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GLOBAL REGULARITY AND TIME DECAY FOR THE 2D MAGNETO-MICROPOLAR SYSTEM WITH FRACTIONAL DISSIPATION AND PARTIAL MAGNETIC DIFFUSION*

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Abstract. This paper focuses on the 2D incompressible magneto-micropolar system with the kinematic dissipation given by the fractional operator $(-\Delta)^\alpha$, the magnetic diffusion by partial Laplacian and the spin dissipation by the fractional operator $(-\Delta)^\gamma$. We prove that this system, with any $0<\alpha<\gamma<1$ and $\alpha+\gamma>1$, always possesses a unique global smooth solution $(\mathbf{u},\ \mathbf{b},\ \mathrm{w})\in H^s(\mathbb{R}^2)(s\geq 3)$ if the initial data is sufficiently smooth. In addition, we study the large-time behavior of these smooth solutions and obtain optimal large-time decay rates.

Keywords. Magneto-micropolar system; fractional operator; partial dissipation; large-time decay.

AMS subject classifications. 76D03; 35B40; 35B65.

1. Introduction

The micropolar fluid model is a generalization of the classical Navier-Stokes equations when the microstructure of the fluid particles is not ignored, which was first introduced by Eringen in [21]. When one considers the micropolar fluid moving into a magnetic field, one gets the more complete magnetic-micropolar fluids. The generalized incompressible magneto-micropolar fluid flow in the whole 3D space is governed by the following equations,

$$\begin{cases} \partial_t \mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+(\mu+\chi)(-\Delta)^\alpha\mathbf{u}=-\nabla\pi+\mathbf{b}\cdot\nabla\mathbf{b}+2\chi\nabla\times\mathbf{w}, \\ \partial_t\mathbf{b}+\mathbf{u}\cdot\nabla\mathbf{b}+\nu(-\Delta)^\beta\mathbf{b}=\mathbf{b}\cdot\nabla\mathbf{u}, \\ \partial_t\mathbf{w}+\mathbf{u}\cdot\nabla\mathbf{w}+\kappa(-\Delta)^\gamma\mathbf{w}-\lambda\nabla\nabla\cdot\mathbf{w}+4\chi\mathbf{w}=2\chi\nabla\times\mathbf{u}, \\ \nabla\cdot\mathbf{u}=0,\ \nabla\cdot\mathbf{b}=0, \end{cases} \tag{1.1}$$

where $\mathbf{u}=\mathbf{u}(x,t)\in\mathbb{R}^3$ denotes the fluid velocity, $\mathbf{b}=\mathbf{b}(x,t)\in\mathbb{R}^3$ the magnetic field, $\mathbf{w}=\mathbf{w}(x,t)\in\mathbb{R}^3$ the micro-rotational field and $\pi(x,t)$ the scalar pressure with $x\in\mathbb{R}^3, t\geq 0$. The positive parameter μ denotes the kinematic viscosity, χ the vortex viscosity, $\frac{1}{\nu}$ the magnetic Reynolds number, κ and λ the angular viscosities. $\alpha,\ \beta,\ \gamma\geq 0$ are the given constants. The fractional Laplacian operator $(-\Delta)^s$ is defined via the Fourier transform

$$\widehat{(-\Delta)^s f}(\xi)=|\xi|^{2s}\hat{f}(\xi).$$

When $\alpha=\beta=\gamma=1$, (1.1) becomes the standard magneto-micropolar equations. We define

$$\mathbf{u}=(u_1,\ u_2,\ 0),\ \mathbf{b}=(b_1,\ b_2,\ 0),\ \mathbf{w}=(0,\ 0,\ \mathrm{w}),$$

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then (1.1) becomes the 2D generalized magneto-micropolar fluid flow which can be stated as

$$\begin{cases} \partial_t \mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+(\mu+\chi)(-\Delta)^{\alpha}\mathbf{u}=-\nabla\pi+\mathbf{b}\cdot\nabla\mathbf{b}+2\chi\nabla^{\perp}\mathrm{w}, \\ \partial_t \mathbf{b}+\mathbf{u}\cdot\nabla\mathbf{b}+\nu(-\Delta)^{\beta}\mathbf{b}=\mathbf{b}\cdot\nabla\mathbf{u}, \\ \partial_t \mathrm{w}+\mathbf{u}\cdot\nabla\mathrm{w}+\kappa(-\Delta)^{\gamma}\mathrm{w}+4\chi\mathrm{w}=2\chi\nabla\times\mathbf{u}, \\ \nabla\cdot\mathbf{u}=0,\ \nabla\cdot\mathbf{b}=0, \end{cases} \tag{1.2}$$

where $\nabla\times\mathbf{u}=\partial_1 u_2-\partial_2 u_1$ and $\nabla^{\perp}=(\partial_2,\ -\partial_1)$. If we ignore the magnetic field effects in the fluid motion, i.e. $\mathbf{b}=0$, the 2D magneto-micropolar problem (1.2) reduces to the micropolar fluid equations. Physically, micropolar fluids represent fluids consisting of rigid, randomly oriented (or spherical) particles suspended in a viscous medium, where the deformation of fluid particles is ignored. This is a kind of non-Newtonian fluid model. Magneto-micropolar fluids have been used in modeling a variety of physical phenomena involving suspensions of rigid particles in fluids, such as human blood, polymeric suspensions, and so on. It has been applied intensively in physiological and engineering problems. One can refer to [34, 40] for more information on these type of fluids and the references therein. The existences of weak and strong solutions were studied by Galdi and Rionero [25] and Yamaguchi [64]. Lukaszewicz [34] established the global well-posedness for the micropolar equations with full viscosity. In particular, the authors in [15,19] investigated the global regularity of solutions to (1.2) with zero angular viscosity (i.e. $\alpha=1$ and $\gamma=0$) and with only angular viscosity (i.e. $\alpha=0$ and $\gamma=1$), respectively. Very recently, Dong-Wu-Xu-Ye [18] obtained the global regularity for the 2D micropolar equations with $0<\alpha,\ \beta<1$ and $\alpha+\beta>1$. The global well-posedness and large-time decay for the 2D micropolar equations were studied by Dong-Li-Wu in [15]. There are many important global regularity results for the 2D micropolar fluid flows (see, e.g., [13, 15, 19, 63] and the references therein).

If the micro-rotation field was ignored and the vortex viscosity $\chi=0$, then the system (1.2) reduces to the 2D generalized magnetohydrodynamic equations (GMHD), which describe the motion of electrically conducting fluids such as plasmas, liquid metals, and electrolytes (see e.g., [44]). Very recently, Dong-Li-Wu in [17] obtained the global regularity for the 2D MHD equations with partial hyperresistivity. The global regularity and time decay for the 2D magnetohydrodynamic equations with fractional dissipation and partial magnetic diffusion was established by Dong-Jia-Li-Wu in [16]. The global regularity of solutions for the classical MHD equations (i.e. $\alpha=\beta=1$) has been established in [54]. The global regularity issue for the 2D MHD system has attracted much attention (see, e.g., [4, 7, 31, 55, 66–68]). However, whether or not there exists a global unique classical solution to generalized magnetohydrodynamic equations (GMHD), with $\alpha=0,\ \beta=1$ or $\alpha=1,\ \beta=0$ is still a challenging open problem. Recently, 2D GMHD has been attracted a lot of attention (see, e.g. [4, 22, 31, 55, 57, 58, 70]). In particular, with the works in [4, 22, 31], we have known that the 2D GMHD equations with $\alpha>0,\ \beta>1$ and $\alpha=0,\ \beta>1$ have a unique global classical solution. There have been significant recent developments on the MHD equations with partial or fractional dissipation. For details, one can refer to, for example, [2, 4–6, 9, 10, 12, 20, 22–24, 28–30, 37, 38, 49, 56–61] and the references therein.

The magneto-micropolar system (1.2) was considered in [63], which describes the motion of electrically conducting micropolar fluids in the presence of a magnetic field.

However, the global regularity of the strong solution with large initial data or finite time singularity to the 3D magneto-micropolar fluid equations (1.1) with full viscosity (i.e. $\alpha=\beta=\gamma=1$) is still an open problem. Fortunately, the local existence and uniqueness of strong solutions, the global existence of strong solutions for small initial data, the global existence of weak solutions were obtained in [43, 45, 48]. Recently, the global regularity for (1.2) with partial dissipation was obtained by Regmi and Wu in [47]. Ma [42] extended their results to other mixed partial viscosities cases. The global well-posedness for (1.2) with $\alpha=\beta=1$, $\gamma=0$ was obtained by Yamazaki in [65] and the initial-boundary value problem for 2D magneto-micropolar equations with zero angular viscosity was studied by Wang-Xu-Liu in [62]. Cheng-Liu [14] established the global regularity of the 2D magnetic micropolar fluid flows with mixed partial viscosity. Shang-Wu [52] studied the global regularity for system (1.2) with $\alpha=2$, $\beta=0$, $\gamma=0$ or $\alpha>0$, $\beta=\gamma=1$ or $\alpha+\beta\geq 2$, $\gamma=0$. Very recently, Shang-Zhao [53] obtained the global regularity for (1.2) with $\alpha=0$, $\beta>1$, $\gamma=1$. The global regularity for the 2D magneto-micropolar equations with partial and fractional dissipations was investigated by Yuan-Qiao in [69]. Guterres, Nunes and Perusato [26] obtained the decay rates for the magneto-micropolar system in $L^2(\mathbb{R}^n)(n=2,3)$. Lin-Zhang [39] investigated local well-posedness for 2D incompressible magneto-micropolar boundary layer system. There are some scholars who have studied the initial-boundary value problem. For example, Jiu-Liu-Wu and Yu [27] established the initial-and boundary-value problem for 2D micropolar equations with only angular velocity dissipation. Liu-Wang [35] studied the initial-boundary value problem for 2D micropolar equations without angular viscosity. There are other references about the initial-boundary value problem and the references therein. The global regularity problem and decay estimates for two classes of two-dimensional magneto-micropolar equations with partial dissipation were obtained by Shang-Gu in [50]. Regmi [46] studied the global existence and regularity of classical solutions to the 2D incompressible magneto-micropolar equations with partial dissipation. Global well-posedness for the 2D incompressible magneto-micropolar fluid system with partial viscosity established by Lin-Xiang in [36]. Recently, Shang-Wu [52] investigated the global regularity for 2D fractional magneto-micropolar equations.

In [52], the authors remarked that (1.1) with the fractional Laplacian operators is physically relevant. Replacing the standard Laplacian operators, these fractional diffusion operators model the so-called anomalous diffusion, a much studied topic in physics, probability and finance. In this paper, we focus on the 2D magneto-micropolar system with fractional dissipation and partial magnetic diffusion. For simplicity, we take $\mu=\chi=\dfrac{1}{2}$ and $\kappa=\nu=1$. More precisely,

$$\begin{cases} \partial_t \mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+(-\Delta)^{\alpha}\mathbf{u}=-\nabla\pi+\mathbf{b}\cdot\nabla\mathbf{b}+\nabla^{\perp}\mathrm{w}, \\ \partial_t b_1+\mathbf{u}\cdot\nabla b_1-\partial_{22}b_1=\mathbf{b}\cdot\nabla u_1, \\ \partial_t b_2+\mathbf{u}\cdot\nabla b_2-\partial_{11}b_2=\mathbf{b}\cdot\nabla u_2, \\ \partial_t \mathrm{w}+\mathbf{u}\cdot\nabla\mathrm{w}+(-\Delta)^{\gamma}\mathrm{w}+2\mathrm{w}=\nabla\times\mathbf{u}, \\ \nabla\cdot\mathbf{u}=0,\ \nabla\cdot\mathbf{b}=0, \\ \mathbf{u}(x,\ 0)=u_0(x),\quad \mathbf{b}(x,\ 0)=b_0(x),\quad \mathrm{w}(x,\ 0)=\mathrm{w}_0(x), \end{cases} \tag{1.3}$$

where $0<\alpha<\gamma<1$ and $\alpha+\gamma>1$. The first goal of this paper is to establish the global well-posedness for the system (1.3) with any sufficiently smooth initial data $(\mathbf{u}_0,\ \mathbf{b}_0,\ \mathrm{w}_0)$. The second goal is to study the large-time behavior of these smooth solutions and obtain optimal large-time decay rates. More precisely, the main results of this paper are stated as follows:

THEOREM 1.1. *Consider the system (1.3) with $0<\alpha<\gamma<1$ and $\alpha+\gamma>1$. Assume the initial data $(\mathbf{u}_0,\ \mathbf{b}_0,\ w_0)\in H^s(\mathbb{R}^2)$ with $s\geq 3$ and $\nabla\cdot\mathbf{u}_0=\nabla\cdot\mathbf{b}_0=0$. Then system (1.3) has a unique global solution $(\mathbf{u},\ \mathbf{b},\ w)$ satisfying, for any $T>0$,*

$$\begin{aligned}&\mathbf{u}\in C([0,\ \infty);\ H^s(\mathbb{R}^2))\cap L^2(0,\ T;\ H^{s+\alpha}(\mathbb{R}^2)),\\&\mathbf{b}\in C([0,\ \infty);\ H^s(\mathbb{R}^2)),\ \nabla\mathbf{b}\in L^2(0,\ T;\ H^s(\mathbb{R}^2)),\\&w\in C([0,\ \infty);\ H^s(\mathbb{R}^2))\cap L^2(0,\ T;\ H^{s+\gamma}(\mathbb{R}^2)).\end{aligned}\tag{1.4}$$

THEOREM 1.2. *Consider the system (1.3) with $0<\alpha<\frac{1}{2}$, $0<\gamma<1$ and $\alpha+\gamma>1$. Assume the initial data $(\mathbf{u}_0,\ \mathbf{b}_0,\ w_0)\in H^s(\mathbb{R}^2)$ with $s\geq 3$ and $\nabla\cdot\mathbf{u}_0=\nabla\cdot\mathbf{b}_0=0$. Let $(\mathbf{u},\ \mathbf{b},\ w)$ be the corresponding solution to the system (1.3) as stated in Theorem 1.1. If the initial data $(\mathbf{u}_0,\ \mathbf{b}_0,\ w_0)$ satisfies*

$$\begin{aligned}&|\hat{\mathbf{u}}_0(\xi)|\leq C\sqrt{|\xi|},\quad |\hat{w}_0(\xi)|\leq C\sqrt{|\xi|},\\&\|\hat{b}_{01}(\xi)\|_{L^2_{\xi_1}}\leq C\sqrt{|\xi|_2},\quad \|\hat{b}_{02}(\xi)\|_{L^2_{\xi_2}}\leq C\sqrt{|\xi|_1}.\end{aligned}\tag{1.5}$$

Then $(\mathbf{u},\ \mathbf{b},\ w)$ obeys, for any $t>0$,

$$\begin{aligned}&\|\mathbf{b}(t)\|_{L^2}\leq C(1+t)^{-\frac{1}{2}},\quad \|\nabla\mathbf{b}(t)\|_{L^2}\leq C(1+t)^{-1};\\&\|\mathbf{u}(t)\|_{L^2}\leq C(1+t)^{-\frac{1}{2}},\quad \|\nabla\mathbf{u}(t)\|_{L^2}\leq C(1+t)^{-\frac{1}{2}};\\&\|w(t)\|_{L^2}\leq C(1+t)^{-\frac{3}{4}},\quad \|\nabla w(t)\|_{L^2}\leq C(1+t)^{-\frac{3}{4}}.\end{aligned}\tag{1.6}$$

The rest of this paper is constructed as follows. In Section 2, we will give some notation and preliminaries. In Section 3, we will prove Theorem 1.1. Section 4 supplies the proof of Theorem 1.2.

2. Notation and preliminaries

For convenience, before we prove our main result, we will give some notations which are used throughout this paper. We denote

$$\|f\|_{L^p(\mathbb{R}^2)}=\|f\|_p,\ \frac{\partial f}{\partial x_i}=\partial_i f,$$
$$\int f dxdy=\iint_{\mathbb{R}^2} f dxdy,$$

and

$$\|f_1,f_2,\cdots,f_n\|^2_{L^2(\mathbb{R}^2)}=\|f_1\|_2^2+\|f_2\|_2^2+\cdots+\|f_n\|_2^2.$$

Next, we will give some auxiliary lemmas. First, we recall the classical commutator estimate (see, e.g., [33]).

Lemma 2.1. *Assume that $s>0$. Let $1<r<\infty$ and $\frac{1}{r}=\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{p_2}+\frac{1}{q_2}$ with $q_1,p_2\in(1,\infty)$ and $p_1,q_2\in[1,\infty]$. Then,*

$$\|[\Lambda^s,\ f]g\|_r\leq C(\|\nabla f\|_{p_1}\|\Lambda^{s-1}g\|_{q_1}+\|\Lambda^s f\|_{p_2}\|g\|_{q_2}),\tag{2.1}$$

where C is a constant depending on the indices s, r, p_1, q_1, p_2 and q_2.

The following lemma can be found in [32].

Lemma 2.2. *Assume that $0<s<1$ and $1<p<\infty$. Then,*

$$\|\Lambda^s(fg)-f\Lambda^s g-g\Lambda^s f\|_p\leq C\|g\|_\infty\|\Lambda^s f\|_p.\tag{2.2}$$

The next lemma is very useful to establish the global bound of $\|\nabla\mathbf{b}\|_{L^p}$.

Lemma 2.3. *Assume that $\beta>0$, $t>0$. Consider the following equations,*

$$\begin{cases}\partial_t\mathbf{u}+(-\Delta)^\beta\mathbf{u}=f,\\ \mathbf{u}(x,\ 0)=\mathbf{u}_0(x).\end{cases}\tag{2.3}$$

Then its solution can be expressed as

$$\mathbf{u}(x,\ t)=K_\beta(\cdot,\ t)*\mathbf{u}_0+\int_0^t K_\beta(\cdot,\ t-\tau)*f(\cdot,\ \tau)d\tau,$$

where the kernel function is defined via the Fourier transform

$$K_\beta(x,\ t)=\int_{\mathbb{R}^n}e^{-t|\xi|^{2\beta}}e^{ix\cdot\xi}d\xi,$$

and $K_\beta(x,\ t)$ satisfies the following properties:
(i) For any $t>0$,

$$K_\beta(x,\ t)=t^{-\frac{n}{2\beta}}K_\beta(xt^{-\frac{1}{2\beta}},\ 1).$$

(ii) For any integer $m>0$, $1\leq r\leq\infty$ and any $t>0$,

$$\|\nabla^m K_\beta(x,\ t)\|_{L^r(\mathbb{R}^n)}\leq Ct^{-\frac{m}{2\beta}-\frac{n}{2\beta}(1-\frac{1}{r})}.$$

In particular, when $\beta=1$, $K_1(x,\ t)$ is the classical heat equation kernel. One can refer to [41] for the proof. We omit it here.

The following lemma generalizes the Kato-Ponce inequality, which requires m to be an integer (see, e.g., [32]). It also holds for any real number $m\geq 2$. One can refer to [18] for the proof. We omit it here.

Lemma 2.4. *Let $0<s<\varrho<1$, $m\geq 2$,and $p,\ q,\ r\in(1,\ \infty)^3$ satisfying $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$. Then, there exists a constant $C=C(s,\varrho,m,p,q,r)$ such that*

$$\||f|^{m-2}f\|_p+\|\Lambda^s(|f|^{m-2}f)\|_p\leq C\|f\|_{B^\varrho_{q,p}}\|f\|^{m-2}_{r(m-2)}.\tag{2.4}$$

Next, we recall the maximal regularity property for the heat operator (see, e.g., [1]).

LEMMA 2.5. *Let $G_d(x,t)$, is the heat kernel of d-dimensional heat equation*

$$G_d(x,t)=(4\pi t)^{-\frac{d}{2}}e^{-\frac{|x|^2}{4t}},$$

and define the operator A as

$$Af=\int_0^t\int_{\mathbb{R}^d}G_d(x,t)\Delta_x f(x-y,\ t-s)dyds.$$

Then, for any $T>0$ and $p,q\in(1,\ \infty)$, the operator A maps $L^p(0,T;L^q(\mathbb{R}^d))$ to $L^p(0,T;L^q(\mathbb{R}^d))$.

The next lemma gives the L^p-L^q decay estimates of the heat operator associated with the fractional Laplacian, and which can be found in [51].

LEMMA 2.6. *Let $\alpha>0$, $\mu>0$, $1\le p\le q\le\infty$ and $m\ge 0$. Then the L^p-L^q estimate on the semigroup $e^{-\mu(-\Delta)^\alpha t}$ is valid for any $t>0$,*

$$\|\nabla^m e^{-\mu(-\Delta)^\alpha t}\|_q\le Ct^{-\frac{m}{2\alpha}-\frac{1}{\alpha}(\frac{1}{p}-\frac{1}{q})}. \tag{2.5}$$

3. Proof of Theorem1.1

In this section, we will prove Theorem 1.1, the global existence and uniqueness of smooth solution to (1.3). The key component of the proof is the global a priori estimate of $\|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_{H^s}$ with $s\ge 3$. We will divide it into several steps. The first subsection will construct the global H^1 bound and $\int_0^T\|\mathrm{w}\|_\infty dt<\infty$. The second subsection will establish L^q-bounds for the vorticity $\Omega=\nabla\times\mathbf{u}$, $\nabla\mathrm{w}$ and $\Delta\mathbf{b}$ for any $1<q<\infty$. Finally, we will prove the global bound for $\|\nabla\mathbf{u}\|_\infty$, $\|\nabla\mathbf{b}\|_\infty$ and finish the proof of Theorem 1.1.

3.1. Global H^1 bound for $(\mathbf{u},\ \mathbf{b},\ \mathbf{w})$.

We will establish the global bounds for $\|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_{H^1}$.

PROPOSITION 3.1. *Assume that α, γ and $(\mathbf{u}_0,\ \mathbf{b}_0,\ w_0)$ satisfy the conditions stated in Theorem 1.1. Then system (1.3) has a global solution $(\mathbf{u},\ \mathbf{b},\ w)$ that satisfies, for any $T>0$,*

$$\begin{aligned}&\|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_2^2+\int_0^T\|\Lambda^\alpha\mathbf{u},\ \nabla\mathbf{b},\ \Lambda^\gamma\mathrm{w}\|_2^2dt\le C,\\ &\|\Omega,\ j,\ \nabla\mathrm{w}\|_2^2+\int_0^T\|\Lambda^\alpha\Omega,\ \Delta\mathbf{b},\ \Lambda^{1+\gamma}\mathrm{w}\|_2^2dt\le C,\end{aligned} \tag{3.1}$$

where $C>0$ is a constant, depending on T and the initial data and $j=\nabla\times\mathbf{b}=\partial_1 b_2-\partial_2 b_1$ is the current density and $\Lambda=(-\Delta)^{\frac{1}{2}}$ denotes the Zygmund operator.

Proof. We start with the energy inequality. Multiplying the equations $(1.3)_{1\sim 4}$ by $\mathbf{u}$, b_1, b_2 and w, respectively and taking the L^2 inner product, integrating by parts, using the divergence-free conditions $\nabla\cdot\mathbf{u}=0$ and $\nabla\cdot\mathbf{b}=0$, adding the resulting equations together, yield that

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_2^2+\|\Lambda^\alpha\mathbf{u},\ \nabla\mathbf{b},\ \Lambda^\gamma\mathrm{w}\|_2^2+2\|\mathrm{w}\|_2^2$$

$$=2\int \Omega \mathrm{w}\,dxdy \le C\|\Lambda^{\alpha}\mathbf{u}\|_2\|\Lambda^{1-\alpha}\mathrm{w}\|_2 \le \frac{1}{2}\|\Lambda^{\alpha}\mathbf{u},\ \Lambda^{\gamma}\mathrm{w}\|_2^2 + C\|\mathrm{w}\|_2^2, \tag{3.2}$$

where we have used the condition $\alpha+\gamma>1$ and the following facts that

$$\int \mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}\ dxdy + \int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\ dxdy = 0,$$

and

$$\|\nabla\mathbf{b}\|_2^2 \le C\|\partial_1 b_2,\ \partial_2 b_1\|_2^2.$$

Applying Gronwall's inequality, we obtain the L^2 bound for $\mathbf{u}$, $\mathbf{b}$, w as follows

$$\|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_2^2 + \int_0^T \|\Lambda^{\alpha}\mathbf{u},\ \nabla\mathbf{b},\ \Lambda^{\gamma}\mathrm{w}\|_2^2\,dt \le C. \tag{3.3}$$

To establish the global H^1 bound, we consider the equation of the vorticity $\Omega = \nabla\times\mathbf{u}$ and the current density $j = \nabla\times\mathbf{b}$, combining the Equation $(1.3)_4$, which satisfy

$$\begin{cases} \partial_t\Omega + \mathbf{u}\cdot\nabla\Omega + \Lambda^{2\alpha}\Omega + \Delta\mathrm{w} = \mathbf{b}\cdot\nabla j, \\ \partial_t j + \mathbf{u}\cdot\nabla j - \partial_{111}b_2 + \partial_{222}b_1 = \mathbf{b}\cdot\nabla\Omega + Q(\nabla\mathbf{u},\ \nabla\mathbf{b}), \\ \partial_t\mathrm{w} + \mathbf{u}\cdot\nabla\mathrm{w} + 2\mathrm{w} + \Lambda^{2\gamma}\mathrm{w} = \Omega, \end{cases} \tag{3.4}$$

where

$$Q(\nabla\mathbf{u},\ \nabla\mathbf{b}) = 2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) - 2\partial_1 u_1(\partial_1 b_2 + \partial_2 b_1).$$

Multiplying the equations $(3.4)_1$, $(3.4)_2$ and $(3.4)_3$ by Ω, j and $\Lambda^{2(2\gamma-1)}\mathrm{w}$, respectively and integrating by parts yield

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\Omega,\ j,\ \Lambda^{2\gamma-1}\mathrm{w}\|_2^2 + \|\Lambda^{\alpha}\Omega,\ \Delta\mathbf{b},\ \Lambda^{3\gamma-1}\mathrm{w}\|_2^2 + 2\|\Lambda^{2\gamma-1}\mathrm{w}\|_2^2 \\ =& -\int \Delta\mathrm{w}\Omega\,dxdy + \int Q(\nabla\mathbf{u},\ \nabla\mathbf{b})j\,dxdy \\ &+ \int \Omega\Lambda^{2(2\gamma-1)}\mathrm{w}\,dxdy - \int [\Lambda^{2\gamma-1},\ \mathbf{u}\cdot\nabla]\mathrm{w}\Lambda^{2\gamma-1}\mathrm{w}\,dxdy \\ =& I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{3.5}$$

where we have used the facts

$$\int \mathbf{b}\cdot\nabla j\cdot\Omega\,dxdy + \int \mathbf{b}\cdot\nabla\Omega\cdot j\,dxdy = 0,$$

and

$$\begin{aligned} \int (-\partial_{111}b_2 + \partial_{222}b_1)j\,dxdy =& \int (-\partial_{111}b_2\partial_1 b_2 + \partial_{111}b_2\partial_2 b_1 \\ &+ \partial_{222}b_1\partial_1 b_2 - \partial_{222}b_1\partial_2 b_1)\,dxdy \\ =& \int (|\partial_{11}b_1|^2 + |\partial_{22}b_1|^2 + |\partial_{11}b_2|^2 + |\partial_{22}b_2|^2)\,dxdy = \|\Delta\mathbf{b}\|_2^2. \end{aligned}$$

Using Hölder's and Young's inequalities, the term I_1 can be bounded as

$$I_1 = -\int \Delta \mathrm{w} \Omega \, dxdy \le \|\Lambda^\alpha \Omega\|_2 \|\Lambda^{2-\alpha} \mathrm{w}\|_2 \le \|\Lambda^\alpha \Omega\|_2 \|\mathrm{w}\|_2^{\frac{\alpha+3\gamma-3}{3\gamma-1}} \|\Lambda^{3\gamma-1} \mathrm{w}\|_2^{\frac{2-\alpha}{3\gamma-1}}$$
$$\le \frac{1}{2}\|\Lambda^\alpha \Omega\|_2^2 + \frac{1}{6}\|\Lambda^{3\gamma-1}\mathrm{w}\|_2^2 + C\|\mathrm{w}\|_2^2,$$

where we need to choose α, γ satisfying $\alpha+3\gamma>3$. Due to the divergence-free condition $\nabla\cdot\mathbf{b}=0$, applying Hölder's and the Gagliardo-Nirenberg inequalities, we can estimate the term I_2 as follows

$$I_2 = \int Q(\nabla \mathbf{u},\ \nabla \mathbf{b}) j \, dxdy \le \|\Omega\|_2 \|j\|_4^2 \le \frac{1}{2}\|\Delta \mathbf{b}\|_2^2 + C\|\Omega\|_2^2 \|j\|_2^2.$$

One can easily check that $\|\nabla j\|_2^2 \le \|\Delta \mathbf{b}\|_2^2$. Similarly, I_3 can be bounded as the term I_1 for $\gamma < 2(2\gamma-1) < 3\gamma-1$,

$$I_3 = \int \Omega \Lambda^{2(2\gamma-1)} \mathrm{w} \, dxdy \le \|\Omega\|_2 \|\Lambda^\gamma \mathrm{w}\|_2^{\frac{1-\gamma}{2\gamma-1}} \|\Lambda^{3\gamma-1}\mathrm{w}\|_2^{\frac{3\gamma-2}{2\gamma-1}}$$
$$\le \frac{1}{6}\|\Lambda^{3\gamma-1}\mathrm{w}\|_2^2 + C\|\Lambda^\gamma \mathrm{w}\|_2^2 + C\|\Omega\|_2^2.$$

Employing Hölder's and Sobolev's inequalities and Lemma 2.1, I_4 can be estimated as

$$I_4 = -\int [\Lambda^{2\gamma-1},\ \mathbf{u}\cdot\nabla]\mathrm{w} \Lambda^{2\gamma-1}\mathrm{w} \, dxdy$$
$$\le (\|\nabla \mathbf{u}\|_2 \|\Lambda^{2\gamma-1}\mathrm{w}\|_q + \|\Lambda^{2\gamma-1}\mathbf{u}\|_{p_1} \|\nabla \mathrm{w}\|_{q_1}) \|\Lambda^{2\gamma-1}\mathrm{w}\|_r$$
$$\le C\|\Omega\|_2 \|\Lambda^\gamma \mathrm{w}\|_2 \|\Lambda^{3\gamma-1}\mathrm{w}\|_2 \le \frac{1}{6}\|\Lambda^{3\gamma-1}\mathrm{w}\|_2^2 + C\|\Omega\|_2^2 \|\Lambda^\gamma \mathrm{w}\|_2^2,$$

where the indices are given by

$$q = \frac{2}{1-\gamma},\ r = \frac{2}{\gamma},\ p_1 = \frac{2}{2\gamma-1},\ q_1 = \frac{2}{3-3\gamma}.$$

Inserting the estimates for $I_1 \sim I_4$ into (3.5), yields

$$\frac{d}{dt}\|\Omega,\ j,\ \Lambda^{2\gamma-1}\mathrm{w}\|_2^2 + \|\Lambda^\alpha \Omega,\ \Delta \mathbf{b},\ \Lambda^{3\gamma-1}\mathrm{w}\|_2^2 + \|\Lambda^{2\gamma-1}\mathrm{w}\|_2^2$$
$$\le C\|\Lambda^\gamma \mathrm{w},\ j\|_2^2 \|\Omega\|_2^2 + C\|\Lambda^\gamma \mathrm{w}\|_2^2,$$

which together with (3.3) and Gronwall's inequality imply that

$$\|\Omega,\ j,\ \Lambda^{2\gamma-1}\mathrm{w}\|_2^2 + \int_0^T \|\Lambda^\alpha \Omega,\ \Delta \mathbf{b},\ \Lambda^{3\gamma-1}\mathrm{w}\|_2^2 dt \le C. \tag{3.6}$$

Multiplying the Equation $(3.4)_3$ by $\Lambda^{2(\alpha+\gamma)}\mathrm{w}$ and integrating over $\mathbb{R}^2$, we have

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{\alpha+\gamma}\mathrm{w}\|_2^2 + \|\Lambda^{\alpha+2\gamma}\mathrm{w}\|_2^2 + 2\|\Lambda^{\alpha+\gamma}\mathrm{w}\|_2^2$$
$$= \int \Omega \Lambda^{2(\alpha+\gamma)}\mathrm{w} \, dxdy - \int [\Lambda^{\alpha+\gamma},\ \mathbf{u}\cdot\nabla]\mathrm{w}\Lambda^{\alpha+\gamma}\mathrm{w} \, dxdy$$
$$= I_5 + I_6. \tag{3.7}$$

Applying the same method as that for the case of I_1, one can easily find that

$$I_5=\int \Omega\Lambda^{2(\alpha+\gamma)}\mathrm{w}\,dxdy\le\frac{1}{4}\|\Lambda^{\alpha+2\gamma}\mathrm{w}\|_2^2+C\|\Lambda^{\alpha}\Omega\|_2^2.$$

Using Hölder's and Sobolev's inequalities and Lemma 2.1, for $\alpha+2\gamma>2$, one has

$$\begin{aligned}I_6&=-\int[\Lambda^{\alpha+\gamma},\ \mathbf{u}\cdot\nabla]\mathrm{w}\Lambda^{\alpha+\gamma}\mathrm{w}\,dxdy\\&\le C(\|\nabla\mathbf{u}\|_{\frac{2}{\gamma}}\|\Lambda^{\alpha+\gamma}\mathrm{w}\|_{\frac{2}{1-\gamma}}+\|\Lambda^{\alpha+\gamma}\mathbf{u}\|_2\|\nabla\mathrm{w}\|_\infty)\|\Lambda^{\alpha+\gamma}\mathrm{w}\|_2\\&\le C(\|\Lambda^{\alpha}\Omega\|_2\|\Lambda^{\alpha+2\gamma}\mathrm{w}\|_2\|\Lambda^{\alpha+\gamma}\mathrm{w}\|_2\\&\quad+\|\Lambda^{\alpha+\gamma-1}\Omega\|_2\|\mathrm{w}\|_2^{\frac{\alpha+2\gamma-2}{\alpha+2\gamma}}\|\Lambda^{\alpha+2\gamma}\mathrm{w}\|_2^{\frac{2}{\alpha+2\gamma}}\|\Lambda^{\alpha+\gamma}\mathrm{w}\|_2)\\&\le\frac{1}{4}\|\Lambda^{\alpha+2\gamma}\mathrm{w}\|_2^2+C\|\Omega,\ \Lambda^{\alpha}\Omega\|_2^2\|\Lambda^{\alpha+\gamma}\mathrm{w}\|_2^2+C\|\mathrm{w}\|_2^2.\end{aligned}$$

Combining the above two estimates with (3.6), (3.7), applying Gronwall's inequality, we infer that

$$\|\Lambda^{\alpha+\gamma}\mathrm{w}\|_2^2+\int_0^T\|\Lambda^{\alpha+2\gamma}\mathrm{w}\|_2^2\,dt\le C, \tag{3.8}$$

where C is a constant only depending on $T>0$ and the initial data. According to (3.8) and Sobolev's inequality, one can find that

$$\int_0^T\|\nabla\mathrm{w}\|_\infty\,dt\le C, \tag{3.9}$$

for any $T>0$. Applying the operator ∇ to each side of the Equation $(3.4)_3$ and dotting with the resulting equation by $\nabla\mathrm{w}$ leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\mathrm{w}\|_2^2+\|\Lambda^{1+\gamma}\mathrm{w}\|_2^2+\|\nabla\mathrm{w}\|_2^2\\&=\int\nabla\Omega\nabla\mathrm{w}\,dxdy-\int\nabla(\mathbf{u}\cdot\nabla\mathrm{w})\nabla\mathrm{w}\,dxdy=I_7+I_8.\end{aligned}\tag{3.10}$$

Using Hölder's and Sobolev's inequalities, one can easily find that

$$I_7=\int\nabla\Omega\nabla\mathrm{w}\,dxdy\le\|\Lambda^{1-\gamma}\Omega\|_2\|\Lambda^{1+\gamma}\mathrm{w}\|_2\le\frac{1}{2}\|\Lambda^{1+\gamma}\mathrm{w}\|_2^2+C\|\Omega,\ \Lambda^{\alpha}\Omega\|_2^2.$$

By Hölder's inequality, we have

$$I_8=\int\nabla(\mathbf{u}\cdot\nabla\mathrm{w})\nabla\mathrm{w}\,dxdy\le\|\nabla\mathrm{w}\|_\infty\|\nabla\mathbf{u}\|_2\|\nabla\mathrm{w}\|_2\le\|\nabla\mathrm{w}\|_2^2+C\|\nabla\mathrm{w}\|_\infty^2\|\Omega\|_2^2.$$

Inserting the estimates of $I_7\sim I_8$ and (3.6), (3.9), into (3.10), we obtain

$$\|\nabla\mathrm{w}\|_2^2+\int_0^T\|\Lambda^{1+\gamma}\mathrm{w}\|_2^2\,dt\le C, \tag{3.11}$$

which together with (3.6) implies that

$$\|\Omega,\ j,\ \nabla\mathrm{w}\|_2^2+\int_0^T\|\Lambda^{\alpha}\Omega,\ \Delta\mathbf{b},\ \Lambda^{1+\gamma}\mathrm{w}\|_2^2\,dt\le C. \tag{3.12}$$

□

3.2. L^q bounds for $\Delta\mathbf{b}$, Ω. In this section, we will establish the global bounds for $\|\Delta\mathbf{b}\|_{L_t^2L^q}$, $\|\Omega\|_{L_t^\infty L^q}$.

Proposition 3.2. *Assume that α, γ and $(\mathbf{u}_0,\ \mathbf{b}_0,\ w_0)$ satisfies the conditions stated in Theorem 1.1. Then system (1.3) has a global solution $(\mathbf{u},\ \mathbf{b},\ w)$ satisfying, for any $T>0$ and $1<q<\infty$,*

$$\Lambda^{1+\alpha}\mathbf{b}\in L^\infty(0,\ T;L^2(\mathbb{R}^2)),\quad \Delta\mathbf{b}\in L^2(0,\ T;L^q(\mathbb{R}^2)),\quad \Omega\in L^\infty(0,\ T;L^q(\mathbb{R}^2)). \tag{3.13}$$

Proof. Multiplying the Equation $(1.3)_2$ and $(1.3)_3$ by $\Lambda^{2+2\alpha}b_1$ and $\Lambda^{2+2\alpha}b_2$, respectively and integrating over $\mathbb{R}^2$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{1+\alpha}\mathbf{b}\|_2^2+\|\Lambda^{2+\alpha}\mathbf{b}\|_2^2=\int(\mathbf{b}\cdot\nabla\mathbf{u}-\mathbf{u}\cdot\nabla\mathbf{b})\Lambda^{2+2\alpha}\mathbf{b}\,dxdy, \tag{3.14}$$

where we have used the fact

$$\|\Lambda^{2+\alpha}\mathbf{b}\|_2^2\le C\|\Lambda^{1+\alpha}\partial_1 b_2,\ \Lambda^{1+\alpha}\partial_2 b_1\|_2^2.$$

Applying Proposition 3.1, Hölder's and Young's inequalities, we find that

$$\begin{aligned}
&\int(\mathbf{b}\cdot\nabla\mathbf{u}-\mathbf{u}\cdot\nabla\mathbf{b})\Lambda^{2+2\alpha}\mathbf{b}\,dxdy\\
\le&C(\|\Lambda^\alpha(\mathbf{b}\cdot\nabla\mathbf{u})\|_2+\|\Lambda^\alpha(\mathbf{u}\cdot\nabla\mathbf{b})\|_2)\|\Lambda^{2+\alpha}\mathbf{b}\|_2\\
\le&C(\|\Lambda^\alpha\mathbf{b}\|_\infty\|\nabla\mathbf{u}\|_2+\|\mathbf{b}\|_\infty\|\Lambda^\alpha\nabla\mathbf{u}\|_2)\|\Lambda^{2+\alpha}\mathbf{b}\|_2\\
&+C(\|\Lambda^\alpha\mathbf{u}\|_{\frac{2}{\alpha}}\|\nabla\mathbf{b}\|_{\frac{2}{1-\alpha}}+\|\mathbf{u}\|_\infty\|\Lambda^\alpha\nabla\mathbf{b}\|_2)\|\Lambda^{2+\alpha}\mathbf{b}\|_2\\
\le&\frac{1}{4}\|\Lambda^{2+\alpha}\mathbf{b}\|_2^2+C\|\mathbf{b},\ \Lambda^2\mathbf{b}\|_2^2\|\Omega\|_2^2+C\|\mathbf{b},\ \Lambda^{1+\alpha}\mathbf{b}\|_2^2\|\Lambda^\alpha\Omega\|_2^2\\
&+\frac{1}{4}\|\Lambda^{2+\alpha}\mathbf{b}\|_2^2+C\|\mathbf{u},\ \Lambda^{1+\alpha}\mathbf{u}\|_2^2\|\Lambda^{1+\alpha}\mathbf{b}\|_2^2+C\|\Omega\|_2^2\|\Lambda^{1+\alpha}\mathbf{b}\|_2^2\\
\le&\frac{1}{2}\|\Lambda^{2+\alpha}\mathbf{b}\|_2^2+C(1+\|\Lambda^\alpha\Omega\|_2^2)\|\Lambda^{1+\alpha}\mathbf{b}\|_2^2.
\end{aligned}$$

Therefore Gronwall's inequality gives

$$\|\Lambda^{1+\alpha}\mathbf{b}\|_2^2+\int_0^T\|\Lambda^{2+\alpha}\mathbf{b}\|_2^2dt\le C, \tag{3.15}$$

where C is a constant depending on any $T>0$ and the initial data. Furthermore, Sobolev's inequality and Proposition 3.1 imply that

$$\|\mathbf{b}\|_\infty^2\le C\|\mathbf{b},\ \Lambda^{1+\alpha}\mathbf{b}\|_2^2\le C. \tag{3.16}$$

Next, we will prove the L^q-bounds for $\Delta\mathbf{b}$. We will make full use of the special structure of the nonlinear terms in the equation of $\mathbf{b}$ and the integral form of b_1 and b_2. Due to the divergence-free conditions $\nabla\cdot\mathbf{u}=\nabla\cdot\mathbf{b}=0$, we find that

$$\mathbf{b}\cdot\nabla u_1-\mathbf{u}\cdot\nabla b_1=\partial_2(b_2u_1-u_2b_1), \tag{3.17}$$

and

$$\mathbf{b}\cdot\nabla u_2-\mathbf{u}\cdot\nabla b_2=\partial_1(b_1u_2-u_1b_2). \tag{3.18}$$

In addition, we write the equations $(1.3)_2$ and $(1.3)_3$ in the integral form which was previously considered in [16].

$$\begin{aligned} b_1=&\int_{\mathbb{R}}G_1(y_2,\ t)b_{01}(x_1,\ x_2-y_2)dy_2\\ &+\int_0^t\int_{\mathbb{R}}G_1(y_2,\ \tau)(\mathbf{b}\cdot\nabla u_1-\mathbf{u}\cdot\nabla b_1)(x_1,\ x_2-y_2,\ t-\tau)dy_2d\tau, \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} b_2=&\int_{\mathbb{R}}G_1(y_1,\ t)b_{02}(x_1-y_1,\ x_2)dy_1\\ &+\int_0^t\int_{\mathbb{R}}G_1(y_1,\ \tau)(\mathbf{b}\cdot\nabla u_2-\mathbf{u}\cdot\nabla b_2)(x_1-y_1,\ x_2,\ t-\tau)dy_1d\tau, \end{aligned} \tag{3.20}$$

We are the first to estimate $\|\partial_{22}b_1\|_{L_t^2L^q}$ for $q\in(2,\infty)$. Taking the L^q-norm with respect to x_1 and then L^q-norm with respect to x_2, we obtain

$$\begin{aligned} &\int_0^t\|\partial_{22}(\int_{\mathbb{R}}G_1(y_2,\ t)b_{01}(x_1,\ x_2-y_2)dy_2)\|_q^2d\tau\\ \le&\int_0^t\|G_1(y_2,\ t)\partial_{22}b_{01}(x_1,\ x_2-y_2)dy_2\|_q^2d\tau\\ \le&C\int_0^t\|G_1(y_2,\ t)\|_1^2\|\partial_{22}b_{01}\|_q^2d\tau\\ \le&Ct\|b_{01}\|_{H^3}^2, \end{aligned} \tag{3.21}$$

where we have used the fact $\|G_1(y_2,\ t)\|_1=1$. Similarly, by Lemma 2.5, we have

$$\begin{aligned} &\int_0^t\left\|\partial_{22}\int_0^\tau\int_{\mathbb{R}}G_1(y_2,\ s)(\mathbf{b}\cdot\nabla u_1-\mathbf{u}\cdot\nabla b_1)(x_1,\ x_2-y_2,\ \tau-s)dy_2ds\right\|_q^2d\tau\\ \le&C\int_0^t\|\mathbf{b}\cdot\nabla u_1-\mathbf{u}\cdot\nabla b_1\|_q^2d\tau\le C\int_0^t(\|\mathbf{b}\|_\infty\|\nabla u_1\|_q^2+\|\mathbf{u}\|_{2q}^2\|\nabla b_1\|_{2q}^2)d\tau\\ \le&C\int_0^t(\|\Omega\|_q^2+\|\mathbf{u},\ \Omega\|_2^2\|\mathbf{b},\ \Delta\mathbf{b}\|_2^2)d\tau\le C\int_0^t\|\Omega\|_q^2d\tau+C\int_0^t(1+\|\Delta\mathbf{b}\|_2^2)d\tau\\ \le&C\int_0^t\|\Omega\|_q^2d\tau+C(t+1), \end{aligned} \tag{3.22}$$

which together with (3.21) yields

$$\|\partial_{22}b_1\|_{L_t^2L^q}\le C\int_0^t\|\Omega\|_q^2d\tau+C(t+1). \tag{3.23}$$

Due to the special structure of (3.17), we can use a similar method as above to bound $\|\partial_{12}b_1\|_{L_t^2L^q}$ as follows

$$\int_0^t\|\partial_{12}(\int_{\mathbb{R}}G_1(y_2,\ t)b_{01}(x_1,\ x_2-y_2)dy_2)\|_q^2d\tau\le Ct\|b_{01}\|_{H^3}^2, \tag{3.24}$$

and

$$\begin{aligned}
&\int_0^t \left\|\partial_{12}\int_0^\tau \int_{\mathbb{R}} G_1(y_2,\, s)(\mathbf{b}\cdot\nabla u_1-\mathbf{u}\cdot\nabla b_1)(x_1,\, x_2-y_2,\, \tau-s)\,dy_2ds\right\|_q^2 d\tau\\
=&\int_0^t \left\|\int_0^\tau \int_{\mathbb{R}} G_1(y_2,\, s)\partial_{22}(\partial_1(b_2u_1-u_2b_1))(x_1,\, x_2-y_2,\, \tau-s)\,dy_2ds\right\|_q^2 d\tau\\
\le& C\int_0^t \|\partial_1(b_2u_1-u_2b_1))\|_q^2 d\tau \le C\int_0^t (\|\mathbf{b}\|_\infty\|\nabla\mathbf{u}\|_q^2+\|\mathbf{u}\|_{2q}^2\|\nabla\mathbf{b}\|_{2q}^2)d\tau\\
\le& C\int_0^t (\|\Omega\|_q^2+\|\mathbf{u},\,\Omega\|_2^2\|\mathbf{b},\,\Delta\mathbf{b}\|_2^2)d\tau \le C\int_0^t \|\Omega\|_q^2 d\tau + C\int_0^t (1+\|\Delta\mathbf{b}\|_2^2)d\tau\\
\le& C\int_0^t \|\Omega\|_q^2 d\tau + C(t+1).
\end{aligned} \tag{3.25}$$

Combining (3.24) and (3.25), we have

$$\|\partial_{12}b_1\|_{L_t^2L^q} \le C\int_0^t \|\Omega\|_q^2 d\tau + C(t+1). \tag{3.26}$$

Applying similar methods to the Equation (3.20), we obtain

$$\|\partial_{11}b_2\|_{L_t^2L^q},\ \|\partial_{12}b_2\|_{L_t^2L^q} \le C\int_0^t \|\Omega\|_q^2 d\tau + C(t+1), \tag{3.27}$$

which together with (3.23) and (3.26), lead to

$$\begin{aligned}
\|\Delta\mathbf{b}\|_{L_t^2L^q} &\le C(\|\partial_{11}b_1\|_{L_t^2L^q}+\|\partial_{22}b_1\|_{L_t^2L^q}+\|\partial_{11}b_2\|_{L_t^2L^q}+\|\partial_{22}b_2\|_{L_t^2L^q})\\
&\le C\int_0^t \|\Omega\|_q^2 d\tau + C(t+1).
\end{aligned} \tag{3.28}$$

In addition, due to the divergence-free condition $\nabla\cdot\mathbf{b}=0$, we have

$$\|\nabla j\|_{L_t^2L^q} \le C\|\Delta\mathbf{b}\|_{L_t^2L^q} \le C\int_0^t \|\Omega\|_q^2 d\tau + C(t+1). \tag{3.29}$$

Clearly, if we have the global bound for $\|\Omega\|_{L_t^2L^q}$, which implies

$$\|\Delta\mathbf{b}\|_{L_t^2L^q} < C(t+1).$$

Furthermore, Sobolev's inequality gives

$$\|\nabla\mathbf{b}\|_{L_t^1L_\infty} < C(t+1).$$

Next, we will prove the crucial estimate for $\|\Omega\|_q$. Due to the term $\Delta\mathrm{w}$ in $(3.4)_1$, we can not bound $\|\nabla\Omega\|_q$ directly. To overcome this difficulty, we work with the combined quantity

$$Z=\Omega+\Lambda^{2(1-\gamma)}\mathrm{w},$$

which was considered previously in [18]. For readers' convenience, we give the details. Applying $\Lambda^{2(1-\gamma)}$ to both sides of the Equation $(3.4)_3$ leads to

$$\partial_t\Lambda^{2(1-\gamma)}\mathrm{w}+\mathbf{u}\cdot\nabla\Lambda^{2(1-\gamma)}\mathrm{w}+\Lambda^2\mathrm{w}+2\Lambda^{2(1-\gamma)}\mathrm{w}=\Lambda^{2(1-\gamma)}\Omega-[\Lambda^{2(1-\gamma)},\,\mathbf{u}\cdot\nabla]\mathrm{w}.$$

Combining the Equation (3.4)$_1$ yields

$$\begin{aligned}\partial_t Z+\mathbf{u}\cdot\nabla Z+\Lambda^{2\alpha}Z=&-2\Lambda^{2(1-\gamma)}\mathrm{w}+\Lambda^{2(1-\gamma+\alpha)}\mathrm{w}\\&+\Lambda^{2(1-\gamma)}\Omega-[\Lambda^{2(1-\gamma)},\ \mathbf{u}\cdot\nabla]\mathrm{w}+\mathbf{b}\cdot\nabla j.\end{aligned}\tag{3.30}$$

Although (3.30) appears to be more complex than (3.4)$_1$, it eliminates the most regularity demanding term $\Delta\mathrm{w}$. We can establish the estimate for $\|Z\|_q$ and then obtain the global bound for $\|\Omega\|_{L^1_tL^q}$. Due to the Proposition 3.1, (3.8) and $Z=\Omega+\Lambda^{2(1-\gamma)}\mathrm{w}$, one can easily infer that

$$\|Z\|_2\le\|\Omega\|_2+\|\Lambda^{2(1-\gamma)}\mathrm{w}\|_2\le\|\Omega\|_2+C\|\mathrm{w}\|_2^{\frac{\alpha+3\gamma-3}{\alpha+\gamma}}\|\Lambda^{\alpha+\gamma}\mathrm{w}\|_2^{\frac{2-2\gamma}{\alpha+\gamma}}\le C.\tag{3.31}$$

Employing Sobolev's inequality and noting that $\Lambda^\alpha Z=\Lambda^\alpha\Omega+\Lambda^{\alpha+2(1-\gamma)}\mathrm{w}$, one has

$$\begin{aligned}\int_0^t\|\Lambda^\alpha Z\|_2^2\,dt\le&\int_0^t\|\Lambda^\alpha\Omega\|_2^2\,dt+\int_0^t\|\Lambda^{\alpha+2(1-\gamma)}\mathrm{w}\|_2^2\,dt\\\le&\int_0^t\|\Lambda^\alpha\Omega\|_2^2\,dt+C\int_0^t\|\mathrm{w}\|_2^{\frac{2(3\gamma-\alpha-1)}{1+\gamma}}\|\Lambda^{1+\gamma}\mathrm{w}\|_2^{\frac{2(\alpha+2-2\gamma)}{1+\gamma}}\,dt\le C.\end{aligned}\tag{3.32}$$

Furthermore,

$$\|Z\|_2^2+\int_0^t\|Z\|^2_{\frac{2}{1-\alpha}}\,dt\le C.\tag{3.33}$$

Next, we will prove, for any $2\le q<\dfrac{2\alpha}{1-\gamma}$, Z obeys

$$\|Z\|_q^q+\int_0^t\|Z\|^q_{\frac{q}{1-\alpha}}\,dt\le C.\tag{3.34}$$

Multiplying the Equation (3.30) by $|Z|^{q-2}Z$ and integrating over $\mathbb{R}^2$, we obtain

$$\begin{aligned}&\frac{1}{q}\frac{d}{dt}\|Z\|_q^q+C_1\|Z\|^q_{\dot B^{\frac{2\alpha}{q}}_{q,q}}+C_2\|Z\|^q_{\frac{q}{1-\alpha}}+C_3\|\Lambda^\alpha(|Z|^{\frac q2})\|_2^2\\\le&\int(-2\Lambda^{2(1-\gamma)}\mathrm{w}+\Lambda^{2(\alpha+1-\gamma)}\mathrm{w})|Z|^{q-2}Z\,dxdy+\int\Lambda^{2(1-\gamma)}\Omega|Z|^{q-2}Z\,dxdy\\&-\int[\Lambda^{2(1-\gamma)},\ \mathbf{u}\cdot\nabla]\mathrm{w}|Z|^{q-2}Z\,dxdy+\int\mathbf{b}\cdot\nabla j|Z|^{q-2}Z\,dxdy\\=&J_1+J_2+J_3+J_4,\end{aligned}\tag{3.35}$$

where $\dot B^s_{p,q}$ represents the homogeneous Besov space and we also have used the following facts, for any $2\le q<\infty$ and $0<s<1$,

$$\int\Lambda^{2s}f|f|^{q-2}f\,dxdy\ge C(s,q)\|\Lambda^s(|f|^{\frac q2})\|_2^2,\tag{3.36}$$

$$\int\Lambda^{2s}f|f|^{q-2}f\,dxdy\ge C(s,q)\|f\|^q_{\frac{q}{1-s}},\tag{3.37}$$

and

$$\int \Lambda^{2s} f|f|^{q-2} f\,dxdy \geq C(s,q)\|f\|^q_{\dot{B}^{\frac{2\alpha}{q}}_{q,q}}. \tag{3.38}$$

One can refer to [3] for the inequality (3.36) and (3.37) follows from (3.36) via Sobolev's inequality. The inequality (3.38) can be found in [8]. Applying Hölder's inequality and the Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}
J_1 &= \int (-2\Lambda^{2(1-\gamma)}\mathrm{w} + \Lambda^{2(\alpha+1-\gamma)}\mathrm{w})|Z|^{q-2}Z\,dxdy \\
&\leq C\|\Lambda^\alpha(|Z|^{\frac{q}{2}})\|_2\|\Lambda^{-\alpha}((-2\Lambda^{2(1-\gamma)}\mathrm{w}+\Lambda^{2(\alpha+1-\gamma)}\mathrm{w})|Z|^{\frac{q}{2}-2}Z)\|_2 \\
&\leq C\|\Lambda^\alpha(|Z|^{\frac{q}{2}})\|_2\|(-2\Lambda^{2(1-\gamma)}\mathrm{w}+\Lambda^{2(\alpha+1-\gamma)}\mathrm{w})|Z|^{\frac{q}{2}-2}Z\|_{\frac{2}{1+\alpha}} \\
&\leq C\|\Lambda^\alpha(|Z|^{\frac{q}{2}})\|_2\|-2\Lambda^{2(1-\gamma)}\mathrm{w}+\Lambda^{2(\alpha+1-\gamma)}\mathrm{w}\|_{\frac{2}{1+\alpha-\gamma}}\||Z|^{\frac{q}{2}-2}Z\|_{\frac{2}{\gamma}} \\
&\leq C\|\Lambda^\alpha(|Z|^{\frac{q}{2}})\|_2(\|\mathrm{w}\|_2+\|\Lambda^{\alpha+2\gamma}\mathrm{w}\|_2)\|Z\|^{\frac{q}{2}-1}_{\frac{q-2}{\gamma}} \\
&\leq \frac{C_3}{4}\|\Lambda^\alpha(|Z|^{\frac{q}{2}})\|_2^2 + C\|\mathrm{w},\ \Lambda^{\alpha+2\gamma}\mathrm{w}\|_2^2(\|Z\|_2^{q-2}+\|Z\|_q^{q-2}).
\end{aligned}$$

Due to $Z=\Omega+\Lambda^{2(1-\gamma)}\mathrm{w}$, we can bound J_2 as

$$\begin{aligned}
J_2 &= \int \Lambda^{2(1-\gamma)}\Omega|Z|^{q-2}Z\,dxdy \\
&= \int \Lambda^{4(1-\gamma)}\mathrm{w}|Z|^{q-2}Z\,dxdy + \int \Lambda^{2(1-\gamma)}Z|Z|^{q-2}Z\,dxdy \\
&= J_{21}+J_{22}.
\end{aligned}$$

Obviously, we have $4-4\gamma<\alpha+2\gamma$. Therefore, we can bound the term J_{21} similarly as J_1,

$$J_{21} \leq \frac{C_3}{4}\|\Lambda^\alpha(|Z|^{\frac{q}{2}})\|_2^2 + C\|\mathrm{w},\ \Lambda^{\alpha+2\gamma}\mathrm{w}\|_2^2(\|Z\|_2^{q-2}+\|Z\|_q^{q-2}).$$

Next, we will estimate the difficult term J_{22}. For any $2\leq q<\dfrac{2\alpha}{1-\gamma}$, one can easily check that $2(1-\gamma)-\dfrac{2\alpha}{q}<\dfrac{2\alpha}{q}$. Then we can choose $0<s<\sigma<1$ satisfying

$$2(1-\gamma)-\frac{2\alpha}{q}<s<\sigma<\frac{2\alpha}{q}.$$

According to Lemma 2.4, we obtain

$$\begin{aligned}
J_{22} &\leq C\|\Lambda^{2(1-\gamma)-s}Z\|_q\|\Lambda^s(|Z|^{q-2}Z)\|_{\frac{q}{q-1}} \\
&\leq C(\|Z\|_{\dot{B}^{\frac{2\alpha}{q}}_{q,q}}+\|Z\|_q)\|Z\|_{B^\sigma_{q,\frac{q}{q-1}}}\||Z|^{q-2}\|_{\frac{q}{q-2}} \\
&\leq C(\|Z\|_{\dot{B}^{\frac{2\alpha}{q}}_{q,q}}+\|Z\|_q)\|Z\|_{B^\sigma_{q,\frac{q}{q-1}}}\|Z\|_q^{q-2} \\
&\leq C(\|Z\|_{\dot{B}^{\frac{2\alpha}{q}}_{q,q}}+\|Z\|_q)\|Z\|_{B^{\frac{2\alpha}{q}}_{q,q}}\|Z\|_q^{q-2}
\end{aligned}$$

$$\leq \frac{C_1}{2}\|Z\|^q_{\dot{B}^{\frac{2\alpha}{q}}_{q,q}} + C\|Z\|^q_q,$$

where we have used the facts

$$\dot{B}^{\frac{2\alpha}{q}}_{q,q} \hookrightarrow \dot{W}^{2(1-\gamma)-s,q}, \quad B^{\frac{2\alpha}{q}}_{q,q} \hookrightarrow B^{\sigma}_{q,\frac{q}{q-1}}.$$

By Lemma 2.1 and Hölder's inequality, J_3 can be bounded as

$$\begin{aligned} J_3 = -\int [\Lambda^{2(1-\gamma)},\ \mathbf{u}\cdot\nabla]\mathrm{w}|Z|^{q-2}Z\,dxdy &\leq \|[\Lambda^{2(1-\gamma)},\ \mathbf{u}\cdot\nabla]\mathrm{w}\|_q \||Z|^{q-2}Z\|_{\frac{q}{q-1}} \\ \leq C\|\nabla \mathrm{w}\|_\infty \|\Lambda^{2(1-\gamma)}\mathbf{u}\|_q \|Z\|^{q-1}_q &\leq C\|\mathbf{u},\ \Lambda^\alpha\Omega\|^2_2(1+\|Z\|^q_q), \end{aligned}$$

where we have used the fact and (3.9), for $\alpha+2\gamma>2$,

$$\|\Lambda^{2(1-\gamma)}\mathbf{u}\|_q \leq C(\|\mathbf{u}\|_2 + \|\Lambda^\alpha\Omega\|_2).$$

Finally, we will estimate J_4. Using Hölder's inequality, one has

$$J_4 = \int \mathbf{b}\cdot\nabla j|Z|^{q-2}Z\,dxdy \leq \|\mathbf{b}\|_\infty \|\nabla j\|_q \|Z\|^{q-1}_q.$$

According to (3.29), we have

$$\begin{aligned} J_4 &\leq C(\|\Omega\|_q+1)\|Z\|^{q-1}_q \leq C(\|Z\|_q + \|\Lambda^{2(1-\gamma)}\mathrm{w}\|_q+1)\|Z\|^{q-1}_q \\ &\leq C\|Z\|^q_q + C(\|\mathrm{w},\ \Lambda^{\alpha+\gamma}\mathrm{w}\|^2_2+1)\|Z\|^{q-1}_q. \end{aligned}$$

Inserting the estimates for $J_1 \sim J_4$ into (3.35), we find that

$$\begin{aligned} &\frac{1}{q}\frac{d}{dt}\|Z\|^q_q + \frac{C_1}{2}\|Z\|^q_{\dot{B}^{\frac{2\alpha}{q}}_{q,q}} + C_2\|Z\|^q_{\frac{q}{1-\alpha}} + \frac{C_3}{2}\|\Lambda^\alpha(|Z|^{\frac{q}{2}})\|^2_2 \\ \leq & C(1+\|\Lambda^\alpha\Omega,\ \Lambda^{\alpha+\gamma}\mathrm{w}\|^2_2)(1+\|Z\|^q_q), \end{aligned}$$

which together with Gronwall's inequality implies

$$\|Z\|^q_q + \int_0^t \|Z\|^q_{\frac{q}{1-\alpha}}\,dt \leq C. \tag{3.39}$$

In addition, combining with $Z = \Omega + \Lambda^{2(1-\gamma)}\mathrm{w}$, one has

$$\|\Omega\|_q \leq \|Z\|_q + \|\Lambda^{2(1-\gamma)}\mathrm{w}\|_q \leq \|Z\|_q + \|\mathrm{w},\ \Lambda^{\alpha+\gamma}\mathrm{w}\|^2_2 \leq C, \tag{3.40}$$

which together with (3.29) leads to

$$\|\Delta\mathbf{b}\|_{L^2_tL^q} \leq C(t+1). \tag{3.41}$$

Furthermore, Sobolev's inequality gives, for any $T>0$,

$$\int_0^T \|\nabla\mathbf{b}\|_\infty\,dt \leq C. \tag{3.42}$$

□

3.3. L^∞-bound for $\nabla\mathbf{u}$. In this section, we will establish the global bound for $\|\nabla\mathbf{u}\|_{L^1_tL^\infty}$. This crucial global bound allows us to obtain the global bound for $\|(\mathbf{u},\ \mathbf{b},\ \mathrm{w})\|_{H^s}$ with $s>2$.

PROPOSITION 3.3. *Assume that α, γ and $(\mathbf{u}_0,\ \mathbf{b}_0,\ w_0)$ satisfies the conditions stated in Theorem 1.1. Then system (1.3) has a global solution $(\mathbf{u},\ \mathbf{b},\ w)$ satisfying, for any $T>0$,*

$$\|\Lambda^2\mathbf{u},\ \Lambda^2\mathbf{b},\ \Lambda^2\mathrm{w}\|_2^2+\int_0^T\|\Lambda^{1+\alpha}\Omega,\ \Lambda^2 j,\ \Lambda^{2+\gamma}\mathrm{w}\|_2^2\,dt\le C. \tag{3.43}$$

As a special consequence, for any $T>0$,

$$\int_0^T\|\nabla\mathbf{u}\|_\infty\,dt\le C, \tag{3.44}$$

where C depends only on T and the initial data.

Proof. Multiplying the Equations $(1.3)_{1\sim4}$ by $\Lambda^4\mathbf{u}$, $\Lambda^4 b_1$, $\Lambda^4 b_2$ and $\Lambda^4\mathrm{w}$, respectively and integrating over $\mathbb{R}^2$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^2\mathbf{u},\ \Lambda^2\mathbf{b},\ \Lambda^2\mathrm{w}\|_2^2+\|\Lambda^{1+\alpha}\Omega,\ \Lambda^2 j,\ \Lambda^{2+\gamma}\mathrm{w}\|_2^2+2\|\Lambda^2\mathrm{w}\|_2^2\\
=&2\int\Omega\Lambda^4\mathrm{w}\,dxdy+\int(\mathbf{b}\cdot\nabla\mathbf{b})\Lambda^4\mathbf{u}\,dxdy+\int(\mathbf{b}\cdot\nabla\mathbf{u})\Lambda^4\mathbf{b}\,dxdy\\
&-\int[\Lambda^2,\ \mathbf{u}\cdot\nabla]\mathbf{u}\Lambda^2\mathbf{u}\,dxdy-\int[\Lambda^2,\ \mathbf{u}\cdot\nabla]\mathbf{b}\Lambda^2\mathbf{b}\,dxdy-\int[\Lambda^2,\ \mathbf{u}\cdot\nabla]\mathrm{w}\Lambda^2\mathrm{w}\,dxdy\\
=&K_1+K_2+K_3+K_4+K_5+K_6.
\end{aligned} \tag{3.45}$$

Noting that $2<3-\alpha<2+\gamma$, applying Hölder's inequality, one has

$$\begin{aligned}
K_1=2\int\Omega\Lambda^4\mathrm{w}\,dxdy&\le C\|\Lambda^{1+\alpha}\Omega\|_2\|\Lambda^{3-\alpha}\mathrm{w}\|_2\\
&\le\frac{1}{4}\|\Lambda^{1+\alpha}\Omega\|_2^2+\frac{1}{4}\|\Lambda^{2+\gamma}\mathrm{w}\|_2^2+C\|\Lambda^2\mathrm{w}\|_2^2.
\end{aligned}$$

By Lemma 2.2, using Hölder's inequality, we find that

$$\begin{aligned}
K_2=\int(\mathbf{b}\cdot\nabla\mathbf{b})\Lambda^4\mathbf{u}\,dxdy&=\int\Lambda^2(\mathbf{b}\cdot\nabla\mathbf{b})\Lambda^2\mathbf{u}\,dxdy\\
&\le C(\|\mathbf{b}\|_\infty\|\Lambda^2 j\|_2+\|\nabla\mathbf{b}\|_\infty\|\Lambda^2\mathbf{b}\|_2)\|\Lambda^2\mathbf{u}\|_2\\
&\le\frac{1}{6}\|\Lambda^2 j\|_2^2+C\|\Lambda^2\mathbf{u},\ \Lambda^2\mathbf{b}\|_2^2.
\end{aligned}$$

Similarly, integrating by parts, we have

$$\begin{aligned}
K_3=&\int(\mathbf{b}\cdot\nabla\mathbf{u})\Lambda^4\mathbf{b}\,dxdy\\
=&\int((\Lambda^2(\mathbf{b}\cdot\nabla\mathbf{u})-\mathbf{b}\cdot\Lambda^2\nabla\mathbf{u}-\nabla\mathbf{b}\cdot\Lambda^2\mathbf{u})+\mathbf{b}\cdot\Lambda^2\nabla\mathbf{u}+\nabla\mathbf{b}\cdot\Lambda^2\mathbf{u})\Lambda^2\mathbf{b}\,dxdy\\
\le&C(\|\nabla\mathbf{u}\|_\infty\|\Lambda^2\mathbf{b}\|_2+\|\nabla\mathbf{b}\|_\infty\|\Lambda^2\mathbf{u}\|_2)\|\Lambda^2\mathbf{b}\|_2+C\|\mathbf{b}\|_\infty\|\Lambda^2 j\|_2\|\Lambda^2\mathbf{u}\|_2\\
\le&\frac{1}{6}\|\Lambda^2 j\|_2^2+C(1+\|\Lambda^\alpha\Omega\|_2^2)\|\Lambda^2\mathbf{u},\ \Lambda^2\mathbf{b}\|_2^2.
\end{aligned}$$

Applying Hölder's inequality and Lemma 2.1, we can estimate K_4 as

$$K_4=-\int[\Lambda^2,\ \mathbf{u}\cdot\nabla]\mathbf{u}\Lambda^2\mathbf{u}dxdy\leq C\|\nabla\mathbf{u}\|_{\frac{q}{1-\alpha}}\|\Lambda^2\mathbf{u}\|^2_{\frac{2q}{\alpha+q-1}}$$
$$\leq\|\Omega\|_{\frac{q}{1-\alpha}}\|\mathbf{u}\|_2^{\frac{2(\alpha q+\alpha-1)}{q(\alpha+2)}}\|\Lambda^{2+\alpha}\mathbf{u}\|_2^{\frac{2(2q-\alpha+1)}{q(\alpha+2})}\leq\frac{1}{4}\|\Lambda^{1+\alpha}\Omega\|_2^2+C.$$

By Lemma 2.1, one has

$$\begin{aligned}K_5=&-\int[\Lambda^2,\ \mathbf{u}\cdot\nabla]\mathbf{b}\Lambda^2\mathbf{b}dxdy\\
\leq& C(\|\nabla\mathbf{u}\|_{\frac{2}{1-\alpha}}\|\Lambda^2\mathbf{b}\|_{\frac{2}{\alpha}}+\|\nabla\mathbf{b}\|_\infty\|\Lambda^2\mathbf{u}\|_2)\|\Lambda^2\mathbf{b}\|_2\\
\leq& C(\|\Omega\|_{\frac{2}{1-\alpha}}\|\mathbf{b}\|_2^{\frac{\alpha}{3}}\|\Lambda^2 j\|_2^{\frac{2\alpha}{3}}\|\Lambda^2\mathbf{b}\|_2+C\|\Lambda^2\mathbf{u},\ \Lambda^2\mathbf{b}\|_2^2\\
\leq&\frac{1}{6}\|\Lambda^2 j\|_2^2+C\|\Omega\|^{\frac{6}{\alpha}}_{\frac{2}{1-\alpha}}\|\mathbf{b}\|_2^2+C\|\Lambda^2\mathbf{u},\ \Lambda^2\mathbf{b}\|_2^2\\
\leq&\frac{1}{6}\|\Lambda^2 j\|_2^2+C(1+\|\Lambda^2\mathbf{u},\ \Lambda^2\mathbf{b}\|_2^2).\end{aligned}$$

Similarly,

$$\begin{aligned}K_6=&-\int[\Lambda^2,\ \mathbf{u}\cdot\nabla]\mathrm{w}\Lambda^2\mathrm{w}dxdy\\
\leq& C(\|\nabla\mathbf{u}\|_{\frac{2}{1-\alpha}}\|\Lambda^2\mathrm{w}\|_{\frac{2}{\alpha}}+\|\nabla\mathrm{w}\|_\infty\|\Lambda^2\mathbf{u}\|_2)\|\Lambda^2\mathrm{w}\|_2\\
\leq& C(\|\Omega\|_{\frac{2}{1-\alpha}}\|\mathrm{w}\|_2^{\frac{\alpha+\gamma-1}{\alpha+\gamma}}\|\Lambda^{2+\gamma}\mathrm{w}\|_2^{\frac{1}{\alpha+\gamma}}\|\Lambda^2\mathrm{w}\|_2+C\|\nabla\mathrm{w}\|_\infty\|\Lambda^2\mathbf{u},\ \Lambda^2\mathrm{w}\|_2^2\\
\leq&\frac{1}{4}\|\Lambda^{2+\gamma}\mathrm{w}\|_2^2+C(1+\|\nabla\mathrm{w}\|_\infty)(\|\Lambda^2\mathbf{u},\ \Lambda^2\mathrm{w}\|_2^2+1).\end{aligned}$$

Inserting the estimates for $K_1\sim K_6$ into (3.45), we obtain

$$\begin{aligned}&\frac{d}{dt}\|\Lambda^2\mathbf{u},\ \Lambda^2\mathbf{b},\ \Lambda^2\mathrm{w}\|_2^2+\|\Lambda^{1+\alpha}\Omega,\ \Lambda^2 j,\ \Lambda^{2+\gamma}\mathrm{w}\|_2^2+2\|\Lambda^2\mathrm{w}\|_2^2\\
\leq& C(1+\|\Lambda^\alpha\Omega\|_2^2+\|\nabla\mathrm{w}\|_\infty)(\|\Lambda^2\mathbf{u},\ \Lambda^2\mathbf{u},\ \Lambda^2\mathrm{w}\|_2^2+1).\end{aligned}$$

Applying Gronwall's inequality yields

$$\|\Lambda^2\mathbf{u},\ \Lambda^2\mathbf{b},\ \Lambda^2\mathrm{w}\|_2^2+\int_0^T\|\Lambda^{1+\alpha}\Omega,\ \Lambda^2 j,\ \Lambda^{2+\gamma}\mathrm{w}\|_2^2dt\leq C,\tag{3.46}$$

which together with Sobolev's inequality implies (3.44). Using Proposition 3.1, Proposition 3.2 and Proposition 3.3, we can prove Theorem 1.1. Multiplying the Equations $(1.3)_{1\sim4}$ by $\Lambda^{2s}\mathbf{u}$, $\Lambda^{2s}b_1$, $\Lambda^{2s}b_2$ and $\Lambda^{2s}\mathrm{w}$, respectively and taking the L^2-inner product and adding with $(3.1)_1$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_{H^s}^2+\|\Lambda^\alpha\mathbf{u},\ \nabla\mathbf{b},\ \Lambda^\gamma\mathrm{w}\|_2^2+2\|\mathrm{w}\|_{H^s}^2\\
=&2\int\Omega\Lambda^{2s}\mathrm{w}dxdy+\int[\Lambda^s,\ \mathbf{b}\cdot\nabla]\mathbf{b}\Lambda^s\mathbf{u}dxdy+\int[\Lambda^s,\ \mathbf{b}\cdot\nabla]\mathbf{u}\Lambda^s\mathbf{b}dxdy\\
&-\int[\Lambda^s,\ \mathbf{u}\cdot\nabla]\mathbf{u}\Lambda^s\mathbf{u}dxdy-\int[\Lambda^s,\ \mathbf{u}\cdot\nabla]\mathbf{b}\Lambda^s\mathbf{b}dxdy-\int[\Lambda^s,\ \mathbf{u}\cdot\nabla]\mathrm{w}\Lambda^s\mathrm{w}dxdy\end{aligned}$$

$$=L_1+L_2+L_3+L_4+L_5+L_6, \tag{3.47}$$

where we have used the fact

$$\|\Lambda^s\nabla\mathbf{b}\|_2^2\leq C\|\Lambda^s\partial_1 b_2,\ \Lambda^s\partial_2 b_1\|_2^2.$$

Using Hölder's inequality, one has

$$\begin{aligned}L_1=2\int\Omega\Lambda^{2s}\mathrm{w}\,dxdy&\leq\|\Lambda^{s-\gamma}\Omega\|_2\|\Lambda^{s+\gamma}\mathrm{w}\|_2\\&\leq C\|\mathbf{u}\|_2^{\frac{\alpha+\gamma-1}{\alpha+s}}\|\Lambda^{s+\alpha}\mathbf{u}\|_2^{\frac{s+1-\gamma}{\alpha+s}}\|\Lambda^{s+\gamma}\mathrm{w}\|_2\\&\leq\frac{1}{2}\|\Lambda^{s+\gamma}\mathrm{w}\|_2^2+\frac{1}{2}\|\Lambda^{s+\alpha}\mathbf{u}\|_2^2+C\|\mathbf{u}\|_{H^s}^2\\&\leq\frac{1}{2}\|\Lambda^{\alpha}\mathbf{u},\ \Lambda^{\gamma}\mathrm{w}\|_{H^s}^2+C\|\mathbf{u}\|_{H^s}^2.\end{aligned}$$

The remaining terms in (3.47) can be bounded by

$$C(\|\nabla\mathbf{u}\|_\infty+\|\nabla\mathbf{b}\|_\infty+\|\nabla\mathrm{w}\|_\infty)\|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_{H^s}^2.$$

Inserting the estimates for $L_1\sim L_6$ into (3.47) and applying Gronwall's inequality yield, for any $T>0$,

$$\|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_{H^s}^2+\int_0^T\|\Lambda^\alpha\mathbf{u},\ \nabla\mathbf{b},\ \Lambda^\gamma\mathrm{w}\|_2^2dt\leq C. \tag{3.48}$$

Next, we will prove the uniqueness. One can obtain it by a standard method. We first find the solution to a regularized system. To this end, we will give some notation as follows. For $\varepsilon>0$, we denote the standard mollifier by j_ε, namely

$$j_\varepsilon(x)=\varepsilon^{-2}j(\varepsilon^{-1}|x|),$$

with

$$j\in C_0^\infty(\mathbb{R}^2),\ j(x)=j(|x|),\ suppj\subset\{x||x|<1\},\ \int_{\mathbb{R}^2}j(x)dx=1.$$

For any locally integrable function v, we define the mollification $\mathcal{J}_\varepsilon v$ by

$$\mathcal{J}_\varepsilon v=j_\varepsilon*v.$$

Assume $\mathbb{P}$ is the Leray projection. One can establish a solution $(\mathbf{u}^\varepsilon,\ \mathbf{b}^\varepsilon,\ \mathrm{w}^\varepsilon)$ to the following system

$$\begin{cases}\partial_t\mathbf{u}^\varepsilon+\mathbb{P}\mathcal{J}_\varepsilon((\mathcal{J}_\varepsilon\mathbf{u}^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon\mathbf{u}^\varepsilon))+(-\Delta)^\alpha\mathcal{J}_\varepsilon^2\mathbf{u}^\varepsilon=\mathbb{P}\mathcal{J}_\varepsilon((\mathcal{J}_\varepsilon\mathbf{b}^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon\mathbf{b}^\varepsilon))+\nabla^\perp(\mathcal{J}_\varepsilon\mathrm{w}^\varepsilon),\\ \partial_t b_1^\varepsilon+\mathcal{J}_\varepsilon((\mathcal{J}_\varepsilon\mathbf{u}^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon b_1^\varepsilon))-\mathcal{J}_\varepsilon^2\partial_{22}b_1^\varepsilon=\mathcal{J}_\varepsilon((\mathcal{J}_\varepsilon\mathbf{b}^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon u_1^\varepsilon)),\\ \partial_t b_2^\varepsilon+\mathcal{J}_\varepsilon((\mathcal{J}_\varepsilon\mathbf{u}^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon b_2^\varepsilon))-\mathcal{J}_\varepsilon^2\partial_{11}b_2^\varepsilon=\mathcal{J}_\varepsilon((\mathcal{J}_\varepsilon\mathbf{b}^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon u_2^\varepsilon)),\\ \partial_t\mathrm{w}^\varepsilon+\mathcal{J}_\varepsilon((\mathcal{J}_\varepsilon\mathbf{u}^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon\mathrm{w}^\varepsilon))+\mathcal{J}_\varepsilon^2(-\Delta)^\gamma\mathrm{w}^\varepsilon+2\mathrm{w}^\varepsilon=\nabla\times(\mathcal{J}_\varepsilon\mathbf{u}^\varepsilon),\\ \nabla\cdot\mathbf{u}^\varepsilon=0,\ \nabla\cdot\mathbf{b}^\varepsilon=0,\\ \mathbf{u}^\varepsilon(x,\ 0)=u_0(x)*j_\varepsilon,\quad\mathbf{b}^\varepsilon(x,\ 0)=b_0(x)*j_\varepsilon,\quad\mathrm{w}^\varepsilon(x,\ 0)=\mathrm{w}_0(x)*j_\varepsilon,\end{cases} \tag{3.49}$$

where $0<\alpha<\gamma<1$ and $\alpha+\gamma>1$. According the proofs of Propositions 3.1, 3.2 and 3.3, one will obtain the global estimate as follows

$$\|\mathbf{u}^\varepsilon(t),\ \mathbf{b}^\varepsilon(t),\ \mathrm{w}^\varepsilon(t)\|_{H^s}^2\leq C,$$

for any $t>0$. Therefore, the global existence of the classical solution $(\mathbf{u},\ \mathbf{b},\ \mathrm{w})$ to the system (1.3) can be obtained by the standard compactness argument. The uniqueness can also be obtained by a standard process. We omit the details. This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

In this section, we will prove the Theorem 1.2. The proof of Theorem 1.2 will be divided into three stages. The first step is to show that

$$(1+t)\|\nabla\mathbf{u}(t),\ \nabla\mathbf{b}(t),\ \mathrm{w}(t)\|_2^2\to 0,\quad \text{as}\quad t\to\infty.$$

The second step will prove the global bounds for $\mathbf{b}$ namely,

$$\|\nabla\mathbf{b}(t)\|_2\leq C(1+t)^{-1},\qquad \|\mathbf{b}(t)\|_2\leq C(1+t)^{-\frac{1}{2}}.$$

Finally, we will show the decay rates for $\|\mathbf{u}(t)\|_2$ and $\|\mathrm{w}(t)\|_2$.

4.1. L^∞-bound for $\nabla\mathbf{u}$.

In this subsection we will show that $\|\nabla\mathbf{u}(t),\ \nabla\mathbf{b}(t),\ \nabla\mathrm{w}(t)\|_2$ decays faster than $(1+t)^{-\frac{1}{2}}$ as $t\to\infty$. More precisely, we will prove the following proposition.

Proposition 4.1. *Assume that $(\mathbf{u}_0,\ \mathbf{b}_0,\ w_0)\in H^1$. Then system (1.3) has a global solution $(\mathbf{u},\ \mathbf{b},\ w)$ satisfying,*

$$\lim_{t\to\infty} t\|\nabla\mathbf{u}(t),\ \nabla\mathbf{b}(t),\ \nabla w(t)\|_2^2=0. \tag{4.1}$$

Proof. According to Proposition 3.1, for any $0\leq t_0\leq t\leq\infty$, we have

$$\|\mathbf{u}(t),\ \mathbf{b}(t),\ \mathrm{w}(t)\|_2^2+\int_{t_0}^t\|\Lambda^\alpha\mathbf{u},\ \nabla\mathbf{b},\ \Lambda^\gamma\mathrm{w}\|_2^2ds\leq\|\mathbf{u}(t_0),\ \mathbf{b}(t_0),\ w(t_0)\|_2^2 \tag{4.2}$$

and

$$\begin{aligned}&\|\Omega(t),\ j(t),\ \nabla\mathrm{w}(t)\|_2^2+\int_{t_0}^t\|\Lambda^\alpha\Omega,\ \Delta\mathbf{b},\ \Lambda^{1+\gamma}\mathrm{w}\|_2^2ds\\ \leq&\|\Omega(t_0),\ j(t_0),\ \nabla\mathrm{w}(t_0)\|_2^2\exp^{C(1+\|\mathbf{u}_0,\ \mathbf{b}_0,\ \mathrm{w}_0\|_2^2)}.\end{aligned} \tag{4.3}$$

Furthermore, one has

$$\int_0^\infty\|\nabla\mathbf{b}(t)\|_2^2dt\leq C\|\mathbf{u}_0,\ \mathbf{b}_0,\ w_0\|_2^2,$$

and

$$\int_0^\infty\|\nabla\mathbf{u}(t)\|_2^2dt\leq\int_0^\infty\|\Lambda^\alpha\mathbf{u}(t),\ \Lambda^\alpha\Omega(t)\|_2^2dt\leq C\|\mathbf{u}_0,\ \mathbf{b}_0,\ w_0\|_{H^1}^2.$$

By Sobolev's inequality, we infer that

$$\int_0^\infty\|\nabla\mathrm{w}(t)\|_2^2dt\leq\int_0^\infty\|\Lambda^\gamma\mathrm{w}(t),\ \Lambda^{1+\gamma}\mathrm{w}(t)\|_2^2dt\leq C\|\mathbf{u}_0,\ \mathbf{b}_0,\ \mathrm{w}_0\|_{H^1}^2.$$

In particular,

$$\int_{\frac{t}{2}}^{t} \|\nabla \mathbf{u}(t),\ \nabla \mathbf{b}(t),\ \nabla \mathrm{w}(t)\|_2^2\, ds \to 0 \quad \text{as} \quad t\to\infty,$$

which together with (4.3) leads to

$$\frac{t}{2}\exp^{-C(1+\|\mathbf{u}_0,\ \mathbf{b}_0,\ \mathrm{w}_0\|_2^2)}\|\nabla \mathbf{u}(t),\ \nabla \mathbf{b}(t),\ \nabla \mathrm{w}(t)\|_2^2 \le \int_{\frac{t}{2}}^{t} \|\nabla \mathbf{u}(t),\ \nabla \mathbf{b}(t),\ \nabla \mathrm{w}(t)\|_2^2\, ds.$$

Furthermore, we obtain the desired decay rate

$$\lim_{t\to\infty}(1+t)\|\nabla \mathbf{u}(t),\ \nabla \mathbf{b}(t),\ \nabla \mathrm{w}(t)\|_2^2 = 0.$$

This completes the proof of Proposition 4.1. □

4.2. Optimal decay rates for b and ∇b. In this section we will make use of the special structure of the nonlinear terms in the equation of $\mathbf{b}$ and the integral form of b_1, b_2 to derive the optimal decay rates for $\mathbf{b}$ and $\nabla\mathbf{b}$.

PROPOSITION 4.2. *Assume the same conditions as those stated in Theorem 1.2. Then system (1.3) has a global solution* $(\mathbf{u},\ \mathbf{b},\ \mathrm{w})$ *satisfying,*

$$\|\mathbf{b}(t)\|_2 \le C(1+t)^{-\frac{1}{2}}, \qquad \|\nabla\mathbf{b}(t)\| \le C(1+t)^{-1}. \tag{4.4}$$

Proof. We write the first equation of (1.3) in the integral form,

$$\begin{aligned}\mathbf{u}(t) &= e^{-(-\Delta)^\alpha t}\mathbf{u}_0 + \int_0^t e^{-(-\Delta)^\alpha (t-\tau)}(\mathbb{P}(\mathbf{b}\cdot\nabla\mathbf{b} - \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla^\perp \mathrm{w})\, d\tau \\ &= e^{-(-\Delta)^\alpha t}\mathbf{u}_0 + \int_0^{\frac{t}{2}} e^{-(-\Delta)^\alpha (t-\tau)}(\mathbb{P}(\mathbf{b}\cdot\nabla\mathbf{b} - \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla^\perp \mathrm{w})\, d\tau \\ &\quad + \int_{\frac{t}{2}}^t e^{-(-\Delta)^\alpha (t-\tau)}(\mathbb{P}(\mathbf{b}\cdot\nabla\mathbf{b} - \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla^\perp \mathrm{w})\, d\tau, \end{aligned} \tag{4.5}$$

where $\mathbb{P}$ denotes the Leray projection onto divergence-free vector fields. We can eliminate the pressure term by it. We split the time integral into two parts to estimate. For $t\ge 1$, using Plancherel's theorem and (1.5), we have

$$\|e^{-(-\Delta)^\alpha t}\mathbf{u}_0\|_2 = \|e^{-|\xi|^{2\alpha}t}\widehat{\mathbf{u}}_0\|_2 \le C\|e^{-|\xi|^{2\alpha}t}\sqrt{|\xi|}\|_2 \le Ct^{-\frac{3}{4\alpha}}.$$

While $t<1$, one has

$$\|e^{-(-\Delta)^\alpha t}\mathbf{u}_0\|_2 \le \|\mathbf{u}_0\|_2,$$

therefore,

$$\|e^{-(-\Delta)^\alpha t}\mathbf{u}_0\|_2 \le C(1+t)^{-\frac{3}{4\alpha}}, \tag{4.6}$$

where C only depends on $\mathbf{u}_0$. Thanks to Lemma 2.5, we obtain

$$\left\|\int_0^{\frac{t}{2}} e^{-(-\Delta)^\alpha (t-\tau)}(\mathbb{P}(\mathbf{b}\cdot\nabla\mathbf{b} - \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla^\perp \mathrm{w})\, d\tau\right\|_2$$

$$=\left\|\int_0^{\frac{t}{2}}\nabla e^{-(-\Delta)^\alpha(t-\tau)}(\mathbb{P}(\mathbf{b}\otimes\mathbf{b}-\mathbf{u}\otimes\mathbf{u})-\mathrm{w})\,d\tau\right\|_2$$
$$\le C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{\alpha}}(\|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_2^2+1)\,d\tau.$$

For any $2<\dfrac{1}{\alpha}<s<\dfrac{2}{\alpha}$, integrating by parts and according to Lemma 2.5, we find that

$$\begin{aligned}&\left\|\int_{\frac{t}{2}}^{t}e^{-(-\Delta)^\alpha(t-\tau)}(\mathbb{P}(\mathbf{b}\cdot\nabla\mathbf{b}-\mathbf{u}\cdot\nabla\mathbf{u})+\nabla^\perp\mathrm{w})\,d\tau\right\|_2\\
&=\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{\alpha}(\frac{2+s}{2s}-\frac{1}{2})}\|\mathbf{b}\cdot\nabla\mathbf{b}-\mathbf{u}\cdot\nabla\mathbf{u}\|_{\frac{2s}{2+s}}\,d\tau+\left\|\int_{\frac{t}{2}}^{t}\nabla e^{-(-\Delta)^\alpha(t-\tau)}\mathrm{w}\,d\tau\right\|_2\\
&\le C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{s\alpha}}(\|\mathbf{b}\|_s\|\nabla\mathbf{b}\|_2+\|\mathbf{u}\|_s\|\nabla\mathbf{u}\|_2)\,d\tau+\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{\alpha}(\frac{2+s}{2s}-\frac{1}{2})}\|\mathrm{w}\|_{\frac{2s}{2+s}}\,d\tau\\
&\le C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{s\alpha}}(\|\mathbf{b}\|_2^{\frac{2}{s}}\|\nabla\mathbf{b}\|_2^{2-\frac{2}{s}}+\|\mathbf{u}\|_2^{\frac{2}{s}}\|\nabla\mathbf{u}\|_2^{2-\frac{2}{s}}+\|\mathrm{w}\|_2^{\frac{2}{s}}\|\nabla\mathrm{w}\|_2^{1-\frac{2}{s}})\,d\tau.\end{aligned}$$

Inserting these estimates into (4.5), we obtain

$$\begin{aligned}\|\mathbf{u}(t)\|_2\le{}&C(1+t)^{-\frac{3}{4\alpha}}+C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{\alpha}}(\|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_2^2+1)\,d\tau\\
&+C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{s\alpha}}(\|\mathbf{b}\|_2^{\frac{2}{s}}\|\nabla\mathbf{b}\|_2^{2-\frac{2}{s}}+\|\mathbf{u}\|_2^{\frac{2}{s}}\|\nabla\mathbf{u}\|_2^{2-\frac{2}{s}}+\|\mathrm{w}\|_2^{\frac{2}{s}}\|\nabla\mathrm{w}\|_2^{1-\frac{2}{s}})\,d\tau.\end{aligned}\tag{4.7}$$

Next we will estimate $\|\mathbf{b}(t)\|_2$. We write the integral form of b_1 in the Equation $(1.3)_2$, which was considered in [16]; for readers' convenience, we give the details as follows

$$b_1(x_1,\ x_2,\ t)=G_1(x_2,\ t)*b_{01}+\int_0^t G_1(x_2,\ t-\tau)*(\mathbf{b}\cdot\nabla u_1-\mathbf{u}\cdot\nabla b_1)(\tau)\,d\tau,\tag{4.8}$$

where G_1 denotes the 1D heat kernel. One can easily check that for $t<1$,

$$\|G_1(x_2,\ t)*b_{01}\|_2\le\|b_{01}\|_2.$$

For $t\ge1$, using Plancherel's theorem and (1.5), we find that

$$\begin{aligned}\|G_1(x_2,\ t)*b_{01}\|_2&=\|\widehat{G_1}(\xi_2,\ t)\widehat{b}_{01}(\xi_1,\ \xi_2)\|_2\le C\|\widehat{G_1}(\xi_2,\ t)\|\widehat{b}_{01}(\xi_1,\ \xi_2)\|_{L^2_{\xi_1}}\|_{L^2_{\xi_2}}\\
&\le C\|e^{-|\xi_2|^2t}\sqrt{|\xi_2|}\|L^2_{\xi_2}\le Ct^{-\frac{1}{2}}.\end{aligned}$$

Therefore,

$$\|G_1(x_2,\ t)*b_{01}(x_1,\ x_2)\|_2\le C(1+t)^{-\frac{1}{2}}.\tag{4.9}$$

Applying Hölder's inequality and Lemma 2.6, we can estimate the second term in (4.8) as

$$\left\|\int_0^t G_1(x_2,\ t-\tau)*(\mathbf{b}\cdot\nabla u_1-\mathbf{u}\cdot\nabla b_1)(x_1,\ x_2,\ \tau)\,d\tau\right\|_2$$

$$
\begin{aligned}
&\leq C\int_0^t \Big\|\partial_2 G_1(x_2,\ t-\tau)\|(b_2u_1-u_2b_1)(x_1,\ x_2,\ \tau)\|_{L^2_{x_1}}\Big\|_{L^2_{x_2}}\,d\tau\\
&\leq C\int_0^t (t-\tau)^{-\frac{1}{2}}\|b_2u_1-u_2b_1\|_2\,d\tau \leq C\int_0^t (t-\tau)^{-\frac{1}{2}}\|\mathbf{b}\|_{\frac{2}{\alpha}}\|\mathbf{u}\|_{\frac{2}{1-\alpha}}\,d\tau\\
&\leq C\int_0^t (t-\tau)^{-\frac{1}{2}}\|\mathbf{b}\|_2^{\alpha}\|\nabla\mathbf{b}\|_2^{1-\alpha}\|\Lambda^{\alpha}\mathbf{u}\|_2\,d\tau, \qquad (4.10)
\end{aligned}
$$

which together with (4.9) implies

$$
\|b_1(t)\|_2 \leq C(1+t)^{-\frac{1}{2}} + C\int_0^t (t-\tau)^{-\frac{1}{2}}\|\mathbf{b}\|_2^{\alpha}\|\nabla\mathbf{b}\|_2^{1-\alpha}\|\Lambda^{\alpha}\mathbf{u}\|_2\,d\tau. \qquad (4.11)
$$

Using similar methods to the integral form of b_2,

$$
b_2(x_1,\ x_2,\ t) = G_1(x_1,\ t)*b_{02} + \int_0^t G_1(x_1,\ t-\tau)*(\mathbf{b}\cdot\nabla u_2 - \mathbf{u}\cdot\nabla b_2)(\tau)\,d\tau,
$$

one can easily to check

$$
\|b_2(t)\|_2 \leq C(1+t)^{-\frac{1}{2}} + C\int_0^t (t-\tau)^{-\frac{1}{2}}\|\mathbf{b}\|_2^{\alpha}\|\nabla\mathbf{b}\|_2^{1-\alpha}\|\Lambda^{\alpha}\mathbf{u}\|_2\,d\tau. \qquad (4.12)
$$

Therefore,

$$
\begin{aligned}
\|\mathbf{b}(t)\|_2 &\leq C(1+t)^{-\frac{1}{2}} + C\int_0^t (t-\tau)^{-\frac{1}{2}}\|\mathbf{b}\|_2^{\alpha}\|\nabla\mathbf{b}\|_2^{1-\alpha}\|\Lambda^{\alpha}\mathbf{u}\|_2\,d\tau\\
&\leq C(1+t)^{-\frac{1}{2}} + C\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1}{2}}\|\mathbf{b}\|_2^{\alpha}\|\nabla\mathbf{b}\|_2^{1-\alpha}\|\Lambda^{\alpha}\mathbf{u}\|_2\,d\tau\\
&\quad + C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}}\|\mathbf{b}\|_2^{\alpha}\|\nabla\mathbf{b}\|_2^{1-\alpha}\|\mathbf{u}\|_2^{1-\alpha}\|\nabla\mathbf{u}\|_2^{\alpha}\,d\tau. \qquad (4.13)
\end{aligned}
$$

Next we will estimate $\|\mathrm{w}(t)\|_2$. Define $\Psi = e^{2t}\mathrm{w}$, applying Duhamel's principle, we obtain the integral form of w in the Equation $(1.3)_4$, namely

$$
\begin{aligned}
\mathrm{w}(t) &= e^{-2t}e^{-(-\Delta)^{\gamma}t}\mathrm{w}_0 + \int_0^t e^{-2(t-\tau)}e^{-(-\Delta)^{\gamma}(t-\tau)}(\nabla\times\mathbf{u} - \mathbf{u}\cdot\nabla\mathrm{w})\,d\tau\\
&= e^{-2t}e^{-(-\Delta)^{\gamma}t}\mathrm{w}_0 + \int_0^{\frac{t}{2}} e^{-2(t-\tau)}e^{-(-\Delta)^{\gamma}(t-\tau)}(\nabla\times\mathbf{u} - \mathbf{u}\cdot\nabla\mathrm{w})\,d\tau\\
&\quad + \int_{\frac{t}{2}}^t e^{-2(t-\tau)}e^{-(-\Delta)^{\gamma}(t-\tau)}(\nabla\times\mathbf{u} - \mathbf{u}\cdot\nabla\mathrm{w})\,d\tau. \qquad (4.14)
\end{aligned}
$$

For $t \geq 1$, using Plancherel's theorem and (1.5) leads to

$$
\|e^{-2t}e^{-(-\Delta)^{\gamma}t}\mathrm{w}_0\|_2 = \|e^{-2t}e^{-|\xi|^{2\gamma}t}\widehat{\mathrm{w}}_0\|_2 \leq Ce^{-2t}\|e^{-|\xi|^{2\gamma}t}\sqrt{|\xi|}\|_2 \leq Ct^{-\frac{3}{4\gamma}}.
$$

While $t < 1$, one has

$$
\|e^{-2t}e^{-(-\Delta)^{\gamma}t}\mathrm{w}_0\|_2 \leq e^{-2t}\|\mathrm{w}_0\|_2 \leq C,
$$

therefore,

$$\|e^{-2t}e^{-(-\Delta)^{\gamma}t}\mathrm{w}_0\|_2 \le C(1+t)^{-\frac{3}{4\gamma}}. \tag{4.15}$$

By Lemma 2.6, one has

$$\begin{aligned}
&\left\|\int_0^{\frac{t}{2}} e^{-2(t-\tau)}e^{-(-\Delta)^{\gamma}(t-\tau)}(\nabla\times\mathbf{u}-\mathbf{u}\cdot\nabla\mathrm{w})\,d\tau\right\|_2\\
\le& C\left\|\int_0^{\frac{t}{2}} e^{-2(t-\tau)}\nabla e^{-(-\Delta)^{\gamma}(t-\tau)}(\mathbf{u}-\mathbf{u}\otimes\mathrm{w})\,d\tau\right\|_2\\
\le& C\int_0^{\frac{t}{2}} e^{-2(t-\tau)}(t-\tau)^{-\frac{1}{\gamma}}(1+\|\mathbf{u},\ \mathrm{w}\|_2^2)\,d\tau\\
\le& Ce^{-t}\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{\gamma}}(1+\|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_2^2)\,d\tau.
\end{aligned}$$

Similarly, by Lemma 2.5, for any $2<s<\dfrac{2}{\gamma}$, we infer that

$$\begin{aligned}
&\left\|\int_{\frac{t}{2}}^{t} e^{-2(t-\tau)}e^{-(-\Delta)^{\gamma}(t-\tau)}(\nabla\times\mathbf{u}-\mathbf{u}\cdot\nabla\mathrm{w})\,d\tau\right\|_2\\
\le& C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{\gamma}(\frac{2+s}{2s}-\frac{1}{2})}\|\nabla\times\mathbf{u}-\mathbf{u}\cdot\nabla\mathrm{w}\|_{\frac{2s}{s+2}}\,d\tau\\
\le& C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{s\gamma}}\|\mathbf{u}\|_2^{\frac{2}{s}}\|\nabla\mathbf{u}\|_2^{1-\frac{2}{s}}(1+\|\nabla\mathrm{w}\|_2)\,d\tau.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathrm{w}(t)\|_2\le& C(1+t)^{-\frac{3}{4\gamma}}+Ce^{-t}\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{\gamma}}(1+\|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_2^2)\,d\tau\\
&+C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{s\gamma}}\|\mathbf{u}\|_2^{\frac{2}{s}}\|\nabla\mathbf{u}\|_2^{1-\frac{2}{s}}(1+\|\nabla\mathrm{w}\|_2)\,d\tau.
\end{aligned} \tag{4.16}$$

Combining (4.7), (4.13) with (4.16), we obtain

$$\begin{aligned}
&\|\mathbf{u}(t)\|_2+\|\mathbf{b}(t)\|_2+\|\mathrm{w}(t)\|_2\\
\le& C(1+t)^{-\frac{3}{4\gamma}}+C(1+t)^{-\frac{1}{2}}+C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{\alpha}}(\|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_2^2+1)\,d\tau\\
&+C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{2}}\|\mathbf{b}\|_2^{\alpha}\|\nabla\mathbf{b}\|_2^{1-\alpha}\|\Lambda^{\alpha}\mathbf{u}\|_2\,d\tau\\
&+Ce^{-t}\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{\gamma}}(1+\|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_2^2)\,d\tau\\
&+C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{s\alpha}}(\|\mathbf{b}\|_2^{\frac{2}{s}}\|\nabla\mathbf{b}\|_2^{2-\frac{2}{s}}+\|\mathbf{u}\|_2^{\frac{2}{s}}\|\nabla\mathbf{u}\|_2^{2-\frac{2}{s}}+\|\mathrm{w}\|_2^{\frac{2}{s}}\|\nabla\mathrm{w}\|_2^{1-\frac{2}{s}})\,d\tau\\
&+C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{2}}\|\mathbf{b}\|_2^{\alpha}\|\nabla\mathbf{b}\|_2^{1-\alpha}\|\mathbf{u}\|_2^{1-\alpha}\|\nabla\mathbf{u}\|_2^{\alpha}\,d\tau
\end{aligned}$$

$$+C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{s\gamma}}\|\mathbf{u}\|_2^{\frac{2}{s}}\|\nabla\mathbf{u}\|_2^{1-\frac{2}{s}}(1+\|\nabla \mathrm{w}\|_2)\,d\tau. \tag{4.17}$$

First, we show that, under the condition $0<\alpha<\frac{1}{2}$ and for any small $\varepsilon>0$, the corresponding solution $(\mathbf{u},\ \mathbf{b},\ \mathrm{w})$ to system (1.3) satisfies

$$\|\mathbf{u}(t)\|_2+\|\mathbf{b}(t)\|_2+\|\mathrm{w}(t)\|_2\le C(1+t)^{-\frac{1}{2}+\varepsilon}. \tag{4.18}$$

We will use iterative methods to achieve it. We will first show, for any $t\ge 0$,

$$\|\mathbf{u}(t)\|_2+\|\mathbf{b}(t)\|_2+\|\mathrm{w}(t)\|_2\le C(1+t)^{-(\frac{1}{2}-\frac{\alpha}{2})}. \tag{4.19}$$

For notation convenience, denoting

$$\mathcal{N}_1(t)=\sup_{0\le\tau\le t}\{(1+\tau)^{(\frac{1}{2}-\frac{\alpha}{2})}\|\mathbf{u}(\tau)\|_2+\|\mathbf{b}(\tau)\|_2+\|\mathrm{w}(\tau)\|_2\},$$

and

$$\psi(t)=t^{\frac{1}{2}}(\|\mathbf{u}(t)\|_2+\|\mathbf{b}(t)\|_2+\|\mathrm{w}(t)\|_2).$$

Due to (4.17), we have

$$\begin{aligned}
\mathcal{N}_1(t)\le& C(1+t)^{\frac{1}{2}-\frac{\alpha}{2}-\frac{3}{4\gamma}}+C(1+t)^{-\frac{\alpha}{2}}+C(1+t)^{\frac{1}{2}-\frac{\alpha}{2}}\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{\alpha}}(\|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_2^2+1)\,d\tau\\
&+C(1+t)^{\frac{1}{2}-\frac{\alpha}{2}}\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{2}}\|\mathbf{b}\|_2^{\alpha}\|\nabla\mathbf{b}\|_2^{1-\alpha}\|\Lambda^{\alpha}\mathbf{u}\|_2\,d\tau\\
&+C(1+t)^{\frac{1}{2}-\frac{\alpha}{2}}e^{-t}\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{\gamma}}(1+\|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_2^2)\,d\tau\\
&+C\mathcal{N}_1(t)^{\frac{2}{s}}(1+t)^{\frac{1}{2}-\frac{\alpha}{2}}\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{s\alpha}}(\tau^{-1+\frac{1}{s}-\frac{2}{s}(\frac{1}{2}-\frac{\alpha}{2})}\psi(\tau)^{2-\frac{2}{s}}\\
&+\tau^{-\frac{1}{2}+\frac{1}{s}-\frac{2}{s}(\frac{1}{2}-\frac{\alpha}{2})}\psi(\tau)^{1-\frac{2}{s}})\,d\tau+C\mathcal{N}_1(t)(1+t)^{\frac{1}{2}-\frac{\alpha}{2}}\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{2}}\tau^{-1+\frac{\alpha}{2}}\psi(\tau)\,d\tau\\
&+C\mathcal{N}_1(t)^{\frac{2}{s}}(1+t)^{\frac{1}{2}-\frac{\alpha}{2}}\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{s\gamma}}(\tau^{-\frac{1}{2}+\frac{1}{s}-\frac{2}{s}(\frac{1}{2}-\frac{\alpha}{2})}\psi(\tau)^{1-\frac{2}{s}}\\
&+\tau^{-1+\frac{1}{s}-\frac{2}{s}(\frac{1}{2}-\frac{\alpha}{2})}\psi(\tau)^{2-\frac{2}{s}})\,d\tau\\
=&\sum_{i=1}^{8}M_i.
\end{aligned} \tag{4.20}$$

One can easily check that $M_1,\ M_2\le C$. For $0<\alpha<\frac{1}{2}$, we can estimate M_3 as

$$\begin{aligned}
M_3=&C(1+t)^{\frac{1}{2}-\frac{\alpha}{2}}\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{\alpha}}(\|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_2^2+1)\,d\tau\\
\le& C(\|\mathbf{u}_0,\ \mathbf{b}_0,\ \mathrm{w}_0\|_2^2+1)(1+t)^{\frac{3}{2}-\frac{\alpha}{2}-\frac{1}{\alpha}}\to 0\quad\text{as}\quad t\to\infty.
\end{aligned} \tag{4.21}$$

According to (4.2) and (4.3), applying Hölder's inequality, M_4 can be bounded as

$$
\begin{aligned}
M_4 &= C(1+t)^{\frac{1}{2}-\frac{\alpha}{2}} \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1}{2}} \|\mathbf{b}\|_2^{\alpha} \|\nabla \mathbf{b}\|_2^{1-\alpha} \|\Lambda^{\alpha}\mathbf{u}\|_2 \, d\tau \\
&\leq C(\|\mathbf{u}_0,\ \mathbf{b}_0,\ \mathrm{w}_0\|_2^2)^{\frac{\alpha}{2}} \left(\int_0^{\frac{t}{2}} \|\nabla \mathbf{b}\|_2^2 \, d\tau\right)^{\frac{1-\alpha}{2}} \left(\int_0^{\frac{t}{2}} \|\Lambda^{\alpha}\mathbf{u}\|_2^2 \, d\tau\right)^{\frac{1}{2}} \leq C. \qquad (4.22)
\end{aligned}
$$

Clearly, $\lim\limits_{t\to\infty} M_5 = 0$. Due to $\dfrac{1}{s\alpha} > \dfrac{1}{2}$ and $\lim\limits_{t\to\infty} \psi(t) = 0$, one has

$$
\begin{aligned}
\lim_{t\to\infty} M_6 &= \lim_{t\to\infty} \{C\mathcal{N}_1(t)^{\frac{2}{s}} (1+t)^{\frac{1}{2}-\frac{\alpha}{2}} \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{s\alpha}} (\tau^{-1+\frac{1}{s}-\frac{2}{s}(\frac{1}{2}-\frac{\alpha}{2})} \psi(\tau)^{2-\frac{2}{s}} \\
&\quad + \tau^{-\frac{1}{2}+\frac{1}{s}-\frac{2}{s}(\frac{1}{2}-\frac{\alpha}{2})} \psi(\tau)^{1-\frac{2}{s}}) \, d\tau\} \\
&= 0. \qquad (4.23)
\end{aligned}
$$

Similarly, due to $\dfrac{1}{s\gamma} > \dfrac{1}{2}$ and $\lim\limits_{t\to\infty} \psi(t) = 0$, one can easily check that

$$
\lim_{t\to\infty} M_7 = \lim_{t\to\infty} M_8 = 0. \qquad (4.24)
$$

Combining (4.21) $\sim$ (4.24) with (4.20), we find that

$$
\mathcal{N}_1(t) \leq C + C\mathcal{N}_1(t)^{\frac{2}{s}} + \frac{1}{2}\mathcal{N}_1(t) \leq C + \frac{1}{2}\mathcal{N}_1(t),
$$

which implies $\mathcal{N}_1(t) \leq C$. In the second step we will use the estimate (4.19) to show the higher-order decay, for any $t \geq 0$,

$$
\|\mathbf{u}(t)\|_2 + \|\mathbf{b}(t)\|_2 + \|\mathrm{w}(t)\|_2 \leq C(1+t)^{-(\frac{1}{2}-\frac{\alpha}{2})(1+\alpha)}, \qquad (4.25)
$$

whose proof is similar as (4.19). For simplicity, we denote $\rho = (\dfrac{1}{2} - \dfrac{\alpha}{2})(1+\alpha)$ and define

$$
\mathcal{N}_2(t) = \sup_{0\leq\tau\leq t} \{(1+\tau)^{\rho} \|\mathbf{u}(\tau)\|_2 + \|\mathbf{b}(\tau)\|_2 + \|\mathrm{w}(\tau)\|_2\}.
$$

Applying similar methods as (4.20), we have

$$
\begin{aligned}
\mathcal{N}_1(t) \leq\, & C(1+t)^{\rho-\frac{3}{4\gamma}} + C(1+t)^{-\frac{1}{2}+\rho} + C(1+t)^{\rho} \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1}{\alpha}} (\|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_2^2 + 1) \, d\tau \\
& + C(1+t)^{\rho} \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1}{2}} \|\mathbf{b}\|_2^{\alpha} \|\nabla \mathbf{b}\|_2^{1-\alpha} \|\Lambda^{\alpha}\mathbf{u}\|_2 \, d\tau \\
& + C(1+t)^{\rho} e^{-t} \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1}{\gamma}} (1 + \|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_2^2) \, d\tau \\
& + C\mathcal{N}_2(t)^{\frac{2}{s}} (1+t)^{\rho} \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{s\alpha}} (\tau^{-1+\frac{1}{s}-\frac{2}{s}\rho} \psi(\tau)^{2-\frac{2}{s}} + \tau^{-\frac{1}{2}+\frac{1}{s}-\frac{2}{s}\rho} \psi(\tau)^{1-\frac{2}{s}}) \, d\tau \\
& + C\mathcal{N}_2(t)(1+t)^{\rho} \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} \tau^{-1+\frac{\alpha}{2}} \psi(\tau) \, d\tau
\end{aligned}
$$

$$+C\mathcal{N}_2(t)^{\frac{2}{s}}(1+t)^\rho\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{1}{s\gamma}}(\tau^{-\frac{1}{2}+\frac{1}{s}-\frac{2}{s}\rho}\psi(\tau)^{1-\frac{2}{s}}+\tau^{-1+\frac{1}{s}-\frac{2}{s}\rho}\psi(\tau)^{2-\frac{2}{s}})\,d\tau. \tag{4.26}$$

Combining with the estimate (4.19), most terms on the right-hand side of (4.26) can be bounded similarly as done previously. We only need to estimate the following term

$$(1+t)^\rho\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{2}}\|\mathbf{b}\|_2^\alpha\|\nabla\mathbf{b}\|_2^{1-\alpha}\|\Lambda^\alpha\mathbf{u}\|_2\,d\tau.$$

Using Hölder inequality, we can bound it as

$$\begin{aligned}
&(1+t)^\rho\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{2}}\|\mathbf{b}\|_2^\alpha\|\nabla\mathbf{b}\|_2^{1-\alpha}\|\Lambda^\alpha\mathbf{u}\|_2\,d\tau\\
\le& C(1+t)^\rho(1+t)^{-\frac{1}{2}}\left(\int_0^{\frac{t}{2}}\|\mathbf{b}\|_2^2\,d\tau\right)^{\frac{\alpha}{2}}\left(\int_0^{\frac{t}{2}}\|\nabla\mathbf{b}\|_2^2\,d\tau\right)^{\frac{1-\alpha}{2}}\left(\int_0^{\frac{t}{2}}\|\Lambda^\alpha\mathbf{u}\|_2^2\,d\tau\right)^{\frac{1}{2}}\\
\le& C(1+t)^{-\frac{\alpha^2}{2}}\left(\int_0^{\frac{t}{2}}(1+\tau)^{-1+\alpha}\,d\tau\right)^{\frac{\alpha}{2}}\left(\int_0^{\frac{t}{2}}\|\nabla\mathbf{b}\|_2^2\,d\tau\right)^{\frac{1-\alpha}{2}}\left(\int_0^{\frac{t}{2}}\|\Lambda^\alpha\mathbf{u}\|_2^2\,d\tau\right)^{\frac{1}{2}}\\
\le& C(1+t)^{-\frac{\alpha^2}{2}}(1+t)^{\frac{\alpha^2}{2}}\left(\int_0^{\frac{t}{2}}\|\nabla\mathbf{b}\|_2^2\,d\tau\right)^{\frac{1-\alpha}{2}}\left(\int_0^{\frac{t}{2}}\|\Lambda^\alpha\mathbf{u}\|_2^2\,d\tau\right)^{\frac{1}{2}}\le C.
\end{aligned}$$

Combining these estimates with (4.26), we can infer that $\mathcal{N}_2(t)\le C$. Doing this process again and again, for any natural number N, we can show that

$$\|\mathbf{u}(t)\|_2+\|\mathbf{b}(t)\|_2+\|\mathrm{w}(t)\|_2\le C(1+t)^{-(\frac{1}{2}-\frac{\alpha}{2})(1+\alpha+\alpha^2+\cdots+\alpha^N)}.$$

Due to

$$\lim_{N\to\infty}(1+\alpha+\alpha^2+\cdots+\alpha^N)=\frac{1}{1-\alpha}.$$

Therefore, for given $\varepsilon>0$, we have

$$\|\mathbf{u}(t)\|_2+\|\mathbf{b}(t)\|_2+\|\mathrm{w}(t)\|_2\le C(1+t)^{-\frac{1}{2}+\varepsilon}. \tag{4.27}$$

Next, we will show the improved decay rate for $\|\nabla\mathbf{b}\|_2$, which can be handled similarly as [16]. For readers' convenience, we give the details. Invoking the integral form of b_1, we have

$$\begin{aligned}
\|\partial_2 b_1\|_2\le&\|\partial_2G_1(x_2,\,t)*b_{01}\|_2+\int_0^t\|\partial_2G_1(x_2,\,t-\tau)*(\mathbf{b}\cdot\nabla u_1-\mathbf{u}\cdot\nabla b_1)(\tau)\|_2\,d\tau\\
\le&\|\partial_2G_1(x_2,\,t)*b_{01}\|_2\\
&+\int_0^{\frac{t}{2}}\|\partial_{22}G_1(x_2,\,t-\tau)*\|(b_2u_1-u_2b_1)(x_1,\,x_2,\,\tau)\|_{L^2_{x_1}}\|_{L^2_{x_2}}\,d\tau\\
&+\int_{\frac{t}{2}}^t\|\partial_{22}|\partial_2|^{-\frac{1}{4}}G_1(x_2,\,t-\tau)*\||\partial_2|^{\frac{1}{4}}(b_2u_1-u_2b_1)(x_1,\,x_2,\,\tau)\|_{L^2_{x_1}}\|_{L^2_{x_2}}\,d\tau
\end{aligned}$$

$$=A_1+A_2+A_3.$$

Using the Plancherel's theorem and (1.5), one has

$$A_1=\|\partial_2 G_1(x_2,\ t)*b_{01}\|_2\le C(1+t)^{-1}.$$

Define

$$\varphi(t)=(1+t)^{\frac{1}{2}}(\|\nabla\mathbf{u}(t)\|_2+\|\nabla\mathbf{b}(t)\|_2+\|\nabla\mathrm{w}(t)\|_2).$$

Thanks to Lemma 2.5, for any small $\varepsilon>0$, we can estimate A_2 as

$$\begin{aligned}A_2&\le C\int_0^{\frac{t}{2}}(t-\tau)^{-1}\|b_2u_1-u_2b_1\|_2\,d\tau\le\int_0^{\frac{t}{2}}(t-\tau)^{-1}\|\mathbf{u}\|_2^{\frac12}\|\nabla\mathbf{u}\|_2^{\frac12}\|\mathbf{b}\|_2^{\frac12}\|\nabla\mathbf{b}\|_2^{\frac12}\,d\tau\\&\le C\int_0^{\frac{t}{2}}(t-\tau)^{-1}(1+\tau)^{-\frac34+\frac{\varepsilon}{2}}\varphi(\tau)^{\frac12}\|\nabla\mathbf{b}\|_2^{\frac12}\,d\tau.\end{aligned}$$

Similarly, employing Hölder's inequality, we have

$$\begin{aligned}A_3&=\int_{\frac{t}{2}}^{t}\|\partial_{22}|\partial_2|^{-\frac14}G_1(x_2,\ t-\tau)*\||\partial_2|^{\frac14}(b_2u_1-u_2b_1)(x_1,\ x_2,\ \tau)\|_{L^2_{x_1}}\|_{L^2_{x_2}}\,d\tau\\&\le C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac78}\||\partial_2|^{\frac14}(b_2u_1-u_2b_1)\|_2\,d\tau\\&\le C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac78}(\||\partial_2|^{\frac14}\mathbf{u}\|_4\|\mathbf{b}\|_4+\||\partial_2|^{\frac14}\mathbf{b}\|_4\|\mathbf{u}\|_4)\,d\tau\\&\le C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac78}(\|\mathbf{u}\|_2^{\frac14}\|\nabla\mathbf{u}\|_2^{\frac34}\|\mathbf{b}\|_2^{\frac12}\|\nabla\mathbf{b}\|_2^{\frac12}+\|\mathbf{b}\|_2^{\frac14}\|\nabla\mathbf{b}\|_2^{\frac34}\|\mathbf{u}\|_2^{\frac12}\|\nabla\mathbf{u}\|_2^{\frac12})\,d\tau\\&\le C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac78}((1+\tau)^{-\frac34+\frac{3\varepsilon}{4}}\varphi(\tau)^{\frac34}\|\nabla\mathbf{b}\|_2^{\frac12}+(1+\tau)^{-\frac58+\frac{3\varepsilon}{4}}\varphi(\tau)^{\frac12}\|\nabla\mathbf{b}\|_2^{\frac34})\,d\tau.\end{aligned}$$

Combining the estimates for $A_1,\ A_2,\ A_3$, we obtain

$$\begin{aligned}\|\partial_2b_1\|_2\le&\ C(1+t)^{-1}+C\int_0^{\frac{t}{2}}(t-\tau)^{-1}(1+\tau)^{-\frac34+\frac{\varepsilon}{2}}\varphi(\tau)^{\frac12}\|\nabla\mathbf{b}\|_2^{\frac12}\,d\tau\\&+C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac78}((1+\tau)^{-\frac34+\frac{3\varepsilon}{4}}\varphi(\tau)^{\frac34}\|\nabla\mathbf{b}\|_2^{\frac12}+(1+\tau)^{-\frac58+\frac{3\varepsilon}{4}}\varphi(\tau)^{\frac12}\|\nabla\mathbf{b}\|_2^{\frac34})\,d\tau.\end{aligned}\tag{4.28}$$

Using similar methods as those used for b_1 to the integral form of b_2, one can easily check that $\|\partial_1b_2\|_2$ has the same bound of (4.28). Due to the divergence-free condition $\nabla\cdot\mathbf{b}=0$, we have

$$\|\nabla\mathbf{b}\|_2\le C(\|\partial_2b_1\|_2+\|\partial_1b_2\|_2),\tag{4.29}$$

Therefore, define

$$\mathcal{N}_3(t)=\sup_{0\le\tau\le t}\{(1+\tau)\|\nabla\mathbf{b}(\tau)\|_2\},$$

which satisfies

$$
\begin{aligned}
\mathcal{N}_3(t)\le & C+C\mathcal{N}_3(t)^{\frac{1}{2}}(1+t)\int_0^{\frac{t}{2}}(t-\tau)^{-1}(1+\tau)^{-\frac{5}{4}+\frac{\varepsilon}{2}}\varphi(\tau)^{\frac{1}{2}}\,d\tau\\
&+C\mathcal{N}_3(t)^{\frac{1}{2}}(1+t)\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{7}{8}}((1+\tau)^{-\frac{5}{4}+\frac{3\varepsilon}{4}}\varphi(\tau)^{\frac{3}{4}}\,d\tau\\
&+C\mathcal{N}_3(t)^{\frac{3}{4}}(1+t)\int_{\frac{t}{2}}^t(1+\tau)^{-\frac{11}{8}+\frac{3\varepsilon}{4}}\varphi(\tau)^{\frac{1}{2}}\,d\tau\\
\le & C+C\mathcal{N}_3(t)^{\frac{1}{2}}+C\mathcal{N}_3(t)^{\frac{3}{4}}\le C+\frac{1}{2}\mathcal{N}_3(t),
\end{aligned}
$$

which implies,

$$
\mathcal{N}_3(t)\le C\quad\text{ or }\quad\|\nabla\mathbf{b}\|_2\le C(1+t)^{-1}.
$$

Furthermore, according to (4.17), we obtain

$$
\begin{aligned}
&\|\mathbf{u}(t)\|_2+\|\mathbf{b}(t)\|_2+\|\mathrm{w}(t)\|_2\\
\le & C(1+t)^{-\frac{3}{4\gamma}}+C(1+t)^{-\frac{1}{2}}+C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{\alpha}}(\|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_2^2+1)\,d\tau\\
&+C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{2}}\|\mathbf{b}\|_2^{\alpha}\|\nabla\mathbf{b}\|_2^{1-\alpha}\|\Lambda^\alpha\mathbf{u}\|_2\,d\tau+Ce^{-t}\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{\gamma}}(1+\|\mathbf{u},\ \mathbf{b},\ \mathrm{w}\|_2^2)\,d\tau\\
&+C\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{1}{s\alpha}}(\|\mathbf{b}\|_2^{\frac{2}{s}}\|\nabla\mathbf{b}\|_2^{2-\frac{2}{s}}+\|\mathbf{u}\|_2^{\frac{2}{s}}\|\nabla\mathbf{u}\|_2^{2-\frac{2}{s}}+\|\mathrm{w}\|_2^{\frac{2}{s}}\|\nabla\mathrm{w}\|_2^{1-\frac{2}{s}})\,d\tau\\
&+C\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{1}{2}}\|\mathbf{b}\|_2^{\alpha}\|\nabla\mathbf{b}\|_2^{1-\alpha}\|\mathbf{u}\|_2^{1-\alpha}\|\nabla\mathbf{u}\|_2^{\alpha}\,d\tau\\
&+C\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{1}{s\gamma}}\|\mathbf{u}\|_2^{\frac{2}{s}}\|\nabla\mathbf{u}\|_2^{1-\frac{2}{s}}(1+\|\nabla\mathrm{w}\|_2)\,d\tau\\
\le & C(1+t)^{-\frac{3}{4\gamma}}+C(1+t)^{-\frac{1}{2}}+C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{\alpha}}((1+\tau)^{-1+2\varepsilon}+1)\,d\tau\\
&+C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{3}{2}+\frac{\alpha}{2}+\varepsilon}\,d\tau+Ce^{-t}\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{\gamma}}(1+(1+\tau)^{-1+2\varepsilon})\,d\tau\\
&+C\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{1}{s\alpha}}(1+\tau)^{-\frac{1}{2}+\frac{2\varepsilon}{s}}\,d\tau+C\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{1}{s\gamma}}(1+\tau)^{-\frac{1}{2}+\frac{2\varepsilon}{s}}\,d\tau\\
\le & C(1+t)^{-\frac{1}{2}}.
\end{aligned}\tag{4.30}
$$

This completes the proof of the Proposition 4.2.

Further, one can use a similar method as Theorem 3.1 in [11] to obtain $\|w\|_2\le C(1+t)^{-\frac{3}{2}}$.

Next, we will improve the decay rate for $\|\nabla w\|_2$ through three steps.

Firstly, multiplying the equations $(1.3)_{1\sim4}$ by $(1+t)^2\mathbf{u}$, $(1+t)^2b_1$, $(1+t)^2b_2$ and $(1+t)^2\mathrm{w}$, respectively and taking the L^2 inner product, integrating by parts, using the divergence-free conditions $\nabla\cdot\mathbf{u}=0$ and $\nabla\cdot\mathbf{b}=0$, adding the resulting equations together and integrating from 0 to t, yield that

$$
\int_0^t(1+\tau)^2\|\Lambda^\alpha\mathbf{u},\ \nabla\mathbf{b},\ \Lambda^\gamma\mathrm{w}\|_2^2d\tau\le C(1+t)^{-1}.\tag{4.31}
$$

Secondly, multiplying the equations $(3.4)_{1\sim 3}$ by $(1+t)^2\Omega$, $(1+t)^2 j$ and $(1+t)^2\Lambda^{2(2\gamma-1)}\mathrm{w}$, respectively and taking the L^2 inner product, integrating by parts, adding the resulting equations together and integrating from 0 to t, combining with (4.31), one has

$$\int_0^t (1+\tau)^2\|\Lambda^\alpha\Omega,\ \Delta\mathbf{b},\ \Lambda^{3\gamma-1}\mathrm{w}\|_2^2\,d\tau \le C(1+t)^{-1}, \tag{4.32}$$

where we have used the fact $\|\Omega\|_2^2 \le C\|\Lambda^\alpha\mathbf{u},\Lambda^\alpha\Omega\|_2^2$.

In the end, multiplying the equations (3.10) by $(1+t)^3\nabla\mathrm{w}$, respectively and taking the L^2 inner product, adding the resulting equations together and integrating from 0 to t, combining with (4.31),(4.32), one has

$$(1+t)^3\|\nabla\mathrm{w}\|_2^2 + \int_0^t (1+\tau)^3\|\Lambda^{1+\gamma}\mathrm{w},\nabla\mathrm{w}\|_2^2\,d\tau \le C. \tag{4.33}$$

Furthermore, using Proposition 4.1 and Proposition 4.2, we complete the proof of Theorem 1.2. □

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