COMMUN. MATH. SCI.
Vol. 20, No. 5, pp. 1193–1209

GLOBAL EXISTENCE AND LARGE TIME BEHAVIOR OF STRONG SOLUTIONS FOR NONHOMOGENEOUS HEAT CONDUCTING NAVIER-STOKES EQUATIONS WITH LARGE INITIAL DATA AND VACUUM*

XIN ZHONG†

Abstract. We are concerned with an initial boundary value problem of two-dimensional nonhomogeneous heat conducting Navier-Stokes equations in bounded domains. Applying delicate energy estimates and Desjardins' interpolation inequality, we derive the global existence and uniqueness of strong solutions. Furthermore, we also show large-time decay rates of the solution. Note that the initial data can be arbitrarily large and the initial density allows vacuum states.

Keywords. Nonhomogeneous heat conducting Navier-Stokes equations; global strong solution; large-time behavior; vacuum.

AMS subject classifications. 76D05; 76D03.

1. Introduction

Let $\Omega\subset\mathbb{R}^2$ be a bounded smooth domain. We study nonhomogeneous heat conducting Navier-Stokes equations (see [18, p. 23])

$$\begin{cases} \rho_t+\operatorname{div}(\rho\mathbf{u})=0,\\ (\rho\mathbf{u})_t+\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})-\mu\Delta\mathbf{u}+\nabla P=\mathbf{0},\\ c_v[(\rho\theta)_t+\operatorname{div}(\rho\mathbf{u}\theta)]-\kappa\Delta\theta=2\mu|\mathfrak{D}(\mathbf{u})|^2,\\ \operatorname{div}\mathbf{u}=0, \end{cases} \tag{1.1}$$

with the initial condition

$$(\rho,\rho\mathbf{u},\rho\theta)(x,0)=(\rho_0,\rho_0\mathbf{u}_0,\rho_0\theta_0)(x),\quad x\in\Omega, \tag{1.2}$$

and the boundary condition

$$\mathbf{u}=\mathbf{0},\ \frac{\partial\theta}{\partial\mathbf{n}}=0,\ x\in\partial\Omega,\ t>0, \tag{1.3}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$. Here $\rho,\mathbf{u},\theta,P$ are the fluid density, velocity, absolute temperature, and pressure, respectively. $\mathfrak{D}(\mathbf{u})$ denotes the deformation tensor given by

$$\mathfrak{D}(\mathbf{u})=\frac{1}{2}(\nabla\mathbf{u}+(\nabla\mathbf{u})^{tr}).$$

The positive constant μ is the viscosity coefficient of the fluid, while c_v and κ are the heat capacity and the ratio of the heat conductivity coefficient over the heat capacity, respectively.

*Received: January 09, 2021; Accepted (in revised form): November 09, 2021. Communicated by Zhen Lei.
This research was partially supported by National Natural Science Foundation of China (Nos. 11901474, 12071359), Exceptional Young Talents Project of Chongqing Talent (No. cstc2021ycjh-bgzxm0153), and the Innovation Support Program for Chongqing Overseas Returnees (No. cx2020082).
†School of Mathematics and Statistics, Southwest University, Chongqing 400715, People's Republic of China (xzhong1014@amss.ac.cn).

Nonhomogeneous Navier-Stokes equations describe a fluid which is obtained by mixing two miscible fluids that are incompressible and that have different densities. It may also describe a fluid containing a melted substance. Due to their prominent roles in modeling many phenomena in physics, the nonhomogeneous Navier-Stokes equations have been extensively studied, mathematically. Given $0\le\rho_0\in L^\infty$, and $\mathbf{u}_0$ satisfying $\operatorname{div}\mathbf{u}_0=0$, $\sqrt{\rho_0}\mathbf{u}_0\in L^2$, Lions [16] proved that there exists a global weak solution (see also [20] and the references therein for an overview of results on weak solutions). For the initial density allowing vacuum, Choe and Kim [3] proposed a compatibility condition on the initial data and investigated the local existence of strong solutions, which was later improved by the authors [13,14] without using such a compatibility condition. The global (with general large data) existence of strong solutions with nonnegative density on two-dimensional (2D) bounded domains and in the whole space $\mathbb{R}^2$ was established by Huang and Wang [11] and Lü, Shi, and Zhong [19], respectively. Meanwhile, there are also very interesting investigations about the global existence of strong solutions to the 3D nonhomogeneous Navier-Stokes equations under some smallness assumptions, please refer to [4, 5, 10, 12, 17, 26]. The mathematical studies on the nonhomogeneous Navier-Stokes equations between 2D and 3D case appear highly different.

Recently, some attention was focused on the nonhomogeneous heat conducting Navier-Stokes Equations (1.1). We refer the reader to [18, Chapter 2] for the detailed derivation of such a system. Cho and Kim [2] proved the local existence of strong solutions for such a model with vacuum under compatibility conditions. Later, Zhong [27] showed global strong solutions of 3D initial boundary value problems with vacuum, provided that some smallness condition holds true. Then Wang et al. [22] improved [27] for the system with the external force. At the same time, Xu and Yu [23,24] extended the global existence result in [27] to the 3D nonhomogeneous heat conducting Navier-Stokes equations with density-temperature-dependent viscosity and vacuum. Since global well-posedness to the nonhomogeneous heat conducting Navier-Stokes equations with general large data in three dimensions is still an open problem, the goal of this paper is to establish the global existence and large-time behavior of strong solutions to the 2D problem (1.1)–(1.3) with general large initial data. The initial density is allowed to vanish.

Before stating our main result, we first explain the notations used throughout this paper. For $1\le p\le\infty$ and integer $k\ge 0$, the standard Sobolev spaces are denoted by

$$\begin{cases} L^p=L^p(\Omega),\ W^{k,p}=W^{k,p}(\Omega),\ H^k=H^{k,2}(\Omega),\\ H_0^1=\{u\in H^1|u=0 \text{ on } \partial\Omega\},\ H_{\mathbf{n}}^2=\{u\in H^2|\nabla u\cdot\mathbf{n}=0 \text{ on } \partial\Omega\}. \end{cases}$$

Our main result reads as follows.

THEOREM 1.1. *For constant $q\in(2,\infty)$, assume that the initial data $(\rho_0\ge 0,\mathbf{u}_0,\theta_0\ge 0)$ satisfies*

$$\rho_0\in W^{1,q}(\Omega),\ \mathbf{u}_0\in H_0^1(\Omega),\ \theta_0\in H_{\mathbf{n}}^2(\Omega),\ \operatorname{div}\mathbf{u}_0=0, \tag{1.4}$$

then the problem (1.1)–(1.3) has a unique global strong solution $(\rho\ge 0,\mathbf{u},\theta\ge 0)$ such that for any $0<\tau<T<\infty$ and $2\le r<q$,

$$\begin{cases} \rho_t\in L^\infty(0,T;L^r),\ \rho\in L^\infty(0,T;W^{1,q}),\\ \nabla\mathbf{u}\in L^\infty(0,T;L^2)\cap L^\infty(\tau,T;H^1)\cap L^2(\tau,T;H^2),\\ \nabla\theta\in L^\infty(\tau,T;H^1)\cap L^2(\tau,T;H^2),\\ \nabla P\in L^\infty(\tau,T;L^2)\cap L^2(\tau,T;H^1),\\ \sqrt{\rho}\theta,\ t\nabla\theta\in L^\infty(0,T;L^2),\ t\nabla\mathbf{u}\in L^\infty(0,T;H^1),\\ t\sqrt{\rho}\mathbf{u}_t,\ t\sqrt{\rho}\theta_t\in L^\infty(0,T;L^2)\cap L^2(0,T;L^2),\\ e^{\frac{\sigma}{2}t}\nabla\mathbf{u},\ t\nabla\mathbf{u}_t,\ t\nabla\theta_t,\ \nabla\theta\in L^2(0,T;L^2), \end{cases} \tag{1.5}$$

where $\sigma \triangleq \frac{\mu}{d^2\|\rho_0\|_{L^\infty}}$ *with* d *being the diameter of* Ω*. Moreover, there exists a positive constant* C *depending only on* Ω*,* μ*,* c_v*,* κ*,* q*, and the initial data such that, for* $t\ge 1$*,*

$$\begin{cases} \|\mathbf{u}(\cdot,t)\|_{H^2}^2+\|\nabla P(\cdot,t)\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}_t(\cdot,t)\|_{L^2}^2\le Ce^{-\sigma t}, \\ \|\nabla\theta(\cdot,t)\|_{L^2}^2+\|\sqrt{\rho}\theta_t(\cdot,t)\|_{L^2}^2\le Ct^{-2}. \end{cases} \tag{1.6}$$

Remark 1.1. Our Theorem 1.1 holds for arbitrarily large initial data which is in sharp contrast to [27] where some smallness condition on the initial data is needed in order to obtain the global existence of strong solutions to the 3D nonhomogeneous heat conducting Navier-Stokes equations.

Remark 1.2. Compared with nonhomogeneous Navier-Stokes equations without heat effect [11], we remove the classical compatibility condition used in [11] via appropriate time-weighted techniques. Furthermore, we can obtain decay estimates of the solution. We emphasize that it seems difficult to show large-time behavior by applying the methods used in [11].

We now make some comments on the key ingredients of the analysis in this paper. The local existence and uniqueness of strong solutions to the problem (1.1)–(1.3) follows from the works in the literature such as [13] (see Lemma 2.1). Thus our efforts are devoted to establishing global *a priori* estimates on strong solutions to the system (1.1) in suitable higher-order norms. It should be pointed out that compared with the related works in the literature, the proof of Theorem 1.1 is much more involved due to the absence of the positive lower bound for the initial density as well as the absence of the smallness and the compatibility conditions for the initial data. Consequently, some new ideas are needed to overcome these difficulties.

First, applying the upper bounds on the density (see (3.2)) and the Poincaré inequality, we derive that $\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2$ decays with the rate of $e^{-\sigma t}$ for some $\sigma>0$ depending only on $\mu, \|\rho_0\|_{L^\infty}$, and the diameter of Ω (see (3.10)). Next, we need to obtain the bound of $\|\nabla\mathbf{u}\|_{L^2}^2$. However, the presence of vacuum prevents us from achieving this goal. To overcome this difficulty, we make use of Desjardins' interpolation inequality (see Lemma 2.4) to obtain time-weighted estimate on the $L^\infty(0,T;L^2)$-norm of the gradient of the velocity (see (3.11) and (3.12)). Indeed, the *time-weighted estimate* is a crucial technique in dropping the compatibility condition on the initial data (see [19] for example). Next, due to the structure of the system (1.1), the basic energy estimate provides us with

$$\int\Big(c_v\rho\theta+\frac{1}{2}\rho|\mathbf{u}|^2\Big)dx=\int\Big(c_v\rho_0\theta_0+\frac{1}{2}\rho_0|\mathbf{u}_0|^2\Big)dx,$$

and there isn't any useful dissipation estimate on θ. To overcome this difficulty, we recover the crucial dissipation estimate of the form $\int_0^T\|\nabla\theta\|_{L^2}^2dt$ (see Lemma 3.3), which is needed to get the time-weighted estimate on $\|\nabla\theta\|_{L^2}^2$ (see (3.42)), while this time-weighted estimate on $\|\nabla\theta\|_{L^2}^2$ in turn affects the derivation of time-weighted estimate of $\|\sqrt{\rho}\theta_t\|_{L^2}^2$ (see (3.51)). Next, with time-weighted estimates on the velocity at hand, we can then obtain that $\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2$ decays as t^{-2} for large time (see (3.28)). In fact, all these decay-in-time rates play an important role in obtaining the desired uniform bound (with respect to time) on the $L^1(0,T;L^\infty)$-norm of $\nabla\mathbf{u}$ (see (3.48)). Finally, using these *a priori* estimates, we establish the time-independent higher order estimates on the solution $(\rho,\mathbf{u},P,\theta)$, see Lemmas 3.5 and 3.7 for details.

The rest of this paper is organized as follows. In Section 2, we collect some elementary facts and inequalities that will be used later. Section 3 is devoted to the a priori estimates. Finally, we give the proof of Theorem 1.1 in Section 4.

2. Preliminaries

In this section, we will recall some known facts and elementary inequalities that will be frequently used later.

We begin with the local existence and uniqueness of strong solutions whose proof can be performed by using standard energy methods (see, e.g., [13]).

LEMMA 2.1. *Assume that* $(\rho_0,\mathbf{u}_0,\theta_0)$ *satisfies* (1.4). *Then there exist a small time* $T>0$ *and a unique strong solution* $(\rho,\mathbf{u},\theta)$ *to the problem* (1.1)–(1.3) *in* $\Omega\times(0,T]$.

Next, the following Gagliardo-Nirenberg inequality (see [9, Theorem 10.1, p. 27]) will be useful in the next section.

LEMMA 2.2. *Let* $\Omega\subset\mathbb{R}^2$ *be a bounded smooth domain. Assume that* $1\le q,r\le\infty$, *and* j,m *are arbitrary integers satisfying* $0\le j<m$. *If* $v\in W^{m,r}(\Omega)\cap L^q(\Omega)$, *then we have*

$$\|D^j v\|_{L^p}\le C\|v\|_{L^q}^{1-a}\|v\|_{W^{m,r}}^a,$$

where

$$-j+\frac{2}{p}=(1-a)\frac{2}{q}+a\Big(-m+\frac{2}{r}\Big),$$

and

$$a\in\begin{cases}[\frac{j}{m},1), & \text{if } m-j-\frac{2}{r} \text{ is a nonnegative integer},\\ [\frac{j}{m},1], & \text{otherwise.}\end{cases}$$

The constant C *depends only on* m,j,q,r,a, *and* Ω. *In particular, we have*

$$\|v\|_{L^4}^4\le C\|v\|_{L^2}^2\|v\|_{H^1}^2, \tag{2.1}$$

which will be frequently used in the next section.

Next, we give some regularity properties for the following Stokes system:

$$\begin{cases}-\mu\Delta\mathbf{u}+\nabla P=\mathbf{F}, & x\in\Omega,\\ \operatorname{div}\mathbf{u}=0, & x\in\Omega,\\ \mathbf{u}=\mathbf{0}, & x\in\partial\Omega.\end{cases} \tag{2.2}$$

LEMMA 2.3. *Suppose that* $\mathbf{F}\in L^r(\Omega)$ *with* $1<r<\infty$. *Let* $(\mathbf{u},P)\in H_0^1\times L^2$ *be the unique weak solution to the problem* (2.2), *then* $(\mathbf{u},P)\in W^{2,r}\times W^{1,r}$ *and there exists a constant* C *depending only on* Ω *and* r *such that*

$$\|\mathbf{u}\|_{W^{2,r}}+\|P\|_{W^{1,r}/\mathbb{R}}\le C\|\mathbf{F}\|_{L^r}.$$

Proof. See [1, Proposition 4.3]. □

Finally, the following interpolation inequality was first obtained by Desjardins [6, Lemma 1] for periodic domains. Later, following Desjardins' idea, Ye [25, Lemma 2.2] extended it to the bounded domains.

LEMMA 2.4. *Let* $\Omega\subset\mathbb{R}^2$ *be a bounded smooth domain. Suppose that* $0\le\rho\le\bar\rho$ *and* $\mathbf{u}\in H_0^1(\Omega)$, *then we have*

$$\|\sqrt{\rho}\mathbf{u}\|_{L^4}^2\le C(\bar\rho,\Omega)(1+\|\sqrt{\rho}\mathbf{u}\|_{L^2})\|\nabla\mathbf{u}\|_{L^2}\sqrt{\ln(2+\|\nabla\mathbf{u}\|_{L^2}^2)}. \tag{2.3}$$

3. *A priori* estimates

In this section, we will establish some necessary *a priori* bounds for strong solutions $(\rho,\mathbf{u},\theta)$ to the problem (1.1)–(1.3) to extend the local strong solution. Thus, let $T>0$ be a fixed time and $(\rho,\mathbf{u},\theta)$ be the strong solution to (1.1)–(1.3) on $\Omega\times(0,T]$ with initial data $(\rho_0,\mathbf{u}_0,\theta_0)$ satisfying (1.4). In what follows, we write

$$\int \cdot dx = \int_\Omega \cdot dx.$$

Sometimes we use $C(f)$ to emphasize the dependence on f.

Before proceeding, we rewrite another equivalent form of the system (1.1) as the following

$$\begin{cases}\rho_t+\mathbf{u}\cdot\nabla\rho=0,\\ \rho\mathbf{u}_t+\rho\mathbf{u}\cdot\nabla\mathbf{u}-\mu\Delta\mathbf{u}+\nabla P=\mathbf{0},\\ c_v[\rho\theta_t+\rho\mathbf{u}\cdot\nabla\theta]-\kappa\Delta\theta=2\mu|\mathfrak{D}(\mathbf{u})|^2,\\ \operatorname{div}\mathbf{u}=0.\end{cases} \tag{3.1}$$

We begin with the following standard energy estimate and the L^∞-norm estimate of the density.

Lemma 3.1. *It holds that*

$$\sup_{0\le t\le T}\|\rho\|_{L^\infty}\le\|\rho_0\|_{L^\infty}, \tag{3.2}$$

and

$$\begin{aligned}&\sup_{0\le t\le T}\left(c_v\|\rho\theta\|_{L^1}+\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+e^{\sigma t}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2\right)+\int_0^T e^{\sigma t}(\mu\|\nabla\mathbf{u}\|_{L^2}^2)dt\\ &\le 2c_v\|\rho_0\theta_0\|_{L^1}+2\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2,\end{aligned} \tag{3.3}$$

where $\sigma\triangleq\frac{\mu}{d^2\|\rho_0\|_{L^\infty}}$ *with* d *being the diameter of* Ω.

Proof.

(1) Since $(3.1)_1$ is a transport equation, we then directly obtain the desired (3.2). Moreover, applying standard maximum principle (see [8, p. 43]) to (3.1) along with $\rho_0,\theta_0\ge 0$ shows

$$\inf_{\Omega\times[0,T]}\rho(x,t)\ge 0,\quad \inf_{\Omega\times[0,T]}\theta(x,t)\ge 0.$$

(2) Multiplying $(3.1)_2$ by $\mathbf{u}$ and integrating (by parts) over Ω, we derive that

$$\frac{1}{2}\frac{d}{dt}\int\rho|\mathbf{u}|^2dx+\mu\int|\nabla\mathbf{u}|^2dx=0. \tag{3.4}$$

Integrating $(3.1)_3$ with respect to the spatial variable and using (1.3) give rise to

$$c_v\frac{d}{dt}\int\rho\theta dx-2\mu\int|\mathfrak{D}(\mathbf{u})|^2dx=0. \tag{3.5}$$

By virtue of $(3.1)_4$ and integration by parts, we have

$$2\mu\int|\mathfrak{D}(\mathbf{u})|^2dx=\frac{\mu}{2}\int(\partial_iu^j+\partial_ju^i)(\partial_iu^j+\partial_ju^i)dx=\mu\int|\nabla\mathbf{u}|^2dx, \tag{3.6}$$

which, together with (3.5) and (3.4), yields

$$\frac{d}{dt}\int\left(c_v\rho\theta+\frac{1}{2}\rho|\mathbf{u}|^2\right)dx=0. \tag{3.7}$$

Integrating (3.7) with respect to time leads to

$$\int\left(c_v\rho\theta+\frac{1}{2}\rho|\mathbf{u}|^2\right)dx=\int\left(c_v\rho_0\theta_0+\frac{1}{2}\rho_0|\mathbf{u}_0|^2\right)dx. \tag{3.8}$$

(3) It follows from the Poincaré inequality (see [21, (A.3), p. 266]) and (3.2) that

$$\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2\le\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^2}^2\le\|\rho_0\|_{L^\infty}d^2\|\nabla\mathbf{u}\|_{L^2}^2,$$

where d is the diameter of Ω. Hence, we get

$$\frac{1}{d^2\|\rho_0\|_{L^\infty}}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2\le\|\nabla\mathbf{u}\|_{L^2}^2. \tag{3.9}$$

Consequently, letting $\sigma\triangleq\frac{\mu}{d^2\|\rho_0\|_{L^\infty}}$, then we derive from (3.4) and (3.9) that

$$\frac{d}{dt}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\sigma\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\mu\|\nabla\mathbf{u}\|_{L^2}^2\le 0.$$

This implies that

$$\frac{d}{dt}e^{\sigma t}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+e^{\sigma t}\mu\|\nabla\mathbf{u}\|_{L^2}^2\le 0.$$

Integrating the above inequality over $[0,T]$ leads to

$$\sup_{0\le t\le T}\left(e^{\sigma t}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2\right)+\int_0^T e^{\sigma t}(\mu\|\nabla\mathbf{u}\|_{L^2}^2)dt\le\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2, \tag{3.10}$$

which, combined with (3.8), gives (3.3).

□

Next, the following lemma concerns the key time-weighted estimates on the $L^\infty(0,T;L^2)$-norm of the gradient of the velocity.

LEMMA 3.2. *Let σ be as in Lemma 3.1, then there exists a positive constant C depending only on Ω, μ, $\|\rho_0\|_{L^\infty}$, and $\|\nabla\mathbf{u}_0\|_{L^2}$ such that for $i\in\{0,1,2\}$,*

$$\sup_{0\le t\le T}\left(t^i\|\nabla\mathbf{u}\|_{L^2}^2\right)+\int_0^T t^i\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2dt\le C, \tag{3.11}$$

$$\sup_{0\le t\le T}\left(e^{\sigma t}\|\nabla\mathbf{u}\|_{L^2}^2\right)+\int_0^T e^{\sigma t}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2dt\le C. \tag{3.12}$$

Proof.

(1) Multiplying $(3.1)_2$ by $\mathbf{u}_t$ and integrating the resulting equation over Ω, we get that

$$\frac{\mu}{2}\frac{d}{dt}\int|\nabla\mathbf{u}|^2dx+\int\rho|\mathbf{u}_t|^2dx=-\int\rho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_tdx. \tag{3.13}$$

By Hölder's and Gagliardo-Nirenberg inequalities, we have

$$\begin{aligned}\left|-\int \rho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t dx\right| &\le \frac{1}{2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\frac{1}{2}\|\sqrt{\rho}\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2\\ &\le \frac{1}{2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\|\sqrt{\rho}\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{H^2}. \end{aligned}\tag{3.14}$$

Substituting (3.14) into (3.13), we derive that

$$\mu\frac{d}{dt}\|\nabla\mathbf{u}\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\le C\|\sqrt{\rho}\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{H^2}.\tag{3.15}$$

Recall that $(\mathbf{u},P)$ satisfies the following Stokes system

$$\begin{cases}-\mu\Delta\mathbf{u}+\nabla P=-\rho\mathbf{u}_t-\rho\mathbf{u}\cdot\nabla\mathbf{u}, & x\in\Omega,\\ \operatorname{div}\mathbf{u}=0, & x\in\Omega,\\ \mathbf{u}=\mathbf{0}, & x\in\partial\Omega.\end{cases}\tag{3.16}$$

Applying Lemma 2.3 with $\mathbf{F}=-\rho\mathbf{u}_t-\rho\mathbf{u}\cdot\nabla\mathbf{u}$, we obtain from (3.2) that

$$\begin{aligned}\|\mathbf{u}\|_{H^2}+\|\nabla P\|_{L^2}&\le C\left(\|\rho\mathbf{u}_t\|_{L^2}+\|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}\right)\\ &\le C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}+C\|\sqrt{\rho}\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}\|_{L^4}\\ &\le C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}+C\|\sqrt{\rho}\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\mathbf{u}\|_{H^2}^{\frac{1}{2}}\\ &\le C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}+C\|\sqrt{\rho}\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^2}+\frac{1}{2}\|\mathbf{u}\|_{H^2},\end{aligned}$$

and thus

$$\|\mathbf{u}\|_{H^2}+\|\nabla P\|_{L^2}\le C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}+C\|\sqrt{\rho}\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^2}.\tag{3.17}$$

Inserting (3.17) into (3.15) and applying Cauchy-Schwarz inequality, we deduce that

$$\frac{d}{dt}\|\nabla\mathbf{u}\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\le C\|\sqrt{\rho}\mathbf{u}\|_{L^4}^4\|\nabla\mathbf{u}\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^4,\tag{3.18}$$

and so by virtue of (2.3) and (3.3), we have

$$\frac{d}{dt}\|\nabla\mathbf{u}\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\le C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^2\left(1+\ln(2+\|\nabla\mathbf{u}\|_{L^2}^2)\right).\tag{3.19}$$

Setting

$$f(t)\triangleq 2+\|\nabla\mathbf{u}\|_{L^2}^2,$$

then we deduce from (3.19) that

$$f'(t)\le C\|\nabla\mathbf{u}\|_{L^2}^2 f(t)+C\|\nabla\mathbf{u}\|_{L^2}^2 f(t)\ln f(t),$$

which yields that

$$(\log f(t))'\le C\|\nabla\mathbf{u}\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2\log(f(t)).$$

This, combined with Gronwall's inequality and (3.3), leads to

$$\sup_{0\le t\le T}\|\nabla\mathbf{u}\|_{L^2}^2\le C. \tag{3.20}$$

Integrating (3.19) with respect to t together with (3.20) and (3.3) leads to

$$\int_0^T\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2dt\le C.$$

For $i\in\{1,2\}$, we can obtain similar results. Here we omit the details for simplicity.

(2) Multiplying (3.19) by $e^{\sigma t}$ and applying (3.20), we derive that

$$\frac{d}{dt}\left(e^{\sigma t}\|\nabla\mathbf{u}\|_{L^2}^2\right)+e^{\sigma t}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\le Ce^{\sigma t}\|\nabla\mathbf{u}\|_{L^2}^2+\sigma e^{\sigma t}\|\nabla\mathbf{u}\|_{L^2}^2\le Ce^{\sigma t}\|\nabla\mathbf{u}\|_{L^2}^2. \tag{3.21}$$

Integrating (3.21) over $[0,T]$ together with (3.10) leads to (3.12).

□

REMARK 3.1. It follows from (2.3), (3.8), and (3.11) that

$$\sup_{0\le t\le T}\|\sqrt{\rho}\mathbf{u}\|_{L^4}^2\le C. \tag{3.22}$$

Next, we improve the regularity of the temperature θ as follows.

LEMMA 3.3. *There exists a positive constant C depending only on Ω, μ, c_v, κ, and the initial data such that*

$$\sup_{0\le t\le T}\|\sqrt{\rho}\theta\|_{L^2}^2+\int_0^T\|\nabla\theta\|_{L^2}^2dt\le C. \tag{3.23}$$

Proof. Multiplying $(3.1)_3$ by θ and integrating the resulting equation over Ω yield

$$\begin{aligned}c_v\frac{d}{dt}\|\sqrt{\rho}\theta\|_{L^2}^2+2\kappa\|\nabla\theta\|_{L^2}^2&\le C\int|\nabla\mathbf{u}|^2\theta dx\\&\le C\|\theta\|_{L^4}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^4}\\&\le C\|\theta\|_{H^1}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\|\nabla\mathbf{u}\|_{H^1}^{\frac12}\\&\le C\|\theta\|_{H^1}\|\nabla\mathbf{u}\|_{L^2}^{\frac32}\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac12}+\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\right)\end{aligned} \tag{3.24}$$

due to Sobolev's inequality, (2.1), (3.17), and (3.22). To estimate $\|\theta\|_{H^1}$, denote by $\bar\theta\triangleq\frac{1}{|\Omega|}\int\theta dx$ (the average of θ), then we obtain from (3.2) and the Poincaré inequality that

$$|\bar\theta|\int\rho dx\le\left|\int\rho\theta dx\right|+\left|\int\rho(\theta-\bar\theta)dx\right|\le\|\rho\theta\|_{L^1}+C\|\nabla\theta\|_{L^2},$$

which, together with (3.8) and the fact that $\left|\int v dx\right|+\|\nabla v\|_{L^2}$ is an equivalent norm to the usual one in $H^1(\Omega)$, implies that

$$\|\theta\|_{H^1}\le C+C\|\nabla\theta\|_{L^2}, \tag{3.25}$$

$$\|\theta\|_{H^1}\le C\|\sqrt{\rho}\theta\|_{L^2}+C\|\nabla\theta\|_{L^2}. \tag{3.26}$$

Hence, we obtain from (3.24), (3.26), and (3.11) that

$$
\begin{aligned}
c_v \frac{d}{dt}\|\sqrt{\rho}\theta\|_{L^2}^2 + 2\kappa\|\nabla\theta\|_{L^2}^2 &\le C\big(\|\sqrt{\rho}\theta\|_{L^2} + \|\nabla\theta\|_{L^2}\big)\|\nabla \mathbf{u}\|_{L^2}^{\frac{3}{2}}\big(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{1}{2}} + \|\nabla \mathbf{u}\|_{L^2}^{\frac{1}{2}}\big) \\
&\le \varepsilon\|\nabla\theta\|_{L^2}^2 + C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2\|\sqrt{\rho}\theta\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2,
\end{aligned}
$$

which implies after choosing ε suitably small that

$$
\frac{d}{dt}\|\sqrt{\rho}\theta\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 \le C\|\nabla\mathbf{u}\|_{L^2}^2\|\sqrt{\rho}\theta\|_{L^2}^2 + C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2. \tag{3.27}
$$

Hence, we obtain the desired (3.23) from (3.27), Gronwall's inequality, (3.10), and (3.11). ∎

LEMMA 3.4. *There exists a positive constant C depending only on Ω, μ, c_v, κ, and the initial data such that for $i \in \{1,2\}$,*

$$
\sup_{0\le t\le T}\big[t^i\big(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\big)\big] + \int_0^T t^i\big(\|\nabla\mathbf{u}_t\|_{L^2}^2 + \|\sqrt{\rho}\theta_t\|_{L^2}^2\big)dt \le C. \tag{3.28}
$$

Moreover, for σ as that in Lemma 3.1, one has, for $\zeta(T)\triangleq\min\{1,T\}$, that

$$
\sup_{\zeta(T)\le t\le T}\big[e^{\sigma t}\big(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\big)\big] + \int_{\zeta(T)}^T e^{\sigma t}\|\nabla\mathbf{u}_t\|_{L^2}^2 dt \le C. \tag{3.29}
$$

Proof.

(1) Differentiating $(3.1)_2$ with respect to t, we arrive at

$$
\rho\mathbf{u}_{tt} + \rho\mathbf{u}\cdot\nabla\mathbf{u}_t - \mu\Delta\mathbf{u}_t = -\nabla P_t + \rho_t(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) - \rho\mathbf{u}_t\cdot\nabla\mathbf{u}. \tag{3.30}
$$

Multiplying (3.30) by $\mathbf{u}_t$ and integrating (by parts) over Ω and using $(1.1)_1$ yield

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int\rho|\mathbf{u}_t|^2dx + \mu\int|\nabla\mathbf{u}_t|^2dx &= \int \operatorname{div}(\rho\mathbf{u})|\mathbf{u}_t|^2dx + \int\operatorname{div}(\rho\mathbf{u})\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_tdx \\
&\quad - \int\rho\mathbf{u}_t\cdot\nabla\mathbf{u}\cdot\mathbf{u}_tdx \triangleq J_1 + J_2 + J_3.
\end{aligned} \tag{3.31}
$$

By virtue of Hölder's inequality, Sobolev's inequality, (3.2), and (3.11), we deduce that

$$
\begin{aligned}
|J_1| &= \left|-\int\rho\mathbf{u}\cdot\nabla|\mathbf{u}_t|^2dx\right| \\
&\le 2\|\rho\|_{L^\infty}^{\frac{1}{2}}\|\mathbf{u}\|_{L^6}\|\sqrt{\rho}\mathbf{u}_t\|_{L^3}\|\nabla\mathbf{u}_t\|_{L^2} \\
&\le C\|\rho\|_{L^\infty}^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\mathbf{u}_t\|_{L^6}^{\frac{1}{2}}\|\nabla\mathbf{u}_t\|_{L^2} \\
&\le C\|\rho\|_{L^\infty}^{\frac{3}{4}}\|\nabla\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{u}_t\|_{L^2}^{\frac{3}{2}} \\
&\le \frac{\mu}{6}\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2; \\
|J_2| &= \left|-\int\rho\mathbf{u}\cdot\nabla(\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t)dx\right| \\
&\le \int\big(\rho|\mathbf{u}||\nabla\mathbf{u}|^2|\mathbf{u}_t| + \rho|\mathbf{u}|^2|\nabla^2\mathbf{u}||\mathbf{u}_t| + \rho|\mathbf{u}|^2|\nabla\mathbf{u}||\nabla\mathbf{u}_t|\big)\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\le C\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^4}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}+C\|\mathbf{u}\|_{L^\infty}^2\|\nabla^2\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}\\
&\quad +C\|\mathbf{u}\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2}\\
&\le C\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}\\
&\quad +C\|\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{H^2}\|\nabla^2\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}+C\|\mathbf{u}\|_{H^2}\|\nabla\mathbf{u}_t\|_{L^2}\\
&\le \frac{\mu}{6}\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\mathbf{u}\|_{H^2}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\|\mathbf{u}\|_{H^2}^2;
\end{aligned}
$$

$$
\begin{aligned}
|J_3|&\le\|\nabla\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^4}^2\le C\|\nabla\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac12}\|\sqrt{\rho}\mathbf{u}_t\|_{L^6}^{\frac32}\\
&\le C\|\rho\|_{L^\infty}^{\frac34}\|\nabla\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac12}\|\nabla\mathbf{u}_t\|_{L^2}^{\frac32}\\
&\le \frac{\mu}{6}\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2.
\end{aligned}
$$

Substituting the above estimates into (3.31), we derive that

$$
\frac{d}{dt}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\mu\|\nabla\mathbf{u}_t\|_{L^2}^2\le C\|\mathbf{u}\|_{H^2}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\|\mathbf{u}\|_{H^2}^2. \tag{3.32}
$$

By (3.17), (3.10), (3.11), (3.3), and (2.3), we have that, for $i\in\{1,2\}$,

$$
\int_0^T\|\mathbf{u}\|_{H^2}^2dt\le C\int_0^T\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2dt+C\int_0^T\|\nabla\mathbf{u}\|_{L^2}^2dt\le C, \tag{3.33}
$$

$$
\int_0^T t^i\|\mathbf{u}\|_{H^2}^2dt\le C\int_0^T t^i\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2dt+C\int_0^T t^i\|\nabla\mathbf{u}\|_{L^2}^2dt\le C. \tag{3.34}
$$

Then we obtain from (3.32) multiplied by t^i ($i\in\{1,2\}$), Gronwall's inequality, (3.33), (3.34), and (3.11) that

$$
\sup_{0\le t\le T}\left(t^i\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right)+\int_0^T t^i\|\nabla\mathbf{u}_t\|_{L^2}^2dt\le C. \tag{3.35}
$$

(2) Multiplying $(3.1)_3$ by θ_t and integrating the resulting equation over Ω yield that

$$
\begin{aligned}
\frac{\kappa}{2}\frac{d}{dt}\int|\nabla\theta|^2dx+c_v\int\rho|\theta_t|^2dx&=-c_v\int\rho(\mathbf{u}\cdot\nabla\theta)\theta_tdx+2\mu\int|\mathfrak{D}(\mathbf{u})|^2\theta_tdx\\
&\triangleq I_1+I_2.
\end{aligned}\tag{3.36}
$$

By Hölder's inequality and (3.2), we get that

$$
|I_1|\le c_v\|\rho\|_{L^\infty}^{\frac12}\|\sqrt{\rho}\theta_t\|_{L^2}\|\mathbf{u}\|_{L^\infty}\|\nabla\theta\|_{L^2}\le\frac{c_v}{2}\|\sqrt{\rho}\theta_t\|_{L^2}^2+C\|\mathbf{u}\|_{H^2}^2\|\nabla\theta\|_{L^2}^2. \tag{3.37}
$$

From (3.25), Hölder's inequality, (2.1), and (3.11), one has

$$
\begin{aligned}
I_2&=2\mu\frac{d}{dt}\int|\mathfrak{D}(\mathbf{u})|^2\theta dx-2\mu\int(|\mathfrak{D}(\mathbf{u})|^2)_t\theta dx\\
&\le 2\mu\frac{d}{dt}\int|\mathfrak{D}(\mathbf{u})|^2\theta dx+C\int\theta|\nabla\mathbf{u}||\nabla\mathbf{u}_t|dx\\
&\le 2\mu\frac{d}{dt}\int|\mathfrak{D}(\mathbf{u})|^2\theta dx+C\|\theta\|_{L^4}\|\nabla\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}_t\|_{L^2}\\
&\le 2\mu\frac{d}{dt}\int|\mathfrak{D}(\mathbf{u})|^2\theta dx+C\|\theta\|_{H^1}\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\|\nabla\mathbf{u}\|_{H^1}^{\frac12}\|\nabla\mathbf{u}_t\|_{L^2}
\end{aligned}
$$

$$\le 2\mu\frac{d}{dt}\int|\mathfrak{D}(\mathbf{u})|^2\theta dx + C\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^4 + C\|\nabla\mathbf{u}\|_{H^1}^2. \tag{3.38}$$

Substituting (3.37) and (3.38) into (3.36), we obtain that

$$B'(t) + c_v\|\sqrt{\rho}\theta_t\|_{L^2}^2 \le C\big(\|\nabla\theta\|_{L^2}^2 + \|\mathbf{u}\|_{H^2}^2\big)\|\nabla\theta\|_{L^2}^2 + C\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{H^1}^2, \tag{3.39}$$

where

$$B(t) \triangleq \int\big(\kappa|\nabla\theta|^2 - 4\mu|\mathfrak{D}(\mathbf{u})|^2\theta\big)\,dx$$

satisfies

$$\frac{\kappa}{2}\|\nabla\theta\|_{L^2}^2 - C\|\nabla\mathbf{u}\|_{L^2}^2 - C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 \le B(t) \le C\|\nabla\theta\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2 + C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2, \tag{3.40}$$

due to

$$\begin{aligned}4\mu\int|\mathfrak{D}(\mathbf{u})|^2\theta dx &\le C\|\theta\|_{L^2}\|\nabla\mathbf{u}\|_{L^4}^2\\ &\le C\|\theta\|_{H^1}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}\\ &\le C(1+\|\nabla\theta\|_{L^2})\|\nabla\mathbf{u}\|_{L^2}(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2})\\ &\le \frac{\kappa}{2}\|\nabla\theta\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2 + C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2.\end{aligned}$$

Multiplying (3.39) by t^i $(i\in\{1,2\})$ together with (3.40) leads to

$$\begin{aligned}&\frac{d}{dt}(t^iB(t)) + c_vt^i\|\sqrt{\rho}\theta_t\|_{L^2}^2\\ &\le C\big(\|\nabla\theta\|_{L^2}^2 + \|\mathbf{u}\|_{H^2}^2\big)(t^i\|\nabla\theta\|_{L^2}^2) + Ct^i\|\nabla\mathbf{u}_t\|_{L^2}^2 + Ct^i\|\nabla\mathbf{u}\|_{H^1}^2\\ &\quad + Cit^{i-1}\big(\|\nabla\theta\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^2}^2 + \|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\big),\end{aligned} \tag{3.41}$$

which, combined with Gronwall's inequality, (3.40), (3.35), (3.11), (3.23), (3.10), and (3.34), yields

$$\sup_{0\le t\le T}\big(t^i\|\nabla\theta\|_{L^2}^2\big) + \int_0^T t^i\|\sqrt{\rho}\theta_t\|_{L^2}^2 dt \le C. \tag{3.42}$$

This along with (3.35) gives rise to the desired (3.28).

(3) Multiplying (3.32) by $e^{\sigma t}$, we get from (3.17) and (3.22) that

$$\begin{aligned}&\frac{d}{dt}\big(e^{\sigma t}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\big) + \mu e^{\sigma t}\|\nabla\mathbf{u}_t\|_{L^2}^2\\ &\le C\|\mathbf{u}\|_{H^2}^2\big(e^{\sigma t}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\big) + Ce^{\sigma t}\|\mathbf{u}\|_{H^2}^2 + \sigma e^{\sigma t}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\\ &\le C\|\mathbf{u}\|_{H^2}^2\big(e^{\sigma t}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\big) + Ce^{\sigma t}\|\nabla\mathbf{u}\|_{L^2}^2 + Ce^{\sigma t}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2.\end{aligned}$$

This, combined with Gronwall's inequality, (3.33), (3.10), and (3.12), implies (3.29). □

LEMMA 3.5. *Let q be as in Theorem 1.1, then there exists a positive constant C depending only on Ω, μ, c_v, κ, q, and the initial data such that for $r\in[2,q)$,*

$$\sup_{0\le t\le T}\big(\|\rho\|_{W^{1,q}} + \|\rho_t\|_{L^r}\big) \le C. \tag{3.43}$$

Proof.

(1) It follows from Sobolev's embedding theorem, Lemma 2.3, and (3.2) that

$$\|\nabla\mathbf{u}\|_{L^\infty}\le C\|\mathbf{u}\|_{W^{2,3}}\le C(\|\rho\mathbf{u}_t\|_{L^3}+\|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^3})\le C\|\rho\mathbf{u}_t\|_{L^3}+C\|\mathbf{u}\|_{H^2}^2. \quad (3.44)$$

By Hölder's inequality, Sobolev's inequality, and (3.2), we have

$$\|\rho\mathbf{u}_t\|_{L^3}\le\|\rho\|_{L^\infty}^{\frac12}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac12}\|\sqrt{\rho}\mathbf{u}_t\|_{L^6}^{\frac12}\le C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac12}\|\nabla\mathbf{u}_t\|_{L^2}^{\frac12},$$

which, together with Hölder's inequality, implies for any $0\le a<b<\infty$,

$$\begin{aligned}\int_a^b\|\rho\mathbf{u}_t\|_{L^3}dt&\le C\int_a^b t^{-\frac38}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac12}\cdot t^{\frac38}\|\nabla\mathbf{u}_t\|_{L^2}^{\frac12}dt\\&\le C\Big[\int_a^b t^{-\frac12}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac23}dt\Big]^{\frac34}\times\Big[\int_a^b t^{\frac32}\|\nabla\mathbf{u}_t\|_{L^2}^2dt\Big]^{\frac14}.\end{aligned} \quad (3.45)$$

As a consequence, if $T\le1$, we obtain from (3.45) and (3.35) that

$$\begin{aligned}&\int_0^T\|\rho\mathbf{u}_t\|_{L^3}dt\\&\le C\Big[\int_0^T t^{-\frac12}\cdot t^{-\frac13}t^{\frac13}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac23}dt\Big]^{\frac34}\times\Big[\int_0^T t^{\frac12}\|\nabla\mathbf{u}_t\|_{L^2}\cdot t\|\nabla\mathbf{u}_t\|_{L^2}dt\Big]^{\frac14}\\&\le C\Big(\sup_{0\le t\le T}t\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\Big)^{\frac14}\Big(\int_0^T t^{-\frac56}dt\Big)^{\frac34}\Big(\int_0^T t\|\nabla\mathbf{u}_t\|_{L^2}^2dt\Big)^{\frac18}\Big(\int_0^T t^2\|\nabla\mathbf{u}_t\|_{L^2}^2dt\Big)^{\frac18}\\&\le CT^{\frac18}\le C.\end{aligned} \quad (3.46)$$

If $T>1$, one deduces from (3.46), (3.45), and (3.35) that

$$\begin{aligned}&\int_0^T\|\rho\mathbf{u}_t\|_{L^3}dt\\&=\int_0^1\|\rho\mathbf{u}_t\|_{L^3}dt+\int_1^T\|\rho\mathbf{u}_t\|_{L^3}dt\\&\le C+C\Big[\int_1^T t^{-\frac12}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac23}dt\Big]^{\frac34}\times\Big[\int_1^T t^{\frac12}\|\nabla\mathbf{u}_t\|_{L^2}\cdot t\|\nabla\mathbf{u}_t\|_{L^2}dt\Big]^{\frac14}\\&\le C+C\Big(\sup_{1\le t\le T}t^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\Big)^{\frac14}\Big(\int_1^T t^{-\frac12}\cdot t^{-\frac23}dt\Big)^{\frac34}\Big(\int_1^T t\|\nabla\mathbf{u}_t\|_{L^2}^2dt\Big)^{\frac18}\\&\quad\times\Big(\int_1^T t^2\|\nabla\mathbf{u}_t\|_{L^2}^2dt\Big)^{\frac18}\\&\le C+C\Big(1-T^{-\frac16}\Big)^{\frac34}\le C.\end{aligned} \quad (3.47)$$

Hence, we derive from (3.44), (3.46), (3.47), and (3.33) that

$$\int_0^T\|\nabla\mathbf{u}\|_{L^\infty}dt\le C. \quad (3.48)$$

(2) Taking spatial derivative ∇ of the transport Equation $(3.1)_1$ leads to

$$\partial_t\nabla\rho+\mathbf{u}\cdot\nabla^2\rho+\nabla\mathbf{u}\cdot\nabla\rho=\mathbf{0}.$$

Thus standard energy methods yield for any $q\in(2,\infty)$,

$$\frac{d}{dt}\|\nabla\rho\|_{L^q}\le C(q)\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^q},$$

which, combined with Gronwall's inequality and (3.48), gives

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^q}\le C. \tag{3.49}$$

Noticing the following fact

$$\|\rho_t\|_{L^r}=\|\mathbf{u}\cdot\nabla\rho\|_{L^r}\le\|\nabla\rho\|_{L^q}\|\mathbf{u}\|_{L^{\frac{qr}{q-r}}}\le\|\nabla\rho\|_{L^q}\|\nabla\mathbf{u}\|_{L^2},$$

which, together with (3.49) and (3.11), yields

$$\sup_{0\le t\le T}\|\rho_t\|_{L^r}\le C. \tag{3.50}$$

Thus, the desired (3.43) follows from (3.2), (3.49), and (3.50).

□

LEMMA 3.6. *Let q be as in Theorem 1.1, then there exists a positive constant C depending only on Ω, μ, c_v, κ, q, and the initial data such that*

$$\sup_{0\le t\le T}\left(t^2\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)+\int_0^T t^2\|\nabla\theta_t\|_{L^2}^2dt\le C. \tag{3.51}$$

Proof. Differentiating $(3.1)_3$ with respect to t and using $(1.1)_1$, we arrive at

$$c_v[\rho\theta_{tt}+\rho\mathbf{u}\cdot\nabla\theta_t]-\kappa\Delta\theta_t=c_v\operatorname{div}(\rho\mathbf{u})\theta_t-c_v\rho_t\mathbf{u}\cdot\nabla\theta-c_v\rho\mathbf{u}_t\cdot\nabla\theta+2\mu(|\mathfrak{D}(\mathbf{u})|^2)_t. \tag{3.52}$$

Multiplying (3.52) by θ_t and integrating (by parts) over Ω yield

$$\begin{aligned}
&\frac{c_v}{2}\frac{d}{dt}\int\rho|\theta_t|^2dx+\kappa\int|\nabla\theta_t|^2dx\\
&=c_v\int\operatorname{div}(\rho\mathbf{u})|\theta_t|^2dx+c_v\int\operatorname{div}(\rho\mathbf{u})(\mathbf{u}\cdot\nabla\theta)\theta_tdx-c_v\int\rho(\mathbf{u}_t\cdot\nabla\theta)\theta_tdx\\
&\quad+2\mu\int(|\mathfrak{D}(\mathbf{u})|^2)_t\theta_tdx\\
&\triangleq\bar{J}_1+\bar{J}_2+\bar{J}_3+\bar{J}_4.
\end{aligned} \tag{3.53}$$

Similarly to (3.26), one deduces

$$\|\theta_t\|_{H^1}\le C\|\sqrt{\rho}\theta_t\|_{L^2}+C\|\nabla\theta_t\|_{L^2}. \tag{3.54}$$

By Hölder's inequality, Sobolev's inequality, (2.1), (3.2), (3.54), (3.17), (3.11), and (3.43), we have

$$\begin{aligned}
|\bar{J}_1|&=\left|-c_v\int\rho\mathbf{u}\cdot\nabla|\theta_t|^2dx\right|\\
&\le 2c_v\|\rho\|_{L^\infty}^{\frac12}\|\mathbf{u}\|_{L^\infty}\|\sqrt{\rho}\theta_t\|_{L^2}\|\nabla\theta_t\|_{L^2}\\
&\le\frac{\kappa}{8}\|\nabla\theta_t\|_{L^2}^2+C\|\mathbf{u}\|_{H^2}^2\|\sqrt{\rho}\theta_t\|_{L^2}^2;
\end{aligned}$$

$$
\begin{aligned}
|\bar{J}_2| \le & c_v \int |\rho_t||\mathbf{u}||\nabla\theta||\theta_t| dx \\
\le & c_v \|\rho_t\|_{L^{\frac{2(q-1)}{q-2}}} \|\mathbf{u}\|_{L^\infty} \|\nabla\theta\|_{L^2} \|\theta_t\|_{L^{2(q-1)}} \\
\le & C\|\mathbf{u}\|_{H^2} \|\nabla\theta\|_{L^2} \big(\|\sqrt{\rho}\theta_t\|_{L^2} + \|\nabla\theta_t\|_{L^2}\big) \\
\le & \frac{\kappa}{8}\|\nabla\theta_t\|_{L^2}^2 + C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2\|\mathbf{u}\|_{H^2}^2; \\
|\bar{J}_3| \le & \|\rho\|_{L^\infty}\|\nabla\theta\|_{L^2}\|\theta_t\|_{L^4}\|\mathbf{u}_t\|_{L^4} \\
\le & C\|\nabla\theta\|_{L^2}(\|\sqrt{\rho}\theta_t\|_{L^2} + \|\nabla\theta_t\|_{L^2})\|\nabla\mathbf{u}_t\|_{L^2} \\
\le & \frac{\kappa}{8}\|\nabla\theta_t\|_{L^2}^2 + C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2\|\nabla\mathbf{u}_t\|_{L^2}^2; \\
|\bar{J}_4| \le & C\int |\nabla\mathbf{u}||\nabla\mathbf{u}_t|\theta_t dx \\
\le & C\|\nabla\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}_t\|_{L^2}\|\theta_t\|_{L^4} \\
\le & C\|\nabla\mathbf{u}\|_{H^1}\|\nabla\mathbf{u}_t\|_{L^2}(\|\sqrt{\rho}\theta_t\|_{L^2} + \|\nabla\theta_t\|_{L^2}) \\
\le & C\big(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2} + 1\big)\|\nabla\mathbf{u}_t\|_{L^2}(\|\sqrt{\rho}\theta_t\|_{L^2} + \|\nabla\theta_t\|_{L^2}) \\
\le & \frac{\kappa}{8}\|\nabla\theta_t\|_{L^2}^2 + C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\sqrt{\rho}\theta_t\|_{L^2}^2.
\end{aligned}
$$

Substituting the above estimates into (3.53), we derive that

$$
\begin{aligned}
c_v\frac{d}{dt}\|\sqrt{\rho}\theta_t\|_{L^2}^2 + \kappa\|\nabla\theta_t\|_{L^2}^2 \le & C\|\mathbf{u}\|_{H^2}^2\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\big(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\big)\|\nabla\mathbf{u}_t\|_{L^2}^2 \\
& + C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2\|\mathbf{u}\|_{H^2}^2. \quad (3.55)
\end{aligned}
$$

Multiplying (3.55) by t^2 yields

$$
\begin{aligned}
& c_v\frac{d}{dt}\big(t^2\|\sqrt{\rho}\theta_t\|_{L^2}^2\big) + \kappa t^2\|\nabla\theta_t\|_{L^2}^2 \\
\le & C\|\mathbf{u}\|_{H^2}^2\big(t^2\|\sqrt{\rho}\theta_t\|_{L^2}^2\big) + Ct\big(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\big)\big(t\|\nabla\mathbf{u}_t\|_{L^2}^2\big) \\
& + Ct^2\|\sqrt{\rho}\theta_t\|_{L^2}^2 + Ct^2\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\big(t^2\|\nabla\theta\|_{L^2}^2\big)\|\mathbf{u}\|_{H^2}^2 + 2c_v t\|\sqrt{\rho}\theta_t\|_{L^2}^2, \quad (3.56)
\end{aligned}
$$

which, combined with Gronwall's inequality, (3.33), (3.35), and (3.42), leads to the desired (3.51). □

LEMMA 3.7. *Let q be as in Theorem 1.1, then there exists a positive constant C depending only on Ω, μ, c_v, κ, q, and the initial data such that*

$$
\sup_{0\le t\le T}\big[t^2\big(\|\mathbf{u}\|_{H^2}^2 + \|\nabla P\|_{L^2}^2\big)\big] + \int_0^T t^2\big(\|\mathbf{u}\|_{H^3}^2 + \|\nabla P\|_{H^1}^2\big)dt \le C. \quad (3.57)
$$

Moreover, for σ as that in Lemma 3.1 and $\zeta(T)$ as that in (3.29), one has

$$
\sup_{\zeta(T)\le t\le T}\big[e^{\sigma t}\big(\|\mathbf{u}\|_{H^2}^2 + \|\nabla P\|_{L^2}^2\big)\big] \le C, \quad (3.58)
$$

and

$$
\sup_{0\le t\le T}\big(t^2\|\theta\|_{H^2}^2\big) + \int_0^T t^2\|\theta\|_{H^3}^2 dt \le C(T). \quad (3.59)
$$

Proof.

(1) From (3.17) and (3.22), we have

$$\|\mathbf{u}\|_{H^2}^2+\|\nabla P\|_{L^2}^2\le C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2. \tag{3.60}$$

This along with (3.35) and (3.11) yields

$$\sup_{0\le t\le T}\left[t^2\left(\|\mathbf{u}\|_{H^2}^2+\|\nabla P\|_{L^2}^2\right)\right]\le C. \tag{3.61}$$

We derive from (3.60), (3.29), and (3.12) that

$$\sup_{\zeta(T)\le t\le T}\left[e^{\sigma t}\left(\|\mathbf{u}\|_{H^2}^2+\|\nabla P\|_{L^2}^2\right)\right]\le C. \tag{3.62}$$

We infer from (3.16), (3.2), (3.11), (3.49), and Sobolev's inequality that

$$\begin{aligned}
&\|\mathbf{u}\|_{H^3}^2+\|\nabla P\|_{H^1}^2\\
&\le C\left(\|\rho\mathbf{u}_t\|_{H^1}^2+\|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{H^1}^2\right)+C\|\mathbf{u}\|_{H^1}^2\\
&\le C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\|\mathbf{u}\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^2}^2+C\|\nabla(\rho\mathbf{u}_t)\|_{L^2}^2+C\|\nabla(\rho\mathbf{u}\cdot\nabla\mathbf{u})\|_{L^2}^2+C\|\mathbf{u}\|_{H^1}^2\\
&\le C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\|\mathbf{u}\|_{H^2}^2+C\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\mathbf{u}\|_{H^2}^2\|\mathbf{u}\|_{H^2}^2,
\end{aligned} \tag{3.63}$$

which, together with (3.11), (3.34), (3.35), (3.61), and (3.33), yields

$$\int_0^T t^2\left(\|\mathbf{u}\|_{H^3}^2+\|\nabla P\|_{H^1}^2\right)dt\le C. \tag{3.64}$$

(2) It follows from $(3.1)_3$ and (1.3) that

$$\begin{cases}-\kappa\Delta\theta=2\mu|\mathfrak{D}(\mathbf{u})|^2-c_v\rho\theta_t-c_v\rho\mathbf{u}\cdot\nabla\theta, & \text{in } \Omega,\\ \frac{\partial\theta}{\partial\mathbf{n}}=0, & \text{on } \partial\Omega.\end{cases} \tag{3.65}$$

Hence the standard H^2-estimate for Neumann problem to the elliptic equation (see, e.g., [15]) gives rise to

$$\begin{aligned}
\|\theta\|_{H^2}^2&\le C\left(\||\nabla\mathbf{u}|^2\|_{L^2}^2+\|\rho\theta_t\|_{L^2}^2+\|\rho\mathbf{u}\cdot\nabla\theta\|_{L^2}^2+\|\theta\|_{H^1}^2\right)\\
&\le C\|\nabla\mathbf{u}\|_{L^4}^4+C\|\rho\|_{L^\infty}\|\sqrt{\rho}\theta_t\|_{L^2}^2+C\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^4}^2\|\nabla\theta\|_{L^4}^2+C\|\theta\|_{H^1}^2\\
&\le C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}\|_{H^1}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\theta\|_{L^2}\|\nabla\theta\|_{H^1}+C\|\theta\|_{H^1}^2\\
&\le C\|\nabla\mathbf{u}\|_{H^1}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^2+C\|\sqrt{\rho}\theta\|_{L^2}^2+\frac{1}{2}\|\theta\|_{H^2}^2,
\end{aligned}$$

due to (2.1), (3.11), and (3.26). Thus, one gets

$$\|\theta\|_{H^2}^2\le C\|\nabla\mathbf{u}\|_{H^1}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^2+C\|\sqrt{\rho}\theta\|_{L^2}^2, \tag{3.66}$$

which along with (3.61), (3.51), (3.42), and (3.23) yields

$$\sup_{0\le t\le T}\left(t^2\|\theta\|_{H^2}^2\right)\le C(T). \tag{3.67}$$

Applying (3.65), regularity theory of elliptic equation, and estimates we have shown, it is not difficult to obtain that

$$\int_0^T t^2\|\theta\|_{H^3}^2dt\le C(T). \tag{3.68}$$

This finishes the proof of Lemma 3.7.

□

4. Proof of Theorem 1.1

With the *a priori* estimates in Section 3 at hand, we are now in a position to prove Theorem 1.1.

By Lemma 2.1, there exists a $T_*>0$ such that the problem (1.1)–(1.3) has a unique local strong solution $(\rho,\mathbf{u},\theta)$ on $\Omega\times(0,T_*]$. We plan to extend the local solution to all time.

Set

$$T^*=\sup\{T \mid (\rho,\mathbf{u},\theta) \text{ is a strong solution on } \Omega\times(0,T]\}. \tag{4.1}$$

First, for any $0<\tau<T_*<T\leq T^*$ with T finite, one deduces from (3.11), (3.28), (3.57), (3.59), and [7, Theorem 4, p. 304] that

$$\nabla\mathbf{u},\ \nabla\theta\in C([\tau,T];H^1). \tag{4.2}$$

Moreover, it follows from (3.43) that

$$\rho\in C([0,T];W^{1,q}). \tag{4.3}$$

Owing to (3.2) and (3.11), we get

$$\rho\mathbf{u}_t=\sqrt{\rho}\cdot\sqrt{\rho}\mathbf{u}_t\in L^2(0,T;L^2).$$

From (3.50), (3.10), and Sobolev's inequality, one has

$$\rho_t\mathbf{u}\in L^2(0,T;L^2).$$

Thus, we arrive at

$$(\rho\mathbf{u})_t=\rho\mathbf{u}_t+\rho_t\mathbf{u}\in L^2(0,T;L^2). \tag{4.4}$$

From (3.2) and (3.8), we have

$$\rho\mathbf{u}=\sqrt{\rho}\cdot\sqrt{\rho}\mathbf{u}\in L^\infty(0,T;L^2),$$

which, combined with (4.4), yields

$$\rho\mathbf{u}\in C([0,T];L^2). \tag{4.5}$$

Similarly, we can derive

$$\rho\theta\in C([0,T];L^2). \tag{4.6}$$

Finally, if $T^*<\infty$, it follows from (4.2), (4.3), and (3.11), that

$$(\rho,\mathbf{u},\theta)(x,T^*)=\lim_{t\to T^*}(\rho,\mathbf{u},\theta)(x,t)$$

satisfies the initial condition (1.4) at $t=T^*$. Thus, taking $(\rho,\mathbf{u},\theta)(x,T^*)$ as the initial data, Lemma 2.1 implies that one can extend the strong solutions beyond T^*. This contradicts the assumption of T^* in (4.1). Furthermore, the estimates as those in (1.5) and (1.6) follow from Lemmas 3.1–3.7. The proof of Theorem 1.1 is complete.

REFERENCES

[1] C. Amrouche and V. Girault, *Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension*, Czechoslov. Math. J., 44:109–140, 1994. 2

[2] Y. Cho and H. Kim, *Existence result for heat-conducting viscous incompressible fluids with vacuum*, J. Korean Math. Soc., 45:645–681, 2008. 1

[3] H.J. Choe and H. Kim, *Strong solutions of the Navier-Stokes equations for nonhomogeneous incompressible fluids*, Commun. Partial Differ. Equ., 28:1183–1201, 2003. 1

[4] W. Craig, X. Huang, and Y. Wang, *Global wellposedness for the 3D inhomogeneous incompressible Navier-Stokes equations*, J. Math. Fluid Mech., 15:747–758, 2013. 1

[5] R. Danchin and P.B. Mucha, *The incompressible Navier-Stokes equations in vacuum*, Commun. Pure Appl. Math., 72:1351–1385, 2019. 1

[6] B. Desjardins, *Regularity results for two-dimensional flows of multiphase viscous fluids*, Arch. Ration. Mech. Anal., 137:135–158, 1997. 2

[7] L.C. Evans, *Partial Differential Equations*, Second Edition, Amer. Math. Soc., Providence, RI, 2010. 4

[8] E. Feireisl, *Dynamics of Viscous Compressible Fluids*, Oxford University Press, Oxford, 2004. 1

[9] A. Friedman, *Partial Differential Equations*, Dover Books on Mathematics, New York, 2008. 2

[10] C. He, J. Li, and B. Lü, *Global well-posedness and exponential stability of 3D Navier-Stokes equations with density-dependent viscosity and vacuum in unbounded domains*, Arch. Ration. Mech. Anal., 239:1809–1835, 2021. 1

[11] X. Huang and Y. Wang, *Global strong solution to the 2D nonhomogeneous incompressible MHD system*, J. Differ. Equ., 254:511–527, 2013. 1, 1.2

[12] X. Huang and Y. Wang, *Global strong solution of 3D inhomogeneous Navier-Stokes equations with density-dependent viscosity*, J. Differ. Equ., 259:1606–1627, 2015. 1

[13] J. Li, *Local existence and uniqueness of strong solutions to the Navier-Stokes equations with nonnegative density*, J. Differ. Equ., 263:6512–6536, 2017. 1, 1, 2

[14] Z. Liang, *Local strong solution and blow-up criterion for the 2D nonhomogeneous incompressible fluids*, J. Differ. Equ., 258:2633–2654, 2015. 1

[15] G.M. Lieberman, *Oblique Derivative Problems for Elliptic Equations*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013. 2

[16] P.L. Lions, *Mathematical Topics in Fluid Mechanics, Vol. I: Incompressible Models*, Oxford University Press, Oxford, 1996. 1

[17] Y. Liu, *Global existence and exponential decay of strong solutions to the Cauchy problem of 3D density-dependent Navier-Stokes equations with vacuum*, Discrete Contin. Dyn. Syst. Ser. B, 26:1291–1303, 2021. 1

[18] G. Łukaszewicz and P. Kalita, *Navier-Stokes Equations. An Introduction with Applications*, Springer, Cham., 2016. 1, 1

[19] B. Lü, X. Shi, and X. Zhong, *Global existence and large time asymptotic behavior of strong solutions to the Cauchy problem of 2D density-dependent Navier-Stokes equations with vacuum*, Nonlinearity, 31(6):2617–2632, 2018. 1, 1

[20] J. Simon, *Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure*, SIAM J. Math. Anal., 21:1093–1117, 1990. 1

[21] M. Struwe, *Variational methods. Applications to nonlinear Partial Differential Equations and Hamiltonian systems*, Fourth Edition, Springer-Verlag, Berlin, 2008. 3

[22] W. Wang, H. Yu, and P. Zhang, *Global strong solutions for 3D viscous incompressible heat conducting Navier-Stokes flows with the general external force*, Math. Meth. Appl. Sci., 41:4589–4601, 2018. 1

[23] H. Xu and H. Yu, *Global regularity to the Cauchy problem of the 3D heat conducting incompressible Navier-Stokes equations*, J. Math. Anal. Appl., 464:823–837, 2018. 1

[24] H. Xu and H. Yu, *Global strong solutions to the 3D inhomogeneous heat-conducting incompressible fluids*, Appl. Anal., 98:622–637, 2019. 1

[25] Z. Ye, *Blow-up criterion of strong solution with vacuum for the 2D nonhomogeneous density-temperature-dependent Boussinesq equations*, Z. Anal. Anwend., 39:83–101, 2020. 2

[26] J. Zhang, *Global well-posedness for the incompressible Navier-Stokes equations with density-dependent viscosity coefficient*, J. Differ. Equ., 259:1722–1742, 2015. 1

[27] X. Zhong, *Global strong solution for 3D viscous incompressible heat conducting Navier-Stokes flows with non-negative density*, J. Differ. Equ., 263:4978–4996, 2017. 1, 1.1