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CUCKER-SMALE FLOCKING UNDER HIERARCHICAL LEADERSHIP WITH TIME-DELAY AND A FREE-WILL LEADER*

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Abstract. We study the discrete-time Cucker-Smale model under hierarchical leadership with a constant time delay. The overall leader of the flock is assumed to have a free-will acceleration. The strength of the interaction is measured by a parameter $\beta \geq 0$. Under some suitable constraints on the acceleration, we prove that unconditional convergence to flocking can be achieved for arbitrarily large constant delay as long as $\beta \leq \frac{1}{2}$. The convergence rate is also provided depending on the leader's acceleration.

Keywords. Cucker-Smale model; flocking; hierarchical leadership; time delay; free-will.

AMS subject classifications. 39A12; 34K25; 93A13.

1. Introduction

Collective behaviors in many-body systems are ubiquitous in our nature. The terminology "*flocking*" represents a collective phenomenon in which the ensemble of flocking particles (agents) organize into an ordered motion using only limited information and simple rules. The typical examples of flocking include flocking of birds, schooling of fish and herding of sheep. The flocking problem has gained considerable attention from researchers in diverse disciplines such as computer science, biology, physics, control engineering, etc. [4, 20, 21, 24, 26]. Mathematical abstraction and rigorous analysis are acknowledged as the foundation for the study of the flocking problem. After the seminal work by Vicsek *et al.* [27], several mathematical models for flocking have been proposed in the literature, see, e.g., the survey papers [1, 28]. Among them, our focus in this paper lies on the flocking model introduced by Cucker and Smale [7] in 2007.

Let us first recall the original flocking model in [7]. Consider a flock of k agents moving in the Euclidean space $\mathbb{R}^d$. Let x_i and v_i be the position and velocity of the i-th agent, respectively. The discrete-time Cucker-Smale (C-S, for short) model is given by

$$\begin{cases} x_i[n+1] = x_i[n] + h v_i[n], \ n \in \mathbb{N}, \\ v_i[n+1] = h \sum_{j=1}^{k} a_{ij}(x[n])(v_j[n] - v_i[n]) + v_i[n], \end{cases}$$

where h is the time step, $x_i[n] := x_i(hn)$, $v_i[n] := v_i(hn)$, and the weight function $a_{ij}(x)$ quantifies the influence of agent j over i. In [7], it takes the form

$$a_{ij}(x) = \frac{H}{(1 + \|x_i - x_j\|^2)^\beta}, \tag{1.1}$$

where $H > 0$ and $\beta \geq 0$ are system parameters, and $\|\cdot\|$ denotes the standard 2-norm in $\mathbb{R}^d$. The corresponding continuous-time model is also studied in [7]. The weight $a_{ij}(x)$

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is a decreasing function of the distance between agents. A notable feature of the C-S model is that convergence to flocking can be established depending on the initial state of the flock and system parameters only. More precisely, emergence of flocking exhibits a threshold phenomenon, i.e., convergence to flock occurs for any initial state when $\beta \leq 1/2$, while for $\beta > 1/2$, such convergence occurs only for some restricted class of initial configurations. Extensions of the Cucker-Smale results have been made in different perspectives, e.g., avoiding collisions [5,30], kinetic and hydrodynamic description [2,15], noisy effects [14,16], directed networks [6,10,13], mean-field limit [18] and time-delay effects [3,11,29], etc.

In 2007, a generalization of the C-S model was made in [25] by incorporating the structure of *hierarchical leadership* (HL) into the model. A detailed discussion for background on hierarchy can be found in [31]. In contrast with the all-to-all interactions in [7], the HL structure leads to that the weights $a_{ij}(x)$ are zero for certain pairs (i,j). In particular, Shen [25] developed two elegant methods to deal with the discrete- and continuous-time cases, respectively. Further extensions of the discrete-time C-S model under HL were considered in [8,9,17]. In real applications, time delay is inevitable, and it can cause oscillation, divergence, and even instability. The effect of time delays on the C-S model has been studied in the literature. For example, in [19,22], the C-S model with time-delays under all-to-all interactions was considered. The effect of time-delay on the C-S model under HL was further studied in [23]. More precisely, it was proved in [23] that for the continuous-time C-S model under HL, flocking is achieved under the same conditions as in the undelayed case when communication delay appears in velocity information, and processing delay appears in position information. The continuous dynamics with delay under hierarchical leadership of (x_i,v_i) is governed by the following continuous system:

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \displaystyle\sum_{j\in\mathcal{L}(i)} a_{ij}(x(t-\tau))(v_j(t-\tau)-v_i(t)), \end{cases} \tag{1.2}$$

where $a_{ij}(x)$ is defined as in (1.1) . In this paper, we are interested in the asymptotic flocking behavior of discretization of (1.2). Therefore, it is very natural to ask whether similar results can be rigorously proved for the corresponding discrete-time model.

We provide a positive answer to the above questions in the present paper. To be precise, we discretize the continuous-time system (1.2) by the one-step forward Euler scheme. We assume that the overall leader can have a free-will acceleration instead of merely moving with a constant velocity. Under some growth assumption on the free-will acceleration, we prove that when $\beta \leq 1/2$, convergence to flocking occurs for all initial states and transmission delays. Our result shows the robustness of flocking in the studied discrete-time model with respect to time delay, similar to the results for continuous-time model in [23].

The rest of this paper is organized as follows. In Section 2, we introduce the discrete-time C-S model under HL to be studied and present our main result. In Section 3, we provide the proof of our flocking result. Finally, we conclude the paper in Section 4.

2. Preliminaries and main result

Before we state our main result, we first introduce the studied model.

2.1. The model. We recall the formal definition of hierarchical leadership first introduced in [25].

DEFINITION 2.1. *A flock of* $(k+1)$ *agents labeled* $\{0,1,\dots,k\}$ *is said to be under* hierarchical leadership *if, for all* $x\in(R^d)^{k+1}$*, the adjacency matrix* $A_x=(a_{ij}(x))$ *satisfies:*

(i) $a_{ij}\neq 0$ *implies that* $j<i$*; and*

(ii) *for all* $i>1$*, the set* $\mathcal{L}(i)=\{j:\ a_{ij}>0\}$ *is non-empty.*

We call $\mathcal{L}(i)$ *the leader set of agent* i*. Such a flock is called an* HL-flock.

We see that agent 0 is not influenced by any other agents, and we call it the overall leader of the HL flock. For an HL-flock $\{0,1,\dots,k\}$, we assume that the agents need time to process perceived information, which is denoted by a constant transmission delay $\tau\in\mathbb{N}$. In discrete time setting, for $1\le i\le k$, the dynamics of agent i is specified by

$$\begin{cases} x_i[n+1]=x_i[n]+hv_i[n],\ n\in\mathbb{N},\\ v_i[n+1]=h\displaystyle\sum_{j\in\mathcal{L}(i)}a_{ij}(x[n-\tau])(v_j[n-\tau]-v_i[n])+v_i[n],\\ x_i[n]=x_i^{in}[n],\ v_i[n]=v_i^{in}[n],\ -\tau\le n\le 0,\end{cases}\tag{2.1}$$

where the weight function $a_{ij}(x[n-\tau])$ is defined as in (1.1), i.e.,

$$a_{ij}(x[n-\tau])=\frac{H}{(1+\|x_i[n-\tau]-x_j[n-\tau]\|^2)^\beta}.\tag{2.2}$$

We assume that the overall leader agent 0 has a free-will acceleration and its behavior is given by

$$\begin{cases} x_0[n+1]=x_0[n]+hv_0[n],\ n\in\mathbb{N},\\ v_0[n+1]=v_0[n]+hf[n].\\ x_0[n]=x_0^{in}[n],\ v_0[n]=v_0^{in}[n],\ -\tau\le n\le 0.\end{cases}\tag{2.3}$$

For the case without delay, i.e., $\tau=0$, system (2.1) becomes the classical C-S model under HL, which has been studied in the literature, see, e.g., [8, 9, 25]. Before we state our main result, we first give the concept of flocking. In what follows, we fix a solution $\{(x_i[n],v_i[n])\}_{i=0}^k$ of systems (2.1) and (2.3).

DEFINITION 2.2. We say that the HL-flock $\{0,1,\dots,k\}$ converges to flocking asymptotically if the following conditions hold:

$$\lim_{n\to\infty}\max_{0\le i,j\le k}\|v_i[n]-v_j[n]\|=0,\qquad \sup_{0\le n<\infty}\max_{0\le i,j\le k}\|x_i[n]-x_j[n]\|<\infty.$$

We are now in a position to state our main result.

THEOREM 2.1. *Consider the HL-flock* $\{0,1,\dots,k\}$ *with* $\beta\le 1/2$ *in* (2.2)*. Assume that* h *satisfies* $0<h\le\frac{1}{kH}$ *and that the free-will acceleration* $f[n]$ *is such that*

$$\|f[n]\|=O((1+n)^{-p}),\quad p>k.\tag{2.4}$$

Then the HL-flock converges to flocking asymptotically. In addition, we have

$$\max_{0\le i,j\le k}\|v_i[n]-v_j[n]\|=O((1+n)^{-p+k}).$$

REMARK 2.1.
(1) We see that Theorem 2.1 shows that when $\beta \le 1/2$, convergence to flocking occurs independently of the initial state of the flock. Our result strengthens the corresponding undelayed result in [8, 9, 25], where flocking occurs unconditionally for $\beta < 1/2$. This shows that the flocking behavior in systems (2.1) and (2.3) is robust with respect to the time delay τ. In addition, when the free-will acceleration $f[n]$ satisfies

$$f[n] = O(e^{-pn}), \quad p > 0,$$

then we can use the same arguments to obtain

$$\max_{0 \le i,j \le k} \|v_i[n] - v_j[n]\| = O\left(e^{-\tilde{p}n}\right) \text{ with some } \tilde{p} > 0.$$

(2) By the same arguments, Theorem 2.1 also holds true for the case with a bounded time-varying delay $\tau[n]$.

3. Proof of Theorem 2.1

In this section, we present the proof of Theorem 2.1. To this end, we next prove some stepping stones towards the proof.

LEMMA 3.1. *Suppose that $0 < h \le \frac{1}{kH}$ and (2.4) holds. Then we have for all $n \ge 0$ and $0 \le i \le k$,*

$$\|v_i[n]\| \le V_0,$$

where the constant V_0 is defined as

$$V_0 := \max\left\{ \max_{\substack{0 \le i \le k \\ -\tau \le m \le 0}} \|v_i^{in}[m]\|,\ \|v_0^{in}[0]\| + h\sum_{m=0}^{\infty} \|f[m]\| \right\}.$$

Proof. For the overall leader agent 0, we observe that for all $n \in \mathbb{N}$,

$$\begin{aligned}
\|v_0[n]\| &= \|v_0[n-1] + hf[n-1]\| \\
&\le \|v_0[n-1]\| + h\|f[n-1]\| \\
&\le \|v_0^{in}[0]\| + h\sum_{m=0}^{n-1} \|f[m]\| \\
&\le V_0.
\end{aligned}$$

This shows that the statement holds true for agent 0. For $1 \le i \le k$, we next show that the statement holds for every time n by induction on $n \ge 0$. It is trivially true when $n = 0$. Assume that it holds true for $n \in \{0, 1, 2, ..., m-1\}$. For $n = m$, it follows from $(2.1)_2$ that

$$\begin{aligned}
\|v_i[m]\| &= \left\| h\sum_{j \in \mathcal{L}(i)} a_{ij}(x[m-1-\tau])(v_j[m-1-\tau] - v_i[m-1]) + v_i[m-1] \right\| \\
&= \Bigg\| h\sum_{j \in \mathcal{L}(i)} a_{ij}(x[m-1-\tau])v_j[m-1-\tau] \\
&\qquad + \left(1 - h\sum_{j \in \mathcal{L}(i)} a_{ij}(x[m-1-\tau])\right) v_i[m-1] \Bigg\|
\end{aligned}$$

$$
\begin{aligned}
&\le h \sum_{j\in\mathcal{L}(i)} a_{ij}(x[m-1-\tau])\|v_j[m-1-\tau]\| \\
&\qquad + \left(1-h\sum_{j\in\mathcal{L}(i)} a_{ij}(x[m-1-\tau])\right)\|v_i[m-1]\| \\
&\le V_0,
\end{aligned}
$$

where the first inequality is due to

$$1-h\sum_{j\in\mathcal{L}(i)} a_{ij}(x[m-1-\tau]) \ge 1-khH \ge 0,$$

and the last one follows by induction hypothesis. Thus, we complete the proof. ∎

The following proposition considers the case of two agents.

PROPOSITION 3.1. *Assume that $x[n], v[n] \in R^d$ (corresponding to $x_1[n]-x_0[n]$ and $v_1[n]-v_0[n]$ for an HL-flocking with two agents) satisfy the system*

$$
\begin{cases}
x[n+1] = x[n] + hv[n], \ n \ge \tau, \\
v[n+1] = (1-ha(x[n-\tau],n))v[n] + \varepsilon[n], \\
x[n] = x^{in}[n], \ v[n] = v^{in}[n], \ 0 \le n \le \tau,
\end{cases}
\tag{3.1}
$$

where $a(x,n)$, $v[n]$ and $\varepsilon[n]$ satisfy the conditions

(1) $\frac{R}{(M^2+\|x[n-\tau]\|^2)^\beta} \le a(x[n-\tau],n) \le K \le \frac{1}{h}$, *for some $R>0$, $M>0$, $K>0$, and $\beta \le \frac{1}{2}$;*

(2) $\|v[n]\| \le D_0$, *for $D_0 > 0$ and all $n \ge 0$;*

(3) $\|\varepsilon[n]\| = O((1+n)^{-\mu})$ *with $\mu > 1$, i.e., $\exists\ A > 0$ such that $\|\varepsilon[n]\| \le A(1+n)^{-\mu}$ for all $n \ge \tau$.*

Then, there exists $B_0 > 0$ such that $\|x[n]\| \le B_0$ for all $n \ge 0$ and $\|v[n]\| = O((1+n)^{-\mu+1})$.

We next present two lemmas to be used in the proof of Proposition 3.1.

LEMMA 3.2. *Under the hypotheses of Proposition 3.1, we have, for all $n \ge \tau$,*

$$\|v[n+1]\| \le \left(1-h\psi\left(\sqrt{2}\|x[n]\|\right)\right)\|v[n]\| + \|\varepsilon[n]\|,$$

where

$$\psi(r) = \frac{R}{(M^2+2C_\tau^2+r^2)^\beta} \tag{3.2}$$

with

$$C_\tau := h\tau V_0 + \max_{0\le m<\tau} \|x^{in}[m]\|.$$

Proof. It follows from $(3.1)_2$ that

$$
\begin{aligned}
\|v[n+1]\| &= \|(1-ha(x[n-\tau],n))v[n] + \varepsilon[n]\| \\
&\le (1-ha(x[n-\tau],n))\|v[n]\| + \|\varepsilon[n]\|,
\end{aligned}
\tag{3.3}
$$

where we use the fact that $ha(x[n-\tau],n) \leq hK \leq 1$ by hypothesis (1) in Proposition 3.1. On the other hand, we use $(3.1)_1$ to see that for $n \geq 2\tau$,

$$\begin{aligned} x[n] &= x[n-1] + hv[n-1] \\ &= x[n-2] + hv[n-2] + hv[n-1] \\ &= x[n-\tau] + h\sum_{m=1}^{\tau} v[n-m], \end{aligned}$$

which, together with Lemma 3.1, implies that for $n \geq 2\tau$,

$$\begin{aligned} \|x[n-\tau]\| &= \|x[n] - h\sum_{m=1}^{\tau} v[n-m]\| \\ &\leq \|x[n]\| + h\tau V_0. \end{aligned}$$

Thus, we obtain that for all $n \geq \tau$,

$$\begin{aligned} \|x[n-\tau]\| &\leq \|x[n]\| + h\tau V_0 + \max_{0\leq m<\tau} \|x^{in}[m]\| \\ &= \|x[n]\| + C_\tau. \end{aligned}$$

We combine the above inequality and hypothesis (1) of Proposition 3.1 to see that

$$\begin{aligned} a(x[n-\tau],n) &\geq \frac{R}{(M^2 + \|x[n-\tau]\|^2)^\beta} \\ &\geq \frac{R}{\left(M^2 + (\|x[n]\| + C_\tau)^2\right)^\beta} \\ &\geq \frac{R}{(M^2 + 2\|x[n]\|^2 + 2C_\tau^2)^\beta} \\ &= \psi\left(\sqrt{2}\|x[n]\|\right). \end{aligned} \tag{3.4}$$

We then complete the proof by substituting (3.4) into (3.3). ∎

Motivated directly by the nonlinear functionals introduced in [12], we define the functional $\mathcal{L}[n]$ as follows:

$$\mathcal{L}[n] := \|v[n]\| + \sum_{i=1}^{n} \psi\left(\sqrt{2}\|x[i-1]\|\right)(\|x[i]\| - \|x[i-1]\|), \quad n \geq \tau,$$

where $\psi(r)$ is given by (3.2).

LEMMA 3.3. *Under the hypotheses of Proposition 3.1, we have, for all $n \geq \tau$,*

$$\mathcal{L}[n+1] \leq \mathcal{L}[n] + \|\varepsilon[n]\|.$$

Proof. It follows from Lemma 3.2 and $(3.1)_1$ that we have for $n \geq \tau$,

$$\begin{aligned} &\mathcal{L}[n+1] - \mathcal{L}[n] \\ &= \|v[n+1]\| - \|v[n]\| + \psi\left(\sqrt{2}\|x[n]\|\right)(\|x[n+1]\| - \|x[n]\|) \\ &\leq -h\psi\left(\sqrt{2}\|x[n]\|\right)\|v[n]\| + \|\varepsilon[n]\| + \psi\left(\sqrt{2}\|x[n]\|\right)(\|x[n+1]\| - \|x[n]\|) \end{aligned}$$

$$
\begin{aligned}
&= -\psi\left(\sqrt{2}\|x[n]\|\right)\|x[n+1]-x[n]\| + \|\varepsilon[n]\| + \psi\left(\sqrt{2}\|x[n]\|\right)(\|x[n+1]\| - \|x[n]\|) \\
&\le -\psi\left(\sqrt{2}\|x[n]\|\right)\|x[n+1]-x[n]\| + \|\varepsilon[n]\| + \psi\left(\sqrt{2}\|x[n]\|\right)\|x[n+1]-x[n]\| \\
&= \|\varepsilon[n]\|.
\end{aligned}
$$

□

The following lemma gives a useful lower bound for $\mathcal{L}[n]$ in terms of the integral of ψ.

LEMMA 3.4. *For each $n \ge \tau$, we have*

$$\mathcal{L}[n] \ge \|v[n]\| + \int_{\|x[0]\|}^{\|x[n]\|} \psi(\sqrt{2}r)dr.$$

Proof. The proof is the same as that for [12, Proposition 3.2]. To avoid repetition, we omit it here. □

Proof. **(Proof of Proposition 3.1.)** First, we apply Lemma 3.3 and hypothesis (3) to derive that for all $n \ge \tau$,

$$\mathcal{L}[n+1] \le \mathcal{L}[\tau] + \sum_{m=\tau}^{n} \|\varepsilon[m]\| \le \mathcal{L}[\tau] + \sum_{m=\tau}^{\infty} \|\varepsilon[m]\| < \infty.$$

That is, $\mathcal{L}[n]$ is uniformly bounded with n. Recalling the definition of (3.2) with $\beta \le \frac{1}{2}$, we can always find a constant $U_0 > 0$ such that

$$\mathcal{L}[\tau] + \sum_{m=\tau}^{\infty} \|\varepsilon[m]\| = \int_{\|x[0]\|}^{U_0} \psi(\sqrt{2}r)dr.$$

This equality and Lemma 3.4 yield that for $n \ge \tau$,

$$\int_{\|x[0]\|}^{\|x[n]\|} \psi(\sqrt{2}r)dr \le \int_{\|x[0]\|}^{U_0} \psi(\sqrt{2}r)dr.$$

Therefore, we have $\|x[n]\| \le B_0 := \max\{U_0, \max_{0 \le m < \tau} \|x^{in}[m]\|\}$ for all $n \ge 0$. For the estimate of velocity decay, we use Lemma 3.1 and Lemma 3.2 to see that for $n \ge \tau$,

$$
\begin{aligned}
\|v[n+1]\| &\le \left(1 - h\psi\left(\sqrt{2}\|x[n]\|\right)\right)\|v[n]\| + \|\varepsilon[n]\| \\
&\le \left(1 - h\psi\left(\sqrt{2}B_0\right)\right)\|v[n]\| + \|\varepsilon[n]\| \\
&\le \left(1 - h\psi\left(\sqrt{2}B_0\right)\right)^{\lceil \frac{n}{2} \rceil}\left\|v\left[n - \left\lceil \frac{n}{2} \right\rceil\right]\right\| + \sum_{m=1}^{\lceil \frac{n}{2} \rceil}\left(1 - h\psi\left(\sqrt{2}B_0\right)\right)^{m-1}\|\varepsilon[n-m]\| \\
&\le \left(1 - h\psi\left(\sqrt{2}B_0\right)\right)^{\lceil \frac{n}{2} \rceil} V_0 + \sum_{m=1}^{\lceil \frac{n}{2} \rceil}\|\varepsilon[n-m]\| \\
&\le \left(1 - h\psi\left(\sqrt{2}B_0\right)\right)^{\lceil \frac{n}{2} \rceil} V_0 + \sum_{m=\lceil \frac{n}{2} \rceil - 1}^{n-1}\|\varepsilon[m]\| \\
&= O((1+n)^{-\mu+1}),
\end{aligned}
$$

where the last equality follows from hypothesis (3) of Proposition 3.1. This completes the proof. ∎

We are now in a position to provide the proof of Theorem 2.1.

Proof. **(Proof of Theorem 2.1.)** We prove the statement by induction on sub-flock $\{0,1,\dots,\ell\}$ with $\ell=1,\dots,k$.

• Initial step. For the sub-flock $\{0,1\}$. From the definition of HL-flock (Definition 2.1), we know that $\mathcal{L}(1)=\{0\}$, and by (2.3) we have for $n\geq\tau$,

$$v_0[n]=v_0[n-1]+hf[n-1]=v_0[n-\tau]+h\sum_{m=n-\tau}^{n-1}f[m]. \tag{3.5}$$

Let $x[n]=x_1[n]-x_0[n]$ and $v[n]=v_1[n]-v_0[n]$. Then we have

$$\begin{aligned}x[n+1]&=x_1[n+1]-x_0[n+1]\\&=x_1[n]+hv_1[n]-x_0[n]-hv_0[n]\\&=x[n]+hv[n],\end{aligned}$$

and by (3.5),

$$\begin{aligned}v[n+1]&=v_1[n+1]-v_0[n+1]\\&=ha_{10}(x[n-\tau])(v_0[n-\tau]-v_1[n])+v_1[n]-v_0[n]-f[n]h\\&=ha_{10}(x[n-\tau])\left(v_0[n]-h\sum_{m=n-\tau}^{n-1}f[m]-v_1[n]\right)+v[n]-f[n]h\\&=(1-ha_{10}(x[n-\tau]))v[n]-h\left(ha_{10}(x[n-\tau])\sum_{m=n-\tau}^{n-1}f[m]+f[n]\right).\end{aligned}$$

Combining the above two equations, we obtain

$$\begin{cases}x[n+1]=x[n]+hv[n]\\v[n+1]=(1-ha_{10}(x[n-\tau]))v[n]+\varepsilon[n]\end{cases} \tag{3.6}$$

where

$$a_{10}(x[n-\tau])=\frac{H}{(1+\|x_1[n-\tau]-x_0[n-\tau]\|^2)^\beta},$$

and

$$\varepsilon[n]=-h\left(ha_{10}(x[n-\tau])\sum_{m=n-\tau}^{n-1}f[m]+f[n]\right).$$

By Lemma 3.1, we see that $\|v[n]\|\leq\|v_1[n]\|+\|v_0[n]\|\leq 2V_0$ for all $n\geq 0$. We apply Proposition 3.1 with $R=K=H$, $M=1$, $D_0=2V_0$ and $\mu=p$ to establish the uniform boundedness of $\|x[n]\|$ and $\|v[n]\|=O\left((1+n)^{-p+1}\right)$.

• Induction step. Suppose that the statement holds for the sub-flock $\{0,1,\dots,\ell-1\}$ with $2\leq\ell\leq k$. We will show that it holds as well for the sub-flock $\{0,1,\dots,\ell-1,\ell\}$. We first note that the induction hypothesis implies that

$$\max_{0\leq i,j\leq \ell-1}\|v_i[n]-v_j[n]\|=O\left((1+n)^{-p+\ell-1}\right), \tag{3.7}$$

which implies that

$$
\begin{aligned}
&\max_{0\le i,j\le \ell-1}\|v_i[n]-v_j[n-\tau]\|\\
&\le \max_{0\le i,j\le \ell-1}\left(\|v_0[n]-v_j[n-\tau]\|+\|v_i[n]-v_0[n]\|\right)\\
&=\max_{0\le j\le \ell-1}\left\|v_0[n-\tau]+h\sum_{m=n-\tau}^{n-1}f[m]-v_j[n-\tau]\right\|+\max_{0\le i\le \ell-1}\|v_i[n]-v_0[n]\|\\
&\le \max_{0\le j\le \ell-1}\|v_0[n-\tau]-v_j[n-\tau]\|+h\sum_{m=n-\tau}^{n-1}\|f[m]\|+\max_{0\le i\le \ell-1}\|v_i[n]-v_0[n]\|\\
&=O\left((1+n)^{-p+\ell-1}\right).
\end{aligned}
\tag{3.8}
$$

Consider the average velocity of the leaders of agent ℓ:

$$
\bar{v}_\ell[n]=\frac{1}{d_\ell}\sum_{i\in\mathcal{L}(\ell)}v_i[n]
$$

where d_ℓ is the cardinality of the set $\mathcal{L}(\ell)$, i.e., $d_\ell=\#\mathcal{L}(\ell)$. For each $1\le j\le \ell-1$, $n\ge\tau$, it follows from (3.8) that

$$
\begin{aligned}
\|v_j[n-\tau]-\bar{v}_\ell[n]\|&=\left\|\frac{1}{d_\ell}\sum_{i\in\mathcal{L}(\ell)}(v_j[n-\tau]-v_i[n])\right\|\\
&\le\frac{1}{d_\ell}\sum_{i\in\mathcal{L}(\ell)}\|v_j[n-\tau]-v_i[n]\|\\
&=O\left((1+n)^{-p+\ell-1}\right).
\end{aligned}
\tag{3.9}
$$

Similarly, define

$$
\bar{x}_\ell[n]=\frac{1}{d_\ell}\sum_{i\in\mathcal{L}(\ell)}x_i[n]
$$

and let $x[n]=x_\ell[n]-\bar{x}_\ell[n]$, $v[n]=v_\ell[n]-\bar{v}_\ell[n]$. Thus,

$$
x[n+1]=x[n]+hv[n]
$$

and

$$
\begin{aligned}
v[n+1]&=v_\ell[n+1]-\bar{v}_\ell[n+1]\\
&=\sum_{j\in\mathcal{L}(\ell)}ha_{\ell j}(x[n-\tau])(v_j[n-\tau]-v_\ell[n])+v_\ell[n]-\frac{1}{d_\ell}\sum_{i\in\mathcal{L}(\ell)}v_i[n+1]\\
&=\sum_{j\in\mathcal{L}(\ell)}ha_{\ell j}(x[n-\tau])(\bar{v}_\ell[n]-v_\ell[n])+\sum_{j\in\mathcal{L}(\ell)}ha_{\ell j}(x[n-\tau])(v_j[n-\tau]-\bar{v}_\ell[n])\\
&\quad+v_\ell[n]-\frac{1}{d_\ell}\sum_{i\in\mathcal{L}(\ell)}\left(\sum_{j\in\mathcal{L}(i)}ha_{ij}(x[n-\tau])(v_j[n-\tau]-v_i[n])+v_i[n]\right)\\
&=\left(1-\sum_{j\in\mathcal{L}(\ell)}ha_{\ell j}(x[n-\tau])\right)v[n]+\sum_{j\in\mathcal{L}(\ell)}ha_{\ell j}(x[n-\tau])(v_j[n-\tau]-\bar{v}_\ell[n])
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{d_\ell}\sum_{i\in\mathcal{L}(\ell)}\sum_{j\in\mathcal{L}(i)} ha_{ij}(x[n-\tau])(v_j[n-\tau]-v_i[n])\\
&=: (1-ha(x[n-\tau],n))v[n]+\varepsilon[n].
\end{aligned}
$$

Hence we again obtain the system:

$$
\begin{aligned}
x[n+1]&=x[n]+hv[n]\\
v[n+1]&=(1-ha(x[n-\tau],n))v[n]+\varepsilon[n].
\end{aligned}
$$

We need to show that all the hypotheses of Proposition 3.1 are satisfied. It first follows from (3.8) and (3.9) and the fact that $a_{ij}(x)\le H$ that

$$
\|\varepsilon[n]\|=O\left((1+n)^{-p+\ell-1}\right)
$$

with $p-\ell+1>1$. This implies that the hypothesis (3) of Proposition 3.1 is fulfilled.

We next deal with

$$
a(x[n-\tau],n)=\sum_{j\in\mathcal{L}(\ell)} a_{\ell j}(x[n-\tau])=\sum_{j\in\mathcal{L}(\ell)}\frac{H}{(1+\|x_j[n-\tau]-x_\ell[n-\tau]\|^2)^\beta}.
$$

It is clear that $a(x)\le kH\le \frac{1}{h}$ for all $x\in\mathbb{R}^d$. On the other hand, define $g(s)=\frac{H}{(1+s)^\beta}$ with $s\ge 0$. Then $g(s)$ is convex, and

$$
\frac{1}{d_\ell}\sum_{j\in\mathcal{L}(\ell)} g(s_j)\ge g\left(\frac{1}{d_\ell}\sum_{j\in\mathcal{L}(\ell)} s_j\right).
$$

Let $s_j=\|x_j[n-\tau]-x_l[n-\tau]\|^2$. Then

$$
\begin{aligned}
a(x[n-\tau],n)&=\sum_{j\in\mathcal{L}(\ell)}\frac{H}{(1+\|x_j[n-\tau]-x_\ell[n-\tau]\|^2)^\beta}\\
&\ge \frac{d_\ell H}{(1+\frac{1}{d_\ell}\sum_{j\in\mathcal{L}(\ell)}\|x_j[n-\tau]-x_\ell[n-\tau]\|^2)^\beta}.
\end{aligned}\tag{3.10}
$$

By the definition of $\bar{x}_\ell[n]$, we note that

$$
\begin{aligned}
&\frac{1}{d_\ell}\sum_{j\in\mathcal{L}(\ell)}\|x_j[n-\tau]-x_\ell[n-\tau]\|^2\\
&=\|x_\ell[n-\tau]-\bar{x}_\ell[n-\tau]\|^2+\frac{1}{d_\ell}\sum_{j\in\mathcal{L}(\ell)}\|x_j[n-\tau]-\bar{x}_\ell[n-\tau]\|^2\\
&=\|x[n-\tau]\|^2+\frac{1}{d_\ell}\sum_{j\in\mathcal{L}(\ell)}\|x_j[n-\tau]-\bar{x}_\ell[n-\tau]\|^2
\end{aligned}\tag{3.11}
$$

and additionally,

$$
\begin{aligned}
&\frac{1}{d_\ell}\sum_{j\in\mathcal{L}(\ell)}\|x_j[n-\tau]-\bar{x}_\ell[n-\tau]\|^2\\
&\le \frac{1}{d_\ell}\sum_{j\in\mathcal{L}(\ell)}\left(\|x_j(0)-\bar{x}_\ell(0)\|+\sum_{m=0}^{n-\tau-1}\|(x_j[m+1]-\bar{x}_\ell[m+1])-(x_j[m]-\bar{x}_\ell[m])\|\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{d_\ell} \sum_{j \in \mathcal{L}(\ell)} \left(\|x_j(0) - \bar{x}_\ell(0)\| + h \sum_{m=0}^{n-\tau-1} \|v_j[m] - \bar{v}_\ell[m]\| \right)^2 \\
&\overset{(3.9)}{\leq} \frac{1}{d_\ell} \sum_{j \in \mathcal{L}(\ell)} \left(\|x_j(0) - \bar{x}_\ell(0)\| + h \sum_{m=0}^{n-\tau-1} O\left((1+m)^{-p+\ell-1}\right) \right)^2 \\
&=: M_\ell^2 - 1.
\end{aligned} \tag{3.12}
$$

Combining (3.10), (3.11) and (3.12), we have

$$
a(x[n-\tau], n) \geq \frac{d_\ell H}{(M_\ell^2 + \|x[n-\tau]\|^2)^\beta},
$$

which gives the hypothesis (1) of Proposition 3.1. To check the hypothesis (2), we use Lemma 3.1 to see

$$
\begin{aligned}
\|v[n]\| = \|v_l[n] - \bar{v}_\ell[n]\| &= \frac{1}{d_\ell} \left\| \sum_{i \in \mathcal{L}(\ell)} (v_\ell[n] - v_i[n]) \right\| \\
&\leq \frac{1}{d_\ell} \sum_{i \in \mathcal{L}(\ell)} \|v_\ell[n] - v_i[n]\| \\
&\leq \frac{1}{d_\ell} \sum_{i \in \mathcal{L}(\ell)} 2V_0 = 2V_0.
\end{aligned}
$$

Finally, we may apply Proposition 3.1, now with $v_1 = \bar{v}_\ell$, $v_2 = v_\ell$, $K = kH$, $R = d_\ell H$, $M = M_\ell$, $D_0 = 2V_0$, and $\mu = p - \ell + 1$ to establish the uniform boundedness of $\|x[n]\|$ and

$$
\|v[n]\| = \|v_\ell[n] - \bar{v}_\ell[n]\| = O\left((1+n)^{-p+\ell}\right).
$$

Using this bound and (3.9), we deduce that, for any $1 \leq j \leq \ell - 1$.

$$
\begin{aligned}
\|v_j[n] - v_\ell[n]\| &\leq \|v_j[n] - \bar{v}_\ell[n]\| + \|v_\ell[n] - \bar{v}_\ell[n]\| \\
&= O\left((1+n)^{-p+\ell-1}\right) + O\left((1+n)^{-p+\ell}\right) \\
&= O\left((1+n)^{-p+\ell}\right)
\end{aligned}
$$

which, together with (3.7), leads to

$$
\max_{0 \leq i,j \leq \ell} \|v_i[n] - v_j[n]\| = O\left((1+n)^{-p+\ell}\right).
$$

For $0 \leq i, j \leq \ell$, the uniform boundedness of $\|x_i[n] - x_j[n]\|$ follows from induction hypothesis and the relation: $0 \leq j \leq \ell - 1$,

$$
\begin{aligned}
\|x_j[n] - x_\ell[n]\| &\leq \|x_j[n] - \bar{x}_\ell[n]\| + \|\bar{x}_\ell[n] - x_\ell[n]\| \\
&\leq \frac{1}{d_\ell} \sum_{i \in \mathcal{L}(\ell)} \|x_j[n] - x_i[n]\| + \|x[n]\|.
\end{aligned}
$$

This completes the proof of Theorem 2.1. ∎

4. Conclusion

This paper studied the flocking dynamics of discrete-time C-S model under hierarchical leadership under the effect of a constant time delay. It is assumed that the overall leader of the flock has a reasonable free-will acceleration. We show that for the case with long-rang communication weight, i.e., $\beta \leq 1/2$, convergence to flocking can be guaranteed for any size of the time delay. The result reveals that the asymptotic flocking is robust with respect to the transmission delay. This tolerance may be due to the efficiency of the hierarchical leadership structure. This work opens some further research directions. For example, it is of interest to consider the effect of processing delay in velocity information on convergence of flocking behavior. Also, more general delays, e.g., time-varying unbounded delays or heterogenous delays, should be studied.

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