COMMUN. MATH. SCI.
Vol. 20, No. 4, pp. 1081–1102

A FINITE ELEMENT METHOD FOR DIRICHLET BOUNDARY CONTROL OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract. This paper introduces a new variational formulation for Dirichlet boundary control problem of elliptic partial differential equations, based on an observation that the state and adjoint state are related through the control on the boundary of the domain, and that such a relation may be imposed in the variational formulation of the adjoint state. Well-posedness (unique solvability and stability) of the new variational problem is established in the $H^1(\Omega)\times H_0^1(\Omega)$ spaces for the respective state and adjoint state. A finite element method based on this formulation is analyzed. It is shown that the conforming $k-$th order finite element approximations to the state and the adjoint state, in the respective L^2 and H^1 norms, converge at the rate of order $k-1/2$ on quasi-uniform meshes. Numerical examples are presented to validate the theory.

Keywords. Dirichlet boundary control problem; new variational formulation; finite element method; *a priori* error estimates.

AMS subject classifications. 65K10; 65N30; 65N21; 49M25; 49K20.

1. Introduction

Let $\Omega\subset\mathbb{R}^d, d\geq 2$, be a bounded polygonal or polyhedral domain with Lipshitz boundary $\Gamma=\partial\Omega$. Consider the following Dirichlet boundary control problem of elliptic partial differential equations (PDEs):

$$\min\ J(u),\quad J(u)=\frac{1}{2}||y-y_d||^2_{L^2(\Omega)}+\frac{\gamma}{2}||u||^2_{L^2(\Gamma)}, \tag{1.1}$$

where the regularization parameter $\gamma>0$ and y is the solution of the Poisson equation with nonhomogeneous Dirichlet boundary conditions:

$$-\triangle y=f\quad \text{in } \Omega, \tag{1.2}$$

$$y=u\quad \text{on } \Gamma. \tag{1.3}$$

After the pioneering works of Falk [19] and Geveci [21], there were some efforts on the error estimates for finite element approximation to control problems governed by PDEs. Arada et al. in [4, 10] derived error estimates for the control in the L^∞ and L^2 norms for semilinear elliptic control problem. The articles [20, 25] studied the error estimates of finite element approximation for some important flow control problems. Casas [10] carried out the study of the Neumann boundary control problem; Wang, Yang, and Xie [38] developed a Nitsche-extended finite element method for control problems of elliptic interface equations. However, these works are mainly contributions to the distributed control.

*Received: February 08, 2021; Accepted (in revised form): October 29, 2021. Communicated by Shi Jin.
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This work is supported in part by the Natural Science Foundation of Chongqing under Grant cstc2018jcyjAX490, the Education Science Foundation of Chongqing under Grant KJZD-K201900701, the Group Building Scientific Innovation Project for universities in Chongqing (CXQT21021), and the Joint Training Base Construction Projection for Graduate Students in Chongqing (JDLHPYJD202101 6).
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It is well known that the Dirichlet boundary control plays an important role in many applications such as flow control problems and has been a hot topic for decades. However, the Dirichlet boundary control problems are extremely difficult to solve from both the theoretical and the numerical points of view, because the Dirichlet boundary data does not directly enter a standard variational setting for the PDEs. On the one hand, the traditional finite element method (see, e.g., [3, 11, 15, 33, 37]) deals with the state variable (y) using its weak formulation, e.g., allowing for solution $y \in L^2(\Omega)$; on the other hand, the attempt of the first order optimality condition involves the normal derivative of the adjoint state (z) on the boundary of the domain. Therefore, it is crucial to obtain this normal derivative numerically by an additional equation. But in doing so the problem becomes complicated in both theoretical analysis and numerical practice. Note that the regularity of the solution and error estimates for finite element approximates have been studied in [2, 3, 32].

To avoid the difficulty described above, there are two ways to deal with the control variable. One is to replace the L^2 norm in the cost functional with the $H^{1/2}$ norm, thus, *a priori* estimate of the numerical error of the control is obtained by using piecewise linear elements (see [34]); and the other is to approximate the nonhomogeneous Dirichlet boundary condition with a Robin boundary condition or weak boundary penalization. These techniques were further developed later in [13, 23, 24]. However, the former approach changed the problem and the latter had to deal with the penalization which is computationally expensive.

Recently, Gong and Yan considered the mixed finite element method in [22], where the optimal control and the adjoint state were involved in a variational form in a natural sense, which makes its theoretical analysis straightforward, but the corresponding fluxes of the two states (y and z) are required to be introduced. Apel et al. [3] considered a standard finite element method on a special class of meshes and achieved a superlinear convergence rate for the control. Very recently, Hu et al. [27] considered a hybridizable discontinuous Galerkin method and obtained optimal *a priori* error estimates for the control.

Based on the fact that the state and adjoint state are related through the control on the boundary of the domain, and that such a relation may be imposed in the variational formulation of the adjoint state, i.e., one can eliminate the boundary control by the control law (the control is the normal derivative of the adjoint state on the boundary (up to a factor)), a new variational formulation for Dirichlet boundary control problem of elliptic PDEs is introduced. This idea is different from that in the literature e.g., [3,11,13,15,23,24,33,34,37], where both the original equation and an extra equation were taken into account in variational formulations. Its well-posedness (unique solvability and stability) is established in the $H^1(\Omega) \times H_0^1(\Omega)$ spaces for the respective state and adjoint state. A finite element method based on this formulation is analyzed. It is shown that the conforming $k-$th order finite element approximations to the state and the adjoint state, in the respective L^2 and H^1 norms converge at the rate of order $k-1/2$ on a quasi-uniform mesh. Note that the order of convergence is optimal for the control, and that the new variational setting cannot be incorporated into the framework of mixed problems, since LBB condition is not satisfied.

In fact, the new discrete variational problem can also be derived by eliminating the discrete normal derivative in terms of the discrete control rule (analogous to the continuous control rule) given by Casas and Raymond in [11], where the concept of the discrete normal derivative has been introduced and the discrete first order optimality conditions have been achieved. This procedure indeed follows the route of "discretize-then-optimize-then-eliminate", it is different from the route of "optimize-then-eliminate-then-discretize" adopted by us in this paper. We shall remark that eliminating the control variable does not only make the system simple, but also brings a convenience to define the residual functional with the help of the new discrete variational formulation when the residual-type *a posteriori* estimators are considered (see Remark 4.1).

This paper is organized as follows. In Section 2, we introduce a new variational setting based on an observation. Section 3 is devoted to the unique solvability and stability of the variational problem. In Section 4, we introduce finite element approximation to the variational setting and prove a preliminary result, which will be used in the *a priori* error estimation in Section 5. In Section 6, we analyze the stability of the discrete control in $L^2(\Gamma)$ norm and $H^{1/2}(\Gamma)$ norm in the sense that the restriction of the discrete state on the boundary is considered as an approximation of the control. Finally, numerical tests are provided in Section 7 to support our theory.

2. A variational formulation

For any bounded open subset ω of Ω with Lipschitz boundary ι, let $L^2(\iota)$ and $H^m(\omega)$ be the standard Lebesgue and Sobolev spaces equipped with standard norms $\|\cdot\|_\iota = \|\cdot\|_{L^2(\iota)}$ and $\|\cdot\|_{m,\omega} = \|\cdot\|_{H^m(\omega)}$, $m\in\mathbb{N}$. Note that $H^0(\omega)=L^2(\omega)$. Denote by $|\cdot|_{m,\omega}$ the semi-norm in $H^m(\omega)$. Similarly, denote by $(\cdot,\cdot)_\iota$ and $(\cdot,\cdot)_\omega$ the L^2 inner products on ι and ω, respectively. We shall omit the symbol Ω in the notations above if $\omega=\Omega$.

It is well known that the Dirichlet boundary control problem in (1.1)-(1.3) is equivalent to the optimality system (the first order optimality conditions):

$$-\triangle y = f \quad \text{in} \quad \Omega, \tag{2.1}$$

$$y = u \quad \text{on} \quad \Gamma, \tag{2.2}$$

$$-\triangle z = y - y_d \quad \text{in} \quad \Omega, \tag{2.3}$$

$$z = 0 \quad \text{on} \quad \Gamma, \tag{2.4}$$

$$u = \frac{1}{\gamma}\frac{\partial z}{\partial \mathbf{n}} \quad \text{on} \quad \Gamma, \tag{2.5}$$

where, y_d and f are sufficiently smooth prescribed functions, $\gamma>0$ is a regularization parameter, $\mathbf{n}$ is the outer normal unit vector. Note that these equations must be understood in a weak sense. We refer to the references [11,27,33].

To see the idea of variational setting, we consider the following several cases under an assumption that the domain and known data are respectively satisfied with these cases:

Case one: the control $u\in L^2(\Gamma)$, so y belongs to $H^{1/2}(\Omega)$. Owing to the control law, (2.5) yields $\partial z/\partial\mathbf{n}|_\Gamma \in L^2(\Gamma)$, which needs the adjoint state $z\in H^{3/2}(\Omega)$.

Case two: $y\in H^1(\Omega), y|_\Gamma = u\in H^{1/2}(\Gamma)$, the Equation (2.5) means $\partial z/\partial\mathbf{n}|_\Gamma \in H^{1/2}(\Gamma)$, which demands $z\in H^2(\Omega)$.

Case three: $y\in L^2(\Omega), y|_\Gamma = u\in H^{-1/2}(\Gamma)$ (the dual space of $H^{1/2}(\Gamma)$), the Equation (2.5) indicates $\partial z/\partial\mathbf{n}|_\Gamma \in H^{-1/2}(\Gamma)$, which requires $z\in H^1(\Omega)$.

Case four: $y\in H^{3/2}(\Omega), y|_\Gamma = u\in H^1(\Gamma)$, the Equation (2.5) shows $\partial z/\partial\mathbf{n}|_\Gamma \in H^1(\Gamma)$, which expects $z\in H^{5/2}(\Omega)$.

Since natural functional analytical setting of this problem uses $L^2(\Gamma)$ as a "control space", Case one is an ideal choice for the control u, state y, and adjoint state z.

However, it is difficult to bring these characteristics of y and z into their respective variational formulations if (2.2) and (2.5) are regarded as two independent equations. For Cases two and three, it is convenient to incorporate the spaces of y and z into their respective variational formulation, but doing so expands or narrows down the space of the control u, and can not provide the variational formulation of u if (1.3) and (2.5) are still regarded as two independent equations. Case four further not only enlarges the space of u, but also requires a higher regularity on y and z, and brings an unexpected difficulty to variation and computation.

These cases show that it is difficult to keep the compatibility of the spaces of u,y and z and incorporate them into their respective variational formulations. Based on the fact that the state y and adjoint state z are connected by the control u on the boundary of the domain, and that such connection $\frac{1}{\gamma}\frac{\partial z}{\partial \mathbf{n}}|_\Gamma = y|_\Gamma$ can be absorbed into the variational formulation of z as a boundary condition, i.e., we can eliminate the control by the control law to obtain a coupled system of the state and adjoint state. Although the control u is eliminated in form, it can be in essence reflected by the state y.

Based on this idea, multiplying both sides of (2.1) by $\psi \in H_0^1(\Omega)$, and applying integration by parts, we attain

$$\int_\Omega \nabla y \cdot \nabla \psi d\mathbf{x} = \int_\Omega f \psi d\mathbf{x}. \tag{2.6}$$

Similarly, multiplying both sides of (2.3) by $\phi \in H^1(\Omega)$, and applying integration by parts, yield

$$\int_\Omega \nabla z \cdot \nabla \phi d\mathbf{x} - \int_\Gamma \frac{\partial z}{\partial \mathbf{n}} \phi ds = \int_\Omega (y - y_d)\phi d\mathbf{x}. \tag{2.7}$$

Eliminating u from a combination of (2.2) and (2.5), yields

$$\frac{\partial z}{\partial \mathbf{n}}|_\Gamma = \gamma y|_\Gamma. \tag{2.8}$$

Substituting (2.8) into (2.7), we get

$$\int_\Omega \nabla z \cdot \nabla \phi d\mathbf{x} - \int_\Gamma \gamma y \phi ds = \int_\Omega (y - y_d)\phi d\mathbf{x}. \tag{2.9}$$

Collecting (2.6) and (2.9), gives the following variational formulation: Find $(y,z) \in H^1(\Omega) \times H_0^1(\Omega)$ such that

$$(\nabla y, \nabla \psi) = (f, \psi) \quad \forall\ \psi \in H_0^1(\Omega), \tag{2.10}$$

$$(\nabla z, \nabla \phi) - (\gamma y, \phi)_\Gamma - (y, \phi) = -(y_d, \phi) \quad \forall\ \phi \in H^1(\Omega). \tag{2.11}$$

In what follows, we clarify the unique solvability of the variational problem in (2.10)-(2.11). For a 2D convex polygonal domain, we recall a regularity result of May, Rannacher, and Vexler in [33] below, which gives conditions on the domain and data to guarantee the regularity of the solutions. To this end, let $\omega_{\max}$ be the maximum interior angle of the polygonal domain Ω, and denote p_*^Ω by

$$p_*^\Omega = 2\omega_{\max}/(2\omega_{\max} - \pi), \tag{2.12}$$

including the special case $p_*^\Omega = \infty$ for $\omega_{\max} = \pi/2$. For a higher dimensional convex polygonal domain, we do not attempt to provide conditions on the regularity of the solutions, because we put an emphasis on a variational setting and the corresponding finite element approximation. Of course, the regularity theory is more complicated in the three-dimensional case.

LEMMA 2.1. ([33], Lemma 2.9). *Suppose that $f \in L^2(\Omega)$ and $y_d \in L^{p_*^d}(\Omega), p_*^d > 2$, and that $\Omega \subset \mathbb{R}^2$ is a bounded convex domain with polygonal boundary Γ. Let $p_*^\Omega \geq 2$ be defined by (2.12) and $p_* := \min(p_*^d, p_*^\Omega)$. Then, the solutions (y,u) of the optimization problem (1.1)-(1.3) and the associated adjoint state determined by (2.3) have the regularity properties*

$$(y,u,z) \in H^{3/2-1/p}(\Omega) \times H^{1-1/p}(\Gamma) \times (H_0^1(\Omega) \cap W_p^2(\Omega)), \quad 2 \leq p < p_*.$$

Since the optimal system in (2.1)-(2.5) is equivalent to the problem (1.1)-(1.3), and since the coupled system (2.10)-(2.11) is obtained by eliminating the control variable, the regularities of the solutions of the system (2.10)-(2.11) are guaranteed in terms of Lemma 2.1 in case of $p = 2$.

3. Unique solvability and stability

This section establishes unique solvability for the variational problem in (2.10)-(2.11), and stability estimates of the control, state and adjoint state.

THEOREM 3.1. *For $f \in H^{-1}(\Omega)$ and $y_d \in L^2(\Omega)$, the system (2.10)-(2.11) is uniquely solvable, and is stable in the sense that there exists a positive constant C_γ, depending on γ, such that*

$$\gamma^{1/2} ||u||_{0,\Gamma} + ||y|| + ||\nabla z|| \leq C_\gamma (||f||_{-1} + ||y_d||). \tag{3.1}$$

Proof. The existence of the solutions of this coupled system is an immediate corollary of Theorem 2.3 in [11].

We now prove its stability. By (2.11) with $\psi = z$, we obtain

$$\begin{aligned} ||\nabla z||^2 &= (\gamma y, z)_\Gamma + (y,z) - (y_d, z) \\ &\leq \gamma ||y||_{-1/2,\Gamma} ||z||_{1/2,\Gamma} + ||y||_{-1} ||z||_1 + ||y_d||_{-1} ||z||_1 \\ &\leq C(\gamma ||y||_{0,\Gamma} ||z||_1 + \|y\| \, \|z\|_1 + \|y_d\| \, \|z\|_1), \end{aligned}$$

which, together with the Poincáre inequality, implies

$$||z||_1 \leq C(\gamma ||y||_{0,\Gamma} + ||y|| + ||y_d||). \tag{3.2}$$

It follows from (2.11) with $\psi = y$, (2.10), the Cauchy-Schwarz and Young inequalities, and (3.2) that

$$\begin{aligned} \gamma ||y||_{0,\Gamma}^2 + ||y||^2 &= (\nabla z, \nabla y) + (y_d, y) \\ &= (f,z) + (y_d, y) \leq ||f||_{-1} ||z||_1 + \|y_d\| \, \|y\| \\ &\leq C ||f||_{-1} (\gamma ||y||_{0,\Gamma} + ||y|| + ||y_d||) + \|y_d\| \, \|y\| \\ &\leq C(||f||_{-1} + ||y_d||)(\gamma ||y||_{0,\Gamma} + ||y||) + C\left(||f||_{-1}^2 + ||y_d||^2\right), \end{aligned}$$

which implies

$$\gamma ||y||_{0,\Gamma}^2 + ||y||^2 \leq C_\gamma \left(||f||_{-1}^2 + ||y_d||^2\right).$$

Now, (3.1) is a direct consequence of (3.2) and the fact that $u=y$ on Γ, and the uniqueness of the solution follows from (3.1) immediately, since the corresponding homogeneous system has vanishing solution. This completes the proof of the theorem. ∎

THEOREM 3.2. *Assume that Ω is a convex domain with Lipshitz boundary. For $f\in H^{-1}(\Omega)$ and $y_d\in L^2(\Omega)$, there exists a positive constant C_γ dependent on γ such that*

$$||\nabla y||\leq C_\gamma(||f||_{-1}+||y_d||). \tag{3.3}$$

Proof. By the standard $H^1(\Omega)$ a priori estimate of the problem in (2.1)-(2.2), and by the Equation (2.5), we have

$$\begin{aligned}||\nabla y||&\leq C\left(||f|_{-1}+||u||_{1/2,\Gamma}\right)\\ &\leq C\left(||f|_{-1}+\frac{1}{\gamma}\left\|\frac{\partial z}{\partial \mathbf{n}}\right\|_{1/2,\Gamma}\right).\end{aligned} \tag{3.4}$$

The standard $H^2(\Omega)$ a priori estimate (see, e.g., [12,33]) of the problem in (2.2)-(2.3) gives

$$\begin{aligned}||z||_2+\left\|\frac{\partial z}{\partial \mathbf{n}}\right\|_{1/2,\Gamma}&\leq C||y-y_d||\\ &\leq C(||y||+||y_d||).\end{aligned} \tag{3.5}$$

Now, (3.3) is a direct consequence of (3.4), (3.5), and (3.1). ∎

REMARK 3.1. Due to $z\in H_0^1(\Omega)$, the Pioncaré inequality implies $||z||_1\leq C||\nabla z||$. Hence, under the assumption of Theorem 3.2, there holds the following stable estimate:

$$\gamma^{1/2}||u||_{0,\Gamma}+||y||_1+||z||_1\leq C_\gamma(||f||_{-1}+||y_d|).$$

4. Finite element approximation and preliminary result

We introduce the discrete formulation of (2.10)-(2.11). To this end, let $\mathcal{T}_h$ be a shape regular partition of Ω into triangles (tetrahedra for $d=3$) or parallelograms (parallelepiped for $d=3$). With each element $K\in\mathcal{T}_h$, we denote $\rho(K)$ as the diameter of the set K, and define the size of the mesh by $h=\max_{K\in\mathcal{T}_h}\rho(K)$. About the partition, we also assume that there exists a constant $\rho>0$ such that $h/\rho(K)\leq\rho$ for all $K\in\mathcal{T}_h$ and $h>0$, i.e., the mesh $\mathcal{T}_h$ is quasi-uniform.

Denote $P_k(K)$ to be the space of polynomials of total degree at most k if K is a simplex, or the space of polynomials with degree at most k for each variable if K is a parallelogram/parallelepiped. Define the finite element space by

$$V_h:=\{v_h\in C(\overline{\Omega}):v_h|_K\in P_k(K),\quad \forall K\in\mathcal{T}_h\}.$$

Furthermore, set $V_h^0=V_h\cap H_0^1(\Omega)$, and denote V_h^∂ as the trace space corresponding to V_h, i.e., $V_h^\partial=V_h|_\Gamma$.

In the rest of this paper, we denote by C a constant independent of mesh size with a different context in a different occurrence, and also use the notation $A\lesssim F$ to represent $A\leq CF$ with a generic constant $C>0$ independent of mesh size. In addition, $A\approx F$ abbreviates $A\lesssim F\lesssim A$.

The discrete form reads: Find $(y_h,z_h)\in V_h\times V_h^0$ such that

$$(\nabla y_h,\nabla\psi_h)=(f,\psi_h)\qquad\forall\ \psi_h\in V_h^0, \tag{4.1}$$

$$(\nabla z_h, \nabla \phi_h) - (\gamma y_h, \phi_h)_\Gamma - (y_h, \phi_h) = -(y_d, \phi_h) \quad \forall\ \phi_h \in V_h. \tag{4.2}$$

THEOREM 4.1. *The discrete variational problem in (4.1)-(4.2) has a unique solution* $(y_h, z_h) \in V_h \times V_h^0$.

Proof. Since the existence of the solutions is equivalent to its uniqueness for a finite-dimensional system, it is sufficient to prove that the corresponding homogeneous system has trivial solution. To this end, let $f = 0$ and $y_d = 0$ in (4.1) and (4.2), respectively, we get

$$(\nabla y_h, \nabla \psi_h) = 0 \quad \forall\ \psi_h \in V_h^0, \tag{4.3}$$

$$(\nabla z_h, \nabla \phi_h) - (\gamma y_h, \phi_h)_\Gamma - (y_h, \phi_h) = 0 \quad \forall\ \phi_h \in V_h. \tag{4.4}$$

Taking $\phi_h = y_h$ in (4.4) leads to

$$(\nabla z_h, \nabla y_h) - (\gamma y_h, y_h)_\Gamma - ||y_h||^2 = 0. \tag{4.5}$$

Noticing $z_h \in V_h^0$ gives $(\nabla z_h, \nabla y_h) = 0$. Combining this with (4.5), yields

$$\int_\Gamma \gamma y_h^2 ds + ||y_h||^2 = 0,$$

which, altogether with the assumption $\gamma > 0$, results in $y_h = 0$.

Inserting $y_h = 0$ into (4.4), gives

$$(\nabla z_h, \nabla \phi_h) = 0 \quad \forall\ \phi_h \in V_h, \tag{4.6}$$

which, in turn, yields $(\nabla z_h, \nabla z_h) = 0$, by choosing $\phi_h = z_h$. Noticing $z_h \in V_h^0$, we get $z_h = 0$. Thus, the corresponding homogeneous system has vanishing solution. ◻

REMARK 4.1. To obtain the analogous discrete control rule to (2.5), Casas and Raymond [11] have introduced the so-called discrete normal derivative $\partial_{\mathbf{n}}^h z_h \in V_h^\partial$ solving

$$\left(\partial_{\mathbf{n}}^h z_h, w_h\right)_\Gamma = (\nabla z_h, \nabla w_h) - (y_h - y_d), \quad \forall w_h \in V_h, \tag{4.7}$$

and have obtained the discrete first order optimality conditions (see Theorem 4.3 in [11]): Find $(z_h, \partial_{\mathbf{n}}^h z_h) \in V_h^0 \times V_h^\partial$ such that

$$(\nabla z_h, \nabla w_h) = (y_h - y_d), \quad \forall w_h \in V_h^0, \tag{4.8}$$

$$(\gamma u_h, \chi_h)_\Gamma = (\partial_{\mathbf{n}}^h z_h, \chi_h)_\Gamma, \quad \forall \chi_h \in V_h^\partial. \tag{4.9}$$

Note that (4.8) is a special case of (4.7) with $w_h \in V_h^0$. Owing to (4.9), we have the discrete control rule $\gamma u_h = \partial_{\mathbf{n}}^h z_h$. Eliminating the discrete normal derivative $\partial_{\mathbf{n}}^h z_h$ from (4.7), and noticing the discrete trace condition $u_h = y_h|_\Gamma$, we can still derive the discrete variational problem in (4.1)-(4.2). This procedure indeed follows the route of "discretize-then-optimize-then-eliminate", where the concept of the discrete normal derivative is needed to be introduced, it is different from the route of "optimize-then-eliminate-then-discretize" adopted by us in this paper. We further remark that eliminating the control variable not only makes the system simple, but also brings a convenience to define the residual functional according to (4.1)-(4.2) when the residual-type a posteriori estimators are considered.

LEMMA 4.1. *Assume that $\theta_1^b \in V_h$ and vanishes at all interior nodes of the mesh, and let h be the size of the quasi-uniform mesh $\mathcal{T}_h$. There holds the following estimate*

$$||\nabla\theta_1^b|| \lesssim h^{-1/2}||\theta_1^b||_{L^2(\Gamma)}. \tag{4.10}$$

Proof. Denote Ω_h^b as the set of elements with at least one vertex on the boundary. Since θ_1^b vanishes at any node of an element whose vertices are completely contained in the interior of the domain Ω, it's restriction on the element is zero, which implies

$$||\nabla\theta_1^b||^2 = \sum_{K\in\Omega_h^b} ||\nabla\theta_1^b||_K^2. \tag{4.11}$$

For the sake of simplicity, we consider only triangular element in two dimensions as an example, since the same proof is easily extended to the other types of elements and the three-dimensional case. There are three cases as follows:

(1) Two vertices of an element K lie on the boundary, i.e., K has an edge E contained in Γ ($E = \partial K \cap \Gamma$) (see (Case 1) in Figure 4.1). Assuming $||\nabla\theta_1^b||_K = 0$ indicates that θ_1^b is a constant over the element K, and since θ_1^b vanishes at internal node of K (there exists at least an internal node such as internal vertex), this shows that θ_1^b is zero over K, and that $||\nabla\theta_1^b||_K$ is a norm of θ_1^b over K. Further assuming $||\theta_1^b||_E = 0$, this leads to the fact that θ_1^b vanishes over E, and that θ_1^b vanishes at nodes of E, and that θ_1^b is zero over K. Therefore, $||\theta_1^b||_E$ is another norm of θ_1^b over K. Since any two norms are equivalent to each other over a finite-dimension space, we attain

$$||\nabla\theta_1^b||_K \approx C_K||\theta_1^b||_E, \tag{4.12}$$

where the positive constant C_K depends on the size h_K of K (and number of dimensions of $V_h|_K$). To see the dependence on the size of K, we apply the scaling argument.

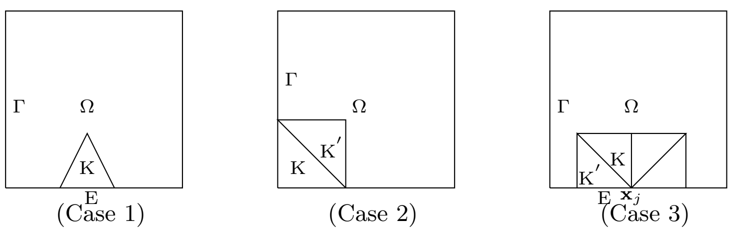

FIG. 4.1. *Three cases of location of an element in Ω_h^b for triangular element in two dimensions.*

To this end, for any element $K \in \mathcal{T}_h$ there exists a bijection $F_K : \hat{K} \to K$, where $\hat{K}$ is the reference element. Denote by DF_K the Jacobian matrix and let $J_K = |det(DF_K)|$. It is easy to see that for all element types, the mapping definition and shape-regularity and quasi-uniformity of the grids imply that

$$||DF_K^{-1}||_{0,\infty,\hat{K}} \approx h_K^{-1}, \quad ||J_K||_{0,\infty,\hat{K}} \approx h_K^d, \quad ||DF_K||_{0,\infty,\hat{K}} \approx h_K,$$

which, results in

$$||\nabla\theta_1^b||_K^2 = \int_K \nabla\theta_1^b \cdot \nabla\theta_1^b d\mathbf{x}$$

$$
\begin{aligned}
&= \int_{\hat{K}} DF_K^{-1} \hat{\nabla}\hat{\theta}_1^b \cdot DF_K^{-1} \hat{\nabla}\hat{\theta}_1^b J_K d\hat{\mathbf{x}} \\
&\approx h_K^{d-2} ||\hat{\nabla}\hat{\theta}_1^b||_{\hat{K}}^2. \qquad (4.13)
\end{aligned}
$$

Let E be an edge (side) of K, and $\hat{E}$ be an edge (side) of $\hat{K}$ with respect to E. Similarly, we have

$$||\theta_1^b||_E^2 = \int_E (\theta_1^b)^2 ds = \int_{\hat{E}} \frac{|E|}{|\hat{E}|} (\hat{\theta}_1^b)^2 d\hat{s} = \frac{|E|}{|\hat{E}|} ||\hat{\theta}_1^b||_{\hat{E}}^2 \approx h_K^{d-1} ||\hat{\theta}_1^b||_{\hat{E}}^2. \qquad (4.14)$$

We have from (4.12)

$$||\hat{\nabla}\hat{\theta}_1^b||_{\hat{K}} \approx ||\hat{\theta}_1^b||_{\hat{E}}. \qquad (4.15)$$

A combination of (4.13), (4.14), and (4.15), yields

$$||\nabla\theta_1^b||_K \lesssim h_K^{-1/2} ||\theta_1^b||_E. \qquad (4.16)$$

(2) Three vertices of an element K lie on the boundary Γ, i.e., K has two edges contained in Γ (see (Case 2) in Figure 4.1). Suppose that one can always choose an element K' that has an internal vertex and a common edge with K. Now consider θ_1^b over $K \cup K'$. Repeating the proof of Case (1), we have

$$||\nabla\theta_1^b||_{K\cup K'} \approx C_{K\cup K'} ||\theta_1^b||_{\Gamma\cap\partial(K\cup K')}, \qquad (4.17)$$

where $C_{K\cup K'}$ relies on the size $h_{K\cup K'}$, of $K \cup K'$. Using the scaling argument again, we easily obtain

$$||\nabla\theta_1^b||_{K\cup K'}^2 \approx h_K^{d-2} ||\hat{\nabla}\hat{\theta}_1^b||_{\widehat{K\cup K'}}^2, \qquad (4.18)$$

$$||\theta_1^b||_{\partial(K\cup K')\cap\Gamma}^2 \approx h_K^{d-1} ||\hat{\theta}_1^b||_{\widehat{\partial K\cap\Gamma}}^2. \qquad (4.19)$$

(4.17) indicates that $||\hat{\nabla}\hat{\theta}_1^b||_{\widehat{K\cup K'}} \approx ||\hat{\theta}_1^b||_{\widehat{\partial K\cap\Gamma}}$. Hence, we obtain from a combination of (4.18) and (4.19)

$$||\nabla\theta_1^b||_{K\cup K'} \lesssim h_K^{-1/2} ||\theta_1^b||_{\partial K\cup\Gamma}. \qquad (4.20)$$

(3) Only one vertex $\mathbf{x}_j$, of K lies on the boundary Γ (see (Case 3) in Figure 4.1). Suppose that one can always choose an element K' such that $\partial(K\cup K')$ contains a boundary edge E and K' has a common edge with K, i.e., $E \subset \partial(K\cup K') \cap \Gamma$. Similarly to Case (2) or (1), we easily obtain

$$||\nabla\theta_1^b||_K \le ||\nabla\theta_1^b||_{K\cup K'} \lesssim h_K^{-1/2} ||\theta_1^b||_{\partial(K\cup K')\cap\Gamma}. \qquad (4.21)$$

In fact, in this case, we can also consider θ_1^b over the patch $\omega_{\mathbf{x}_j}$ (the set of elements sharing $\mathbf{x}_j$ with K), of $\mathbf{x}_j$. By using the scaling argument, we can obtain

$$||\nabla\theta_1^b||_K \le ||\nabla\theta_1^b||_{\omega_{\mathbf{x}_j}} \lesssim h_K^{-1/2} ||\theta_1^b||_{\partial(\omega_{\mathbf{x}_j})\cap\Gamma}. \qquad (4.22)$$

Collecting (4.16) and (4.20)-(4.22), we obtain from (4.11)

$$\begin{aligned}||\nabla\theta_1^b||^2 &= \sum_{K\in\Omega_h^b:\ \text{Case (1)}} ||\nabla\theta_1^b||_K^2 + \sum_{K\in\Omega_h^b:\ \text{Case (2)}} ||\nabla\theta_1^b||_K^2 + \sum_{K\in\Omega_h^b:\ \text{Case (3)}} ||\nabla\theta_1^b||_K^2 \\ &\lesssim \sum_{K\in\Omega_h^b:\ \text{Case (1)}} h_K^{-1}||\theta_1^b||_{\partial K\cap\Gamma}^2 + \sum_{K\in\Omega_h^b:\ \text{Case (2)}} h_K^{-1}||\theta_1^b||_{\partial K\cap\Gamma}^2 \\ &\quad + \sum_{K\in\Omega_h^b:\ \text{Case (3)}} h_K^{-1}||\theta_1^b||_{\partial(\omega_{\mathbf{x}_j})\cap\Gamma}^2 \\ &\lesssim h^{-1}||\theta_1^b||_\Gamma^2,\end{aligned}$$

which results in the desired estimate (4.10). □

5. Analysis of error

Since the control u is equal to the restriction of the state y on the boundary, i.e., $u=y|_\Gamma$, it is natural that the restriction of an approximation y_h of y on the boundary is also an approximation of u. Therefore, $||y-y_h||_{0,\Gamma}$ can be used to measure the numerical error of the control, and in this sense, we write $||u-u_h||_{0,\Gamma}=||y-y_h||_{0,\Gamma}$.

THEOREM 5.1. *Assume that $(y,z)\in H^1(\Omega)\times H_0^1(\Omega)$ and $(y_h,z_h)\in V_h\times V_h^0$ be the solutions to (2.10)-(2.11) and (4.1)-(4.2), respectively. For $y\in H^k(\Omega), z\in H^{k+1}(\Omega)\cap H_0^1(\Omega)$, and for the numerical error of the state variable y, there exists a positive constant C_γ depending on γ such that*

$$||y-y_h||+||\gamma^{1/2}(y-y_h)||_{0,\Gamma}\le C_\gamma h^{k-1/2}(|y|_k+|z|_{k+1}). \tag{5.1}$$

Proof. Denote by $R_h: H^1(\Omega)\to V_h$ the Ritz projection operator

$$(\nabla(R_hv),\nabla v_h)=(\nabla v,\nabla v_h),\quad (v-R_hv,1)=0,\quad \forall v_h\in V_h. \tag{5.2}$$

Recall the properties of the Ritz projection [7,8] as the following

$$||v-R_hv||\lesssim h^k|v|_k, ||\nabla(v-R_hv)||\lesssim h^{k-1}|v|_k,\quad \forall\ v\in H^m(\Omega), 0<k\le m\le 3. \tag{5.3}$$

Setting $\eta_1=y-R_hy$ and $\theta_1=R_hy-y_h$, gives $y-y_h=\eta_1+\theta_1$. We have from triangle inequality and (5.3)

$$||y-y_h||\le||\eta_1||+||\theta_1||\lesssim h^k|y|_k+||\theta_1||. \tag{5.4}$$

The trace inequality and the properties, (5.3), of the Ritz projection imply that

$$\begin{aligned}||\gamma^{1/2}(y-y_h)||_{0,\Gamma} &\le \gamma^{1/2}||\eta_1||_{0,\Gamma}+||\gamma^{1/2}\theta_1||_{0,\Gamma} \\ &\lesssim \gamma^{1/2}||\eta_1||^{1/2}||\eta_1||_1^{1/2}+||\gamma^{1/2}\theta_1||_{0,\Gamma} \\ &\lesssim \gamma^{1/2}\left(||\eta_1||+||\nabla\eta_1||^{1/2}||\eta_1||^{1/2}\right)+||\gamma^{1/2}\theta_1||_{0,\Gamma)} \\ &\lesssim \gamma^{1/2}h^{k-1/2}|y|_k+||\gamma^{1/2}\theta_1||_{0,\Gamma}.\end{aligned} \tag{5.5}$$

(5.4) and (5.5) indicate that we only need to estimate $||\theta_1||$ and $||\gamma^{1/2}\theta_1||_{0,\Gamma}$ in order to estimate $||y-y_h||+||\gamma^{1/2}(y-y_h)||_{0,\Gamma}$. To this end, let $R_h^0: H_0^1(\Omega)\to V_h^0$ be the Ritz projection operator by

$$(\nabla(R_h^0v),\nabla v_h)=(\nabla v,\nabla v_h)\quad \forall\ v_h\in V_h^0.$$

Again, recall the properties of the Ritz projection [7,8] as the following

$$||\nabla(v-R_h^0 v)||\lesssim h^{k-1}|v|_k, \quad \forall\ v\in H^m(\Omega)\cap H_0^1(\Omega), 0<k\le m\le 3. \tag{5.6}$$

Setting $\eta_2=z-R_h^0 z, \theta_2=R_h^0 z-z_h$, gives $z-z_h=\eta_2+\theta_2$. From (2.11) and (4.2), we obtain the following orthogonality

$$(\nabla(z-z_h),\nabla\phi_h)-(\gamma(y-y_h),\phi_h)_\Gamma-(y-y_h,\phi_h)=0, \quad \forall\ \phi_h\in V_h. \tag{5.7}$$

Especially taking $\phi_h=\theta_1\in V_h$ in (5.7), yields

$$(\nabla\eta_2+\nabla\theta_2,\nabla\theta_1)-(\gamma(\eta_1+\theta_1),\theta_1)_\Gamma-(\eta_1+\theta_1,\theta_1)=0,$$

which results in

$$||\gamma^{1/2}\theta_1||_{0,\Gamma}^2+||\theta_1||^2=(\nabla\eta_2,\nabla\theta_1)+(\nabla\theta_2,\nabla\theta_1)-(\gamma\eta_1,\theta_1)_\Gamma-(\eta_1,\theta_1). \tag{5.8}$$

From (2.10) and (4.1), we get the following orthogonal property

$$(\nabla(y-y_h),\nabla\psi_h)=0, \quad \forall\ \psi_h\in V_h^0. \tag{5.9}$$

Taking $\psi_h=\theta_2\in V_h^0$ in (5.9), yields

$$(\nabla\theta_2,\nabla\theta_1)=-(\nabla\eta_1,\nabla\theta_2)=0. \tag{5.10}$$

In the second step above, we apply the orthogonal property of the Ritz projection, because of $\theta_2\in V_h^0\subset V_h$. Combining (5.8) with (5.10), we attain

$$||\gamma^{1/2}\theta_1||_{0,\Gamma}^2+||\theta_1||^2=(\nabla\eta_2,\nabla\theta_1)-(\gamma\eta_1,\theta_1)_\Gamma-(\eta_1,\theta_1). \tag{5.11}$$

In what follows, we estimate each term on the right-hand side of (5.11). In terms of the proof of (5.4) and (5.5), we immediately obtain the estimates of the last two terms on the right-hand side of (5.11)

$$|-(\eta_1,\theta_1)|\lesssim h^k|y|_k||\theta_1|| \tag{5.12}$$

and

$$|-(\gamma\eta_1,\theta_1)_\Gamma|\lesssim\gamma^{1/2}h^{k-1/2}|y|_k||\gamma^{1/2}\theta_1||_{0,\Gamma}. \tag{5.13}$$

To estimate the first term on the right-hand side of (5.11), we decompose θ_1 into θ_1^i and θ_1^b, where the value of θ_1^i at the internal node equals to that of θ_1 at the corresponding node, and the value of θ_1^i at the boundary node is zero; the value of θ_1^b at the internal node is zero, and the value of θ_1^b at boundary node equals to that of θ_1 at the corresponding node. Obviously, $\theta_1=\theta_1^i+\theta_1^b$.

Noticing $\theta_1^i\in V_h^0, \theta_1^b\in V_h$, we have from the definition of the Ritz projection

$$\begin{aligned}(\nabla\eta_2,\nabla\theta_1) &= (\nabla\eta_2,\nabla\theta_1^i+\nabla\theta_1^b)\\ &= (\nabla\eta_2,\nabla\theta_1^b)\le\|\nabla\eta_2\|\,\|\nabla\theta_1^b\|.\end{aligned} \tag{5.14}$$

We further derive from Lemma 4.1, together with $\theta_1^b=\theta_1$ on the boundary Γ

$$\begin{aligned}(\nabla\eta_2,\nabla\theta_1) &\lesssim h^{-1/2}\|\nabla\eta_2\|\,\|\theta_1^b\|_{0,\Gamma}\\ &= h^{-1/2}\gamma^{-1/2}\|\nabla\eta_2\|\,\|\gamma^{1/2}\theta_1\|_{0,\Gamma}.\end{aligned} \tag{5.15}$$

By combining (5.11)-(5.13) with (5.15), and applying the properties, (5.6), of the Ritz projection, and Young inequality, we obtain

$$\begin{aligned}||\gamma^{1/2}\theta_1||_{0,\Gamma}^2+||\theta_1||^2 &\le Ch^{-1/2}\gamma^{-1/2}\|\nabla\eta_2\|\,\|\gamma^{1/2}\theta_1\|_{0,\Gamma}\\ &\quad +Ch^k|y|_k||\theta_1||+C\gamma^{1/2}h^{k-1/2}|y|_k||\gamma^{1/2}\theta_1||_{0,\Gamma}\\ &\le Ch^{-1/2}\gamma^{-1/2}h^k|z|_{k+1}||\gamma^{1/2}\theta_1||_{0,\Gamma}\\ &\quad +Ch^k|y|_k||\theta_1||+C\gamma^{1/2}h^{k-1/2}|y|_k||\gamma^{1/2}\theta_1||_{0,\Gamma}\\ &\le C\gamma^{-1}h^{2k-1}|z|_{k+1}^2+||\gamma^{1/2}\theta_1||_{0,\Gamma}^2/4+Ch^{2k}|y|_k^2\\ &\quad +||\theta_1||^2/2+C\gamma h^{2k-1}|y|_k^2+||\gamma^{1/2}\theta_1||_{0,\Gamma}^2/4,\end{aligned}$$

which, implies

$$||\gamma^{1/2}\theta_1||_{0,\Gamma}^2+||\theta_1||^2\le C_\gamma h^{2k-1}\left(|z|_{k+1}^2+|y|_k^2\right). \tag{5.16}$$

Collecting (5.4), (5.5), and (5.16), we get

$$||\gamma^{1/2}(y-y_h)||_{0,\Gamma}^2+||y-y_h||^2\le C_\gamma h^{2k-1}\left(|z|_{k+1}^2+|y|_k^2\right),$$

which results in the desired estimate (5.1). ∎

REMARK 5.1. As pointed at the beginning of this section, we understand $||u-u_h||_{L^2(\Gamma)}$ as $||y-y_h||_{0,\Gamma}$. For $y\in H^k(\Omega), z\in H_0^1(\Omega)\cap H^{k+1}(\Omega)$, Theorem 5.1 gives the control an estimate

$$||u-u_h||_{0,\Gamma}\le C_\gamma h^{k-1/2}(|y|_k+|z|_{k+1}).$$

Owing to $y\in H^1(\Omega)$ and $u\in H^{1/2}(\Gamma)$, yields the optimal order 1/2 of convergence for the control.

THEOREM 5.2. *Assume that* $(y,z)\in H^1(\Omega)\times H_0^1(\Omega)$ *and* $(y_h,z_h)\in V_h\times V_h^0$ *be the solutions to* (2.10)-(2.11) *and* (4.1)-(4.2)*, respectively. For* $y\in H^k(\Omega), z\in H^{k+1}(\Omega)\cap H_0^1(\Omega)$*, the numerical error of the adjoint state* z *is bounded by*

$$||\nabla(z-z_h)||\le C_\gamma h^{k-1/2}(|y|_k+|z|_{k+1}). \tag{5.17}$$

Proof. Recalling the decomposition of the error $z-z_h$ in the proof of Theorem 5.1, we obtain from the orthogonal property of the Ritz projection

$$||\nabla(z-z_h)||^2=||\nabla\eta_2||^2+||\nabla\theta_2||^2, \tag{5.18}$$

which, together with the property (5.6) of the Ritz projection, results in,

$$||\nabla(z-z_h)||\lesssim h^k|z|_{k+1}+||\nabla\theta_2||. \tag{5.19}$$

The inequality (5.19) means that it is sufficient to only estimate $||\nabla\theta_2||$ in order to estimate $||\nabla(z-z_h)||$.

Taking $\phi_h=\theta_2\in V_h^0$ in (5.7), yields

$$(\nabla\eta_2+\nabla\theta_2,\nabla\theta_2)-(y-y_h,\theta_2)=0,$$

which, together with the orthogonal relation $(\nabla\eta_2,\nabla\theta_2)=0$, results in,

$$||\nabla\theta_2||^2=(y-y_h,\theta_2)\le||y-y_h||||\theta_2||. \tag{5.20}$$

Applying the Poincaré inequality, we obtain from (5.20)

$$||\nabla\theta_2|| \lesssim ||y-y_h||. \tag{5.21}$$

Combining (5.19) with (5.21), we obtain

$$||\nabla(z-z_h)|| \lesssim h^k|z|_{k+1} + ||y-y_h||, \tag{5.22}$$

which, together with (5.1), results in the desired estimate (5.17). □

REMARK 5.2. According to Lemma 2.1, the H^2 regularity for the state cannot be reached on polygonal/polyhedral domain. Hence, these estimates above are restricted to the case of $k=1$. However, the H^k regularity for the state can be reached for domains with sufficiently smooth boundary. Since the Dirichlet boundary control problem is completely different from the Dirichlet boundary value problem, it is non-trivial to generalize analytical technique for high order element (including isoparametric-equivalent element) for the Dirichlet boundary value problem to the Dirichlet boundary control problem. Here are two remedies.

In first remedy, let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $\mathcal{T}_h$ be a "triangulation" of Ω, where each triangle at the boundary has at most one curved side. The finite element spaces V_h and V_h^0 are defined by

$$V_h = \left\{ v \in C(\bar{\Omega}) : v|_T \in P_k \right\} \quad \text{and} \quad V_h^0 = V_h \cap H_0^1(\Omega), \text{ respectively.}$$

By using standard interpolation error estimates, we can easily verify that the properties of the Ritz projection over V_h (and V_h^0) are still true. Assume that the "triangulation" $\mathcal{T}_h$ guarantees Lemma 4.1. Indeed, this is easily realised by assuming that there exists $\rho > 0$ such that for each triangle $T \in \mathcal{T}_h$ one can find two concentric circular discs D_1 and D_2 such that

$$D_1 \subseteq T \subseteq D_2 \quad \text{and} \quad \frac{\text{diam} D_2}{\text{diam} D_1} \leq \rho.$$

Since $\partial\Omega$ is smooth, for h small enough, we have $h_e < 2\text{diam}T < 2\text{diam}D_2$ (curved side $e \subset \partial T$, h_e denotes the arc length of e). This indicates Lemma 4.1 is still valid. Therefore, the results of Theorems 5.1-5.2 are applicable to high order curved-triangle Lagrange element.

In the second remedy, recall that we have a polyhedral approximation, Ω_h to Ω, and an isoparametric mapping F^h such that $F^h(\Omega_h)$ closely approximates Ω, and denote $\tilde{V}_h$ as a base finite element space defined on Ω_h, the resulting space,

$$V_h := \left\{ v((F^h)^{-1}(\mathbf{x})) : \mathbf{x} \in F^h(\Omega_h),\ v \in \tilde{V}_h \right\},$$

is an isoparametric-equivalent finite element space (we refer to [14, 16] for the details). Let $V_h^0 = V_h \cap H_0^1(F^h(\Omega_h))$. If we impose the control rule on $\partial(F^h(\Omega_h))$, i.e., $\dfrac{1}{\gamma}\dfrac{\partial z}{\partial \mathbf{n}} = u$ on $\partial(F^h(\Omega_h))$, this shows that we are considering the problem (1.1)-(1.3) on the domain $F^h(\Omega_h)$. The only difference is that we substitute the domain Ω in the previous context with $F^h(\Omega_h)$. Since the corresponding Ritz projection $R_h : H^1(F^h(\Omega_h)) \to V_h$ ($R_h^0 : H_0^1(F^h(\Omega_h)) \to V_h^0$) still possesses the same approximation properties as (5.3)((5.6)), and hence the result of Lemma 4.1 can be achieved by a similar proof. Therefore, by repeating the proof of Theorem 5.1, we can obtain the following estimate

$$||y-y_h||_{0,F^h(\Omega_h)}+||\gamma^{1/2}(y-y_h)||_{0,\partial(F^h(\Omega_h))}\leq C_\gamma h^{k-\frac{1}{2}}(|y|_{k,F^h(\Omega_h)}+|z|_{k+1,F^h(\Omega_h)}) \tag{5.23}$$

under the assumption that $y\in H^k(F^h(\Omega_h)), z\in H^{k+1}(F^h(\Omega_h))$.

Furthermore, we will assume there is an auxiliary mapping $F:\Omega_h\to\Omega$ and that $F_i^h=I^hF_i$ for each component of the mapping. Here I^hv denotes the isoparametric interpolation by $I^hv(F^h(\mathbf{x}))=\tilde{I}^h\tilde{v}(\mathbf{x})$ for all $\mathbf{x}\in\Omega_h$ where $\tilde{v}(\mathbf{x})=v(F^h(\mathbf{x}))$ for all $\mathbf{x}\in\Omega_h$ and $\tilde{I}_h$ is the global interpolation for the base finite element space, $\tilde{V}_h$ (we refer to [14,16] for the details). Thus, the mapping $\Phi^h:\Omega\to F^h(\Omega_h)$ defined by $\Phi^h(\mathbf{x})=F^h(F^{-1}(\mathbf{x}))$ gives an estimate for (5.23)

$$||y-\hat{y}_h||_{0,\Omega}+||\gamma^{1/2}(y-y_h)||_{0,\Omega}\leq C_\gamma h^{k-1/2}(|y|_{k,\Omega}+|z|_{k+1,\Omega})$$

under the assumptions that the Jacobian J_{Φ^h} is independent of h and that $y\in H^k(\Omega), z\in H^{k+1}(\Omega)$, where $\hat{y}_h=y_h(\Phi^h(\mathbf{x})),\mathbf{x}\in\Omega$.

It is well known that the L^2 norm of numerical error is controlled by the H^1 norm for conforming finite element approximation to the standard Laplacian equation, and that the L^2 norm of numerical error is of order one higher than the H^1 norm. The following Theorem 5.3 shows that $||y-y_h||$ is still controlled by $||y-y_h||_1$, but isn't of order one higher than $||y-y_h||_1$. This will be testified by numerical experiments in Section 7.

THEOREM 5.3. *Assume that* $(y,z)\in H^1(\Omega)\times H_0^1(\Omega)$ *and* $(y_h,z_h)\in V_h\times V_h^0$ *be the solutions to* (2.10)-(2.11) *and* (4.1)-(4.2), *respectively. There holds*

$$||y-y_h||\lesssim||\nabla(y-y_h)||. \tag{5.24}$$

Proof. Consider the following Neumann boundary-value problem

$$\begin{cases}-\triangle w=y(\mathbf{x})-y_h(\mathbf{x}) & \text{in } \Omega,\\ \dfrac{\partial w}{\partial\mathbf{n}}=\gamma(y(\mathbf{x})-y_h(\mathbf{x}))|_\Gamma & \text{on } \Gamma.\end{cases} \tag{5.25}$$

The continuous weak formulation for the problem (5.25) reads: Find $w\in H^1(\Omega)$ such that

$$(\nabla w,\nabla\psi)=(\gamma(y-y_h),\psi)_\Gamma+(y-y_h,\psi)\quad\forall\ \psi\in H^1(\Omega). \tag{5.26}$$

We get the following orthogonality from a combination of (2.11) and (4.2)

$$(\nabla(z-z_h),\nabla v_h)-(\gamma(y-y_h),v_h)_\Gamma-(y-y_h,v_h)=0,\quad\forall\ v_h\in V_h.$$

Owing to $v_h=1\in V_h$, the above identity implies

$$\int_\Omega(y(\mathbf{x})-y_h(\mathbf{x}))d\mathbf{x}+\int_\Gamma\gamma(y(\mathbf{x})-y_h(\mathbf{x}))ds=0,$$

which shows that the problem (5.25) satisfies the consistent condition. Therefore, the weak formulation (5.26) has a unique solution in the sense that the solutions differ by a constant, and satisfies the following estimate

$$||\nabla w||\lesssim||y-y_h||+\gamma||y-y_h||_{-1/2,\Gamma}. \tag{5.27}$$

Taking $\psi = y$ and $\psi = y_h$, respectively, in (5.26), yields

$$(\nabla w, \nabla y) = (\gamma(y - y_h), y)_\Gamma + (y - y_h, y) \tag{5.28}$$

and

$$(\nabla w, \nabla y_h) = (\gamma(y - y_h), y_h)_\Gamma + (y - y_h, y_h). \tag{5.29}$$

A combination of (5.28) and (5.29) leads

$$||\gamma^{1/2}(y - y_h)||_{0,\Gamma}^2 + ||y - y_h||^2 = (\nabla w, \nabla(y - y_h)) \le \|\nabla w\| \|\nabla(y - y_h)\|. \tag{5.30}$$

We obtain from (5.27)

$$||\nabla w|| \lesssim ||y - y_h|| + \gamma ||y - y_h||_{0,\Gamma}. \tag{5.31}$$

A combination of (5.30) and (5.31) yields

$$||\gamma^{1/2}(y - y_h)||_{0,\Gamma}^2 + ||y - y_h||^2 \lesssim (||y - y_h|| + \gamma ||y - y_h||_{0,\Gamma}) ||\nabla(y - y_h)||,$$

which, results in

$$||y - y_h|| \le \left(||\gamma^{1/2}(y - y_h)||_{0,\Gamma}^2 + ||y - y_h||^2\right)^{1/2} \lesssim ||\nabla(y - y_h)||.$$

We complete the proof of (5.24). ∎

Remark 5.3. In terms of the proof of Theorem 5.3, for a function $v \in H^1(\Omega)$ satisfying

$$\int_\Omega v d\mathbf{x} + \int_\Gamma v ds = 0,$$

there holds an analogue of the Poincaré inequality

$$||v||_1 \lesssim ||\nabla v||.$$

6. Stability for discrete solution

Since the control is firstly concerned in practice for the optimal control problem, this section is specially devoted to an analysis of the stability for the control in the sense that the restriction of the discrete state y_h on the boundary is an approximation of the control u. Recall the following "inverse estimate" for finite element functions $\chi_h \in V_h^\partial$:

$$|\chi_h|_{H^{1/2}(\Gamma)} \lesssim h^{-1/2} ||\chi_h||_{L^2(\Gamma)}. \tag{6.1}$$

Indeed, this can be found in [33] or be proven by combining estimates in [7, 14] with standard results from interpolation theory. We define the L^2 projection $P_h^\partial : L^2(\Gamma) \to V_h^\partial$ by

$$(q - P_h^\partial q, \chi_h) = 0, \quad \forall \chi_h \in V_h^\partial.$$

By standard results for finite elements we have the error estimate (see [7, 11, 14])

$$||q - P_h^\partial q||_{0,\Gamma} + h^{1/2} |P_h^\partial q|_{1/2,\Gamma} \lesssim h^{1/2} |q|_{1/2,\Gamma}, \quad \forall q \in H^{1/2}(\Gamma). \tag{6.2}$$

THEOREM 6.1. *Assume that $f \in H^{-1}(\Omega), y_d \in L^2(\Omega)$, the domain Ω is convex, and its boundary Γ is Lipschitz continuous. There exists a positive constant C_γ depending on γ such that*

$$||\gamma^{1/2} y_h||_{0,\Gamma} + ||y_h|| \leq C_\gamma (||f||_{-1} + ||y_d||). \tag{6.3}$$

Proof. Taking $\phi_h = y_h$ and $\psi_h = z_h$ in (4.2) and (4.1), respectively, gives

$$\begin{aligned} ||\gamma^{1/2} y_h||_{0,\Gamma}^2 + ||y_h||^2 &= (\nabla z_h, \nabla y_h) + (y_d, y_h) \\ &= (f, z_h) + (y_d, y_h) \\ &= (f, z_h - z) + (f, z) + (y_d, y_h) \\ &\leq ||f|_{-1}|||z - z_h||_1 + ||f||_{-1}||z||_1 + \|y_d\| \, \|y_h\|. \end{aligned} \tag{6.4}$$

Noticing $z - z_h \in H_0^1(\Omega)$, we obtain from the Poincaré inequality, (5.17) with $k = 1$, and (3.5)

$$\begin{aligned} ||z - z_h||_1 &\lesssim ||\nabla(z - z_h)|| \leq C_\gamma h^{1/2}(||y||_1 + ||z||_2) \\ &\leq C_\gamma h^{1/2}(||y||_1 + ||y - y_d||) \leq C_\gamma h^{1/2}(||y||_1 + ||y_d||). \end{aligned} \tag{6.5}$$

Combining (6.4) with (6.5), together with the Young inequality, gives

$$||\gamma^{1/2} y_h||_{0,\Gamma}^2 + ||y_h||^2 \leq C_\gamma \left(||y||_1^2 + ||y_d||^2 + ||f||_{-1}^2 + ||z||_1^2\right). \tag{6.6}$$

Applying the stable estimates (3.1) and (3.3) of the state y and adjoint state z, respectively, we get

$$||\gamma^{1/2} y_h||_{0,\Gamma}^2 + ||y_h||^2 \leq C_\gamma \left(||y_d||^2 + ||f||_{-1}^2\right),$$

which results in the desired estimate (6.3). ∎

THEOREM 6.2. *Under the assumption of Theorem 6.1, the discrete solutions admit the uniform bound*

$$|y_h|_{1/2,\Gamma} \leq C_\gamma (||f||_{-1} + ||y_d||). \tag{6.7}$$

Proof. From triangle inequality, "inverse estimate" (6.1), and the property (6.2) of the L^2 projection operator P_h^∂, we get

$$\begin{aligned} |y_h|_{1/2,\Gamma} &\leq |y_h - P_h^\partial y|_{1/2,\Gamma} + |P_h^\partial y - y|_{1/2,\Gamma} + |y|_{1/2,\Gamma} \\ &\lesssim h^{-1/2}||y_h - P_h^\partial y||_{0,\Gamma} + |y|_{1/2,\Gamma} \\ &\leq h^{-1/2}\left(||y_h - y||_{0,\Gamma} + ||y - P_h^\partial y||_{0,\Gamma}\right) + |y|_{1/2,\Gamma} \\ &\lesssim h^{-1/2}||y_h - y||_{0,\Gamma} + h^{-1/2}h^{1/2}|y|_{1/2,\Gamma} + |y|_{1/2,\Gamma)}. \end{aligned} \tag{6.8}$$

From (5.1) with $k = 1$ and (3.5), we have

$$||y_h - y||_{0,\Gamma} \leq C_\gamma h^{1/2}(||\nabla y|| + |z|_2) \leq C_\gamma h^{1/2}(||y||_1 + ||y_d||). \tag{6.9}$$

A combination of (6.8) and (6.9) yields

$$|y_h|_{1/2,\Gamma} \leq C_\gamma \left(||y||_1 + ||y_d|| + |y|_{1/2,\Gamma}\right) \leq C_\gamma (||y||_1 + ||y_d||). \tag{6.10}$$

The desired estimate (6.7) follows from a combination of (6.10), (3.1) and (3.3). ∎

7. Numerical experiments

In this section, we test the performance of finite element approximation to the variational formulation developed in this paper with two model problems. The actual solution of the first model problem is known, and the true solution of the second example is unknown, and two settings of the regularization parameter γ will be considered here, We are thus able to study the convergence rate of the state y and adjoint state z, as well as the control variable u over quasi-uniform mesh, and to study the relation between the singularity of the actual solution and the regularization parameter in Example two. Note that we shall employ the piecewise linear elements in both examples. Let $\{\psi_i\}$ and $\{\phi_j\}$ be respectively the basis of V_h^0 and V_h, then the algebraic system with respect to (4.1)-(4.2) has the following form

$$\begin{pmatrix} A & O \\ B & C \end{pmatrix}\begin{pmatrix} Y \\ Z \end{pmatrix}=\begin{pmatrix} F \\ G \end{pmatrix}.$$

7.1. Example one.

We consider the problem (1.1)-(1.2) over a unit square $\Omega=(0,1)\times(0,1)$ with

$$f=-\frac{4}{\gamma}, y_d=\left(2+\frac{1}{\gamma}\right)\left(x_1^2-x_1+x_2^2-x_2\right).$$

The exact solutions are given by

$$u=\frac{x_1^2-x_1+x_2^2-x_2}{\gamma}, y=\frac{x_1^2-x_1+x_2^2-x_2}{\gamma}, z=\left(x_1^2-x_1\right)\left(x_2^2-x_2\right).$$

It is easy to verify that the control u, state y, and adjoint state z satisfy

$$u=y|_\Gamma=\frac{1}{\gamma}\frac{\partial z}{\partial \mathbf{n}}\Big|_\Gamma.$$

Here, we consider two settings, $\gamma=1$ and $\gamma=0.01$, of the regularization parameter.

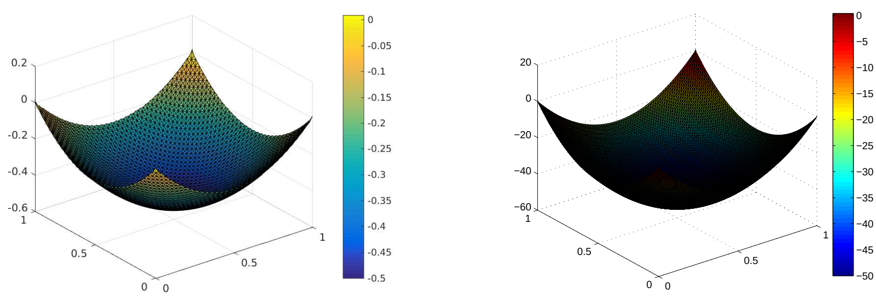

FIG. 7.1. *Left: regularization parameter $\gamma=1$, an approximation to the state variable y over the mesh with 8192 elements generated by uniform refinement of iterations 5. Right: regularization parameter $\gamma=0.01$, an approximation to the state variable y over the mesh with 32768 elements generated by uniform refinement of iterations 6.*

We start with an initial mesh consisting of 8 congruent right triangles. Figure 7.1 reports an approximation solution of the state variable y over the mesh with 8192 elements, which are generated by uniform refinement of iterations 5 for regularization parameter $\gamma=1$ (left), and over the mesh with 32768 elements, which are generated

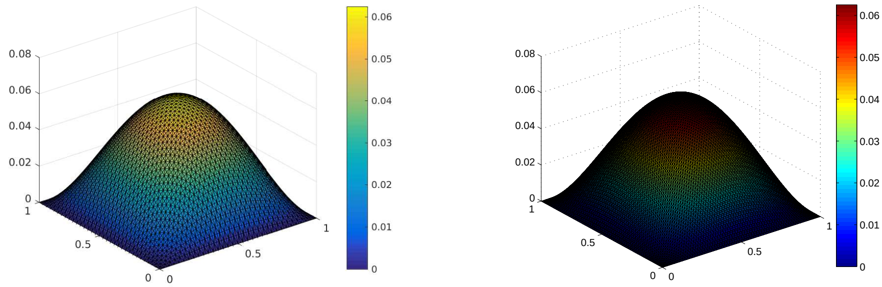

FIG. 7.2. *An approximation to the adjoint state z over the mesh with 8192 elements for $\gamma=1$ (left) and over the mesh with 32768 elements for $\gamma=0.01$ (right).*

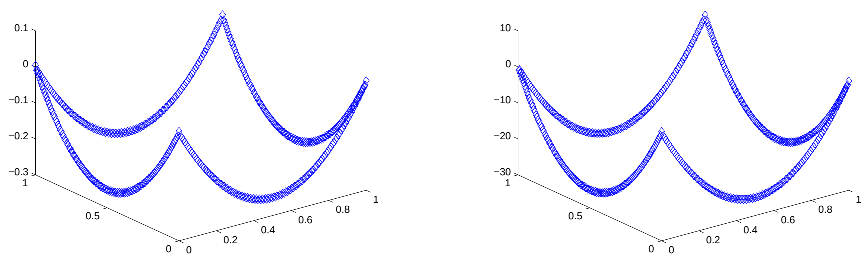

FIG. 7.3. *An approximation to the control variable u, i.e., a restriction of y_h to the boundary Γ, over the mesh with 32768 elements generated by uniform refinement of iterations 6 for regularization parameter $\gamma=1$ (left) and $\gamma=0.01$ (right).*

by uniform refinement of iterations 6 for regularization parameter $\gamma=0.01$ (right). In Figure 7.2, we depict the pictures of an approximation solution of the adjoint state z over the mesh with 8192 elements for $\gamma=1$ (left) and over the mesh with 32768 elements for $\gamma=0.01$ (right). Figure 7.3 shows a restriction (which is regarded as an approximation solution of the control variable u) of y_h on the boundary over the mesh with 32768 elements for $\gamma=1$ (left) and for $\gamma=0.01$ (right).

Table 7.1 shows, respectively, the exact errors $\|\nabla(y-y_h)\|, \|\nabla(z-z_h)\|$ and $\|u-u_h\|_{0,\Gamma}$ for the regularization parameter $\gamma=1$. It is observed that they have the rate of convergence of order one for linear element, which is order half higher than theoretical results. Table 7.2 reports the true errors of the state and adjoint state in L^2 norm for $\gamma=1$. It can be seen that $||y-y_h||$ has the rate of convergence of order 1.5 at least, and that the speed of convergence of $||z-z_h||$ is close to 2. Table 7.3 provides the exact errors of $||y-y_h||, ||\nabla(z-z_h)||$ and $||u-u_h||_{0,\Gamma}$ for the regularization parameter $\gamma=0.01$, and a similar rate of convergence as that for $\gamma=1$ can be observed.

In addition, comparing Table 7.1 with Table 7.2, we can see that the speed of convergence of $||y-y_h||$ is order half higher than $||\nabla(y-y_h)||$, and that the rate of convergence of $||z-z_h||$ is order one higher than $||\nabla(z-z_h)||$.

7.2. Example two. We consider a 2D example over a square domain $\Omega=(0,1/4)\times(0,1/4)\subset\mathbb{R}^2$. The data is chosen as

$$f=0, \quad y_d=\left(x_1^2+x_2^2\right)^s,$$

h	$\|\|\nabla(y-y_h)\|\|$	order$_y$	$\|\|\nabla(z-z_h)\|\|$	order$_z$	$\|\|u-u_h\|\|_{0,\Gamma}$	order$_u$
0.7071	0.7187	–	0.1069	–	0.1901	–
0.3536	0.3603	0.9964	0.0539	0.9881	0.0663	1.5200
0.1768	0.1928	0.9021	0.0278	0.9552	0.0345	0.9424
0.0884	0.0898	1.1023	0.0140	0.9897	0.0154	1.1637
0.0442	0.0446	1.0097	0.0070	1.000	0.0066	1.2224

TABLE 7.1. *Numerical data of $\gamma=1$ for Example 1: h – maximum size of quasi-uniform mesh; $||\nabla(y-y_h)||$ – numerical error for the state variable y; order$_y$ – the speed of convergence for y; $||\nabla(z-z_h)||$ – numerical error for the adjoint state variable z; order$_z$ – the speed of convergence for z; $||u-u_h||_{0,\Gamma}$ – numerical error for the control variable u; order$_u$ – the speed of convergence for u.*

h	0.7071	0.3536	0.1768	0.0884	0.0442	0.0221
$\|\|y-y_h\|\|$	0.0897	0.0250	0.0078	0.0025	7.86e-004	2.55e-004
order$_y$	–	1.8436	1.6804	1.6415	1.6686	1.6259
$\|\|z-z_h\|\|$	0.0181	0.0054	0.0014	3.55e-004	8.74e-005	2.10e-005
order$_z$	–	1.7453	1.9475	1.9775	2.0234	2.0604

TABLE 7.2. *Numerical data of $\gamma=1$ for Example 1: h – maximum size of quasi-uniform mesh; $||y-y_h||$ – numerical error for the state variable y;* order$_y$ *– the speed of convergence for y in L^2 norm; $||z-z_h||$ – numerical error for the adjoint state variable z;* order$_z$ *– the speed of convergence for z in L^2 norm.*

h	$\|\|y-y_h\|\|$	order$_y$	$\|\|\nabla(z-z_h)\|\|$	order$_z$	$\|\|u-u_h\|\|_{0,\Gamma}$	order$_u$
0.7071	2.9637	–	0.1134	–	9.0693	–
0.3536	0.7594	1.9649	0.0561	1.0156	2.5525	1.8295
0.1768	0.2101	1.8538	0.0279	1.0077	0.8519	1.5832
0.0884	0.0663	1.6640	0.0140	0.9948	0.3384	1.3320
0.0442	0.0191	1.7954	0.0070	1.000	0.1187	1.5114

TABLE 7.3. *Numerical data of $\gamma=0.01$ for Example 1: h – maximum size of quasi-uniform mesh; $||y-y_h||$ – numerical error for the state variable y; order$_y$ – the speed of convergence for y in L^2 norm; $||\nabla(z-z_h)||$ – numerical error for the adjoint state variable z; order$_z$ – the speed of convergence for z; $||u-u_h||_{0,\Gamma}$ – numerical error for the control variable u; order$_u$ – the speed of convergence for u.*

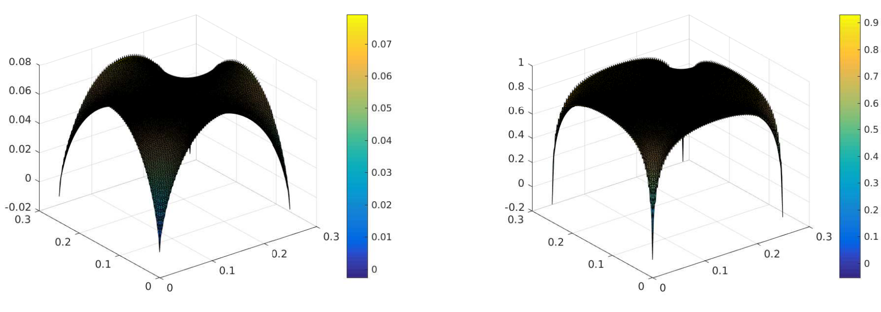

FIG. 7.4. *An approximation solution to the state variable y over the mesh generated by uniform refinement of iteration 6 (with 32768 elements) for the regularization parameter $\gamma=1$ (left) and $\gamma=0.01$ (right).*

where $s=10^{-5}$. Since we do not have an explicit expression for the exact solution, the "reference solution" has been calculated over a fine mesh with 131072 elements. Here, we also consider two settings, $\gamma=1$ and $\gamma=0.01$, of the regularization parameter.

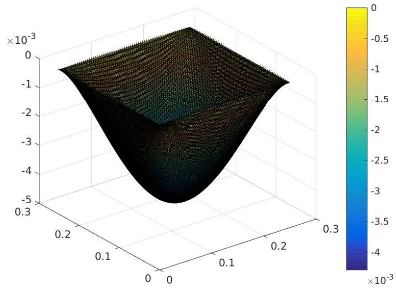

FIG. 7.5. *An approximation solution to the adjoint state z over the mesh generated by uniform refinement of iteration 6 (with 32768 elements) for the regularization parameter $\gamma=1$ (left) and $\gamma=0.01$ (right).*

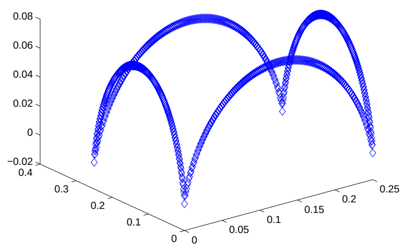

FIG. 7.6. *An approximation to the control variable u, i.e., the restriction of y_h on the boundary Γ, over the mesh generated by uniform refinement of iteration 7 (with 131072 elements) for the regularization parameter $\gamma=1$ (left) and $\gamma=0.01$ (right).*

h	$\|\|y-y_h\|\|$	order$_y$	$\|\|u-u_h\|\|_{0,\Gamma}$	order$_u$	$\|\|\nabla(y-y_h)\|\|_{0,\Gamma}$	order$_{\nabla y}$
0.1768	0.0117	–	0.3212	–	0.4462	–
0.0884	0.0034	1.7833	0.1663	0.9497	0.2596	0.7812
0.0442	0.0011	1.6280	0.0851	0.9659	0.1683	0.6249
0.0221	3.61e-004	1.6078	0.0445	0.9356	0.1106	0.6057
0.0111	1.18e-004	1.6185	0.0236	0.9178	0.0723	0.6124

TABLE 7.4. *Numerical data of $\gamma=1$ for Example 2: h – maximum size of quasi-uniform mesh; $||y-y_h||$ – numerical error for the state variable y in L^2 norm; order$_y$ – the speed of convergence for y ; $||u-u_h||_{0,\Gamma}$ – numerical error for the control variable u; order$_u$ – the speed of convergence for u; $||\nabla(y-y_h)||$ – numerical error for the state variable y in H^1 seminorm; order$_{\nabla y}$ – the speed of convergence for y in H^1 seminorm*

We still start with an initial mesh consisting of 8 congruent right triangles. Figures 7.4 and 7.5 show an approximation solution to the state y and adjoint state z over the mesh generated by uniform refinement of iteration 6 (with 32768 elements) for two different values of the regularization parameter, namely, $\gamma=1$ (left) and $\gamma=0.01$ (right). Figure 7.6 reports the restriction of an approximation of the state on the boundary, i.e., an approximation solution of the control u, over the mesh generated by uniform refinement of iteration 7 (with 131072 elements) for two different values of the regularization parameter, namely, $\gamma=1$ (left) and $\gamma=0.01$ (right).

From Figures 7.4 and 7.6, we observe that the control changes quickly at the four corners of the boundary Γ. Furthermore, we remark that the control for the regularization parameter $\gamma=0.01$ changes more sharply at the four corners of the boundary than for $\gamma=1$, and that the singularity of the exact solution for $\gamma=0.01$ is stronger than for $\gamma=1$.

From Table 7.4, we observe that the numerical error of the state y in L^2 norm has the speed of convergence of order 1.6, and that the speed of convergence of $||u-u||_{0,\Gamma}$ is close to one. However, the numerical error $||\nabla(y-y_h)||$ has a slow speed of convergence, this is due to the low regularity of the exact solutions. In fact, the exact control u has strong singularity at four corners of the boundary, which indicates that adaptive mesh based on a posteriori error estimator is efficient for this type of problems; we refer to the articles [1, 5, 6, 9, 17, 18, 26, 28–31, 35, 36] about adaptive finite element methods on the base of a posteriori error estimates.

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