = \int_\Omega \left(\frac{\alpha}{2} |\nabla \varphi|^2 + \Psi(\varphi) \right) dx. \tag{3.4}$$

Indeed, by $\|\varphi_0\|_{L^\infty(\Omega)} \le 1$, we easily infer that $\Psi(\varphi_0) \in L^1(\Omega)$. The assumption on the total mass $|\overline{\varphi}_0| < 1$, however, prevents the existence of the pure phases (i.e. $\varphi_0 \equiv 1$ or $\varphi_0 \equiv -1$). Besides, we notice that any solution satisfies the mass conservation property, namely

$$\overline{\varphi}(t) = \overline{\varphi}_0(t) \qquad \forall t \ge 0.$$

Remark 3.2. Note that $\mathbf{u} \in C([0,T], \mathbf{H}_\sigma)$, $\varphi \in C([0,T],H)$ and $\theta \in C([0,T],H)$ by standard results. Thus the initial conditions make sense in L^2. Moreover, it can be shown (see Remark 3.13 or [1]) that $\phi \in C([0,T];V)$.

REMARK 3.3. As is customary, the pressure term π is dropped in the weak formulation. The pressure can be recovered (up to a constant) thanks to the classical de Rham's theorem (see [9, Sec.V.1.5] or [47]): There exists, up to an additive constant, the pressure π in $W^{-1,\infty}(0,T;H)$.

REMARK 3.4. Due to regularity estimates, since $\mu \in L^2(0,T;V)$ we immediately deduce from the definition of μ itself and from [28, Thm.A.2] that $\varphi \in L^2(0,T;W^{2,p}(\Omega))$, $2 \leq p < \infty$.

Let us now introduce the definition of strong solution.

DEFINITION 3.2. *A weak solution to CHB_κ is a strong solution if*

- $\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{V}_\sigma) \cap L^2(0,T;\boldsymbol{W}_\sigma) \cap H^1(0,T;\boldsymbol{H}_\sigma)$;
- $\pi \in L^2(0,T;V)$;
- $\varphi \in L^\infty(0,T;W^{2,p}(\Omega) \cap V_2) \cap H^1(0,T;V)$, *with* $2 \leq p < \infty$;
- $\mu \in L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega) \cap V_2) \cap H^1(0,T;V')$;
- $\theta \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)) \cap H^1(0,T;H)$ *and* $\theta = g$ *almost everywhere on* $\partial\Omega \times (0,T)$ *in the sense of traces.*

Therefore this solution satisfies (1.5) almost everywhere in $\Omega \times (0,T)$.

REMARK 3.5. Again we can recover the pressure as in Remark 3.3, but in this case, arguing as in [47], we can obtain higher regularity. Indeed, we have

$$\mathbf{f} = \mu \nabla \varphi - \partial_t \mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{u} + \theta \mathbf{e}_2 \in L^2(0,T;[L^2(\Omega)]^2),$$

where $\theta \in L^2(0,T;H)$. Then we deduce that the pressure π satisfies

$$\int_0^T \|\pi\|^2_{W^{1,2}} \leq C \int_0^T \|\mathbf{f}\|^2 < \infty,$$

therefore $\pi \in L^2(0,T;V)$ and $-\nu \Delta \mathbf{u} + \nabla \pi = \mathbf{f}$ almost everywhere in $\Omega \times (0,T)$.

REMARK 3.6. Since we have that $\mu \in L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega)) \cap H^1(0,T;V')$, we also get $\partial_n \mu = 0$ almost everywhere on $\partial\Omega \times (0,T)$.

$\mathbf{CHB}_0$ system. We now introduce the notions of weak and strong solutions to CHB_0 or, equivalently, to the incompressible approximation of compressible NSCH system (see (1.3)). Concerning the weak solution, we have

DEFINITION 3.3. *Let $\nu > 0$. Given $T > 0$, a triple ($\boldsymbol{u}$, φ, θ) is a weak solution to CHB_0 on $[0,T]$ if*

- $\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{H}_\sigma) \cap L^2(0,T;\boldsymbol{V}_\sigma) \cap H^1(0,T;\boldsymbol{V}'_\sigma)$;
- $\pi \in W^{-1,\infty}(0,T;H)$;
- $\varphi \in L^\infty(0,T;V) \cap L^4(0,T;V_2) \cap L^2(0,T;W^{2,p}(\Omega)) \cap H^1(0,T;V')$, *where* $2 \leq p < \infty$;
- $\varphi \in L^\infty(\Omega \times (0,T))$ *and* $|\varphi(x,t)| < 1$ *for a.a.* $(x,t) \in \Omega \times (0,T)$;
- $\theta \in L^\infty(0,T;H) \cap L^\infty(\Omega \times (0,T)) \cap H^1(0,T;V'_\theta)$;

$$< \partial_t \boldsymbol{u}, \boldsymbol{w} > + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{w}) + (\nu \nabla \boldsymbol{u}, \nabla \boldsymbol{w}) = -(\varphi \nabla \mu, \boldsymbol{w}) + (\theta, \boldsymbol{e}_2 \cdot \boldsymbol{w}) \quad \forall \boldsymbol{w} \in \boldsymbol{V}_\sigma \quad (3.5)$$
$$< \partial_t \varphi, v > + (\nabla \mu, \nabla v) + (\boldsymbol{u} \cdot \nabla \varphi, v) = 0 \qquad \forall v \in V, \quad (3.6)$$
$$< \partial_t \theta, \xi > - (\boldsymbol{u}\theta, \nabla \xi) = 0 \qquad \forall \xi \in V_\theta, \quad (3.7)$$

for almost every $t \in (0,T)$;

- $\mu = -\alpha\Delta\varphi + \Psi'(\varphi)$ *a.e. in* $\Omega\times(0,T)$ *with* $\mu\in L^2(0,T;V)$;
- $\boldsymbol{u}(0)=\boldsymbol{u}_0, \qquad \varphi(0)=\varphi_0 \qquad \theta(0)=\theta_0,$

REMARK 3.7. The initial condition $\theta(0)=\theta_0$ is still meant in the strong sense. Indeed, thanks to the DiPerna-Lions theory of renormalized solutions for the transport equation, it can be shown that any weak solution to the transport equation satisfies $\theta\in C([0,T];L^p(\Omega))$, for any $1\le p<\infty$ (see, e.g., [9, Thm.VI.1.3]).

The notion of strong solution reads as follows.

DEFINITION 3.4. *A weak solution to* CHB_0 *(see Definition 3.3) is a strong solution if*

- $\boldsymbol{u}\in L^\infty(0,T;\boldsymbol{V}_\sigma)\cap L^2(0,T;\boldsymbol{W}_\sigma)\cap L^2(0,T;[W^{2,p}(\Omega)]^2)\cap H^1(0,T;[L^p(\Omega)]^2)$, *for any* $2\le p<\infty$;
- $\pi\in L^2(0,T;V)$;
- $\varphi\in L^\infty(0,T;W^{2,q}(\Omega)\cap V_2)\cap H^1(0,T;V)$, *for any* $2\le q<\infty$;
- $|\varphi(x,t)|<1$ *a.e.* $(x,t)\in\Omega\times(0,T)$;
- $\mu\in L^\infty(0,T;V)\cap L^2(0,T;H^3(\Omega)\cap V_2)\cap H^1(0,T;V')$;
- $\theta\in L^\infty(\Omega\times(0,T))\cap L^\infty(0,T;H^2(\Omega))\cap W^{1,\infty}(0,T;H)$.

Thus the solution satisfies (1.4) almost everywhere in $\Omega\times(0,T)$.

REMARK 3.8. The pressure π can be recovered as above (see Remarks 3.3 and 3.5).

We can now state our main results.

3.2. Well-posedness of CHB$_\kappa$.

Let us begin with the existence (and uniqueness) of strong solutions to CHB$_\kappa$.

The additional hypotheses to (1)-(5) are the following:

(6) $\varphi_0\in V_2$;

(7) $\mu_0=-\alpha\Delta\varphi_0+\Psi'(\varphi_0)\in V$;

(8) $\mathbf{u}_0\in\mathbf{V}_\sigma$;

(9) $g\in L^2(0,T;H^{3/2}(\partial\Omega))\cap H^1(0,T;H^{1/2}(\partial\Omega))$;

(10) $\theta_0\in V$ and $\theta_0=g(0)$ on $\partial\Omega$ in the sense of traces.

REMARK 3.9. On account of [49], since $\mu_0\in V$, we notice, from [28, Thm.A.2], with $f=\mu_0+\alpha_0\varphi_0\in V$, that $\varphi_0\in W^{2,p}(\Omega)$ for every $p\ge 2$ and $F'(\varphi_0)\in L^p(\Omega)$ for every $p\ge2$. Moreover, from property (2.3) of F and again by [28, Thm.A.2] (see also [27, Lemma 5.1]), since $f\in V$, we deduce that $F''(\varphi_0)\in L^p(\Omega)$ for every $p\ge2$. Since φ_0 belongs to V and $F'\in C^2((-1,1))$, we can apply the chain rule to obtain

$$\nabla F'(\varphi_0)=F''(\varphi_0)\nabla\varphi_0,$$

but then $\|\nabla F'(\varphi_0)\|_{L^p(\Omega)}\le\|F''(\varphi_0)\|_{L^{2p}(\Omega)}\|\nabla\varphi_0\|_{L^{2p}(\Omega)}<\infty$, since $\nabla\varphi_0\in W^{1,q}(\Omega)$ for every $q\ge2$. Therefore we get $F'(\varphi_0)\in W^{1,p}(\Omega)$ for every $p\ge2$, implying that $F'(\varphi_0)\in L^\infty(\Omega)$ and thus we obtain that the initial field φ_0 is strictly separated. Indeed, there exists $\tilde\delta>0$ such that

$$\|\varphi_0\|_{C(\overline{\Omega})}\le 1-\tilde\delta.$$

The existence of a strong solution is given by

THEOREM 3.1. *Let hypotheses (1)-(5) and (6)-(10) be fulfilled. For any given $T>0$, there exists a triple $(\boldsymbol{u}, \varphi, \theta)$ which is a strong solution to CHB_κ according to Definition 3.2. Moreover, if $\theta_0 \in L^\infty(\Omega)\cap V_\theta$ and $\boldsymbol{u}_0 \in \boldsymbol{W}_\sigma$, then we have the additional regularity:*

- $\boldsymbol{u}\in L^\infty(0,T;\boldsymbol{V}_\sigma)\cap L^2(0,T;[W^{2,q}(\Omega)]^2)$ *with* $\partial_t \boldsymbol{u}\in L^2(0,T;[L^q(\Omega)]^2)$, *for every* $q\geq 2$,
- $\theta \in L^\infty(\Omega\times(0,T))$,

and

$$\|\theta\|_{L^\infty(\Omega\times(0,T))}\leq C, \qquad \|\partial_t \boldsymbol{u}\|_{L^2(0,T;[L^q(\Omega)]^2)}+\|\boldsymbol{u}\|_{L^2(0,T;[W^{2,q}(\Omega)]^2)}\leq C, \tag{3.8}$$

independently of κ.

REMARK 3.10. If $\theta_0 \in L^\infty(\Omega)$, for a strong solution we infer that

$$\|\theta\|_{L^\infty(\Omega\times(0,T))}\leq C(\|\theta_0\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\partial\Omega\times(0,T))}),$$

also in the case $g\neq 0$, as long as $g\in L^\infty(\partial\Omega\times(0,T))$, which is ensured by assumption (I_4). This can be deduced from [35, Ch.III, Thm. 7.2] or more simply by the following argument. Let us define $\chi = \|\theta\|_{L^\infty(\Gamma_T)}$, where $\Gamma_T = \partial\Omega\times[0,T]\cup\Omega\times\{0\}$ and set $\xi = (\theta-\chi)^+$ in (3.3). Note that, by construction, $\xi(0)=0$ and $\xi=0$ on $\partial\Omega$ for almost any $t\in[0,T]$. Therefore we can integrate by parts without considering the boundary terms, and, recalling that $(\mathbf{u}\cdot\nabla\theta, (\theta-\chi)^+)=0$ due to the zero-divergence property of $\mathbf{u}$, we obtain, owing to the regularity of θ and $\mathbf{u}$,

$$\frac{1}{2}\frac{d}{dt}\|(\theta-\chi)^+\|^2+\kappa\|\nabla(\theta-\chi)^+\|^2=0,$$

which implies that $\theta\leq\|\theta\|_{L^\infty(\Gamma_T)}$ almost everywhere in $\Omega\times(0,T)$. Applying the same argument using $\xi=(\theta+\chi)^-$, we eventually reach the desired conclusion. If $\theta_0\in L^\infty(\Omega)$ and $g\equiv 0$, we easily infer that, also for a weak solution, $\|\theta\|_{L^q(\Omega)}\leq\|\theta_0\|_{L^q(\Omega)}$ for almost any $t\in(0,T)$ and for every $q\geq 2$, implying that $\|\theta\|_{L^\infty(\Omega\times(0,T))}\leq\|\theta_0\|_{L^\infty(\Omega)}$.

REMARK 3.11. Given a strong solution, there exists $T_0>0$ sufficiently small such that the solution is strictly separated on $[0,T_0]$. Indeed, $\varphi\in L^2(0,T;H^2(\Omega))\cap H^1(0,T;V)$ for any $T>0$. Then, by [9, Thm.II.5.14], we get $\varphi\in C([0,T];[H^2(\Omega),V]_{\frac{1}{2}})$, but $[H^2(\Omega),V]_{\frac{1}{2}}=H^{3/2}(\Omega)$ (with equivalent norms) and $H^{3/2}(\Omega)\hookrightarrow C(\overline{\Omega})$. Thus we infer $\varphi\in C([0,T];C(\overline{\Omega}))$ and, by Remark 3.9, there exists $T_0>0$ such that $\|\varphi\|_{C(\overline{\Omega})}<1-\tilde{\delta}$ for every $t\in[0,T_0]$.

We now state a stability estimate for the strong solutions, which entails uniqueness.

THEOREM 3.2. *Consider two sets of initial data $(\boldsymbol{u}_{01},\varphi_{01},\theta_{01})$ and $(\boldsymbol{u}_{02},\varphi_{02},\theta_{02})$ satisfying the assumptions (1)-(5) and (6)-(10), with the same Dirichlet boundary datum g, and denote by $(\boldsymbol{u}_1,\varphi_1,\theta_1)$ and $(\boldsymbol{u}_2,\varphi_2,\theta_2)$ the corresponding strong solutions. Then the following continuous dependence estimate holds.*

$$\begin{aligned}&\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|+\|\varphi_1(t)-\varphi_2(t)\|+\|\theta_1(t)-\theta_2(t)\|\\ &\leq C(\|\boldsymbol{u}_{01}-\boldsymbol{u}_{02}\|+\|\varphi_{01}-\varphi_{02}\|+\|\theta_{01}-\theta_{02}\|) \qquad \forall t\in[0,T].\end{aligned} \tag{3.9}$$

Let us introduce the total energy E of the Cahn-Hilliard-Boussinesq system (3.4)

$$E(\mathbf{u},\varphi,\theta):=\frac{1}{2}\|\mathbf{u}\|^2+\mathcal{E}(\varphi)+\frac{1}{2}\|\theta-\theta_g\|^2. \tag{3.10}$$

REMARK 3.12. We notice that (3.10) does not take into account the energy coming from an external source, which is here represented by the boundary datum g.

As we shall see, the existence of a weak solution can be deduced by constructing a suitable sequence of strong solutions. More precisely, we have

THEOREM 3.3. *Let hypotheses (1)-(5) be satisfied. For any given $T>0$, there exists a triple ($\boldsymbol{u}$, φ, θ) which is a weak solution to CHB_κ according to Definition 3.1. Moreover, the following energy identity holds*

$$\begin{aligned}&\frac{d}{dt}E(\boldsymbol{u},\varphi,\theta)+\nu\|\nabla\boldsymbol{u}\|^2+\kappa\|\nabla\Theta\|^2+\|\nabla\mu\|^2\\&=(\theta,\boldsymbol{u}\cdot\boldsymbol{e}_2)-(\partial_t\theta_g,\Theta)-\kappa(\nabla\theta_g,\nabla\Theta)-(\boldsymbol{u}\cdot\nabla\theta_g,\Theta)\quad a.e.\ in\ (0,T),\end{aligned}\tag{3.11}$$

where $\Theta=\theta-\theta_g$.

REMARK 3.13. From the energy identity (3.11), we deduce the regularity $\varphi\in C([0,T];V)$ (see, e.g., [1]). Actually, the energy identity entails $\varphi\in AC([0,T];V)$. Moreover, we have that the function $t\mapsto\int_\Omega F(\varphi(t))dx$ is bounded for all $t\geq0$. Therefore we also have

$$\sup_{t\geq0}\|\varphi(t)\|_{L^\infty(\Omega)}\leq1.$$

The weak solution is also unique. Indeed, we have

THEOREM 3.4. *Let $(\boldsymbol{u}_i,\varphi_i,\theta_i)$, $i=1,2$ be weak solutions given by Theorem 3.3 corresponding to initial data $(\boldsymbol{u}_{0i},\varphi_{0i},\theta_{0i})$ and boundary datum g. If $\overline{\varphi}_{01}=\overline{\varphi}_{02}$, then we obtain the estimate*

$$\begin{aligned}&\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_{\boldsymbol{V}'_\sigma}+\|\varphi_1(t)-\varphi_2(t)\|_{H'}+\|\theta_1(t)-\theta_2(t)\|_{\boldsymbol{V}'_\theta}\\&\leq C(\|\boldsymbol{u}_{01}-\boldsymbol{u}_{02}\|_{\boldsymbol{V}'_\sigma}+\|\varphi_{01}-\varphi_{02}\|_{H'}+\|\theta_{01}-\theta_{02}\|_{\boldsymbol{V}'_\theta})\qquad\forall t\in[0,T],\end{aligned}\tag{3.12}$$

for some positive constant C. If also the initial data coincide, we have $(\boldsymbol{u}_1,\varphi_1,\theta_1)=(\boldsymbol{u}_2,\varphi_2,\theta_2)$ almost everywhere on $[0,T]$.

Let us now state the corresponding results for CHB_0.

3.3. Well-posedness of $\mathbf{CHB}_0$. Our assumptions are:

(11) $\nu>0$;

(12) $\varphi_0\in V\cap L^\infty(\Omega)$ with $\|\varphi_0\|_{L^\infty(\Omega)}\leq1$ and $|\overline{\varphi}_0|<1$;

(13) $\mathbf{u}_0\in\mathbf{H}_\sigma$;

(14) $\theta_0\in H\cap L^\infty(\Omega)$.

The existence and uniqueness of a weak solution is given by

THEOREM 3.5. *Let (11)-(14) hold. For any given $T>0$, there exists a unique triple ($\boldsymbol{u}$, φ, θ), which is a weak solution to CHB_0 according to Definition 3.3. Moreover, we have the following additional regularity:*

$$\|\boldsymbol{u}\|_{L^2(0,T;[L^\infty(\Omega)]^2)}+\sup_{p\geq2}\int_0^T\frac{\|\nabla\boldsymbol{u}\|_{[L^p(\Omega)]2}}{p}ds\leq C,\tag{3.13}$$

for some positive constant C.

The existence of a strong solution requires assumptions (11)-(12) and

(15) $\varphi_0 \in V_2$;

(16) $\mu_0 = -\alpha\Delta\varphi_0 + \Psi'(\varphi_0) \in V$;

(17) $\mathbf{u}_0 \in \mathbf{W}_\sigma$;

(18) $\theta_0 \in H^2(\Omega)$.

THEOREM 3.6. *Let (11)-(12) and (15)-(18) hold. For any given $T>0$, there exists a triple ($\boldsymbol{u}$, φ, θ) which is the unique strong solution to CHB$_0$ according to Definition 3.4.*

Uniqueness of a strong solution comes directly from Theorem 3.5. In addition we can prove a weak-strong continuous dependence estimate.

THEOREM 3.7. *Let $(\boldsymbol{u}_1,\varphi_1,\theta_1)$ be a strong solution given by Theorem 3.6 and corresponding to the initial data $(\varphi_{01},\boldsymbol{u}_{01},\theta_{01})$. Then let $(\boldsymbol{u}_2,\varphi_2,\theta_2)$ be a weak solution given by Theorem 3.5 and corresponding to the initial data $(\varphi_{02},\boldsymbol{u}_{02},\theta_{02})$. If $\overline{\varphi}_{01} = \overline{\varphi}_{02}$ then*

$$\begin{aligned} &\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_{\boldsymbol{V}'_\sigma} + \|\varphi_1(t)-\varphi_2(t)\|_{H'} + \|\theta_1(t)-\theta_2(t)\|_{V'_\theta} \\ &\leq C(\|\boldsymbol{u}_{01}-\boldsymbol{u}_{02}\|_{\boldsymbol{V}'_\sigma} + \|\varphi_{01}-\varphi_{02}\|_{H'} + \|\theta_{01}-\theta_{02}\|_{V'_\theta}) \qquad \forall t\in[0,T], \end{aligned} \tag{3.14}$$

for some positive constant C.

We can also prove an error estimate which quantifies the velocity of convergence of the strong solutions to CHB$_\kappa$ to the (unique) strong solution to CHB$_0$ as $\kappa \to 0$.

THEOREM 3.8. *Consider the initial data $(\boldsymbol{u}_0,\varphi_0,\theta_0)$ satisfying the assumptions (11)-(12) and (15)-(18). For any $\kappa>0$, let moreover $g \equiv \theta_{0|\partial\Omega}$. Denote by $(\boldsymbol{u}_\kappa,\varphi_\kappa,\theta_\kappa)$ and $(\boldsymbol{u},\varphi,\theta)$ the corresponding (unique) strong solutions to CHB$_\kappa$ and to CHB$_0$. Then the following estimate holds*

$$\sup_{t\in[0,T]}\|\boldsymbol{u}(t)-\boldsymbol{u}_\kappa(t)\| + \sup_{t\in[0,T]}\|\varphi(t)-\varphi_\kappa(t)\| + \sup_{t\in[0,T]}\|\theta(t)-\theta_\kappa(t)\| \leq C(\sqrt{\kappa}+\kappa), \tag{3.15}$$

with $C=C(T)$ independent of κ.

Finally we state some regularization properties of the weak solutions as well as the strict separation property.

3.4. Regularization and strict separation property. The following result shows that a weak solution to CHB$_\kappa$ regularizes instantaneously, that is, a weak solution gets strong in finite time. It also deals with the strict separation property. We have

THEOREM 3.9. *Let $R>0$, $m=\overline{\varphi}_0 \in (-1,1)$ and $\tau>0$ be given. Suppose $g\equiv 0$ (and thus $\theta_g \equiv 0$) and assume that $(\boldsymbol{u}_0,\varphi_0,\theta_0)$ satisfies (1)-(5) with $E(\boldsymbol{u}_0,\varphi_0,\theta_0) \leq R$. If $(\boldsymbol{u},\varphi,\theta)$ is the corresponding weak solution to CHB$_\kappa$ then there exist two positive constants $M_1 = M_1(R,m,\tau)$ and $M_2 = M_2(R,m,\tau)$, independent of the initial datum, such that*

$$\sup_{t\geq\tau}\|\boldsymbol{u}(t)\|_{\boldsymbol{V}_\sigma} + \sup_{t\geq\tau}\|\mu(t)\|_V + \sup_{t\geq\tau}\|\theta(t)\|_{V_\theta} \leq M_1, \tag{3.16}$$

and

$$\begin{aligned} &\|\boldsymbol{u}\|_{L^2(t,t+1;\boldsymbol{W}_\sigma)} + \|\partial_t\boldsymbol{u}\|_{L^2(t,t+1;\boldsymbol{H}_\sigma)} + \|\partial_t\varphi\|_{L^2(t,t+1;V)} \\ &\quad + \|\theta\|_{L^2(t,t+1;V_\theta^2)} + \|\partial_t\theta\|_{L^2(t,t+1;H)} \leq M_2 \quad \forall t\geq\tau. \end{aligned} \tag{3.17}$$

In addition, for any $p\geq 2$, there exists a positive constant $M_3 = M_3(R,m,\tau,p)$ such that

$$\|\varphi\|_{L^\infty(\tau,\infty,W^{2,p}(\Omega))} + \|F''(\varphi)\|_{L^\infty(\tau,\infty,L^p(\Omega))} \leq M_3, \tag{3.18}$$

and there exists $\delta=\delta(R,m,\tau)>0$ *and* $M_4=M_4(R,m,\tau)$ *such that*

$$\sup_{t\geq\tau}\|\varphi\|_{C(\overline{\Omega})}\leq 1-\delta. \tag{3.19}$$

Further regularity estimates are given by

THEOREM 3.10. *Under the same assumptions of Theorem 3.9, there exist two positive constants* $M_4=M_4(R,m,\tau)$ *and* $M_5=M_5(R,m,\tau)$*, independent of the initial datum, such that*

$$\|\partial_t \boldsymbol{u}\|_{L^\infty(\tau,\infty,\boldsymbol{H}_\sigma)}+\|\partial_t\varphi\|_{L^\infty(\tau,\infty,H)}+\|\partial_t\theta\|_{L^\infty(\tau,\infty,H)}\leq M_4, \tag{3.20}$$

$$\|\partial_t \boldsymbol{u}\|_{L^2(t,t+1;\boldsymbol{V}_\sigma)}+\|\partial_t\varphi\|_{L^2(t,t+1;H^2(\Omega))}+\|\partial_t\theta\|_{L^2(t,t+1;V_\theta)}\leq M_5 \qquad \forall t\geq\tau, \tag{3.21}$$

$$\|\boldsymbol{u}\|_{L^\infty(\tau,\infty,\boldsymbol{W}_\sigma)}+\|\varphi\|_{L^\infty(\tau,\infty,H^4(\Omega))}\leq M_6. \tag{3.22}$$

REMARK 3.14. If we consider $(\mathbf{u}_0,\varphi_0,\theta_0)$ as an initial datum satisfying (6)-(10), since the solution is strong from $t=0$, we know (Remark 3.11) that there exists $T_0>0$ such that, for $\tilde{\delta}$ in Remark 3.9, $\|\varphi\|_{C(\overline{\Omega})}<1-\tilde{\delta}$ for every $t\in[0,T_0]$. Holding (3.19) for any $\tau>0$, we choose $\tau=T_0$: defining $\delta=\min\{\tilde{\delta},\delta(T_0)\}$, we obtain

$$\sup_{t\geq 0}\|\varphi(t)\|_{C(\overline{\Omega})}\leq 1-\delta,$$

noticing that δ depends only on the initial data and T_0: any strong solution is strictly separated from the initial time $t=0$.

Concerning CHB_0, due to its non-dissipative nature, we can establish the separation property for strong solutions only and on a finite time interval $[0,T]$.

THEOREM 3.11. *Assume that* $(\boldsymbol{u}_0,\varphi_0,\theta_0)$ *satisfies the assumptions of Theorem 3.6. If* $(\boldsymbol{u},\varphi,\theta)$ *is the corresponding (unique) strong solution then, for every* $T>0$*, there exists* $\delta=\delta(T)>0$ *such that*

$$\sup_{0\leq t\leq T}\|\varphi\|_{C(\overline{\Omega})}\leq 1-\delta, \tag{3.23}$$

meaning that, for any fixed $T>0$*, any strong solution is strictly separated on* $[0,T]$.

REMARK 3.15. We point out that, for both CHB_0 and CHB_κ, further results about the existence of solutions with corresponding stability estimates can be proven. In particular the existence of solutions such that $(1.4)_1$-$(1.4)_3$ and $(1.5)_1$-$(1.5)_3$, respectively, are satisfied almost everywhere in $\Omega\times(0,T)$, whereas $(1.4)_4$ and $(1.5)_4$, respectively, are satisfied only in the weak formulation can be shown.

4. Proofs of Section 3.2

Proof. (**Proof of Theorem 3.1.**) The proof is based on a Galerkin approximation for the problem combined with the regularized potential Ψ_λ, provided that the initial datum φ_0 is suitably regularized. Within this proof, $C>0$ stands for a constant independent of t,n,λ,r which may vary from line to line. Following [28], we introduce the globally Lipschitz function $h_r:\mathbb{R}\to\mathbb{R}$, $r\in\mathbb{N}$ such that

$$h_r(z)=\begin{cases}-r, & z<-r,\\ z, & z\in[-r,r],\\ r, & z>r.\end{cases}$$

Then we define $\tilde{\mu}_{0,r} = h_r \circ \tilde{\mu}_0$, where $\tilde{\mu}_0 = -\alpha\Delta\varphi_0 + F'(\varphi_0) = \mu_0 + \alpha_0\varphi_0$. Since $\tilde{\mu}_0 \in V$, on account of, e.g., [45], we have $\tilde{\mu}_{0,r} \in V$, for any $r > 0$, and $\nabla\tilde{\mu}_{0,r} = \nabla\tilde{\mu}_0 \cdot \chi_{[-r,r]}(\tilde{\mu}_0)$. This in turn gives

$$\|\tilde{\mu}_{0,r}\|_1 \leq \|\tilde{\mu}_0\|_1. \tag{4.1}$$

For $r \in \mathbb{N}$ we consider the Neumann problem

$$\begin{cases} -\alpha\Delta\varphi_{0,r} + F'(\varphi_{0,r}) = \tilde{\mu}_{0,r} & \text{in } \Omega \\ \partial_{\mathbf{n}}\varphi_{0,r} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Recalling [28, Lemma A.1], we know that there exists a unique (strong) solution to (4.2) such that $\varphi_{0,r} \in V_2$, $F'(\varphi_{0,r}) \in H$. In addition, by [28, Thm.A.2] and (4.1) we get

$$\|\varphi_{0,r}\|_{V_2} \leq C(1 + \|\tilde{\mu}_0\|). \tag{4.3}$$

Since $\tilde{\mu}_{0,r} \to \tilde{\mu}_0$ in H then $\varphi_{0,r} \to \varphi_0$ in V (see [28, Lemma A.1]). As a consequence, there exists an $\tilde{m} \in (0,1)$, independent of r, and $\tilde{k}$ sufficiently large such that

$$\|\varphi_{0,r}\|_1 \leq 1 + \|\varphi_0\|_1, \quad |\overline{\varphi}_{0,r}| \leq \tilde{m} < 1 \quad \forall r > \tilde{k}. \tag{4.4}$$

In addition, from [28, Thm.A.2], with $f = \tilde{\mu}_{0,r}$, we obtain

$$\|F'(\varphi_{0,r})\|_{L^\infty(\Omega)} \leq \|\tilde{\mu}_{0,r}\|_{L^\infty(\Omega)} \leq r.$$

In conclusion, since $\varphi_{0,r} \in C(\overline{\Omega})$, we can say that there exists $\delta = \delta(r) > 0$ such that

$$\|\varphi_{0,r}\|_{L^\infty(\Omega)} \leq 1 - \delta. \tag{4.5}$$

Observe now that F'' is continuous on $(-1,1)$, thus bounded on compact sets, so that

$$\nabla F'(\varphi_{0,r}) = F''(\varphi_{0,r})\nabla\varphi_{0,r} \in H.$$

Then, being $F'(\varphi_{0,r}) \in H$, we deduce that $F'(\varphi_{0,r}) \in V$. Thus $\Delta\varphi_{0,r} \in V$ and $\varphi_{0,r} \in H^3(\Omega)$. Finally, for any $\lambda \in (0,\lambda^*)$, where $\lambda^* = min\{\frac{1}{2}\delta(r), \overline{\lambda}\}$, since $F(z) = F_\lambda(z)$ for all $z \in [-1+\lambda, 1-\lambda]$ (see [20]), we infer from (4.5) that $-\alpha\Delta\varphi_{0,r} + F'_\lambda(\varphi_{0,r}) = \tilde{\mu}_{0,r}$, which entails

$$\|-\alpha\Delta\varphi_{0,r} + F'_\lambda(\varphi_{0,r})\|_1 \leq \|\tilde{\mu}_0\|_1.$$

Before introducing the Galerkin approximation, we need to lift the Dirichlet boundary datum of the temperature. We set $\Theta = \theta - \theta_g$ (so $\Theta_0 = \theta_0 - \theta_g(0) \in V_\theta$). We know that $g \in L^2(0,T;H^{3/2}(\partial\Omega)) \cap C([0,T];H^{1/2}(\partial\Omega))$ (see, e.g., [9, Prop.II.5.11]). Thus, on account of Section 2.4, we have $\theta_g \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$, $\partial_t\theta_g \in L^2(0,T;V)$.

We now consider the family $\{\mathbf{w}_j\}_{j\geqslant 1}$ of the eigenfunctions of the Stokes operator $\mathbf{A}$ (see, e.g., [47]) as a Galerkin base in $\mathbf{V}_\sigma$ and the family $\{\psi_j\}_{j\geqslant 1}$ of the eigenfunctions of the Laplace operator with homogeneous Neumann boundary conditions as a Galerkin base in V. Also, we consider the family $\{v_j\}_{j\geqslant 1}$ of the eigenfunctions of the Laplace operator with homogeneous Dirichlet boundary conditions as a Galerkin base in V_θ. Then we define the n-dimensional subspaces

$$\mathbb{W}_n := Span(\mathbf{w}_1, \dots, \mathbf{w}_n), \quad \mathbb{Z}_n := Span(\psi_1, \dots, \psi_n), \quad \mathbb{V}_n := Span(v_1, \dots, v_n)$$

where $\psi_1 \equiv 1/\sqrt{|\Omega|}$ and the related orthogonal projectors on these subspaces in $\mathbf{H}_\sigma$ and H, respectively, that is, $P_n := P_{\mathbb{W}_n}$, $\tilde{P}_n := P_{\mathbb{Z}_n}$ and $\hat{P}_n := P_{\mathbb{V}_n}$. We then look for four functions of the form

$$\mathbf{u}^n_{r,\lambda}(t) = \sum_{i=1}^n \hat{\alpha}_i(t)\mathbf{w}_i \in \mathbb{W}_n \qquad \varphi^n_{r,\lambda}(t) = \sum_{i=1}^n \beta_i(t)\psi_i \in \mathbb{Z}_n$$
$$\mu^n_{r,\lambda}(t) = \sum_{i=1}^n \gamma_i(t)\psi_i \in \mathbb{Z}_n \qquad \Theta^n_{r,\lambda}(t) = \sum_{i=1}^n \delta_i(t)v_i \in \mathbb{V}_n,$$

where $\hat{\alpha}_i, \beta_i, \gamma_i, \delta_i$ are real-valued functions and $\theta_n = \Theta_n + \theta_g$, which solve the following problem

$$\begin{aligned}&(\partial_t \mathbf{u}^n_{r,\lambda}, \mathbf{w}) + b(\mathbf{u}, \mathbf{u}^n_{r,\lambda}, \mathbf{w}) + \nu(\nabla \mathbf{u}^n_{r,\lambda}, \nabla \mathbf{w}) \\ &= -(\varphi^n_{r,\lambda}\nabla\mu^n_{r,\lambda}, \mathbf{w}) + (\Theta^n_{r,\lambda}, \mathbf{e}_2\cdot\mathbf{w}) + (\theta_g, \mathbf{e}_2\cdot\mathbf{w}) \quad \forall \mathbf{w}\in\mathbb{W}_n\end{aligned} \tag{4.6}$$

$$(\partial_t \varphi^n_{r,\lambda}, v) + (\nabla\mu^n_{r,\lambda}, \nabla v) + (\mathbf{u}^n_{r,\lambda}\cdot\nabla\varphi^n_{r,\lambda}, v) = 0 \quad \forall v \in \mathbb{Z}_n \tag{4.7}$$

$$\begin{aligned}&(\partial_t \Theta^n_{r,\lambda}, \xi) + \kappa(\nabla\Theta^n_{r,\lambda}, \nabla\xi) + (\mathbf{u}^n_{r,\lambda}\cdot\nabla\Theta^n_{r,\lambda}, \xi) \\ &= -<\partial_t\theta_g, \xi> - \kappa(\nabla\theta_g, \nabla\xi) - (\mathbf{u}^n_{r,\lambda}\cdot\nabla\theta_g, \xi) \quad \forall \xi\in\mathbb{V}_n\end{aligned} \tag{4.8}$$

$$\mathbf{u}^n_{r,\lambda}(0) = P_n(\mathbf{u}_0),\ \varphi^n_{r,\lambda}(0) = \tilde{P}_n(\varphi_{0,r}),\ \Theta^n_{r,\lambda}(0) = \hat{P}_n(\Theta_0) \tag{4.9}$$

$$\mu^n_{r,\lambda} = \tilde{P}_n(-\alpha\Delta\varphi^n_{r,\lambda} + \Psi'_\lambda(\varphi^n_{r,\lambda})) = -\alpha\Delta\varphi^n_{r,\lambda} + \tilde{P}_n(\Psi'_\lambda(\varphi^n_{r,\lambda})) \tag{4.10}$$

for every $t \in (0,T)$.

We notice that $\tilde{P}_n(-\alpha\Delta\varphi_n) = -\alpha\Delta\varphi_n$ because the linear operator $-\Delta$ commutes with the orthogonal projector $\tilde{P}_n$. Moreover, the basis chosen for V is still a complete family in $\{u \in H^3(\Omega) : \partial_\mathbf{n} u = 0 \text{ on } \partial\Omega\}$, then we have that

$$\varphi^n_{r,\lambda}(0) \to \varphi_{0,r} \quad \text{in } H^3(\Omega).$$

In turn, by the embedding $H^3(\Omega) \hookrightarrow L^\infty(\Omega)$, we get

$$\varphi^n_{r,\lambda}(0) \to \varphi_{0,r} \quad \text{in } L^\infty(\Omega).$$

Hence there exists $\overline{n} = \overline{n}(r)$ such that

$$\|\varphi^n_{r,\lambda}(0)\|_\infty \le \frac{1}{2}\delta(r) + \|\varphi_{0,r}\|_\infty \le 1 - \frac{1}{2}\delta(r) \quad \forall n > \overline{n}. \tag{4.11}$$

For any $r > \tilde{k}$ ($\tilde{k}$ independent of n and λ) we fix $\lambda \in (0, \lambda^*(r))$ and $n > \overline{n}(r)$. Since the function $\Psi'_\lambda(s)$ is locally Lipschitz, we can locally solve the Cauchy problem for the system in the unknowns $\hat{\alpha}_i$, β_i, δ_i and find a unique maximal solution $\hat{\alpha}^{(n)} \in C^1([0,t_n), \mathbb{R}^n)$, $\beta^{(n)} \in C^1([0,t_n), \mathbb{R}^n)$, $\delta^{(n)} \in C^1([0,t_n), \mathbb{R}^n)$. Then, from Equation (4.10) we deduce $\gamma^{(n)} \in C^1([0,t_n), \mathbb{R}^n)$.

We can now derive some uniform estimates in order to guarantee that $t_n = +\infty$. First of all, we have the mass conservation property: From Equation (4.7), considering $v \equiv 1$ as test function ($v \in \mathbb{Z}_n \quad \forall n \ge 1$) and integrating by parts the third term we get

$$\int_\Omega \partial_t \varphi^n_{r,\lambda} = |\Omega| \frac{d\overline{\varphi}^n_{r,\lambda}}{dt} = 0$$

thus $\overline{\varphi}^n_{r,\lambda}=\overline{\varphi}_0$.

Consider Equation (4.7) first. We can use $\mu^n_{r,\lambda}\in\mathbb{Z}_n$ as a test function. Integrating by parts and using the boundary conditions we get

$$(\partial_t\varphi^n_{r,\lambda},\mu^n_{r,\lambda})+(\nabla\mu^n_{r,\lambda},\nabla\mu^n_{r,\lambda})-(\mathbf{u}^n_{r,\lambda}\cdot\nabla\mu^n_{r,\lambda},\varphi^n_{r,\lambda})=0, \tag{4.12}$$

which gives (see (4.10))

$$\frac{d}{dt}\left(\frac{\alpha}{2}\|\nabla\varphi^n_{r,\lambda}\|^2+\int_\Omega\Psi_\lambda(\varphi^n_{r,\lambda})\right)+\|\nabla\mu^n_{r,\lambda}\|^2-(\mathbf{u}^n_{r,\lambda}\cdot\nabla\mu^n_{r,\lambda},\varphi^n_{r,\lambda})=0. \tag{4.13}$$

Let us now test equation (4.8) with $\xi=\Theta^n_{r,\lambda}$. We find

$$\frac{d}{dt}\frac{1}{2}\|\Theta^n_{r,\lambda}\|^2+\kappa\|\nabla\Theta^n_{r,\lambda}\|^2\le -<\partial_t\theta_g,\Theta^n_{r,\lambda}>-\kappa(\nabla\theta_g,\nabla\Theta^n_{r,\lambda})-(\mathbf{u}^n_{r,\lambda}\cdot\nabla\theta_g,\Theta^n_{r,\lambda}). \tag{4.14}$$

Recalling that $\|\partial_t\theta_g\|\le\|\partial_t\theta_g\|_1\le C\|\partial_t g\|_{1/2,\partial\Omega}$ and using Young's inequality, we have

$$-<\partial_t\theta_g,\Theta^n_{r,\lambda}>\le\|\partial_t\theta_g\|\ \|\Theta^n_{r,\lambda}\|\le C\|\Theta^n_{r,\lambda}\|^2+C\|\partial_t g\|^2_{1/2,\partial\Omega}.$$

Similarly, we get

$$-(\kappa\nabla\theta_g,\nabla\Theta^n_{r,\lambda})\le\frac{\kappa}{8}\|\nabla\Theta^n_{r,\lambda}\|^2+C\|\nabla\theta_g\|^2\le\frac{\kappa}{4}\|\nabla\Theta^n_{r,\lambda}\|^2+C\|g\|^2_{1/2,\partial\Omega}. \tag{4.15}$$

Ladyzhenskaya's inequality, together with the Sobolev embedding $V_\theta\hookrightarrow L^4(\Omega)$ and Young's inequality, yield

$$\begin{aligned}-(\mathbf{u}^n_{r,\lambda}\cdot\nabla\theta_g,\Theta^n_{r,\lambda})&\le\|\mathbf{u}^n_{r,\lambda}\|_{L^4(\Omega)}\|\nabla\theta_g\|\ \|\Theta^n_{r,\lambda}\|_{L^4(\Omega)}\\&\le\|\mathbf{u}^n_{r,\lambda}\|^{1/2}\|\nabla\mathbf{u}^n_{r,\lambda}\|^{1/2}\|\nabla\theta_g\|\ \|\nabla\Theta^n_{r,\lambda}\|\\&\le\frac{\nu_*}{2}\|\nabla\mathbf{u}^n_{r,\lambda}\|^2+\frac{\kappa}{4}\|\nabla\Theta^n_{r,\lambda}\|^2+C\|\mathbf{u}^n_{r,\lambda}\|^2\|g\|^4_{1/2,\partial\Omega}.\end{aligned} \tag{4.16}$$

Then we test Equation (4.6) with $\mathbf{u}^n_{r,\lambda}$. This gives

$$\frac{d}{dt}\frac{1}{2}\|\mathbf{u}^n_{r,\lambda}\|^2+\nu\|\nabla\mathbf{u}^n_{r,\lambda}\|^2+(\mathbf{u}^n_{r,\lambda}\cdot\nabla\mu^n_{r,\lambda},\varphi^n_{r,\lambda})\le(\Theta^n_{r,\lambda},\mathbf{e}_2\cdot\mathbf{u}^n_{r,\lambda})+(\theta_g,\mathbf{e}_2\cdot\mathbf{u}^n_{r,\lambda}). \tag{4.17}$$

Observe that, thanks to standard inequalities, we get

$$(\theta_g,\mathbf{e}_2\cdot\mathbf{u}^n_{r,\lambda})\le\|\theta_g\|\ \|\mathbf{u}^n_{r,\lambda}\|\le\frac{1}{2}\|\theta_g\|_1^2+\frac{1}{2}\|\mathbf{u}^n_{r,\lambda}\|^2\le\frac{1}{2}\|g\|^2_{1/2,\partial\Omega}+\frac{1}{2}\|\mathbf{u}^n_{r,\lambda}\|^2$$

and

$$(\Theta^n_{r,\lambda},\mathbf{e}_2\cdot\mathbf{u}^n_{r,\lambda})\le\frac{1}{2}\|\Theta^n_{r,\lambda}\|^2+\frac{1}{2}\|\mathbf{u}^n_{r,\lambda}\|^2.$$

We can now add up (4.13), (4.14), and (4.17). Setting

$$E^n_{r,\lambda}=\frac{1}{2}\|\mathbf{u}^n_{r,\lambda}\|^2+\frac{1}{2}\|\Theta^n_{r,\lambda}\|^2+\frac{\alpha}{2}\|\nabla\varphi^n_{r,\lambda}\|^2+\int_\Omega(\Psi_\lambda(\varphi^n_{r,\lambda})+\hat{C}), \tag{4.18}$$

for some $\hat{C}>0$ suitably large in order to have $E^n_{r,\lambda}(t)\ge0$, and

$$\mathcal{D}^n_{r,\lambda}=\|\nabla\mu^n_{r,\lambda}\|^2+\frac{\nu}{2}\|\nabla\mathbf{u}^n_{r,\lambda}\|^2+\frac{\kappa}{2}\|\nabla\Theta^n_{r,\lambda}\|^2,$$

we then infer

$$\frac{d}{dt}E^n_{r,\lambda}+\mathcal{D}^n_{r,\lambda}$$
$$\leq \frac{1}{2}\|g\|^2_{1/2,\partial\Omega}+C(\|\Theta^n_{r,\lambda}\|^2+\|\mathbf{u}^n_{r,\lambda}\|^2+\|\partial_t g\|^2_{1/2,\partial\Omega}+\|g\|^2_{1/2,\partial\Omega}+\|\mathbf{u}^n_{r,\lambda}\|^2\|g\|^4_{1/2,\partial\Omega}).$$

Thus we have

$$\frac{d}{dt}E^n_{r,\lambda}+\mathcal{D}^n_{r,\lambda}\leq C(1+\|g\|^4_{1/2,\partial\Omega})\ E^n_{r,\lambda}+C(\|g\|^2_{1/2,\partial\Omega}+\|\partial_t g\|^2_{1/2,\partial\Omega}). \tag{4.19}$$

Observe now that

$$\mathcal{Q}:=C(1+\|g\|^4_{1/2,\partial\Omega})\in L^1(0,t_n), \qquad \mathcal{R}:=C(\|g\|^2_{1/2,\partial\Omega}+\|\partial_t g\|^2_{1/2,\partial\Omega})\in L^1(0,t_n).$$

Therefore we can apply Gronwall's lemma and find

$$E^n_{r,\lambda}\leq E^n_{r,\lambda}(0)e^{\int_0^t \mathcal{Q}(r)dr}+\int_0^t e^{\int_s^t \mathcal{Q}(r)dr}\mathcal{R}(s)ds. \tag{4.20}$$

Remembering that $\Theta_0=\theta_0-\theta_g(0)$, we obtain

$$E^n_{r,\lambda}(0)=\frac{1}{2}\|P_n(\mathbf{u}_0)\|^2+\frac{1}{2}\|\hat{P}_n(\Theta_0)\|^2+\frac{\alpha}{2}\|\nabla\tilde{P}_n(\varphi_0)\|^2+\int_\Omega(\Psi_\lambda(\tilde{P}_n(\varphi_0))+\hat{C}).$$

Recall that $\Psi_\lambda(z)\leq\Psi(z) \quad \forall z\in[-1,1]$. From (4.11) we deduce $\Psi_\lambda(\varphi^n_{r,\lambda}(0))\leq \Psi(\varphi^n_{r,\lambda}(0))\leq K=\max_{s\in[-1,1]}\Psi(s)$. Hence, using the properties of the orthogonal projectors and owing to (4.4), we deduce

$$\begin{aligned}E^n_{r,\lambda}(\mathbf{u}^n_{r,\lambda}(0),\varphi^n_{r,\lambda}(0),\Theta^n_{r,\lambda}(0))=&\frac{1}{2}\|P_n(\mathbf{u}_0)\|^2+\frac{\alpha}{2}\|\nabla\tilde{P}_n(\varphi_{0,r})\|^2\\&+\frac{1}{2}\|\hat{P}_n(\Theta_0)\|^2+\int_\Omega\Psi_\lambda(\varphi^n_{r,\lambda}(0))\\\leq&\frac{1}{2}\|\mathbf{u}_0\|^2+\frac{\alpha}{2}\|\varphi_0\|^2_1+\frac{1}{2}\|\Theta_0\|^2+(K+\hat{C})|\Omega|,\end{aligned} \tag{4.21}$$

so that

$$\frac{1}{2}\|\mathbf{u}^n_{r,\lambda}\|^2+\frac{1}{2}\|\Theta^n_{r,\lambda}\|^2+\frac{\alpha}{2}\|\nabla\varphi^n_{r,\lambda}\|^2\leq C.$$

Using now Poincaré's inequality (C_0 being the Poincaré's constant) and the conservation of mass, we get

$$\|\varphi^n_{r,\lambda}\|\leq\|\varphi^n_{r,\lambda}-\overline{\varphi}^n_{r,\lambda}\|+\|\overline{\varphi}^n_{r,\lambda}\|\leq C_0\|\nabla\varphi^n_{r,\lambda}\|+\|\overline{\varphi}_0\|\leq C(1+|\overline{\varphi}_0|). \tag{4.22}$$

Therefore we have

$$\|\varphi^n_{r,\lambda}\|+\|u^n_{r,\lambda}\|+\|\Theta^n_{r,\lambda}\|\leq C$$

and this entails $t_n=+\infty$ for every $n\geq 1$, i.e., problem (4.6)-(4.9) has a unique global-in-time solution, and (4.20) is satisfied for every $t\geq 0$. In particular, for every $0<T<+\infty$, we have

$$\|\Theta^n_{r,\lambda}\|_{L^\infty(0,T;H)}\leq C, \qquad \|\mathbf{u}^n_{r,\lambda}\|_{L^\infty(0,T;\mathbf{H}_\sigma)}\leq C. \tag{4.23}$$

Also, from (4.19), integrating in time over (0,T), applying inequality (4.21) and recalling that $E^n_{r,\lambda} \geq 0$, we obtain that

$$\int_0^T \|\nabla \mu^n_{r,\lambda}\|^2 + \frac{\nu}{2}\int_0^T \|\nabla \mathbf{u}^n_{r,\lambda}\|^2 + \frac{k}{2}\int_0^T \|\nabla \Theta^n_{r,\lambda}\|^2 \leq C(1+T).$$

Hence we have

$$\|\mathbf{u}^n_{r,\lambda}\|_{L^2(0,T;\mathbf{V}_\sigma)} \leq C(1+\sqrt{T}), \qquad \sqrt{\kappa}\|\Theta^n_{r,\lambda}\|_{L^2(0,T;V_\theta)} \leq C(1+\sqrt{T}) \tag{4.24}$$

for any $0<T<+\infty$.

On account of (4.22), we also have

$$\|\varphi^n_{r,\lambda}\|_{L^\infty(0,T;V)} \leq C. \tag{4.25}$$

A higher-order estimate for $\varphi^n_{r,\lambda}$ can be obtained by multiplying (4.10) by $-\Delta\varphi^n_{r,\lambda}$ and integrating over Ω. This gives

$$(\nabla \mu^n_{r,\lambda}, \nabla \varphi^n_{r,\lambda}) = \alpha \|\Delta \varphi^n_{r,\lambda}\|^2 + (\Psi''_\lambda(\varphi^n_{r,\lambda})\nabla\varphi^n_{r,\lambda}, \nabla\varphi^n_{r,\lambda}).$$

Then, on account of (2.4) of Ψ_λ, we deduce

$$\alpha\|\Delta\varphi^n_{r,\lambda}\|^2 \leq \alpha_0 \|\nabla\varphi^n_{r,\lambda}\|^2 + \|\nabla\mu^n_{r,\lambda}\| \; \|\nabla\varphi^n_{r,\lambda}\| \leq C\left(1+\|\nabla\mu^n_{r,\lambda}\|\right),$$

which yields

$$\alpha^2\int_0^T \|\varphi^n_{r,\lambda} - \overline{\varphi}^n_{r,\lambda}\|^4_{H^2(\Omega)} \leq C^4\alpha^2 \int_0^T \|\Delta\varphi^n_{r,\lambda}\|^4 \leq CT + C\int_0^T \|\nabla\mu^n_{r,\lambda}\|^2 \leq CT.$$

Thus we eventually get

$$\|\varphi^n_{r,\lambda}\|_{L^4(0,T;V_2)} \leq \|\varphi^n_{r,\lambda} - \overline{\varphi}^n_{r,\lambda}\|_{L^4(0,T;V_2)} + \|\overline{\varphi}_0\|_{L^4(0,T;V_2)} \leq C(1+\sqrt{T}). \tag{4.26}$$

Let us now find an estimate for $\overline{\mu}^n_{r,\lambda}$. We multiply (4.10) by $\varphi^n_{r,\lambda} - \overline{\varphi}^n_{r,\lambda}$ and integrate over Ω, finding

$$(\mu^n_{r,\lambda}, \varphi^n_{r,\lambda} - \overline{\varphi}^n_{r,\lambda}) = \alpha\|\nabla\varphi^n_{r,\lambda}\|^2 + (F'_\lambda(\varphi^n_{r,\lambda}), \varphi^n_{r,\lambda} - \overline{\varphi}^n_{r,\lambda}) - \alpha_0(\varphi^n_{r,\lambda}, \varphi^n_{r,\lambda} - \overline{\varphi}^n_{r,\lambda}).$$

Observing that $(\overline{\mu}^n_{r,\lambda}, \varphi^n_{r,\lambda} - \overline{\varphi}^n_{r,\lambda}) = 0$ and applying standard inequalities, we deduce

$$\begin{aligned}(F'_\lambda(\varphi^n_{r,\lambda}), \varphi^n_{r,\lambda} - \overline{\varphi}^n_{r,\lambda}) &= (\mu^n_{r,\lambda} - \overline{\mu}^n_{r,\lambda}, \varphi^n_{r,\lambda} - \overline{\varphi}^n_{r,\lambda}) - \alpha\|\nabla\varphi^n_{r,\lambda}\|^2 + \alpha_0(\varphi^n_{r,\lambda}, \varphi^n_{r,\lambda} - \overline{\varphi}^n_{r,\lambda})\\ &\leq C_0^2\left(\|\nabla\mu^n_{r,\lambda}\| \; \|\nabla\varphi^n_{r,\lambda}\|\right) - \alpha\|\nabla\varphi^n_{r,\lambda}\|^2\\ &\quad + 2\alpha_0\left(\|\varphi^n_{r,\lambda}\|^2 + \|\varphi^n_{r,\lambda} - \overline{\varphi}^n_{r,\lambda}\|^2\right) \leq C(1+\|\nabla\mu^n_{r,\lambda}\|).\end{aligned}$$

Therefore we have (cf. (2.5))

$$\begin{aligned}|\overline{\mu}^n_{r,\lambda}| &= \frac{1}{|\Omega|}\int_\Omega |\Psi'_\lambda(\varphi^n_{r,\lambda})| \leq \frac{1}{|\Omega|}\left(\int_\Omega |F'_\lambda(\varphi^n_{r,\lambda})| + \alpha_0\int_\Omega |\varphi^n_{r,\lambda}|\right)\\ &\leq \frac{1}{|\Omega|}\left(\int_\Omega |F'_\lambda(\varphi^n_{r,\lambda})| + \alpha_0\sqrt{|\Omega|}\; \|\varphi^n_{r,\lambda}\|\right) \leq \frac{C}{|\Omega|}\left(\left|\int_\Omega F'_\lambda(\varphi^n_{r,\lambda})(\varphi^n_{r,\lambda} - \overline{\varphi}^n_{r,\lambda})\right| + 1\right)\\ &\leq C\left(1+\|\nabla\mu^n_{r,\lambda}\|\right).\end{aligned}$$

Hence we infer

$$\|\mu^n_{r,\lambda}\|_{L^2(0,T;V)} \le C(1+\sqrt{T}). \tag{4.27}$$

Let us now find the bounds for the time derivatives. Equation (4.8) can be rewritten in the form

$$\frac{d\Theta^n_{r,\lambda}}{dt} + \hat{P}^*_n(\mathbf{u}_n \cdot \nabla\Theta^n_{r,\lambda} + \bar{\mathcal{A}}(\Theta^n_{r,\lambda}) + \frac{d\theta_g}{dt} + \mathbf{u}^n_{r,\lambda}\cdot\nabla\theta_g + \bar{\mathcal{A}}(\theta_g)) = 0 \quad \text{in } V'_\theta \tag{4.28}$$

where $\hat{P}^*_n : \mathbb{V}'_n \to V'_\theta$ is the adjoint of the orthogonal projector $\hat{P}_n$. Hence $\|\hat{P}^*_n\|_{\mathcal{L}(\mathbb{V}'_n, V'_\theta)} \le 1$ for every $n \ge 1$. The linear operator $\bar{\mathcal{A}} : \mathbb{V}_n \to \mathbb{V}'_n$ is defined by $<\bar{\mathcal{A}}(\Theta^n_{r,\lambda}), \xi> = (\kappa\nabla\Theta^n_{r,\lambda}, \nabla\xi)$ for every $\xi \in \mathbb{V}_n$.

Observe first that

$$\|\hat{P}^*_n(\bar{\mathcal{A}}(\Theta^n_{r,\lambda}))\|_{V'_\theta} \le \|\bar{\mathcal{A}}(\Theta^n_{r,\lambda})\|_{\mathbb{V}'_n} \le \kappa\|\nabla\Theta^n_{r,\lambda}\|$$
$$\|\hat{P}^*_n(\bar{\mathcal{A}}(\theta_g))\|_{V'_\theta} \le \|\bar{\mathcal{A}}(\theta_g)\|_{\mathbb{V}'_n} \le \kappa\|\nabla\theta_g\| \le C\|g\|_{1/2,\partial\Omega}.$$

Concerning the transport terms, we have

$$|<\mathbf{u}^n_{r,\lambda}\cdot\nabla\Theta^n_{r,\lambda}, \xi>| = |(\mathbf{u}^n_{r,\lambda}\cdot\nabla\Theta^n_{r,\lambda}, \xi)| \le \|\mathbf{u}^n_{r,\lambda}\|_{[L^4(\Omega)]^2}\|\Theta^n_{r,\lambda}\|_{L^4(\Omega)}\|\nabla\xi\|,$$

for every $\xi \in \mathbb{V}_n$. Thus we find (see (4.23)

$$\begin{aligned}
&\|\hat{P}^*_n(\mathbf{u}^n_{r,\lambda}\cdot\nabla\Theta^n_{r,\lambda})\|_{V'_\theta} \le \|\mathbf{u}^n_{r,\lambda}\|_{[L^4(\Omega)]^2}\|\Theta^n_{r,\lambda}\|_{L^4(\Omega)} \le \frac{1}{2}(\|\mathbf{u}^n_{r,\lambda}\|^2_{[L^4(\Omega)]^2} + \|\Theta^n_{r,\lambda}\|^2_{L^4(\Omega)})\\
&\le \frac{C}{2}\left(\|\mathbf{u}^n_{r,\lambda}\|\ \|\nabla\mathbf{u}^n_{r,\lambda}\| + \|\mathbf{u}^n_{r,\lambda}\|^2 + \|\Theta^n_{r,\lambda}\|\ \|\nabla\Theta^n_{r,\lambda}\| + \|\Theta^n_{r,\lambda}\|^2\right)\\
&\le C(1 + \|\nabla\mathbf{u}^n_{r,\lambda}\| + \|\nabla\Theta^n_{r,\lambda}\|).
\end{aligned}$$

Arguing similarly, we get

$$\begin{aligned}
|<\mathbf{u}^n_{r,\lambda}\cdot\nabla\theta_g, \xi>| = |(\mathbf{u}^n_{r,\lambda}\cdot\nabla\theta_g, \xi)| &\le \|\mathbf{u}^n_{r,\lambda}\|_{[L^4(\Omega)]^2}\ \|\nabla\theta_g\|\ \|\xi\|_{L^4(\Omega)}\\
&\le C\|\mathbf{u}^n_{r,\lambda}\|_{[L^4(\Omega)]^2}\ \|\nabla\theta_g\|\ \|\nabla\xi\|,
\end{aligned}$$

which yields

$$\begin{aligned}
\|\hat{P}^*_n(\mathbf{u}^n_{r,\lambda}\cdot\nabla\theta_g)\|_{V'_\theta} \le \|\mathbf{u}^n_{r,\lambda}\|_{[L^4(\Omega)]^2}\|\nabla\theta_g\| &\le \|\mathbf{u}^n_{r,\lambda}\|^{1/2}\|\nabla\mathbf{u}^n_{r,\lambda}\|^{1/2}\|\nabla\theta_g\|\\
&\le C\left(\|\nabla\mathbf{u}^n_{r,\lambda}\| + \|\nabla\theta_g\|^2\|\mathbf{u}^n_{r,\lambda}\|\right)\\
&\le C\left(\|\nabla\mathbf{u}^n_{r,\lambda}\| + \|\nabla\theta_g\|^2\right).
\end{aligned}$$

On the other hand, we also have

$$\|\hat{P}^*_n(\partial_t\theta_g)\|_{V'_\theta} \le \|\partial_t\theta_g\|_{\mathbb{V}'_n} \le C_0\|\partial_t\theta_g\| \le C\|\partial_t g\|_{1/2,\partial\Omega}.$$

Summing up, we deduce

$$\left\|\frac{d\Theta^n_{r,\lambda}}{dt}\right\|_{L^2(0,T,V'_\theta)} \le C(1+\sqrt{T}). \tag{4.29}$$

Let us rewrite Equation (4.7) as follows

$$\frac{d\varphi^n_{r,\lambda}}{dt} + \tilde{P}^*_n(\mathbf{u}^n_{r,\lambda}\cdot\nabla\varphi^n_{r,\lambda} + \tilde{\mathcal{A}}(\mu^n_{r,\lambda})) = 0 \quad \text{in } V', \tag{4.30}$$

where $\tilde{\mathcal{A}}(\mu^n_{r,\lambda}):\mathbb{Z}_n\to\mathbb{Z}_n'$ is defined by $<\tilde{\mathcal{A}}(\mu^n_{r,\lambda}),\psi>=(\nabla\mu^n_{r,\lambda},\nabla\psi)$ for any $\psi\in\mathbb{Z}_n$. Observe that

$$\|\tilde{P}^*_n(\tilde{\mathcal{A}}(\mu^n_{r,\lambda}))\|_{V'}\leq\|\tilde{\mathcal{A}}(\mu^n_{r,\lambda})\|_{\mathbb{Z}_n'}\leq\|\nabla\mu^n_{r,\lambda}\|.$$

On the other hand, we have

$$\begin{aligned}|<\mathbf{u}^n_{r,\lambda}\cdot\nabla\varphi^n_{r,\lambda},\psi>|&=|(\mathbf{u}^n_{r,\lambda}\cdot\nabla\varphi^n_{r,\lambda},\psi)|\leq\|\mathbf{u}^n_{r,\lambda}\|_{[L^4(\Omega)]^2}\|\varphi^n_{r,\lambda}\|_{L^4(\Omega)}\|\nabla\psi\|\\&\leq C^2\|\mathbf{u}^n_{r,\lambda}\|_V\|\varphi^n_{r,\lambda}\|_V\|\psi\|_V\leq C^2\sqrt{C_0^2+1}\|\nabla\mathbf{u}^n_{r,\lambda}\|\ \|\varphi^n_{r,\lambda}\|_V\|\psi\|_V,\end{aligned}$$

so that

$$\|\tilde{P}^*_n(\mathbf{u}^n_{r,\lambda}\cdot\nabla\varphi^n_{r,\lambda})\|_{V'}\leq\|\mathbf{u}^n_{r,\lambda}\cdot\nabla\varphi^n_{r,\lambda}\|_{\mathbb{Z}_n'}\leq C\|\nabla\mathbf{u}^n_{r,\lambda}\|.$$

Therefore we find

$$\left\|\frac{d\varphi^n_{r,\lambda}}{dt}\right\|_{V'}\leq C(\|\nabla\mathbf{u}^n_{r,\lambda}\|+\|\nabla\mu^n_{r,\lambda}\|)$$

that implies

$$\left\|\frac{d\varphi^n_{r,\lambda}}{dt}\right\|_{L^2(0,T;V')}\leq C\sqrt{T}. \tag{4.31}$$

Let us now estimate the time derivative of the approximated velocity field. Equation (4.6) can be rewritten in the form

$$\frac{d\mathbf{u}^n_{r,\lambda}}{dt}+P^*_n(\mathcal{B}(\mathbf{u}^n_{r,\lambda},\mathbf{u}^n_{r,\lambda})+\mathcal{A}(\mathbf{u}^n_{r,\lambda})+\varphi^n_{r,\lambda}\nabla\mu^n_{r,\lambda}-\Theta^n_{r,\lambda}\mathbf{e}_2-\theta_g\mathbf{e}_2)=0\quad\text{in }\mathbf{V}_\sigma', \tag{4.32}$$

where $\mathcal{A}:\mathbb{W}_n\subset\mathbf{V}_\sigma\to\mathbb{W}_n'$ is defined by $<\mathcal{A}(\mathbf{u}^n_{r,\lambda}),\mathbf{w}>=(\nu\nabla\mathbf{u}^n_{r,\lambda},\nabla\mathbf{w})$ for every $\mathbf{w}\in\mathbb{W}_n$ and $\mathcal{B}:\mathbb{W}_n\times\mathbb{W}_n\to\mathbb{W}_n'$ is defined by $<\mathcal{B}(\mathbf{u}^n_{r,\lambda},\mathbf{u}^n_{r,\lambda}),\mathbf{w}>=b(\mathbf{u}^n_{r,\lambda},\mathbf{u}^n_{r,\lambda},\mathbf{w})$ for every $\mathbf{w}\in\mathbb{W}_n$. It is easy to check that

$$\|P^*_n\mathcal{A}(\mathbf{u}^n_{r,\lambda})\|_{\mathbf{V}_\sigma'}\leq\|\mathcal{A}(\mathbf{u}^n_{r,\lambda})\|_{\mathbb{W}_n'}\leq\nu\|\nabla\mathbf{u}^n_{r,\lambda}\|.$$

Concerning $\mathcal{B}$ we have (see (2.6)), for any $\mathbf{w}\in\mathbb{W}_n$,

$$\begin{aligned}|<\mathcal{B}(\mathbf{u}^n_{r,\lambda},\mathbf{u}^n_{r,\lambda}),\mathbf{w}>|&\leq\|\mathbf{u}^n_{r,\lambda}\|^{\frac12}\|\mathbf{u}^n_{r,\lambda}\|_1^{\frac12}\|\mathbf{u}^n_{r,\lambda}\|^{\frac12}\|\mathbf{u}^n_{r,\lambda}\|_1^{\frac12}\|\mathbf{w}\|_1\\&\leq C\ (C_0^2+1)\ \|\nabla\mathbf{u}^n_{r,\lambda}\|\ \|\mathbf{u}^n_{r,\lambda}\|\ \|\nabla\mathbf{w}\|\end{aligned}$$

and this yields

$$\|P^*_n(\mathcal{B}(\mathbf{u}^n_{r,\lambda},\mathbf{u}^n_{r,\lambda}))\|_{\mathbf{V}_\sigma'}\leq\|\mathcal{B}(\mathbf{u}^n_{r,\lambda},\mathbf{u}^n_{r,\lambda})\|_{\mathbf{W}_n'}\leq C\ (C_0^2+1)\ \|\mathbf{u}^n_{r,\lambda}\|\ \|\nabla\mathbf{u}^n_{r,\lambda}\|\leq C\ \|\nabla\mathbf{u}^n_{r,\lambda}\|.$$

Consider now the Korteweg force. For any $\mathbf{w}\in\mathbf{W}_n$, using a Sobolev embedding and Poincaré's inequality we find

$$|<\varphi^n_{r,\lambda}\nabla\mu^n_{r,\lambda},\mathbf{w}>|\leq\|\varphi^n_{r,\lambda}\|_{L^4(\Omega)}\|\nabla\mu^n_{r,\lambda}\|\ \|\mathbf{w}\|_{L^4(\Omega)}\leq C\|\varphi^n_{r,\lambda}\|_V\|\nabla\mu^n_{r,\lambda}\|\ \|\nabla\mathbf{w}\|,$$

which entails

$$\|P^*_n(\varphi^n_{r,\lambda}\nabla\mu^n_{r,\lambda})\|_{\mathbf{V}_\sigma'}\leq\|\varphi^n_{r,\lambda}\nabla\mu^n_{r,\lambda}\|_{\mathbb{W}_n'}\leq C\|\varphi^n_{r,\lambda}\|_V\|\nabla\mu^n_{r,\lambda}\|\leq C\|\nabla\mu^n_{r,\lambda}\|.$$

Concerning the temperature terms in (4.32), for every $\mathbf{w}\in\mathbb{W}_n$, using Poincaré's inequality we find

$$|<\Theta^n_{r,\lambda}\mathbf{e}_2,\mathbf{w}>|\leq\|\Theta^n_{r,\lambda}\|\ \|\mathbf{w}\|\leq C_0\ \|\Theta^n_{r,\lambda}\|\ \|\nabla\mathbf{w}\|.$$

This gives

$$\|P_n^*(\Theta^n_{r,\lambda}\mathbf{e}_2)\|_{\mathbf{V}'_\sigma}\leq\|\Theta^n_{r,\lambda}\mathbf{e}_2\|_{\mathbb{W}'_n}\leq C_0\|\Theta^n_{r,\lambda}\|\leq C.$$

Moreover, we easily obtain

$$\|P_n^*(\theta_g\mathbf{e}_2)\|_{\mathbf{V}'_\sigma}\leq\|\theta_g\mathbf{e}_2\|_{\mathbb{W}'_n}\leq C_0\|\theta_g\|\leq C\|\theta_g\|_1\leq C\|g\|_{1/2,\partial\Omega}.$$

Collecting the previous bounds, we deduce

$$\left\|\frac{d\mathbf{u}^n_{r,\lambda}}{dt}\right\|_{\mathbf{V}'_\sigma}\leq C\left[(\nu+1)\|\nabla\mathbf{u}^n_{r,\lambda}\|+\|\nabla\mu^n_{r,\lambda}\|+\|g\|_{1/2,\partial\Omega}+1\right].$$

We thus infer

$$\left\|\frac{d\mathbf{u}^n_{r,\lambda}}{dt}\right\|_{L^2(0,T;\mathbf{V}'_\sigma)}\leq C(1+\sqrt{T}). \tag{4.33}$$

Let us now find higher-order bounds. Multiplying Equation (4.10) by $\partial_t\mu^n_{r,\lambda}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mu^n_{r,\lambda}\|^2+(\partial_t\mu^n_{r,\lambda},\partial_t\varphi^n_{r,\lambda})+(\partial_t\mu^n_{r,\lambda},\mathbf{u}^n_{r,\lambda}\cdot\nabla\varphi^n_{r,\lambda})=0. \tag{4.34}$$

Observe that

$$\begin{aligned}\alpha_0\|\partial_t\varphi^n_{r,\lambda}\|^2=\alpha_0(\nabla\partial_t\varphi^n_{r,\lambda},\nabla\bar{A}_0^{-1}\partial_t\varphi^n_{r,\lambda})&\leq\alpha_0\|\nabla\partial_t\varphi^n_{r,\lambda}\|\ \|\nabla\bar{A}_0^{-1}\partial_t\varphi^n_{r,\lambda}\|\\&\leq\frac{\alpha}{2}\|\nabla\partial_t\varphi^n_{r,\lambda}\|^2+\frac{\alpha_0^2}{2\alpha}\|\nabla\bar{A}_0^{-1}\partial_t\varphi^n_{r,\lambda}\|^2\\&=\frac{\alpha}{2}\|\nabla\partial_t\varphi^n_{r,\lambda}\|^2+\frac{\alpha_0^2}{2\alpha}\|\partial_t\varphi^n_{r,\lambda}\|^2_*.\end{aligned}$$

Then we deduce (see (2.4))

$$\begin{aligned}(\partial_t\mu^n_{r,\lambda},\partial_t\varphi^n_{r,\lambda})&=\alpha\|\nabla\partial_t\varphi^n_{r,\lambda}\|^2+(\Psi''_\lambda(\varphi^n_{r,\lambda})\partial_t\varphi^n_{r,\lambda},\partial_t\varphi^n_{r,\lambda})\\&\geq\alpha\|\nabla\partial_t\varphi^n_{r,\lambda}\|^2-\alpha_0\|\partial_t\varphi^n_{r,\lambda}\|^2\\&\geq\frac{\alpha}{2}\|\nabla\partial_t\varphi^n_{r,\lambda}\|^2-C\|\partial_t\varphi^n_{r,\lambda}\|^2_*.\end{aligned}$$

Moreover, we have

$$\begin{aligned}(\partial_t\mu^n_{r,\lambda},\mathbf{u}^n_{r,\lambda}\cdot\nabla\varphi^n_{r,\lambda})=&\frac{d}{dt}[(\mathbf{u}^n_{r,\lambda}\cdot\nabla\varphi^n_{r,\lambda},\mu^n_{r,\lambda})]\\&-(\partial_t\mathbf{u}^n_{r,\lambda}\cdot\nabla\varphi^n_{r,\lambda},\mu^n_{r,\lambda})-(\mathbf{u}^n_{r,\lambda}\cdot\nabla\partial_t\varphi^n_{r,\lambda},\mu^n_{r,\lambda}).\end{aligned} \tag{4.35}$$

Observe now that

$$(\mu^n_{r,\lambda},\mathbf{u}^n_{r,\lambda}\cdot\nabla\partial_t\varphi^n_{r,\lambda})\leq\|\mu^n_{r,\lambda}\|_{L^6(\Omega)}\ \|\mathbf{u}^n_{r,\lambda}\|_{[L^3(\Omega)]^2}\ \|\nabla\partial_t\varphi^n_{r,\lambda}\|$$

$$\begin{aligned}&\le \frac{\alpha}{4}\|\nabla\partial_t\varphi^n_{r,\lambda}\|^2+C\|\mu^n_{r,\lambda}\|^2_{L^6(\Omega)}\ \|\mathbf{u}^n_{r,\lambda}\|^2_{[L^3(\Omega)]^2}\\ &\le \frac{\alpha}{4}\|\nabla\partial_t\varphi^n_{r,\lambda}\|^2+C\left(1+\|\nabla\mu^n_{r,\lambda}\|^2\right)\|\mathbf{u}^n_{r,\lambda}\|^2_{[L^3(\Omega)]^2}.\end{aligned}$$

Also, recall that in V_0' the norm $\|\cdot\|_*$ is equivalent to the canonical one and $\|\cdot\|_{V_0'}\le\|\cdot\|_{V'}$. This gives

$$\|\partial_t\varphi^n_{r,\lambda}\|_*\le C\left(\|\nabla\mathbf{u}^n_{r,\lambda}\|+\|\nabla\mu^n_{r,\lambda}\|\right). \tag{4.36}$$

Adding then (4.34) and (4.35) together, we find

$$\begin{aligned}&\frac{d}{dt}\left\{(\mu^n_{r,\lambda},\mathbf{u}^n_{r,\lambda}\cdot\nabla\varphi^n_{r,\lambda})+\frac12\|\nabla\mu^n_{r,\lambda}\|^2\right\}=(\partial_t\mathbf{u}^n_{r,\lambda}\cdot\nabla\varphi^n_{r,\lambda},\mu^n_{r,\lambda})\\ &\quad+(\mathbf{u}^n_{r,\lambda}\cdot\nabla\partial_t\varphi^n_{r,\lambda},\mu^n_{r,\lambda})-(\partial_t\mu^n_{r,\lambda},\partial_t\varphi^n_{r,\lambda})\\ &\le\frac{\alpha}{4}\|\nabla\partial_t\varphi^n_{r,\lambda}\|^2+C(1+\|\nabla\mu^n_{r,\lambda}\|^2)\|\mathbf{u}^n_{r,\lambda}\|^2_{[L^3(\Omega)]^2}\\ &\quad-\frac{\alpha}{2}\|\nabla\partial_t\varphi^n_{r,\lambda}\|^2+C\|\partial_t\varphi^n_{r,\lambda}\|_*^2+(\partial_t\mathbf{u}^n_{r,\lambda}\cdot\nabla\varphi^n_{r,\lambda},\mu^n_{r,\lambda}).\end{aligned}$$

Hence we obtain (see (4.36))

$$\begin{aligned}&\frac{d}{dt}\left\{(\mu^n_{r,\lambda},\mathbf{u}^n_{r,\lambda}\cdot\nabla\varphi^n_{r,\lambda})+\frac12\|\nabla\mu^n_{r,\lambda}\|^2\right\}+\frac{\alpha}{4}\|\nabla\partial_t\varphi^n_{r,\lambda}\|^2\\ &\le C\left(1+\|\mathbf{u}^n_{r,\lambda}\|^2_{[L^3(\Omega)]^2}\right)\left(1+\|\nabla\mu^n_{r,\lambda}\|^2+\|\nabla\mathbf{u}^n_{r,\lambda}\|^2\right)\\ &\quad+(\partial_t\mathbf{u}^n_{r,\lambda}\cdot\nabla\varphi^n_{r,\lambda},\mu^n_{r,\lambda}).\end{aligned}\tag{4.37}$$

Let us take $\mathbf{w}=\partial_t\mathbf{u}^n_{r,\lambda}$ in Equation (4.6). This gives

$$\begin{aligned}&\|\partial_t\mathbf{u}^n_{r,\lambda}\|^2+b(\mathbf{u}^n_{r,\lambda},\mathbf{u}^n_{r,\lambda},\partial_t\mathbf{u}^n_{r,\lambda})+\nu(\nabla\mathbf{u}^n_{r,\lambda},\nabla\partial_t\mathbf{u}^n_{r,\lambda})\\ &=(\mu^n_{r,\lambda}\nabla\varphi^n_{r,\lambda},\partial_t\mathbf{u}^n_{r,\lambda})+(\Theta^n_{r,\lambda},\mathbf{e}_2\cdot\partial_t\mathbf{u}^n_{r,\lambda})+(\theta_g,\mathbf{e}_2\cdot\partial_t\mathbf{u}^n_{r,\lambda}).\end{aligned}\tag{4.38}$$

Using Ladyzhenskaya's inequality, the Sobolev embedding $\mathbf{V}_\sigma\hookrightarrow[L^4(\Omega)]^2$, (2.9), and Young's inequality, we deduce

$$\begin{aligned}|b(\mathbf{u}^n_{r,\lambda},\mathbf{u}^n_{r,\lambda},\partial_t\mathbf{u}^n_{r,\lambda})|&\le\|\mathbf{u}^n_{r,\lambda}\|_{[L^4(\Omega)]^2}\|\nabla\mathbf{u}^n_{r,\lambda}\|_{[L^4(\Omega)]^4}\ \|\partial_t\mathbf{u}^n_{r,\lambda}\|\\ &\le C\|\mathbf{u}^n_{r,\lambda}\|^{1/2}\|\nabla\mathbf{u}^n_{r,\lambda}\|^{1/2}\|\nabla\mathbf{u}^n_{r,\lambda}\|^{1/2}\|\nabla\mathbf{u}^n_{r,\lambda}\|_1^{1/2}\|\partial_t\mathbf{u}^n_{r,\lambda}\|\\ &\le C\|\mathbf{u}^n_{r,\lambda}\|^{1/2}\|\nabla\mathbf{u}^n_{r,\lambda}\|\ \|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|^{1/2}\|\partial_t\mathbf{u}^n_{r,\lambda}\|\\ &\le C\|\nabla\mathbf{u}^n_{r,\lambda}\|\ \|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|^{1/2}\|\partial_t\mathbf{u}^n_{r,\lambda}\|\\ &\le\frac16\|\partial_t\mathbf{u}^n_{r,\lambda}\|^2+C\left(\|\nabla\mathbf{u}^n_{r,\lambda}\|^4+\|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|^2\right).\end{aligned}$$

Then clearly

$$\nu(\nabla\mathbf{u}^n_{r,\lambda},\nabla\partial_t\mathbf{u}^n_{r,\lambda})=\frac{d}{dt}\frac{\nu}{2}\|\nabla\mathbf{u}^n_{r,\lambda}\|^2.$$

Moreover, thanks to Hölder's inequality and the Sobolev embeddings $V\hookrightarrow L^6(\Omega)$ and $V_2\hookrightarrow W^{1,3}(\Omega)$, we have

$$(\mu^n_{r,\lambda}\nabla\varphi^n_{r,\lambda},\partial_t\mathbf{u}^n_{r,\lambda})\le\|\mu^n_{r,\lambda}\|_{L^6(\Omega)}\|\nabla\varphi\|_{[L^3(\Omega)]^2}\|\partial_t\mathbf{u}^n_{r,\lambda}\|$$

$$\begin{aligned}&\le C\|\mu^n_{r,\lambda}\|_1\ \|\varphi^n_{r,\lambda}\|_{H^2(\Omega)}\|\partial_t\mathbf{u}^n_{r,\lambda}\|\\&\le C(1+\|\nabla\mu^n_{r,\lambda}\|)\ \|\varphi^n_{r,\lambda}\|_{H^2(\Omega)}\|\partial_t\mathbf{u}^n_{r,\lambda}\|\\&\le\frac{1}{6}\|\partial_t\mathbf{u}^n_{r,\lambda}\|^2+C\|\varphi^n_{r,\lambda}\|^2_{H^2(\Omega)}\left(1+\|\nabla\mu^n_{r,\lambda}\|^2\right).\end{aligned}\tag{4.39}$$

Moreover, it is easy to check that

$$|(\Theta^n_{r,\lambda},\mathbf{e}_2\cdot\partial_t\mathbf{u}^n_{r,\lambda})|\le\frac{1}{6}\|\partial_t\mathbf{u}^n_{r,\lambda}\|^2+C\tag{4.40}$$

$$|(\theta_g,\mathbf{e}_2\cdot\partial_t\mathbf{u}^n_{r,\lambda})|\le\frac{1}{6}\|\partial_t\mathbf{u}^n_{r,\lambda}\|^2+C\|\theta_g\|^2\le\frac{1}{6}\|\partial_t\mathbf{u}^n_{r,\lambda}\|^2+C\|g\|^2_{1/2,\partial\Omega}.\tag{4.41}$$

Collecting the above estimates we find

$$\begin{aligned}&\frac{d}{dt}\frac{\nu}{2}\|\nabla\mathbf{u}^n_{r,\lambda}\|^2+\|\partial_t\mathbf{u}^n_{r,\lambda}\|^2\\&\le C\left[\|\nabla\mathbf{u}^n_{r,\lambda}\|^4+\|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|^2+\|\varphi^n_{r,\lambda}\|^2_{H^2(\Omega)}(1+\|\nabla\mu^n_{r,\lambda}\|^2)+\|g\|^2_{1/2,\partial\Omega}+1\right].\end{aligned}\tag{4.42}$$

Take now $\mathbf{w}=\mathbf{A}\mathbf{u}^n_{r,\lambda}$ in Equation (4.6). Observe that $\mathbf{A}\mathbf{u}^n_{r,\lambda}\in L^2(0,T,\mathbf{H}_\sigma)$ and there exists $p^n_{r,\lambda}\in L^2(0,T;V)$ such that $-\Delta\mathbf{u}^n_{r,\lambda}+\nabla p^n_{r,\lambda}=\mathbf{A}\mathbf{u}^n_{r,\lambda}$ almost everywhere in $\Omega\times(0,T)$ (see, e.g., [47]). Also, observe that $(\partial_t\mathbf{u}^n_{r,\lambda},\nabla p^n_{r,\lambda})=0$. Hence we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}^n_{r,\lambda}\|^2+b(\mathbf{u}^n_{r,\lambda},\mathbf{u}^n_{r,\lambda},\mathbf{A}\mathbf{u}^n_{r,\lambda})+\nu\|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|^2\\&=(\mu^n_{r,\lambda}\nabla\varphi^n_{r,\lambda},\mathbf{A}\mathbf{u}^n_{r,\lambda})+(\Theta^n_{r,\lambda},\mathbf{e}_2\cdot\mathbf{A}\mathbf{u}^n_{r,\lambda})+(\theta_g,\mathbf{e}_2\cdot\mathbf{A}\mathbf{u}^n_{r,\lambda}),\end{aligned}$$

recalling that

$$(-\nu\Delta\mathbf{u}^n_{r,\lambda},\mathbf{A}\mathbf{u}^n_{r,\lambda})=\nu\|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|^2.$$

Arguing as above (see (4.42) and (4.39)), we obtain

$$\begin{aligned}|b(\mathbf{u}^n_{r,\lambda},\mathbf{u}^n_{r,\lambda},\mathbf{A}\mathbf{u}^n_{r,\lambda})|&\le\|\mathbf{u}^n_{r,\lambda}\|_{[L^4(\Omega)]^2}\|\nabla\mathbf{u}^n_{r,\lambda}\|_{[L^4(\Omega)]^4}\ \|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|\\&\le C\|\mathbf{u}^n_{r,\lambda}\|^{1/2}\|\nabla\mathbf{u}^n_{r,\lambda}\|^{1/2}\|\nabla\mathbf{u}^n_{r,\lambda}\|^{1/2}\|\nabla\mathbf{u}^n_{r,\lambda}\|_1^{1/2}\|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|\\&\le C\|\mathbf{u}^n_{r,\lambda}\|^{1/2}\|\nabla\mathbf{u}^n_{r,\lambda}\|\ \|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|^{3/2}\\&\le C\|\nabla\mathbf{u}^n_{r,\lambda}\|\ \|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|^{3/2}\le\frac{\nu}{8}\|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|^2+C\|\nabla\mathbf{u}^n_{r,\lambda}\|^4\\(\mu^n_{r,\lambda}\nabla\varphi^n_{r,\lambda},\mathbf{A}\mathbf{u}^n_{r,\lambda})&\le\|\mu^n_{r,\lambda}\|_{L^6(\Omega)}\|\nabla\varphi\|_{[L^3(\Omega)]^2}\|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|\\&\le C\|\mu^n_{r,\lambda}\|_1\ \|\varphi^n_{r,\lambda}\|_{H^2(\Omega)}\|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|\\&\le C(1+\|\nabla\mu^n_{r,\lambda}\|)\ \|\varphi^n_{r,\lambda}\|_{H^2(\Omega)}\|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|\\&\le\frac{\nu}{8}\|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|^2+C\|\varphi^n_{r,\lambda}\|^2_{H^2(\Omega)}\left(1+\|\nabla\mu^n_{r,\lambda}\|^2\right).\end{aligned}$$

Recalling (4.40) and (4.41), we deduce

$$|(\Theta^n_{r,\lambda},\mathbf{e}_2\cdot\mathbf{A}\mathbf{u}^n_{r,\lambda})|\le\frac{\nu}{8}\|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|^2+C\tag{4.43}$$

and

$$|(\theta_g,\mathbf{e}_2\cdot\mathbf{A}\mathbf{u}^n_{r,\lambda})|\le\frac{\nu}{8}\|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|^2+C\|\theta_g\|^2\le\frac{\nu}{8}\|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|^2+C\|g\|^2_{1/2,\partial\Omega}.$$

Collecting the above estimates, we infer

$$\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}^n_{r,\lambda}\|^2+\frac{\nu}{2}\|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|^2\le C(\|\nabla \mathbf{u}^n_{r,\lambda}\|^4+\|\varphi^n_{r,\lambda}\|^2_{H^2(\Omega)}\left(1+\|\nabla\mu^n_{r,\lambda}\|^2\right)+\|g\|^2_{1/2,\partial\Omega}+1). \tag{4.44}$$

If we multiply (4.42) by $\overline{\omega}=\frac{\nu}{4C}$ and then add it to (4.44), we obtain

$$\begin{aligned}&\frac{1+\overline{\omega}\nu}{2}\frac{d}{dt}\|\nabla \mathbf{u}^n_{r,\lambda}\|^2+\frac{\nu}{4}\|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|^2+\overline{\omega}\|\partial_t\mathbf{u}^n_{r,\lambda}\|^2\\ &\le C(\|\nabla \mathbf{u}^n_{r,\lambda}\|^4+\|\varphi^n_{r,\lambda}\|^2_{H^2(\Omega)}(1+\|\nabla\mu^n_{r,\lambda}\|^2)+\|g\|^2_{1/2,\partial\Omega}+1).\end{aligned}$$

This inequality, added to (4.37), gives

$$\begin{aligned}&\frac{d\Lambda}{dt}+\frac{\nu}{4}\|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|^2+\overline{\omega}\|\partial_t\mathbf{u}^n_{r,\lambda}\|^2+\frac{\alpha}{4}\|\nabla\partial_t\varphi^n_{r,\lambda}\|^2\\ &\le C\left(1+\|\mathbf{u}^n_{r,\lambda}\|^2_{[L^3(\Omega)]^2}\right)\left(1+\|\nabla\mu^n_{r,\lambda}\|^2+\|\nabla \mathbf{u}^n_{r,\lambda}\|^2\right)\\ &\quad+(\partial_t\mathbf{u}^n_{r,\lambda}\cdot\nabla\varphi^n_{r,\lambda},\mu^n_{r,\lambda})+C(\|\nabla \mathbf{u}^n_{r,\lambda}\|^4+\|\varphi^n_{r,\lambda}\|^2_{H^2(\Omega)}(1+\|\nabla\mu^n_{r,\lambda}\|^2)\\ &\quad+\|g\|^2_{1/2,\partial\Omega}+1),\end{aligned} \tag{4.45}$$

where

$$\Lambda:=(\mu^n_{r,\lambda},\mathbf{u}^n_{r,\lambda}\cdot\nabla\varphi^n_{r,\lambda})+\frac{1}{2}\|\nabla\mu^n_{r,\lambda}\|^2+\frac{1+\overline{\omega}\nu}{2}\|\nabla \mathbf{u}^n_{r,\lambda}\|^2. \tag{4.46}$$

Using Hölder's inequality, the Sobolev embeddings $V\hookrightarrow L^6(\Omega)$ and $V_2\hookrightarrow W^{1,3}(\Omega)$, we get

$$\begin{aligned}(\partial_t\mathbf{u}^n_{r,\lambda}\cdot\nabla\varphi^n_{r,\lambda},\mu^n_{r,\lambda})&\le\|\partial_t\mathbf{u}^n_{r,\lambda}\|\ \|\nabla\varphi^n_{r,\lambda}\|_{[L^3(\Omega)]^2}\|\mu^n_{r,\lambda}\|_{L^6(\Omega)}\\ &\le\frac{\overline{\omega}}{2}\|\partial_t\mathbf{u}^n_{r,\lambda}\|^2+C\|\varphi^n_{r,\lambda}\|^2_{H^2(\Omega)}\|\mu^n_{r,\lambda}\|^2_1\\ &\le\frac{\overline{\omega}}{2}\|\partial_t\mathbf{u}^n_{r,\lambda}\|^2+C\|\varphi^n_{r,\lambda}\|^2_{H^2(\Omega)}(1+\|\nabla\mu^n_{r,\lambda}\|^2).\end{aligned}$$

Thus, we infer from (4.45) that

$$\begin{aligned}&\frac{d\Lambda}{dt}+\frac{\nu}{4}\|\mathbf{A}\mathbf{u}^n_{r,\lambda}\|^2+\frac{\overline{\omega}}{2}\|\partial_t\mathbf{u}^n_{r,\lambda}\|^2+\frac{\alpha}{4}\|\nabla\partial_t\varphi^n_{r,\lambda}\|^2\\ &\le C\left(1+\|\mathbf{u}^n_{r,\lambda}\|^2_{[L^3(\Omega)]^2}\right)\left(1+\|\nabla\mu^n_{r,\lambda}\|^2+\|\nabla \mathbf{u}^n_{r,\lambda}\|^2\right)\\ &\quad+C(\|\nabla \mathbf{u}^n_{r,\lambda}\|^4+\|\varphi^n_{r,\lambda}\|^2_{H^2(\Omega)}(1+\|\nabla\mu^n_{r,\lambda}\|^2)+\|g\|^2_{1/2,\partial\Omega}+1).\end{aligned} \tag{4.47}$$

Let us show that Λ is bounded from below. Indeed, we have

$$\begin{aligned}(\mathbf{u}^n_{r,\lambda}\cdot\nabla\varphi^n_{r,\lambda},\mu^n_{r,\lambda})&\le\|\mathbf{u}^n_{r,\lambda}\|_{L^4(\Omega)}\|\nabla\varphi^n_{r,\lambda}\|\ \|\mu^n_{r,\lambda}\|_{L^4(\Omega)}\\ &\le C\|\mathbf{u}^n_{r,\lambda}\|^{1/2}\|\nabla \mathbf{u}^n_{r,\lambda}\|^{1/2}\|\mu^n_{r,\lambda}\|_1\\ &\le C\|\mathbf{u}^n_{r,\lambda}\|^{1/2}\|\nabla \mathbf{u}^n_{r,\lambda}\|^{1/2}(1+\|\nabla\mu^n_{r,\lambda}\|)\\ &\le\frac{1}{4}\|\nabla \mathbf{u}^n_{r,\lambda}\|^2+\frac{1}{4}\|\nabla\mu^n_{r,\lambda}\|^2+C.\end{aligned}$$

Thus we deduce

$$\Lambda\ge\frac{1}{4}\|\nabla \mathbf{u}^n_{r,\lambda}\|^2+\frac{1}{4}\|\nabla\mu^n_{r,\lambda}\|^2-C'$$

for some $C'>0$. Then, setting $\tilde{\Lambda}=\Lambda+C'$, we also have

$$\tilde{\Lambda}\leq C\left(1+\|\nabla\mu_{r,\lambda}^n\|^2+\|\nabla\mathbf{u}_{r,\lambda}^n\|^2\right).$$

Recalling now that

$$\|\varphi_{r,\lambda}^n\|_{H^2(\Omega)}^2\leq C(1+\|\nabla\mu_{r,\lambda}^n\|)$$

we infer from (4.47) the following

$$\frac{d}{dt}\tilde{\Lambda}\leq C\left(1+\|g\|_{1/2,\partial\Omega}^2+\tilde{\Lambda}^2\right), \tag{4.48}$$

and we know that $\tilde{\Lambda}\in L^1(0,T)$. Thus Gronwall's lemma yields, for every $t\in[0,T]$,

$$\begin{aligned}\tilde{\Lambda}(t)&\leq\tilde{\Lambda}(0)e^{C\int_0^t\tilde{\Lambda}(s)ds}+\int_0^t e^{C\int_s^t\tilde{\Lambda}(\tau)d\tau}C\left(1+\|g(s)\|_{1/2,\partial\Omega}^2\right)ds\\&\leq e^{C(T)}\left[\tilde{\Lambda}(0)+C(T)\right].\end{aligned} \tag{4.49}$$

Here $C(T)>0$ is independent of n,λ,κ but it depends on T. Let us estimate $\tilde{\Lambda}(0)$. Thanks to Hölder's inequality and using Sobolev embeddings $\mathbf{V}_\sigma\hookrightarrow[L^3(\Omega)]^2$ and $V\hookrightarrow L^6(\Omega)$, we find

$$\begin{aligned}\tilde{\Lambda}(0)&=(\mu_{r,\lambda}^n(0),\mathbf{u}_{r,\lambda}^n(0)\cdot\nabla\varphi_{r,\lambda}^n(0))+\frac{1}{2}\|\nabla\mu_{r,\lambda}^n(0)\|^2+\frac{1+\overline{\omega}\nu}{2}\|\nabla\mathbf{u}_{r,\lambda}^n(0)\|^2+C'\\&\leq(\mu_{r,\lambda}^n(0),P_n(\mathbf{u}_0)\cdot\nabla\tilde{P}_n(\varphi_{0,r}))+\frac{1}{2}\|\nabla\mu_{r,\lambda}^n(0)\|^2+\frac{1+\overline{\omega}\nu}{2}\|\nabla P_n(\mathbf{u}_0)\|^2+C'\\&\leq\|P_n(\mathbf{u}_0)\|_{[L^3(\Omega)]^2}\|\mu_{r,\lambda}^n(0)\|_{L^6(\Omega)}\|\nabla\tilde{P}_n(\varphi_{0,r})\|\\&\quad+\frac{1}{2}\|\nabla\mu_{r,\lambda}^n(0)\|^2+\frac{1+\overline{\omega}\nu}{2}\|\nabla\mathbf{u}_0\|^2+C'\\&\leq\|\nabla P_n(\mathbf{u}_0)\|\;\|\mu_{r,\lambda}^n(0)\|_1\;\|\nabla\varphi_{0,r}\|+\frac{1}{2}\|\nabla\mu_{r,\lambda}^n(0)\|^2+\frac{1+\overline{\omega}\nu}{2}\|\nabla\mathbf{u}_0\|^2+C'\\&\leq\|\nabla\mathbf{u}_0\|\;\|\mu_{r,\lambda}^n(0)\|_1\;\|\nabla\varphi_{0,r}\|+\frac{1}{2}\|\nabla\mu_{r,\lambda}^n(0)\|^2+\frac{1+\overline{\omega}\nu}{2}\|\nabla\mathbf{u}_0\|^2+C'.\end{aligned}$$

Observe now that (see (4.4))

$$\begin{aligned}\|\nabla\mathbf{u}_0\|\;\|\mu_{r,\lambda}^n(0)\|_1\;\|\nabla\varphi_{0,r}\|&\leq\|\nabla\mathbf{u}_0\|\;\|\mu_{r,\lambda}^n(0)\|_1(1+\|\varphi_0\|_1)\\&\leq\frac{1}{2}\|\nabla\mathbf{u}_0\|^2+C\;\|\mu_{r,\lambda}^n(0)\|_1^2\;\left(1+\|\varphi_0\|_1^2\right).\end{aligned}$$

We are left to control $\|\mu_{r,\lambda}^n(0)\|_1$.

Recalling the orthogonality of the projector $\tilde{P}_n$, the definition of Ψ_λ and $\tilde{\mu}_{0,r}=-\alpha\Delta\varphi_{0,r}+F_\lambda'(\varphi_{0,r})$ with $\|\tilde{\mu}_{0,r}\|_1\leq\|\tilde{\mu}_0\|_1$ (see (4.1)), we get

$$\begin{aligned}\|\mu_{r,\lambda}^n(0)\|_1&=\|\tilde{P}_n(-\alpha\Delta\varphi_{r,\lambda}^n(0)+\Psi_\lambda'(\varphi_{r,\lambda}^n(0)))\|_1\\&\leq\|-\alpha\Delta\varphi_{r,\lambda}^n(0)+\Psi_\lambda'(\varphi_{r,\lambda}^n(0))\|_1\\&\leq\|-\alpha\Delta\varphi_{r,\lambda}^n(0)+F_\lambda'(\varphi_{r,\lambda}^n(0))\|_1+\alpha_0\|\varphi_{r,\lambda}^n(0)\|_1\\&\leq\|-\alpha\Delta\varphi_{r,\lambda}^n(0)+F_\lambda'(\varphi_{r,\lambda}^n(0))+\alpha\Delta\varphi_{0,r}-F_\lambda'(\varphi_{0,r})\|_1+\|\tilde{\mu}_{0,r}\|_1+\alpha_0\|\varphi_{r,\lambda}^n(0)\|_1\\&\leq\|\varphi_{r,\lambda}^n(0)-\varphi_{0,r}\|_{H^3(\Omega)}+\|F_\lambda'(\varphi_{r,\lambda}^n(0))-F_\lambda'(\varphi_{0,r})\|_1+C\;(\|\tilde{\mu}_0\|_1+\|\varphi_0\|_1).\end{aligned}$$

We know that $\varphi^n_{r,\lambda}(0)=\tilde{P}_n(\varphi_{0,r})\to\varphi_{0,r}$ in $H^3(\Omega)$ as $n\to\infty$. Thus the first term on the right-hand side is bounded and $\|\varphi^n_{r,\lambda}(0)\|_{H^3(\Omega)}\le C$ for n sufficiently large. On the other hand, exploiting the fact that F_λ and its first and second derivatives coincide with the corresponding ones of F on $[-1+\lambda^*,1-\lambda^*]$, we have

$$\begin{aligned}&\|\nabla(F'_\lambda(\varphi^n_{r,\lambda}(0))-F'_\lambda(\varphi_{0,r}))\|\le\|F''_\lambda(\varphi^n_{r,\lambda}(0))\nabla\varphi^n_{r,\lambda}(0)-F''_\lambda(\varphi_{0,r})\nabla\varphi_{0,r}\|\\ \le&\|F''_\lambda(\varphi_{0,r})\nabla(\varphi^n_{r,\lambda}(0)-\varphi_{0,r})\|+\|(F''_\lambda(\varphi^n_{r,\lambda}(0))-F''_\lambda(\varphi_{0,r}))\nabla\varphi_{0,r}\|\\ \le&C\left(\max_{z\in[-1+\lambda^*,1-\lambda^*]}|F''(z)|+\max_{z\in[-1+\lambda^*,1-\lambda^*]}|F'''(z)|\right)\|\varphi^n_{r,\lambda}(0)-\varphi_{0,r}\|_1.\end{aligned}$$

The maxima are finite, being $F\in\mathcal{C}^3(-1,1)$, but they depend on λ^* and thus on r. However, the norm $\|\varphi^n_{r,\lambda}(0)-\varphi_{0,r}\|_1$ goes to zero as $n\to+\infty$. Therefore we can always choose, for any given r, a sufficiently large n so that the estimated difference $\|\nabla(F'_\lambda(\varphi^n_{r,\lambda}(0))-F'_\lambda(\varphi_{0,r}))\|$ is eventually arbitrarily small. We can thus infer that, for any fixed $r>r_0$, $\lambda\in(0,\lambda^*(r_0))$ and $n>\overline{n}(r_0,\lambda^*(r_0))$:

$$\tilde{\Lambda}(0)\le C.$$

In view of (4.49) we deduce that

$$\sup_{t\in[0,T]}\|\nabla\mathbf{u}^n_{r,\lambda}(t)\|+\sup_{t\in[0,T]}\|\nabla\mu^n_{r,\lambda}(t)\|\le C(T). \tag{4.50}$$

We also obtain

$$\int_0^T\left(\|\mathbf{A}\mathbf{u}^n_{r,\lambda}(t)\|^2+\|\partial_t\mathbf{u}^n_{r,\lambda}(t)\|^2+\|\nabla\partial_t\varphi^n_{r,\lambda}(t)\|^2\right)dt\le C(T). \tag{4.51}$$

The above estimates allow us to find higher-order bounds for the temperature approximation. Take $\partial_t\Theta^n_{r,\lambda}$ as a test function in Equation (4.8). This gives

$$\begin{aligned}\frac{k}{2}\frac{d}{dt}\|\nabla\Theta^n_{r,\lambda}\|^2+\|\partial_t\Theta^n_{r,\lambda}\|^2=&-(\mathbf{u}^n_{r,\lambda}\cdot\nabla\Theta^n_{r,\lambda},\partial_t\Theta^n_{r,\lambda})-(\mathbf{u}^n_{r,\lambda}\cdot\nabla\theta_g,\partial_t\Theta^n_{r,\lambda})\\&+\kappa(\Delta\theta_g,\partial_t\Theta^n_{r,\lambda})-(\partial_t\theta_g,\partial_t\Theta^n_{r,\lambda}).\end{aligned} \tag{4.52}$$

Also we can take $\Delta\Theta^n_{r,\lambda}\in\mathbb{V}_n$ as a test function and obtain

$$\begin{aligned}&(\partial_t\Theta^n_{r,\lambda},\Delta\Theta^n_{r,\lambda})+\kappa(\nabla\Theta^n_{r,\lambda},\nabla\Delta\Theta^n_{r,\lambda})+(\mathbf{u}^n_{r,\lambda}\cdot\nabla\Theta^n_{r,\lambda},\Delta\Theta^n_{r,\lambda})\\=&-(\mathbf{u}^n_{r,\lambda}\cdot\nabla\theta_g,\Delta\Theta^n_{r,\lambda})+\kappa(\Delta\theta_g,\Delta\Theta^n_{r,\lambda})-(\partial_t\theta_g,\Delta\Theta^n_{r,\lambda}),\end{aligned} \tag{4.53}$$

which can be rewritten as follows

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\Theta^n_{r,\lambda}\|^2+\kappa\|\Delta\Theta^n_{r,\lambda}\|^2\\=&(\mathbf{u}^n_{r,\lambda}\cdot\nabla\Theta^n_{r,\lambda},\Delta\Theta^n_{r,\lambda})+(\mathbf{u}^n_{r,\lambda}\cdot\nabla\theta_g,\Delta\Theta^n_{r,\lambda})-\kappa(\Delta\theta_g,\Delta\Theta^n_{r,\lambda})+(\partial_t\theta_g,\Delta\Theta^n_{r,\lambda}).\end{aligned}$$

Observe that (cf. (4.50))

$$\|\mathbf{u}^n_{r,\lambda}\cdot\nabla\Theta^n_{r,\lambda}\|^2\le\|\mathbf{u}^n_{r,\lambda}\|^2_{[L^4(\Omega)]^2}\|\nabla\Theta^n_{r,\lambda}\|^2_{[L^4(\Omega)]^2}\le C(T)\|\nabla\Theta^n_{r,\lambda}\|^2_{[L^4(\Omega)]^2}. \tag{4.54}$$

and

$$\|\mathbf{u}^n_{r,\lambda}\cdot\nabla\theta_g\|^2\le\|\mathbf{u}^n_{r,\lambda}\|^2_{[L^4(\Omega)]^2}\|\nabla\theta_g\|^2_{[L^4(\Omega)]^2}\le\|\nabla\mathbf{u}^n_{r,\lambda}\|^2\|\theta_g\|^2_{H^2(\Omega)}\le C(T)\|\theta_g\|^2_{H^2(\Omega)}.$$

We can thus obtain, recalling that $\|\Delta\theta_g\| \le \|\theta_g\|_{H^2(\Omega)} \le C\|g\|_{H^{3/2}(\partial\Omega)}$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\Theta^n_{r,\lambda}\|^2 + \kappa\|\Delta\Theta^n_{r,\lambda}\|^2 \\
&\le \|\mathbf{u}^n_{r,\lambda}\cdot\nabla\Theta^n_{r,\lambda}\|\|\Delta\Theta^n_{r,\lambda}\| + \|\mathbf{u}^n_{r,\lambda}\cdot\nabla\theta_g\|\|\Delta\Theta^n_{r,\lambda}\| + \kappa\|\Delta\theta_g\|\|\Delta\Theta^n_{r,\lambda}\| + \|\partial_t\theta_g\|\|\Delta\Theta^n_{r,\lambda}\| \\
&\le C(T)\left(\|\nabla\Theta^n_{r,\lambda}\|^2_{[L^4(\Omega)]^2} + \kappa^2\|g\|^2_{H^{3/2}(\partial\Omega)}\right) + \frac{3\kappa}{8}\|\Delta\Theta^n_{r,\lambda}\|^2 + C\|\partial_t g\|^2_{H^{1/2}(\partial\Omega)}. \qquad (4.55)
\end{aligned}$$

Similar arguments applied to (4.52) entail

$$\begin{aligned}
&\frac{k}{2}\frac{d}{dt}\|\nabla\Theta^n_{r,\lambda}\|^2 + \|\partial_t\Theta^n_{r,\lambda}\|^2 \\
&\le \|\mathbf{u}^n_{r,\lambda}\cdot\nabla\Theta^n_{r,\lambda}\|\ \|\partial_t\Theta^n_{r,\lambda}\| + \|\mathbf{u}^n_{r,\lambda}\cdot\nabla\theta_g\|\ \|\partial_t\Theta^n_{r,\lambda}\| \\
&\quad + \|\partial_t\theta_g\|\ \|\partial_t\Theta^n_{r,\lambda}\| + \kappa\|\Delta\theta_g\|\ \|\partial_t\Theta^n_{r,\lambda}\| \\
&\le \frac{1}{2}\|\partial_t\Theta^n_{r,\lambda}\|^2 + C(\|\mathbf{u}^n_{r,\lambda}\cdot\nabla\theta_g\|^2 + \|\nabla\Theta^n_{r,\lambda}\|^2_{[L^4(\Omega)]^2} + \|\partial_t\theta_g\|^2 + \kappa^2\|\Delta\theta_g\|^2) \\
&\le \frac{1}{2}\|\partial_t\Theta^n_{r,\lambda}\|^2 + C\left(\|\nabla\Theta^n_{r,\lambda}\|^2_{[L^4(\Omega)]^2} + \|\partial_t g\|^2_{H^{1/2}(\partial\Omega)}\right) + C(T)\kappa^2\|g\|^2_{H^{3/2}(\partial\Omega)}. \qquad (4.56)
\end{aligned}$$

Using now Young's inequality, for a given $\delta > 0$, we find (see [52, 53])

$$\|\nabla\Theta^n_{r,\lambda}\|^2_{[L^4(\Omega)]^2} \le C\left(\|\nabla\Theta^n_{r,\lambda}\|\ \|\Theta^n_{r,\lambda}\|_{H^2(\Omega)} + \|\nabla\Theta^n_{r,\lambda}\|^2\right) \le \delta\|\Delta\Theta^n_{r,\lambda}\|^2 + C\|\nabla\Theta^n_{r,\lambda}\|^2. \qquad (4.57)$$

Then, choosing $\delta = \frac{\kappa}{8C(T)}$ for (4.55) and $\delta = \frac{\kappa}{4C}$ for (4.56), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\Theta^n_{r,\lambda}\|^2 + \frac{\kappa}{2}\|\Delta\Theta^n_{r,\lambda}\|^2 \\
&\le C(T)\left(\|\nabla\Theta^n_{r,\lambda}\|^2 + \kappa^2\|g\|^2_{H^{3/2}(\partial\Omega)}\right) + C\|\partial_t g\|^2_{H^{1/2}(\partial\Omega)}, \qquad (4.58)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{k}{2}\frac{d}{dt}\|\nabla\Theta^n_{r,\lambda}\|^2 + \frac{1}{2}\|\partial_t\Theta^n_{r,\lambda}\|^2 \\
&\le \frac{\kappa}{4}\|\Delta\Theta^n_{r,\lambda}\|^2 + C\left(\|\nabla\Theta^n_{r,\lambda}\|^2 + \|\partial_t g\|^2_{H^{1/2}(\partial\Omega)}\right) + C(T)\kappa^2\|g\|^2_{H^{3/2}(\partial\Omega)}. \qquad (4.59)
\end{aligned}$$

Adding (4.58) and (4.59) together, we obtain

$$\begin{aligned}
&\left(\frac{1}{2} + \frac{\kappa}{2}\right)\frac{d}{dt}\|\nabla\Theta^n_{r,\lambda}\|^2 + \frac{1}{2}\|\partial_t\Theta^n_{r,\lambda}\|^2 + \frac{\kappa}{4}\|\Delta\Theta^n_{r,\lambda}\|^2 \\
&\le C(T)\left(\|\nabla\Theta^n_{r,\lambda}\|^2 + \kappa^2\|g\|^2_{H^{3/2}(\partial\Omega)}\right) + C\|\partial_t g\|^2_{H^{1/2}(\partial\Omega)} \qquad (4.60)
\end{aligned}$$

and Gronwall's lemma yields

$$\|\Theta^n_{r,\lambda}\|_{L^\infty(0,T;V_\theta)} \le C(T). \qquad (4.61)$$

Then, integrating (4.60) in time over $(0,T)$ we also find

$$\int_0^T \|\partial_t\Theta^n_{r,\lambda}(s)\|^2 ds + \int_0^T \|\Theta^n_{r,\lambda}(s)\|^2_{V_\theta^2} ds \le C(T). \qquad (4.62)$$

Summing up, we have obtained the following bounds which are uniform with respect to n, r, and λ (see (4.23)-(4.27), (4.29), (4.31), (4.33), (4.50), (4.51), (4.61) and (4.62)):

- $\mathbf{u}^n_{r,\lambda}$ is uniformly bounded in $L^\infty(0,T;\mathbf{V}_\sigma)\cap L^2(0,T;\mathbf{W}_\sigma)\cap H^1(0,T;\mathbf{H}_\sigma)$
- $\varphi^n_{r,\lambda}$ is uniformly bounded in $L^\infty(0,T;V)\cap L^4(0,T;V_2)\cap H^1(0,T;V)$
- $\Theta^n_{r,\lambda}$ is uniformly bounded in $L^\infty(0,T;V_\theta)\cap L^2(0,T;V^2_\theta)\cap H^1(0,T;H)$
- $\mu^n_{r,\lambda}$ is uniformly bounded in $L^\infty(0,T;V)$.

Using weak and weak* compactness, and working first on n, then on λ and finally on r, we can extract a subsequence

$$\{(\mathbf{u}^{n_h}_{r_h,\lambda_h},\varphi^{n_h}_{r_h,\lambda_h},\Theta^{n_h}_{r_h,\lambda_h})\}_{h\in\mathbb{N}}$$

which suitably converges as $h\to\infty$ to a triple $(\mathbf{u},\varphi,\Theta)$ such that

$$\begin{aligned}&\mathbf{u}\in L^\infty(0,T;\mathbf{V}_\sigma)\cap L^2(0,T;\mathbf{W}_\sigma)\cap H^1(0,T;\mathbf{H}_\sigma)\\&\varphi\in L^\infty(0,T;V)\cap L^4(0,T;V_2)\cap H^1(0,T;V)\\&\theta=\Theta+\theta_g\in L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega))\cap H^1(0,T;H).\end{aligned}$$

Also, using a standard strong compactness argument, we can suppose that the above sequence is such that $\{(\mathbf{u}^{n_h}_{r_h,\lambda_h},\varphi^{n_h}_{r_h,\lambda_h},\Theta^{n_h}_{r_h,\lambda_h})\}_{h\in\mathbb{N}}$ converges strongly in $L^2(0,T;\mathbf{H}_\sigma)\times(L^2(0,T;H))^2$ to $(\mathbf{u},\ \varphi,\ \Theta)$. The above convergences are enough to prove that $(\mathbf{u},\ \varphi,\ \theta)$ is a weak solution. In particular, we recall that (see [20, 24, 28]), we have

$$|E_\eta|\le\frac{C}{\min\{F'(1-\eta),|F'(\eta-1)|\}},\tag{4.63}$$

where $E_\eta=\{(x,t)\in\Omega\times[0,T]:|\varphi(x,t)|>1-\eta\}$. Then, as $\eta\to0^+$, we deduce that the set $\{(x,t)\in\Omega\times[0,T]:|\varphi(x,t)|\ge1\}$ has zero measure as needed. Then, on account of the properties of F'_λ, we infer $\mu=-\alpha\Delta\varphi+\Psi'(\varphi)$ almost everywhere in $\Omega\times(0,T)$. Moreover, the obtained regularities suffice to show that $(\mathbf{u},\ \varphi,\ \theta)$ satisfies the equations of CHB$_\kappa$ almost everywhere. We are left to prove the additional regularity properties stated in Definition 3.2. Observe that $\partial_t\varphi+\mathbf{u}\cdot\nabla\varphi\in L^2(0,T;V)$. Thus we infer $\mu\in L^2(0,T;H^3(\Omega))$ and $\partial_{\mathbf{n}}\mu=0$ almost everywhere on $\partial\Omega\times(0,T)$. On the other hand, recalling [27, Cor.4.1], $\mu\in L^\infty(0,T;V)$ implies that $\varphi\in L^\infty(0,T;W^{2,p}(\Omega))$ and $F'(\varphi)\in L^\infty(0,T;L^p(\Omega))$ for any $2\le p<\infty$. Also, thanks to condition (2.3) and to (2.8), we deduce $F''(\varphi)\in L^\infty(0,T;L^p(\Omega))$ for any $p\in[2,\infty)$ (see [27, Lemma 5.1]). Finally, arguing as in [28, Sec. 4, step 7], we obtain $\partial_t\mu\in L^2(0,T;V')$ so that $\mu\in C([0,T];V)$.

Observe now that actually we can prove that estimates (4.23)-(4.27), (4.31), (4.33), (4.50), (4.51) and (4.63) still hold with C independent of κ, thanks to

$$g\in L^2(0,T;H^{3/2}(\partial\Omega))\cap H^1(0,T;H^{1/2}(\partial\Omega))\hookrightarrow C([0,T];H^1(\partial\Omega))\hookrightarrow L^\infty(\partial\Omega\times(0,T)).$$

Indeed, the only two estimates depending on κ which need to be modified are (4.15) and (4.16), but by a simple integration by parts we have, supposing, e.g., $\kappa\le1$,

$$-(\kappa\nabla\theta_g,\nabla\Theta^n_{r,\lambda})=(\kappa\Delta\theta_g,\Theta^n_{r,\lambda})\le C\|\Theta^n_{r,\lambda}\|^2+C\|\Delta\theta_g\|^2\le C\|\Theta^n_{r,\lambda}\|^2+C\|g\|^2_{3/2,\partial\Omega}.$$

Concerning (4.16) we have, by Sobolev embeddings,

$$\begin{aligned}-(\mathbf{u}^n_{r,\lambda}\cdot\nabla\theta_g,\Theta^n_{r,\lambda})&\le\|\mathbf{u}^n_{r,\lambda}\|_{L^4(\Omega)}\|\nabla\theta_g\|_{[L^4(\Omega)]^2}\ \|\Theta^n_{r,\lambda}\|\\&\le\|\mathbf{u}^n_{r,\lambda}\|^{1/2}\|\nabla\mathbf{u}^n_{r,\lambda}\|^{1/2}\|\theta_g\|_{H^2(\Omega)}\ \|\Theta^n_{r,\lambda}\|\\&\le\frac{\nu_*}{2}\|\nabla\mathbf{u}^n_{r,\lambda}\|^2+C\|g\|^2_{3/2,\partial\Omega}\|\Theta^n_{r,\lambda}\|^2+C\|\mathbf{u}^n_{r,\lambda}\|^2.\end{aligned}$$

In addition, for g with the same regularity, $\theta_0 \in L^\infty(\Omega)\cap V_\theta$ and $\mathbf{u}_0 \in \mathbf{W}_\sigma$ we can deduce a higher-order regularity for θ and $\mathbf{u}$ independent of κ as well. As above, let us argue formally. From now on $C>0$ does not depend on κ. We first observe that, due to Remark 3.10, since $g \in L^\infty(\partial\Omega\times(0,T))$, we have

$$\|\theta\|_{L^\infty(\Omega\times(0,T))} \le C(\|\theta_0\|_{L^\infty}, \|g\|_{L^\infty(\partial\Omega\times(0,T)}) \le C. \tag{4.64}$$

If $\mathbf{u}$ is a strong solution and $\mathbf{u}_0 \in \mathbf{W}_\sigma$, then we can write

$$\begin{cases} \partial_t \mathbf{u} + \nabla p - \nu\Delta\mathbf{u} = \mathbf{h} := -(\mathbf{u}\cdot\nabla)\mathbf{u} + \mu\nabla\varphi + \theta\mathbf{e}_2 \\ \text{div } \mathbf{u} = 0 \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{a.e. in } \Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega\times(0,T). \end{cases}$$

Let us show a higher-order regularity estimate for $\mathbf{u}$: We start proving that $\mathbf{h} \in L^2(0,T;[L^p(\Omega)]^2$. Observe first that, owing to (4.64),

$$\|\theta\mathbf{e}_2\|_{L^2(0,T;[L^p(\Omega)]^2)} \le C. \tag{4.65}$$

Also, we have

$$\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L^2(0,T;[L^p(\Omega)]^2)} \le \|\mathbf{u}\|_{L^\infty(0,T;[L^{2p}(\Omega)]^2)} \|\nabla\mathbf{u}\|_{L^2(0,T;[L^{2p}(\Omega)]^4)}.$$

By the Sobolev embedding $\mathbf{V}_\sigma \hookrightarrow L^{2p}(\Omega)$ for all $p \in [1,\infty)$ (see also (4.50)), we obtain

$$\|\mathbf{u}\|_{L^\infty(0,T;[L^{2p}(\Omega)]^2)} \le C\|\nabla\mathbf{u}\|_{L^\infty(0,T;[L^2(\Omega)]^4)} \le C.$$

Moreover, due to Sobolev embedding $\mathbf{W}_\sigma \hookrightarrow [W^{1,2p}(\Omega)]^2$ and to (4.51), we get

$$\|\nabla\mathbf{u}\|_{L^2(0,T;[L^{2p}(\Omega)]^4)} \le C\|\mathbf{A}\mathbf{u}\|_{L^2(0,T;[L^2(\Omega)]^2)} \le C.$$

Thus we conclude that

$$\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L^2(0,T;[L^p(\Omega)]^2)} \le C. \tag{4.66}$$

Consider now the Korteweg force. We have

$$\|\mu\nabla\varphi\|_{L^2(0,T;[L^p(\Omega)]^2)} \le \|\mu\|_{L^\infty(0,T;L^{2p}(\Omega))} \|\nabla\varphi\|_{L^2(0,T;[L^{2p}(\Omega)]^2)}.$$

Recalling (4.50) we deduce

$$\|\mu\|_{L^\infty(0,T;L^{2p}(\Omega))} \le C\|\mu\|_{L^\infty(0,T;V)} \le C.$$

Moreover, due to Sobolev embedding $V_2 \hookrightarrow W^{1,2p}(\Omega)$ for all $p \in [1,\infty)$ and to (4.26), we get

$$\|\nabla\varphi\|_{L^2(0,T;[L^{2p}(\Omega)]^2)} \le C\|\varphi\|_{L^2(0,T;V_2)} \le C.$$

Thus we conclude that

$$\|\mu\nabla\varphi\|_{L^2(0,T;[L^p(\Omega)]^2)} \le C. \tag{4.67}$$

From (4.65), (4.66), (4.67) we can conclude that $\mathbf{h} \in L^2(0,T;[L^p(\Omega)]^2)$. Then, by the maximal regularity theory of the Stokes system (see, e.g., [25] and [44]), since $\mathbf{u}_0 \in \mathbf{W}_\sigma$

(actually, considering Besov spaces, the initial datum $\mathbf{u}_0$ can be even less regular, see, e.g., [54, Lemma 2.6]), we have that, for any $p\in(2,\infty)$,

$$\|\partial_t\mathbf{u}\|_{L^2(0,T;[L^p(\Omega)]^2)}+\|\mathbf{u}\|_{L^2(0,T;[W^{2,p}(\Omega)]^2)}\le C\left(\|\mathbf{h}\|_{L^2(0,T;[L^p(\Omega)]^2)}+\|\mathbf{u}_0\|_{\mathbf{W}_\sigma}\right).$$

The proof is finished. □

REMARK 4.1. In the case $g\equiv 0$ the Galerkin scheme is simpler. Indeed, arguing formally, we have

$$\frac{d}{dt}\|\theta\|^2+\beta_0\|\theta\|^2\le 0 \tag{4.68}$$

where $\beta_0=2\frac{\kappa}{C_0}$ and C_0 is the Poincaré's constant. Hence Gronwall's inequality gives

$$\|\theta\|^2\le\|\theta_0\|^2e^{-\beta_0 t}. \tag{4.69}$$

Moreover, we get (see (3.10))

$$\frac{d}{dt}E(t)+\|\nabla\mu\|^2+\frac{\nu}{2}\|\nabla\mathbf{u}\|^2+\kappa\|\nabla\theta\|^2\le\frac{C_0^2}{2\nu}\|\theta\|^2. \tag{4.70}$$

Therefore the following energy estimate holds

$$E(t)+\int_0^t\|\nabla\mu\|^2ds+\int_0^t\frac{\nu}{2}\|\nabla\mathbf{u}\|^2ds+\int_0^t k\|\nabla\theta\|^2ds\le E(0)+\frac{C_0^2}{2\nu\beta_0}\|\theta_0\|^2(1-e^{-\beta_0 t}), \tag{4.71}$$

for every $t\in[0,T]$. This implies that estimates (4.23)-(4.27), (4.31), (4.33), (4.50), (4.51) and (4.63) still hold with C independent of κ.

Proof. (**Proof of Theorem 3.2.**) Let us set $\mathbf{u}=\mathbf{u}_1-\mathbf{u}_2$, $\varphi=\varphi_1-\varphi_2$ and $\theta=\theta_1-\theta_2$. We also define $\mu=-\alpha\Delta\varphi+\Psi'(\varphi_1)-\Psi'(\varphi_2)$. Recalling the weak formulation, we can write

$$\begin{aligned}&<\partial_t\mathbf{u},\mathbf{w}>+b(\mathbf{u}_1,\mathbf{u},\mathbf{w})+b(\mathbf{u},\mathbf{u}_2,\mathbf{w})+\nu(\nabla\mathbf{u},\nabla\mathbf{w})\\&=\alpha(\nabla\varphi_1\otimes\nabla\varphi,\nabla\mathbf{w})+\alpha(\nabla\varphi\otimes\nabla\varphi_2,\nabla\mathbf{w})+(\theta,\mathbf{e}_2\cdot\mathbf{w})\qquad\forall\mathbf{w}\in\mathbf{V}_\sigma &(4.72)\\&<\partial_t\varphi,v>+(\nabla\mu,\nabla v)+(\mathbf{u}_1\cdot\nabla\varphi,v)+(\mathbf{u}\cdot\nabla\varphi_2,v)=0\qquad\forall v\in V &(4.73)\\&<\partial_t\theta,\xi>+\kappa(\nabla\theta,\nabla\xi)-(\mathbf{u}_1\theta,\nabla\xi)-(\mathbf{u}\theta_2,\nabla\xi)=0\qquad\forall\xi\in V_\theta. &(4.74)\end{aligned}$$

Here we have used an alternative expression of the Korteweg force (see the Introduction). We take $\mathbf{w}=\mathbf{u}$, $v=\varphi$ and $\xi=\theta$. Then we add together the resulting identities. This gives

$$\frac{d}{dt}\mathcal{H}_1+\nu\|\nabla\mathbf{u}\|^2+\kappa\|\nabla\theta\|^2+(\nabla\mu,\nabla\varphi)=\sum_{j=1}^{6}\mathcal{I}_j,$$

where $\mathcal{H}_1=\frac{1}{2}\|\mathbf{u}\|^2+\frac{1}{2}\|\varphi\|^2+\frac{1}{2}\|\theta\|^2$ and

$$\begin{aligned}&\mathcal{I}_1=-b(\mathbf{u},\mathbf{u}_2,\mathbf{u}), &&\mathcal{I}_2=\alpha(\nabla\varphi_1\otimes\nabla\varphi,\nabla\mathbf{u}), &&\mathcal{I}_3=\alpha(\nabla\varphi\otimes\nabla\varphi_2,\nabla\mathbf{u}),\\&\mathcal{I}_4=(\mathbf{u}\varphi_2,\nabla\varphi), &&\mathcal{I}_5=(\theta,\mathbf{e}_2\cdot\mathbf{u}), &&\mathcal{I}_6=(\mathbf{u}\theta_2,\nabla\theta),\end{aligned}$$

observing that $(\mathbf{u}_1\varphi,\nabla\varphi)=0$. Recalling then the proof of Theorem 3.1 and [28, Thm.A.2], we have that

$$\|\mathbf{u}_i\|_{L^\infty(0,T;\mathbf{V}_\sigma)}+\|\varphi_i\|_{L^\infty(0,T;W^{2,3}(\Omega))}+\|\Psi''(\varphi_i)\|_{L^\infty(0,T;L^3(\Omega))}\le C,\quad i=1,2 \tag{4.75}$$

for some constant $C>0$ also depending on T. On the other hand, observe that

$$\|\varphi\|_1^2\le\|\Delta\varphi\|\|\varphi\|+\|\varphi\|^2. \tag{4.76}$$

Also, integrating by parts, we get

$$(\nabla\mu,\nabla\varphi)=\alpha\|\Delta\varphi\|^2-(\Psi'(\varphi_1),\Delta\varphi)+(\Psi'(\varphi_2),\Delta\varphi).$$

Using now (4.75) and classical embeddings we deduce

$$\begin{aligned}(\Psi'(\varphi_1)-\Psi'(\varphi_2),\Delta\varphi)&=\left(\varphi\int_0^1\{s\Psi''(\varphi_1)+(1-s)\Psi''(\varphi_2)\}\,ds,\Delta\varphi\right)\\&\le(\|\Psi''(\varphi_1)\|_{L^3(\Omega)}+\|\Psi''(\varphi_2)\|_{L^3(\Omega)})\|\varphi\|_{L^6(\Omega)}\|\Delta\varphi\|\le C\|\varphi\|_1\|\Delta\varphi\|\end{aligned}$$

Therefore, using (4.76) and Young's inequality twice, we find

$$(\nabla\mu,\nabla\varphi)\ge\alpha\|\Delta\varphi\|^2-C\|\varphi\|_1\|\Delta\varphi\|\ge\frac{\alpha}{2}\|\Delta\varphi\|^2-C\|\varphi\|^2.$$

On account of Sobolev embedding $\mathbf{V}_\sigma\hookrightarrow[L^6(\Omega)]^2$, from (4.75) we deduce

$$\mathcal{I}_1\le\|\mathbf{u}\|\ \|\nabla\mathbf{u}_2\|_{[L^3(\Omega)]^4}\|\mathbf{u}\|_{[L^6(\Omega)]^2}\le\frac{\nu}{4}\|\nabla\mathbf{u}\|^2+C\|\mathbf{u}\|^2\|\nabla\mathbf{u}_2\|^2_{[L^3(\Omega)]^4}.$$

By (4.75), the embedding $W^{2,3}(\Omega)\hookrightarrow W^{1,\infty}(\Omega)$ and (4.76), we infer

$$\mathcal{I}_2+\mathcal{I}_3\le\alpha(\|\nabla\varphi_1\|_\infty+\|\nabla\varphi_2\|_\infty)\|\nabla\varphi\|\ \|\nabla\mathbf{u}\|\le\frac{\nu}{4}\|\nabla\mathbf{u}\|^2+\frac{\alpha}{8}\|\Delta\varphi\|^2+C\|\varphi\|^2.$$

By standard embeddings, (4.75) and (4.76), we get

$$\mathcal{I}_4\le\|\varphi_2\|_\infty\|\mathbf{u}\|\ \|\nabla\varphi\|\le\frac{\alpha}{8}\|\Delta\varphi\|^2+C\left(\|\varphi\|^2+\|\varphi_2\|^2_\infty\|\mathbf{u}\|^2\right),$$

$$\mathcal{I}_5\le\|\theta\|\ \|\mathbf{u}\|\le\frac{1}{2}\|\theta\|^2+\frac{1}{2}\|\mathbf{u}\|^2.$$

Furthermore, since $H^2(\Omega)\hookrightarrow W^{1,4}(\Omega)$, by Young's inequality:

$$\begin{aligned}\mathcal{I}_6=(\mathbf{u}\cdot\nabla\theta_2,\theta)&\le\|\nabla\theta_2\|_{[L^4(\Omega)]^2}\ \|\mathbf{u}\|_{[L^4(\Omega)]^2}\ \|\theta\|\\&\le\frac{\nu}{4}\|\nabla\mathbf{u}\|^2+C\|\theta\|^2\|\theta_2\|^2_{H^2(\Omega)}.\end{aligned}$$

Adding up all the terms, we obtain

$$\frac{d}{dt}\mathcal{H}_1+\frac{\nu}{4}\|\nabla\mathbf{u}\|^2+\kappa\|\nabla\theta\|^2+\frac{\alpha}{4}\|\Delta\varphi\|^2\le C\mathcal{R}_1\mathcal{H}_1, \tag{4.77}$$

where $\mathcal{R}_1:=1+\|\nabla\mathbf{u}_2\|^2_{[L^3(\Omega)]^4}+\|\varphi_2\|^2_\infty+\|\theta_2\|^2_{H^2(\Omega)}\in L^1(0,T)$. Thus Gronwall's lemma yields (3.9). The proof is finished. ∎

Proof. (**Proof of Theorem 3.3.**) The proof is divided into two steps.

Approximating the initial data. We need to approximate the initial data to apply the existence result of Theorem 3.1 and then find suitable estimates which allow to recover a weak solution. First of all, by the density of $\mathbf{V}_\sigma$ in $\mathbf{H}_\sigma$, we can find a sequence $\{\mathbf{u}_{0,m}\}_m \subset \mathbf{V}_\sigma$ such that $\mathbf{u}_{0,m} \to \mathbf{u}_0$ in $\mathbf{H}_\sigma$ as $m \to \infty$ so that $\mathbf{u}_{0,m}$ is uniformly bounded in $\mathbf{H}_\sigma$. Concerning φ_0, we consider the following two-step approximation.

(1) We introduce the Lipschitz function $h_m : \mathbb{R} \to \mathbb{R}$, $m \in \mathbb{N}_0$, such that

$$h_m(z) = \begin{cases} 1 - \frac{1}{m} & \text{if } z > 1 - \frac{1}{m} \\ z & \text{if } -1 + \frac{1}{m} \leq z \leq 1 - \frac{1}{m} \\ -1 + \frac{1}{m} & \text{if } z < -1 + \frac{1}{m} \end{cases}$$

and define $\varphi^m := h_m(\varphi_0) \in V$. By the properties of the composition, we have $\nabla \varphi^m = \nabla \varphi_0 \chi_{[-1+\frac{1}{m}, 1-\frac{1}{m}]}(\varphi_0)$. Then

$$\|\nabla \varphi^m\| \leq \|\nabla \varphi_0\|, \tag{4.78}$$

for every $m \in \mathbb{N}_0$. Moreover, by Lebesgue's dominated convergence theorem, we have, as $m \to \infty$,

$$\varphi^m \to \varphi_0 \qquad \text{in } H. \tag{4.79}$$

By the cutoff properties, we get $\|\varphi^m\|_{L^\infty(\Omega)} \leq 1 - \frac{1}{m}$, for every $m \in \mathbb{N}_0$. Observe that $|\overline{\varphi}^m| \to |\overline{\varphi}_0|$. We know that there exists $\delta > 0$ such that $|\overline{\varphi}_0| < 1 - \delta$. Then there exists $\overline{m} > 0$ such that $|\overline{\varphi}^m| < 1 - \delta$ for every $m > \overline{m}$.

(2) Let us now introduce the sequence $\{\varphi_{0,m}\}_{m \in \mathbb{N}_0}$ which approximates φ_0. The function $\varphi_{0,m}$ is the unique solution to the problem

$$\begin{cases} -\frac{1}{m} \Delta \varphi_{0,m} + \varphi_{0,m} = \varphi^m & \text{in } \Omega \\ \partial_{\mathbf{n}} \varphi_{0,m} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.80}$$

From the elliptic regularity (see, e.g., [39]) we obtain

$$\|\varphi_{0,m}\|_{H^3(\Omega)} \leq C(m) \|\varphi^m\|_V.$$

Exploiting this regularity we can take the gradient of (4.80), multiply by $\nabla \varphi_{0,m} \in H^2(\Omega)$ and integrate by parts. This gives

$$\frac{1}{m} \|\Delta \varphi_{0,m}\|^2 + \|\nabla \varphi_{0,m}\|^2 = (\nabla \varphi^m, \nabla \varphi_{0,m}) \leq \frac{1}{2} \|\nabla \varphi_{0,m}\|^2 + \frac{1}{2} \|\nabla \varphi^m\|^2.$$

We then deduce

$$\|\nabla \varphi_{0,m}\| \leq \|\nabla \varphi^m\| \leq \|\nabla \varphi_0\|, \tag{4.81}$$

for every $m \in \mathbb{N}_0$, where in the last estimate we exploited (4.78). If we now test (4.80) against $\varphi_{0,m} - \varphi^m$ and integrate by parts, we obtain

$$\|\varphi^m - \varphi_{0,m}\|^2 = \frac{1}{m} (\nabla \varphi_{0,m}, \nabla(\varphi_{0,m} - \varphi^m)),$$

but, from (4.78) and (4.81), $(\nabla\varphi_{0,m},\nabla(\varphi_{0,m}-\varphi^m))\le 2\|\nabla\varphi_0\|^2$, independently of m, and thus we deduce

$$\|\varphi_{0,m}-\varphi_0\|\le\|\varphi^m-\varphi_0\|+\|\varphi^m-\varphi_{0,m}\|\le\|\varphi^m-\varphi_0\|+\frac{2}{m}\|\nabla\varphi_0\|^2,$$

which implies, since $\varphi_0\in V$ and by (4.79), that $\varphi_{0,m}\to\varphi_0$ strongly in H. Thanks to this result and from (4.81), we immediately have $\nabla\varphi_{0,m}\rightharpoonup\nabla\varphi_0$, which, together with (4.81) and V being a Hilbert space, implies also $\nabla\varphi_{0,m}\to\nabla\varphi_0$ and thus we get $\varphi_{0,m}\to\varphi_0$ in V. For what concerns the mean value, integrating Equation (4.80) over Ω and using the boundary condition, we get $\overline{\varphi}_{0,m}=\overline{\varphi}^m$, thus there exists $\overline{m}>0$ such that

$$|\overline{\varphi}_{0,m}|=|\overline{\varphi}^m|<1-\delta, \tag{4.82}$$

for every $m>\overline{m}$, with δ independent of m. We now show that $\varphi_{0,m}$ enjoys the separation property for every $m\in\mathbb{N}_0$. We know that $-1+\frac{1}{m}\le\varphi^m\le 1-\frac{1}{m}$ almost everywhere in Ω by the cutoff properties. We consider $v=\left(\varphi_{0,m}-\left(1-\frac{1}{m}\right)\right)^+\in V$ and we rewrite system (4.80) as follows

$$\begin{cases}-\frac{1}{m}\Delta\left(\varphi_{0,m}-\left(1-\frac{1}{m}\right)\right)+\varphi_{0,m}-\left(1-\frac{1}{m}\right)=\varphi^m-\left(1-\frac{1}{m}\right) & \text{in } \Omega\\ \partial_{\mathbf{n}}\left(\varphi_{0,m}-\left(1-\frac{1}{m}\right)\right)=0 & \text{on } \partial\Omega.\end{cases}$$

Multiplying the above equation by v and integrating by parts, we deduce

$$\frac{1}{m}\int_{\{x\in\Omega:\ \varphi_{0,m}(x)-(1-\frac{1}{m})\ge 0\}}\left|\nabla\left(\varphi_{0,m}-\left(1-\frac{1}{m}\right)\right)\right|^2dx+\left\|\left(\varphi_{0,m}-\left(1-\frac{1}{m}\right)\right)^+\right\|^2$$
$$=\int_\Omega\left(\varphi_m-\left(1-\frac{1}{m}\right)\right)\left(\varphi_{0,m}-\left(1-\frac{1}{m}\right)\right)^+dx\le 0,$$

implying that $\varphi_{0,m}\le 1-\frac{1}{m}$ almost everywhere in Ω.
Consider now $w=\left(\varphi_{0,m}-\left(-1+\frac{1}{m}\right)\right)^-\in V$. A similar argument entails $\varphi_{0,m}\ge -1+\frac{1}{m}$ almost everywhere in Ω. Therefore, we have $\|\varphi_{0,m}\|_{L^\infty(\Omega)}\le 1-\frac{1}{m}$. From this property and from the $H^3(\Omega)$-regularity of $\varphi_{0,m}$, we conclude that $\mu_{0,m}=-\alpha\Delta\varphi_{0,m}+\Psi'(\varphi_{0,m})\in V$. Therefore $\varphi_{0,m}$ satisfies the assumptions of Theorem 3.1.

Regarding θ, we first approximate the boundary datum. Using Lemma 2.1, there exists a sequence of functions $g_m\in C^\infty([0,T];H^{3/2}(\partial\Omega))$ such that $g_m\to g$ in $H^1(0,T;H^{1/2}(\partial\Omega))$ and, in particular, $g_m\to g$ in $L^\infty(0,T;H^{1/2}(\partial\Omega))$. Consider now the lift operator $\theta_{g,m}$ with g_m as boundary datum. Then we have that $\theta_{g,m}\in H^1(0,T;V)$. Moreover $\theta_{g,m}\to\theta_g$ in $L^\infty(0,T;V)$ and $\partial_t\theta_{g,m}\to\partial_t\theta_g$ in $L^2(0,T;V)$, where θ_g is the lift operator with boundary datum g. Hence we also have $\theta_{g,m}\to\theta_g$ in $C([0,T];H)$ so that $\theta_{g,m}(0)\to\theta_g(0)$ in H. We then exploit the density of $D(A_0)=V_\theta^2$ in H to find a sequence $\{\Theta_{0,m}\}_{m\in\mathbb{N}_0}\subset V_\theta^2$ such that $\Theta_{0,m}\to\theta_0-\theta_g(0)$ in H. Thus the approximating initial datum $\theta_{0,m}=\Theta_{0,m}+\theta_{g,m}(0)\in V$ respects the compatibility condition $\theta_{0,m}=g_m(0)$ on $\partial\Omega$, satisfies the assumptions of Theorem 3.1, and $\theta_{0,m}\to\theta_0$ in H.

Existence of a weak solution. Let us consider CHB$_\kappa$, with initial conditions $(\mathbf{u}_{0,m},\varphi_{0,m},\theta_{0,m})$ and g_m as Dirichlet boundary condition for the temperature, supposing $m>\overline{m}$ previously defined. By Theorems 3.1 and 3.2, there exists a unique strong solution to CHB$_\kappa$, say, $(\mathbf{u}_m,\varphi_m.\theta_m)$. Set $\Theta_m=\theta_m-\theta_{g,m}\in V_\theta$ for almost any $t\in(0,T)$

and observe that, by construction, $\Theta_m(0)=\Theta_{0,m}$. Consider the weak formulation of the problem. Exploiting the regularity of the strong solution, we repeat verbatim the first part of the proof of Theorem 3.1. From now on, $C>0$ stands for a constant independent of m and t, which may vary from line to line.

Setting $E_m:=\frac{1}{2}\|\mathbf{u}_m\|^2+\frac{1}{2}\|\Theta_m\|^2+\frac{\alpha}{2}\|\nabla\varphi_m\|^2+\int_\Omega(\Psi(\varphi_m)+\hat{C})$, we obtain

$$\frac{d}{dt}E_m+\mathcal{D}_m\leq C(1+\|g_m\|^4_{1/2,\partial\Omega})\ E_m+C(\|g_m\|^2_{1/2,\partial\Omega}+\|\partial_t g_m\|^2_{1/2,\partial\Omega}),$$

where $\mathcal{D}_m:=\|\nabla\mu_m\|^2+\frac{\nu}{2}\|\nabla\mathbf{u}_m\|^2+\frac{\kappa}{2}\|\nabla\Theta_m\|^2$. Thus, on account of the properties of g_m, we have that $\mathcal{Q}:=C(1+\|g_m\|^4_{1/2,\partial\Omega})$ and $\mathcal{R}=C(\|g_m\|^2_{1/2,\partial\Omega}+\|\partial_t g_m\|^2_{1/2,\partial\Omega})$ are bounded in $L^1(0,T)$ uniformly with respect to m. In addition, we know that $||\varphi_{0,m}||_{L^\infty(\Omega)}\leq 1-\frac{1}{m}$ for every $m\in\mathbb{N}_0$. Thus $\Psi(\varphi_{0,m})\leq K=\max_{s\in[-1,1]}\Psi(s)$, independently on m. Hence we have

$$E_m(0)=\frac{1}{2}\|\mathbf{u}_{0,m}\|^2+\frac{\alpha}{2}\|\nabla\varphi_{0,m}\|^2+\frac{1}{2}\|\Theta_{0,m}\|^2+\int_\Omega\left(\Psi(\varphi_{0,m})+\hat{C}\right)dx\leq C.$$

We also recall that there exists $C>0$ such that (see [24] for a proof),

$$\int_\Omega|F'(\varphi_m)|dx\leq C\left|\int_\Omega F'(\varphi_m)(\varphi_m-\overline{\varphi}_m)dx\right|+C. \tag{4.83}$$

Indeed, C could depend only on $\overline{\varphi}_{0,m}=\overline{\varphi}_m$, but, since we have $|\overline{\varphi}_m|\leq 1-\delta$ independently of m, for $m>\overline{m}$, we can choose C in such a way that it is independent of m. Summing up, the above inequalities and Gronwall's lemma allow us to find the following uniform bounds (see the proof of Theorem 3.1)

$$\|\Theta_m\|_{L^\infty(0,T;H)\cap L^2(0,T;V_\theta)}\leq C,\qquad \|\mathbf{u}_m\|_{L^\infty(0,T;\mathbf{H}_\sigma)\cap L^2(0,T;\mathbf{V}_\sigma)}\leq C \tag{4.84}$$

$$\|\varphi_m\|_{L^\infty(0,T;V)\cap L^4(0,T;V_2)}\leq C,\qquad \|\mu_m\|_{L^2(0,T;V)}\leq C, \tag{4.85}$$

$$\int_0^T\|F'(\varphi_m)\|^2\leq C,\qquad \left\|\left(\frac{d\mathbf{u}_m}{dt},\frac{d\varphi_m}{dt},\frac{d\Theta_m}{dt}\right)\right\|_{L^2(0,T;\mathbf{V}'_\sigma\times V'\times V'_\theta)}\leq C, \tag{4.86}$$

for any given $T>0$. Thanks to a standard compactness argument (see the proof of Theorem 3.1), we find $(\mathbf{u},\varphi,\Theta)$, such that

$$\mathbf{u}\in L^\infty(0,T;\mathbf{H}_\sigma)\cap L^2(0,T;\mathbf{V}_\sigma)\cap H^1(0,T;\mathbf{V}'_\sigma) \tag{4.87}$$

$$\varphi\in L^\infty(0,T;V)\cap L^4(0,T;V_2)\cap H^1(0,T;V') \tag{4.88}$$

$$\theta=\Theta+\theta_g\in L^\infty(0,T;H)\cap L^2(0,T;V)\cap H^1(0,T;V'_\theta+V'), \tag{4.89}$$

which is a suitable limit, up to a subsequence, of the sequence of the strong solutions. The convergences, including the ones of the approximating data, are enough to prove that $(\mathbf{u},\varphi,\Theta)$ is a weak solution to our problem according to Definition 3.1 (see again the proof of Theorem 3.1). We are left to prove the energy identity (3.11). Consider again the lift operator θ_g and test Equations (3.1)-(3.3) with $\mathbf{w}=\mathbf{u}$, $v=\mu$, $\xi=\theta-\theta_g$, respectively. This gives, for almost any $t\in(0,T)$,

$$\frac{d}{dt}\mathcal{E}(\varphi)+\|\nabla\mu\|^2-(\mathbf{u}\cdot\nabla\mu,\varphi)=0,$$

$$\frac{d}{dt}\frac{1}{2}\|\mathbf{u}\|^2+\nu\|\nabla\mathbf{u}\|^2+(\mathbf{u}\cdot\nabla\mu,\varphi)=(\theta,\mathbf{e}_2\cdot\mathbf{u}),$$

$$\frac{d}{dt}\frac{1}{2}\|\theta-\theta_g\|^2+\kappa\|\nabla(\theta-\theta_g)\|^2=-(\partial_t\theta_g,\theta-\theta_g)-\kappa(\nabla\theta_g,\nabla(\theta-\theta_g))-(\mathbf{u}\cdot\nabla\theta_g,\theta-\theta_g).$$

Adding up the above identities we get (3.11). The proof is finished. ☐

REMARK 4.2. In the case $g\equiv 0$, recalling Remark 4.1, it is not difficult to realize that energy estimate (4.71) also holds for weak solutions with a constant independent of κ (it can be deduced also from (3.11)). This is true also for estimates (4.84)-(4.86).

REMARK 4.3. Suppose that $\nu:\mathbb{R}^2\to\mathbb{R}$ and $\kappa:\mathbb{R}\to\mathbb{R}$ are globally Lipschitz functions such that $0<\nu_*\le\nu(z_1,z_2)\le\nu^*$ for every $(z_1,z_2)\in\mathbb{R}^2$ and $0<k_*\le\kappa(z)\le k^*$ for every $z\in\mathbb{R}$ for some positive values ν_*, ν^*, k_* and k^*. Then the existence of a weak solution can be extended to the case where ν depends on φ and θ, while κ depends on θ. However, in this case, we can no longer take advantage of strong solutions but we need to apply directly a Galerkin scheme. This scheme, if suitably adapted, can also yield the existence of a weak solution in dimension three.

Proof. (**Proof of Theorem 3.4.**) Let us set $\mathbf{u}=\mathbf{u}_1-\mathbf{u}_2$, $\varphi=\varphi_1-\varphi_2$, $\theta=\theta_1-\theta_2$ and $\mu=-\alpha\Delta\varphi+\Psi'(\varphi_1)-\Psi'(\varphi_2)$. Then consider again equations (4.72)-(4.74). From now on, $C>0$ stands for a constant, depending on T, which may vary from line to line. We know that, for $i=1,2$,

$$\|\mathbf{u}_i(t)\|\le C,\qquad \|\varphi_i(t)\|_V\le C,\qquad \|\varphi_i(t)\|_{L^\infty(\Omega)}\le 1,\qquad \|\theta_i(t)\|\le C \tag{4.90}$$

for almost any $t\in(0,T)$. Let us rewrite Equation (4.73) as follows

$$<\partial_t\varphi,v>+(\nabla\mu,\nabla v)-(\mathbf{u}_1\varphi,\nabla v)-(\mathbf{u}\varphi_2,\nabla v)=0\qquad\forall v\in V.$$

We have $\overline{\varphi}(t)=\overline{\varphi}(0)=0$ for all $t\in[0,T]$. Take $v=\bar{A}_0^{-1}\varphi$. This gives

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|_*^2+(\mu,\varphi)=\mathcal{I}_1+\mathcal{I}_2, \tag{4.91}$$

where

$$\mathcal{I}_1=(\mathbf{u}_1\varphi,\nabla\bar{A}_0^{-1}\varphi),\qquad \mathcal{I}_2=(\mathbf{u}\varphi_2,\nabla\bar{A}_0^{-1}\varphi).$$

From (2.2) we get, almost everywhere in $\Omega\times(0,T)$,

$$\Psi'(\varphi_1)-\Psi'(\varphi_2)\ge-\tilde{\alpha}(\varphi_1-\varphi_2)=-\tilde{\alpha}\varphi.$$

Hence, we have

$$(\mu,\varphi)=-\alpha(\Delta\varphi,\varphi)+(\Psi'(\varphi_1)-\Psi'(\varphi_2),\varphi)\ge\alpha\|\nabla\varphi\|^2-\tilde{\alpha}\|\varphi\|^2$$

and also

$$\tilde{\alpha}\|\varphi\|^2=\tilde{\alpha}(\nabla\varphi,\nabla\bar{A}_0^{-1}\varphi)\le\frac{\alpha}{2}\|\nabla\varphi\|^2+C\|\varphi\|_*^2,$$

so that

$$(\mu,\varphi)\ge\frac{\alpha}{2}\|\nabla\varphi\|^2-C\|\varphi\|_*^2. \tag{4.92}$$

Then, from (4.91) we infer

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|_*^2+\frac{\alpha}{2}\|\nabla\varphi\|^2\le C\|\varphi\|_*^2+\mathcal{I}_1+\mathcal{I}_2.$$

Using Cauchy-Schwartz's and Young's inequalities, Sobolev embeddings and (4.90), we find

$$\mathcal{I}_1 \le \|\varphi\|_{L^6(\Omega)} \|\mathbf{u}_1\|_{[L^3(\Omega)]^2} \|\varphi\|_* \le \frac{\alpha}{8}\|\nabla\varphi\|^2 + C\|\nabla\mathbf{u}_1\|^2\|\varphi\|_*^2,$$
$$\mathcal{I}_2 \le \|\varphi_2\|_{L^\infty(\Omega)}\|\mathbf{u}\|\ \|\varphi\|_* \le \frac{\nu}{8}\|\mathbf{u}\|^2 + C\|\varphi\|_*^2.$$

Taking now $\mathbf{v} = \mathbf{A}^{-1}\mathbf{u}$ in (4.72), $\mathbf{A}$ being the Stokes operator, we get

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_\flat^2 + \nu\|\mathbf{u}\|^2 = \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5,$$

where

$$\mathcal{I}_3 = -b(\mathbf{u}_1, \mathbf{u}, \mathbf{A}^{-1}\mathbf{u}) - b(\mathbf{u}, \mathbf{u}_2, \mathbf{A}^{-1}\mathbf{u}),$$
$$\mathcal{I}_4 = \alpha(\nabla\varphi_1 \otimes \nabla\varphi, \nabla\mathbf{A}^{-1}\mathbf{u}) + \alpha(\nabla\varphi \otimes \nabla\varphi_2, \nabla\mathbf{A}^{-1}\mathbf{u}),$$
$$\mathcal{I}_5 = (\theta, \mathbf{e}_2 \cdot \mathbf{A}^{-1}\mathbf{u}).$$

Thanks to (4.90) and to standard inequalities and embeddings, we deduce

$$\begin{aligned}\mathcal{I}_3 &\le (\|\mathbf{u}_1\|_{[L^4(\Omega)]^2} + \|\mathbf{u}_2\|_{[L^4(\Omega)]^2})\ \|\mathbf{u}\|\ \|\nabla\mathbf{A}^{-1}\mathbf{u}\|_{[L^4(\Omega)]^4}\\ &\le C\left(\|\mathbf{u}_1\|^{1/2}\|\nabla\mathbf{u}_1\|^{1/2} + \|\mathbf{u}_2\|^{1/2}\|\nabla\mathbf{u}_2\|^{1/2}\right)\ \|\mathbf{u}\|^{\frac{3}{2}}\ \|\nabla\mathbf{A}^{-1}\mathbf{u}\|^{1/2}\\ &\le \frac{\nu}{8}\|\mathbf{u}\|^2 + C\|\mathbf{u}\|_\flat^2\left(\|\nabla\mathbf{u}_1\|^{1/2} + \|\nabla\mathbf{u}_2\|^{1/2}\right)^4 \le \frac{\nu}{8}\|\mathbf{u}\|^2 + C\|\mathbf{u}\|_\flat^2\left(\|\nabla\mathbf{u}_1\|^2 + \|\nabla\mathbf{u}_2\|^2\right).\end{aligned}$$

Arguing similarly, we get

$$\begin{aligned}\mathcal{I}_4 &\le C\left(\|\varphi_1\|_{H^2(\Omega)} + \|\varphi_2\|_{H^2(\Omega)}\right)\ \|\nabla\varphi\|\ \|\nabla\mathbf{A}^{-1}\mathbf{u}\|^{1/2}\|\mathbf{u}\|^{1/2}\\ &\le \frac{\alpha}{4}\|\nabla\varphi\|^2 + \frac{\nu}{8}\|\mathbf{u}\|^2 + C\left(\|\varphi_1\|_{H^2(\Omega)}^4 + \|\varphi_2\|_{H^2(\Omega)}^4\right)\|\mathbf{u}\|_\flat^2.\end{aligned}$$

Moreover, we have

$$\mathcal{I}_5 \le \frac{\kappa}{6}\|\theta\|^2 + C\|\mathbf{u}\|_\flat^2.$$

Consider now Equation (4.74) and take $\xi = A_0^{-1}\theta$. We obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_*^2 + \kappa\|\theta\|^2 = \mathcal{I}_6 + \mathcal{I}_7, \tag{4.93}$$

where

$$\mathcal{I}_6 = (\mathbf{u}_1\theta, \nabla A_0^{-1}\theta), \qquad \mathcal{I}_7 = (\mathbf{u}\theta_2, \nabla A_0^{-1}\theta).$$

Recalling the treatment of $\mathcal{I}_3$, owing to (2.10) and (4.90), we find

$$\mathcal{I}_6 \le C\|\theta\|^{3/2}\ \|\nabla\mathbf{u}_1\|^{1/2}\ \|\nabla A_0^{-1}\theta\|^{1/2} \le \frac{\kappa}{6}\|\theta\|^2 + C\|\theta\|_\sharp^2\ \|\nabla\mathbf{u}_1\|^2.$$

On the other hand, by Ladyzhenskaya and Young's inequalities, (2.10) and (4.90), we have

$$\mathcal{I}_7 = -(\mathbf{u}\cdot\theta_2, \nabla A_0^{-1}\theta) \le \|\mathbf{u}\|\ \|\theta_2\|_{L^4(\Omega)}\ \|\nabla A_0^{-1}\theta\|_{[L^4(\Omega)]^2}$$

$$\begin{aligned}&\leq C\|\mathbf{u}\|\ \|\theta_2\|^{1/2}\|\nabla\theta_2\|^{1/2}\|\nabla A_0^{-1}\theta\|^{1/2}\|A_0^{-1}\theta\|_{H^2(\Omega)}^{1/2}\\&\leq C\|\mathbf{u}\|\|\nabla\theta_2\|^{1/2}\|\theta\|_\sharp^{1/2}\|\theta\|^{1/2}\\&\leq\frac{\nu}{8}\|\mathbf{u}\|^2+\frac{\kappa}{3}\|\theta\|^2+C\|\nabla\theta_2\|^2\|\theta\|_\sharp^2,\end{aligned}\tag{4.94}$$

for a suitable $C>0$ independent of t. Set now

$$\begin{aligned}&\mathcal{H}_2:=\frac{1}{2}\|\mathbf{u}\|_\flat^2+\frac{1}{2}\|\varphi\|_*^2+\frac{1}{2}\|\theta\|_\sharp^2,\\&\mathcal{R}_2:=1+\|\nabla\mathbf{u}_1\|^2+\|\nabla\mathbf{u}_2\|^2+\|\varphi_1\|_{H^2(\Omega)}^4+\|\varphi_2\|_{H^2(\Omega)}^4+\|\nabla\theta_2\|^2.\end{aligned}$$

Collecting and adding together the above estimates we get

$$\frac{d}{dt}\mathcal{H}_2+\frac{\nu}{2}\|\mathbf{u}\|^2+\frac{\kappa}{3}\|\theta\|^2+\frac{\alpha}{8}\|\nabla\varphi\|^2\leq C\mathcal{R}_2\mathcal{H}_2.\tag{4.95}$$

and Gronwall's lemma entails (3.12). This ends the proof. ∎

5. Proofs of Section 3.3

Proof. (**Proof of Theorem 3.5.**) The idea of the proof is to find uniform-in-κ estimates for the weak solution to CHB$_\kappa$, with given initial data and $g\equiv 0$, and then pass to the limit as $\kappa\to 0$. On account of Remark 4.2, we already know that estimates (4.84)-(4.86), and (4.63) are uniform in κ. We are only left to consider a uniform estimate for the time derivative of the temperature. From now on we will denote by $(\mathbf{u}_\kappa,\varphi_\kappa,\theta_\kappa)$ the weak solution to CHB$_\kappa$, for a fixed $0<\kappa\leq 1$. Moreover, $C>0$ will indicate a constant independent of κ which may vary from line to line. First observe that, for almost any $t\in(0,T)$, being $g\equiv 0$, by Remark 3.10, we get

$$\|\theta_\kappa\|\leq\|\theta_0\|,\qquad \|\theta_\kappa\|_{L^\infty(\Omega)}\leq\|\theta_0\|_{L^\infty(\Omega)}.$$

Then consider the weak formulation

$$<\partial_t\theta_\kappa,\xi>+(\mathbf{u}_\kappa\cdot\nabla\theta_\kappa,\xi)+\kappa(\nabla\theta_\kappa,\nabla\xi)=0\qquad\forall\xi\in V_\theta.\tag{5.1}$$

Observe that, for every $\xi\in V_\theta$,

$$|(\mathbf{u}_\kappa\cdot\nabla\theta_\kappa,\xi)|\leq\|\theta_\kappa\|_{L^\infty(\Omega)}\|\mathbf{u}_\kappa\|\ \|\nabla\xi\|\leq\|\theta_0\|_{L^\infty(\Omega)}\|\mathbf{u}_\kappa\|\ \|\nabla\xi\|\leq C\|\nabla\xi\|,$$

since $\mathbf{u}_k$ is uniformly bounded in $L^\infty(0,T;\mathbf{H}_\sigma)$. Thus, we get

$$\int_0^T\|\partial_t\theta_\kappa\|_{V_\theta'}^2\leq 2\kappa^2\int_0^T\|\nabla\theta_\kappa\|^2+2C^2T\leq 2\kappa\int_0^T\|\nabla\theta_\kappa\|^2+2C^2T\leq C(T),$$

where in the last estimate we exploited the bound $\sqrt{\kappa}\|\theta\|_{L^2(0,T;V_\theta)}\leq C$.

Summing up, we have that

- $\mathbf{u}_\kappa$ is uniformly bounded in $L^\infty(0,T;\mathbf{H}_\sigma)\cap L^2(0,T;\mathbf{V}_\sigma)\cap H^1(0,T;\mathbf{V}_\sigma')$,
- φ_κ is uniformly bounded in $L^\infty(0,T;V)\cap L^4(0,T;V_2)\cap H^1(0,T;V')$,
- θ_κ is uniformly bounded in $L^\infty(0,T;H)\cap L^\infty(\Omega\times(0,T))\cap H^1(0,T;V_\theta')$.

Then, by classical compactness arguments, we obtain a candidate weak solution $(\mathbf{u},\varphi,\theta)$ such that

$$\mathbf{u}\in L^\infty(0,T;\mathbf{H}_\sigma)\cap L^2(0,T;\mathbf{V}_\sigma)\cap H^1(0,T;\mathbf{V}_\sigma')$$

$$\varphi \in L^\infty(0,T;V)\cap L^4(0,T;V_2)\cap H^1(0,T;V')$$
$$\theta \in L^\infty(0,T;H)\cap L^\infty(\Omega\times(0,T))\cap H^1(0,T;V_\theta').$$

We just need to show the convergence in the transport-diffusion equation, the other convergences being as above. Let us multiply the equation by $\chi \in C_0^\infty(0,T)$ and integrate in time between 0 and T, after integration by parts in space. We obtain, up to a non-relabeled subsequence,

$$\int_0^T (<\partial_t\theta_\kappa,\xi> + \kappa(\nabla\theta_\kappa,\nabla\xi) - (\mathbf{u}_\kappa\theta_\kappa,\nabla\xi))\chi(t)dt \qquad \forall \xi \in V_\theta.$$

We notice that

$$\left|\int_0^T (\mathbf{u}_\kappa\theta_\kappa,\nabla\xi)\chi(t)dt - \int_0^T (\mathbf{u}\theta,\nabla\xi)\chi(t)dt\right|$$
$$\le \max_{t\in[0,T]}|\chi(t)| \left\{ \|\mathbf{u}_\kappa - \mathbf{u}\|_{L^2(0,T;\mathbf{H}_\sigma)}\|\nabla\xi\| \; \|\theta_\kappa\|_{L^\infty(\Omega\times(0,T))} + \left|\int_0^T\int_\Omega (\mathbf{u}\cdot\nabla\xi)(\theta-\theta_\kappa)dxdt\right|\right\}$$

so, since $\|\theta_\kappa\|_{L^\infty(\Omega\times(0,T))} \le C$, by the strong convergence $\mathbf{u}_\kappa \to \mathbf{u}$ in $L^2(0,T;\mathbf{H}_\sigma)$ (up to a subsequence), the first term in the right-hand side vanishes as $\kappa\to 0$. Regarding the second term, it vanishes by the weak* convergence $\theta_\kappa \overset{*}{\rightharpoonup} \theta$ in $L^\infty(\Omega\times(0,T))$, since $\mathbf{u}\cdot\nabla\xi \in L^1(\Omega\times(0,T))$. Concerning the diffusion term, we have

$$\left|\kappa\int_0^T (\nabla\theta_\kappa,\chi(t)\nabla\xi)dt\right| \le \sqrt{\kappa}\,\left(\sqrt{\kappa}\|\theta_\kappa\|_{L^2(0,T;V_\theta)}\right)\|\chi(t)\nabla\xi\|_{L^2(0,T;H)} \le \sqrt{\kappa}C \underset{\kappa\to 0}{\to} 0.$$

In addition, we have $\partial_t\theta_\kappa \rightharpoonup \partial_t\theta$ in $L^2(0,T;V_\theta')$. Thus we can pass to the limit as $\kappa\to 0$ and by standard density arguments (see, e.g., [9, Lemma V.1.2]) we deduce that

$$\int_0^T \{<\partial_t\theta,w> - (\mathbf{u}\theta,\nabla w)\}\,dt = 0 \qquad \forall w \in L^2(0,T;V_\theta). \tag{5.2}$$

We conclude that $(\mathbf{u},\varphi,\theta)$ is a weak solution to CHB_0 according to Definition 3.3. To get the additional regularity (3.13), we observe that, ν being constant, we can apply [32, Thm.1.1] with, using the same notation as in [32], $g\equiv 0$ and $h = \phi\nabla\mu + \theta\mathbf{e}_2 \in L^2(\Omega\times(0,T))$ (where the term $\nabla(\phi\mu)$ has been added to the pressure).

To show uniqueness, let us consider $(\mathbf{u}_i,\varphi_i,\theta_i)$, $i=1,2$, two weak solutions with the same initial data, and set $\mathbf{u}=\mathbf{u}_1-\mathbf{u}_2$, $\varphi=\varphi_1-\varphi_2$, $\theta=\theta_1-\theta_2$ and $\mu = -\alpha\Delta\varphi + \Psi'(\varphi_1) - \Psi'(\varphi_2)$. Then we have

$$<\partial_t\mathbf{u},\mathbf{w}> + b(\mathbf{u}_1,\mathbf{u},\mathbf{w}) + b(\mathbf{u},\mathbf{u}_2,\mathbf{w}) + \nu(\nabla\mathbf{u},\nabla\mathbf{w})$$
$$= \alpha(\nabla\varphi_1\otimes\nabla\varphi,\nabla\mathbf{w}) + \alpha(\nabla\varphi\otimes\nabla\varphi_2,\nabla\mathbf{w}) + (\theta,\mathbf{e}_2\cdot\mathbf{w}) \qquad \forall \mathbf{w}\in\mathbf{V}_\sigma \tag{5.3}$$
$$<\partial_t\varphi,v> + (\nabla\mu,\nabla v) + (\mathbf{u}_1\cdot\nabla\varphi,v) + (\mathbf{u}\cdot\nabla\varphi_2,v) = 0 \qquad \forall v\in V \tag{5.4}$$
$$<\partial_t\theta,\xi> - (\mathbf{u}_1\cdot\theta,\nabla\xi) - (\mathbf{u}\cdot\theta_2,\nabla\xi) = 0 \qquad \forall \xi\in V_\theta. \tag{5.5}$$

We then take $\mathbf{w} = \mathbf{A}^{-1}\mathbf{u}$, $v = \bar{A}_0^{-1}\varphi$ and $\xi = A_0^{-1}\theta$, respectively. These choices yield

$$\frac{d}{dt}\mathcal{H}_3(t) + \nu\|\mathbf{u}\|^2 + (\mu,\varphi) = \sum_{j=1}^6 \mathcal{I}_j, \tag{5.6}$$

where $\mathcal{H}_3 := \frac{1}{2}\|\mathbf{u}\|_\flat^2 + \frac{1}{2}\|\varphi\|_*^2 + \frac{1}{2}\|\theta\|_\sharp^2$ and

$$\begin{aligned}
&\mathcal{I}_1 = (\mathbf{u}_1\varphi, \nabla \bar{A}_0^{-1}\varphi), \qquad \mathcal{I}_2 = (\mathbf{u}\varphi_2, \nabla \bar{A}_0^{-1}\varphi),\\
&\mathcal{I}_3 = -b(\mathbf{u}_1, \mathbf{u}, \mathbf{A}^{-1}\mathbf{u}) - b(\mathbf{u}, \mathbf{u}_2, \mathbf{A}^{-1}\mathbf{u}),\\
&\mathcal{I}_4 = \alpha(\nabla\varphi_1 \otimes \nabla\varphi, \nabla\mathbf{A}^{-1}\mathbf{u}) + \alpha(\nabla\varphi\otimes\nabla\varphi_2, \nabla\mathbf{A}^{-1}\mathbf{u}),\\
&\mathcal{I}_5 = (\theta, \mathbf{e}_2\cdot\mathbf{A}^{-1}\mathbf{u}), \qquad \mathcal{I}_6 = (\mathbf{u}\theta_2, \nabla A_0^{-1}\theta), \qquad \mathcal{I}_7 = (\mathbf{u}_1\theta, \nabla A_0^{-1}\theta).
\end{aligned}$$

From now on, $C>0$ is a generic constant depending at most on the data, on the structural parameters, on Ω and on T. C may vary from line to line. Due to the regularity of a weak solution, we have, for almost any $t\in(0,T)$, $i=1,2$,

$$\|\mathbf{u}_i(t)\| \le C, \quad \|\varphi_i(t)\|_V \le C, \quad \|\varphi_i(t)\|_{L^\infty(\Omega)} \le 1, \quad \|\theta_i(t)\|_{L^\infty(\Omega)} \le C. \tag{5.7}$$

The terms $\mathcal{I}_1$-$\mathcal{I}_4$ can be estimated as the ones in the proof of Theorem 3.4 with the corresponding numbering, thanks to (5.7). Thus we obtain

$$\begin{aligned}
&\mathcal{I}_1 \le \frac{\alpha}{8}\|\nabla\varphi\|^2 + C\|\nabla\mathbf{u}_1\|^2\|\varphi\|_*^2,\\
&\mathcal{I}_2 \le \frac{\nu}{8}\|\mathbf{u}\|^2 + C\|\varphi\|_*^2,\\
&\mathcal{I}_3 \le \frac{\nu}{8}\|\mathbf{u}\|^2 + C\|\mathbf{u}\|_\flat^2\left(\|\nabla\mathbf{u}_1\|^2 + \|\nabla\mathbf{u}_2\|^2\right),\\
&\mathcal{I}_4 \le \frac{\alpha}{4}\|\nabla\varphi\|^2 + \frac{\nu}{8}\|\mathbf{u}\|^2 + C\left(\|\varphi_1\|_{H^2(\Omega)}^4 + \|\varphi_2\|_{H^2(\Omega)}^4\right)\|\mathbf{u}\|_\flat^2.
\end{aligned}$$

Then, recalling the equivalence of the norms $\|\cdot\|_\sharp$ and $\|\cdot\|_{V_\theta'}$ and using (5.7), we find

$$\begin{aligned}
&\mathcal{I}_5 = <\theta, \mathbf{e}_2\cdot\mathbf{A}^{-1}\mathbf{u}> \le \|\theta\|_{V_\theta'}\|\nabla\mathbf{A}^{-1}\mathbf{u}\| \le C\|\theta\|_\sharp^2 + \|\mathbf{u}\|_\flat^2,\\
&\mathcal{I}_6 \le \frac{\nu}{8}\|\mathbf{u}\|^2 + C\|\theta_2\|_{L^\infty(\Omega)}^2\|\theta\|_\sharp^2 \le \frac{\nu}{8}\|\mathbf{u}\|^2 + C\|\theta\|_\sharp^2.
\end{aligned}$$

Consider now $\mathcal{I}_7$. To treat this term we will follow closely [32, Sec.5]. Recalling the two-dimensional interpolation inequality, for $r\ge 2$,

$$\|f\|_{L^r(\Omega)} \le \sqrt{r}\|f\|^{2/r}\|f\|_1^{1-2/r},$$

an integration by parts together with (2.10) gives, for any $p\in[2,\infty)$,

$$\begin{aligned}
\mathcal{I}_7 &= -(\mathbf{u}_1\Delta A_0^{-1}\theta, \nabla A_0^{-1}\theta) = (\nabla A_0^{-1}\theta, \nabla\mathbf{u}_1\cdot\nabla A_0^{-1}\theta) \le \|\nabla\mathbf{u}_1\|_{[L^p(\Omega)]^2}\||\nabla A_0^{-1}\theta|^2\|_{L^{\frac{p}{p-1}}(\Omega)}\\
&\le \|\nabla\mathbf{u}_1\|_{[L^p(\Omega)]^2}\|\nabla A_0^{-1}\theta\|^2_{\left[L^{\frac{2p}{p-1}}(\Omega)\right]^2} \le \frac{2p}{p-1}\|\nabla\mathbf{u}_1\|_{[L^p(\Omega)]^2}\|\nabla A_0^{-1}\theta\|^{2-\frac{2}{p}}\|\nabla A_0^{-1}\theta\|_1^{\frac{2}{p}}\\
&\le \frac{2Cp}{p-1}\|\nabla\mathbf{u}_1\|_{[L^p(\Omega)]^2}\|\nabla A_0^{-1}\theta\|^{2-\frac{2}{p}}\|\theta\|^{\frac{2}{p}} \le C\|\nabla\mathbf{u}_1\|_{[L^p(\Omega)]^2}\|\theta\|_\sharp^{2-\frac{2}{p}},
\end{aligned}$$

where we exploited $-\Delta A_0^{-1}\theta = \theta$ almost everywhere in $\Omega\times(0,T)$, the fact that $\frac{2p}{p-1}\le 4$. Note that C does not depend on p and (5.7). Therefore, setting

$$\mathcal{R}_3 = 1 + \|\nabla\mathbf{u}_1\|^2 + \|\nabla\mathbf{u}_2\|^2 + \|\varphi_1\|_{H^2(\Omega)}^4 + \|\varphi_2\|_{H^2(\Omega)}^4,$$

we get from (5.6) (see also (4.92))

$$\frac{d}{dt}\mathcal{H}_3 + \frac{\nu}{2}\|\mathbf{u}\|^2 + \frac{\alpha}{8}\|\nabla\varphi\|^2 \le C\mathcal{R}_3\mathcal{H}_3 + Cp\frac{\|\nabla\mathbf{u}_1\|_{[L^p(\Omega)]^2}}{p}\mathcal{H}_3^{1-\frac{1}{p}}. \tag{5.8}$$

Applying now the well-known Yudovich's argument (see [32] and [50]), we can easily conclude the proof of uniqueness. First, integrate (5.8) in time over $(0,s)$ to get, for any $0\le s\le t\le T$, recalling that $\mathcal{H}_3(0)=0$,

$$\mathcal{H}_3(s)\le C\int_0^t \mathcal{R}_3(\tau)\mathcal{H}_3(\tau)d\tau + Cp\int_0^t \frac{\|\nabla \mathbf{u}_1(\tau)\|_{[L^p(\Omega)]^2}}{p}\mathcal{H}_3{}^{1-\frac{1}{p}}(\tau)d\tau,$$

i.e.,

$$\sup_{s\in[0,t]}\mathcal{H}_3(s)\le C\int_0^t \mathcal{R}_3(\tau)\mathcal{H}_3(\tau)d\tau + Cp\int_0^t \frac{\|\nabla \mathbf{u}_1(\tau)\|_{[L^p(\Omega)]^2}}{p}\mathcal{H}_3{}^{1-\frac{1}{p}}(\tau)d\tau. \tag{5.9}$$

Then let $0<t_\star\le T$ sufficiently small such that

$$C\int_0^{t_\star}\mathcal{R}_3(\tau)d\tau\le\frac{1}{2},$$

which is possible since $\mathcal{R}_3\in L^1(0,T)$, and set

$$\widetilde{\mathcal{H}}_3(t)=\sup_{s\in[0,t]}\mathcal{H}_3(s).$$

We infer from (5.9), since $\mathcal{H}_3(\tau)\le\widetilde{\mathcal{H}}_3(\tau)$ for any $\tau\ge 0$,

$$\widetilde{\mathcal{H}}_3(t)\le Cp\int_0^t\left(1+\frac{\|\nabla \mathbf{u}_1(\tau)\|_{[L^p(\Omega)]^2}}{p}\right)\widetilde{\mathcal{H}}_3{}^{1-\frac{1}{p}}(\tau)d\tau, \tag{5.10}$$

for any $t\in[0,t_\star]$. Setting $\Xi=p\int_0^t\left(1+\frac{\|\nabla \mathbf{u}_1(\tau)\|_{[L^p(\Omega)]^2}}{p}\right)\widetilde{\mathcal{H}}_3{}^{1-\frac{1}{p}}(\tau)d\tau$ and $\Xi_\varepsilon=\Xi+\varepsilon$, for $\varepsilon>0$, we get

$$\frac{d}{dt}\Xi_\varepsilon^{\frac{1}{p}}=\frac{1}{p}\Xi_\varepsilon^{\frac{1}{p}-1}\frac{d}{dt}\Xi_\varepsilon\le C\left(1+\frac{\|\nabla \mathbf{u}_1\|_{[L^p(\Omega)]^2}}{p}\right),$$

which means, integrating in time over $(0,t)$, $t\le t_\star$, and recalling that $\Xi_\varepsilon(0)=\varepsilon$,

$$\Xi_\varepsilon(t)\le\left(\varepsilon^{\frac{1}{p}}+C\int_0^t\left(1+\frac{\|\nabla \mathbf{u}_1(\tau)\|_{[L^p(\Omega)]^2}}{p}\right)d\tau\right)^p,$$

This implies, letting $\varepsilon\to 0$,

$$\Xi(t)\le\left(C\int_0^t\left(1+\frac{\|\nabla \mathbf{u}_1(\tau)\|_{[L^p(\Omega)]^2}}{p}\right)d\tau\right)^p\le\left(C\int_0^t\left(1+\sup_{p\ge 2}\frac{\|\nabla \mathbf{u}_1(\tau)\|_{[L^p(\Omega)]^2}}{p}\right)d\tau\right)^p,$$

ending up, recalling (5.10), with

$$\widetilde{\mathcal{H}}_3(t)\le C\left(C\int_0^t\left(1+\sup_{p\ge 2}\frac{\|\nabla \mathbf{u}_1(\tau)\|_{[L^p(\Omega)]^2}}{p}\right)d\tau\right)^p.$$

Therefore, if we adjust $t_\star$ so that also

$$C\int_0^{t_\star}\left(1+\sup_{p\ge 2}\frac{\|\nabla \mathbf{u}_1(\tau)\|_{[L^p(\Omega)]^2}}{p}\right)d\tau\le\frac{1}{2},$$

which is possible due to (3.13), we immediately infer, letting $p\to\infty$, $\mathcal{H}_3(t)=0$ for every $t\le t_\star$. We then extend this result to any $t\in[0,T]$ by means of a standard connectivity argument and by the continuity in time of the solution, so that $(\mathbf{u},\varphi,\theta)=(0,0,0)$ for any $t\in[0,T]$, implying uniqueness for any $T>0$ and thus concluding the proof. □

Proof. (**Proof of Theorem 3.6.**) Within this proof, $C>0$ is a constant independent of κ and t, which may vary from line to line. We consider the (unique) strong solutions $(\mathbf{u}_\kappa,\varphi_\kappa,\theta_\kappa)$, with the same initial data of the assumptions and with $g\equiv\theta_{0|\partial\Omega}\in H^{3/2}(\partial\Omega)$. Note that in this way assumption (I_4) is trivially satisfied and also the compatibility condition (I_5), i.e., $\theta_\kappa(0)_{|\partial\Omega}=g(0)$, is fulfilled. Therefore, from the end of the proof of Theorem 3.1, estimates (4.23)-(4.27), (4.31), (4.33), (4.50), (4.51) and (4.63) hold with C independent of κ. Moreover, since $\theta_0\in H^2(\Omega)\hookrightarrow L^\infty(\Omega)$, estimate (3.8) holds with C independent of κ as well. Our aim is now to show some other uniform-in-κ estimates to be able to pass to the limit as $\kappa\to 0$ in a suitable sense (see [14]). We observe that θ_κ solves

$$\begin{aligned}
&\partial_t\theta_\kappa+\mathbf{u}_\kappa\cdot\nabla\theta_\kappa-\kappa\Delta\theta_\kappa=0 \qquad \text{a.e. in } \Omega\times(0,T)\\
&\theta_\kappa=g \qquad \text{on } \partial\Omega\times(0,T)\\
&\theta_\kappa(0)=\theta_0 \qquad \text{in } \Omega.
\end{aligned}\tag{5.11}$$

Let us proceed formally. The argument can be justified by using a semi-Galerkin scheme (i.e. keeping $\mathbf{u}_\kappa$ given). We resort again to the lift operator (see Section 2.4), write $\theta_\kappa=\Theta_\kappa+\theta_g$ and note that $\Delta\theta_g=0$ in $\Omega\times(0,T)$ and, since g is independent of time, $\partial_t\theta_g=0$. We apply Δ to the first equation, multiply it by $\Delta\Theta_\kappa$ and integrate it over Ω, recalling that $\Delta\Theta_\kappa=0$ (due to the fact that g is independent of time) and $\mathbf{u}_\kappa=0$ on $\partial\Omega\times(0,T)$. We find

$$\frac{d}{dt}\|\Delta\Theta_\kappa\|^2+(\Delta(\mathbf{u}_\kappa\cdot\nabla\Theta_\kappa),\Delta\Theta_\kappa)+(\Delta(\mathbf{u}_\kappa\cdot\nabla\theta_g),\Delta\Theta_\kappa)+\kappa\|\nabla(\Delta\Theta_\kappa)\|^2=0,$$

but

$$\Delta(\mathbf{u}_\kappa\cdot\nabla\Theta_\kappa)=\Delta\mathbf{u}_\kappa\cdot\nabla\Theta_\kappa+2\nabla\mathbf{u}_\kappa:D^2\Theta_\kappa+\mathbf{u}_\kappa\cdot\nabla(\Delta\Theta_\kappa)\tag{5.12}$$

and note that $(\mathbf{u}_\kappa\cdot\nabla(\Delta\Theta_\kappa),\Delta\Theta_\kappa)=0$, whereas, similarly,

$$\Delta(\mathbf{u}_\kappa\cdot\nabla\theta_g)=\Delta\mathbf{u}_\kappa\cdot\nabla\theta_g+2\nabla\mathbf{u}_\kappa:D^2\theta_g+\mathbf{u}_\kappa\cdot\nabla(\Delta\theta_g)=\Delta\mathbf{u}_\kappa\cdot\nabla\theta_g+2\nabla\mathbf{u}_\kappa:D^2\theta_g.$$

Thus we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\Delta\Theta_\kappa\|^2+\kappa\|\nabla(\Delta\Theta_\kappa)\|^2\le\left(\|\Delta\mathbf{u}_\kappa\|_{L^4(\Omega)}\|\nabla\Theta_\kappa\|_{L^4(\Omega)}+2\|\nabla\mathbf{u}_\kappa\|_{L^\infty(\Omega)}\|D^2\Theta_\kappa\|\right)\|\Delta\Theta_\kappa\|\\
&\quad+\left(\|\Delta\mathbf{u}_\kappa\|_{L^4(\Omega)}\|\nabla\theta_g\|_{L^4(\Omega)}+2\|\nabla\mathbf{u}_\kappa\|_{L^\infty(\Omega)}\|D^2\theta_g\|\right)\|\Delta\Theta_\kappa\|\\
&\le C(1+\|\Delta\mathbf{u}_\kappa\|_{L^4(\Omega)}+\|\nabla\mathbf{u}_\kappa\|_{L^\infty(\Omega)})\|\Delta\Theta_\kappa\|^2+C(\|\Delta\mathbf{u}_\kappa\|^2_{L^4(\Omega)}+\|\nabla\mathbf{u}_\kappa\|^2_{L^\infty(\Omega)})\|\theta_g\|^2_{H^2(\Omega)}\\
&\le C(1+\|\Delta\mathbf{u}_\kappa\|_{L^4(\Omega)}+\|\nabla\mathbf{u}_\kappa\|_{L^\infty(\Omega)})\|\Delta\Theta_\kappa\|^2\\
&\quad+C(\|\Delta\mathbf{u}_\kappa\|^2_{L^4(\Omega)}+\|\nabla\mathbf{u}_\kappa\|^2_{L^\infty(\Omega)})\|g\|^2_{H^{3/2}(\partial\Omega)},
\end{aligned}\tag{5.13}$$

where in the last inequality we have used the embedding $H^2(\Omega)\hookrightarrow W^{1,4}(\Omega)$. Recalling that $\|\Delta\mathbf{u}_\kappa\|_{L^4(\Omega)}\le C\|\mathbf{u}_\kappa\|_{W^{2,4}(\Omega)}$ and $[W^{2,4}(\Omega)]^2\hookrightarrow[W^{1,\infty}(\Omega)]^2$, from (5.13), by (3.8), we deduce

$$\frac{d}{dt}\|\Delta\Theta_\kappa\|^2+\kappa\|\nabla(\Delta\Theta_\kappa)\|^2\le C\left(1+\|\mathbf{u}_\kappa\|_{W^{2,4}(\Omega)}\right)\|\Delta\Theta_\kappa\|^2+C\|\mathbf{u}_\kappa\|^2_{W^{2,4}(\Omega)}\|g\|^2_{H^{3/2}(\partial\Omega)}.$$

Thus, since $\|\mathbf{u}_\kappa\|_{L^2(0,T;W^{2,4}(\Omega)} \le C$ and g independent of time, Gronwall's lemma entails

$$\|\Theta_\kappa\|_{L^\infty(0,T;V_\theta^2)} \le C,$$

therefore, recalling that $\theta_g \in L^\infty(0,T;H^2(\Omega))$, we have

$$\|\theta_\kappa\|_{L^\infty(0,T;H^2(\Omega))} \le C. \tag{5.14}$$

By comparison in the equation for θ_κ, we infer

$$\|\partial_t\theta_\kappa\| \le \kappa\|\Delta\theta_\kappa\| + \|\nabla\theta_\kappa\|_{L^4(\Omega)}\|\mathbf{u}_\kappa\|_{L^4(\Omega)} \le C,$$

where we have exploited (5.14), $\|\mathbf{u}_\kappa\|_{L^\infty(0,T;\mathbf{V}_\sigma)} \le C$ and standard Sobolev embeddings. Therefore we obtain $\|\partial_t\theta_\kappa\|_{L^\infty(0,T;H)} \le C$. Summing up, for the original problem, the following uniform-in-κ estimates hold:

- $\mathbf{u}_\kappa$ is uniformly bounded in $L^\infty(0,T;\mathbf{V}_\sigma) \cap L^2(0,T;\mathbf{W}_\sigma) \cap L^2(0,T;[W^{2,p}(\Omega)]^2) \cap H^1(0,T;[L^p(\Omega)]^2)$, for any $p \in [2,\infty)$,
- φ_κ is uniformly bounded in $L^\infty(0,T;V) \cap L^4(0,T;V_2) \cap H^1(0,T;V)$,
- μ_κ is uniformly bounded in $L^\infty(0,T;V)$,
- θ_κ is uniformly bounded in $L^\infty(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;H^2(\Omega)) \cap W^{1,\infty}(0,T;H)$.

By a classical compactness argument, we obtain a candidate strong solution to CHB_0 as $\kappa \to 0$, say $(\mathbf{u},\varphi,\theta)$, with the following regularity

$$\begin{aligned}
&\mathbf{u} \in L^\infty(0,T;\mathbf{V}_\sigma) \cap L^2(0,T;\mathbf{W}_\sigma) \cap L^2(0,T;[W^{2,p}(\Omega)]^2) \cap H^1(0,T;[L^p(\Omega)]^2),\ p \in [2,\infty)\\
&\varphi \in L^\infty(0,T;V) \cap L^4(0,T;V_2) \cap H^1(0,T;V)\\
&\mu \in L^\infty(0,T;V)\\
&\theta \in L^\infty(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;H^2(\Omega)) \cap W^{1,\infty}(0,T;H).
\end{aligned}$$

Using a standard argument we can pass to the limit as $\kappa \to 0$, obtaining that $(\mathbf{u},\ \varphi,\ \theta)$ is a solution to CHB_0 that satisfies the system almost everywhere in $\Omega \times (0,T)$. Thanks to the same arguments used in the proof of Theorem 3.1, we can prove the additional regularity properties for φ and $\mathbf{u}$ stated in Theorem 3.6. The proof is finished. ∎

Proof. (**Proof of Theorem 3.7.**) In order to prove Theorem 3.7, let us consider the same setting as the one adopted in the proof of uniqueness of weak solutions (Theorem 3.5). We start from (5.6). All the estimates are indeed the same: the only difference is in the term $\mathcal{I}_7$. In particular, an integration by parts gives

$$\mathcal{I}_7 = -(\mathbf{u}_1\Delta A_0^{-1}\theta, \nabla A_0^{-1}\theta) = (\nabla A_0^{-1}\theta, \nabla\mathbf{u}_1 \cdot \nabla A_0^{-1}\theta) \le \|\nabla\mathbf{u}_1\|_{L^\infty(\Omega)}\|\theta\|_\sharp^2,$$

where we exploited $-\Delta A_0^{-1}\theta = \theta$ almost everywhere in $\Omega \times (0,T)$. Therefore, we get from (5.6) (see also (4.92))

$$\frac{d}{dt}\mathcal{H}_3 + \frac{\nu}{2}\|\mathbf{u}\|^2 + \frac{\alpha}{8}\|\nabla\varphi\|^2 \le C\mathcal{R}_4\mathcal{H}_3,$$

where $\mathcal{R}_4 = \mathcal{R}_3 + \|\nabla\mathbf{u}_1\|_{L^\infty(\Omega)}$. Observe that $\mathcal{R}_4 \in L^1(0,T)$, since $\mathbf{u}_1 \in L^2(0,T;[W^{2,4}(\Omega)]^2) \hookrightarrow L^1(0,T;[W^{1,\infty}(\Omega)]^2)$, $\mathbf{u}_1$ being a strong solution. Hence estimate (3.14) follows through Gronwall's lemma. This concludes the proof. ∎

Proof. (**Proof of Theorem 3.8.**) First of all, on account of the assumptions on the initial data and on g, due to Theorems 3.1, 3.4, 3.6 and 3.7, there exist unique strong solutions to CHB_κ, for each $\kappa>0$ as well as a unique strong solution to CHB_0. Thus, given $\kappa>0$, we consider the strong solution $(\mathbf{u}_\kappa,\varphi_\kappa,\theta_\kappa)$ to CHB_κ and the strong solution $(\mathbf{u},\varphi,\theta)$ to CHB_0 corresponding to the same initial data. Note that, for any $\kappa>0$, the boundary datum g satisfies assumptions $(I)_4$ and $(I)_5$. Setting $\mathbf{U}=\mathbf{u}-\mathbf{u}_\kappa$, $\Phi=\varphi-\varphi_\kappa$, $\Theta=\theta-\theta_\kappa$, $\mu=-\alpha\Delta\Phi+\Psi'(\varphi)-\Psi'(\varphi_\kappa)$ and following [12], we have that $(\mathbf{U},\Phi,\Theta)$ solves the following identities

$$
\begin{aligned}
&<\partial_t\mathbf{U},\mathbf{w}>+b(\mathbf{u},\mathbf{U},\mathbf{w})+b(\mathbf{U},\mathbf{u}_\kappa,\mathbf{w})+\nu(\nabla\mathbf{U},\nabla\mathbf{w})\\
&=\alpha(\nabla\varphi\otimes\nabla\Phi,\nabla\mathbf{w})+\alpha(\nabla\Phi\otimes\nabla\varphi_\kappa,\nabla\mathbf{w})+(\Theta,\mathbf{e}_2\cdot\mathbf{w})\qquad\forall\mathbf{w}\in\mathbf{V}_\sigma \qquad (5.15)\\
&<\partial_t\Phi,v>+(\nabla\mu,\nabla v)+(\mathbf{u}\cdot\nabla\Phi,v)+(\mathbf{U}\cdot\nabla\varphi_\kappa,v)=0\qquad\forall v\in V \qquad (5.16)\\
&<\partial_t\Theta,\xi>+\kappa(\nabla\Theta,\nabla\xi)-\kappa(\nabla\theta,\nabla\xi)+\kappa\int_{\partial\Omega}(\nabla\theta_\kappa\cdot\mathbf{n})\xi dS \qquad (5.17)\\
&\quad-(\mathbf{u}\Theta,\nabla\xi)-(\mathbf{U}\theta_\kappa,\nabla\xi)=0\qquad\forall\xi\in V.
\end{aligned}
$$

We test the three equations with Φ, $\mathbf{U}$ and Θ, respectively. This yields

$$
\frac{d}{dt}\mathcal{H}_4+\nu\|\nabla\mathbf{U}\|^2+\kappa\|\nabla\Theta\|^2+(\nabla\mu,\nabla\Phi)=\sum_{j=1}^{6}\mathcal{I}_j,
$$

where $\mathcal{H}_4=\frac12\|\mathbf{U}\|^2+\frac12\|\Phi\|^2+\frac12\|\Theta\|^2$ and

$$
\begin{aligned}
&\mathcal{I}_1=-b(\mathbf{U},\mathbf{u}_\kappa,\mathbf{U}), &&\mathcal{I}_2=\alpha(\nabla\varphi\otimes\nabla\Phi,\nabla\mathbf{U})+\alpha(\nabla\Phi\otimes\nabla\varphi_\kappa,\nabla\mathbf{U}),\\
&\mathcal{I}_3=(\mathbf{u}\Phi,\nabla\Phi)+(\mathbf{U}\varphi_\kappa,\nabla\Phi), &&\mathcal{I}_4=(\Theta,\mathbf{e}_2\cdot\mathbf{U}),\\
&\mathcal{I}_5=-(\mathbf{U}\cdot\nabla\theta_\kappa,\Theta), &&\mathcal{I}_6=\kappa(\nabla\theta,\nabla\Theta)-\kappa\int_{\partial\Omega}(\nabla\theta_\kappa\cdot\mathbf{n})\Theta dS.
\end{aligned}
$$

From now on, $C>0$ is a constant independent of κ that may vary from line to line. On account of the existence and uniqueness (in particular from the proof of Theorem 3.6), we know that our solutions satisfy the following bounds

$$
\begin{aligned}
&\|\mathbf{u}\|_{L^\infty(0,T;\mathbf{V}_\sigma)}+\|\mathbf{u}\|_{L^\infty(0,T;[L^3(\Omega)]^2)}+\|\varphi\|_{L^\infty(0,T;V)}\\
&\quad+\|\varphi\|_{L^\infty(0,T;W^{2,3}(\Omega))}+\|\Psi''(\varphi)\|_{L^\infty(0,T;L^3(\Omega))}\le C, \qquad (5.18)\\
&\|\mathbf{u}_\kappa\|_{L^\infty(0,T;\mathbf{V}_\sigma)}+\|\mathbf{u}_\kappa\|_{L^\infty(0,T;[L^3(\Omega)]^2)}+\|\varphi_\kappa\|_{L^\infty(0,T;V)}+\|\varphi_\kappa\|_{L^\infty(0,T;W^{2,3}(\Omega))}\\
&\quad+\|\Psi''(\varphi_\kappa)\|_{L^\infty(0,T;L^3(\Omega))}+\|\theta_\kappa\|_{L^\infty(\Omega\times(0,T))}+\|\theta_\kappa\|_{L^\infty(0,T;H^2(\Omega))}\le C. \qquad (5.19)
\end{aligned}
$$

Following step by step the proof of Theorem 3.2, we can get

$$
\begin{aligned}
\mathcal{I}_1&\le\frac{\nu}{4}\|\nabla\mathbf{U}\|^2+C\|\mathbf{U}\|^2\|\nabla\mathbf{u}_\kappa\|^2_{[L^3(\Omega)]^4},\\
\mathcal{I}_2&\le\frac{\nu}{4}\|\nabla\mathbf{U}\|^2+\frac{\alpha}{8}\|\Delta\Phi\|^2+C\|\Phi\|^2,\\
\mathcal{I}_3&\le\frac{\alpha}{8}\|\Delta\Phi\|^2+C(\|\Phi\|^2+\|\varphi_\kappa\|^2_\infty\|\mathbf{U}\|^2).
\end{aligned}
$$

Moreover, using standard arguments (see also (5.19)), we obtain

$$
\begin{aligned}
&\mathcal{I}_4\le\|\Theta\|\ \|\mathbf{U}\|\le\frac12\|\Theta\|^2+\frac12\|\mathbf{U}\|^2,\\
&\mathcal{I}_5\le\|\mathbf{U}\|_{[L^4(\Omega)]^2}\|\nabla\theta_\kappa\|_{[L^4(\Omega)]^2}\|\Theta\|\le\frac{\nu}{4}\|\nabla\mathbf{U}\|^2+C\|\theta_\kappa\|^2_{H^2(\Omega)}\|\Theta\|^2\le\frac{\nu}{4}\|\nabla\mathbf{U}\|^2+C\|\Theta\|^2.
\end{aligned}
$$

In conclusion, by the properties of the trace operator, together with the embedding $V \hookrightarrow H^{1/2}(\Omega)$,

$$\begin{aligned}\mathcal{I}_6 &\leq \kappa\left(\|\nabla\theta\|\|\nabla\Theta\| + \|\nabla\theta_\kappa\|_{[L^2(\partial\Omega)]^2}\|\Theta\|_{L^2(\partial\Omega)}\right)\\ &\leq \kappa\left(\|\nabla\theta\|\|\nabla\Theta\| + C\|\theta_\kappa\|_{H^{3/2}(\Omega)}(\|\Theta\| + \|\nabla\Theta\|)\right)\\ &\leq \frac{\kappa}{2}\|\nabla\theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2 + C(\kappa+\kappa^2)\|\theta_\kappa\|^2_{H^{3/2}(\Omega)} + C\|\Theta\|^2.\end{aligned}$$

Adding together the above estimates and recalling (4.92), we find

$$\frac{d}{dt}\mathcal{H}_4 + \frac{\nu}{4}\|\nabla\mathbf{U}\|^2 + \frac{\kappa}{2}\|\Theta\|^2 + \frac{\alpha}{4}\|\Delta\Phi\|^2 \leq C\mathcal{R}_5\mathcal{H}_4 + \frac{\kappa}{2}\|\nabla\theta\|^2 + C(\kappa+\kappa^2)\|\theta_\kappa\|^2_{H^{3/2}(\Omega)},$$

where $\mathcal{R}_5 := 1 + \|\nabla\mathbf{u}_\kappa\|^2_{[L^3(\Omega)]^4} + \|\varphi_\kappa\|^2_\infty \in L^1(0,T)$. Hence Gronwall's lemma gives (recall that $\mathcal{H}_4(0) = 0$)

$$\mathcal{H}_4 \leq \frac{\kappa}{2}\int_0^T e^C\|\nabla\theta\|^2 ds + C(\kappa+\kappa^2)\int_0^T e^C\|\theta_\kappa\|^2_{H^{3/2}(\Omega)} ds.$$

Since we have $\|\theta\|_{L^2(0,T;V_\theta)} \leq C$ and by (5.19), recalling the embedding $H^2(\Omega) \hookrightarrow H^{3/2}(\Omega)$, we infer that (3.15) holds and the proof is finished. □

6. Proofs of Section 3.4

Proof. (**Proof of Theorem 3.9.**) The proof follows closely the corresponding one in [28]. Let $(\mathbf{u}, \varphi, \theta)$ be the global weak solution with initial condition $(\mathbf{u}_0, \varphi_0, \theta_0)$ given by Theorem 3.3. Due to the regularity properties of weak solution we have that for any $\tau > 0$ there exists $\tau_0 \in (0,\tau)$ such that $(\mathbf{u}(\tau_0), \varphi(\tau_0), \theta(\tau_0))$ satisfies assumptions of Theorem 3.1. Moreover, recalling (4.71) and Remark 4.2, we have

$$E(\tau_0) \leq E(0) + \frac{C_0^2}{2\nu\beta_0}\|\theta_0\|^2(1 - e^{-\beta_0\tau_0}) \leq R + k_0 \leq R_1, \qquad \overline{\varphi}(\tau_0) = m.$$

Thus we have a global strong solution on the time interval $[\tau_0, +\infty)$, which coincides with the weak solution due to Theorem 3.4, corresponding to the initial datum $(\mathbf{u}(\tau_0), \varphi(\tau_0), \theta(\tau_0))$. From (4.68) we have

$$\|\theta\|^2 \leq \|\theta(\tau_0)\|^2 e^{-\beta_0(t-\tau_0)} \qquad \forall t \geq \tau_0. \tag{6.1}$$

In the following the positive constants denoted by c_i, $i \in \mathbb{N}$, depend on R and possibly τ, but are independent of t and the specific initial data. Due to Gronwall's lemma applied to the energy estimate (4.70) together with (6.1), we have that, for every $t \geq \tau_0$,

$$\begin{aligned}E(\mathbf{u}(t), \varphi(t), \theta(t)) &\leq E(\mathbf{u}(\tau_0), \varphi(\tau_0), \theta(\tau_0))\\ &\quad + \frac{C_0^2}{2\nu\beta_0}\|\theta(\tau_0)\|^2(1 - e^{-\beta_0(t-\tau_0)}) \leq c_0.\end{aligned} \tag{6.2}$$

We then have, by the same estimate (6.1) applied to (4.70),

$$\frac{d}{dt}E(t) + \|\nabla\mu\|^2 + \frac{\nu}{2}\|\nabla\mathbf{u}\|^2 + \kappa\|\nabla\theta\|^2 \leq \frac{C_0^2}{2\nu}\|\theta(\tau_0)\|^2 e^{-\beta_0(t-\tau_0)}, \tag{6.3}$$

for every $t \geq \tau_0$. Integrating (6.3) on $(t, t+1)$ we deduce:

$$E(\mathbf{u}(t+1), \varphi(t+1), \theta(t+1))$$

$$+\frac{\nu}{2}\int_t^{t+1}\|\nabla\mathbf{u}(s)\|^2ds+\kappa\int_t^{t+1}\|\nabla\theta(s)\|^2ds+\int_t^{t+1}\|\nabla\mu(s)\|^2ds \tag{6.4}$$

$$\leq E(\mathbf{u}(t),\varphi(t),\theta(t))+\frac{C_0^2e^{\tau\beta_0}}{2\nu\beta_0}\|\theta(\tau_0)\|^2(1-e^{-\beta_0}),\qquad \forall t\geq\tau_0, \tag{6.5}$$

since $e^{-\beta_0 t}\leq 1$ and $e^{\tau_0\beta_0}\leq e^{\tau\beta_0}$. Thus, thanks to (6.2), since $|\Psi(s)|\geq -C$ for some $C>0$ independent of t and every $s\in[-1,1]$, we get

$$\frac{\nu}{2}\int_t^{t+1}\|\nabla\mathbf{u}(s)\|^2ds+\kappa\int_t^{t+1}\|\nabla\theta(s)\|^2ds+\int_t^{t+1}\|\nabla\mu(s)\|^2ds\leq c_1\qquad\forall t\geq\tau_0.$$

In conclusion, we write, for every $t\geq\tau_0$,

$$E(\mathbf{u}(t),\varphi(t),\theta(t))+\int_t^{t+1}\left(\frac{\nu}{2}\|\nabla\mathbf{u}(s)\|^2+\kappa\|\nabla\theta(s)\|^2+\|\nabla\mu(s)\|^2\right)ds\leq c_2. \tag{6.6}$$

The following global-in-time estimates are formal, but they can be made rigorous by repeating verbatim the proof of Theorem 3.1 in the Galerkin setting (note that the energy identity also holds in the Galerkin scheme). On account of (4.47), here we can find

$$\frac{d}{dt}\Lambda+\frac{\nu}{4}\|\mathbf{A}\mathbf{u}\|^2+\frac{\overline{\omega}}{2}\|\partial_t\mathbf{u}\|^2+\frac{\alpha}{4}\|\nabla\partial_t\varphi\|^2\leq c_3(1+\Lambda^2), \tag{6.7}$$

where $\Lambda:=(\mu,\mathbf{u}\cdot\nabla\varphi)+\frac{1}{2}\|\nabla\mu\|^2+\frac{1+\overline{\omega}\nu}{2}\|\nabla\mathbf{u}\|^2$. Recalling (6.6), we have

$$\int_t^{t+1}\Lambda(s)ds\leq c_4,\qquad\forall t\geq\tau_0. \tag{6.8}$$

Hence, observing that $\Lambda\geq\frac{1}{4}\left(\|\nabla\mathbf{u}\|^2+\|\nabla\mu\|^2\right)-C$, for some $C>0$ independent of t, we can apply the uniform Gronwall lemma (see [48, Ch.3, Lemma 1.1]), to (6.8) to obtain $\Lambda(t)\leq c_5$ for every $t\geq\tau$. This entails

$$\|\mathbf{u}\|_{L^\infty(\tau,\infty;\mathbf{V}_\sigma)}+\|\mu\|_{L^\infty(\tau,\infty;V)}\leq c_6. \tag{6.9}$$

Integrating (6.7) in time, on $(t,t+1)$, we deduce

$$\|\mathbf{u}\|_{L^2(t,t+1;\mathbf{W}_\sigma)}+\|\partial_t\mathbf{u}\|_{L^2(t,t+1;\mathbf{H}_\sigma)}+\|\partial_t\varphi\|_{L^2(t,t+1;V)}\leq c_7\qquad\forall t\geq\tau.$$

Arguing formally, we also obtain (see (4.60))

$$\left(\frac{1}{2}+\frac{\kappa}{2}\right)\frac{d}{dt}\|\nabla\theta\|^2+\frac{1}{2}\|\partial_t\theta\|^2+\frac{\kappa}{4}\|\Delta\theta\|^2\leq C\|\nabla\theta\|^2\qquad\forall t\geq\tau_0, \tag{6.10}$$

with C depending on c_i, $i=0,\dots,7$, but not on t. Therefore, due to (6.6) we infer, by uniform Gronwall's lemma, $\|\theta\|_{L^\infty(\tau,\infty;V_\theta)}\leq c_8$, for every $t\geq\tau$. Then, integrating (6.10) in time on $(t,t+1)$, we get

$$\|\theta\|_{L^2(t,t+1;V_\theta^2)}+\|\partial_t\theta\|_{L^2(t,t+1;H)}\leq c_9\qquad\forall t\geq\tau.$$

Taking the time regularity of the strong solutions into account, (3.16) and (3.17) follow. Then (3.18) is deduced from (6.9) and [28, Thm.A.2]. Therefore, we know that

$\varphi \in L^\infty(\tau,\infty;W^{2,p}(\Omega))$ and $F'(\varphi)\in L^\infty(\tau,\infty;L^p(\Omega))$ for any $p\in[2,\infty)$. Then, on account of (2.8), we deduce $F''(\varphi)\in L^\infty(\tau,\infty;L^p(\Omega))$, for $p\in(2,\infty)$ (see [27, Lemma 5.1]). Thus, by the chain rule, $F'(\varphi)\in L^\infty(\tau,\infty;W^{1,p}(\Omega))$ for any $p\in(2,\infty)$. Fix now $p>2$. Thanks to the embedding $W^{1,p}(\Omega)\hookrightarrow L^\infty(\Omega)$, there exists $C>0$ depending on the norms of the initial data, the parameters of the problem, the domain and F, such that $\|F'(\varphi)\|_{L^\infty(\Omega\times(\tau,\infty))}\le c_{10}$. This entails the existence of $\delta>0$ such that

$$\sup_{t\geq\tau}\|\varphi(t)\|_{C(\overline{\Omega})}\le 1-\delta, \tag{6.11}$$

ending the proof. ∎

Proof. (**Proof of Theorem 3.10.**) By replacing τ with $\frac{\tau}{2}$ in Theorem 3.1, we can assume that the solution $(\mathbf{u},\ \varphi,\ \theta)$ satisfies the uniform estimates of Theorem 3.9 on the interval $[\frac{\tau}{2},\infty)$. We need some other higher-order a priori estimates on the solution (see [27] for an analogous proof). In the sequel, c_i, $i\in\mathbb{N}$, stands for a positive constant depending on R,m,τ, but not on the specific initial data. Given $h>0$, repeating verbatim the proof of Theorem 3.2, in which the difference $(\mathbf{u}_1-\mathbf{u}_2,\varphi_1-\varphi_2,\theta_1-\theta_2)$ is replaced by the difference quotients $(\partial_t^h\mathbf{u},\partial_t^h\varphi,\partial_t^h\theta)$, we deduce the following, for $t\ge\frac{\tau}{2}$

$$\frac{d}{dt}\mathcal{H}_2+\frac{\nu}{4}\|\nabla\partial_t^h\mathbf{u}\|^2+\kappa\|\nabla\partial_t^h\theta\|^2+\frac{\alpha}{4}\|\Delta\partial_t^h\varphi\|^2\le C\mathcal{R}\mathcal{H}_2, \tag{6.12}$$

where

$$\mathcal{H}_2:=\frac{1}{2}\|\partial_t^h\mathbf{u}\|^2+\frac{1}{2}\|\partial_t^h\varphi\|^2+\frac{1}{2}\|\partial_t^h\theta\|^2,\quad \mathcal{R}:=1+\|\nabla\mathbf{u}\|^2_{[L^3(\Omega)]^4}+\|\varphi\|^2_\infty+\|\theta\|^2_{V_\theta^2}.$$

Here $C>0$ does not depend on h, but depends on M_1 and M_3 (see Theorem 3.9). By Sobolev embeddings, we obtain (see also (3.17))

$$\int_t^{t+1}(\mathcal{H}_2(s)+\mathcal{R}(s))\,ds\le c_0\qquad\forall t\ge\frac{\tau}{2},$$

where c_0 depends on M_2 but not on h. We can thus apply the uniform Gronwall's lemma to (6.12), with $r=\frac{\tau}{2}$. This gives

$$\|\partial_t^h\mathbf{u}\|+\|\partial_t^h\varphi\|+\|\partial_t^h\theta\|\le c_1\qquad\forall t\ge\tau,$$

so that

$$\|\partial_t^h\mathbf{u}\|_{L^2(t,t+1;\mathbf{V}_\sigma)}+\|\partial_t^h\varphi\|^2_{L^2(t,t+1;H^2(\Omega))}+\|\partial_t^h\theta\|_{L^2(t,t+1;V_\theta)}\le c_2\qquad\forall t\ge\tau.$$

A passage to the limit as $h\to 0$ entails (3.20) and (3.21).

We now prove the separation property. First of all, we recall that $\|\mathbf{u}\cdot\nabla\varphi\|_{L^\infty(\tau,\infty,H)}\le c_3$ owing to (3.16) and (3.18). Then, due to (3.20) we deduce $\partial_t\varphi+\mathbf{u}\cdot\nabla\varphi\in L^\infty(\tau,\infty;H)$; thus the regularity theory for the Neumann problem gives $\|\mu\|_{L^\infty(\tau,\infty,H^2(\Omega))}\le c_4$ so that $\|\mu\|_{L^\infty(\Omega\times(\tau,\infty))}\le c_5$. From (3.18), (3.19) and $F'\in C^3([-1+\delta,1-\delta])$ we deduce that $\|F'(\varphi)\|_{L^\infty(\tau,\infty,V_2)}\le c_6$. Hence, well-known regularity results imply $\|\varphi\|_{L^\infty(\tau,\infty,H^4(\Omega))}\le c_7$. Concerning $\mathbf{u}$, setting $\mathbf{f}:=\mu\nabla\varphi-\partial_t\mathbf{u}-(\mathbf{u}\cdot\nabla)\mathbf{u}+\theta\mathbf{e}_2$, and arguing as in [28, Thm.4.4], we find $\|\mathbf{u}\|_{L^\infty(\tau,\infty,[W^{2,p}(\Omega)]^2)}\le c_8=c_8(p)$, since $\mathbf{f}\in L^\infty(\tau,\infty,[L^p(\Omega)]^2)$, with $p\in(1,2)$. Then, using $[W^{2,\frac{4}{3}}(\Omega)]^2\hookrightarrow[W^{1,4}(\Omega)]^2$ we can get $\mathbf{f}\in L^\infty(\tau,\infty,[L^2(\Omega)]^2)$ and recover $\|\mathbf{u}\|_{L^\infty(\tau,\infty;\mathbf{W}_\sigma)}\le c_9$. Therefore (3.22) holds and the proof is concluded. ∎

Proof. (**Proof of Theorem 3.11.**) Let $T>0$ be given and let $(\mathbf{u},\ \varphi,\ \theta)$ be a strong solution to CHB_0 on $[0,T]$ according to Definition 3.4. We can exploit the regularity stated in Definition 3.4, though, in this case, δ depends on T. We know that $\varphi\in L^\infty(0,T;W^{2,p}(\Omega))$ and $F'(\varphi)\in L^\infty(0,T;L^p(\Omega))$ for any $p\in[2,\infty)$. Similarly as before, we deduce $F''(\varphi)\in L^\infty(0,T;L^p(\Omega))$, for $p\in(2,\infty)$. Thus the chain rule gives $F'(\varphi)\in L^\infty(0,T;W^{1,p}(\Omega))$ for any $p\in(2,\infty)$. For $p>2$, due to the embedding $W^{1,p}(\Omega)\hookrightarrow L^\infty(\Omega)$, there exists $C>0$ depending on the norms of the initial data, the parameters of the problem, the domain, F and T, such that $\|F'(\varphi)\|_{L^\infty(\Omega\times(0,T))}\le C$, which implies, together with Remarks 3.9 and 3.11, the existence of $\delta=\delta(T)>0$ such that (3.23) holds. The proof is finished. ∎

Acknowledgments. The authors are grateful to the anonymous referees who significantly contributed to improve some results of this paper. They also wish to thank Andrea Giorgini and Hao Wu for their observations, which contributed to the improvement of this work. The first author is a member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Appendix. Approximating the NSCH system (1.1). Here we show how to formally deduce system (1.3) from NSCH system (1.1) introduced in [40]. We use a perturbation argument ([26]): Let $(\rho^*,\mathbf{u}^*,\varphi^*,p^*)=(c_1,0,c_2,p^*)$, $c_1\neq 0$, $c_2\in(-1,1)$, be a stationary solution to (1.1). We now write the system for the perturbation $(\rho+\rho^*,\ \mathbf{u},\ \varphi,\ p+p^*)$. From $(1.1)_3$ we obtain

$$\partial_t(\rho+\rho^*)+\mathbf{u}\cdot\nabla(\rho+\rho^*)=-(\rho+\rho^*)\mathrm{div}\ \mathbf{u},$$

so that

$$\partial_t\rho+\mathbf{u}\cdot\nabla\rho=-\rho^*\mathrm{div}\ \mathbf{u}-\rho\mathrm{div}\ \mathbf{u}.$$

This equation holds for any $\rho^*=c\in\mathbb{R}^+$. Therefore we obtain

$$\begin{cases}\partial_t\rho+\mathbf{u}\cdot\nabla\rho=0\\ \mathrm{div}\ \mathbf{u}=0.\end{cases}\tag{A.1}$$

From $(1.1)_1$ we deduce

$$\begin{aligned}&(\rho+\rho^*)\partial_t\mathbf{u}+(\rho+\rho^*)(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla(p+p^*)-\mathrm{div}(\nu(\varphi)D\mathbf{u})-\nabla(\mathrm{div}\ \mathbf{u})\\ &=-\alpha\ \mathrm{div}((\rho+\rho^*)\nabla\varphi\otimes\nabla\varphi)+(\rho+\rho^*)\mathbf{g}.\end{aligned}$$

We recall that, $(\rho^*,\mathbf{u}^*,\varphi^*,p^*)$ being a stationary solution to (1.1), the hydrostatic balance $\nabla p^*=\rho^*\mathbf{g}$ holds. Thus we get, remembering (A.1) and dividing by ρ^*,

$$\begin{aligned}&\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\frac{\rho}{\rho^*}\partial_t\mathbf{u}+\frac{\rho}{\rho^*}(\mathbf{u}\cdot\nabla)\mathbf{u}+\frac{1}{\rho^*}\nabla p-\mathrm{div}\left(\frac{\nu(\varphi)}{\rho^*}D\mathbf{u}\right)\\ &=-\alpha\ \mathrm{div}(\nabla\varphi\otimes\nabla\varphi)-\alpha\ \mathrm{div}\left(\frac{\rho}{\rho^*}\nabla\varphi\otimes\nabla\varphi\right)+\frac{\rho}{\rho^*}\mathbf{g}.\end{aligned}$$

Since ρ^* is arbitrary, we can take it arbitrarily large, such that $\frac{\rho}{\rho^*}\approx 0$, and we can neglect all the terms with this coefficient in front, except the gravitational one, because it is linear and by means of an energy budget argument. Therefore, we find

$$\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\frac{1}{\rho^*}\nabla p-\mathrm{div}\left(\frac{\nu(\varphi)}{\rho^*}D\mathbf{u}\right)=-\alpha\ \mathrm{div}(\nabla\varphi\otimes\nabla\varphi)+\frac{\rho}{\rho^*}\mathbf{g}.\tag{A.2}$$

In conclusion, dividing by ρ^*, we infer from $(1.1)_2$ the following

$$\left(1+\frac{\rho}{\rho^*}\right)\partial_t\varphi+\left(1+\frac{\rho}{\rho^*}\right)\mathbf{u}\cdot\nabla\varphi=\Delta\left(-\frac{\alpha}{\rho^*}\frac{1}{1+\frac{\rho}{\rho^*}}\operatorname{div}\left((1+\frac{\rho}{\rho^*})\nabla\varphi\right)+\frac{1}{\rho^*}\Psi'(\varphi)\right).$$

By using again $\frac{\rho}{\rho^*}\approx 0$, we deduce

$$\partial_t\varphi+\mathbf{u}\cdot\nabla\varphi=\Delta\left(-\frac{\alpha}{\rho^*}\Delta\varphi+\frac{1}{\rho^*}\Psi'(\varphi)\right). \tag{A.3}$$

Putting together equations (A.1), (A.2) and (A.3), we are then led to formulate (1.3), which can thus be interpreted as an incompressible approximation of (1.1).

REFERENCES

[1] H. Abels, *On a diffuse interface model for two-phase flows of viscous, incompressible fluids with matched densities*, Arch. Ration. Mech. Anal., 194:463–506, 2009. 1, 3.2, 3.13

[2] H. Abels, *Strong well-posedness of a diffuse interface model for a viscous, quasi-incompressible two-phase flow*, SIAM J. Math. Anal., 44:316–340, 2012. 1

[3] H. Abels and E. Feireisl, *On a diffuse interface model for a two-phase flow of compressible viscous fluids*, Indiana Univ. Math. J., 57:659–698, 2008. 1

[4] D.M. Anderson, G.B. McFadden, and A.A. Wheeler, *Diffuse-interface methods in fluid mechanics*, Annu. Rev. Fluid Mech., 30:139–165, 1998. 1

[5] A. Berti, V. Berti, and D. Grandi, *Well-posedness of an isothermal diffusive model for binary mixtures of incompressible fluids*, Nonlinearity, 24:3143–3164, 2011. 1

[6] A. Biswas, C. Foias, and A. Larios, *On the attractor for the semi-dissipative Boussinesq equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 34:381–405, 2017. 1

[7] J.L. Boldrini, and S.A. Lorca, *The initial value problem for a generalized Boussinesq model*, Nonlinear Anal., 36:457–480, 1999. 1, 2.4

[8] F. Boyer, *Mathematical study of multi-phase flow under shear through order parameter formulation*, Asymptot. Anal., 20:175–212, 1999. 1

[9] F. Boyer and P. Fabrie, *Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models*, Springer-Verlag, New York, 2013. 2.3, 3.3, 3.7, 3.11, 4, 5

[10] J.R. Cannon and E. DiBenedetto, *The initial value problem for the Boussinesq equations with data in L^p*, in R. Rautmann (eds.), Approximation Methods for Navier-Stokes problems, Lect. Notes in Math., Springer-Verlag, Berlin, 771:129–144, 1980. 1

[11] D.H. Chae and O. Imanuvilov, *Generic solvability of the axisymmetric 3D Euler equations and the 2D Boussinesq equations*, J. Differ. Equ., 156:1–17, 1999. 1

[12] D.H. Chae, *Global regularity for the 2D Boussinesq equations with partial viscosity terms*, Adv. Math., 203:497–513, 2006. 1, 1, 5

[13] L. Cherfils, E. Feireisl, M. Michlek, A. Miranville, M. Petcu, and D. Prak, *The compressible Navier-Stokes-Cahn-Hilliard equations with dynamic boundary conditions*, Math. Model. Meth. Appl. Sci., 29:2557–2584, 2019. 1

[14] J. Fan and Z. Jiang, *Vanishing heat conductivity limit for the 2D Cahn-Hilliard-Boussinesq system*, Bound. Value Probl., 2011:54, 2011. 1, 5

[15] J. Fan and Y. Zhou, *Blow-up criteria of smooth solutions for the Cahn-Hilliard-Boussinesq system with zero viscosity in a bounded domain*, Abstr. Appl. Anal., 2012:13, 2012. 1

[16] J. Fan and Y. Zhou, *The vanishing viscosity limit for a 2D Cahn-Hilliard-Navier-Stokes system with a slip boundary condition*, Nonlinear Anal. Real World Appl., 14:1130–1134, 2013. 1

[17] H.O. Fattorini, *Infinite Dimensional Optimization and Control Theory*, Cambridge Univ. Press, Cambridge, 1999. 2.3

[18] P.C. Fife, *Models for phase separation and their mathematics*, Electron. J. Differ. Equ., 48:1–26, 2000. 1

[19] S. Frigeri and M. Grasselli, *Global and trajectory attractors for a nonlocal Cahn-Hilliard-Navier-Stokes system*, J. Dyn. Differ. Equ., 24:827–856, 2012. 2.1, 5

[20] S. Frigeri and M. Grasselli, *Nonlocal Cahn-Hilliard-Navier-Stokes systems with singular potential*, Dyn. Partial Differ. Equ., 4:273–304, 2012. 4, 4

[21] C.G. Gal and M. Grasselli, *Trajectory and global attractors for binary mixtures fluid flows in 3D*, Chin. Ann. Math. Ser. B, 31:655–678, 2010. 1

[22] C.G. Gal and M. Grasselli, *Asymptotic behavior of a Cahn-Hilliard-Navier-Stokes system in 2D*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27:401–436, 2010. 1

[23] C.G. Gal, M. Grasselli, and A. Miranville, *Cahn-Hilliard-Navier-Stokes systems with moving contact lines*, Calc. Var. Partial Differ. Equ., 55:1–47, 2016. 1

[24] C.G. Gal, A. Giorgini, and M. Grasselli, *The nonlocal Cahn-Hilliard equation with singular potential: well-posedness, regularity and strict separation property*, J. Differ. Equ., 263:5253–5297, 2017. 5, 4, 4

[25] M. Geissert, M. Hess, M. Hieber, C. Schwarz, and K. Stavrakidis, *Maximal L^p-L^q-estimates for the Stokes equation: a short proof of Solonnikov Theorem*, J. Math. Fluid Mech., 12:47–60, 2010. 4

[26] A. Giorgini, private communication, 2019. 1, 6

[27] A. Giorgini, M. Grasselli, and A. Miranville, *The Cahn-Hilliard-Oono equation with singular potential*, Math. Model. Meth. Appl. Sci., 27:2485–2510, 2017. 3.9, 4, 6, 6

[28] A. Giorgini, A. Miranville, and R. Temam, *Uniqueness and regularity for the Navier-Stokes-Cahn-Hilliard system*, SIAM J. Math. Anal., 51:2535–2574, 2019. 1, 1, 2.1, 2.3, 2.3, 3.4, 3.9, 4, 4, 4, 4, 4, 4, 4, 6, 6, 6

[29] A. Giorgini and R. Temam, *Weak and strong solutions to the nonhomogeneous incompressible Navier-Stokes-Cahn-Hilliard system*, J. Math. Pures Appl., 144:194–249, 2020. 1

[30] M.E. Gurtin, D. Polignone, and J. Viñals, *Two-phase binary fluids and immiscible fluids described by an order parameter*, Math. Model. Meth. Appl. Sci., 6:815–831, 1996. 1

[31] B.I. Halperin and P.C. Hohenberg, *Theory of dynamic critical phenomena*, Rev. Mod. Phys., 49:435–479, 1977. 1

[32] L. He, *Smoothing estimates of 2D incompressible Navier-Stokes equations in bounded domains with applications*, J. Funct. Anal., 262:3430–3464, 2012. 5, 5, 5

[33] T. Hmidi and S. Keraani, *On the global well-posedness of the two-dimensional Boussinesq system with a zero diffusivity*, Adv. Differ. Equ., 12:461–480, 2007. 1

[34] W. Hua, I. Kukavica, and M. Ziane, *On the regularity for the Boussinesq equations in a bounded domain*, J. Math. Phys., 54:081507, 2013. 1

[35] O.A. Ladyženskaja, V.A. Solonnikov, and N.N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monogr., Amer. Math. Soc., Providence, RI, 23, 1968. 3.10

[36] M.J. Lai, R.H. Pan, and K. Zhao, *Initial boundary value problem for 2D viscous Boussinesq equations*, Arch. Ration. Mech. Anal., 199:739–760, 2011. 1

[37] A. Larios, E. Lunasin, and E.S. Titi, *Global well-posedness for the 2D Boussinesq system with anisotropic viscosity and without heat diffusion*, J. Differ. Equ., 255:2636–2654, 2013. 1

[38] Z. Li and D. Zhou, *Global well-posedness for the 2D Boussinesq equations with zero viscosity*, J. Math. Anal. Appl., 447:1072–1079, 2017. 1

[39] J.-L. Lions and E. Magenes, *Nonhomogeneous Boundary Value Problems and Applications*, Vols. I-III, Springer-Verlag, Berlin, 1972. 2.4, 2

[40] J. Lowengrub and L. Truskinovsky, *Quasi-incompressible Cahn-Hilliard fluids and topological transitions*, Proc. R. Soc. Lond. A., 454:2617–2654, 1998. 1, 6

[41] C. Ma, W. Gu, and J. Sun, *Global well-posedness for the 2D Cahn-Hilliard-Boussinesq and a related system on bounded domains*, Bound. Value Probl., 2017:119, 2017. 1

[42] A. Miranville, *The Cahn-Hilliard Equation: Recent Advances and Applications*, CBMS-NSF Regional Conf. Ser. in Appl. Math., SIAM, Philadelphia, PA., 2019. 1

[43] T. Nagai, T. Senba, and K. Yoshida, *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis*, Funkcial. Ekvac., 40:411–433, 1997. 2.3

[44] H. Sohr, *The Navier-Stokes Equations. An Elementary Functional Analytic Approach*, Birkhäuser Basel, 2001. 4

[45] G. Stampacchia, *Équations Elliptiques du Second Ordre à Coefficients Discontinus*, Presses de l'Université de Montréal, 1966. 4

[46] V.N. Starovoitov, *The dynamics of a two-component fluid in the presence of capillary forces*, Math. Notes, 62:244–254, 1997. 1

[47] R. Temam, *Navier-Stokes Equations. Theory and Numerical Analysis*, North-Holland, Amsterdam, 1984. 3.3, 3.5, 4, 4

[48] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New York, 1997. 2.3, 6

[49] H. Wu, private communication, 2020. 3.9

[50] V. Yudovich, *Non-stationary flows of an ideal incompressible fluid*, Ž Vyčisl. Mat i Mat. Fiz., 3:1032–1066, 1963. 5

[51] K. Zhao, *2D inviscid heat conductive Boussinesq equations on a bounded domain*, Michigan Math. J., 59:329–352, 2010. 1

[52] K. Zhao, *Global regularity for a coupled Cahn-Hilliard-Boussinesq system on bounded domains*,

Quart. Appl. Math. 69:331–356, 2011. 1, 4

[53] K. Zhao, *Large time behavior of a Cahn-Hilliard-Boussinesq system on a bounded domain*, Electron. J. Differ. Equ., 46:1–21, 2011. 1, 4

[54] D. Zhou, *Global regularity of the two-dimensional Boussinesq equations without diffusivity in bounded domains*, Nonlinear Anal. Real World Appl., 43:144–154, 2018. 1, 4