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A SHARP CRITICAL THRESHOLD FOR A TRAFFIC FLOW MODEL WITH LOOK-AHEAD DYNAMICS*

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Abstract. We study a Lighthill-Whitham-Richards (LWR) type traffic flow model, with a nonlocal look-ahead interaction that has a slow-down effect depending on the traffic ahead. We show a *sharp* critical threshold condition on the initial data that distinguishes global smooth solutions and finite-time wave breakdown. It is well-known that the LWR model leads to a finite-time shock formation, representing the creation of traffic jams, for generic smooth initial data with finite mass. Our result shows that the nonlocal slowdown effect can help to prevent shock formations, for a class of subcritical initial data.

Keywords. Nonlocal conservation law; traffic flow; critical threshold; global regularity; shock formation.

AMS subject classifications. 76A30; 35L65; 35L67; 35B51; 35B65.

1. Introduction

We consider the following one-dimensional traffic flow model with a nonlocal flux

$$\begin{cases} \partial_t u + \partial_x\big(u(1-u)e^{-\bar{u}}\big) = 0, & t>0, x\in\mathbb{R}, \\ u(0,x) = u_0(x), & x\in\mathbb{R}. \end{cases} \tag{1.1}$$

Here, $u(t,x)$ represents the vehicle density at time t and location x, with value normalized in the interval $[0,1]$. The velocity of the flow $v=(1-u)e^{-\bar{u}}$ becomes zero when the maximum density is reached. It is also weighted by a nonlocal Arrhenius type slow down factor $e^{-\bar{u}}$, where

$$\bar{u}(t,x) = (K*u)(t,x) = \int_{\mathbb{R}} K(x-y)u(t,y)\,dy, \tag{1.2}$$

with appropriate choices of the kernel K to be discussed later.

We are interested in the local and global wellposedness of this nonlocal macroscopic traffic flow model (1.1)-(1.2). The goal is to understand whether smooth solutions persist in all time, or if there could be a finite-time singularity formation. A typical type of singularity is the formation of shocks, when $\partial_x u$ becomes unbounded at a finite time. This is known as the *wave break-down phenomenon*. In the context of traffic flow, it describes the generation of traffic jams.

1.1. Nonlocal scalar conservation laws.

The traffic flow model (1.1) falls into a class of models in nonlocal scalar conservation laws, which have the form

$$\partial_t u + \partial_x F(u,\bar{u}) = 0, \tag{1.3}$$

where the flux F depends on both the local density u, and the nonlocal quantity $\bar{u}$ defined in (1.2). This class of models has a variety of applications, not only in traffic

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flows [25,30,37], but also in dispersive water waves [14,19,34,43], the collective motion of biological cells [8,16], high-frequency waves in relaxing medium [20,42], the kinematic sedimentation [5,26,44], pressureless gas dynamics [6,31,35], and many more.

The global wellposedness theory for weak solutions of the nonlocal scalar conservation laws (1.3) is under fast development over the past decades. We refer readers to the book [15] by Dafermos for the general theory. In particular, the theory of entropic weak solutions has been established for nonlocal traffic flow models [7,13,23,24], as well as for extensive numerical investigations [1,11,18].

One challenging problem is to understand the wave break-down phenomenon. It has been addressed by Whitham [43, pp. 457]: *"the breaking phenomenon is one of the most intriguing long-standing problems of water wave theory"*. A natural question is, whether smooth solutions of (1.3) persist in all time, or is there a finite-time break-down.

The wave-breakdown phenomenon has been investigated for many nonlocal conservation laws in the class (1.3), e.g. a nonlinear water wave equation [21], a one-dimensional Keller-Segel model [16], the aggregation equations [3,4,10,40]. In a recent work [28], the first author studied the wave break-down phenomenon for general nonlocal conservation laws (1.3). A sufficient condition on initial data is derived which guarantees a finite-time blowup. However, whether the equations admit global smooth solutions for some initial data is much less understood, especially with the presence of the nonlocality.

1.2. Nonlocal traffic models. We focus on the nonlocal traffic models (1.1)-(1.2). They are other examples for the nonlocal conservation law (1.3).

When there is no interaction, namely $K\equiv 0$, the dynamics is the classical Lighthill-Whitham-Richards (LWR) model [33,36]

$$\partial_t u+\partial_x(u(1-u))=0, \tag{1.4}$$

with the maximum velocity normalized to 1. For this local model, it is well-known that there is a finite-time wave break-down for generic smooth initial data.

For uniform interaction $K\equiv 1$, the nonlocal term

$$\bar{u}(t,x)=\int_{\mathbb{R}} u(t,y)dy=\int_{\mathbb{R}} u_0(y)dy=:m$$

is a constant, due to the conservation of mass. Then, the dynamics again becomes LWR model, with velocity $v=(1-u)e^{-m}$.

Another class of choices of K is called the *look-ahead* kernel, where

$$\operatorname{supp}(K)\subseteq(-\infty,0].$$

Under the assumption, the nonlocal term

$$\bar{u}(x)=\int_x^\infty K(x-y)u(t,y)dy$$

only depends on the density ahead. Sopasakis and Katsoulakis (SK) [37] introduce a celebrated traffic model with Arrhenius type look-ahead interactions, where

$$K(x)=\begin{cases}1 & -1<x<0,\\ 0 & \text{otherwise.}\end{cases} \tag{1.5}$$

A family of kernels with look-ahead distance L can be generated by the scaling

$$K_L(x) = K\left(\frac{x}{L}\right). \tag{1.6}$$

Note that when $L \to 0$, the system formally reduces to the local LWR model (1.4).

The wave break-down phenomenon for the SK model is observed in [25], through an extensive numerical study. A different class of linear look-ahead kernels is also introduced, with

$$K(x) = \begin{cases} 2\big(1-(-x)\big) & -1 < x < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.7}$$

Numerical examples suggest that wave break-down happens in finite time, for a class of initial data. However, unlike the LWR model, it is generally unclear for the nonlocal models whether wave break-down happens for generic smooth initial data.

1.3. Critical threshold and wave break-down. In many examples above, whether there is a finite-time wave break-down depends on the choice of initial conditions: subcritical initial datum leads to global smooth solution, while supercritical initial datum leads to a finite-time wave break-down. This is known as the *critical threshold phenomenon*, which has been studied in the context of Eulerian dynamics, including the Euler-Poisson equations [17, 29, 38], the Euler-alignment equations [9, 39, 41], and more systems of conservation laws.

A critical threshold is called *sharp* if all initial data lie in either the subcritical region, or the supercritical region.

For the traffic model (1.1) with nonlocal look-ahead interactions (1.5) or (1.7), a supercritical region has been obtained in [30]. which leads to a finite-time wave break-down. The result is further extended in [27] to a larger class of models with nonlocal convex-concave fluxes. However, the result is not sharp. It is not known whether such blowup happens for generic smooth initial data, like the LWR model.

A challenging open problem is, *whether there exists a class of subcritical initial data, such that the solution of the nonlocal traffic model is globally regular.*

In this paper, we give a positive answer to the question. The nonlocal look-ahead interaction has a remarkable slowdown effect, which competes with the nonlinearity in the LWR model, and thus prevents the generation of traffic jams, for a class of initial data.

1.4. Main result. We study the traffic flow model (1.1) with the following look-ahead interaction

$$K(x) = \begin{cases} 1 & -\infty < x < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.8}$$

The kernel can be viewed as a limit of the SK model (1.5) under scaling (1.6), with look-ahead distance $L \to \infty$.

The corresponding nonlocal term

$$\bar{u}(t,x) = \int_x^\infty u(t,y)dy. \tag{1.9}$$

The main result is stated as follow:

Theorem 1.1 (Sharp critical threshold). *Consider the traffic flow model* (1.1) *with a nonlocal look-ahead kernel* (1.9)*. Suppose the initial data is smooth, with* $u_0 \in L^1 \cap H^s(\mathbb{R})$ *for* $s > 3/2$*, and* $0 \le u_0 \le 1$*. Then, there exists a function* $\sigma : [0,1] \to \mathbb{R}_+$*, uniquely defined in* (3.3)*, such that*

- *if the initial data is* ***subcritical****, satisfying*

$$u_0'(x) \le \sigma(u_0(x)), \quad \forall\, x \in \mathbb{R}, \tag{1.10}$$

  then the solution exists globally in time. Namely, for any $T > 0$*, there exists a unique solution* $u \in C([0,T]; L^1 \cap H^s(\mathbb{R}))$*.*

- *if the initial data is* ***supercritical****, satisfying*

$$\exists\, x_0 \in \mathbb{R} \quad s.t. \quad u_0'(x_0) > \sigma(u_0(x_0)), \tag{1.11}$$

  then the solution must blow up in finite time. More precisely, there exists a finite time $T_* > 0$*, such that*

$$\limsup_{t \to T_*} \|\partial_x u(t,\cdot)\|_{L^\infty} = +\infty.$$

Remark 1.1. To the best of our knowledge, this is the first result for the nonlocal traffic models where global wellposedness for smooth solutions is obtained for a class of subcritical initial data. It is a big improvement to the finite-time blowup results in the existing literature, as it reveals a remarkable property of the nonlocal look-ahead interaction: preventing the loss of regularity.

An example of subcritical initial data is given in Section 4.2. Global regularity is verified through numerical simulation. A striking discovery is that, with the same initial condition, finite-time wave break-downs are observed in both LWR and SK models. This indicates a unique feature of the kernel (1.9).

Remark 1.2. The critical threshold result in Theorem 1.1 is sharp. For nonlocal conservation laws, sharp results are generally difficult to obtain, due to the presence of nonlocality. We utilize a special structure of the kernel (1.9) to obtain a sharp threshold, $\partial_x \bar{u} = -u$. So, this kernel is in some sense more “local”. We perform a refined phase-plane analysis, construct a curve σ, and show that it sharply distinguishes global regularity and finite-time blowup. It is worth noting that the phase-plane dynamics (3.1) is degenerate around a crucial steady state $(0,0)$. We introduce new analytical techniques to overcome such difficulty. The refined analysis around $(0,0)$ plays a critical role, which allows us to find nontrivial subcritical initial data that lead to global regularity of the solutions.

It is possible to extend the result to more general kernels. However, for more “nonlocal” kernels, sharp results are very difficult to obtain. Instead, an intriguing question would be, whether there exist subcritical initial data that lead to global regularity. We will discuss possible generalizations in Section 5.

The rest of the paper is organized as follows. In Section 2, we establish the local wellposedness theory for our nonlocal traffic model (1.1) with (1.9), as well as a criterion to preserve smooth solutions. In Section 3, we show the sharp critical threshold, and prove Theorem 1.1. Some numerical examples are provided in Section 4, which illustrate the behaviors of the solution under subcritical and supercritical initial data. Finally, we make some remarks in Section 5, which may lead to future investigations.

Notations. We denote $H^s(\mathbb{R})$, $s\geq 0$ to be the Sobolev space in $\mathbb{R}$, with the norm $\|\cdot\|_{H^s}$ and the homogeneous semi-norm $\|\cdot\|_{\dot{H}^s}$ defined as

$$\|f\|_{H^s}=\|(I-\Delta)^{s/2}f\|_{L^2}=\left\|\mathcal{F}^{-1}\left[\left(1+|\xi|^2\right)^{s/2}\mathcal{F}f\right]\right\|_{L^2},$$
$$\|f\|_{\dot{H}^s}=\|(-\Delta)^{s/2}f\|_{L^2}=\left\|\mathcal{F}^{-1}\left[|\xi|^s\mathcal{F}f\right]\right\|_{L^2},$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the forward and inverse Fourier transforms in $\mathbb{R}$, respectively. Note that when s is an integer, we have $\|f\|_{\dot{H}^s}=\|\frac{d^s}{dx^s}f\|_{L^2}$ and $\|f\|_{H^s}^2=\|f\|_{\dot{H}^s}^2+\|f\|_{L^2}^2$.

We use the short-cut notation $A\lesssim B$ to represent that there exists a universal constant $C>0$ such that $A\leq CB$. We further denote $A\lesssim_p B$ if the constant $C=C(p)$ depends on parameter p. Throughout the paper, the constant C is repeatedly used, whose value can change line by line.

We denote $\lceil s\rceil$ to be the smallest integer that is greater or equal to s.

We use f' to represent the derivative of f, if f has a single variable; and $\dot{f}$ denotes the material derivative of $f=f(t,x)$ along a characteristic path

$$\dot{f}(t,X(t))=\frac{d}{dt}f(t,X(t))=\partial_t f+((1-2u)e^{-\bar{u}})\partial_x f,$$

where $X(t)$ is defined in (2.4).

2. Local wellposedness and regularity criterion

In this section, we establish the local wellposedness theory for our main system (1.1).

THEOREM 2.1 (Local wellposedness). *Let $s>3/2$. Consider equation (1.1) with (1.9). Suppose the initial data $u_0\in L^1\cap H^s(\mathbb{R})$, and $0\leq u_0\leq 1$. Then, there exists a time $T_*=T_*(u_0)>0$ and a unique solution*

$$u\in C([0,T_*];L^1\cap H^s(\mathbb{R})). \tag{2.1}$$

Moreover, given any time $T>0$, the solution exists in $C([0,T];L^1\cap H^s(\mathbb{R}))$ if and only if

$$\int_0^T\|\partial_x u(t,\cdot)\|_{L^\infty}dt<+\infty. \tag{2.2}$$

The proof is based on standard a priori energy estimates. The main subtlety is on the nonlocal term. As the kernel K in (1.8) has a jump discontinuity at the origin, the nonlocal term does not have enough regularity to be treated directly. To this end, a priori bounds on the nonlocal terms are carefully studied, c.f. Proposition 2.2. Nontrivial commutator estimates are used to obtain the regularity criterion (2.2).

The rest of this section is devoted to the proof of Theorem 2.1. We focus on the *a priori estimates* that can be used to establish Theorem 2.1 in a standard way - for example using Galerkin approximations or vanishing viscosity.

The proof can be extended to general kernels (1.5) and (1.7) with no extra difficulties.

2.1. Conservation of mass.

Assume u is smooth, satisfying (2.1). Integrating (1.1) in x, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}}u(t,x)dx=-\int_{\mathbb{R}}\partial_x(u(1-u)e^{-\bar{u}})dx=0.$$

Therefore, the total mass

$$m := \int_{\mathbb{R}} u(t,x)dx$$

is conserved in time.

2.2. Maximum principle. We next show that there is a maximum density for our traffic model. Rewrite (1.1) as

$$\partial_t u + (1-2u)e^{-\bar{u}}\partial_x u + u^2(1-u)e^{-\bar{u}} = 0. \tag{2.3}$$

Let $X(t,x)$ be the characteristic path originating at x, defined as

$$\frac{d}{dt}X(t,x) = (1-2u(t,X(t,x)))e^{-\bar{u}(t,X(t,x))}, \quad X(0,x) = x. \tag{2.4}$$

It is well-known that X is well-defined, and $X(t,\cdot)$ is invertible if solution u is smooth, in the sense of (2.1). We denote the backward characteristic $X^{-1}(t,\cdot)$ as the inverse function of $X(t,\cdot)$. For simplicity, we shall suppress the x-dependence of X throughout the paper.

From (2.3), we have along each characteristic path

$$\dot{u} = -u^2(1-u)e^{-\bar{u}}. \tag{2.5}$$

The following maximum principle holds.

PROPOSITION 2.1 (Maximum principle). *Let u be a smooth solution of (2.3), with initial condition $0 \le u_0 \le 1$. Then, $0 \le u(t,x) \le 1$ for any $x \in \mathbb{R}$ and $t \ge 0$.*

Proof. For the upper bound, note that $-u^2(1-u)e^{-\bar{u}} \le 0$ when $u \le 1$. We immediately get $u(t,x) \le u_0(X^{-1}(t,x)) \le 1$.

For the lower bound, we argue by contradiction. Suppose there exist a positive time $t > 0$ and a characteristic path such that $u(t,X(t)) < 0$. Then, there must be a time t_0 such that

$$u(t_0, X(t_0)) = 0, \quad u(t_0+, X(t_0+)) < 0.$$

However, equation (2.5) with initial condition $u(t_0, X(t_0)) = 0$ has a unique solution

$$u(t, X(t)) = 0, \quad \forall\, t \ge t_0.$$

This leads to a contradiction. Hence, $u(t,x) > 0$ for any x and $t \ge 0$. ∎

2.3. A priori bounds on the nonlocal term. From the definition of $\bar{u}$ (1.9) and positivity of u, we get the following a priori bound

$$0 \le \bar{u}(t,x) \le m. \tag{2.6}$$

We now bound the nonlocal term $e^{-\bar{u}}$. First, from (2.6), we have

$$e^{-m} \le e^{-\bar{u}} \le 1. \tag{2.7}$$

This shows the nonlocal weight is bounded from above and below, away from zero.

Next, we compute

$$\|\partial_x(e^{-\bar{u}})\|_{L^\infty} = \|u \cdot e^{-\bar{u}}\|_{L^\infty} \le 1. \tag{2.8}$$

For higher derivatives of $e^{-\bar{u}}$, we make use of the following composition estimate.

LEMMA 2.1 (Composition estimate). *Let $s>0$. Suppose $g\in L^\infty\cap \dot{H}^s(\mathbb{R})$ and $f\in C^{\lceil s\rceil}(\mathrm{Range}(g))$. Then, the composition $f\circ g\in L^\infty\cap \dot{H}^s(\mathbb{R})$. Moreover, there exists a constant $C>0$, depending on $s, \|f\|_{C^{\lceil s\rceil}(\mathrm{Range}(g))}$ and $\|g\|_{L^\infty}$, such that*

$$\|f\circ g\|_{\dot{H}^s}\le C\|g\|_{\dot{H}^s}.$$

A proof of Lemma 2.1 will be provided in the Appendix for self-consistency.

PROPOSITION 2.2. *For $s\ge 1$,*

$$\|e^{-\bar{u}}\|_{\dot{H}^s}\lesssim_{s,m}\|u\|_{\dot{H}^{s-1}}.$$

Proof. We apply Lemma 2.1, with $f(x)=e^x$ and $g(x)=-\bar{u}(t,x)$. From (2.6), we know g is bounded, and $g(x)\in[-m,0]$. Therefore, $\|f\|_{C^{\lceil s\rceil}([-m,0])}=1$ for any $s\ge 0$.
Lemma 2.1 then implies

$$\|e^{-\bar{u}}\|_{\dot{H}^s}\le C(m,s)\|g\|_{\dot{H}^s}=C(m,s)\|u\|_{\dot{H}^{s-1}}.$$

The last equality is due to the fact that $\partial_x g=u$. ∎

2.4. L^2 energy estimate. We perform a standard L^2 energy estimate.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|_{L^2}^2&=\int_{\mathbb{R}}u\;\partial_x\big(u(1-u)e^{-\bar{u}}\big)dx=-\int_{\mathbb{R}}\partial_x u\cdot u(1-u)e^{-\bar{u}}dx\\&=\int_{\mathbb{R}}\frac{1}{2}u^2\cdot\partial_x(e^{-\bar{u}})dx+\int_{\mathbb{R}}u^2\cdot\partial_x u\cdot e^{-\bar{u}}dx\\&\le\frac{1}{2}\|u\|_{L^2}^2\|\partial_x(e^{-\bar{u}})\|_{L^\infty}+\|\partial_x u\|_{L^\infty}\|u\|_{L^2}^2\|e^{-\bar{u}}\|_{L^\infty}\\&\lesssim(1+\|\partial_x u\|_{L^\infty})\|u\|_{L^2}^2,\end{aligned}\tag{2.9}$$

where we apply (2.7) and (2.8) in the last inequality.

2.5. H^s energy estimate. Let $\Lambda:=(-\Delta)^{1/2}$ be the pseudo-differential operator. We perform an energy estimate by acting Λ^s on (2.3) and integrating against $\Lambda^s u$. This yields the evolution of the homogeneous H^s semi-norm on u:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|_{\dot{H}^s}^2&=\int_{\mathbb{R}}\Lambda^s u\cdot\Lambda^s\big(-(1-2u)e^{-\bar{u}}\partial_x u-u^2(1-u)e^{-\bar{u}}\big)dx\\&=\int_{\mathbb{R}}\Lambda^s u\cdot(2u-1)e^{-\bar{u}}\cdot\Lambda^s\partial_x u\;dx+\int\Lambda^s u\cdot\big(\big[\Lambda^s,(2u-1)e^{-\bar{u}}\big]\,\partial_x u\big)dx\\&\quad-\int_{\mathbb{R}}\Lambda^s u\cdot\Lambda^s\big(u^2(1-u)e^{-\bar{u}}\big)dx=\mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

Here, the commutator $[\Lambda^s,f]g$ is defined as

$$[\Lambda^s,f]g=\Lambda^s(fg)-f\Lambda^s g.$$

We shall estimate the three terms one by one.
For the first term, apply integration by parts and get

$$\mathrm{I}=\int_{\mathbb{R}}\frac{1}{2}\partial_x\big((\Lambda^s u)^2\big)\cdot(2u-1)e^{-\bar{u}}\;dx=-\frac{1}{2}\int_{\mathbb{R}}(\Lambda^s u)^2\cdot\partial_x((2u-1)e^{-\bar{u}})dx$$

$$\leq \frac{1}{2}\|u\|_{\dot{H}^s}^2 \|\partial_x((2u-1)e^{-\bar{u}})\|_{L^\infty} = \frac{1}{2}\|u\|_{\dot{H}^s}^2 \|(2\partial_x u + (2u-1)u)e^{-\bar{u}}\|_{L^\infty}.$$

Since both u and $\bar{u}$ are bounded, we have

$$\|(2\partial_x u + (2u-1)u)e^{-\bar{u}}\|_{L^\infty} \leq 2\|\partial_x u\|_{L^\infty} + 1.$$

Therefore,

$$\mathrm{I} \leq (1 + \|\partial_x u\|_{L^\infty})\|u\|_{\dot{H}^s}^2. \tag{2.10}$$

For the second term,

$$\mathrm{II} \leq \|u\|_{\dot{H}^s} \left\| \left[\Lambda^s, (2u-1)e^{-\bar{u}}\right] \partial_x u \right\|_{L^2}.$$

Let us state the following two estimates. Both lemmas can be proved using Littlewood-Paley theory.

Lemma 2.2 (Fractional Leibniz rule). *Let $s \geq 0$. There exists a constant $C > 0$, depending only on s, such that*

$$\|fg\|_{\dot{H}^s} \leq C(\|f\|_{L^\infty}\|g\|_{\dot{H}^s} + \|f\|_{\dot{H}^s}\|g\|_{L^\infty}).$$

A proof of the fractional Leibniz rule can be found in [2, Corollary 2.86].

Lemma 2.3 (Commutator estimate). *Let $s \geq 1$. There exists a constant $C > 0$, depending only on s, such that*

$$\|[\Lambda^s, f]g\|_{L^2} \leq C(\|\partial_x f\|_{L^\infty}\|g\|_{\dot{H}^{s-1}} + \|f\|_{\dot{H}^s}\|g\|_{L^\infty}).$$

The commutator estimate is due to Kato and Ponce [22]. See e.g. [32, Remark 1.5] for a version for homogeneous operator Λ^s.

Apply Lemma 2.3 to the commutator in II. We get

$$\begin{aligned} &\left\| \left[\Lambda^s, (2u-1)e^{-\bar{u}}\right] \partial_x u \right\|_{L^2} \\ &\lesssim_s \|(2u-1)e^{-\bar{u}}\|_{L^\infty}\|\partial_x u\|_{\dot{H}^{s-1}} + \|(2u-1)e^{-\bar{u}}\|_{\dot{H}^s}\|\partial_x u\|_{L^\infty} = \mathrm{II}_1 + \mathrm{II}_2. \end{aligned}$$

Due to maximum principle, $|2u-1| \leq 1$. Also, $\|e^{-\bar{u}}\|_{L^\infty} \leq 1$ and $\|\partial_x(e^{-\bar{u}})\|_{L^\infty} \leq 1$ by (2.7) and (2.8). Therefore, II_1 can be easily estimated by

$$\mathrm{II}_1 \lesssim (1 + \|\partial_x u\|_{L^\infty})\|u\|_{\dot{H}^s}.$$

For II_2, we apply Lemma 2.2 and Proposition 2.2,

$$\begin{aligned} \mathrm{II}_2 &\lesssim_s \left(\|2u-1\|_{\dot{H}^s}\|e^{-\bar{u}}\|_{L^\infty} + \|2u-1\|_{L^\infty}\|e^{-\bar{u}}\|_{\dot{H}^s}\right)\|\partial_x u\|_{L^\infty} \\ &\lesssim_{s,m} (\|u\|_{\dot{H}^s} + \|u\|_{\dot{H}^{s-1}})\|\partial_x u\|_{L^\infty}. \end{aligned}$$

Combining the estimates on II_1 and II_2, we obtain

$$\mathrm{II} \lesssim_{s,m} \|\partial_x u\|_{L^\infty}\|u\|_{\dot{H}^s}\|u\|_{H^s}, \tag{2.11}$$

where we have used the fact $\|u\|_{\dot{H}^{s-1}} \leq \|u\|_{H^s}$.

For the third term, we again apply Lemma 2.2 and get

$$\mathrm{III} \lesssim_s \|u\|_{\dot{H}^s}\Big(\|u^2(1-u)\|_{\dot{H}^s}\|e^{-\bar{u}}\|_{L^\infty} + \|u^2(1-u)\|_{L^\infty}\|e^{-\bar{u}}\|_{\dot{H}^s}\Big).$$

Applying Proposition 2.2 to the term $\|e^{-\bar{u}}\|_{\dot{H}^s}$, and the following estimate

$$\|u^2(1-u)\|_{\dot{H}^s} \lesssim_s 2\|u\|_{\dot{H}^s}\|u\|_{L^\infty}\|1-u\|_{L^\infty} + \|u\|_{L^\infty}^2\|1-u\|_{\dot{H}^s} \le 3\|u\|_{\dot{H}^s},$$

we conclude with

$$\mathrm{III} \lesssim_{s,m} \|u\|_{\dot{H}^s}^2. \tag{2.12}$$

Gathering the estimates (2.10), (2.11) and (2.12), we derive

$$\frac{d}{dt}\|u(\cdot,t)\|_{\dot{H}^s}^2 \lesssim_{s,m} \|u\|_{\dot{H}^s}^2 + \|\partial_x u\|_{L^\infty}\|u\|_{\dot{H}^s}\|u\|_{H^s}.$$

Together with the L^2 estimate (2.9), we get the full H^s estimate

$$\frac{d}{dt}\|u(\cdot,t)\|_{H^s}^2 \lesssim_{s,m} (1+\|\partial_x u\|_{L^\infty})\|u\|_{H^s}^2. \tag{2.13}$$

2.6. Proof of Theorem 2.1. Let $s>3/2$, from the Sobolev embedding theorem, we know $\|\partial_x u\|_{L^\infty} \lesssim_s \|u\|_{H^s}$. Define $Y(t)=\|u(t,\cdot)\|_{H^s}^2$. We deduce from (2.13)

$$Y'(t) \lesssim_{s,m} \big(1+Y(t)^{1/2}\big)Y(t).$$

Clearly, there exists a time T_*, depending on $Y(0)$, such that $Y(t)$ exists and is bounded for $t\in[0,T_*]$. This finishes the local existence proof.

To verify the regularity criterion (2.2), we apply the Grönwall inequality to (2.13) and get

$$\|u(T,\cdot)\|_{H^s} \le \|u_0\|_{H^s}\exp\left(C(s,m)\int_0^T (1+\|\partial_x u(t,\cdot)\|_{L^\infty})dt\right).$$

Clearly, $u(t,\cdot)\in H^s$ for all $t\in[0,T]$ as long as (2.2) holds. This concludes the proof of Theorem 2.1.

3. The critical threshold

In this section, we discuss the global behaviors of the solutions. The main goal is to understand whether the criterion (2.2) holds for any time T, namely whether $\partial_x u$ is uniformly bounded in time.

We start with expressing the dynamics of $d:=\partial_x u$ by differentiating (2.3) in x:

$$\partial_t d + (1-2u)e^{-\bar{u}}\partial_x d + e^{-\bar{u}}\big(-2d^2+(3u-5u^2)d+(u^3-u^4)\big)=0.$$

Together with (2.5), we get a coupled dynamics of (d,u) along characteristic paths.

$$\begin{cases} \dot{d} = \big(2d^2-(3u-5u^2)d-u^3(1-u)\big)e^{-\bar{u}}, \\ \dot{u} = -u^2(1-u)e^{-\bar{u}}. \end{cases} \tag{3.1}$$

Note that a classical sufficient condition to avoid the breakdown of the characteristics is that the velocity field is Lipschitz.

$$\big\|\partial_x\big((1-2u)e^{-\bar{u}}\big)\big\|_{L^\infty} = \big\|\big(-2\partial_x u+(1-2u)u\big)e^{-\bar{u}}\big\|_{L^\infty} \le 1+2\|\partial_x u\|_{L^\infty}.$$

Therefore, as long as condition (2.2) holds, the characterstic paths remain valid.

We now perform a phase plane analysis on (d,u) through each characteristic path. It is worth noting that $e^{-\bar{u}}$ is nonlocal. So the values of (d,u) can not be determined solely by information along the characteristic path. However, the ratio

$$\frac{\dot{d}}{\dot{u}} = \frac{2d^2-(3u-5u^2)d-u^3(1-u)}{-u^2(1-u)}$$

is local. Therefore, the trajectories of (d,u) on the phase plane only depend on local information. If we express a trajectory as a function $d=d(u)$, then it will satisfy the ODE

$$d'(u) = \frac{2d^2-(3u-5u^2)d-u^3(1-u)}{-u^2(1-u)}. \tag{3.2}$$

Figure 3.1 illustrates the flow map in the phase plane. In particular, $(0,0)$ is a degenerated hyperbolic point. There is an inward trajectory which separates the plane into two regions. The left region will flow towards $(0,0)$, and the right region will flow towards $d\to\infty$. This indicates the two differernt behaviors: global boundedness versus blowup, respectively. This is so called the *critical threshold phenomenon*.

For the rest of this section, we will show such phenomenon rigorously. This then leads to a proof of Theorem 1.1.

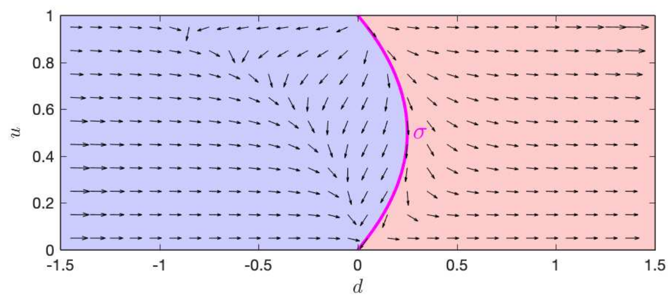

FIG. 3.1. *The flow map and the critical threshold in* (d,u)*-plane*

3.1. The sharp critical threshold. We start with describing the critical threshold that distinguishes the two regions in Figure 3.1 as $d=\sigma(u)$. The main subtlety is to carefully examine the behavior near the degenerate hyperbolic point $(0,0)$.

Before we make a precise definition of the function $\sigma:[0,1]\to\mathbb{R}$, we shall first state its properties as a motivation. First, σ should be a trajectory in the phase plane which passes $(0,0)$, namely it should satisfy the following ODE

$$\sigma'(x) = \frac{2\sigma^2-(3x-5x^2)\sigma-x^3(1-x)}{-x^2(1-x)}, \quad \sigma(0)=0.$$

However, there are infinitely many candidates, since $(0,0)$ is degenerate. We need to pick the one that reveals the hyperbolicity. For all the candidates, we have the following relation on $\sigma'(0)$

$$\begin{aligned}\sigma'(0) &= \lim_{x\to 0+}\frac{2\sigma(x)^2-(3x-5x^2)\sigma(x)-x^3(1-x)}{-x^2(1-x)}\\ &= -2\left(\lim_{x\to 0+}\frac{\sigma(x)}{x}\right)^2+3\lim_{x\to 0+}\frac{\sigma(x)}{x} = -2\sigma'(0)^2+3\sigma'(0).\end{aligned}$$

This implies $\sigma'(0)=1$ or $\sigma'(0)=0$. It turns out the ones with $\sigma'(0)=0$ are not relavant in the study of the critical threshold phenomenon. On the other hand, the following theorem shows that there is only one σ among all candidates that satisfies $\sigma'(0)=1$. We will show later that this is the critical threshold curve we are searching for.

PROPOSITION 3.1 (The critical threshold). *There exists a unique function $\sigma:[0,1]\to\mathbb{R}$, such that*

$$\sigma'(x)=\frac{2\sigma^2-(3x-5x^2)\sigma-x^3(1-x)}{-x^2(1-x)}, \quad \sigma(0)=0, \quad \sigma'(0)=1. \tag{3.3}$$

Proof. We start with the local existence theory. Fix a small $\epsilon>0$. The classical Cauchy-Peano theorem does not apply directly near $x=0$, as

$$F(x,\sigma):=\frac{2\sigma^2-(3x-5x^2)\sigma-x^3(1-x)}{-x^2(1-x)}$$

is not uniformly bounded for $(x,\sigma)\in[0,\epsilon]\times[-\epsilon,\epsilon]$. Instead, we can consider the region

$$A=\left\{(x,\sigma):0\le\sigma\le\frac{5}{4}x,\ \ 0\le x\le\epsilon\right\}.$$

It is easy to check that if $\frac{\sigma}{x}\in[0,\frac{5}{4}]$, then

$$\min\left\{x,\frac{5-50x}{8(1-x)}+x\right\}\le F(x,\sigma)\le\frac{(3-5x)^2}{8(1-x)}+x.$$

Hence, if we pick $\epsilon<\frac{1}{10}$, we would have

$$0\le F(x,\sigma)\le\frac{5}{4}, \quad \forall\ (x,\sigma)\in A. \tag{3.4}$$

Now, we can build a sequence of approximate solutions $\{\sigma_n(x)\}$ for $x\in[0,\epsilon]$. This can be done, for instance, by forward Euler method. Given $n\in\mathbb{Z}_+$, define an equi-distance lattice $\{x_k=\frac{k\epsilon}{n}\}_{k=0}^n$.

(i) $\sigma_n(x)=x, \quad \forall\ x\in[0,x_1]$.

(ii) $\sigma_n(x)=\sigma_n(x_k)+F(x_k,\sigma_n(x_k))(x-x_k), \quad \forall\ x\in[x_k,x_{k+1}], \quad k=1,\cdots,n-1$.

From (3.4), we know $(x,\sigma_n(x))\in A$, for all $x\in[0,\epsilon]$. Hence, $\sigma_n(x)$ is uniformly bounded and equi-continuous in $x\in[0,\epsilon]$. By Arzela-Ascoli theorem, σ_n converges uniformly to σ, up to an extraction of a subsequence. And by its construction, σ is indeed a solution of the ODE in (3.3).

It is clear that $\sigma(0)=0$ since $\sigma_n(0)=0$ for every n. To verify $\sigma'(0)=1$, we show the following statement: the image of the solution $(x,\sigma(x))$ lies inside the cone

$$\{(x,\sigma):(1-2\epsilon)x<\sigma\le x,\ \ 0\le x\le\epsilon\}.$$

Indeed, we check F at the boundary of the cone

$$F(x,\sigma=x)=\frac{1-4x-x^2}{1-x}\le 1,$$
$$F(x,\sigma=(1-2\epsilon)x)=\frac{1+2\epsilon-4x+(-8\epsilon^2+10\epsilon x-x^2)}{1-x}>1-2\epsilon.$$

Therefore, $\sigma'(x)\in(1-2\epsilon,1]$ for all $x\in[0,\epsilon]$. Taking $\epsilon\to 0$, we conclude with $\sigma'(0)=1$.

Next, we discuss the local uniqueness. Let $\sigma^{(1)}$ and $\sigma^{(2)}$ be two different solutions of (3.3). Let $w=\sigma^{(1)}-\sigma^{(2)}$. Note that $\sigma^{(1)}$ and $\sigma^{(2)}$ can not cross for $x\in(0,1)$. Without loss of generality, we assume $w(x)>0$ for $x\in(0,\epsilon]$. (Otherwise, switch $\sigma^{(1)}$ and $\sigma^{(2)}$). Compute

$$w'(x)=\frac{-2(\sigma^{(1)}(x)+\sigma^{(2)}(x))+(3x-5x^2)}{x^2(1-x)}w(x)\le\frac{-4(1-2\epsilon)x+(3x-5x^2)}{x^2(1-x)}w(x)\le 0,$$

for any $x\in(0,\epsilon]$. Since $w(0)=0$, it implies $w(x)\le 0$. This leads to a contradiction.

Once we obtain local wellposedness of σ, global existence and uniqueness for $x\in(0,1)$ follows from the standard Cauchy-Lipschitz theory. ◻

We would like to remark that $\sigma'(0)=1$ is critically used to obtain uniqueness. In particular, the solutions of (3.3) with $\sigma'(0)=0$ are not unique. But those curves are not what we are seeking.

3.2. Global regularity for subcritical initial data. We now prove the first part of Theorem 1.1. The goal is to show that, if the initial data satisfy (1.10), then condition (2.2) holds for any time T. Equivalently, we will show $d=\partial_x u$ is bounded along all characterstic paths.

First, we show an upper bound of d.

Proposition 3.2 (Invariant region). *Let (d,u) satisfy the dynamics (3.1) with initial condition $d_0\le\sigma(u_0)$. Then, $d(t)\le\sigma(u(t))$ for any time $t\ge 0$.*

Proof. We first consider two special cases $u_0=0$ and $u_0=1$. In both cases, $u'=0$ and hence u does not change in time.

For $u_0=0$, the dynamics of d becomes

$$\dot d=2d^2e^{-\bar u}. \tag{3.5}$$

If $d_0\le\sigma(0)=0$, clearly $d(t)\le 0$ for any $t\ge 0$.

For $u_0=1$, the dynamics of d becomes

$$\dot d=2d(d+1)e^{-\bar u}. \tag{3.6}$$

Again, if $d_0\le\sigma(1)=0$, then $d(t)\le 0$ for any $t\ge 0$.

Next, we consider the case $u_0\in(0,1)$. Here, we use the fact that trajectories do not cross. To be more precise, we argue by a contradiction. Suppose there exists a time t such that $d(t)>\sigma(u(t))$. Then, there must exist a time t_0 so that the (d,u) first exits the region at t_0+. By continuity, $d(t_0)=\sigma(u(t_0))$. Starting from $(d(t_0),u(t_0))$, the trajectory satisfies (3.2).

By definition (3.3), $d=\sigma(u)$ is a solution in the phase plane. The standard Cauchy-Lipschitz theorem ensures that (3.2) with initial condition $(d(t_0),u(t_0))$ has a local unique solution. Therefore, the solution has to be $d(t_0+)=\sigma(u_0(t_0+))$. This contradicts the assumption that (d,u) exits the region at t_0+. ◻

Next, we show a lower bound of d. This can be easily observed from Figure 3.1, as the flow is moving to the right as long as $d<-1$.

Proposition 3.3. *Let (d,u) satisfy the dynamics (3.1). Then, for any $t\ge 0$,*

$$d(t)\ge\min\{-1,d_0\}.$$

Proof. We rewrite

$$\dot{d}=2(d-d_-)(d-d_+)e^{-\bar{u}}, \quad d_\pm=\frac{(3u-5u^2)\pm\sqrt{(3u-5u^2)^2+8u^3(1-u)}}{4}.$$

Then, $\dot{d}\geq 0$ if $d\leq d_-$. This implies $d(t)\geq\min\{d_-,d_0\}$. Note that for $u\in[0,1]$, $d_-\geq -1$. Therefore, we obtain the lower bound. ∎

Combining the two bounds, we know that along each characteristic path, d is bounded for all times. Collecting all characterstic paths, we obtain $\|\partial_x u(t,\cdot)\|_{L^\infty}$ is bounded for any $t\geq 0$. Global regularity then follows from Theorem 2.1.

3.3. Finite time breakdown for supercritical initial data. We turn to prove the second part of Theorem 1.1. Suppose the initial data satisfy (1.11). Then, we consider the characteristic path originating at x_0, namely $d_0=u_0'(x_0)$ and $u_0=u_0(x_0)$. So,

$$d_0>\sigma(u_0). \tag{3.7}$$

For $u_0=0$ or $u_0=1$, finite-time blow up can be easily obtained by the Ricatti-type dynamics (3.5) and (3.6). Moreover, as $0\leq u\leq 1$, we must have $d_0=0$ when $u_0=0$ or 1. Therefore, there is no supercritical data with $u_0=0$ or 1.

We focus on the case when $u_0\in(0,1)$. The main idea is illustrated in Figure 3.2. For each trajectory starting at a supercritical initial point (d_0,u_0), u is getting close to 0 as time evolves, unless blowup has already happened. When u becomes close to 0, the dynamics of d becomes close to (3.5). Then, if d is away from 0, the Ricatti-type dynamics will lead to a finite-time blowup.

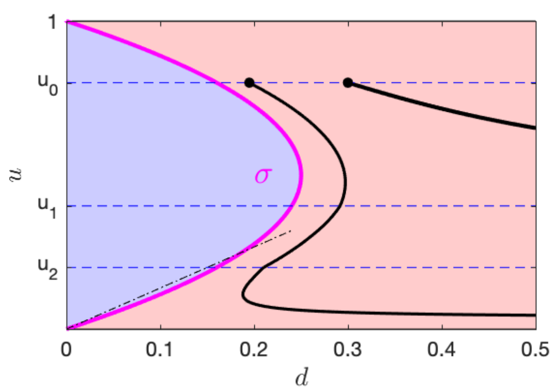

FIG. 3.2. *An illustration of typical trajectories with supercritical initial data* (d_0,u_0). *Case 1: blow up happens before the trajectory reaches* u_1. *Case 2: the trajectory passes* u_1*, but blow up eventually happens in finite time.*

To rigorously justify the idea, we first examine the dynamics of u in (3.1).

PROPOSITION 3.4. *Let* (d,u) *be a solution of* (3.1) *with supercritical initial data* (d_0,u_0)*, satisfying* (3.7). *Then, for any* $u_1\in(0,u_0)$*, there exists a finite time* t_1 *such that, either* $d(t)\to\infty$ *before* t_1*, or* $u(t_1)\leq u_1$.

Proof. Using the bound on the nonlocal term (2.7), we get

$$\dot{u} \leq -e^{-m}u^2(1-u).$$

As long as (d,u) is bounded, the characterstic path stays valid.

The following comparison principle holds. Let $\eta=\eta(t)$ satisfy the ODE

$$\eta' = -e^{-m}\eta^2(1-\eta), \quad \eta(0)=u_0. \tag{3.8}$$

Then, $u(t) \leq \eta(t)$. Indeed,

$$\dot{u}(t)-\eta'(t) \leq e^{-m}\left(-u^2(1-u)+\eta^2(1-\eta)\right) = e^{-m}(-2\xi+3\xi^2)(u-\eta),$$

where $\xi \in (u,\eta) \subset [0,1]$ and therefore $-2\xi+3\xi^2 \in [-\frac{1}{3},1)$ is bounded. This implies

$$u(t)-\eta(t) \leq (u(0)-\eta(0))e^{-\frac{1}{3}e^{-m}t} = 0.$$

The dynamics η in (3.8) can be solved explicitly

$$\left.\left(\frac{1}{\eta}+\log\frac{1-\eta}{\eta}\right)\right]_{u_0}^{\eta(t)} = e^{-m}t.$$

Therefore, $\eta(t_1)=u_1$ at

$$t_1 = e^m\left(\frac{1}{u_1}+\log\frac{1-u_1}{u_1}-\frac{1}{u_0}-\log\frac{1-u_0}{u_0}\right) < +\infty.$$

Applying the comparison principle, we end up with $u(t_1) \leq u_1$. ∎

Proposition 3.4 distinguishes the two cases illustrated in Figure 3.2. Either blowup happens before u reaches u_1, which takes finite time, or the trajectory passes u_1. We shall focus on the latter case from now on.

Next, we argue that by picking a small enough $u_1>0$, the dynamics (3.1) will lead to a blowup in finite time, as long as d stays away from zero.

PROPOSITION 3.5. *Let (d,u) be a solution of (3.1). Suppose d is uniformly bounded away from zero, namely there exists a $C_*>0$ such that*

$$d(t) \geq C_*, \quad \forall\, t \geq 0. \tag{3.9}$$

Then, there exists a $u_1>0$, depending on C_*, such that, with the initial condition $(d(t_1),u(t_1)=u_1)$, the solution has to blow up in finite time.

Proof. As $u(t_1)=u_1$, we know $u(t) \leq u_1$ for any $t \geq t_1$. Then, we get

$$\dot{d} \geq e^{-\bar{u}}(2d^2-3u_1d-u_1^3) = 2e^{-\bar{u}}(d-d_-)(d-d_+), \quad d_\pm = \frac{3\pm\sqrt{9+8u_1}}{4}u_1.$$

Pick $u_1 = C_*/4$, then

$$d(t_1) \geq C_* = 4u_1 > 2d_+.$$

This implies $d(t) > 2d_+$ for all $t \geq t_1$. We can then use (2.7) to bound the nonlocal term and get

$$\dot{d} \geq 2e^{-m}(d-d_-)(d-d_+), \quad \forall\, t \geq t_1. \tag{3.10}$$

Then, by a comparison principle (similar as the one used in Proposition 3.4), the solution satisfies

$$d(t) \geq \frac{d_- e^{2e^{-m}(d_+-d_-)(t-t_1)}(d(t_1)-d_+) - d_+(d(t_1)-d_-)}{e^{2e^{-m}(d_+-d_-)(t-t_1)}(d(t_1)-d_+) - (d(t_1)-d_-)},$$

where the right-hand side is the exact solution of the ODE (3.10) with an equal sign. It blows up at

$$T_* = t_1 + \frac{1}{2e^{-m}(d_+-d_-)} \log \frac{d(t_1)-d_-}{d(t_1)-d_+} < t_1 + \frac{2e^m}{C_*} < +\infty.$$

Therefore, d has to blow up no later than T_*. ◻

We are left with showing the uniform lower bound on d, *i.e.* condition (3.9), for any supercritical initial data. We shall work with trajectories in the phase plane.

Let us denote by $d = d(u)$ the trajectory that goes through (d_0, u_0). As both d and σ satisfy (3.2), we compute

$$(d(u)-\sigma(u))' = \frac{2(d(u)+\sigma(u)) - (3u-5u^2)}{-u^2(1-u)}(d(u)-\sigma(u)) =: A(u)(d(u)-\sigma(u)).$$

Since (d_0, u_0) satisfy (3.7), we get $d(u_0) - \sigma(u_0) > 0$. $A(u)$ is bounded as long as u stays away from 0 and 1. Therefore, we obtain

$$d(u) \geq \sigma(u) \geq 0, \quad \forall\ u \in (0,1).$$

Moreover, for any $u \in (0, u_0)$, we can estimate A by

$$A(u) \leq \frac{3-5u}{u(1-u)} \leq \frac{3}{u}.$$

Integrating in $[u, u_0]$, we get

$$d(u) \geq d(u) - \sigma(u) = (d_0 - \sigma(u_0)) \exp\left[-\int_u^{u_0} A(u) du\right] \geq \frac{(d_0-\sigma(u_0))}{u_0^3} u^3. \tag{3.11}$$

Unfortunately, this bound is not uniform in $(0, u_0]$. We need an enhanced estimate. Let $u_2 > 0$ such that

$$\sigma(u) \geq \frac{3}{4}u, \quad \forall\ u \in [0, u_2]. \tag{3.12}$$

Note that such u_2 exists as $\sigma'(0) = 1$.

For $u \in (0, u_2]$, using (3.12), we obtain an improved estimate on A as follows.

$$A(u) \leq \frac{4\sigma(u) - (3u-5u^2)}{-u^2(1-u)} \leq \frac{3u - (3u-5u^2)}{-u^2(1-u)} = \frac{-5}{1-u} \leq -5.$$

Since $A(u)$ is negative, we immediately get

$$d(u) \geq d(u) - \sigma(u) \geq d(u_2) - \sigma(u_2), \quad \forall\ u < u_2,\ u \in \mathrm{Dom}(d).$$

This, together with (3.11), shows a uniformly lower bound on d

$$d(u) \geq \frac{d_0 - \sigma(u_0)}{u_0^3} u_2^3, \quad \forall\ u \leq u_0,\ u \in \mathrm{Dom}(d).$$

Condition (3.9) follows immediately, with $C_* = (d_0 - \sigma(u_0)) u_0^3 u_2^{-3}$.

4. Examples and simulations

In this section, we present examples and numerical simulations to illustrate our main critical threshold result, Theorem 1.1.

The numerical method we use is the standard finite volume scheme with upwind flux, in a large enough computational domain. The nonlocal term is calculated using a quadrature rule. We refer readers to [1, 11, 18, 25] for extensive discussions on the numerical implementation.

We shall also compare the numerical results for the three different types of nonlocal interaction kernels. Recall

$$K(x)=\begin{cases}0, & \text{① LWR model: look-ahead distance } L=0,\\ 1_{[-1,0)}(x), & \text{② SK model: look-ahead distance } L=1,\\ 1_{(-\infty,0]}(x), & \text{③ Our model: look-ahead distance } L=\infty,\\ 1, & \text{④ LWR model: globally uniform kernel.}\end{cases} \tag{4.1}$$

Here, 1_A denotes the indicator function of a set A.

4.1. Supercritical initial data.

Many smooth initial data u_0 lie in the supercritical region (1.11). In particular, we argue that all compactly supported smooth functions lie in the supercritical region.

Proposition 4.1. *Let $u_0 \in C^1(\mathbb{R})$ be non-negative and compactly supported. Then, u_0 satisfies the supercritical condition* (1.11).

Proof. We argue by contradiction. Suppose u_0 lies in the subcritical region. Then, we have

$$u_0'(x) \le u_0(x), \quad \forall\, x \in \mathbb{R}. \tag{4.2}$$

Let x_L be the left end point of the support of u_0, namely

$$x_L = \arg\inf_x \{u_0(x) > 0\}.$$

By continuity, we know $u_0(x_L)=0$. Solving the ODE (4.2) with initial condition at x_L yields

$$u_0(x) \le 0, \quad \forall\, x \ge x_L.$$

This contradicts with the definition of x_L. Hence, u_0 can not lie in the subcritical region. It must be supercritical. ∎

As an example, let us take the following smooth and compactly supported initial data.

$$u_0(x)=\begin{cases} e^{-\frac{1}{1-x^2}}, & |x|<1,\\ 0, & |x|\ge 1.\end{cases} \tag{4.3}$$

Figure 4.1 shows the contour plot of $(u_0'(x), u_0(x))$ in the phase plane for all $x \in \mathbb{R}$. Clearly, the curve does not lie in the subcritical region. So, u_0 is supercritical. Theorem 1.1 then implies a finite-time wave break-down.

Figure 4.2 shows the numerical result for the model with initial data (4.3), together with the models introduced in (4.1). The wave break-down can be easily observed, which matches our theoretical result.

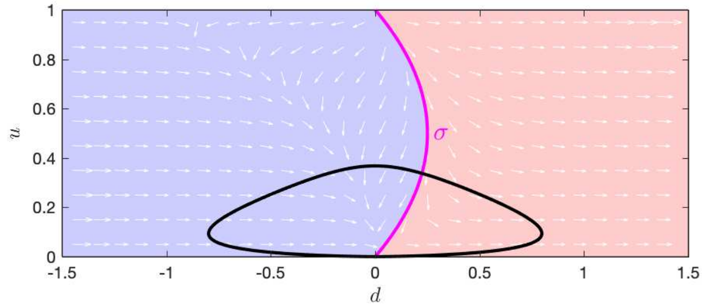

FIG. 4.1. *The contour plot of $(u_0'(x), u_0(x))$ in the phase plane where u_0 is (4.3). This initial condition lies in the supercritical region.*

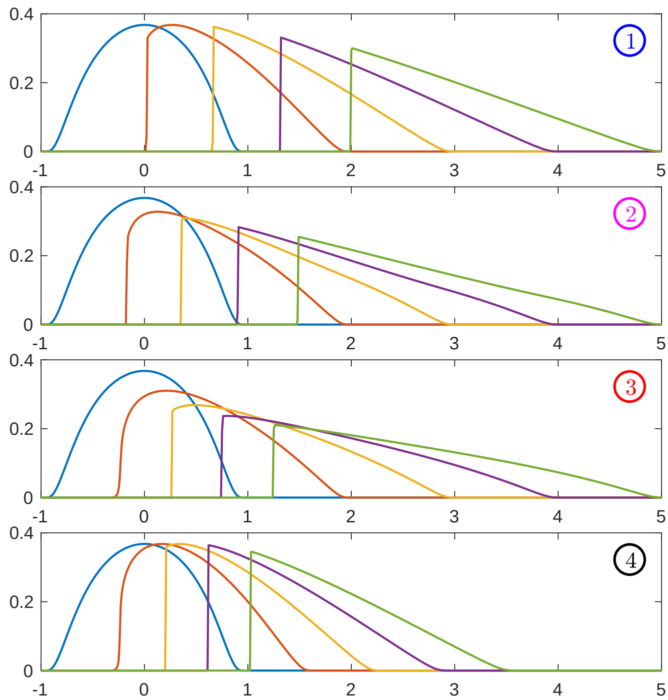

FIG. 4.2. *Snapshots of solutions for the dynamics for the four kernels, with supercritical initial condition (4.3) at time $t = 0,1,2,3,4$.*

Note that since

$$0 \le 1_{[-1,0)}(x) \le 1_{(-\infty,0]}(x) \le 1, \quad \forall\, x \in \mathbb{R}, \tag{4.4}$$

model (1) has the fastest wave speed, while model (4) has the slowest. This is indeed captured in the numerical result.

4.2. Subcritical initial data. We now construct an initial condition u_0 that lies in the subcritical region (1.10).

Due to Proposition 4.1, u_0 can not be compactly supported. Moreover, we need $u_0 \in L^1(\mathbb{R})$. One valid choice is that u_0 decays algebraically when $x \to -\infty$, namely $u_0(x) \sim (-x)^{-\beta}$ for $\beta > 1$. We can check

$$\lim_{x\to-\infty} \frac{u_0'(x)}{u_0(x)} = \lim_{x\to-\infty} \frac{\beta(-x)^{-\beta-1}}{(-x)^{-\beta}} = 0 < 1.$$

Therefore, $(u_0'(x), u_0(x))$ should lie in the subcritical region of the phase plane when x is very negative.

For large x, the choice of u_0 is less critical. As long as $u_0'(x) \le 0$, it always lies in the subcritical region. We can either choose u_0 vanishing for large x, or it decays fast as $x \to +\infty$.

Here is a subcritical initial condition

$$u_0(x) = \begin{cases} 1/x^2, & x \in (-\infty, -3], \\ (3x^5 + 35x^4 + 123x^3 + 81x^2 - 162x + 162)/1458, & x \in (-3, 0], \\ e^{-x}/9, & x \in (0, \infty). \end{cases} \tag{4.5}$$

The middle part is chosen as a polynomial which smoothly connects the two functions, so that $u \in C^2(\mathbb{R})$.

The contour plot of $(u_0'(x), u_0(x))$ is shown in Figure 4.3, which indicates u_0 is subcritical. Therefore, as a consequence of Theorem 1.1, the solution should be globally regular.

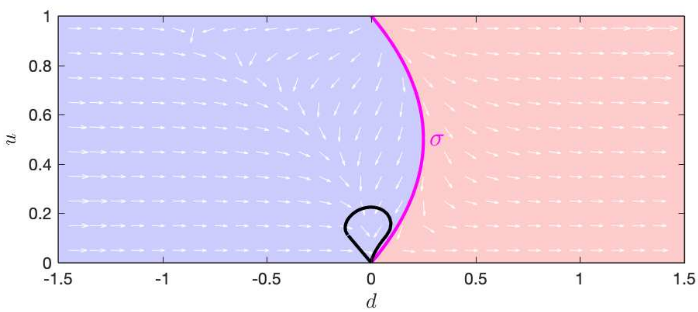

FIG. 4.3. *The contour plot of $(u_0'(x), u_0(x))$ in the phase plane where u_0 is (4.5). This initial condition lies in the subcritical region.*

Figure 4.4 shows the numerical results for all four models with initial condition (4.5). We observe that the solution of our model ③ indeed does not generate shocks.

The wave speeds of the four models behave similar as the supercritical case, due to (4.4). However, very interestingly, our model ③ is the *only* model where there is no finite-time wave break-down. Indeed, we plot the quantity $\|\partial_x u(\cdot,t)\|_{L^\infty} / \|u(\cdot,t)\|_{L^\infty}$ against time t in Figure 4.5. The quantity blows up in finite time for models ①, ② and ④, but remains bounded for our model ③.

5. Further discussion

We have shown a sharp critical threshold for our traffic model (1.1) with look-ahead kernel (1.9). We also compare our model with other classical kernels (4.1) through numerical simulations. The kernel considered in this work has a unique feature that the solution remains globally regular for initial conditions like (4.5).

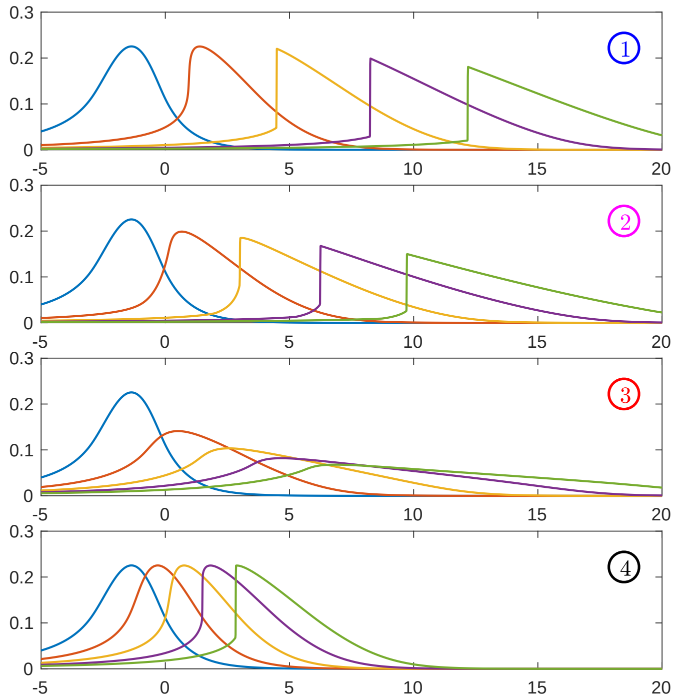

FIG. 4.4. *Snapshots of solutions for the dynamics for the four kernels, with subcritical initial condition* (4.5) *at time* $t = 0, 5, 10, 15, 20$.

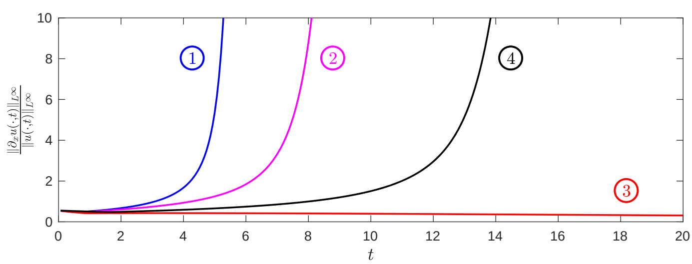

FIG. 4.5. *Numerical indicators of finite time blowup versus global regularity. With initial condition* (4.5)*, only our kernel ③ has a global smooth solution.*

To understand such behavior, we shall focus on the nonlocal slow down factor $e^{-\bar{u}}$. From (4.4), we observe that the model considered in this work has a factor which is neither the largest nor the smallest. Hence, the size of the slow down factor does not matter.

An important feature of the model in this work is that, the slow down factor is monotone increasing. Indeed, we have

$$\partial_x e^{-\bar{u}} = u e^{-\bar{u}} > 0, \quad \forall\ x \ \text{ s.t. } \ u(x) > 0.$$

This implies that the front crowd does not slow down as much as the back crowd. This helps avoid the shock formation, as observed in the example.

For general nonlocal look-ahead kernels, it remains open whether there are subcritical initial data which lead to global regularity. If we consider a family of kernels K_L in (1.6), our result indicates that subcritical initial data exist for $L=\infty$. On the other hand, subcritical initial data does not exist for the LWR model, where $L=0$. For $L\in(0,\infty)$, the problem is open. A conjecture is that subcritical initial data exists for L large enough. This will be left for future investigation.

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Appendix. Composition estimate. In this section, we prove the composition estimate stated in Lemma 2.1. The estimate is useful to control the nonlocal weight $e^{-\bar{u}}$ for our system. We include a proof for self-consistency, as we have been unable to find it in the literature.

Proof. **(Proof of Lemma 2.1.)** We first consider the case when s is an integrer. Express $\partial_x^s(f(g(x)))$ using Faà di Bruno's formula

$$\partial_x^s(f(g(x)))=\sum_{\alpha\in S_s}C_\alpha(x)\prod_{r=1}^{s}(\partial_x^r g(x))^{\alpha_r},$$

where

$$S_s=\left\{\alpha=(\alpha_1,\cdots,\alpha_s)\ :\ a_k\in\mathbb{N},\quad \sum_{r=1}^{s}r\alpha_r=s,\quad \sum_{r=1}^{s}\alpha_r\le s.\right\},$$

and

$$C_\alpha(x)=s!\prod_{r=1}^{s}\left(\frac{1}{\alpha_r!\cdot(r!)^{\alpha_r}}\right)\partial_x^{\nu(\alpha)}f(g(x)),\quad \nu(\alpha)=\sum_{r=1}^{s}\alpha_r.$$

Then,

$$\|\partial_x^s(f\circ g)\|_{L^2}\le\sum_{\alpha\in S_s}\|C_\alpha\|_{L^\infty}\left\|\prod_{r=1}^{s}(\partial_x^r g)^{\alpha_r}\right\|_{L^2}\lesssim\|f\|_{C^s}\sum_{\alpha\in S_s}\left\|\prod_{r=1}^{s}(\partial_x^r g)^{\alpha_r}\right\|_{L^2}.$$

Now, we estimate the last term. Applying Hölder's inequality, we get

$$\left\|\prod_{r=1}^{s}(\partial_x^r g)^{\alpha_r}\right\|_{L^2}\le\prod_{r=1}^{s}\|(\partial_x^r g)^{\alpha_r}\|_{L^{p_r}}=\prod_{r=1}^{s}\|\partial_x^r g\|_{L^{\alpha_r p_r}}^{\alpha_r},$$

where $\{p_r\}_{r=1}^s$ is chosen as $p_r=\frac{2s}{r\alpha_r}$. So we have

$$\sum_{r=1}^{s}\frac{1}{p_r}=\frac{1}{2s}\sum_{r=1}^{s}r\alpha_r=\frac{1}{2}.$$

For each term $\|\partial_x^r g\|_{L^{\alpha_r p_r}}$, we apply the Gagliardo-Nirenberg-Sobolev interpolation inequality

$$\|\partial_x^r g\|_{L^{\alpha_r p_r}}=\|\partial_x^r g\|_{L^{\frac{2s}{r}}}\lesssim\|\partial_x^s g\|_{L^2}^{\frac{r}{s}}\|g\|_{L^\infty}^{1-\frac{r}{s}}.$$

Collecting all terms together, we obtain

$$\prod_{r=1}^{s}\|\partial_x^r g\|_{L^{\alpha_r p_r}}^{\alpha_r} \lesssim \|\partial_x^s g\|_{L^2}^{\sum_{r=1}^s \alpha_r \frac{r}{s}} \|g\|_{L^\infty}^{\sum_{r=1}^s \alpha_r (1-\frac{r}{s})} = \|\partial_x^s g\|_{L^2}\|g\|_{L^\infty}^{\nu(\alpha)-1}.$$

This concludes the proof.

Next, we discuss the case when s is not an integer. For $s\in(0,1)$, one can directly apply the chain rule for fractional derivatives [12, Proposition 3.1]

$$\|f\circ g\|_{\dot{H}^s} \le C\|\partial_x f\|_{L^\infty}\|g\|_{\dot{H}^s},$$

where C is a constant depending on s and $\|g\|_{L^\infty}$.

For $s>1$, we can combine the estimate for $\lfloor s\rfloor$ (the largest integer that is smaller or equal to s) and the fractional chain rule for $s-\lfloor s\rfloor$. The details are left to the readers. □

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