COMMUN. MATH. SCI. © 2022 International Press
Vol. 20, No. 5, pp. 1243–1278

LACK OF EXACT CONTROLLABILITY OF A HIGHER-ORDER BBM-SYSTEM*

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Abstract. The two-way propagation of a certain class of long-crested water waves is governed approximately by systems of equations of Boussinesq type. These equations have been put forward in various forms by many authors and their higher-order generalizations arise when modelling the propagation of waves on large lakes, oceans and in other contexts. Considered here is a class of such systems which couple two higher-order Benjamin-Bona-Mahony type equations. Our aim is to investigate the controllability properties of the linearized model posed on a bounded interval. More precisely, we study whether the solutions can be driven to a given state at a given final time by means of controls acting on the right endpoint of the interval. We show that the model is approximately controllable but not spectrally controllable. This means that any state can be steered arbitrarily close to another state, but no finite linear combination of eigenfunctions, other than zero, can be steered to zero. Our proofs rely strongly on a careful spectral analysis of the operator associated with the state equations.

Keywords. Higher order Boussinesq system; controllability; Fourier expansion; nonharmonic analysis.

AMS subject classifications. 35Q53; 93B05; 93C20; 46B15.

1. Introduction

The field of dispersive equations has received increasing attention since the pioneering works of Stokes, Boussinesq and Korteweg and de Vries in the nineteenth century. It pertains to a modern line of research which is important both scientifically and for potential applications. On the one hand, the mathematical theoretical research of dispersive equations is important for applied sciences since it has provided solid foundations for the verification and applicability of these models. On the other hand, this theoretical research has proved to be very valuable for mathematics itself. Such equations have presented very difficult and interesting challenges, motivating the development of many new ideas and techniques within mathematical analysis.

Starting in the latter half of the 1960s, there have been many advances in the study of the water wave phenomena initially observed by Boussinesq et al. and numerous other applications have been found since then. For instance, the two-way propagation of a certain class of long-crested water waves is governed approximately by systems of Boussinesq type equations. First introduced by Boussinesq in the 1870's, these equations have been put forward in various forms by many authors and, in recent years, the following family of Boussinesq systems was formulated and analyzed by Bona, Chen and Saut [5,6]:

$$\begin{aligned}&\eta_t+\omega_x+a\omega_{xxx}-b\eta_{txx}+a_1\omega_{xxxxx}+b_1\eta_{txxxx}=-(\eta\omega)_x+b(\eta\omega)_{xxx}-\alpha(\eta\omega_{xx})_x,\\&\omega_t+\eta_x+c\eta_{xxx}-d\omega_{txx}+c_1\eta_{xxxxx}+d_1\omega_{txxxx}\\&=-\omega\omega_x-c(\omega\omega_x)_{xx}-(\eta\eta_{xx})_x+\beta\omega_x\omega_{xx}+\rho\omega\omega_{xxx}.\end{aligned}\tag{1.1}$$

Here, the dependent variables $\eta=\eta(x,t)$ and $w=\omega(x,t)$ are real-valued functions of the variables x and t and subscripts indicate partial differentiation. The parameters

*Received: January 10, 2021; Accepted (in revised form): November 13, 2021. Communicated by Alberto Bressan.

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a,b,c,d,a_1,c_1,b_1,d_1 are required to fulfill the relations

$$\begin{aligned} &a+b=\frac{1}{2}(\theta^2-\frac{1}{3}), \quad c+d=\frac{1}{2}(\theta^2-\frac{1}{2}),\\ &a_1-b_1=-\frac{1}{2}(\theta^2-\frac{1}{3})b+\frac{5}{24}(\theta^2-\frac{1}{5})^2,\\ &c_1-d_1=\frac{1}{2}(1-\theta^2)c+\frac{5}{24}(1-\theta^2)(\theta^2-\frac{1}{5}),\\ &\alpha=a+b-\frac{1}{3}, \beta=c+d-1, \rho=c+d, \end{aligned} \tag{1.2}$$

where $\theta\in[0,1]$. Conditions (1.2) come from the physics of the problem and we tacitly assume them to hold throughout the entire paper. Depending on the problem under study, additional restrictions on the sign of these parameters will be imposed later on.

The original system was derived by Boussinesq to describe the two-way propagation of small-amplitude, long wavelength, gravity waves on the surface of water in a canal, but these systems also arise when modeling the propagation of long-crested waves on large lakes or the ocean and in other contexts. The variable, x, is proportional to the distance in the direction of propagation while t is proportional to elapsed time. The quantity $\eta(t,x)+h_0$ corresponds to the total depth of the liquid at the point x and at time t, where h_0 is the undisturbed water depth. The variable $\omega(t,x)$ represents the horizontal velocity at the point $(x,y)=(x,\theta h_0)$, at time t, where y is the vertical coordinate, with $y=0$ corresponding to the channel bottom or sea bed. Thus, ω is the horizontal velocity field at the height θh_0, where θ is a fixed constant in the interval $[0,1]$.

Notice that, when the parameters given in (1.2) are such that $a=a_1=c=c_1=0$, the resulting system couples two higher order Benjamin-Bona-Mahony (BBM) type equations. If $b=b_1=d=d_1=0$, we have a coupled system of two higher order Korteweg-de Vries (KdV) type equations.

1.1. Setting the problem. Despite the success in studying dispersive models, the mathematical theory has been concerned with either the pure initial value problem posed on the entire real line or the periodic-initial value problem posed on the one-dimensional torus. A large body of literature has been concerned with the questions of existence, uniqueness and continuous dependence of solutions corresponding to initial data. The study of initial-boundary value problems with nonhomogeneous boundary conditions has not progressed to the same extent.

In this paper, the goal is to advance the study of the initial-boundary value problems exploring the dynamics of dispersive equations using mathematical analysis from the controllability point of view. Consideration is given to an initial-boundary value problem associated to the linearized Boussinesq system (1.1) when the parameters given in (1.2) are such that $a_1=c_1=0$. Our attention, in particular, is given to the following distributed control system:

$$\begin{cases} \eta_t+\omega_x+a\omega_{xxx}-b\eta_{txx}+b_1\eta_{txxxx}=0 & \text{for } x\in(0,L),\ t>0,\\ \omega_t+\eta_x+c\eta_{xxx}-d\omega_{txx}+d_1\omega_{txxxx}=0 & \text{for } x\in(0,L), t>0,\\ \eta(t,0)=0,\quad \eta(t,L)=f_1(t) & \text{for } \ t\geq 0,\\ \omega(t,0)=0,\quad \omega(t,L)=g_1(t) & \text{for } \ t\geq 0,\\ \eta_x(t,0)=0,\quad \eta_x(t,L)=f_2(t) & \text{for } \ t\geq 0,\\ \omega_x(t,0)=0,\quad \omega_x(t,L)=g_2(t) & \text{for } \ t\geq 0,\\ \eta(0,x)=\eta^0(x);\quad \omega(0,x)=\omega^0(x) & \text{for } x\in(0,L). \end{cases} \tag{1.3}$$

In (1.3), the external forcing terms f_i and g_i, $i=1,2$, are considered as control inputs. The purpose is to see whether one can force the solutions of the system to have certain desired properties by choosing appropriate control inputs acting at one end of the interval. More precisely, we are mainly concerned with the following problems which are fundamental in control theory:

Given $T>0$, initial states (η^0,ω^0) and terminal states (η^1,ω^1) in a certain space, can one find appropriate control inputs f_i and g_i, $i=1,2$, so that the system (1.3) admits a solution (η,ω) which satisfies $(\eta(0,\cdot),\omega(0,\cdot))=(\eta^0,\omega^0)$ and $(\eta(T,\cdot),\omega(T,\cdot))=(\eta^1,\omega^1)$?

If one can always find a control input to guide the system described by (1.3) from any given initial state to any given terminal state, then the system is said to be **exactly controllable**. If the set of all reachable states contains all the eigenfunctions associated to the state operator ($\mathcal{A}:[H_0^2(0,2\pi)]^2\to[H_0^2(0,2\pi)]^2$) the system is said to be **spectrally controllable**.

Given $T>0$, $\varepsilon>0$, initial states (η^0,ω^0) and terminal states (η^1,ω^1) in a certain space, can one find appropriate control inputs f_i and g_i, $i=1,2$, so that the system (1.3) admits a solution (η,ω) which satisfies $||(\eta(T,\cdot),\omega(T,\cdot))-(\eta^1,\omega^1)||_H<\varepsilon$, for a certain space H?

This means that the set of reachable states is dense in H and, in this case, the system is said to be **approximately controllable**.

Our analysis does not depend on formulas (1.2) nor on other particular relations between the coefficients. However, in order to provide the tools needed to deal with the problem, some sign conditions have to be imposed. More precisely, we shall be mainly concerned with the case

$$\begin{cases} b_1>0, b\geq 0, d_1>0, d\geq 0\\ a=c>0,\ a_1=c_1=0.\end{cases} \tag{1.4}$$

Assumptions (1.4) allow us to prove well-posedness and controllability results in some well chosen Sobolev spaces $H^s(0,L)$ and $H^s(0,T)$, respectively.

1.2. Main results. Observe that exact controllability is an essentially stronger notion than approximate controllability. In other words, exact controllability always implies approximate controllability. The converse statement is generally false. In what concerns system (1.3), our results can be summarized as follows:

- The approximate controllability holds for any $T>0$. In more details, we prove that there exist control inputs $f_i,g_i\in H^1(0,T)$, $i=1,2$, such that the set of reachable states is dense in $[L^2(0,L)]^2$, for any $(\eta^0,\omega^0)\in[H^{-2}(0,L)]^2$ and $T>0$.
- On the other hand, we give a negative result for the first problem introduced above.
- System (1.3) is not spectrally controllable if $(\eta^0,\omega^0)\in[H_0^2(0,L)]^2$. This means that no finite linear nontrivial combination of eigenvectors of the operator associated with the state equations can be driven to zero in finite time by using controls $f_i,g_i\in H^1(0,T)$, $i=1,2$.

REMARK 1.1. The following remarks are in order.

(i) When $(\eta^0,\omega^0)\in[H^{-2}(0,L)]^2$, the solution of (1.3) has to be understood in a weak sense. For instance, it can be defined by transposition. With this approach, we have to impose that $f_i,g_i\in H^1(0,T)$, $i=1,2$ in order to obtain a well-posedness result.

(ii) Throughout the work, it will become clear that the lack of exact controllability of the model comes from the existence of a limit point in the spectrum of the operator associated with the state equations, a phenomenon already noticed in [15] for the single linear BBM equation.

By means of a series expansion of the solution in terms of the eigenvectors of the state operator, the approximate controllability is reduced to a unique continuation problem of the eigenvectors. In what concerns the lack of exact controllability, it is addressed through a spectral problem which is solved combining Paley-Wiener theorem and the asymptotic behavior of the eigenvalues. Such an approach requires a careful spectral analysis of the operator associated to the state equation. Indeed, it provides important developments to justify the use of eigenvector expansions for the solutions, as well as, the asymptotic behavior of the eigenvalues. However, due to the structure of the system, the eigenvalues can not be computed explicitly. To overcome this difficulty, we prove that they are *close* to the eigenvalues of a well chosen differential operator. This is done by using less common two-dimensional versions of the shooting method and Rouché's theorem. Our approach was inspired by the techniques presented in [2] and [17]. In [17], the same strategy was successfully used to study the stabilization of a linear Boussinesq system of BBM-BBM type ($a=a_1=c=c_1=b_1=d_1=0$) when a localized damping term acts on one equation only. By considering homogeneous Dirichlet boundary conditions, the authors prove that the energy associated to the model converges to zero as time goes to infinity. In the conservative case, i.e., in the absence of the damping term, the results obtained in [17] were properly adapted in [2] to study the controllability problems we address here. This approach does not apply directly in our case, since we are dealing with a higher order Boussinesq system. Therefore, further developments are required.

Before closing this section we emphasize that the problems we address here remain open for the corresponding nonlinear models, including for the single BBM equation. To our knowledge, the only result on the subject was obtained in [20] for the BBM equation on the torus $\mathbb{T}=\mathbb{R}/(2\pi\mathbb{Z})$. The authors show that, when an internal control acting on a moving interval is applied in the BBM equation, it is locally exactly controllable in $H^s(\mathbb{T})$, for any $s>0$, and globally exactly controllable in $H^s(\mathbb{T})$, for any $s>1$, in a sufficiently large time depending on the H^s-norms of the initial and terminal states. More comments and open problems will be given in Section 5.

1.3. State of the art. The study of the controllability properties for Boussinesq systems was initiated in [16] by considering the following *abdc* Boussinesq system, also derived in [5,6]:

$$\begin{cases} \eta_t+\omega_x+a\omega_{xxx}-b\eta_{txx}=-(\eta\omega)_x, \\ \omega_t+\eta_x+c\eta_{xxx}-d\omega_{txx}=-\omega\omega_x. \end{cases} \tag{1.5}$$

The constants in (1.5) obey the relations

$$a+b=\frac{1}{2}(\theta^2-\frac{1}{3}), \quad c+d=\frac{1}{2}(\theta^2-\frac{1}{2})\geq 0, \text{ where } \theta\in[0,1]. \tag{1.6}$$

The work [16] deals with the internal controllability and stabilization of (1.5) on the torus. First, the space of the controllable data for the associated linear system is established for each possible value of the four parameters given in (1.6). Then, when $b,d>0$ and $a,c<0$, the local exact controllability of the nonlinear system is shown to hold. As an application of the established exact controllability results, some simple feedback controls are constructed for particular choices of the parameters a,b,c and d, such that the resulting closed-loop systems are exponentially stable. Later on, in [19], the authors investigated the boundary stabilization of the Boussinesq system (1.5) of KdV-KdV type ($b=d=0$) posed on a bounded interval. More precisely, they design a two-parameter family of feedback laws for which the solutions issuing from small data are globally defined and exponentially decreasing in the energy space. More recently, in [8], the exact boundary controllability of the linear Boussinesq system (1.5) of KdV-KdV type was studied. It was discovered that whether the associated linear system is exactly controllable or not depends on the length of the spatial domain. The extension of the exact controllability for the Boussinesq system (1.5) is derived in the energy space in the case of a control of Neumann type. It is obtained by incorporating a boundary feedback in the control in order to ensure the so-called Kato smoothing effect. In addition, proceeding as in [19], a local exponential stability result was also derived.

Concerning the Boussinesq system (1.5) of BBM-BBM type ($a=c=0$), the controllability problems addressed here were studied in [2] for the linearized model. As pointed out above, the results were obtained by making use of the analysis developed in [17] to study the stabilization of the energy associated to the model when a localized damping term acts on one equation only. In the same spirit, the work [4] proposes several dissipation mechanisms leading to systems for which one has both the global existence of solutions and a nonincreasing energy. Following the analysis developed in [21], the authors prove that all the trajectories are attracted by the origin provided that the unique continuation of weak solutions holds. Finally, let us mention the work [18] (see also [3]) in which the stability properties of the nonlinear system, posed on a periodic domain, is addressed when generalized damping operators with nonnegative symbols are introduced in each equation. A similar problem was studied in [9] for the model posed on the whole real axis.

As far as we know, the controllability problem for the full system (1.1) has been only addressed in [1] when the model is posed on a periodic domain. General conditions are given to ensure both the well-posedness and the local exact controllability of the nonlinear problem by means of a control localized in the interior of the domain and acting on one equation only. On the contrary, stabilization problems have been studied in some cases. For instance, in [7] the authors investigate the well-posedness and boundary stabilization of a higher order Boussinesq system of KdV type ($b=b_1=d=d_1=0$), posed on a bounded interval. They design a two-parameter family of feedback laws for which the system is locally well-posed and the solutions of the linearized system are exponentially decreasing in time. More recently, a higher order Boussinesq system of BBM-BBM type ($a=a_1=c=c_1=0$) was considered in [3] (see also [18]). The global well-posedness and time decay rates of solutions were studied when the model is posed on a periodic domain and a general class of damping operator acts in each equation. The authors prove that the solutions of the linearized system decay uniformly or not to zero, depending on the parameters of the damping operators. In the uniform decay case, the result is extended for the nonlinear system.

The present contribution proceeds as follows. In the next section, we establish the well-posedness results for system (1.3). In Section 3, we present some basic results on systems of fourth order differential equations associated to system (1.3). This allows to analyze the spectral properties of the corresponding state operator. Section 4 provides proofs for the results stated above. Finally, in Section 5 we present some remarks and open problems.

2. Well-posedness

In this section we show the well-posedness of the homogeneous and non-homogeneous systems associated with (1.3). Throughout this work, the space $[H_0^2(0,L)]^2$ will be endowed with the inner product

$$\left\langle \begin{pmatrix}\eta\\ \omega\end{pmatrix}, \begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\rangle = \int_0^L (\eta\varphi+\omega\psi)dx + \int_0^L (b\eta_x\varphi_x + d\omega_x\psi_x)dx + \int_0^L (b_1\eta_{xx}\varphi_{xx}+d_1\omega_{xx}\psi_{xx})dx. \tag{2.1}$$

2.1. The homogeneous system.

Let us first consider the following homogeneous system

$$\begin{cases} \eta_t+\omega_x+a\omega_{xxx}-b\eta_{txx}+b_1\eta_{txxxx}=0 & \text{for } x\in(0,L),\ t>0,\\ \omega_t+\eta_x+c\eta_{xxx}-d\omega_{txx}+d_1\omega_{txxxx}=0 & \text{for } x\in(0,L), t>0,\\ \eta(t,0)=\eta(t,L)=\omega(t,0)=\omega(t,L)=0 & \text{for } \ t\geq 0,\\ \eta_x(t,0)=\eta_x(t,L)=\omega_x(t,0)=\omega_x(t,L)=0 & \text{for } \ t\geq 0,\\ \eta(0,x)=\eta^0(x);\ \ \omega(0,x)=\omega^0(x) & \text{for } x\in(0,L). \end{cases} \tag{2.2}$$

System (2.2) can be written in the following vectorial form

$$\begin{pmatrix}\eta\\ \omega\end{pmatrix}_t (t)+\mathcal{A}\begin{pmatrix}\eta\\ \omega\end{pmatrix}(t)=\begin{pmatrix}0\\ 0\end{pmatrix}, \quad \begin{pmatrix}\eta\\ \omega\end{pmatrix}(0)=\begin{pmatrix}\eta^0\\ \omega^0\end{pmatrix}, \tag{2.3}$$

where $\mathcal{A}$ is the operator belonging to $\mathcal{L}\left([H_0^2(0,L)]^2\right)$ defined by

$$\mathcal{A}=\begin{pmatrix} 0 & (1-b\partial_x^2+b_1\partial_x^4)^{-1}(\partial_x+a\partial_x^3)\\ (1-d\partial_x^2+d_1\partial_x^4)^{-1}(\partial_x+c\partial_x^3) & 0\end{pmatrix}. \tag{2.4}$$

Recall that, for $\alpha,\beta>0$ the operator $(1-\alpha\partial_x^2+\beta\partial_x^4)^{-1}$ is defined in the following way:

$$(1-\alpha\partial_x^2+\beta\partial_x^4)^{-1}\phi=v \Leftrightarrow \begin{cases} v-\alpha v_{xx}+\beta v_{xxxx}=\phi & \text{in } \ (0,L)\\ \partial_x^r v(0)=\partial_x^r v(L)=0, & r=0,1. \end{cases} \tag{2.5}$$

Then, if $\phi\in L^2(0,L)$, the elliptic equation (2.5) has a unique solution $v\in H^4(0,L)\cap H_0^2(0,L)$, the operator $(1-\alpha\partial_x^2+\beta\partial_x^4)^{-1}$ is a well-defined, compact operator in $L^2(0,L)$.

REMARK 2.1. Due to the regularizing effect of the operators

$$(1-b\partial_x^2+b_1\partial_x^4)^{-1} \quad \text{and} \quad (1-d\partial_x^2+d_1\partial_x^4)^{-1}$$

it follows that $\mathcal{A}$ takes values in $[H^3(0,L)\cap H_0^2(0,L)]^2$ which is compactly embedded in $[H_0^2(0,L)]^2$. Hence $\mathcal{A}$ is compact.

From the classical semigroup theory, we have the following well-posedness result:

THEOREM 2.1. *For any $(\eta^0,\omega^0)\in[H_0^2(0,L)]^2$, system (2.2) has a unique classical solution $(\eta,\omega)\in C(\mathbb{R};[H_0^2(0,L)]^2)$. Moreover, $(\eta,\omega)\in C^\omega(\mathbb{R};[H_0^2(0,L)]^2)$, the class of analytic functions in $t\in\mathbb{R}$ with values in $[H_0^2(0,L)]^2$.*

Proof. We first show that $\mathcal{A}$ is a skew-adjoint operator in $[H_0^2(0,L)]^2$. For any $\varphi_i,\psi_i\in H_0^2\cap H^4(0,L)$, $i=1,2$, and some integrations by parts, we have from (2.1) that

$$\begin{aligned}
\left\langle \mathcal{A}\begin{pmatrix}\varphi_1\\ \psi_1\end{pmatrix},\begin{pmatrix}\varphi_2\\ \psi_2\end{pmatrix}\right\rangle &= \left\langle \begin{pmatrix}(1-b\partial_x^2+b_1\partial_x^4)^{-1}(\partial_x+a\partial_x^3)\psi_1\\ (1-d\partial_x^2+d_1\partial_x^4)^{-1}(\partial_x+c\partial_x^3)\varphi_1\end{pmatrix},\begin{pmatrix}\varphi_2\\ \psi_2\end{pmatrix}\right\rangle\\
&=\int_0^L(\partial_x+a\partial_x^3)\psi_1\varphi_2dx+\int_0^L(\partial_x+c\partial_x^3)\varphi_1\psi_2dx\\
&=-\int_0^L\psi_1(\partial_x+a\partial_x^3)\varphi_2dx-\int_0^L\varphi_1(\partial_x+c\partial_x^3)\psi_2dx\\
&=-\int_0^L\psi_1(1-d\partial_x^2+d_1\partial_x^4)(1-d\partial_x^2+d_1\partial_x^4)^{-1}(\partial_x+a\partial_x^3)\varphi_2dx\\
&\quad-\int_0^L\varphi_1(1-b\partial_x^2+b_1\partial_x^4)(1-b\partial_x^2+b_1\partial_x^4)^{-1}(\partial_x+c\partial_x^3)\psi_2dx\\
&=-\left\langle \begin{pmatrix}\varphi_1\\ \psi_1\end{pmatrix},\begin{pmatrix}(1-b\partial_x^2+b_1\partial_x^4)^{-1}(\partial_x+c\partial_x^3)\psi_2\\ (1-d\partial_x^2+d_1\partial_x^4)^{-1}(\partial_x+a\partial_x^3)\varphi_2\end{pmatrix}\right\rangle\\
&=-\left\langle \begin{pmatrix}\varphi_1\\ \psi_1\end{pmatrix},\mathcal{A}\begin{pmatrix}\varphi_2\\ \psi_2\end{pmatrix}\right\rangle.
\end{aligned}$$

By a density argument, the identity above holds for any $\varphi_i,\psi_i\in H_0^2(0,L)$, $i=1,2$. Then, the Stone theorem guarantees that $\mathcal{A}$ generates a group of isometries $\{S(t)\}_{t\in\mathbb{R}}$ in $[H_0^2(0,L)]^2$, which allows us to obtain the well-posedness result. The second part of the theorem follows from the fact that $\mathcal{A}$ is a compact operator in $[H_0^2(0,L)]^2$ (see, for instance, [Theorem 11.4.1, Chap. XI in [14]]). ∎

2.2. The nonhomogeneous system. In this subsection, attention will be given to the full system (1.3). We begin with the following result:

THEOREM 2.2. *For any $(\eta^0,\omega^0)\in[H_0^2(0,L)]^2$ and $(f_i,g_i)\in[C_0^1(0,\infty)]^2$, $i=1,2$, system (1.3) has a unique classical solution $(\eta,\omega)\in C([0,\infty);[H_0^2(0,L)]^2)$.*

Proof. Let $\varphi_i,\psi_i\in C^\infty([0,L])$, $i=1,2$, be functions, such that

$$\begin{aligned}
&\varphi_1(0)=\psi_1(0)=\varphi_{1x}(0)=\psi_{1x}(0)=\varphi_{1x}(L)=\psi_{1x}(L)=0,\\
&\phi_1(L)=\psi_1(L)=-1
\end{aligned}$$

and

$$\begin{aligned}
&\varphi_2(0)=\psi_2(0)=\varphi_2(L)=\psi_2(L)=\varphi_{2x}(0)=\psi_{2x}(0)=0,\\
&\phi_{2x}(L)=\psi_{2x}(L)=-1.
\end{aligned}$$

For instance,

$$\varphi_1(x)=\psi_1(x)=-\frac{3}{L^2}x^2+\frac{2}{L^3}x^3\ \text{ and }\ \varphi_2(x)=\psi_2(x)=\frac{1}{L}x^2-\frac{1}{L^2}x^3$$

satisfy the conditions above. Then, if we consider the change of functions

$$\begin{pmatrix}z\\ m\end{pmatrix}=\begin{pmatrix}\eta\\ \omega\end{pmatrix}-\begin{pmatrix}u\\ v\end{pmatrix}+\begin{pmatrix}f_1(t)\varphi_1(x)+f_2(t)\varphi_2(x)\\ g_1(t)\psi_1(x)+g_2(t)\psi_2(x)\end{pmatrix},\tag{2.6}$$

where $(u,v)\in C([0,\infty);[H_0^2(0,L)]^2)$ is the solution of the system

$$\begin{cases} u_t+v_x+av_{xxx}-bu_{txx}+b_1u_{txxxx}=0 & \text{for } x\in(0,L),\ t>0,\\ v_t+u_x+cu_{xxx}-dv_{txx}+d_1v_{txxxx}=0 & \text{for } x\in(0,L), t>0,\\ u(t,0)=u(t,L)=v(t,0)=v(t,L)=0 & \text{for } \ t\geq 0,\\ u_x(t,0)=u_x(t,L)=v_x(t,0)=v_x(t,L)=0 & \text{for } \ t\geq 0,\\ u(0,x)=\eta^0(x);\quad v(0,x)=\omega^0(x) & \text{for } x\in(0,L), \end{cases} \tag{2.7}$$

given by Theorem 2.1, the couple (z,m) solves the problem

$$\begin{cases} z_t+m_x+am_{xxx}-bz_{txx}+b_1z_{txxxx}=F & \text{for } x\in(0,L),\ t>0,\\ m_t+z_x+cz_{xxx}-dm_{txx}+d_1m_{txxxx}=G & \text{for } x\in(0,L), t>0,\\ z(t,0)=z(t,L)=m(t,0)=m(t,L)=0 & \text{for } \ t\geq 0,\\ z_x(t,0)=z_x(t,L)=m_x(t,0)=m_x(t,L)=0 & \text{for } \ t\geq 0,\\ z(0,x)=0;\quad m(0,x)=0 & \text{for } x\in(0,L), \end{cases} \tag{2.8}$$

with F and G given by

$$\begin{aligned} \begin{pmatrix} F(t,x)\\ G(t,x)\end{pmatrix} &= \begin{pmatrix} f_1'(t)[\varphi_1(x)-b\varphi_1^{(2)}(x)+b_1\varphi_1^{(4)}(x)]+g_1(t)[\psi_1'(x)+a\psi_1^{(3)}(x)]\\ g_1'(t)[\psi_1(x)-d\psi_1^{(2)}(x)+d_1\psi_1^{(4)}(x)]+f_1(t)[\varphi_1'(x)+a\varphi_1^{(3)}(x)]\end{pmatrix}\\ &\quad+\begin{pmatrix} f_2'(t)[\varphi_2(x)-b\varphi_2^{(2)}(x)+b_1\varphi_2^{(4)}(x)]+g_2(t)[\psi_2'(x)+a\psi_2^{(3)}(x)]\\ g_2'(t)[\psi_2(x)-d\psi_2^{(2)}(x)+d_1\psi_2^{(4)}(x)]+f_2(t)[\varphi_2'(x)+a\varphi_2^{(3)}(x)]\end{pmatrix}\\ &\in L^1(0,T;[L^2(0,L)]^2), \end{aligned}$$

where (i), $i=2,3,4$, denotes the derivative of order i. With the notation introduced in the previous section, system (2.8) can be written as an abstract evolution equation as follows

$$\begin{cases} W_t+\mathcal{A}W=\mathcal{H}\\ W(0)=0, \end{cases}$$

where $W=(z,m)$ and $\mathcal{H}=\mathcal{A}_0(F,G)\in L^1(0,\infty;[H_0^2\cap H^4(0,L)]^2)$, with $\mathcal{A}_0:[L^2(0,L)]^2\longrightarrow[H_0^2\cap H^4(0,L)]^2$ defined by

$$\mathcal{A}_0=\begin{pmatrix} 0 & (1-b\partial_x^2+b_1\partial_x^4)^{-1}\\ (1-d\partial_x^2+d_1\partial_x^4)^{-1} & 0\end{pmatrix}. \tag{2.9}$$

Since $\mathcal{A}$ generates a group of isometries in $[H_0^2(0,L)]^2$, we have that system (2.8) has a unique solution $W=(z,m)\in C([0,\infty);[H_0^2(0,L)]^2)$. Then, returning to (2.6), we conclude the proof. □

Using the previous well-posedness results, we will study the existence of solutions of the system (1.3) in the sense of transposition (see [10, 11]):

Definition 2.1. *Let $(\eta^0,\omega^0)\in[H^{-2}(0,L)]^2$ and $(f_i,g_i)\in[H^1(0,T)]^2$, $i=1,2$. A solution of system (1.3) is a couple $(\eta,\omega)\in C([0,T];[L^2(0,L)]^2)$, such that, for any $(h,k)\in L^1(0,T;[L^2(0,L)]^2)$, satisfies*

$$\begin{aligned}&\int_0^T\int_0^L(\eta h+\omega k)dxdt+\left\langle\begin{pmatrix}\eta^0\\ \omega^0\end{pmatrix},\begin{pmatrix}u(0)\\ v(0)\end{pmatrix}\right\rangle_{[H^{-2}(0,L)]^2,[H_0^2(0,L)]^2}\\ &=\int_0^T f_1(t)[b_1u_{txxx}+cv_{xx}](t,L)dt+\int_0^T g_1(t)[d_1v_{txxx}+au_{xx}](t,L)dt\\ &\qquad -b_1\int_0^T f_2(t)u_{txx}(t,L)dt-d_1\int_0^T g_2(t)v_{txx}(t,L)dt,\end{aligned}\tag{2.10}$$

where (u,v) is a solution of the adjoint system

$$\begin{cases}u_t+v_x+cv_{xxx}-bu_{txx}+b_1u_{txxxx}=h & \text{for } x\in(0,L),\ t>0,\\ v_t+u_x+au_{xxx}-dv_{txx}+d_1v_{txxxx}=k & \text{for } x\in(0,L),t>0,\\ u(t,0)=u(t,L)=v(t,0)=v(t,L)=0 & \text{for }\ t\geq 0,\\ u_x(t,0)=u_x(t,L)=v_x(t,0)=v_x(t,L)=0 & \text{for }\ t\geq 0,\\ u(T,x)=0;\quad v(T,x)=0 & \text{for } x\in(0,L).\end{cases}\tag{2.11}$$

The existence of solutions for system (2.11) can be proved following the arguments used in the proof of Theorem 2.2. Moreover, due to the regularizing effect of the operator $(1-\alpha\partial_x^2+\beta\partial_x^4)^{-1}$, with $\alpha,\beta>0$, we obtain the following result:

Theorem 2.3. *If $(h,k)\in L^1(0,T;[L^2(0,L)]^2)$, system (2.11) has a unique solution $(u,v)\in C([0,T];[H_0^2)$. Moreover,*

$$\begin{aligned}&||(u,v)||_{L^1(0,T;[H_0^2\cap H^3(0,L)]^2)}+||(u_t,v_t)||_{L^1(0,T;[H_0^2\cap H^4(0,L)]^2)}\\ &\leq C||(h,k)||_{L^1(0,T;[L^2(0,L)]^2)},\end{aligned}\tag{2.12}$$

for some constant $C>0$.

Proof. System (2.11) can be written as an abstract evolution equation as follows

$$\begin{cases}W_t+\mathcal{A}W=\mathcal{F}\\ W(0)=0,\end{cases}$$

where $W=(u,v)$ and $\mathcal{F}=\mathcal{A}_0(h,k)\in L^1(0,\infty;[H_0^2\cap H^4(0,L)]^2)$, with

$\mathcal{A}_0:[L^2(0,L)]^2\longrightarrow[H_0^2\cap H^4(0,L)]^2$ defined by (2.9). Since $\mathcal{A}$ generates a group of isometries in $[H_0^2(0,L)]^2$, we have that system (2.11) has a unique solution $W=(u,v)\in C([0,\infty);[H_0^2(0,L)]^2)$. Moreover, using the equations in (2.11), we deduce that $(u_t,v_t)\in L^1(0,\infty;[H_0^2\cap H^3(0,L)]^2)$ and estimate (2.12) holds. Indeed, first, observe that each term of the equations in (2.11) belongs to $L^2(0,T;H^{-2}(0,L))$. Thus, scaling the first equation by u and the second by v we obtain

$$\frac{1}{2}\frac{d}{dt}||(u(t,\cdot),v(t,\cdot))||^2_{[H_0^2(0,L)]^2}=\int_0^L(hu+kv)dx.\tag{2.13}$$

Integrating the above identity from t up to T, from Young's inequality it follows that

$$||(u(t,\cdot),v(t,\cdot))||^2_{[H_0^2(0,L)]^2}$$

$$\le C\left(||h||_{L^1(0,T;L^2(0,L))}||u||_{C([0,T];L^2(0,L))}+||k||_{L^1(0,T;L^2(0,L))}||v||_{C([0,T];L^2(0,L))}\right)$$
$$\le C\left(\frac{1}{2\epsilon}||(h,k)||^2_{L^1(0,T;[L^2(0,L)]^2)}+\frac{\epsilon}{2}||(u,v)||^2_{C([0,T];[L^2(0,L)]^2)}\right), \tag{2.14}$$

for any $\epsilon>0$, where C is a positive constant. Then, by choosing $\epsilon>0$ sufficiently small in (2.14) we obtain

$$||(u,v)||_{C([0,T];[H_0^2(0,L)]^2)}\le C||(h,k)||_{L^1(0,T;[L^2(0,L)]^2)}, \tag{2.15}$$

for some $C>0$. On the other hand, due to the regularizing effect of the operator $(1-\alpha\partial_x^2+\beta\partial_x^4)^{-1}$, $\alpha,\beta>0$, it follows that

$$(1-b\partial_x^2+b_1\partial_x^4)^{-1}h(t,\cdot),(1-d\partial_x^2+d_1\partial_x^4)^{-1}k(t,\cdot)\in H^4(0,L)$$

and the operator $\mathcal{A}$ takes values in $[H_0^2\cap H^3(0,L)]^2$, which is compactly embedded in $[H_0^2(0,L)]^2$. Thus, combining (2.15) and the equations in (2.11), it follows that

$$\begin{aligned}
&||(u_t(t,\cdot),v_t(t,\cdot))||_{[H^3(0,L)]^2}\\
&\le ||((1-b\partial_x^2+b_1\partial_x^4)^{-1}(\partial_x+a\partial_x^3)u,(1-d\partial_x^2+d_1\partial_x^4)^{-1}(\partial_x+a\partial_x^3)v)||_{[H^3(0,L)]^2}\\
&\quad +C||((1-b\partial_x^2+b_1\partial_x^4)^{-1}h,(1-d\partial_x^2+d_1\partial_x^4)^{-1}k)||_{[H^4(0,L)]^2}\\
&\le C\left(||((\partial_x+a\partial_x^3)u,(\partial_x+a\partial_x^3)v)||_{[H^{-1}(0,L)]^2}+||(h,k)||_{[L^2(0,L)]^2}\right)\\
&\le C\left(||(u,v)||_{[H_0^2(0,L)]^2}+||(h,k)||_{[L^2(0,L)]^2}\right)\\
&\le C\left(||(u,v)||_{C([0,T];[H_0^2(0,L)]^2)}+||(h,k)||_{[L^2(0,L)]^2}\right).
\end{aligned} \tag{2.16}$$

By integrating (2.16) on $(0,T)$ we get $(u_t,v_t)\in L^1(0,T;[H_0^2\cap H^3(0,T)]^2)$. On the other hand, since $(u(t,x),v(t,x))=(\int_0^t u_s(s,x)ds,\int_0^t v_s(s,x)ds)$, (2.16) allows us to deduce that $(u,v)\in L^1(0,T;[H_0^2\cap H^3(0,T)]^2)$ and, therefore, we obtain (2.12). ☐

The next theorem establishes the existence and uniqueness of solutions for system (1.3) in the sense of transposition.

THEOREM 2.4. *Let $(\eta^0,\omega^0)\in[H^{-2}(0,L)]^2$ and $(f_i,g_i)\in[H^1(0,T)]^2$, $i=1,2$. Then, there exists a unique solution $(\eta,\omega)\in C([0,T];[L^2(0,L)]^2)$ of system (1.3) which verifies (2.10).*

Proof. The result is proved in two steps. We first use the Riesz representation theorem to prove the existence of a solution in $L^1(0,T;[L^2(0,L)]^2)$. Then, the continuity in the time variable is proved by using density arguments.

We start by introducing the linear operator $\mathcal{T}:L^1(0,T;[L^2(0,L)]^2)\longrightarrow\mathbb{R}$ as follows

$$\begin{aligned}
\mathcal{T}((h,k))=&-\left\langle\begin{pmatrix}\eta^0\\ \omega^0\end{pmatrix},\begin{pmatrix}u(0)\\ v(0)\end{pmatrix}\right\rangle_{[H^{-2}(0,L)]^2,[H_0^2(0,L)]^2}\\
&+\int_0^T f_1(t)[b_1u_{txxx}+cv_{xx}](t,L)dt+\int_0^T g_1(t)[d_1v_{txxx}+au_{xx}](t,L)dt\\
&-b_1\int_0^T f_2(t)u_{txx}(t,L)dt-d_1\int_0^T g_2(t)v_{txx}(t,L)dt,
\end{aligned} \tag{2.17}$$

where (u,v) is a solution of (2.11). We have that $\mathcal{T}$ is well defined and continuous. Indeed, proceeding as in the proof of Theorem 2.3, we obtain identity (2.13). Then, integrating over $(0,T)$, it follows that

$$||(u(0),v(0))||_{[H_0^2(0,L)]^2} \leq C||(h,k)||_{L^1(0,T;[L^2(0,L)]^2)}, \tag{2.18}$$

for some constant $C>0$. On the other hand, by using the Cauchy-Schwarz inequality, the Sobolev embedding and estimate (2.12), the following estimate holds

$$\begin{aligned}\Bigg| \int_0^T f_1(t)[b_1 u_{txxx} + c v_{xx}](t,L)dt + \int_0^T g_1(t)[d_1 v_{txxx} + a u_{xx}](t,L)dt \\ - b_1 \int_0^T f_2(t) u_{txx}(t,L)dt - d_1 \int_0^T g_2(t) v_{txx}(t,L)dt \Bigg| \\ \leq C\Big(||(f_1,g_1)||_{[H^1(0,T)]^2} + ||(f_2,g_2)||_{[L^2(0,T)]^2} \Big) ||(h,k)||_{L^1(0,T;[L^2(0,L)]^2)},\end{aligned} \tag{2.19}$$

where $C>0$. Finally, (2.18) and (2.19) allow us to conclude that $\mathcal{T} \in \mathcal{L}(L^1(0,T;[L^2(0,L)]^2);\mathbb{R})$.

Then, from the Riesz representation theorem, we obtain the existence of a unique $(\eta,\omega) \in L^\infty(0,T;[L^2(0,L)]^2)$ satisfying (2.10). Moreover,

$$\begin{aligned}&||(\eta,\omega)||_{L^\infty(0,T;[L^2(0,L)]^2)} = ||\mathcal{T}||_{\mathcal{L}(L^1(0,T;[L^2(0,L)]^2);\mathbb{R})} \\ &\leq C\left(||(\eta^0,\omega^0)||_{[H^{-2}(0,L)]^2} + ||(f_1,g_1)||_{[H^1(0,T)]^2} + ||(f_2,g_2)||_{[L^2(0,T)]^2} \right).\end{aligned} \tag{2.20}$$

By using density arguments, starting with more regular data, we can also get the regularity in the time variable. Indeed, since $(f_1,g_1) \in [H^1(0,T)]^2, (f_2,g_2) \in [L^2(0,T)]^2$ and $(\eta^0,\omega^0) \in [H^{-2}(0,L)]^2$ there exist sequences $(f_{1,n},g_{1,n}),(f_{2,n},g_{2,n}) \in [\mathcal{D}(0,T)]^2$ and $(\eta_n^0,\omega_n^0) \in [\mathcal{D}(0,L)]^2$, such that

$$\begin{aligned}(f_{1,n},g_{1,n}) &\longrightarrow (f_1,g_1) \text{ in } [H^1(0,T)]^2, \\ (f_{2,n},g_{2,n}) &\longrightarrow (f_2,g_2) \text{ in } [L^2(0,T)]^2, \\ (\eta_n^0,\omega_n^0) &\longrightarrow (\eta^0,\omega^0) \text{ in } [H^{-2}(0,L)]^2,\end{aligned}$$

when $n \to \infty$. Let us denote by (η_n,ω_n) the solution of the system (1.3), corresponding to the data $(f_{1,n},g_{1,n}),(f_{2,n},g_{2,n})$ and (η_n^0,ω_n^0), given by Theorem 2.2. Then, $(\eta_n,\omega_n) \in C([0,T];[L^2(0,L)]^2)$ and, for each $n \in \mathbb{N}$, the solution (η_n,ω_n) satisfies (2.10). Thus, if (η,ω) is a solution by transposition of (1.3), it follows that $(\eta_n,\omega_n)-(\eta,\omega)$ is a solution by transposition with data $(f_{1,n},g_{1,n})-(f_1,g_1)$, $(f_{2,n},g_{2,n})-(f_2,g_2)$ and $(\eta_n^0,\omega_n^0)-(\eta^0,\omega^0)$. Hence, by (2.20), we obtain

$$\begin{aligned}||(\eta_n,\omega_n)-(\eta,\omega)||_{L^\infty(0,T;[L^2(0,L)]^2)} \leq C\Big(||(\eta_n^0,\omega_n^0)-(\eta^0,\omega^0)||_{[H^{-2}(0,L)]^2} \\ + ||(f_{1,n},g_{1,n})-(f_1,g_1)||_{[H^1(0,T)]^2} + ||(f_{2,n},g_{2,n})-(f_2,g_2)||_{[L^2(0,T)]^2} \Big).\end{aligned}$$

When $n \to \infty$, from the above inequality, we deduce that $(\eta_n,\omega_n) \to (\eta,\omega)$ in $L^\infty(0,T;[L^2(0,L)]^2)$ and, since $(\eta_n,\omega_n) \in C([0,T];[L^2(0,L)]^2)$, it follows that $(\eta,\omega) \in C([0,T];[L^2(0,L)]^2)$. ∎

3. Spectral analysis

This section is devoted to develop a spectral analysis of the operator $\mathcal{A}$ introduced in (2.4). We start by presenting some explicit formula and properties of a family of initial value problems depending on several parameters. These results allow us to obtain the asymptotic behavior of the eigenvalues and eigenfunctions of the differential operator associated to (1.3).

3.1. Study of some initial value problems.

Firstly, we study the properties of the following simple initial value problem, where $\sigma \in \mathbb{C}^*$ is a complex nonzero parameter:

$$\begin{cases} a\sigma\nu_{xxx} - b_1\varphi_{xxxx} = f & \text{for } x \in (0,L), \\ a\sigma\varphi_{xxx} - d_1\nu_{xxxx} = g & \text{for } x \in (0,L), \\ (\varphi(0),\nu(0)) = (\varphi^0,\nu^0), \\ (\varphi_x(0),\nu_x(0)) = (\varphi^1,\nu^1), \\ (\varphi_{xx}(0),\nu_{xx}(0)) = (\varphi^2,\nu^2), \\ (\varphi_{xxx}(0),\nu_{xxx}(0)) = (\varphi^3,\nu^3). \end{cases} \tag{3.1}$$

In (3.1) a, b_1 and d_1 are positive real numbers. We have the following result.

LEMMA 3.1. *Given $(\varphi^0,\varphi^1,\varphi^2,\varphi^3,\nu^0,\nu^1,\nu^2,\nu^3) \in \mathbb{C}^8$ and $(f,g) \in [L^2(0,L)]^2$, there exists a unique solution (φ,ν) to the problem (3.1) given by the formula*

$$\begin{pmatrix} \varphi(x) \\ \nu(x) \end{pmatrix} = \begin{pmatrix} \frac{(b_1d_1)^{\frac{3}{2}}}{[a\sigma]^3}\left[\sinh(\frac{a\sigma x}{\sqrt{b_1d_1}}) - \frac{a\sigma x}{\sqrt{b_1d_1}}\right]\varphi^3 + \frac{b_1d_1^2}{[a\sigma]^3}\left[\left(\cosh(\frac{a\sigma x}{\sqrt{b_1d_1}})-1\right) - \frac{[a\sigma]^2}{b_1d_1}\frac{x^2}{2}\right]\nu^3 \\ \frac{b_1^2d_1}{[a\sigma]^3}\left[\left(\cosh(\frac{a\sigma x}{\sqrt{b_1d_1}})-1\right) - \frac{[a\sigma]^2}{b_1d_1}\frac{x^2}{2}\right]\varphi^3 + \frac{(b_1d_1)^{\frac{3}{2}}}{[a\sigma]^3}\left[\sinh(\frac{a\sigma x}{\sqrt{b_1d_1}}) - \frac{a\sigma x}{\sqrt{b_1d_1}}\right]\nu^3 \end{pmatrix}$$

$$+ \begin{pmatrix} \varphi^2\frac{x^2}{2} + \varphi^1 x + \varphi^0 - \frac{1}{a\sigma}\int_0^x \bar{F}(s)ds \\ \nu^2\frac{x^2}{2} + \nu^1 x + \nu^0 - \frac{1}{a\sigma}\int_0^x \bar{G}(s)ds \end{pmatrix} \tag{3.2}$$

where

$$\bar{F}(x) = \int_0^x\int_0^s \left[\sqrt{\frac{d_1}{b_1}}\sinh(\frac{a\sigma(s-r)}{\sqrt{b_1d_1}})f(r) + \left(\cosh(\frac{a\sigma(s-r)}{\sqrt{b_1d_1}})-1\right)g(r)\right]drds,$$

$$\bar{G}(x) = \int_0^x\int_0^s \left[\left(\cosh(\frac{a\sigma(s-r)}{\sqrt{b_1d_1}})-1\right)f(r) + \sqrt{\frac{b_1}{d_1}}\sinh(\frac{a\sigma(s-r)}{\sqrt{b_1d_1}})g(r)\right]drds.$$

Proof. By setting $(\varphi_{xxx},\nu_{xxx}) = (\tilde{\varphi},\tilde{\nu})$ we deduce that

$$\begin{pmatrix} \tilde{\varphi}_x(x) \\ \tilde{\nu}_x(x) \end{pmatrix} = \begin{pmatrix} 0 & \frac{a\sigma}{b_1} \\ \frac{a\sigma}{d_1} & 0 \end{pmatrix}\begin{pmatrix} \tilde{\varphi}(x) \\ \tilde{\nu}(x) \end{pmatrix} - \begin{pmatrix} \frac{f(x)}{b_1} \\ \frac{g(x)}{d_1} \end{pmatrix}, \quad \begin{pmatrix} \tilde{\varphi}(0) \\ \tilde{\nu}(0) \end{pmatrix} = \begin{pmatrix} \varphi^3 \\ \nu^3 \end{pmatrix};$$

consequently,

$$\begin{pmatrix} \tilde{\varphi}(x) \\ \tilde{\nu}(x) \end{pmatrix} = e^{Ax}\begin{pmatrix} \varphi^3 \\ \nu^3 \end{pmatrix} - \int_0^x e^{A(x-s)}\begin{pmatrix} \frac{f(x)}{b_1} \\ \frac{g(x)}{d_1} \end{pmatrix} ds, \tag{3.3}$$

where

$$e^{Ax} = \begin{pmatrix} \cosh(\frac{a\sigma x}{\sqrt{b_1 d_1}}) & \sqrt{\frac{d_1}{b_1}}\sinh(\frac{a\sigma x}{\sqrt{b_1 d_1}}) \\ \sqrt{\frac{b_1}{d_1}}\sinh(\frac{a\sigma x}{\sqrt{b_1 d_1}}) & \cosh(\frac{a\sigma x}{\sqrt{b_1 d_1}}) \end{pmatrix}.$$

By integrating the equations in (3.1) we obtain

$$\begin{pmatrix} \varphi_{xx}(x) \\ \nu_{xx}(x) \end{pmatrix} = \begin{pmatrix} \varphi^2 - \frac{d_1}{a\sigma}\nu^3 + \frac{d_1}{a\sigma}\tilde{\nu}(x) + \frac{1}{a\sigma}\int_0^x g(s)ds \\ \nu^2 - \frac{b_1}{a\sigma}\varphi^3 + \frac{b_1}{a\sigma}\tilde{\varphi}(x) + \frac{1}{a\sigma}\int_0^x f(s)ds \end{pmatrix} \tag{3.4}$$

and, from (3.3), it follows that

$$\begin{pmatrix} \varphi_{xx}(x) \\ \nu_{xx}(x) \end{pmatrix} = \begin{pmatrix} \varphi^2 + \frac{\sqrt{b_1 d_1}}{a\sigma}\sinh(\frac{a\sigma x}{\sqrt{b_1 d_1}})\varphi^3 + \frac{d_1}{a\sigma}\left(\cosh(\frac{a\sigma x}{\sqrt{b_1 d_1}}) - 1\right)\nu^3 \\ \nu^2 + \frac{b_1}{a\sigma}\left(\cosh(\frac{a\sigma x}{\sqrt{b_1 d_1}}) - 1\right)\varphi^3 + \frac{\sqrt{b_1 d_1}}{a\sigma}\sinh(\frac{a\sigma x}{\sqrt{b_1 d_1}})\nu^3 \end{pmatrix}$$

$$- \frac{1}{a\sigma}\begin{pmatrix} \int_0^x \left[\sqrt{\frac{d_1}{b_1}}\sinh(\frac{a\sigma(x-s)}{\sqrt{b_1 d_1}})f(s) + \left(\cosh(\frac{a\sigma(x-s)}{\sqrt{b_1 d_1}}) - 1\right)g(s)\right]ds \\ \int_0^x \left[\left(\cosh(\frac{a\sigma(x-s)}{\sqrt{b_1 d_1}}) - 1\right)f(s) + \sqrt{\frac{b_1}{d_1}}\sinh(\frac{a\sigma(x-s)}{\sqrt{b_1 d_1}})g(s)\right]ds \end{pmatrix}. \tag{3.5}$$

After integration, we get

$$\begin{pmatrix} \varphi_{x}(x) \\ \nu_{x}(x) \end{pmatrix} = \begin{pmatrix} \varphi^1 + \frac{b_1 d_1}{[a\sigma]^2}\left(\cosh(\frac{a\sigma x}{\sqrt{b_1 d_1}}) - 1\right)\varphi^3 + \frac{d_1\sqrt{b_1 d_1}}{[a\sigma]^2}\sinh(\frac{a\sigma x}{\sqrt{b_1 d_1}})\nu^3 \\ \nu^1 + \frac{b_1\sqrt{b_1 d_1}}{[a\sigma]^2}\sinh(\frac{a\sigma x}{\sqrt{b_1 d_1}})\varphi^3 + \frac{b_1 d_1}{[a\sigma]^2}\left(\cosh(\frac{a\sigma x}{\sqrt{b_1 d_1}}) - 1\right)\nu^3 \end{pmatrix}$$

$$+ \begin{pmatrix} (\varphi^2 - \frac{d_1}{a\sigma}\nu^3)x - \frac{1}{a\sigma}\int_0^x F(s)ds \\ (\nu^2 - \frac{b_1}{a\sigma}\varphi^3)x - \frac{1}{a\sigma}\int_0^x G(s)ds \end{pmatrix}, \tag{3.6}$$

where

$$F(x) = \int_0^x \left[\sqrt{\frac{d_1}{b_1}}\sinh(\frac{a\sigma(x-s)}{\sqrt{b_1 d_1}})f(s) + \left(\cosh(\frac{a\sigma(x-s)}{\sqrt{b_1 d_1}}) - 1\right)g(s)\right]ds,$$
$$G(x) = \int_0^x \left[\left(\cosh(\frac{a\sigma(x-s)}{\sqrt{b_1 d_1}}) - 1\right)f(s) + \sqrt{\frac{b_1}{d_1}}\sinh(\frac{a\sigma(x-s)}{\sqrt{b_1 d_1}})g(s)\right]ds.$$

Finally, by integrating (3.6), we obtain

$$\begin{pmatrix} \varphi(x) \\ \nu(x) \end{pmatrix} = \begin{pmatrix} \varphi^0 + \frac{(b_1 d_1)^{\frac{3}{2}}}{[a\sigma]^3}\sinh(\frac{a\sigma x}{\sqrt{b_1 d_1}})\varphi^3 + \frac{b_1 d_1^2}{[a\sigma]^3}\left(\cosh(\frac{a\sigma x}{\sqrt{b_1 d_1}}) - 1\right)\nu^3 \\ \nu^0 + \frac{b_1^2 d_1}{[a\sigma]^3}\left(\cosh(\frac{a\sigma x}{\sqrt{b_1 d_1}}) - 1\right)\varphi^3 + \frac{(b_1 d_1)^{\frac{3}{2}}}{[a\sigma]^3}\sinh(\frac{a\sigma x}{\sqrt{b_1 d_1}})\nu^3 \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{1}{2}(\varphi^2 - \frac{d_1}{a\sigma}\nu^3)x^2 + (\varphi^1 - \frac{b_1 d_1}{[a\sigma]^2}\varphi^3)x - \frac{1}{a\sigma}\int_0^x \bar{F}(s)ds \\ \frac{1}{2}(\nu^2 - \frac{b_1}{a\sigma}\varphi^3)x^2 + (\nu^1 - \frac{b_1 d_1}{[a\sigma]^2}\nu^3)x - \frac{1}{a\sigma}\int_0^x \bar{G}(s)ds \end{pmatrix}. \tag{3.7}$$

Rearranging the terms in (3.7) we obtain (3.2). □

We define the set

$$Z=\left\{z\in\mathbb{C}:|z|\geq\frac{1}{2},|\Re(z)|\leq 1\right\}$$

and show that the following estimates for the solution (φ,ν) of (3.1) hold if $\sigma\in Z$.

LEMMA 3.2. *Let (φ,ν) be the solution of (3.1). There exists a positive constant $C>0$, such that the following estimates hold for all $x\in[0,L]$ and $\sigma\in Z$:*

$$\sum_{i=0}^{2}\left|\frac{d^i\varphi}{dx^i}(x)\right|\leq|\varphi^0|+C\left(|\varphi^1|+|\varphi^2|\right)+\frac{C^2}{|\sigma|}\left[|\varphi^3|+|\nu^3|+\int_0^x|f(s)|+|g(s)|ds\right], \tag{3.8}$$

$$\sum_{i=0}^{2}\left|\frac{d^i\nu}{dx^i}(x)\right|\leq|\nu^0|+C\left(|\nu^1|+|\nu^2|\right)+\frac{C^2}{|\sigma|}\left[|\varphi^3|+|\nu^3|+\int_0^x|f(s)|+|g(s)|ds\right], \tag{3.9}$$

$$\max\{|\varphi_{xxx}(x)|,|\nu_{xxx}(x)|\}\leq C\left[|\varphi^3|+|\nu^3|+\int_0^x|f(s)|+|g(s)|ds\right]. \tag{3.10}$$

Proof. First, let us note that the following estimates hold for $(\tilde\varphi,\tilde\nu)$ given by (3.3):

$$\begin{aligned}|\tilde\varphi(x)|&\leq\left(|\varphi^3|+\sqrt{\frac{d_1}{b_1}}|\nu^3|\right)e^{|\Re(\sigma)|\frac{ax}{\sqrt{b_1d_1}}}+\int_0^x e^{|\Re(\sigma)|\frac{a(x-s)}{\sqrt{b_1d_1}}}\left[\frac{1}{b_1}|f(s)|+\frac{1}{\sqrt{b_1d_1}}|g(s)|\right]ds\\&\leq\left(|\varphi^3|+\sqrt{\frac{d_1}{b_1}}|\nu^3|+\int_0^x\left[\frac{1}{b_1}|f(s)|+\frac{1}{\sqrt{b_1d_1}}|g(s)|\right]ds\right)e^{|\Re(\sigma)|\frac{ax}{\sqrt{b_1d_1}}}\end{aligned}$$

and

$$\begin{aligned}|\tilde\nu(x)|&\leq\left(\sqrt{\frac{b_1}{d_1}}|\nu^3|+|\varphi^3|\right)e^{|\Re(\sigma)|\frac{ax}{\sqrt{b_1d_1}}}+\int_0^x e^{|\Re(\sigma)|\frac{a(x-s)}{\sqrt{b_1d_1}}}\left[\frac{1}{\sqrt{b_1d_1}}|f(s)|+\frac{1}{d_1}|g(s)|\right]ds\\&\leq\left(\sqrt{\frac{b_1}{d_1}}|\nu^3|+|\varphi^3|+\int_0^x\left[\frac{1}{\sqrt{b_1d_1}}|f(s)|+\frac{1}{d_1}|g(s)|\right]ds\right)e^{|\Re(\sigma)|\frac{ax}{\sqrt{b_1d_1}}},\end{aligned}$$

which allow us to deduce (3.10). Moreover, taking into account formulas (3.4), we obtain

$$\begin{aligned}|\varphi_{xx}(x)|&\leq|\varphi^2|+\frac{C}{|\sigma|}\left[|\varphi^3|+|\nu^3|+\int_0^x|f(s)|+|g(s)|ds\right],\\|\nu_{xx}(x)|&\leq|\nu^2|+\frac{C}{|\sigma|}\left[|\varphi^3|+|\nu^3|+\int_0^x|f(s)|+|g(s)|ds\right].\end{aligned}$$

Then, from the first estimate above and by using that

$$\begin{aligned}|\varphi_x(x)|&\leq|\varphi^1|+\int_0^x|\varphi_{xx}(s)|ds,\\|\varphi(x)|&\leq|\varphi^0|+\int_0^x|\varphi_x(s)|ds,\end{aligned}$$

for all $x\in[0,L]$, we obtain estimate (3.8). This argument also holds for the function v. Thus, we obtain estimate (3.9). ∎

Let us now consider the following slightly more complicated system,

$$\begin{cases} -\xi+b\xi_{xx}-b_1\xi_{xxxx}+\sigma\zeta_x+a\sigma\zeta_{xxx}=0 & \text{for } x\in(0,L),\\ -\zeta+d\zeta_{xx}-d_1\zeta_{xxxx}+\sigma\xi_x+a\sigma\xi_{xxx}=0 & \text{for } x\in(0,L),\\ (\xi(0),\zeta(0))=(\xi^0,\zeta^0), \\ (\xi_x(0),\zeta_x(0))=(\xi^1,\zeta^1), \\ (\xi_{xx}(0),\zeta_{xx}(0))=(\xi^2,\zeta^2), \\ (\xi_{xxx}(0),\zeta_{xxx}(0))=(\xi^3,\zeta^3), \end{cases} \tag{3.11}$$

for which we have the following result.

PROPOSITION 3.1. *There exists a positive constant $C>0$, such that*

$$\|(\xi,\zeta)\|_{[W^{2,\infty}(0,L)]^2}\leq C\left[\sum_{i=0}^{2}\left(|\xi^i|+|\zeta^i|\right)+\frac{1}{|\sigma|}\left(|\xi^3|+|\zeta^3|\right)\right], \tag{3.12}$$

for any $\sigma\in Z$ and any solution (ξ,ζ) of (3.11).

Proof. Let $\sigma\in Z$, and let (ξ,ζ) be a solution of (3.11). Then, (ξ,ζ) satisfies

$$\begin{cases} a\sigma\zeta_{xxx}-b_1\xi_{xxxx}=\xi-\sigma\zeta_x-b\xi_{xx} & \text{for } x\in(0,L),\\ a\sigma\xi_{xxx}-d_1\zeta_{xxxx}=\zeta-\sigma\xi_x-d\zeta_{xx} & \text{for } x\in(0,L),\\ (\xi(0),\zeta(0))=(\xi^0,\zeta^0), \\ (\xi_x(0),\zeta_x(0))=(\xi^1,\zeta^1), \\ (\xi_{xx}(0),\zeta_{xx}(0))=(\xi^2,\zeta^2), \\ (\xi_{xxx}(0),\zeta_{xxx}(0))=(\xi^3,\zeta^3). \end{cases} \tag{3.13}$$

Since (3.13) is a system of type (3.1) with $f=\xi-\sigma\zeta_x-b\xi_{xx}$ and $g=\zeta-\sigma\xi_x-d\zeta_{xx}$, we obtain from Lemma 3.2, a constant $C>0$, such that

$$\begin{aligned}\sum_{i=0}^{2}\left|\frac{d^i\xi}{dx^i}(x)\right|\leq |\xi^0|+C\left(|\xi^1|+|\xi^2|\right)+\frac{C^2}{|\sigma|}\left[|\xi^3|+|\zeta^3|\right]\\ +\frac{C^2}{|\sigma|}\int_0^x 2|\sigma|\sum_{i=0}^{2}\left(\left|\frac{d^i\xi}{dx^i}(s)\right|+\left|\frac{d^i\zeta}{dx^i}(s)\right|\right)ds\end{aligned}$$

and

$$\begin{aligned}\sum_{i=0}^{2}\left|\frac{d^i\zeta}{dx^i}(x)\right|\leq |\zeta^0|+C\left(|\zeta^1|+|\zeta^2|\right)+\frac{C^2}{|\sigma|}\left[|\xi^3|+|\zeta^3|\right]\\ +\frac{C^2}{|\sigma|}\int_0^x 2|\sigma|\sum_{i=0}^{2}\left(\left|\frac{d^i\xi}{dx^i}(s)\right|+\left|\frac{d^i\zeta}{dx^i}(s)\right|\right)ds.\end{aligned}$$

By adding the estimates above, we obtain

$$\begin{aligned}\sum_{i=0}^{2}\left(\left|\frac{d^i\xi}{dx^i}(x)\right|+\left|\frac{d^i\zeta}{dx^i}(x)\right|\right)\leq C\left[\sum_{i=0}^{2}(|\xi^i|+|\zeta^i|)\right]+\frac{C^2}{|\sigma|}(|\xi^3|+|\zeta^3|)\\ +C^2\int_0^x\sum_{i=0}^{2}\left(\left|\frac{d^i\xi}{dx^i}(s)\right|+\left|\frac{d^i\zeta}{dx^i}(s)\right|\right)ds,\end{aligned}$$

for every $x\in[0,L]$ and $\sigma\in Z$. Then, from Gronwall's inequality we have that (ξ,ζ) satisfies (3.12). ∎

The following result compares solutions of (3.11) and (3.1).

PROPOSITION 3.2. *There exists a positive constant $C>0$, such that*

$$\|(\xi,\zeta)-(\varphi,\nu)\|_{[W^{2,\infty}(0,L)]^2}\leq\left(1+\frac{C^2}{|\sigma|}\right)\left[\sum_{i=0}^{2}\left(|\xi^i|+|\zeta^i|\right)+\frac{1}{|\sigma|}\left(|\xi^3|+|\zeta^3|\right)\right], \tag{3.14}$$

for any $\sigma\in Z$ and any initial data $(\xi^0,\xi^1,\xi^2,\xi^3,\zeta^0,\zeta^1,\zeta^2,\zeta^3)\in\mathbb{C}^8$, where (ξ,ζ) and (φ,ν) are the solutions, with precisely these initial data, of equations (3.11) and (3.1) with $f\equiv g\equiv 0$, respectively.

Proof. We define $\theta=\xi-\varphi$, $u=\zeta-\nu$ and note that (θ,u) is a solution of

$$\begin{cases} a\sigma u_{xxx}-b_1\theta_{xxxx}=\xi-\sigma\zeta_x-b\xi_{xx} & \text{for } x\in(0,L),\\ a\sigma\theta_{xxx}-d_1u_{xxxx}=\zeta-\sigma\xi_x-d\zeta_{xx} & \text{for } x\in(0,L),\\ (\theta(0),u(0))=(0,0),\\ (\theta_x(0),u_x(0))=(0,0),\\ (\theta_{xx}(0),u_{xx}(0))=(0,0),\\ (\theta_{xxx}(0),u_{xxx}(0))=(0,0). \end{cases} \tag{3.15}$$

Therefore, from Lemma 3.2 we obtain a constant $C>0$, such that, for every $x\in[0,L]$ and $\sigma\in Z$,

$$\sum_{i=0}^{2}\left(\left|\frac{d^i\theta}{dx^i}(x)\right|+\left|\frac{d^iu}{dx^i}(x)\right|\right)\leq\frac{C^2}{|\sigma|}\left[\int_0^x(|\xi(s)|+|\xi_{xx}(s)|+|\zeta(s)|+|\zeta_{xx}(s)|)ds\right.$$
$$\left.+\int_0^x|\sigma|(|\xi_x(s)|+|\zeta_x(s)|)ds\right].$$

From the estimate above and (3.12) it follows that

$$\sum_{i=0}^{2}\left(\left|\frac{d^i\theta}{dx^i}(x)\right|+\left|\frac{d^iu}{dx^i}(x)\right|\right)\leq\frac{C^2}{|\sigma|}\left[\sum_{i=0}^{2}\left(|\xi^i|+|\zeta^i|\right)+\frac{1}{|\sigma|}\left(|\xi^3|+|\zeta^3|\right)\right]$$
$$+C^2\sum_{i=0}^{2}\left(|\xi^i|+|\zeta^i|\right)+\frac{1}{|\sigma|}\left(|\xi^3|+|\zeta^3|\right).$$

Then, the solutions (ξ,ζ) and (φ,ν) satisfy (3.14). ∎

Finally, we consider systems (3.1) and (3.11) with distinct parameters σ. The difference between the respective solutions are given by the following result.

PROPOSITION 3.3. *Let (φ,ν) and (ξ,ζ) be solutions of (3.1) with $\sigma=\mu$ and (3.11) with $\sigma=\tilde{\mu}$, respectively, and $f\equiv g\equiv 0$. Then, there exists a positive constant $C>0$, such that*

$$\|(\xi,\zeta)-(\varphi,\nu)\|_{[W^{2,\infty}(0,L)]^2}$$

$$\leq C\left[\sum_{i=0}^{2}\left(|\xi^i-\varphi^i|+|\zeta^i-\nu^i|\right)+\frac{1}{|\mu|}\left(|\xi^3-\varphi^3|+|\zeta^3-\nu^3|+|\mu-\tilde{\mu}|(|\varphi^3|+|\nu^3|)\right)\right]. \quad (3.16)$$

Proof. We define $\theta=\xi-\varphi$, $u=\zeta-\nu$, and note that (θ,u) is a solution of

$$\begin{cases} a\mu u_{xxx}-b_1\theta_{xxxx}=\xi-\mu\zeta_x-b\xi_{xx}+a(\tilde{\mu}-\mu)\nu_{xxx} & \text{for } x\in(0,L),\\ a\mu\theta_{xxx}-d_1u_{xxxx}=\zeta-\mu\xi_x-d\zeta_{xx}+a(\tilde{\mu}-\mu)\varphi_{xxx} & \text{for } x\in(0,L),\\ (\theta(0),u(0))=(\xi^0-\varphi^0,\zeta^0-\nu^0),\\ (\theta_x(0),u_x(0))=(\xi^1-\varphi^1,\zeta^1-\nu^1),\\ (\theta_{xx}(0),u_{xx}(0))=(\xi^2-\varphi^2,\zeta^2-\nu^2),\\ (\theta_{xxx}(0),u_{xxx}(0))=(\xi^3-\varphi^3,\zeta^3-\nu^3). \end{cases}$$

Therefore, from Lemma 3.2 we obtain (3.16). ∎

3.2. Spectral analysis of the operator $\mathcal{A}$. Given $b_1,d_1>0$, let us first introduce the operator $\mathcal{B}:(H_0^2(0,2\pi))^2\to(H_0^2(0,2\pi))^2$ given by

$$\mathcal{B}=\begin{pmatrix} 0 & (b_1\partial_x^4)^{-1}(a\partial_x^3)\\ (d_1\partial_x^4)^{-1}(a\partial_x^3) & 0\end{pmatrix}. \quad (3.17)$$

Recall that, for $\alpha>0$, the operator $(-\alpha\partial_x^4)^{-1}:L^2(0,2\pi)\to L^2(0,2\pi)$ defined by

$$(-\alpha\partial_x^4)^{-1}\varphi=v\Leftrightarrow\begin{cases}-\alpha v_{xxxx}=\varphi\\ \partial_x^r v(0)=\partial_x^r v(L)=0, r=0,1,\end{cases}$$

is a well-defined, compact operator in $L^2(0,2\pi)$.

In this section, $\lambda\in\mathbb{C}$ is called an eigenvalue of the operator $\mathcal{A}(\mathcal{B})$ if there exists a nontrivial vector $\Phi=(\varphi,\nu)\in[H_0^2(0,L)]^2$, called an eigenfunction corresponding to λ, such that $\mathcal{A}\Phi=\lambda\Phi$ $(\mathcal{B}\Phi=\lambda\Phi)$. The following two theorems are devoted to the spectral analysis of these operators.

THEOREM 3.1. *The eigenvalues of the operator $\mathcal{B}$ defined by (3.17) are $\tilde{\lambda}_n=1/\tilde{\mu}_n$, where*

$$\tilde{\mu}_n=sgn(n)\frac{\sqrt{b_1d_1}}{aL}((2|n|+1)\pi-2\varepsilon_n)\boldsymbol{i}, \quad (3.18)$$

and $\varepsilon_n\in(0,1)$, with $n\in\mathbb{Z}^*$. Each eigenvalue $\tilde{\lambda}_n$ is double and has two independent eigenfunctions given by

$$\tilde{\Phi}_n^1=\left[\frac{\sqrt{b_1d_1}}{a\tilde{\mu}_n}\right]^3\begin{pmatrix}\mathcal{S}(\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}},x)\\ \sqrt{\frac{b_1}{d_1}}\mathcal{C}(\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}},x)\end{pmatrix},\qquad \tilde{\Phi}_n^2=\left[\frac{\sqrt{b_1d_1}}{a\tilde{\mu}_n}\right]^3\begin{pmatrix}\sqrt{\frac{d_1}{b_1}}\mathcal{C}(\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}},x)\\ \mathcal{S}(\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}},x)\end{pmatrix}, \quad (3.19)$$

where

$$\mathcal{S}\left(\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}},x\right)=\sinh(\frac{a\tilde{\mu}_nx}{\sqrt{b_1d_1}})-\frac{a\tilde{\mu}_nx}{\sqrt{b_1d_1}}+\left[\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}}\right]^3L\left(\left[\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}}L\right]^2-4\right)^{-1}x^2,$$

$$\mathcal{C}\left(\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}},x\right)=\left(\cosh(\frac{a\tilde{\mu}_nx}{\sqrt{b_1d_1}})-1\right) -\left(\left[\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}}\right]^2-\left[\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}}\right]^4L^2\left(\left[\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}}L\right]^2-4\right)^{-1}\right)\frac{x^2}{2}.$$

Moreover, the set $\left\{\tilde{\Phi}_n^j : n \in \mathbb{Z}^*, j \in \{1,2\}\right\}$ *forms an orthogonal basis of* $[H_0^2(0,L)]^2$.

Proof. By using Lemma 3.1, with $\varphi^0 = \varphi^1 = \nu^0 = \nu^1 = 0$ and $f \equiv g \equiv 0$, we deduce that (φ, ν) is an eigenfunction of $\mathcal{B}$ corresponding to the eigenvalue $1/\mu$ if and only if

$$\begin{pmatrix} \varphi(x) \\ \nu(x) \end{pmatrix} = \frac{1}{\kappa^3} \begin{pmatrix} [\sinh(\kappa x) - \kappa x]\varphi^3 + \sqrt{\frac{d_1}{b_1}}\left[(\cosh(\kappa x)-1) - \frac{[\kappa x]^2}{2}\right]\nu^3 \\ \sqrt{\frac{b_1}{d_1}}\left[(\cosh(\kappa x)-1) - \frac{[\kappa x]^2}{2}\right]\varphi^3 + [\sinh(\kappa x) - \kappa x]\nu^3 \end{pmatrix} + \begin{pmatrix} \varphi^2 \frac{x^2}{2} \\ \nu^2 \frac{x^2}{2} \end{pmatrix} \tag{3.20}$$

and

$$\begin{pmatrix} \varphi(L) \\ \nu(L) \end{pmatrix} = \begin{pmatrix} \varphi_x(L) \\ \nu_x(L) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.21}$$

where $\kappa = a\mu/\sqrt{b_1 d_1}$. The data (φ^2, ν^2) can be written as a function of κ and (φ^3, ν^3). Indeed, from (3.20) and (3.21) we obtain the following systems

$$\begin{cases} [\sinh(\kappa L) - \kappa L]\varphi^3 + \sqrt{\frac{d_1}{b_1}}\left[(\cosh(\kappa L)-1) - \frac{[\kappa L]^2}{2}\right]\nu^3 + \kappa^3 \frac{L^2}{2}\varphi^2 = 0 \\ \sqrt{\frac{b_1}{d_1}}\left[(\cosh(\kappa L)-1) - \frac{[\kappa L]^2}{2}\right]\varphi^3 + [\sinh(\kappa L) - \kappa L]\nu^3 + \kappa^3 \frac{L^2}{2}\nu^2 = 0 \end{cases}$$

and

$$\begin{cases} (\cosh(\kappa L)-1)\varphi^3 + \sqrt{\frac{d_1}{b_1}}(\sinh(\kappa L) - \kappa L)\nu^3 + \kappa^2 L\varphi^2 = 0 \\ \sqrt{\frac{b_1}{d_1}}(\sinh(\kappa L) - \kappa L)\varphi^3 + (\cosh(\kappa L)-1)\nu^3 + \kappa^2 L\nu^2 = 0. \end{cases}$$

Thus, we deduce that (φ^2, ν^2) should satisfy

$$\begin{pmatrix} \varphi^2 \\ \nu^2 \end{pmatrix} = \frac{L}{[\kappa L]^2 - 4} \begin{pmatrix} 2 & \sqrt{\frac{d_1}{b_1}} L\kappa \\ \sqrt{\frac{b_1}{d_1}} L\kappa & 2 \end{pmatrix} \begin{pmatrix} \varphi^3 \\ \nu^3 \end{pmatrix}, \tag{3.22}$$

with $\kappa \neq \pm 2/L$. Replacing (3.22) in (3.20) we obtain

$$\begin{pmatrix} \varphi(x) \\ \nu(x) \end{pmatrix} = \frac{1}{\kappa^3} \begin{pmatrix} \mathcal{S}(\kappa, x) & \sqrt{\frac{d_1}{b_1}}\mathcal{C}(\kappa, x) \\ \sqrt{\frac{b_1}{d_1}}\mathcal{C}(\kappa, x) & \mathcal{S}(\kappa, x) \end{pmatrix} \begin{pmatrix} \varphi^3 \\ \nu^3 \end{pmatrix}, \tag{3.23}$$

where

$$\begin{aligned} \mathcal{S}(\kappa, x) &= \sinh(\kappa x) - \kappa x + \frac{[\kappa^3 L]}{[\kappa L]^2 - 4} x^2 \\ \mathcal{C}(\kappa, x) &= (\cosh(\kappa x) - 1) - \left(\kappa^2 - \frac{[\kappa^4 L^2]}{[\kappa L]^2 - 4}\right)\frac{x^2}{2}. \end{aligned}$$

The next steps are devoted to obtaining the eigenvalue associated to the eigenfunction given by (3.23). First, we note that $\mathcal{S}_x(\kappa, L) = \kappa\mathcal{C}(\kappa, L)$ and $\mathcal{C}_x(\kappa, L) = \kappa\mathcal{S}(\kappa, L)$. Then, from (3.23) and the boundary conditions (3.21) we have

$$\begin{pmatrix} \varphi(L) \\ \nu(L) \end{pmatrix} = \frac{1}{\kappa^3} \begin{pmatrix} \mathcal{S}(\kappa, L) & \sqrt{\frac{d_1}{b_1}}\mathcal{C}(\kappa, L) \\ \sqrt{\frac{b_1}{d_1}}\mathcal{C}(\kappa, L) & \mathcal{S}(\kappa, L) \end{pmatrix} \begin{pmatrix} \varphi^3 \\ \nu^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \varphi_x(L) \\ \nu_x(L) \end{pmatrix} = \frac{1}{\kappa^2} \begin{pmatrix} \mathcal{C}(\kappa,L) & \sqrt{\frac{d_1}{b_1}}\mathcal{S}(\kappa,L) \\ \sqrt{\frac{b_1}{d_1}}\mathcal{S}(\kappa,L) & \mathcal{C}(\kappa,L) \end{pmatrix} \begin{pmatrix} \varphi^3 \\ \nu^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The systems above imply that κ is a root of the equation

$$\mathcal{C}(\kappa,L)^2 - \mathcal{S}(\kappa,L)^2 = 0, \tag{3.24}$$

which can be written as

$$\frac{4}{[\kappa L]^2 - 4}\left([\kappa L]\cosh\left(\frac{\kappa L}{2}\right) - 2\sinh\left(\frac{\kappa L}{2}\right)\right)^2 = 0. \tag{3.25}$$

The following result allows us to localize the roots of (3.25).

LEMMA 3.3. *The nontrivial roots* $(z_n)_{n\in\mathbb{Z}^*}$ *of*

$$f(z) = z\cosh\left(\frac{z}{2}\right) - 2\sinh\left(\frac{z}{2}\right) \tag{3.26}$$

satisfy $z_n = \boldsymbol{i}y_n$, *where* $(y_n)_{n\in\mathbb{Z}^*} \subset \mathbb{R}$ *are the roots of the transcendental equation*

$$\tan\left(\frac{y}{2}\right) = \frac{y}{2}. \tag{3.27}$$

Proof. First, we show that (3.26) has no roots z with $\Re(z) \neq 0$: Indeed, if $z = x + iy$ we have that

$$f(x+iy) = f(x,y) = U(x,y) + iV(x,y)$$

where

$$\begin{aligned} U(x,y) &= x\cosh\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right) - \sinh\left(\frac{x}{2}\right)\left(2\cos\left(\frac{y}{2}\right) + y\sin\left(\frac{y}{2}\right)\right), \\ V(x,y) &= \cosh\left(\frac{x}{2}\right)\left(y\cos\left(\frac{y}{2}\right) - 2\sin\left(\frac{y}{2}\right)\right) + x\sinh\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right). \end{aligned}$$

For $y \in \mathbb{R}$ fixed, we define the nonnegative function $K_y(x) := |f(x,y)|^2$. Then,

- $K_y'(x)\Big|_{x=0} = x\cos(y) - x\cosh(x) + \frac{1}{2}(x^2+y^2)\sinh(x)\Big|_{x=0} = 0,$
- $K_y''(x) = \frac{1}{2}(x^2+y^2-2)\cosh(x) + \cos(y) \geq 0$, for all $x \in \mathbb{R}$.

The statement above is proved by noting that $x \mapsto K_y''(x)$ is increasing (decreasing) for $x > 0$ $(x < 0)$ and $K_y''(0) = \frac{1}{2}(y^2-2) + \cos(y) \geq 0$, for all $y \in \mathbb{R}$.

Both statements above imply that, for $y \in \mathbb{R}$ fixed, the convex function $x \mapsto |f(x,y)|^2$ has a global minimum value at $(0,y)$. This shows that (x_0,y_0) is a root of (3.26) if and only if $x_0 = 0$ and y_0 is a root of the real-valued function $g(y) = y\cos\left(\frac{y}{2}\right) - 2\sin\left(\frac{y}{2}\right)$. Then,

$$y\cos\left(\frac{y}{2}\right) - 2\sin\left(\frac{y}{2}\right) = 0 \Leftrightarrow \tan\left(\frac{y}{2}\right) = \frac{y}{2}.$$

∎

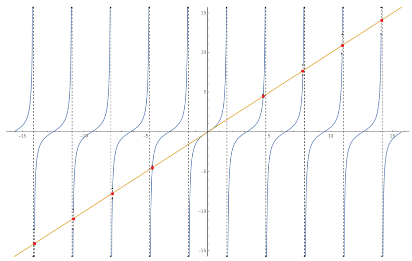

FIG. 3.1. *The distance ε_n between the root x_n of the equation $\tan(x)=x$ and the asymptotic $x=sgn(n)\frac{(2|n|+1)}{2}\pi$ tends to 0, when $|n|\to\infty$.*

By analyzing the graphs of the functions $\tan(x)$ and x (see Figure 3.1), we deduce that the points of intersection $(x_n)_{n\in\mathbb{Z}^*}$, can be written as $x_n=\frac{(2n+1)}{2}\pi-\varepsilon_n$, $x_{-n}=-x_n$, where $\varepsilon_n\in(0,1)$, for all $n\geq 1$.

From the analysis above, we conclude that the roots $(L\kappa_n)_{n\in\mathbb{Z}^*}$ of (3.25) satisfy $L\kappa_n\in i\mathbb{R}$ and $iL\kappa_n=-sgn(n)((2|n|+1)\pi-2\varepsilon_n)$, for all $n\in\mathbb{Z}^*$. Then, the eigenvalues $(1/\tilde{\mu}_n)_{n\in\mathbb{Z}^*}$ satisfy $\tilde{\mu}_n=sgn(n)\frac{\sqrt{b_1d_1}}{aL}((2|n|+1)\pi-2\varepsilon_n)i$, where $\varepsilon_n\in(0,1)$, with $n\in\mathbb{Z}^*$. ◻

REMARK 3.1. If $1/\tilde{\mu}_n$ is an eigenvalue of the operator $\mathcal{B}$, from (3.22) we have that (φ^2,ν^2) satisfies

$$\begin{pmatrix}\varphi^2\\ \nu^2\end{pmatrix}=\frac{b_1d_1L}{[aL\tilde{\mu}_n]^2-4b_1d_1}\begin{pmatrix}2\\ \frac{aL\tilde{\mu}_n}{d_1}\end{pmatrix}\varphi^3+\frac{b_1d_1L}{[aL\tilde{\mu}_n]^2-4b_1d_1}\begin{pmatrix}\frac{aL\tilde{\mu}_n}{b_1}\\ 2\end{pmatrix}\nu^3.$$

By using (3.18) we obtain

$$\begin{aligned}\left|\begin{pmatrix}\varphi^2\\ \nu^2\end{pmatrix}\right|&\leq C'\left(\frac{1}{((2|n|+1)\pi-2\varepsilon_n)^2}+\frac{1}{((2|n|+1)\pi-2\varepsilon_n)}\right)(|\varphi^3|+|\nu^3|)\\ &\leq\frac{\tau}{|\tilde{\mu}_n|}(|\varphi^3|+|\nu^3|),\end{aligned}$$

where τ and C' are positive constants.

We pass to analyze the spectral properties of the operator $\mathcal{A}$. The main difference with respect to $\mathcal{B}$ is that we do not have an explicit representation formula as (3.19) for the eigenfunctions of $\mathcal{A}$. Therefore, in order to prove the next theorem, we use a strategy which combines two-dimensional versions of the shooting method and Rouché's theorem.

THEOREM 3.2. *The eigenvalues of the operator*

$$\mathcal{A}=\begin{pmatrix}0 & (1-b\partial_x^2+b_1\partial_x^4)^{-1}(\partial_x+a\partial_x^3)\\ (1-d\partial_x^2+d_1\partial_x^4)^{-1}(\partial_x+a\partial_x^3) & 0\end{pmatrix}$$

are purely imaginary numbers $(1/\mu_n^j)_{n\in\mathbb{Z}^*,j\in\{1,2\}}$ with the property that

$$\mu_n^j=\tilde{\mu}_n+\mathcal{O}\left(\frac{1}{|n|}\right)\quad(n\in\mathbb{Z}^*,j\in\{1,2\}).\tag{3.28}$$

Moreover, to each eigenvalue $1/\mu_n^j$ *corresponds an eigenfunction* Φ_n^j *given by*

$$\Phi_n^j = \tilde{\Phi}_n^j + \mathcal{O}\left(\frac{1}{|n|}\right) \quad (n \in \mathbb{Z}^*, j \in \{1,2\}), \tag{3.29}$$

with the property that the sequence $(\Phi_n^j)_{n\in\mathbb{Z}^*, j\in\{1,2\}}$ *forms an orthogonal basis of* $[H_0^2(0,L)]^2$.

Proof. According to the proof of Theorem 2.1, $\mathcal{A}$ is a compact skew-adjoint operator in $[H_0^2(0,L)]^2$. Then, it has a sequence of purely eigenvalues tending to zero. In order to localize these eigenvalues, let us define, for given $\delta > 0$ and $N \in \mathbb{N}$, the sets

$$D_n(\delta) = \left\{ (\mu,\gamma,\beta) \in \mathbb{C}^4 : |\mu - \tilde{\mu}_n|^2 + |\gamma|^2 < \frac{\delta^2}{n^2}, |\beta| < \frac{\tau}{|\mu|} \right\},$$
$$\Gamma_n(\delta) = \partial D_n(\delta), \quad (|n| > N),$$
$$D_N = \left\{ (\mu,\gamma,\beta) \in \mathbb{C}^4 : |\Re\mu| \le 1, |\Im\mu| \le \frac{\sqrt{b_1 d_1}}{aL}((2N+2)\pi - 2\varepsilon_N), |\gamma| \le 1, |\beta| \le \frac{\tau}{|\mu|} \right\},$$
$$\Gamma_N = \partial D_N,$$

where τ is given in Remark 3.1 and $\beta \in \mathbb{C}^2$. Also, let us define the maps $F^j, G^j : \mathbb{C}^4 \to \mathbb{C}^4$, $j \in \{1,2\}$, by

$$F^j(\mu,\gamma,\beta^j) = \left(\begin{pmatrix} \varphi^j(\mu,\gamma,\beta^j,L) \\ \nu^j(\mu,\gamma,\beta^j,L) \end{pmatrix}, \begin{pmatrix} \varphi_x^j(\mu,\gamma,\beta^j,L) \\ \nu_x^j(\mu,\gamma,\beta^j,L) \end{pmatrix} \right),$$
$$[2mm] G^j(\mu,\gamma,\beta^j) = \left(\begin{pmatrix} \tilde{\varphi}^j(\mu,\gamma,\beta^j,L) \\ \tilde{\nu}^j(\mu,\gamma,\beta^j,L) \end{pmatrix}, \begin{pmatrix} \tilde{\varphi}_x^j(\mu,\gamma,\beta^j,L) \\ \tilde{\nu}_x^j(\mu,\gamma,\beta^j,L) \end{pmatrix} \right), \tag{3.30}$$

where $\beta^j = (\beta_1^j, \beta_2^j) \in \mathbb{C}^2$, for $j \in \{1,2\}$, and

$$\begin{pmatrix} \varphi^1(\mu,\gamma,\beta^1,\cdot) \\ \nu^j(\mu,\gamma,\beta^1,\cdot) \end{pmatrix}, \begin{pmatrix} \varphi^2(\mu,\gamma,\beta^2,\cdot) \\ \nu^2(\mu,\gamma,\beta^2,\cdot) \end{pmatrix}, \begin{pmatrix} \tilde{\varphi}^1(\mu,\gamma,\beta^1,\cdot) \\ \tilde{\nu}^j(\mu,\gamma,\beta^1,\cdot) \end{pmatrix}, \begin{pmatrix} \tilde{\varphi}^2(\mu,\gamma,\beta^2,\cdot) \\ \tilde{\nu}^2(\mu,\gamma,\beta^2,\cdot) \end{pmatrix}$$

are solutions of the initial values problems

$$\begin{cases} -\varphi^1 + b\varphi_{xx}^1 - b_1\varphi_{xxxx}^1 + \mu\nu_x^1 + a\mu\nu_{xxx}^1 = 0 & \text{for } x \in (0,L), \\ -\nu^1 + d\nu_{xx}^1 - d_1\nu_{xxxx}^1 + \mu\varphi_x^1 + a\mu\varphi_{xxx}^1 = 0 & \text{for } x \in (0,L), \\ (\varphi^1(0), \nu^1(0)) = (0,0), \\ (\varphi_x^1(0), \nu_x^1(0)) = (0,0), \\ (\varphi_{xx}^1(0), \nu_{xx}^1(0)) = (\beta_1^1, \beta_2^1), \\ (\varphi_{xxx}^1(0), \nu_{xxx}^1(0)) = (1,\gamma), \end{cases} \tag{3.31}$$

$$\begin{cases} -\varphi^2 + b\varphi_{xx}^2 - b_1\varphi_{xxxx}^2 + \mu\nu_x^2 + a\mu\nu_{xxx}^2 = 0 & \text{for } x \in (0,L), \\ -\nu^2 + d\nu_{xx}^2 - d_1\nu_{xxxx}^2 + \mu\varphi_x^2 + a\mu\varphi_{xxx}^2 = 0 & \text{for } x \in (0,L), \\ (\varphi^2(0), \nu^2(0)) = (0,0), \\ (\varphi_x^2(0), \nu_x^2(0)) = (0,0), \\ (\varphi_{xx}^2(0), \nu_{xx}^2(0)) = (\beta_1^2, \beta_2^2), \\ (\varphi_{xxx}^2(0), \nu_{xxx}^2(0)) = (\gamma,1), \end{cases} \tag{3.32}$$

$$\begin{cases} a\mu\tilde{\nu}^1_{xxx} - b_1\tilde{\varphi}^1_{xxxx} = 0 & \text{for } x\in(0,L),\\ a\mu\tilde{\varphi}^1_{xxx} - d_1\tilde{\nu}^1_{xxxx} = 0 & \text{for } x\in(0,L),\\ (\tilde{\varphi}^1(0),\tilde{\nu}^1(0)) = (0,0), \\ (\tilde{\varphi}^1_x(0),\tilde{\nu}^1_x(0)) = (0,0), \\ (\tilde{\varphi}^1_{xx}(0),\tilde{\nu}^1_{xx}(0)) = (\beta^1_1,\beta^1_2), \\ (\tilde{\varphi}^1_{xxx}(0),\tilde{\nu}^1_{xxx}(0)) = (1,\gamma), \end{cases} \tag{3.33}$$

$$\begin{cases} a\mu\tilde{\nu}^2_{xxx} - b_1\tilde{\varphi}^2_{xxxx} = 0 & \text{for } x\in(0,L),\\ a\mu\tilde{\varphi}^2_{xxx} - d_1\tilde{\nu}^2_{xxxx} = 0 & \text{for } x\in(0,L),\\ (\tilde{\varphi}^2(0),\tilde{\nu}^2(0)) = (0,0), \\ (\tilde{\varphi}^2_x(0),\tilde{\nu}^2_x(0)) = (0,0), \\ (\tilde{\varphi}^2_{xx}(0),\tilde{\nu}^2_{xx}(0)) = (\beta^2_1,\beta^2_2), \\ (\tilde{\varphi}^2_{xxx}(0),\tilde{\nu}^2_{xxx}(0)) = (\gamma,1), \end{cases} \tag{3.34}$$

respectively.

According to Theorem 3.1 and Remark 3.1, we observe that $1/\tilde{\mu}$ is an eigenvalue of $\mathcal{B}$ if and only if $G^1(\tilde{\mu},0,\tilde{\beta}^1)=0$, where $\tilde{\beta}^1=(\tilde{\beta}^1_1,\tilde{\beta}^1_2)$ satisfies

$$\tilde{\beta}^1_1 = \frac{2b_1d_1L}{(aL\tilde{\mu})^2-4b_1d_1} \quad \text{and} \quad \tilde{\beta}^1_2 = \frac{b_1aL^2\tilde{\mu}}{(aL\tilde{\mu})^2-4b_1d_1},$$

or $G^2(\tilde{\mu},0,\tilde{\beta}^2)=0$, where $\tilde{\beta}^2=(\tilde{\beta}^2_1,\tilde{\beta}^2_2)$ satisfies

$$\tilde{\beta}^2_1 = \frac{d_1aL^2\tilde{\mu}}{(aL\tilde{\mu})^2-4b_1d_1} \quad \text{and} \quad \tilde{\beta}^2_2 = \frac{2b_1d_1L}{(aL\tilde{\mu})^2-4b_1d_1}.$$

Moreover, from the definition (3.30) and (3.31)-(3.32), we deduce that $1/\mu$ is an eigenvalue of $\mathcal{A}$ if and only if there exists $(\gamma,\beta)\in\mathbb{C}^3$, such that $F^1(\mu,\gamma,\beta)=0$ or $F^2(\mu,\gamma,\beta)=0$. Hence, we have reduced the problem of finding the eigenvalues of $\mathcal{A}$ to the problem of determining the zeros of the maps $(F^j)_{j=1,2}$. We analyze only the zeros of the map F^1, since the analysis of those of F^2 is similar. First, we note that the maps F^1 and G^1 are analytic and that

$$\left|F^1(\mu,\gamma,\beta)-G^1(\mu,\gamma,\beta)\right| \le \frac{C_1}{|\mu|} \quad \left(|\Re\mu|\le 1, |\mu|\ge\frac{1}{2}, |\gamma|\le 1, |\beta|\le\frac{\tau}{|\mu|}\right), \tag{3.35}$$

$$\left|G^1(\mu,\gamma,\beta)\right| \ge \frac{\delta C_2}{|\mu|} \quad ((\mu,\gamma,\beta)\in\Gamma_n(\delta)), \tag{3.36}$$

for some positive constants C_1,C_2. Indeed, since $\mu\in Z, |\gamma|\le 1$ and $|\beta|\le\frac{\tau}{|\mu|}$, (3.35) is a direct consequence of Proposition 3.2. On the other hand, since $G^1(\tilde{\mu}_n,0,\tilde{\beta}^1_n)=0$, we can find $C_2>0$, such that

$$|\mu||G^1(\mu,\gamma,\beta)|\ge\delta C_2,$$

for $(\mu,\gamma,\beta)\in\Gamma_n(\delta)$ and we obtain (3.36). It follows from the multidimensional version of Rouché's theorem [12, Theorem 1] (see, also, [13, Theorem 3]) that there exist $\delta>0$ and $N>0$, such that the maps F^1 and G^1 have the same number of zeros in $D_n(\delta)$, for each $|n|\geq N$. Since G^1 has exactly one zero $(\tilde{\mu}_n,0,\tilde{\beta}_n^1)$ in $D_n(\delta)$, then F^1 has a unique zero $(\mu_n^1,\gamma_n^1,\beta_n^1)$ in $D_n(\delta)$. Thus, we have obtained the eigenvalues $(1/\mu_n^1)_{|n|\geq N}$ of $\mathcal{A}$ and proved the corresponding asymptotic estimate (3.28). Arguing as before, we get the existence of a family of zeros $(\mu_n^2,\gamma_n^2,\beta_n^2)_{|n|\geq N}$ for the map F^2. Then, we obtain the other sequence of eigenvalues $(1/\mu_n^2)_{|n|\geq N}$ of $\mathcal{A}$ and the corresponding asymptotic estimate. To obtain the remaining eigenvalues, we note that, since

$$\mathcal{S}\left(\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}},L\right)=\mathcal{C}\left(\frac{a\tilde{\mu}_n}{\sqrt{b_1d_1}},L\right)=0 \quad \text{for all} \quad 1\leq|n|\leq N,$$

then, there exists a positive constant C_3, such that

$$\min\left\{\left|\mathcal{S}\left(\frac{a\mu}{\sqrt{b_1d_1}},L\right)\right|,\left|\mathcal{C}\left(\frac{a\mu}{\sqrt{b_1d_1}},L\right)\right|\right\}\geq C_3$$
$$\text{for } \mu\in\partial\left(|\Re\mu|\leq 1,|\Im\mu|\leq\frac{\sqrt{b_1d_1}}{aL}((2N+2)\pi-2\varepsilon_N)\right).$$

This implies that $\left|G^1(\mu,\gamma,\beta)\right|\geq\frac{\delta C_4}{|\mu|}\quad((\mu,\gamma,\beta)\in\Gamma_N)$ for some $C_4>0$. Combining the last estimate with (3.35) and applying again the multidimensional Rouché's theorem, we obtain the eigenvalues $(1/\mu_n^1)_{|n|\leq N}$ of $\mathcal{A}$ in D_N. From the analysis of the map F^2 we get the existence of the remaining eigenvalues $(1/\mu_n^2)_{|n|\leq N}$.

Let us pass to the analysis of the eigenfunctions. To each eigenvalue $1/\mu_n^j$ corresponds a unique normalized eigenfunction Φ_n^j satisfying (3.31) with $\gamma=\gamma_n^1$ and $\beta=\beta_n^1=(\beta_{1,n}^1,\beta_{2,n}^1)$ or (3.32) with $\gamma=\gamma_n^2$ and $\beta=\beta_n^2=(\beta_{1,n}^2,\beta_{2,n}^2)$, respectively. Since

$$|\gamma_n^j|\leq\frac{\delta}{|n|},\quad|\mu_n^j-\tilde{\mu}_n|\leq\frac{\delta}{|n|}\quad\text{and}\quad|\beta_n^j-\tilde{\beta}_n^j|\leq\tau\left(\frac{1}{|\mu_n^j|}+\frac{1}{|\tilde{\mu}_n^j|}\right)\quad\text{for}\quad j=1,2,$$

then, from Proposition 3.3, we deduce that (3.29) is verified. Finally, since $\mathcal{A}$ is a skew-adjoint operator, these eigenfunctions are orthogonal in $[H_0^2(0,L)]^2$. ☐

4. Controllability

In this section we study some boundary controllability properties of the Boussinesq system. We begin with the following exact controllability problem:

Given $T>0$ and an initial data $(\eta^0,\omega^0)\in[H^{-2}(0,L)]^2$, can we find control inputs $(f_i,g_i)\in[H^1(0,T)]^2$, $i=1,2$, such that the solution (η,ω) of (1.3) satisfies

$$(\eta(T,x),\omega(T,x))=(0,0)\quad\textit{for } x\in(0,L)? \tag{4.1}$$

We have the following characterization of a control driving system (1.3) to the rest.

LEMMA 4.1. *The initial data $(\eta^0,\omega^0)\in[H^{-2}(0,L)]^2$ is controllable to zero in time $T>0$ with controls $(f_i,g_i)\in[H^1(0,T)]^2$, $i=1,2$, if and only if*

$$\left\langle\begin{pmatrix}\eta^0\\ \omega^0\end{pmatrix},\begin{pmatrix}u(0)\\ v(0)\end{pmatrix}\right\rangle_{[H^{-2}(0,L)]^2,[H_0^2(0,L)]^2}$$
$$=\int_0^T f_1(t)[b_1u_{txxx}+cv_{xx}](t,L)dt+\int_0^T g_1(t)[d_1v_{txxx}+au_{xx}](t,L)dt$$

$$-b_1\int_0^T f_2(t)u_{txx}(t,L)dt - d_1\int_0^T g_2(t)v_{txx}(t,L)dt, \tag{4.2}$$

for any solution (u,v) *of the adjoint system*

$$\begin{cases} u_t+v_x+cv_{xxx}-bu_{txx}+b_1u_{txxxx}=0 & \text{for } x\in(0,L),\ t\in(0,T),\\ v_t+u_x+au_{xxx}-dv_{txx}+d_1v_{txxxx}=0 & \text{for } x\in(0,L), t\in(0,T),\\ u(t,0)=u(t,L)=v(t,0)=v(t,L)=0 & \text{for } \ t\in(0,T),\\ u_x(t,0)=u_x(t,L)=v_x(t,0)=v_x(t,L)=0 & \text{for } \ t\in(0,T),\\ u(T,x)=u^T(x);\quad v(T,x)=v^T(x) & \text{for } x\in(0,L), \end{cases} \tag{4.3}$$

with $(u^T,v^T)\in[H_0^2(0,L)]^2$.

Proof. Remark that the change of variables $t\to T-t$ and $x\to L-x$ reduces the system (4.3) to (1.3) with $f_i\equiv g_i\equiv 0$, for $i=1,2$. Then, we can apply to (u,v) the well-posedness results obtained in the previous section.

First, we prove the result for regular solutions. The less regular framework can be proved using density arguments as in the proof of Theorem 2.4. Let (η,ω) be a solution of (1.3) and (u,v) solution of (4.3). After some integrations by parts, we have

$$\begin{aligned} 0=&\int_0^T\int_0^L u\left(\eta_t+\omega_x+a\omega_{xxx}-b\eta_{txx}+b_1\eta_{txxxx}\right)dxdt\\ &+\int_0^T\int_0^L v\left(\omega_t+\eta_x+c\eta_{xxx}-d\omega_{txx}+d_1\omega_{txxxx}\right)dxdt\\ =&\int_0^L[u(T)\eta(T)-u(0)\eta(0)]\,dx+b\int_0^L[u_x(T)\eta_x(T)-u_x(0)\eta_x(0)]\,dx\\ &+b_1\int_0^L[u_{xx}(T)\eta_{xx}(T)-u_{xx}(0)\eta_{xx}(0)]\,dx\\ &+\int_0^L[v(T)\omega(T)-v(0)\omega(0)]\,dx+d\int_0^L[v_x(T)\omega_x(T)-v_x(0)\omega_x(0)]\,dx\\ &+d_1\int_0^L[v_{xx}(T)\omega_{xx}(T)-v_{xx}(0)\omega_{xx}(0)]\,dx\\ &+a\int_0^T u_{xx}(L)g_1dt-b_1\int_0^T u_{txx}(L)f_2dt+b_1\int_0^T u_{txxx}(L)f_1dt\\ &+c\int_0^T v_{xx}(L)f_1dt-d_1\int_0^T v_{txx}(L)g_2dt+d_1\int_0^T v_{txxx}(L)g_1dt. \end{aligned}$$

By using the density of $H_0^2(0,T)$ in $H^{-2}(0,T)$, we can pass the identity above to the limit to obtain

$$\begin{aligned} &\left\langle \begin{pmatrix}\eta^0\\ \omega^0\end{pmatrix},\begin{pmatrix}u(0)\\ v(0)\end{pmatrix}\right\rangle_{[H^{-2}(0,L)]^2,[H_0^2(0,L)]^2}=\left\langle \begin{pmatrix}\eta(T)\\ \omega(T)\end{pmatrix},\begin{pmatrix}u^T\\ v^T\end{pmatrix}\right\rangle_{[H^{-2}(0,L)]^2,[H_0^2(0,L)]^2}\\ &+\int_0^T f_1(t)[b_1u_{txxx}+cv_{xx}](t,L)dt+\int_0^T g_1(t)[d_1v_{txxx}+au_{xx}](t,L)dt\\ &-b_1\int_0^T f_2(t)u_{txx}(t,L)dt-d_1\int_0^T g_2(t)v_{txx}(t,L)dt. \end{aligned} \tag{4.4}$$

Hence, (η^0,ω^0) is controllable to zero in time $T>0$ if and only if (4.2) holds. $\square$

The next result is devoted to show that system (1.3) is not spectrally controllable. This means that no nontrivial finite linear combinations of eigenvectors of the operator $\mathcal{A}$ defined in (2.4) can be driven to zero in finite time by using controls $(f_i,g_i)\in[H^1(0,T)]^2$, $i=1,2$.

THEOREM 4.1. *No eigenfunctions of the operator $\mathcal{A}$ can be driven to zero in finite time.*

Proof. We first note that, according to Theorem 3.2, the operator $\mathcal{A}$ has a sequence of purely imaginary eigenvalues $(1/\mu_n^j)_{n\in\mathbb{Z}^*,j\in\{1,2\}}$. Moreover, the corresponding eigenfunctions $(\Phi_n^j)_{n\in\mathbb{Z}^*,j\in\{1,2\}}$ form an orthogonal basis of $[H_0^2(0,L)]^2$. For each $k\neq 0$, let us consider

$$(\eta_k^0,\omega_k^0)=\Phi_k^j=(\varphi_k^j,\nu_k^j),\quad j=1,2,$$

eigenfunctions of the operator $\mathcal{A}$. In a similar way, if we consider

$$\begin{pmatrix} u_n^T\\ v_n^T\end{pmatrix}=\begin{cases}\Phi_n^j & n\neq k\\ 0 & n=k,\end{cases}$$

the corresponding solution of (4.3) can be written as

$$\begin{pmatrix} u_n\\ v_n\end{pmatrix}=e^{i\lambda_n^j(T-t)}\Phi_n^j,\quad\text{where}\quad i\lambda_n^j=-\frac{1}{\mu_n^j},$$

with $1/\mu_n^j$, $(j=1,2)$ being the eigenvalues of the operator $\mathcal{A}$, given by Theorem 3.2. Moreover,

$$\lim_{|n|\to\infty}\lambda_n^j=0.$$

On the other hand, since the sequence $(\Phi_n^j)_{n\in\mathbb{Z}^*,j\in\{1,2\}}$ forms an orthonormal basis of $[H_0^2(0,L)]^2$, we get

$$\left\langle\begin{pmatrix}\eta_k^0\\ \omega_k^0\end{pmatrix},\begin{pmatrix}u_n(0)\\ v_n(0)\end{pmatrix}\right\rangle_{[H_0^2(0,L)]^2}=\delta_{n,k}^j e^{i\lambda_n^j T},\quad j=1,2.$$

Thus, if (η_k^0,ω_k^0) is controllable to zero in time $T>0$, from (4.2) it follows that

$$\int_0^T e^{i\lambda_n^j(T-t)}\Big[f_1(t)\left(-i\lambda_n^j b_1\varphi_{n,xxx}^j+a\nu_{n,xx}^j\right)(L)+g_1(t)\left(-i\lambda_n^j d_1\nu_{n,xxx}^j+a\varphi_{n,xx}^j\right)(L)$$
$$+b_1 f_2(t)i\lambda_n^j\varphi_{n,xx}^j(L)+d_1 g_2(t)i\lambda_n^j\nu_{n,xx}^j(L)\Big]dt=\delta_{n,k}^j e^{i\lambda_n^j T},\quad j=1,2.\tag{4.5}$$

For $j=1$, the identity above can be written as follows

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} h(t)e^{i\lambda_n^1\left(\frac{T}{2}-t\right)}dt=\delta_{n,k}^1 e^{i\lambda_n^1 T},\tag{4.6}$$

where

$$h(t)=f_1\left(t+\frac{T}{2}\right)\left(-i\lambda_n^1 b_1\varphi_{n,xxx}^1+a\nu_{n,xx}^1\right)(L)+g_1\left(t+\frac{T}{2}\right)\left(-i\lambda_n^1 d_1\nu_{n,xxx}^1+a\varphi_{n,xx}^1\right)(L)$$
$$+i\lambda_n^1 b_1 f_2\left(t+\frac{T}{2}\right)\varphi_{n,xx}^1(L)+i\lambda_n^1 d_1 g_2\left(t+\frac{T}{2}\right)\nu_{n,xx}^1(L).$$

Since $h\in L^2(-\frac{T}{2},\frac{T}{2})$, if we define $F:\mathbb{C}\longrightarrow\mathbb{C}$ by

$$F(z)=\int_{-\frac{T}{2}}^{\frac{T}{2}} h(t)e^{izt},$$

from the Paley-Wiener theorem, we have that F is an entire function. Moreover, since $\lim_{|n|\to\infty}\lambda_n^j=0$, it follows that F is zero on a set with a finite accumulation point. Then, $F\equiv 0$ and, consequently,

$$\begin{aligned} &f_1(t)\left(-i\lambda_n^1 b_1\varphi_{n,xxx}^1+a\nu_{n,xx}^1\right)(L)+g_1(t)\left(-i\lambda_n^1 d_1\nu_{n,xxx}^1+a\varphi_{n,xx}^1\right)(L)\\ &\qquad +b_1 f_2(t)i\lambda_n^1\varphi_{n,xx}^1(L)+d_1 g_2(t)i\lambda_n^1\nu_{n,xx}^1(L)=0, \end{aligned} \tag{4.7}$$

for all $t\in[0,T]$.

For $j=2$, we can use (4.5) and proceed in a similar way to obtain

$$\begin{aligned} &f_1(t)\left(-i\lambda_n^2 b_1\varphi_{n,xxx}^2+a\nu_{n,xx}^2\right)(L)+g_1(t)\left(-i\lambda_n^2 d_1\nu_{n,xxx}^2+a\varphi_{n,xx}^2\right)(L)\\ &\qquad +b_1 f_2(t)i\lambda_n^2\varphi_{n,xx}^2(L)+d_1 g_2(t)i\lambda_n^2\nu_{n,xx}^2(L)=0, \end{aligned} \tag{4.8}$$

for all $t\in[0,T]$.

Thus, by dividing (4.7) and (4.8) by $i\lambda_n^1$ and $i\lambda_n^2$, respectively, we deduce that (f_1,g_1) and (f_2,g_2) should satisfy the system

$$\begin{cases} f_1(t)A_n^1+g_1(t)B_n^1+f_2(t)C_n^1+g_2(t)D_n^1=0\\ f_1(t)A_n^2+g_1(t)B_n^2+f_2(t)C_n^2+g_2(t)D_n^2=0, \end{cases} \tag{4.9}$$

where

$$\begin{aligned} &A_n^j=\frac{a}{i\lambda_n^j}\nu_{n,xx}^j(L)-b_1\varphi_{n,xxx}^j(L),\quad B_n^j=\frac{a}{i\lambda_n^j}\varphi_{n,xx}^j(L)-d_1\nu_{n,xxx}^j(L),\\ &C_n^j=b_1\varphi_{n,xx}^j(L),\quad\text{and}\quad D_n^j=d_1\nu_{n,xx}^j(L),\quad \text{for } j=1,2. \end{aligned}$$

In order to conclude the proof, the following result will be needed:

Lemma 4.2. *For a subsequence, if necessary, the following holds:*

$$\lim_{|n|\to\infty} C_n^j=\lim_{|n|\to\infty} D_n^j=\lim_{|n|\to\infty} A_n^2=\lim_{|n|\to\infty} B_n^1=0,\quad j=1,2, \tag{4.10}$$

$$\lim_{|n|\to\infty} A_n^1=\lim_{|n|\to\infty} B_n^2=\delta_0\frac{\sqrt{b_1d_1}}{L},\quad \textit{for some}\quad \delta_0\in\mathbb{C}^*, \tag{4.11}$$

and

$$\begin{vmatrix} C_n^1 & D_n^1\\ C_n^2 & D_n^2\end{vmatrix}\sim\frac{-L^2b_1d_1}{[(2|n|+1)\pi-2\varepsilon_n]^2+4},\quad \textit{for all}\quad n\in\mathbb{Z}^*. \tag{4.12}$$

By using (4.10) and (4.11) in (4.9) we obtain

$$\begin{aligned} &f_1(t)A_n^1+g_1(t)B_n^1+f_2(t)C_n^1+g_2(t)D_n^1\to\delta_0\frac{\sqrt{b_1d_1}}{L}f_1(t)=0,\\ &f_1(t)A_n^2+g_1(t)B_n^2+f_2(t)C_n^2+g_2(t)D_n^2\to\delta_0\frac{\sqrt{b_1d_1}}{L}g_1(t)=0, \end{aligned}$$

as $|n|\to\infty$. Then, $(f_1,g_1)\equiv(0,0)$ and the system (4.9) becomes simpler:

$$\begin{cases} f_2(t)C_n^1+g_2(t)D_n^1=0 \\ f_2(t)C_n^2+g_2(t)D_n^2=0. \end{cases} \tag{4.13}$$

Hence, from (4.12) we deduce that $(f_1,g_1)\equiv(f_2,g_2)\equiv(0,0)$ is the unique solution of the system (4.9), which contradicts (4.5) and the proof ends. ∎

It remains to prove Lemma 4.2.

Proof. **(Proof of Lemma 4.2.)** We first consider the solutions of the following problems (-$\tilde{\mu}_n\mathcal{B}\Phi_n=\Phi_n$)

$$\begin{cases} -a\tilde{\mu}_n\tilde{\nu}^1_{n,xxx}-b_1\tilde{\varphi}^1_{n,xxxx}=0 & \text{for } x\in(0,L), \\ -a\tilde{\mu}_n\tilde{\varphi}^1_{n,xxx}-d_1\tilde{\nu}^1_{n,xxxx}=0 & \text{for } x\in(0,L), \\ (\tilde{\varphi}^1(0),\tilde{\nu}^1(0))=(0,0), \\ (\tilde{\varphi}^1_{n,x}(0),\tilde{\nu}^1_{n,x}(0))=(0,0), \\ (\tilde{\varphi}^1_{n,xx}(0),\tilde{\nu}^1_{n,xx}(0))=(\tilde{\beta}^1_{1,n},\tilde{\beta}^1_{2,n}), \\ (\tilde{\varphi}^1_{n,xxx}(0),\tilde{\nu}^1_{n,xxx}(0))=(1,0), \end{cases} \tag{4.14}$$

and

$$\begin{cases} -a\tilde{\mu}_n\tilde{\nu}^2_{n,xxx}-b_1\tilde{\varphi}^2_{n,xxxx}=0 & \text{for } x\in(0,L), \\ -a\tilde{\mu}_n\tilde{\varphi}^2_{n,xxx}-d_1\tilde{\nu}^2_{n,xxxx}=0 & \text{for } x\in(0,L), \\ (\tilde{\varphi}^2(0),\tilde{\nu}^2(0))=(0,0), \\ (\tilde{\varphi}^2_{n,x}(0),\tilde{\nu}^2_{n,x}(0))=(0,0), \\ (\tilde{\varphi}^2_{n,xx}(0),\tilde{\nu}^2_{n,xx}(0))=(\tilde{\beta}^2_{1,n},\tilde{\beta}^2_{2,n}), \\ (\tilde{\varphi}^2_{n,xxx}(0),\tilde{\nu}^2_{n,xxx}(0))=(0,1). \end{cases} \tag{4.15}$$

For each $\tilde{\mu}_n=-sgn(n)\frac{\sqrt{b_1d_1}}{aL}((2|n|+1)\pi-2\varepsilon_n)i$ $(n\in\mathbb{Z}^*,\varepsilon_n\in(0,1))$, $(\tilde{\beta}^1_{1,n},\tilde{\beta}^1_{2,n})$ given by

$$\tilde{\beta}^1_{1,n}=\frac{2b_1d_1L}{(aL\tilde{\mu}_n)^2-4b_1d_1},\quad \tilde{\beta}^1_{2,n}=-\frac{b_1aL^2\tilde{\mu}_n}{(aL\tilde{\mu}_n)^2-4b_1d_1} \tag{4.16}$$

and $(\tilde{\beta}^2_{1,n},\tilde{\beta}^2_{2,n})$ such that

$$\tilde{\beta}^2_{1,n}=-\frac{d_1aL^2\tilde{\mu}_n}{(aL\tilde{\mu}_n)^2-4b_1d_1},\quad \tilde{\beta}^2_{2,n}=\frac{2b_1d_1L}{(aL\tilde{\mu}_n)^2-4b_1d_1}, \tag{4.17}$$

the solutions of (4.14) and (4.15) are given by formula (3.19) (replacing $\tilde{\mu}_n$ by $-\tilde{\mu}_n$) and will be denoted by

$$\tilde{\Phi}^1_n=\begin{pmatrix}\tilde{\varphi}^1_n\\ \tilde{\nu}^1_n\end{pmatrix} \quad\text{and}\quad \tilde{\Phi}^2_n=\begin{pmatrix}\tilde{\varphi}^2_n\\ \tilde{\nu}^2_n\end{pmatrix}, \tag{4.18}$$

respectively. We set $\kappa_n = -\frac{a\tilde{\mu}_n}{\sqrt{b_1 d_1}}$. Then, from Theorem 3.1, we get $\mathcal{S}(\kappa_n, L) = \mathcal{C}(\kappa_n, L) = 0$, which implies that

$$\begin{aligned}\sinh(\kappa_n L) &= \kappa_n L - \frac{[\kappa_n L]^3}{[\kappa_n L]^2 - 4} = -\frac{4[\kappa_n L]}{[\kappa_n L]^2 - 4},\\ \cosh(\kappa_n L) - 1 &= -\frac{2[\kappa_n L]^2}{[\kappa_n L]^2 - 4}.\end{aligned}$$

Then,

$$\begin{aligned}\mathcal{S}_{xx}(\kappa_n, L) &= \kappa_n^2\left[\sinh(\kappa_n L) + \frac{2\kappa_n L}{[\kappa_n L]^2 - 4}\right] = -\kappa_n^3\left[\frac{2L}{[\kappa_n L]^2 - 4}\right],\\ \mathcal{C}_{xx}(\kappa_n, L) &= \kappa_n^2\left[(\cosh(\kappa_n L) - 1) + \frac{[\kappa_n L]^2}{[\kappa_n L]^2 - 4}\right] = -\kappa_n^3\left[\frac{\kappa_n L^2}{[\kappa_n L]^2 - 4}\right].\end{aligned}$$

Consequently, the functions $\tilde{\Phi}_n^j$, $j = 1, 2$, satisfy

$$\tilde{\Phi}^1_{n,xx}(L) = \begin{pmatrix}\tilde{\varphi}^1_{n,xx}(L)\\ \tilde{\nu}^1_{n,xx}(L)\end{pmatrix} = \frac{-L}{[\kappa_n L]^2 - 4}\begin{pmatrix}2\\ \kappa_n L\sqrt{\frac{b_1}{d_1}}\end{pmatrix} \tag{4.19}$$

and

$$\tilde{\Phi}^2_{n,xx}(L) = \begin{pmatrix}\tilde{\varphi}^2_{n,xx}(L)\\ \tilde{\nu}^2_{n,xx}(L)\end{pmatrix} = \frac{-L}{[\kappa_n L]^2 - 4}\begin{pmatrix}\kappa_n L\sqrt{\frac{d_1}{b_1}}\\ 2\end{pmatrix}, \tag{4.20}$$

for $n \in \mathbb{N}^*$. Now, we pass to the study of the asymptotic behavior of the eigenvectors of the operator $-\mathcal{A}$. From the proof of Theorem 3.2 we have that, for each eigenvalue $-1/\mu_n^j$, the corresponding eigenfunctions $\Phi_n^1 = (\varphi_n^1, \nu_n^1)$ and $\Phi_n^2 = (\varphi_n^2, \nu_n^2)$ are solutions of

$$\begin{cases}-a\mu_n^1\nu^1_{n,xxx} - b_1\varphi^1_{n,xxxx} = \varphi_n^1 + \mu_n^1\nu^1_{n,x} - b\varphi^1_{n,xx} & \text{for } x \in (0, L),\\ -a\mu_n^1\varphi^1_{n,xxx} - d_1\nu^1_{n,xxxx} = \nu_n^1 + \mu_n^1\varphi^1_{n,x} - d\nu^1_{n,xx} & \text{for } x \in (0, L),\\ (\varphi^1(0), \nu^1(0)) = (0, 0),\\ (\varphi^1_{n,x}(0), \nu^1_{n,x}(0)) = (0, 0),\\ (\varphi^1_{n,xx}(0), \nu^1_{n,xx}(0)) = (\beta^1_{1,n}, \beta^1_{2,n}),\\ (\varphi^1_{n,xxx}(0), \nu^1_{n,xxx}(0)) = (1, \gamma_n^1),\end{cases} \tag{4.21}$$

and

$$\begin{cases}-a\mu_n^2\nu^2_{n,xxx} - b_1\varphi^2_{n,xxxx} = \varphi_n^2 + \mu_n^2\nu^2_{n,x} - b\varphi^2_{n,xx} & \text{for } x \in (0, L),\\ -a\mu_n^2\varphi^2_{n,xxx} - d_1\nu^2_{n,xxxx} = \nu_n^2 + \mu_n^2\varphi^2_{n,x} - d\nu^2_{n,xx} & \text{for } x \in (0, L),\\ (\varphi^2(0), \nu^2(0)) = (0, 0),\\ (\varphi^2_{n,x}(0), \nu^2_{n,x}(0)) = (0, 0),\\ (\varphi^2_{n,xx}(0), \nu^2_{n,xx}(0)) = (\beta^2_{1,n}, \beta^2_{2,n}),\\ (\varphi^2_{n,xxx}(0), \nu^2_{n,xxx}(0)) = (\gamma_n^2, 1),\end{cases} \tag{4.22}$$

respectively. We also note that, according to Theorem 3.2, the data in (4.21) and (4.22) satisfies

$$|\gamma_n^j| \leq \frac{\delta}{|n|}, \quad |\mu_n^j - \tilde{\mu}_n| \leq \frac{\delta}{|n|}.$$

Since $|\beta_n^j - \tilde{\beta}_n^j| \to 0$, as $|n| \to \infty$, for $j = 1,2$, we can extract a subsequence, if necessary, such that

$$\begin{aligned} &|\beta_{1,n}^1 - \tilde{\beta}_{1,n}^1| \leq \frac{\delta}{|n|^2}, \quad |\beta_{2,n}^1 - \tilde{\beta}_{2,n}^1| \leq \frac{\delta}{|n|}, \\ &|\beta_{1,n}^2 - \tilde{\beta}_{1,n}^2| \leq \frac{\delta}{|n|}, \quad |\beta_{2,n}^2 - \tilde{\beta}_{2,n}^2| \leq \frac{\delta}{|n|^2}, \end{aligned} \tag{4.23}$$

for a given positive δ. Therefore, from Proposition 3.3, the eigenfunction (φ_n^1, ν_n^1) satisfies

$$\begin{aligned} &|\varphi_{n,xx}^1(L) - \tilde{\varphi}_{n,xx}^1(L)| + |\nu_{n,xx}^1(L) - \tilde{\nu}_{n,xx}^1(L)| \\ \leq &C\left[\left(|\beta_{1,n}^1 - \tilde{\beta}_{1,n}^1| + |\beta_{2,n}^1 - \tilde{\beta}_{2,n}^1|\right) + \frac{1}{|\mu_n^1|}\left(|\gamma_n^1| + |\mu_n^1 - \tilde{\mu}_n|(1 + |\gamma_n^1|)\right)\right] \\ \leq &C\left[\left(\frac{\delta}{|n|^2} + \frac{\delta}{|n|}\right) + \frac{1}{|\mu_n^1|}\left(\frac{\delta}{|n|} + \frac{\delta}{|n|}(1 + \frac{\delta}{|n|})\right)\right]. \end{aligned}$$

Similarly, the eigenfunction (φ_n^2, ν_n^2) satisfies

$$\begin{aligned} &|\varphi_{n,xx}^2(L) - \tilde{\varphi}_{n,xx}^2(L)| + |\nu_{n,xx}^2(L) - \tilde{\nu}_{n,xx}^2(L)| \\ \leq &C\left[\left(\frac{\delta}{|n|^2} + \frac{\delta}{|n|}\right) + \frac{1}{|\mu_n^2|}\left(\frac{\delta}{|n|} + \frac{\delta}{|n|}(1 + \frac{\delta}{|n|})\right)\right]. \end{aligned}$$

From the estimates above and (4.19)-(4.20), we conclude that

$$\begin{pmatrix} C_n^1 \\ D_n^1 \end{pmatrix} = \begin{pmatrix} b_1\varphi_{n,xx}^1(L) \\ d_1\nu_{n,xx}^1(L) \end{pmatrix} \sim \frac{-L}{[\kappa_n L]^2 - 4}\begin{pmatrix} 2b_1 \\ \kappa_n L\sqrt{b_1 d_1} \end{pmatrix} \tag{4.24}$$

and

$$\begin{pmatrix} C_n^2 \\ D_n^2 \end{pmatrix} = \begin{pmatrix} b_1\varphi_{n,xx}^2(L) \\ d_1\nu_{n,xx}^2(L) \end{pmatrix} \sim \frac{-L}{[\kappa_n L]^2 - 4}\begin{pmatrix} \kappa_n L\sqrt{b_1 d_1} \\ 2d_1 \end{pmatrix}. \tag{4.25}$$

Thus,

$$\begin{vmatrix} C_n^1 & D_n^1 \\ C_n^2 & D_n^2 \end{vmatrix} \sim \frac{L^2 b_1 d_1}{[\kappa_n L]^2 - 4} \neq 0,$$

which gives the behavior of the coefficients C_n^j and D_n^j, for $j = 1,2$.

On the other hand, by integrating the equations in (4.21) over $(0,L)$ we obtain the coefficients A_n^1 and B_n^1:

$$\begin{aligned} A_n^1 &= \left(-a\mu_n^1\nu_{n,xx}^1 - b_1\varphi_{n,xxx}^1\right)(L) = \int_0^L \varphi_n^1(x)dx - a\mu_n^1\beta_{2,n}^1 - b_1, \\ B_n^1 &= \left(-a\mu_n^1\varphi_{n,xx}^1 - d_1\nu_{n,xxx}^1\right)(L) = \int_0^L \nu_n^1(x)dx - a\mu_n^1\beta_{1,n}^1 - d_1\gamma_n^1. \end{aligned} \tag{4.26}$$

The next steps are devoted to study the term on the right-hand side of the equations in (4.26). First, we note that, from Theorem 3.2,

$$\int_0^L \varphi_n^1(x)dx = \int_0^L \tilde{\varphi}_n^1(x)dx + \mathcal{O}\left(\frac{1}{|n|}\right),$$
$$\int_0^L \nu_n^1(x)dx = \int_0^L \tilde{\nu}_n^1(x)dx + \mathcal{O}\left(\frac{1}{|n|}\right).$$

Then, from formula (3.19) we conclude that

$$\lim_{|n|\to\infty} \int_0^L \varphi_n^1(x)dx = \lim_{|n|\to\infty} \int_0^L \nu_n^1(x)dx = 0. \tag{4.27}$$

On the other hand, from (4.23) we get

$$\begin{aligned} a\mu_n^1\beta_{1,n}^1 &= a\left(\tilde{\mu}_n^1 + \mathcal{O}\left(\frac{1}{|n|}\right)\right)\left(\tilde{\beta}_{1,n}^1 + \mathcal{O}\left(\frac{1}{|n|^2}\right)\right) \\ &= a\tilde{\mu}_n^1\tilde{\beta}_{1,n}^1 + a\tilde{\mu}_n^1\mathcal{O}\left(\frac{1}{|n|^2}\right) + a\tilde{\beta}_{1,n}^1\mathcal{O}\left(\frac{1}{|n|}\right) + a\mathcal{O}\left(\frac{1}{|n|^3}\right), \end{aligned} \tag{4.28}$$

and

$$\begin{aligned} a\mu_n^1\beta_{2,n}^1 &= a\left(\tilde{\mu}_n^1 + \mathcal{O}\left(\frac{1}{|n|}\right)\right)\left(\tilde{\beta}_{2,n}^1 + \mathcal{O}\left(\frac{1}{|n|}\right)\right) \\ &= a\tilde{\mu}_n^1\tilde{\beta}_{2,n}^1 + a\tilde{\mu}_n^1\mathcal{O}\left(\frac{1}{|n|}\right) + a\tilde{\beta}_{2,n}^1\mathcal{O}\left(\frac{1}{|n|}\right) + a\mathcal{O}\left(\frac{1}{|n|^2}\right). \end{aligned} \tag{4.29}$$

From (4.16), we note that the right side of (4.28) tends to 0 as $|n|\to\infty$, the last two terms on the right side of (4.29) tend to 0 as $|n|\to\infty$ and, finally, the first two terms in (4.29) satisfy

$$\lim_{|n|\to\infty} a\tilde{\mu}_n^1\tilde{\beta}_{2,n}^1 = -b_1 \quad \text{and} \quad \lim_{|n|\to\infty} a\tilde{\mu}_n^1\mathcal{O}\left(\frac{1}{|n|}\right) = \delta_0\frac{\sqrt{b_1 d_1}}{L},$$

for some $\delta_0 \in \mathbb{C}^*$. Then, from (4.27), (4.28) and (4.29), we conclude that

$$\lim_{|n|\to\infty} A_n^1 = \delta_0\frac{\sqrt{b_1 d_1}}{L} \quad \text{and} \quad \lim_{|n|\to\infty} B_n^1 = 0. \tag{4.30}$$

In order to conclude the proof, we integrate the equations in (4.22) over $(0,L)$ to obtain

$$A_n^2 = \left(-a\mu_n^2\nu_{n,xx}^2 - b_1\varphi_{n,xxx}^2\right)(L) = \int_0^L \varphi_n^2(x)dx - a\mu_n^2\beta_{2,n}^2 - b_1\gamma_n^2,$$
$$B_n^2 = \left(-a\mu_n^2\varphi_{n,xx}^2 - d_1\nu_{n,xxx}^2\right)(L) = \int_0^L \nu_n^2(x)dx - a\mu_n^2\beta_{1,n}^2 - d_1.$$

Then, by arguing as in the previous steps, we deduce that

$$\lim_{|n|\to\infty} A_n^2 = 0 \quad \text{and} \quad \lim_{|n|\to\infty} B_n^2 = \delta_0\frac{\sqrt{b_1 d_1}}{L}.$$

□

REMARK 4.1. $\lambda=0$ is not an eigenvalue of the operator $\mathcal{A}$. Indeed, if (φ,ν) satisfies $\mathcal{A}(\varphi,\nu)=0$, then, it shall be the solution of the uncoupled system

$$\begin{cases} \nu_x + a\nu_{xxx} = 0 & \text{for } x\in(0,L), \\ \varphi_x + a\varphi_{xxx} = 0 & \text{for } x\in(0,L), \\ (\varphi(0),\nu(0)) = (\varphi(L),\nu(L)) = (0,0), \\ (\varphi_x(0),\nu_x(0)) = (\varphi_x(L),\nu_x(L)) = (0,0). \end{cases} \tag{4.31}$$

By setting $\tilde{\nu}=\nu_x$ we obtain $\tilde{\nu}(x)=c_1 e^{\frac{i}{\sqrt{a}}x}+c_2 e^{-\frac{i}{\sqrt{a}}x}$, for some constants c_1,c_2. Then, from the boundary condition $\tilde{\nu}(0)=0$, we deduce that $\tilde{\nu}(x)=2ic_1\sin\left(\frac{x}{\sqrt{a}}\right)$ and the boundary condition $\tilde{\nu}(L)=0$ implies that $2ic_1\sin\left(\frac{L}{\sqrt{a}}\right)=0$. Thus, if $L\neq\sqrt{a}\pi n$, with $n\in\mathbb{Z}^*$, we have that $c_1=0$ and $\nu\equiv const$. Then, from the boundary condition $\nu(0)=0$ we conclude that $\nu\equiv 0$. On the other hand, if $L=\sqrt{a}\pi n$, for some $n\in\mathbb{Z}^*$, we have that $\nu(x)=-2i\sqrt{a}c_1\cos\left(\frac{x}{\sqrt{a}}\right)$ and the condition $\nu(L)=0$ implies that $c_1=0$. Hence, $\nu\equiv 0$. Since the system is uncoupled, we can argue as above to obtain $\varphi\equiv 0$.

Now, we pass to study the approximate controllability of the system (1.3). In order to do that, we introduce the following definition.

DEFINITION 4.1. *System (1.3) is said to be approximately controllable in time $T>0$ if, for every initial data $(\eta^0,\omega^0)\in[H^{-2}(0,L)]^2$, the set of reachable states*

$$R\left(\begin{pmatrix}\eta^0\\ \omega^0\end{pmatrix},T\right)=\left\{\begin{pmatrix}\eta(T,x)\\ \omega(T,x)\end{pmatrix} : \left(\begin{pmatrix}f_1\\ g_1\end{pmatrix},\begin{pmatrix}f_2\\ g_2\end{pmatrix}\right)\in[H^1(0,T)]^2\times[H^1(0,T)]^2\right\}$$

is dense in $[L^2(0,L)]^2$.

The corresponding approximate controllability result reads as follows.

THEOREM 4.2. *System (1.3) is approximately controllable in time $T>0$ with controls and $(f_i,g_i)\in[H^1(0,T)]^2$, $i=1,2$.*

Proof. Due to the linearity of the system (1.3), it is sufficient to prove the result for any $T>0$ and $(\eta^0,\omega^0)=(0,0)$. Thus, we will prove the density of the set $R\left(\begin{pmatrix}0\\0\end{pmatrix},T\right)$ in $[L^2(0,L)]^2$.

Let $(\eta,\omega)\in C([0,T];[L^2(0,L)]^2)$ be the corresponding solution of (1.3) given by Theorem 2.4 and (u,v) be the solution of the adjoint system (4.3). Then, it follows that

$$\begin{aligned} &\left\langle \begin{pmatrix}\eta(T,x)\\ \omega(T,x)\end{pmatrix},\begin{pmatrix}u^T\\ v^T\end{pmatrix}\right\rangle_{[H^{-2}(0,L)]^2,[H_0^2(0,L)]^2} \\ &= -\int_0^T f_1(t)[b_1 u_{txxx}+av_{xx}](t,L)dt - \int_0^T g_1(t)[d_1 v_{txxx}+au_{xx}](t,L)dt \\ &\quad + b_1\int_0^T f_2(t)u_{txx}(t,L)dt + d_1\int_0^T g_2(t)v_{txx}(t,L)dt. \end{aligned} \tag{4.32}$$

Assume that $R\left(\begin{pmatrix}0\\0\end{pmatrix},T\right)$ is not dense in $[H_0^2(0,L)]^2$. In this case, there exists

$(u^T, v^T) \neq (0,0)$ in $[H_0^2(0,L)]^2$, satisfying

$$\left\langle \begin{pmatrix} \eta(T,x) \\ \omega(T,x) \end{pmatrix}, \begin{pmatrix} u^T \\ v^T \end{pmatrix} \right\rangle_{[H^{-2}(0,L)]^2, [H_0^2(0,L)]^2} = 0,$$

for all $(f_i, g_i) \in [H^1(0,T)]^2$, $i = 1,2$. Consequently, from (4.32) we obtain

$$-\left\langle \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} [b_1 u_{txxx} + a v_{xx}](t,L) \\ [d_1 v_{txxx} + a u_{xx}](t,L) \end{pmatrix} \right\rangle_{[L^2(0,T)]^2} + \left\langle \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}, \begin{pmatrix} b_1 u_{txx}(t,L) \\ d_1 v_{txx}(t,L) \end{pmatrix} \right\rangle_{[L^2(0,T)]^2} = 0,$$

for all $(f_i, g_i) \in [H^1(0,T)]^2$, $i = 1,2$. Thus,

$$\begin{pmatrix} [b_1 u_{txxx} + a v_{xx}](t,L) \\ [d_1 v_{txxx} + a u_{xx}](t,L) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_1 u_{txx}(t,L) \\ d_1 v_{txx}(t,L) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \forall t \in (0,T). \tag{4.33}$$

Next, we want to write (4.33) as an infinite sum. From the proof of Theorem 2.1 we know that $\mathcal{A}$ is a skew adjoint operator in $[H_0^2(0,L)]^2$. Hence, it has a sequence of eigenvalues $(i\lambda_n)_{n\in\mathbb{Z}^*} \subset i\mathbb{R}$, each $i\lambda_n = (\mu_n)^{-1}$ with geometric multiplicity at most $\mathcal{M}_n$. The corresponding eigenfunctions form an orthonormal basis for $[H_0^2(0,L)]^2$, which we denote by

$$\bigcup_{n\in\mathbb{Z}^*} \{\Phi_n^k\}_{k=1}^{\mathcal{M}_n} . \{\Phi_n^1, \ldots, \Phi_n^{\mathcal{M}_n}\}.$$

Then, if $(u^T, v^T) \in [H_0^2(0,L)]^2$, we have

$$(u^T, v^T) = \sum_{n\in\mathbb{Z}^*} \sum_{k=1}^{\mathcal{M}_n} \alpha_n^k \Phi_n^k \left(\alpha_n^1 \Phi_n^1 + \ldots + \alpha_n^{\mathcal{M}_n} \Phi_n^{\mathcal{M}_n}\right)$$

and the corresponding solution (u,v) can be written as

$$(u,v) = \sum_{n\in\mathbb{Z}^*} \sum_{k=1}^{\mathcal{M}_n} \alpha_n^k \Phi_n^k e^{i\lambda_n(T-t)} . \left(\alpha_n^1 \Phi_n^1 + \ldots + \alpha_n^{\mathcal{M}_n} \Phi_n^{\mathcal{M}_n}\right) e^{i\lambda_n(T-t)}. \tag{4.34}$$

Thus, from (4.33) and (4.34), it follows that

$$0 = u_{txx}(t,L) = \sum_{n\in\mathbb{Z}^*} -i\lambda_n \sum_{k=1}^{\mathcal{M}_n} \alpha_n^k \varphi_{n,xx}^k(L) e^{i\lambda_n(T-t)}.$$

Since (u,v) is analytic in time (see Theorem 2.1), we can integrate the identity above over $(-S,S)$, for any $S > 0$. Then, for each $m \in \mathbb{Z}^*$, we deduce that

$$0 = \lim_{s\to+\infty} \frac{1}{S} \int_{-S}^{S} u_{txx}(s,L) e^{i\lambda_m s} ds = -i\lambda_m \sum_{k=1}^{\mathcal{M}_m} \alpha_m^k \varphi_{m,xx}^k(L) e^{i\lambda_m T},$$

hence,

$$\sum_{k=1}^{\mathcal{M}_m} \alpha_m^k \varphi_{m,xx}^k(L) = 0. \tag{4.35}$$

Analogously, from $v_{txx}(t,L)=0$, it results that

$$\sum_{k=1}^{\mathcal{M}_m} \alpha_m^k \nu_{m,xx}^k(L)=0. \tag{4.36}$$

On the other hand, from (4.33)-(4.34) we have

$$0=[b_1u_{txxx}+av_{xx}](t,L)=\sum_{n\in\mathbb{Z}^*}\sum_{k=1}^{\mathcal{M}_n}\alpha_n^k\left[-i\lambda_n b_1\varphi_{n,xxx}^k(L)+a\nu_{n,xx}^k(L)\right]e^{i\lambda_n(T-t)}$$

and

$$0=[d_1v_{txxx}+au_{xx}](t,L)=\sum_{n\in\mathbb{Z}^*}\sum_{k=1}^{\mathcal{M}_n}\alpha_n^k\left[-i\lambda_n d_1\nu_{n,xxx}^k(L)+a\varphi_{n,xx}^k(L)\right]e^{i\lambda_n(T-t)}.$$

Next, we proceed as before and use (4.35) and (4.36) to obtain

$$\begin{aligned}0&=\sum_{k=1}^{\mathcal{M}_m}\alpha_m^k[-i\lambda_m b_1\varphi_{m,xxx}^k(L)+a\nu_{m,xx}^k(L)]e^{i\lambda_mT}\\&=\left[-i\lambda_m b_1\sum_{k=1}^{\mathcal{M}_m}\alpha_m^k\varphi_{m,xxx}^k(L)+a\sum_{k=1}^{\mathcal{M}_m}\alpha_m^k\nu_{m,xx}^k(L)\right]e^{i\lambda_mT}\\&=-i\lambda_m b_1\sum_{k=1}^{\mathcal{M}_m}\alpha_m^k\varphi_{m,xxx}^k(L)e^{i\lambda_mT}\end{aligned}$$

and

$$\begin{aligned}0&=\sum_{k=1}^{\mathcal{M}_m}\alpha_m^k[-i\lambda_m d_1\nu_{m,xxx}^k(L)+a\varphi_{m,xx}^k(L)]e^{i\lambda_mT}\\&=\left[-i\lambda_m d_1\sum_{k=1}^{\mathcal{M}_m}\alpha_m^k\nu_{m,xxx}^k(L)+a\sum_{k=1}^{\mathcal{M}_m}\alpha_m^k\varphi_{m,xx}^k(L)\right]e^{i\lambda_mT}\\&=-i\lambda_m d_1\sum_{k=1}^{\mathcal{M}_m}\alpha_m^k\nu_{m,xxx}^k(L)e^{i\lambda_mT},\end{aligned}$$

respectively. Then,

$$\sum_{k=1}^{\mathcal{M}_m}\alpha_m^k\varphi_{m,xxx}^k(L)=\sum_{k=1}^{\mathcal{M}_m}\alpha_m^k\nu_{m,xxx}^k(L)=0. \tag{4.37}$$

Now, for each $m\in\mathbb{Z}^*$, we consider $\Phi_m=(\varphi_m,\nu_m)$ defined as follows

$$\Phi^m=\alpha_m^1\Phi_m^1+\ldots+\alpha_m^{\mathcal{M}_m}\Phi_m^{\mathcal{M}_m}.$$

Thus, from (4.35), (4.36) and (4.37) we have that

$$(\varphi_{m,xx}(L),\nu_{m,xx}(L))=(\varphi_{m,xxx}(L),\nu_{m,xxx}(L))=(0,0)$$

and $\Phi_m = (\varphi_m, \nu_m)$ solves the initial value problem

$$\begin{cases} -\varphi_m + b\varphi_{m,xx} - b_1\varphi_{m,xxxx} + (i\lambda_m)^{-1}\nu_{m,x} + a(i\lambda_m)^{-1}\nu_{m,xxx} = 0 & \text{for } x \in (0,L), \\ -\nu_m + d\nu_{m,xx} - d_1\nu_{m,xxxx} + (i\lambda_m)^{-1}\varphi_{m,x} + a(i\lambda_m)^{-1}\varphi_{m,xxx} = 0 & \text{for } x \in (0,L), \\ (\varphi_m(L), \nu_m(L)) = (0,0), \\ (\varphi_{m,x}(L), \nu_{m,x}(L)) = (0,0), \\ (\varphi_{m,xx}(L), \nu_{m,xx}(L)) = (0,0), \\ (\varphi_{m,xxx}(L), \nu_{m,xxx}(L)) = (0,0). \end{cases}$$

Then, by uniqueness,

$$\Phi_m = \alpha_m^1 \Phi_m^1 + \ldots + \alpha_m^{\mathcal{M}_m} \Phi_m^{\mathcal{M}_m} = (0,0).$$

Since $\{\Phi_m^k\}_{k=1}^{\mathcal{M}_m}$ are linearly independent, it follows that

$$\alpha_m^1 = \ldots = \alpha_m^{\mathcal{M}_m} = 0 \quad \text{for all} \quad m \in \mathbb{Z}^*.$$

Thus, from (4.34) it follows that $(u,v) = (0,0)$ and, in particular, $(u^T, v^T) = (0,0)$. This is a contradiction and the proof ends. □

5. Comments and open problems

We close this paper with some comments and open problems:

- The conditions on the coefficients of the highest order BBM terms ($b_1 > 0$ and $d_1 > 0$) provide a regularizing effect, which is very useful for the well-posedness of the system (1.3). On the other hand, the absence of the coefficients of the highest order KdV terms ($a_1 = c_1 = 0$) is an impediment for the controllability properties to hold. Indeed, from the controllability point of view, KdV type models are known to have a much better behavior (see, for instance, [15, 22]). Therefore, it is an interesting issue to study what can be done in the presence of the highest KdV terms ($a_1 > 0$ and $c_1 > 0$), including the full system (1.1).
- In the spirit of the problem mention above, the controllability issue also remains open when $b_1 = d_1 = 0$ and $a_1, c_1 > 0$, i.e., in the absence of the highest BBM terms. The KdV terms should provide good controllability properties, but in order to study the resulting nonlinear system, more regularity of the solutions is needed.
  As far as we know, the boundary controllability problem was only studied in [8] for the *abcd* system (1.5) when $b = d = 0$, i.e., for the lower-order KdV-KdV system. In what concerns the higher-order KdV model, only the boundary stabilization problem was addressed [7]. However, it is natural to expect that boundary conditions similar to those introduced in [7] also lead to positive exact boundary controllability results, at least for the corresponding linearized system.
- The spectral analysis developed in the previous sections also leads to the study of the stabilization problem when the time t is sufficiently large. By considering homogeneous Dirichlet boundary conditions and a damping term acting in one equation of (1.3), the asymptotic behavior of the energy associated to the model can be studied. Indeed, proceeding as in Section 3, a similar spectral analysis can be developed to construct a Riesz basis of $[H_0^2(0,L)]^2$ consisting

of generalized eigenvalues of the corresponding differential operator. Then, by using arguments similar to those developed in [17], we can conclude that $||(\eta(\cdot,x),\omega(\cdot,x))||_{[H_0^2(0,L)]^2} \to 0$, as $t \to \infty$.

- The program of this work was carried out for a particular choice of boundary control inputs and establishes as a fact that system (1.3) inherits some interesting properties initially observed for the BBM equation. Considerations of this issue for dispersive equations has received considerable attention, specially the problems related to the study of the controllability properties. However, the proof of a general result of lack of spectral controllability for some appropriate evolution operators associated with a compact operator for the part involving derivatives in space is a more difficult task which remains open.

Acknowledgements. We thank the anonymous referees for their valuable comments and remarks which have been taken into account in this revised version of the manuscript. The first author was supported by Capes and CNPq (Brazil). The second author was partially supported by CNPq (Brazil).

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