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INITIAL MIXED-BOUNDARY VALUE PROBLEM FOR ANISOTROPIC FRACTIONAL DEGENERATE PARABOLIC EQUATIONS*

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Abstract. We consider an initial mixed-boundary value problem for anisotropic fractional type degenerate parabolic equations posed in bounded domains. Namely, we consider that the boundary of the domain splits into two parts. In one of them, we impose a Dirichlet boundary condition and in the other part a Neumann condition. Under this mixed-boundary condition, we show the existence of solutions for measurable and bounded non-negative initial data. The nonlocal anisotropic diffusion effect relies on an inverse of a $s-$fractional type elliptic operator, and the solvability is proved for any $s\in(0,1)$.

Keywords. Fractional elliptic operator; initial mixed-boundary value problem; Dirichlet-Neumann homogeneous boundary condition; anisotropic problem.

AMS subject classifications. 35D30; 35K55; 35K61; 35K65.

1. Introduction

We are concerned in this paper with an initial mixed-boundary value problem for a class of anisotropic fractional type degenerate parabolic equations. To this end, let $\Omega\subset\mathbb{R}^n$ be a bounded open set with smooth (C^2) boundary Γ, and denote by ν the outward unit normal vector field on it. We assume that Γ is divided into two parts Γ_0, Γ_1. Then, we consider the following initial mixed-boundary value problem

$$\begin{cases} \partial_t u+\operatorname{div}\mathbf{q}=0 & \text{in } \Omega_T, \\ u|_{\{t=0\}}=u_0 & \text{in } \Omega, \\ u=0 & \text{on } (0,T)\times\Gamma_0, \\ \mathbf{q}\cdot\nu=0 & \text{on } (0,T)\times\Gamma_1, \end{cases} \tag{1.1}$$

where $\Omega_T=(0,T)\times\Omega$, for any real number $T>0$, $u(t,x)$ is a real function, which could be interpreted as a density (concentration, population, etc.) or the thermodynamic temperature, $\mathbf{q}=-u\ A(x)\nabla\mathcal{K}_s u$ is the diffusive fractional flux, and $\mathcal{K}_s$ is the inverse of the s-fractional elliptic operator $\mathcal{L}^s_{\mathcal{B}}$, $(0<s<1)$, see Section 2. The matrix $A(x)=(a_{ij}(x))_{n\times n}$ is assumed symmetric and satisfies

$$a_{ij}\in C(\overline{\Omega})\cap C^{0,1}_{\text{loc}}(\Omega), \quad (i,j=1,\dots,n), \tag{1.2}$$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j\geq\Lambda_1|\xi|^2, \tag{1.3}$$

for all $\xi\in\mathbb{R}^n$ and each $x\in\Omega$, for some ellipticity constant $\Lambda_1>0$. Moreover, the initial data $u_0\in L^\infty(\Omega)$ is a non-negative given function, and we consider homogeneous Dirichlet and Neumann boundary conditions, respectively on Γ_0, Γ_1. This assumption,

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the mixed-boundary condition, brings some difficulties which are discussed through this paper, see for instance, Section 3.1.

The diffusive non-local flux $\mathbf{q}$ in the initial mixed-boundary value problem (1.1) is motivated by the so-called General Fractional Fick's law

$$\mathbf{q}(x,u) := -\kappa(x,u)\ \nabla \mathcal{F} u$$

provided $\kappa(\cdot,u)$ is positive (non-negative in general) defined, where $\mathcal{F}$ is the inverse of a fractional elliptic operator. The first attempt is to consider

$$\mathbf{q}(x,u) := -g(u) A(x) \nabla \mathcal{K}_s u$$

with $g(u)=u$ or $g(u)=u(1-u)$, which from the maximum principle ensures that, $\kappa(\cdot,u)$ is non-negative defined. For the second case, $g(u)=u(1-u)$, it should be also assumed that, $0 \leq u_0 \leq 1$, but we leave this option to future work (see [13]). Moreover, the assumption here $\kappa(x,u)=uA(x)$ makes it clear that the coefficients (a_{ij}), $(i,j=1,\ldots,n)$ describe the anisotropic and the heterogeneous nature of the medium. This is very important to a great many physical theories, for instance, let us mention applications in physical-chemical reactions and biological processes. Although, it is essential to mention that, in another context of porous media diffusion model, Caffarelli and Vazquez [5] introduced for the first time the model (1.1) for a given fractional potential pressure law, that is to say, they considered $\mathbf{q}(u) = -u \nabla \mathcal{K} u$, where $\mathcal{K}$ is the inverse of the $s-$fractional Laplacian in $\mathbb{R}^n$. Hence that paper established a Fractional Darcy's law and under some conditions, they proved existence of weak (non-negative) solutions for the Cauchy problem.

Concerning the elliptic linear operator $\mathcal{L}u := -\text{div}(A(x)\nabla u)$, which is the building block for the construction of the fractional operator $\mathcal{L}^s_{\mathcal{B}}$, we were motivated by the paper of Caffarelli and Stinga [7]. In that paper the authors reproduce Caccioppoli type estimates (for the Dirichlet and also Neumann boundary conditions), which allow them to develop the interior and boundary regularity theory, depending on the smoothness of the matrix $A(x)$ and the source terms. Albeit, we should mention that, different from that paper, here we are focused on the minimal regularity for the matrix $A(x)$, such that, the eigenfunctions $\{\varphi_k\}$ of the problem (2.3) have enough regularity to define conveniently the operator $\mathcal{K}_s$, and also to give a sense of the Neumann boundary condition on Γ_1, that is to say, for each function $\gamma \in H^1_0(0,T; H^1_{\Gamma_0}(\Omega))$

$$\operatorname{ess} \lim_{\tau \to 0^+} \int_0^T \int_{\Gamma_1} \mathbf{q}(\Psi_\tau(r), u(t,\Psi_\tau(r))) \cdot \nu_\tau(\Psi_\tau(r))\ \gamma(t,r)\, dr\, dt = 0,$$

where $\Psi_\tau(r) := r - \tau \nu(r)$, and ν_τ is the unit outward normal field on $\Psi_\tau(\Gamma)$, see the Appendix. Recall that $A(x)$ is (uniformly) continuous up to the boundary, therefore it is bounded in Ω and its restriction on Γ makes sense. This is also important to the $\mathcal{L}_{\mathcal{B}}$ operator's domain definition, see Equation (2.2). Moreover, due to the regularity of the matriz A in $C^{0,1}_{\text{loc}}(\Omega)$, the eigenfunctions $\varphi_k \in H^2(\Omega')$, for all $k \geq 1$ and every Ω' compactly contained in Ω, see Ambrosio, Carlotto, and Massaccesi [1]. We remark that, it is not possible to ensure $H^2(\Omega)$ regularity even if the diffusive matriz A has $C^{0,1}(\Omega)$ smoothness. Indeed, we are considering mixed-boundary conditions and hence Nirenberg's type methods do not apply, since $\varphi_k = 0$ on Γ_0 but not necessarily zero on Γ_1.

Since the paper [5], there exists a considerable list of important correlated results, to mention a few [2,4,6,14,16,20–22]. In particular, along the same problem, Caffarelli, Soria, and Vazquez establish the Hölder regularity of such weak solutions for the case $s\neq 1/2$ in [4], and the case $s=1/2$ has been proved by Caffarelli and Vazquez in [6]. All of these above cited papers are posed in $\mathbb{R}^n$. On the other hand, the authors considered in [12] again $\mathbf{q}(u)=-u\nabla\mathcal{K}u$, but now in the context of heat equation (Fractional Fourier law), and they considered homogeneous Dirichlet boundary condition. Thus the problem was posed in a bounded open subset of $\mathbb{R}^n$. One of the main tasks of that paper was how the boundary condition should be assumed, and it was important to deal with traces at the boundary for any $s\in(0,1)$. The problem here has different difficulties, and a different context. In this way we consider a formulation different from that presented in [12]. Indeed, an important pragmatism concerning the mixed-boundary conditions is that, the (homogeneous) Dirichlet boundary conditions are taken into account in the test functions, and the Neumann boundary conditions are taken into account in the linear form due to boundary integrals. Hence we follow this strategy and direct the reader to Section 3, where the main ideas are well-explained and also Section 4, where the solvability of the initial mixed-boundary value problem (1.1) is shown.

Finally, we would like to stress that the uniqueness property is not established in this paper. First, let us remark that, no uniqueness result has been proven even for the $\mathbb{R}^n$ case with $\mathbf{q}(u)=-u\nabla\mathcal{K}u$. Moreover, along the same model we direct the reader to Serfaty and Vázquez [19] (and references therein), where a counterexample to comparison of densities is constructed, see Section 6.5 (Lack of comparison principle). Hence we may consider a selection principle (or admissibility criteria) in order to attack the issue of uniqueness for (1.1).

1.1. Functional space. From now on, by Ω we denote a bounded open set in $\mathbb{R}^n$ with smooth (C^2) boundary Γ. We assume that $\Gamma=\Gamma_0\cup\Gamma_1$, Γ_0 is a closed set and $\mathcal{H}^{n-1}(\Gamma_0)>0$, where $\mathcal{H}^\theta$ is the usual $\theta-$Hausdorff measure. Moreover, $\Gamma_0\cap\overline{\Gamma}_1$ is a submanifold of codimension greater than 1. Then, we define

$$H^1_{\Gamma_0}(\Omega):=\left\{v\in H^1(\Omega): v=0 \text{ on } \Gamma_0 \text{ in the sense of trace}\right\},$$

endowed with the norm

$$\|v\|_{H^1_{\Gamma_0}(\Omega)}:=\left(\int_\Omega |\nabla v(x)|^2\,dx\right)^{1/2},\quad \text{for each } v\in H^1_{\Gamma_0}(\Omega). \tag{1.4}$$

Since the trace is a continuous operator, we have that $H^1_{\Gamma_0}(\Omega)$ is a Hilbert space with the norm $\|\cdot\|_{H^1(\Omega)}$, which is equivalent to (1.4). Moreover, we define the set

$$C^\infty_{\Gamma_0}(\overline{\Omega}):=\left\{v\in C^\infty(\overline{\Omega}); v=0 \text{ on } \Gamma_0\right\}, \tag{1.5}$$

which is dense in $H^1_{\Gamma_0}(\Omega)$.

Now, we follow Lions and Magenes [15] for the definition of the spaces $H^s(\Omega)$, with $s\in(0,1)$. Indeed, by interpolation between $H^1(\Omega)$ and $L^2(\Omega)$, we have

$$H^s(\Omega)=[H^1(\Omega),L^2(\Omega)]_{1-s}.$$

According to this definition, this space is a Hilbert space with the natural norm given by the interpolation. Moreover, we can define the space $H^s_0(\Omega)$ by

$$H^s_0(\Omega)=\overline{C^\infty_c(\Omega)}^{\|\cdot\|_{H^s(\Omega)}}.$$

Since Ω has a regular boundary, the set $H_0^s(\Omega)$ could be written as an interpolation (see Theorem 11.6 of [15]),

$$H_0^s(\Omega)=[H_0^1(\Omega),L^2(\Omega)]_{1-s},$$

for each $s\in(0,1)\setminus\{1/2\}$. The particular case $s=1/2$ generates the so-called Lions-Magenes space $H_{00}^{1/2}(\Omega)$, which is defined by

$$H_{00}^{1/2}(\Omega):=[H_0^1(\Omega),L^2(\Omega)]_{1/2},$$

which has the following characterization

$$H_{00}^{1/2}(\Omega)=\left\{u\in H^{1/2}(\Omega);\int_\Omega\frac{u(x)^2}{\text{dist}(x,\Gamma)}dx<\infty\right\}.$$

Furthermore, we define the space $H_{\Gamma_0}^s(\Omega)$ by

$$H_{\Gamma_0}^s(\Omega)=\text{ closure of } C_{\Gamma_0}^\infty(\bar\Omega) \text{ in } H^s(\Omega).$$

In particular, for $0<s\le 1/2$ and since Γ is Lipschitz, we have $H_{\Gamma_0}^s(\Omega)=H^s(\Omega)$, which is due to the fact that $C_0^\infty(\Omega)$ is dense in $H^s(\Omega)$ (see [15] Theorem 11.1). On the other hand, if $1/2<s<1$ and Γ is Lipschitz, then the spaces $H_{\Gamma_0}^s(\Omega)$ have a characterization via trace operator (Theorem 9.4 [15]), hence

$$H_{\Gamma_0}^s(\Omega)\equiv\{u\in H^s(\Omega):u=0 \text{ on } \Gamma_0 \text{ in the sense of trace}\}. \tag{1.6}$$

The proof is based on similar arguments as those considered in Theorem 11.5 [15].

Finally, since Ω has a Lipschitz boundary, there exists an equivalent definition given via interpolation. Indeed, due to $H_0^1(\Omega)\subset H_{\Gamma_0}^1(\Omega)\subset H^1(\Omega)$, it follows that, for all $s\in(0,1)$

$$[H_0^1(\Omega),L^2(\Omega)]_{1-s}\subset[H_{\Gamma_0}^1(\Omega),L^2(\Omega)]_{1-s}\subset[H^1(\Omega),L^2(\Omega)]_{1-s}.$$

Therefore, we have

$$\begin{aligned} H_0^s(\Omega)\subset[H_{\Gamma_0}^1(\Omega),L^2(\Omega)]_{1-s}\subset H^s(\Omega), &\quad s\in(0,1)\setminus\{1/2\}\\ H_{00}^{1/2}(\Omega)\subset[H_{\Gamma_0}^1(\Omega),L^2(\Omega)]_{1/2}\subset H^{1/2}(\Omega), &\quad s=1/2. \end{aligned}\tag{1.7}$$

In particular, when $0<s<1/2$ we obtain

$$[H_{\Gamma_0}^1(\Omega),L^2(\Omega)]_{1-s}=H^s(\Omega).$$

On the other hand, using the idea of Theorem 11.6 [15] we may obtain

$$[H_{\Gamma_0}^1(\Omega),L^2(\Omega)]_{1-s}=H_{\Gamma_0}^s(\Omega),\quad \text{for all } s\in(1/2,1).$$

2. Dirichlet-Neumann spectral fractional elliptic operators

In this section, we study some results of Dirichlet-Neumann spectral fractional elliptic operators. We mainly provide the proofs of the new results, in particular we stress Proposition 2.3. One can refer to [3, 7], and [12] for an introduction.

We are mostly interested in fractional powers of a strictly positive self-adjoint operator defined in a domain, which is dense in a (separable) Hilbert space. Therefore, we are going to consider the linear operator $\mathcal{L}u = -\text{div}(A(x)\nabla u)$ equipped with homogeneous mixed Dirichlet-Neumann boundary data, that is to say $\mathcal{B}(u) = 0$ on Γ, where the boundary operator $\mathcal{B}$ is defined as follows

$$\mathcal{B}(u) = \begin{cases} u & \text{on } \Gamma_0, \\ (A\nabla u)\cdot\nu & \text{on } \Gamma_1, \end{cases} \tag{2.1}$$

where $A(x)$ is the symmetric matrix satisfying (1.2) and (1.3).

For convenience, let us denote by $\mathcal{L}_{\mathcal{B}}$, the operator $\mathcal{L}$ subject to Dirichlet-Neumann boundary condition given by (2.1). Observe that $\mathcal{L}_{\mathcal{B}}$ is nonnegative and selfadjoint in

$$D(\mathcal{L}_{\mathcal{B}}) := \big\{u \in H^1(\Omega) : \text{div}(A\nabla u) \in L^2(\Omega), \text{with } \mathcal{B}(u) = 0 \text{ on } \Gamma\big\}. \tag{2.2}$$

Therefore, by the spectral theory, there exists a complete orthonormal basis $\{\varphi_k\}_{k=1}^{\infty}$ of $L^2(\Omega)$, where φ_k satisfies

$$\begin{cases} \mathcal{L}\varphi_k = \lambda_k \varphi_k, & \text{in } \Omega, \\ \mathcal{B}(\varphi_k) = 0, & \text{on } \Gamma. \end{cases} \tag{2.3}$$

It is easy to check that $\{\varphi_k\}_{k=1}^{\infty}$ is also an orthogonal basis of $H^1_{\Gamma_0}(\Omega)$. Moreover, due to the regularity of the matrix $A(x)$, the eigenfunctions $\varphi_k \in H^2(\Omega')$, for all $k \geq 1$ and every Ω' compactly contained in Ω, see Ambrosio et al. [1].

For each $k \geq 1$, it follows that φ_k is an eigenfunction corresponding to λ_k, where one repeats each eigenvalue λ_k according to its (finite) multiplicity

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lambda_k \to \infty \text{ as } k \longrightarrow \infty.$$

Then, we have

$$D(\mathcal{L}_{\mathcal{B}}) = \{u \in L^2(\Omega);\ \sum_{k=1}^{\infty} \lambda_k^2 |\langle u, \varphi_k\rangle|^2 < \infty\},$$

$$\mathcal{L}_{\mathcal{B}}\ u = \sum_{k=1}^{\infty} \lambda_k \langle u, \varphi_k\rangle\ \varphi_k, \quad \text{for each } u \in D(\mathcal{L}_{\mathcal{B}}).$$

Now, applying functional calculus, we define for each $s > 0$, the following fractional elliptic operator $\mathcal{L}^s_{\mathcal{B}}$, given by

$$\mathcal{L}^s_{\mathcal{B}} u := \sum_{k=1}^{\infty} \lambda_k^s \langle u, \varphi_k\rangle\ \varphi_k,$$

and it is well defined in the space of functions

$$D\big(\mathcal{L}^s_{\mathcal{B}}\big) = \Big\{u \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{2s} |\langle u, \varphi_k\rangle|^2 < +\infty\Big\}, \tag{2.4}$$

which is a Hilbert space with the inner product

$$\langle u, v\rangle_s := \langle u, v\rangle + \int_{\Omega} \mathcal{L}^s_{\mathcal{B}} u(x)\, \mathcal{L}^s_{\mathcal{B}} v(x)\, dx.$$

In particular, the norm $|\cdot|_s$ is defined by

$$|u|_s^2 = \|u\|_{L^2(\Omega)}^2 + \|\mathcal{L}_{\mathcal{B}}^s u\|_{L^2(\Omega)}^2. \tag{2.5}$$

Analogously, we can also define $\mathcal{L}_{\mathcal{B}}^{-s} : D(\mathcal{L}_{\mathcal{B}}^{-s}) \subset L^2(\Omega) \to L^2(\Omega)$. The next proposition gives us the main properties of the operators defined above. In particular, we observe that $D(\mathcal{L}_{\mathcal{B}}^{-s}) = L^2(\Omega)$.

PROPOSITION 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, $s \in (0,1)$, and consider the operators $\mathcal{L}_{\mathcal{B}}^s$, and $\mathcal{L}_{\mathcal{B}}^{-s}$. Then, we have:*

(1) *The operator $\mathcal{L}_{\mathcal{B}}^s$ and $\mathcal{L}_{\mathcal{B}}^{-s}$ are self-adjoint. Also $(\mathcal{L}_{\mathcal{B}}^s)^{-1} = \mathcal{L}_{\mathcal{B}}^{-s}$.*

(2) *If $0 \le s_1 < s_2 \le 1$, then*

$$D(\mathcal{L}_{\mathcal{B}}^{s_2}) \hookrightarrow D(\mathcal{L}_{\mathcal{B}}^{s_1}), \text{ and } D(\mathcal{L}_{\mathcal{B}}^{s_2}) \text{ is dense in } D(\mathcal{L}_{\mathcal{B}}^{s_1}).$$

(3) *For each $s, \sigma > 0$ and $u \in D(\mathcal{L}_{\mathcal{B}}^s)$ we have $\mathcal{L}_{\mathcal{B}}^{-\sigma} u \in D(\mathcal{L}_{\mathcal{B}}^{s+\sigma})$.*

Proof. The proof proceeds analogously to that of Proposition 2.1 in [12] and hence we omit it. $\square$

Now, we state a Poincare-type inequality for the $\mathcal{L}_{\mathcal{B}}^s$, and an equivalent norm for $D(\mathcal{L}_{\mathcal{B}}^s)$.

COROLLARY 2.1 (Poincare-type inequality). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then for each $s > 0$, we have*

$$\|u\|_{L^2(\Omega)} \le \lambda_1^{-s} \, \|\mathcal{L}_{\mathcal{B}}^s u\|_{L^2(\Omega)}, \quad \text{for all } u \in D(\mathcal{L}_{\mathcal{B}}^s).$$

Moreover, the norm defined in (2.5) and

$$\|u\|_s^2 := \int_\Omega |\mathcal{L}_{\mathcal{B}}^s u(x)|^2 \, dx \tag{2.6}$$

are equivalent.

REMARK 2.1. As a consequence of the above results, we can consider the inner product in $D(\mathcal{L}_{\mathcal{B}}^s)$, as follows

$$\langle u, v\rangle_s = \int_\Omega \mathcal{L}_{\mathcal{B}}^s u(x) \, \mathcal{L}_{\mathcal{B}}^s v(x) \, dx. \tag{2.7}$$

Now, the aim is to characterize (via interpolation) the space $D(\mathcal{L}_{\mathcal{B}}^s)$. To begin, we consider $u \in D(\mathcal{L}_{\mathcal{B}})$, hence since $\mathcal{L}_{\mathcal{B}}^{1/2}$ is self-adjoint and from the definition of $\mathcal{L}_{\mathcal{B}}$ we have

$$\begin{aligned}\int_\Omega |\mathcal{L}_{\mathcal{B}}^{1/2} u(x)|^2 \, dx &= \int_\Omega \mathcal{L}_{\mathcal{B}}^{1/2} u(x) \, \mathcal{L}_{\mathcal{B}}^{1/2} u(x) \, dx = \int_\Omega \mathcal{L}_{\mathcal{B}} u(x) \, u(x) \, dx \\ &= \int_\Omega -\operatorname{div}(A(x)\nabla u(x)) u(x) \, dx = \int_\Omega A(x)\nabla u(x) \cdot \nabla u(x) \, dx.\end{aligned}$$

On the other hand, using the uniform elliptic condition (see (1.3)), we obtain

$$\Lambda_1 \int_\Omega |\nabla u(x)|^2 \, dx \le \int_\Omega A(x)\nabla u(x) \cdot \nabla u(x) \, dx \le \Lambda_2 \int_\Omega |\nabla u(x)|^2 \, dx,$$

where $\Lambda_2 = \|A\|_\infty$. Therefore

$$\Lambda_1 \|u\|^2_{H^1_{\Gamma_0}(\Omega)} \leq \|\mathcal{L}_{\mathcal{B}}^{1/2} u\|^2_{L^2(\Omega)} \leq \Lambda_2 \|u\|^2_{H^1_{\Gamma_0}(\Omega)}, \tag{2.8}$$

which means the norm $\|\cdot\|_{1/2}$ is equivalent to the norm $\|\cdot\|_{H^1_{\Gamma_0}(\Omega)}$. Consequently, from the density of $D(\mathcal{L}_{\mathcal{B}})$ in $D(\mathcal{L}_{\mathcal{B}}^{1/2})$, and also in $H^1_{\Gamma_0}(\Omega)$, it follows that $D(\mathcal{L}_{\mathcal{B}}^{1/2}) = H^1_{\Gamma_0}(\Omega)$. Similarly, we have the following

PROPOSITION 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. If $s \in (0,1/2]$, then*

$$D(\mathcal{L}_{\mathcal{B}}^s) = \begin{cases} H^{2s}(\Omega), & if \ \ 0 < s < 1/4, \\ [H^1_{\Gamma_0}(\Omega), L^2(\Omega)]_{1/2}, & if \ \ s = 1/4, \\ H^{2s}_{\Gamma_0}(\Omega), & if \ \ 1/4 < s \leq 1/2. \end{cases} \tag{2.9}$$

Proof. The proof follows by applying the discrete version of J-Method for interpolation, see [3] and also [11]. ◻

Now, for each $s \in (0,1)$ we define conveniently the operators

$$\mathcal{K}_s := \mathcal{L}_{\mathcal{B}}^{-s} \quad \text{and} \quad \mathcal{H}_s := \mathcal{L}_{\mathcal{B}}^{-s/2} \equiv \mathcal{K}_s^{1/2}.$$

Then we consider the following:

LEMMA 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, $s \in (0,1)$ and $u \in D(\mathcal{L}_{\mathcal{B}})$, then $\mathcal{K}_s u \in D(\mathcal{L}_{\mathcal{B}})$. In particular, we have in trace sense*

$$\mathcal{K}_s u = 0 \ \ on \ \Gamma_0 \qquad and \qquad A \nabla \mathcal{K}_s u \cdot \nu = 0 \ \ on \ \Gamma_1.$$

Proof. The proof follows directly from Proposition 2.1, item (3). ◻

Here, and subsequently, we denote $\mathbf{L}^2(\Omega) = (L^2(\Omega))^n$. Then we have the following important result.

PROPOSITION 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary.*
(1) *If $u \in H^1_{\Gamma_0}(\Omega)$, then $\nabla \mathcal{K}_s u \in \mathbf{L}^2(\Omega)$ and there exists $C_\Omega > 0$ such that*

$$\int_\Omega |\nabla \mathcal{K}_s u(x)|^2 \, dx \leq C_\Omega \int_\Omega |\nabla u(x)|^2 \, dx. \tag{2.10}$$

Similarly, if $u \in H^1_{\Gamma_0}(\Omega)$, then $\nabla \mathcal{H}_s u \in \mathbf{L}^2(\Omega)$ and

$$\int_\Omega |\nabla \mathcal{H}_s u(x)|^2 \, dx \leq C_\Omega^{1/2} \int_\Omega |\nabla u(x)|^2 \, dx. \tag{2.11}$$

(2) *If $u \in H^1_{\Gamma_0}(\Omega)$, then*

$$\Lambda_1 \int_\Omega |\nabla \mathcal{H}_s u|^2 \, dx \leq \int_\Omega A(x) \nabla \mathcal{K}_s u \cdot \nabla u \, dx \leq \Lambda_2 \int_\Omega |\nabla \mathcal{H}_s u|^2 \, dx.$$

Proof. (1) First, since $u\in H^1_{\Gamma_0}(\Omega)$, it is enough to consider $u\in D(\mathcal{L}_{\mathcal{B}})$ and thus apply a standard density argument. To show item (1), from (2.8) we have

$$\begin{aligned}\int_\Omega |\nabla \mathcal{K}_s u(x)|^2 dx &\le \Lambda_1^{-1}\int_\Omega |\mathcal{L}_{\mathcal{B}}^{1/2}\mathcal{K}_s u(x)|^2 dx = \Lambda_1^{-1}\sum_{k=1}^\infty \lambda_k |\langle \mathcal{K}_s u,\varphi_k\rangle|^2\\ &= \Lambda_1^{-1}\sum_{k=1}^\infty \lambda_k|\lambda_k^{-s}\langle u,\varphi_k\rangle|^2 \le \Lambda_1^{-1}\lambda_1^{-2s}\sum_{k=1}^\infty \lambda_k |\langle u,\varphi_k\rangle|^2\\ &= \Lambda_1^{-1}\lambda_1^{-2s}\int_\Omega |\mathcal{L}_{\mathcal{B}}^{1/2} u(x)|^2 dx \le \Lambda_1^{-1}\Lambda_2\ \lambda_1^{-2s}\int_\Omega |\nabla u(x)|^2 dx<\infty,\end{aligned}$$

and analogously for $\nabla \mathcal{H}_A u$.

(2) Now, we prove item (2). Integrating by parts, we obtain

$$\begin{aligned}&\int_\Omega -\mathrm{div}(A(x)\nabla \mathcal{K}_s u(x))u(x)\,dx\\ &=\int_\Omega A(x)\nabla\mathcal{K}_s u(x)\cdot\nabla u(x)\,dx-\int_\Gamma u(r)\,A(r)\nabla\mathcal{K}_s u(r)\cdot\nu(r)\,dr.\end{aligned}$$

We claim that, the boundary term is zero in the above equation. Let us recall that $\Gamma=\Gamma_0\cup\Gamma_1$, also since $u\in D(\mathcal{L}_{\mathcal{B}})$, we have that $u=0$ on Γ_0. Moreover, from Lemma 2.1 it follows that $A\nabla\mathcal{K}_s u\cdot\nu=0$ on Γ_1. Hence we conclude that the boundary term is zero. Therefore, we obtain

$$\int_\Omega -\mathrm{div}\big(A(x)\nabla\mathcal{K}_s u(x)\big)\,u(x)\,dx=\int_\Omega A(x)\nabla\mathcal{K}_s u(x)\cdot\nabla u(x)\,dx. \tag{2.12}$$

On the other hand, we observe that

$$\begin{aligned}\int_\Omega -\mathrm{div}\big(A(x)\nabla\mathcal{K}_s u(x)\big)u(x)\,dx &= \int_\Omega \mathcal{L}_{\mathcal{B}}(\mathcal{K}_s u(x))\ u(x)dx\\ &=\int_\Omega \mathcal{L}_{\mathcal{B}}^{1-s}u(x)u(x)\,dx,\end{aligned} \tag{2.13}$$

where we have used the definition of $\mathcal{L}_{\mathcal{B}}$ and $\mathcal{K}_s$. Then from (2.12), (2.13) and since $\mathcal{L}_{\mathcal{B}}^{1-s}$ is self-adjoint (Proposition 2.1), it follows that

$$\int_\Omega A(x)\nabla\mathcal{K}_s u(x)\cdot\nabla u(x)\,dx=\int_\Omega |\mathcal{L}_{\mathcal{B}}^{(1-s)/2}u(x)|^2\,dx.$$

Therefore, using the equivalence norm (2.8) together with the definition of $\mathcal{H}_s u$, we have

$$\Lambda_1\int_\Omega |\nabla\mathcal{H}_s u(x)|^2\ dx\le\int_\Omega A(x)\nabla\mathcal{K}_s u(x)\cdot\nabla u(x)dx\le\Lambda_2\int_\Omega|\nabla\mathcal{H}_s u(x)|^2\ dx.$$

∎

3. Initial mixed-boundary value problem

The main issue of this section is to present the definition of weak solutions for the initial mixed-boundary value problem (1.1), and then discuss in details in which sense the initial mixed-boundary data will be considered, for any $s\in(0,1)$.

DEFINITION 3.1. *Given an initial data $u_0 \in L^\infty(\Omega)$ and $0<s<1$, a function*

$$u \in L^2\Big((0,T); D\big(\mathcal{L}_{\mathcal{B}}^{(1-s)/2}\big)\Big) \cap L^\infty(\Omega_T)$$

is called a weak solution of the initial mixed-boundary value problem (1.1)*, when u satisfies*

$$\iint_{\Omega_T} u(t,x)\ (\partial_t\phi - A(x)\nabla\mathcal{K}_s u(t,x)\cdot\nabla\phi)dxdt + \int_\Omega u_0(x)\phi(0)dx = 0, \tag{3.1}$$

for each test function $\phi \in C_c^\infty\big([0,T); C_{\Gamma_0}^\infty(\bar{\Omega})\big)$.

One observes that, the above definition makes sense. Indeed, the first and the last term in (3.1) is well defined, which is due to the fact that, u and u_0 are bounded. The second term also works, it is enough to recall that $A(x)$ is bounded, and since for almost all $t\in(0,T)$, $u(t)\in D\big(\mathcal{L}_{\mathcal{B}}^{(1-s)/2}\big)$, thus from item (3) in Proposition 2.1 and Proposition 2.2, $\mathcal{K}_s u(t) \in H^1_{\Gamma_0}(\Omega)$. Therefore, due to Proposition 2.3

$$\nabla\mathcal{K}_s u(t) \in \mathbf{L}^2(\Omega).$$

3.1. On the initial mixed-boundary data interpretation. The aim of this section is to study the initial mixed-boundary datum interpretation, from the definition of weak solutions as presented by Definition 3.1. We start with the study of the mixed-boundary condition, and then the initial data will be treated at the end of this section.

To follow, we remark first that our definition of weak solutions is given for any $s\in(0,1)$, and hence it is not always possible to recover the boundary conditions in the trace sense. Let us be more precise. The definition of a weak solution

$$u \in L^2\Big((0,T); D\big(\mathcal{L}_{\mathcal{B}}^{(1-s)/2}\big)\Big) \cap L^\infty(\Omega_T)$$

for (1.1) is given by the integral equation (3.1), where it used a convenient space for the test functions, which give us some information about the mixed-boundary condition. Indeed, the homogeneous Dirichlet boundary condition is obtained by the space $D\big(\mathcal{L}_{\mathcal{B}}^{(1-s)/2}\big)$, and the Neumann boundary condition will be state via Coarea and Area Formulas.

Let u be a solution of (1.1) in the sense of Definition 3.1. Firstly, we discuss the Dirichlet condition, and it will be divided into three main steps:

(1) If $0<s<1/2$ we have

$$u \in L^2\left((0,T); H^{1-s}_{\Gamma_0}(\Omega)\right),$$

thanks to Proposition 2.2. In particular, this space naturally encompasses the Dirichlet boundary condition $u=0$ on Γ_0, since the trace is well defined, see (1.6).

(2) Now, we consider $1/2<s<1$. In this case, from Proposition 2.2, we have

$$u \in L^2\left((0,T); H^{1-s}(\Omega)\right).$$

Here, the trace of u on Γ is not well defined, but we could give an interesting characterization. Indeed, applying Theorem 11.2 in [15], see p. 57, since for each $x\in\Omega$, $\text{dist}(x,\Gamma)\le\text{dist}(x,\Gamma_0)$, there exists a positive constant C, such that

$$\int_\Omega \frac{|u(\cdot,x)|^2}{(\text{dist}(x,\Gamma_0))^{2(1-s)}}dx \le \frac{C}{2(1-s)}\|u(\cdot)\|^2_{H^{1-s}(\Omega)}. \tag{3.2}$$

Now, since Γ is a C^2-boundary, there exists a sufficiently small $\delta>0$ such that, each point $x\in\Omega_\delta:=\{x\in\Omega:\ \mathrm{dist}(x,\Gamma)<\delta\}$ has a unique projection $r=\mathbf{r}(x)$ on the boundary Γ. Moreover, for every $x\in\Omega_\delta$ the Jacobian of the change of variables

$$\Omega_\delta\ni x\leftrightarrow(r,\tau)\in\Gamma\times(0,\delta)\quad\text{is equal to } \frac{D(x)}{D(r,\tau)}=1+O(\delta),$$

where $\tau=\mathrm{dist}(x,\Gamma)$. Therefore, we obtain from (3.2)

$$\begin{aligned}&\int_0^\delta\int_{\Gamma_0}\frac{|u(\cdot,(r,\tau))|^2}{(\mathrm{dist}((r,\tau),\Gamma_0))^{2(1-s)}}\,drd\tau+\int_0^\delta\int_{\Gamma_1}\frac{|u(\cdot,(r,\tau))|^2}{(\mathrm{dist}((r,\tau),\Gamma_0))^{2(1-s)}}\,drd\tau\\&\leq\frac{C}{2(1-s)}\|u(\cdot)\|^2_{H^{1-s}(\Omega)},\end{aligned}\tag{3.3}$$

and applying the Coarea Formula, there exists a set of full measures contained in $(0,\delta)$, such that, for each τ in this set

$$\int_\Gamma\frac{|u(\cdot,(r,\tau))|^2}{(\mathrm{dist}((r,\tau),\Gamma_0))^{2(1-s)}}\,dr\leq\frac{C}{2(1-s)}\|u(\cdot)\|^2_{H^{1-s}(\Omega)}.$$

Moreover, for any $r\in\Gamma_0$ it follows that, $\mathrm{dist}((r,\cdot),\Gamma_0)<\delta$. Hence we obtain from (3.3)

$$\limsup_{\delta\to0^+}\Big(\delta^{2s-1}\,\frac{1}{\delta}\int_0^\delta\int_{\Gamma_0}|u(\cdot,(r,\tau))|^2\,drd\tau\Big)\leq C,$$

for some constant $C>0$. Thus defining the following characterization

$$H^{1-s}_{\Gamma_{0(1-2s)}}(\Omega):=\Big\{f\in H^{1-s}(\Omega);\frac{1}{\tau}\int_0^\tau\int_{\Gamma_0}|f(r,\tau')|^2\,drd\tau'=O(\tau^{1-2s})\Big\},\tag{3.4}$$

we have for almost all $t\in(0,T)$ that, $u(t)\in H^{1-s}_{\Gamma_{0(1-2s)}}(\Omega)$, for any $1/2<s<1$.

(3) The case $s=1/2$ is more delicate, since we do not have a precise identification of the domain $D\big(\mathcal{L}_{\mathcal{B}}^{1/4}\big)$. Actually, from Proposition 2.2 and the second equation in (1.7), we obtain

$$H^{1/2}_{00}(\Omega)\subset D\big(\mathcal{L}_{\mathcal{B}}^{1/4}\big)\subset H^{1/2}(\Omega).$$

First, we observe that the space $H^{1/2}(\Omega)$ does not have a well defined trace sense. On the other hand, there exists a notion of weak trace (see Theorem 11.7 in [15]) for $H^{1/2}_{00}(\Omega)$, but the spaces $H^{1/2}_{00}(\Omega)$ and $D\big(\mathcal{L}_{\mathcal{B}}^{1/4}\big)$ are not necessarily equal. Although, we may follow the same strategy of item (2) above, and define the following characterization

$$H^{1/2}_{\Gamma_{00}}(\Omega):=\Big\{f\in H^{1/2}(\Omega);\frac{1}{\tau}\int_0^\tau\int_{\Gamma_0}|f(r,\tau')|^2\,drd\tau'=O(1)\Big\}.$$

Indeed, it is enough to observe that $D\big(\mathcal{L}_{\mathcal{B}}^{1/4}\big)$ is contained in $H^{1-s}(\Omega)$ for any $s\in[1/2,1)$ and the right-hand side of (3.3) is uniformly bounded up to $s=1/2$. Therefore for almost all $t\in(0,T)$, $u(t)\in H^{1/2}_{\Gamma_{00}}(\Omega)$.

To finish the first part of this discussion, we study the Neumann boundary condition $\mathbf{q}(x,u)\cdot\nu=0$ on Γ_1, see (1.1), which is really complicated because it is composed of two terms, that is u and $A\nabla\mathcal{K}_s u\cdot\nu$. In particular, we observe that $\mathbf{q}(x,u)\cdot\nu$ does not have trace on Γ_1 for any $0<s<1$. For instance, if $0<s<1/2$, it follows that $u(t)\in H^{1-s}_{\Gamma_0}(\Omega)\subset H^{1-s}(\Omega)$ a.e. in $(0,T)$, which implies that u has trace on Γ_1 (not necessarily zero). Although, there is no guarantee that $A(x)\nabla\mathcal{K}_s u\cdot\nu$ has trace on Γ, since $\mathcal{K}_s u$ is not sufficiently regular. Similarly, if $1/2<s<1$ then $\mathcal{K}_s u$ is sufficiently regular to have trace on Γ, but as observed before we do not have trace for u. Thus the Neumann boundary condition is not well defined in the strong sense in any case.

On the other hand, Definition 3.1 is sufficiently robust to give a sense of the Neumann boundary condition on Γ_1. More precisely, we state this boundary condition in a weak sense, written as limits of integrals on $(0,T)\times\Gamma_1$. Indeed, we prove that any solution u in the sense of Definition 3.1, satisfies

$$\operatorname*{ess\,lim}_{\tau\to 0^+}\int_0^T\int_{\Gamma_1}\mathbf{q}(\Psi_\tau(r),u(t,\Psi_\tau(r)))\cdot\nu_\tau(\Psi_\tau(r))\ \phi(t,r)\,dr\,dt=0,$$

where $\Psi_\tau(r):=r-\tau\nu(r)$, and ν_τ is the unit outward normal field on $\Psi_\tau(\Gamma)$, see the Appendix.

To prove the above sentence, we consider the following sets: Let $\mathcal{F}$ be a countable dense subset of $C_c^\infty\big((0,T);C^1_{\Gamma_0}(\bar\Omega)\big)$. For each $\gamma\in\mathcal{F}$, we define the set of full measure in $(0,1)$ by

$$F_\gamma=\Big\{\tau\in(0,1)/\tau\text{ is a Lebesgue point of }\mathbf{J}(\tau)\Big\},$$

where $\mathbf{J}(\tau)$ is given by

$$\int_0^T\int_{\Gamma_1}\mathbf{q}(\Psi_\tau(r),u(t,\Psi_\tau(r)))\cdot\nu_\tau(\Psi_\tau(r))J[\Psi_\tau(r)]\gamma(t,r)\,dr\ dt,$$

where $J[\Psi_\tau]$ is the Jacobian of Ψ_τ. Then, we consider

$$F:=\bigcap_{\gamma\in\mathcal{F}}F_\gamma,$$

which is also a set of full measure in $(0,1)$.

Proposition 3.1 (Neumann condition). *Let u be a weak solution for the initial mixed-boundary value problem (1.1), in the sense of Definition 3.1. Then, for each $\gamma\in H^1_0(0,T;H^1_{\Gamma_0}(\Omega))$*

$$\operatorname*{ess\,lim}_{\tau\to 0^+}\int_0^T\int_{\Gamma_1}\mathbf{q}(\Psi_\tau(r),u(t,\Psi_\tau(r)))\cdot\nu_\tau(\Psi_\tau(r))\gamma(t,r)\,dr\,dt=0,$$

where $\Psi_\tau(r):=r-\tau\nu(r)$ and ν_τ is the unit outward normal field in $\Psi_\tau(\Gamma)$.

Proof. First, we define $S:=\Psi(F\times\Gamma)$ and consider

$$\phi(t,x)=\begin{cases}\gamma(t,\Psi^{-1}_{h(x)}(x))\zeta_j(h(x)), & \text{for } x\in S,\\ 0, & \text{for } x\in\Omega\setminus S,\end{cases}$$

where $\gamma\in\mathcal{F}$, $\zeta_j(\tau)=H_j(\tau+\tau_0)-H_j(\tau-\tau_0)$, with $\tau_0\in F$. Therefore, from (3.1) with $\phi(t,x)$ as test function, and applying the Coarea Formula for the function h, we have

$$\begin{aligned}
&\int_0^1 \zeta_j(\tau)\int_0^T\int_{\Psi_\tau(\Gamma)} u(t,r)\partial_t\gamma(t,\Psi_\tau^{-1}(r))\,d\mathcal{H}^{n-1}(r)dtd\tau\\
&=\int_0^1 \zeta_j(\tau)\int_0^T\int_{\Psi_\tau(\Gamma)} \mathbf{q}(r,u(t,r))\cdot\nabla\gamma(t,\Psi_{h(x)}^{-1}(x))(r)\,d\mathcal{H}^{n-1}(r)dtd\tau\\
&\quad+\int_0^1 \zeta_j'(\tau)\int_0^T\int_{\Psi_\tau(\Gamma)} \mathbf{q}(r,u(t,r))\cdot\nu_\tau(r)\gamma(t,\Psi_\tau^{-1}(r))\,d\mathcal{H}^{n-1}(r)dtd\tau,
\end{aligned}$$

where we have used (A.1) and ∇h is parallel to ν_τ $\mathcal{H}^{n-1}$ a.e on $\Psi_\tau(\Gamma)$.

Then, using the Area formula for the function Ψ_τ and passing to the limit in the above equation as $j\to\infty$, recall that τ_0 is a Lebesque point of $\mathbf{J}(\tau)$, moreover $\zeta_j(t)$ converges pointwise to the characteristic function of the interval $[-\tau_0,\tau_0)$ and $\gamma(t,\cdot)=0$ on Γ_0, we obtain

$$\mathbf{J}(\tau_0)=\int_0^{\tau_0}\Phi(\tau)d\tau, \tag{3.5}$$

for all $\tau_0\in F$ and $\gamma\in\mathcal{F}$, where $\Phi(\tau)$ is given by

$$\int_0^T\int_{\Psi_\tau(\Gamma)} u(t,r)\big(\partial_t\gamma(t,\Psi_\tau^{-1}(r))-A(r)\nabla\mathcal{K}_s u(t,r)\cdot\nabla\gamma(t,\Psi_{h(\cdot)}^{-1}(\cdot))(r)\big)d\mathcal{H}^{n-1}(r)dt.$$

On the other hand, since $\mathcal{F}$ is dense in $C_c^\infty\big((0,T);C^1_{\Gamma_0}(\bar\Omega)\big)$, we have that (3.5) holds for $\gamma\in C_c^\infty\big((0,T);C^1_{\Gamma_0}(\bar\Omega)\big)$. Then, for each $\tau\in F$ we have

$$|\mathbf{J}(\tau)|\le C|\Psi((0,\tau)\times\Gamma)|,$$

where C is a positive constant, which does not depend on τ. Moreover, we know that $J[\Psi_\tau]\to 1$ as $\tau\to 0^+$. Therefore, applying the Dominated Convergent Theorem we obtain

$$\operatorname*{ess\,lim}_{\tau\to 0^+}\int_0^T\int_{\Gamma_1}\mathbf{q}(\Psi_\tau(r),u(t,\Psi_\tau(r)))\cdot\nu_\tau(\Psi_\tau(r))\gamma(t,r)\,drdt=0,$$

which completes the proof. □

To finish this section, we characterize the initial boundary condition from Definition 3.1. For this purpose, let $\mathcal{E}$ be a countable dense subset of $C^1_{\Gamma_0}(\bar\Omega)$. For each $\zeta\in\mathcal{E}$, we define the set of full measure in $(0,T)$ by

$$E_\zeta:=\Big\{t\in(0,T)/t \text{ is a Lebesgue point of } \mathrm{I}(t)=\int_\Omega u(t,x)\zeta(x)dx\Big\},$$

and consider

$$E:=\bigcap_{\zeta\in\mathcal{E}}E_\zeta,$$

which is a set of full measure in $(0,T)$.

PROPOSITION 3.2 (Initial condition). *Let u be a weak solution for the initial mixed-boundary value problem (1.1), in the sense of Definition 3.1. Then for all $\zeta \in L^1(\Omega)$*

$$\operatorname{ess}\lim_{t\to 0^+}\int_\Omega u(t,x)\zeta(x)\,dx=\int_\Omega u_0(x)\zeta(x)\,dx. \tag{3.6}$$

Proof. We give only the main ideas of the proof (for more details see [12]). Let us consider $\phi(t,x)=\gamma_j(t)\zeta(x)$, $\gamma_j(t)=H_j(t+t_0)-H_j(t-t_0)$ for any $t_0\in E$ (fixed), and $\zeta\in\mathcal{E}$. Then, substituting ϕ into (3.1) and passing to the limit as $j\to\infty$, (t_0 is Lebesque point of $\mathrm{I}(t)$), we obtain

$$\mathrm{I}(t_0)=\int_\Omega u_0(x)\zeta(x)dx-\int_0^{t_0}\int_\Omega u(x)A(x)\nabla\mathcal{K}_s u(x)\cdot\nabla\zeta(x)dxdt, \tag{3.7}$$

where we have used the Dominated Convergence Theorem. Since $t_0\in E$ is arbitrary, and in view of the density of $\mathcal{E}$ in $L^1(\Omega)$, the proof follows. □

4. Main result

The main result of this section is to show a weak solution of (1.1). To this end, we have the following:

THEOREM 4.1 (Main Theorem). *Let $u_0\in L^\infty(\Omega)$ be a non-negative function. Then, there exists a weak solution $u\in L^2\big((0,T);D\big(\mathcal{L}_{\mathcal{B}}^{(1-s)/2}\big)\big)\cap L^\infty(\Omega_T)$ of the initial mixed-boundary value problem (1.1).*

The proof of this result is given in the next sections.

4.1. Anisotropic parabolic approximation.

In this subsection, we introduce and study the approximate parabolic problem with $\delta,\mu\in(0,1)$, given by

$$\partial_t u_{\mu,\delta}-\delta\,\operatorname{div}(A(x)\nabla u_{\mu,\delta})=\operatorname{div}(\mathbf{q}_\mu(x,u_{\mu,\delta}))\qquad \text{in }\Omega_T, \tag{4.1}$$

$$u_{\mu,\delta}=u_{0\delta}\qquad \text{in }\{t=0\}\times\Omega, \tag{4.2}$$

$$u_{\mu,\delta}=0\qquad \text{on }(0,T)\times\Gamma_0, \tag{4.3}$$

$$\delta A\nabla u_{\mu,\delta}\cdot\nu=-\mathbf{q}_\mu(x,u_{\mu,\delta})\cdot\nu\qquad \text{on }(0,T)\times\Gamma_1, \tag{4.4}$$

where $\mathbf{q}_\mu(x,u):=(\mu+u)A(x)\nabla\mathcal{K}_s u$, and $u_{0\delta}$ is a non-negative regularized initial data such that

$$u_{0,\delta}\to u_0 \text{ strongly in } L^1(\Omega) \text{ as } \delta\to 0,\qquad \|u_{0,\delta}\|_{L^\infty}\le\|u_0\|_{L^\infty},$$

and satisfying suitable compatibility conditions.

Now, we make use of the well known results of existence, uniqueness and uniform L^∞ bounds for parabolic problems with mixed boundary conditions. Therefore, applying Theorem A.1 from the Appendix, for each $\delta,\mu>0$, there exists a unique, namely here strong solution,

$$u_{\mu,\delta}\in C([0,T];H^1_{\Gamma_0}(\Omega))\cap L^2((0,T);H^2(\Omega'))\cap L^\infty(\Omega_T),$$
$$\partial_t u_{\mu,\delta}\in L^2(\Omega_T),$$

for each Ω' compactly contained in Ω. Moreover, one observes that conditions (4.3) and (4.4) are satisfied in the sense of trace.

The following theorem investigates the properties of the solution $u_{\mu,\delta}$ to the (anisotropic) parabolic perturbation (4.1)–(4.4) for fixed $\delta,\mu\in(0,1)$.

THEOREM 4.2. *For each $\mu,\delta>0$, let $u=u_{\mu,\delta}$ be the unique strong solution of (4.1)–(4.4). Then, u satisfies:*

(1) *For all $\phi\in C_c^\infty([0,T):C_{\Gamma_0}^\infty(\bar\Omega))$,*

$$\iint_{\Omega_T}\left(u(t,x)\partial_t\phi(t,x)-\delta A(x)\nabla u\cdot\nabla\phi(t,x)\right)\,dxdt+\int_\Omega u_{0\delta}(x)\ \phi(0,x)\ dx$$
$$=\iint_{\Omega_T}(\mu+u(t,x))A(x)\nabla\mathcal{K}_s u(t,x)\cdot\nabla\phi(t,x)\ dxdt. \tag{4.5}$$

(2) *For all $(t,x)\in\Omega_T$, we have*

$$0\le u(t,x)+\mu\le\|u_0\|_{L^\infty}, \tag{4.6}$$

and the conservation of the "total mass"

$$\int_\Omega u(t,x)\ dx=\int_\Omega u_{0\delta}(x)\ dx\le\|u_0\|_{L^\infty}|\Omega|. \tag{4.7}$$

Proof. (1) Let us show (4.5). First, we observe that, the Equation (4.1) is verified for almost all points $(t,x)\in(0,T)\times\Omega'$, for each Ω' compactly contained in Ω. Therefore, we multiply (4.1) by $\phi(t,x)(1-\zeta_j(h(x)))$ and integrate in Ω_T, where $\phi\in C_c^\infty([0,T);C_{\Gamma_0}^\infty(\bar\Omega))$, and $\zeta_j(h(x))$ is taken as in the proof of Proposition 3.1. We are not going to reproduce here all the details given in Section 3.1, and from now on we omit this procedure. One remarks that, the support of $(1-\zeta_j(h(x)))\subset\Omega$. Then, after integration by parts we obtain

$$\int_0^T\int_\Omega\left\{-u\partial_t\phi+\delta A(x)\nabla u\cdot\nabla\phi+(\mu+u)A(x)\nabla\mathcal{K}_s u\cdot\nabla\phi\right\}(1-\zeta_j)\ dxdt$$
$$=\int_\Omega u_{0\delta}\ \phi(0)\,(1-\zeta_j)\ dx+\int_0^T\int_\Gamma\phi\,(1-\zeta_j)\,(\delta A(r)\nabla u+\mathbf{q}_\mu(r,u))\cdot\nu\,drdt$$
$$+\int_0^1(-\zeta_j'(\tau))\int_0^T\int_{\Psi_\tau(\Gamma)}\phi\,(\delta A(r)\nabla u+\mathbf{q}_\mu(r,u))\cdot\nu_\tau(r)\,drdtd\tau,$$

where we have used the Coarea Formula for the function h in the third integral in the right-hand side of the above equation. Thus, applying the Area formula for the function Ψ_τ, passing to the limit as $j\to\infty$ and making $\tau_0\to0^+$, we have

$$\int_0^T\int_\Omega\left\{-u\partial_t\phi+\delta A(x)\nabla u\cdot\nabla\phi+(\mu+u)A(x)\nabla\mathcal{K}_s u\cdot\nabla\phi\right\}\ dxdt$$
$$=\int_\Omega u_{0\delta}\ \phi(0)\ dx+2\int_0^T\int_\Gamma\phi\,(\delta A(r)\nabla u+\mathbf{q}_\mu(r,u))\cdot\nu\,drdt.$$

Finally, we stress that the boundary term

$$\int_0^T\int_\Gamma\phi\,(\delta A(r)\nabla u+\mathbf{q}_\mu(r,u))\cdot\nu\,drdt=0.$$

Indeed, due to $\Gamma=\Gamma_0\cup\Gamma_1$, $\phi=0$ on Γ_0, and $(\delta A(r)\nabla u+\mathbf{q}_\mu(r,u))\cdot\nu=0$ on Γ_1, see (1.5) and (4.4) respectively.

(2) To show the assertion (4.6), we multiply (4.1) by $\varphi_\varepsilon'(u)$ and integrate in $\Omega_t=(0,t)\times\Omega$, $0<t\leq T$, where

$$\varphi_\varepsilon(z)=\begin{cases}\big((z+\mu)^2+\varepsilon^2\big)^{1/2}-\varepsilon, & \text{for } z\leq-\mu,\\ 0, & \text{for } z\geq-\mu,\end{cases}$$

which converges to $|z+\mu|^-:=\min\{z+\mu,0\}$ as $\varepsilon\to0^+$. Hence from the properties of φ_ε, we obtain

$$\int_\Omega\varphi_\varepsilon(u(t))dx+\iint_{\Omega_t}\varphi_\varepsilon''(u)(\mu+u(x))A(x)\nabla\mathcal{K}_su\cdot\nabla u\;dxd\tau$$
$$+\delta\iint_{\Omega_t}\varphi_\delta''(u_\varepsilon)A(x)\nabla u\cdot\nabla u\,dxd\tau=0,$$

where we have used that, $u_0\geq0$, the boundary conditions in (4.3)-(4.4), and $\varphi_\varepsilon'(0)=0$. On the other hand, we observe

$$\begin{aligned}&\varphi_\varepsilon''(u)(\mu+u(x))A(x)\nabla\mathcal{K}_su\cdot\nabla u+\delta A(x)\nabla u\cdot\nabla u\varphi_\varepsilon''(u)\\&\geq\big\{-|\mu+u(x)||A(x)\nabla\mathcal{K}_su||\nabla u|+\delta\Lambda_1|\nabla u|^2\big\}\varphi_\varepsilon''(u)\\&\geq-\frac{1}{4\delta\Lambda_1}(\mu+u)^2|A(x)\nabla\mathcal{K}_su|^2\varphi_\varepsilon''(u)\\&\geq-\frac{\varepsilon}{4\delta\Lambda_1}|A(x)\nabla\mathcal{K}_su|^2,\end{aligned}$$

where we have used the uniform ellipticity and $(u+\mu)^2\varphi_\varepsilon''(u)\leq\varepsilon$. Consequently,

$$\int_\Omega\varphi_\varepsilon(u(t))dx\leq\frac{\varepsilon}{4\delta\Lambda_1}\int_{\Omega_t}|A(x)\nabla\mathcal{K}_su(\tau,x)|^2\,dx\,d\tau.$$

Then passing the limit as $\varepsilon\to0^+$, we get

$$\int_\Omega|u(t,x)+\mu|^-\;dx\leq0,$$

thus $|u(t,x)+\mu|^-=0$. Similarly, we can show that $|u(t,x)+\mu-\|u\|_\infty|^+=0$, therefore (4.6) is proved.

(3) It remains to prove (4.7). We multiply (4.1) by $\xi_k(x)$ (see the Appendix), and integrate over Ω. Then, after integration by parts and due to $\xi_k=0$ on Γ, we have

$$\frac{\partial}{\partial t}\int_\Omega u(t,x)\xi_k(x)\,dx=-\int_\Omega\delta\;A(x)\nabla u(t,x)\cdot\nabla\xi_k(x)\;dx$$
$$-\int_\Omega(\mu+u(t,x))A(x)\nabla\mathcal{K}_su(t,x)\cdot\nabla\xi_k(x)\;dx.$$

Now, we integrate the above equation over $(0,t)$

$$\int_\Omega\big(u(t,x)-u_{0,\delta}(x)\big)\xi_k(x)\;dx=-\int_0^t\int_\Omega\delta\;A(x)\nabla u(t,x)\cdot\nabla\xi_k(x)\;dx$$

$$
\begin{aligned}
&-\int_0^t\int_\Omega(\mu+u(t,x))A(x)\nabla\mathcal{K}_su(t,x))\cdot\nabla\xi_k(x)\;dxdt' \\
&=-I_1-I_2,
\end{aligned}
\tag{4.8}
$$

with the obvious notation. Let us observe the I_2 term, we have

$$
|I_2|\le(\|u\|_\infty+1)\|A\|_\infty\Big(\iint_{\Omega_T}|\nabla\mathcal{K}_su(t,x)|^2dxdt\Big)^{1/2}\Big(\iint_{\Omega_T}|\nabla\xi_k(x)|^2dxdt\Big)^{1/2},
$$

where we have used Hölder's inequality and the uniform estimates for $u(t,x)$, $A(x)$. Therefore, applying Lemma A.1 we obtain

$$
\lim_{k\to\infty}I_2=0.
$$

Similarly, we have that I_1 goes to zero as $k\to\infty$. Then, passing to the limit as $k\to\infty$ in (4.8), and again applying Lemma A.1 we get (4.7). Hence the proof of the Theorem 4.2 is complete. ∎

Now, let us consider two important estimates of the solution $u_{\delta,\mu}$ for the initial mixed-boundary valued problem (4.1)–(4.4), with fixed $\delta,\mu\in(0,1)$.

PROPOSITION 4.1 (First energy estimate). *Let $u=u_{\mu,\delta}$ be the unique strong solution of (4.1)–(4.4). Then, for all $t\in(0,T)$,*

$$
\int_\Omega\eta(u(t))\,dx+\Lambda_1\delta\int_0^t\int_\Omega\frac{|\nabla u|^2}{\mu+u}\;dx\,dt+\Lambda_1\int_0^t\int_\Omega|\nabla\mathcal{H}_su|^2\,dxdt\le\int_\Omega\eta(u_{0\delta})dx,
\tag{4.9}
$$

where $\eta(\lambda):=(\lambda+\mu)\log(1+(\lambda/\mu))-\lambda$, $(\lambda\ge0)$.

Proof. First, we multiply (4.1) by $\eta'(u)$ and integrate on Ω. Then, after integration by parts, we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_\Omega\eta(u)dx=&-\delta\int_\Omega\frac{1}{\mu+u}\;A(x)\nabla u\cdot\nabla u\;dx-\int_\Omega A(x)\nabla\mathcal{K}_su\cdot\nabla u\,dx\\
&+\int_\Gamma\eta'(u(r))\;\,(\delta A(r)\nabla u(r)+\mathbf{q}_\mu(r,u))\cdot\nu\;dr.
\end{aligned}
$$

One observes that, the boundary terms are zero. Indeed, the proof is similar to Theorem 4.2, where the important point here is that $\eta'(0)=0$ and $u=0$ on Γ_0. Therefore, the boundary terms are zero. Then, we integrate over $(0,t)$, for all $0<t<T$, to obtain

$$
\begin{aligned}
&\int_\Omega\eta(u(t))dx+\delta\int_0^t\int_\Omega\frac{1}{\mu+u(t,x)}\;A(x)\nabla u(t,x)\cdot\nabla u(t,x)\;dxdt\\
&\qquad+\int_0^t\int_\Omega A(x)\nabla\mathcal{K}_su(t,x)\cdot\nabla u(t,x)\,dxdt=\int_\Omega\eta(u_0)\;dx.
\end{aligned}
$$

On the other hand, due to the uniform ellipticity condition, we have

$$
\Lambda_1\int_0^t\int_\Omega\frac{|\nabla u(t,x)|^2}{\mu+u(t,x)}\;dxdt\le\int_0^t\int_\Omega\frac{1}{\mu+u(t,x)}\;A(x)\nabla u(t,x)\cdot\nabla u(t,x)\;dxdt.
$$

For the third term in the left-hand side, we use Proposition 2.3 ($u\in H^1_{\Gamma_0}(\Omega)$), which establishes the first energy estimate. ∎

As a consequence of this last result, we obtain

COROLLARY 4.1. *Under the assumptions of the Proposition 4.1, we have that* $u = u_{\delta,\mu}$ *satisfies*

$$\begin{aligned} \delta \|\nabla u\|^2_{L^2(\Omega_T)} &\leq \|u_0\|_\infty \; \eta(\|u_0\|_\infty) \; |\Omega| \; \Lambda_1^{-1}, \qquad \textit{and} \\ \|\nabla \mathcal{H}_s u\|^2_{L^2(\Omega_T)} &\leq \eta(\|u_0\|_\infty) \; |\Omega| \; \Lambda_1^{-1}, \end{aligned} \tag{4.10}$$

where $|\Omega|$ *is the Lebesgue measure of the set* Ω.

Proof. We only provide the proof for the first inequality in (4.10), the proof of the other one is similar. From (4.9) we have

$$\frac{\Lambda_1 \delta}{\|u_0\|_\infty} \int_0^t \int_\Omega |\nabla u(t,x)|^2 \, dx dt \leq \int_\Omega \eta(u_{0\delta}(x)) dx,$$

where we have used (4.6). Moreover, since $\eta'(\lambda) > 0$, $(\lambda \geq 0)$, it follows that $\eta(\lambda)$ is an increasing function, hence $\eta(u_{0\delta}(x)) \leq \eta(\|u_0\|_\infty)$ for almost all $x \in \Omega$. Consequently, we obtain

$$\int_\Omega \eta(u_{0\delta}(x)) dx \leq \eta(\|u_0\|_\infty) |\Omega|,$$

which completes the proof. ∎

PROPOSITION 4.2 (Second energy estimate). *Under the conditions stated above, we have that* $u = u_{\mu,\delta}$ *satisfies*

$$\begin{aligned} \frac{1}{2} \int_\Omega |\mathcal{H}_s u(t_2,x)|^2 \, dx + \Lambda_1 \delta \int_{t_1}^{t_2} \int_\Omega |\nabla \mathcal{H}_s u|^2 \, dx dt \\ + \Lambda_1 \int_{t_1}^{t_2} \int_\Omega (\mu + u) \left|\nabla \mathcal{K}_s u\right|^2 \, dx dt \leq \frac{1}{2} \int_\Omega \left|\mathcal{H}_s u(t_1,x)\right|^2 \, dx, \end{aligned} \tag{4.11}$$

for all $0 \leq t_1 < t_2 < T$.

Proof. First, we multiply (4.1) by $\mathcal{K}_s u$, and integrate in Ω. Then, we have

$$\begin{aligned} \int_\Omega \frac{\partial u}{\partial t} \mathcal{K}_s u \; dx = &- \delta \int_\Omega A(x) \nabla u \cdot \nabla \mathcal{K}_s u \; dx - \int_\Omega (\mu + u) A(x) \nabla \mathcal{K}_s u \cdot \nabla \mathcal{K}_s u \, dx \\ &+ \int_\Gamma \mathcal{K}_s u \; (\delta A(r) \nabla u + \mathbf{q}_\mu(r,u)) \cdot \nu \; dr. \end{aligned}$$

One observes that, $u(t) \in H^1_{\Gamma_0}(\Omega)$ for each $t \in [0,T)$, thus by Proposition 2.1 it follows that $\mathcal{K}_s u(t) = 0$ on Γ_0. Hence, from the same ideas used above, we have that the boundary terms are zero. Then, integrating over $0 \leq t_1 < t_2 < T$, we obtain

$$\begin{aligned} \frac{1}{2} \int_\Omega |\mathcal{H}_s u(t_2,x)|^2 dx + \delta \int_{t_1}^{t_2} \int_\Omega A(x) \nabla u \cdot \nabla \mathcal{K}_s u \, dx dt \\ + \int_{t_1}^{t_2} \int_\Omega (\mu + u) A(x) \nabla \mathcal{K}_s u \cdot \nabla \mathcal{K}_s u \, dx dt = \frac{1}{2} \int_\Omega |\mathcal{H}_s u(t_1,x)|^2 dx. \end{aligned}$$

From the uniform ellipticity condition, we have an estimate for the third term of the left-hand side

$$\Lambda_1 \int_{t_1}^{t_2} \int_\Omega (\mu+u)|\nabla \mathcal{K}_s u|^2 \, dx \leq \int_{t_1}^{t_2} \int_\Omega (\mu+u) A(x) \nabla \mathcal{K}_s u \cdot \nabla \mathcal{K}_s u \, dx$$

and for the second term, we use Proposition 2.3 ($u \in H^1_{\Gamma_0}(\Omega)$). Therefore we get the second energy estimate (4.11). ∎

Finally, we consider the following:

PROPOSITION 4.3. *Under the above conditions, we have for all $v \in H^1_{\Gamma_0}(\Omega)$*

$$\int_0^T \langle \partial_t u(t), v \rangle dt = -\delta \iint_{\Omega_T} A(x) \nabla u \cdot \nabla v \, dx dt + \iint_{\Omega_T} (\mu+u) A(x) \nabla \mathcal{K}_s u \cdot \nabla v \, dx dt \tag{4.12}$$

where $\langle \cdot,\cdot \rangle$ denotes the pairing between $\left(H^1_{\Gamma_0}(\Omega)\right)^*$ and $H^1_{\Gamma_0}(\Omega)$.

Proof. The proof follows by applying the same techniques considered before, so it is omitted. ∎

4.2. Proof of main theorem. Here we pass to the limit in (4.5), as the two parameters δ, μ go to zero. To this end, we use the first and the second energy estimates together with the Aubin-Lions' Theorem.

4.2.1. Limit transition $\delta \to 0^+$. As a first step, we define $u_\delta := u_{\mu,\delta}$ (fixing $\mu > 0$). The main result in this section is the following

PROPOSITION 4.4. *Let $\{u_\delta\}_{\delta>0}$ be the strong solutions of (4.1)–(4.3). Then, there exists a subsequence of $\{u_\delta\}_{\delta>0}$, which weakly converges to some function $u \in L^2\big((0,T); D\big(\mathcal{L}_\mathcal{B}^{(1-s)/2}\big)\big) \cap L^\infty(\Omega_T)$, satisfying*

$$\begin{aligned}
&\iint_{\Omega_T} u(t,x) \partial_t \varphi(t,x) + \int_\Omega u_0(x) \varphi(0,x) dx \\
&= \iint_{\Omega_T} (\mu + u(t,x)) A(x) \nabla \mathcal{K}_s u(t,x) \cdot \nabla \varphi(t,x) dx dt,
\end{aligned} \tag{4.13}$$

for all test functions $\varphi \in C_c^\infty([0,T); C^\infty_{\Gamma_0}(\bar{\Omega}))$.

The proof's idea of (4.13) is to pass to the limit in (4.5) as $\delta \to 0^+$. First, we consider the following lemmas.

LEMMA 4.1. *Under the hypothesis of Theorem 4.2, there exists a subsequence of $\{u_\delta\}_{\delta>0}$ such that*

$$u_\delta \to u \qquad \text{weakly-}\star \text{ in } L^\infty(\Omega_T),$$

where $u \in L^\infty(\Omega_T)$.

Proof. From (4.6), it follows that $\{u_\delta\}_{\delta>0}$ is (uniformly) bounded in $L^\infty(\Omega_T)$. This proves the lemma. ∎

LEMMA 4.2. *Under the hypothesis of Theorem 4.2, there exist subsequences of $\{\nabla \mathcal{K}_s u_\delta\}_{\delta>0}$ and $\{u_\delta\}_{\delta>0}$ such that*

$$\nabla \mathcal{K}_s u_\delta \to \nabla \mathcal{K}_s u, \qquad \text{weakly in } \mathbf{L}^2(\Omega_T),$$

$$u_\delta \to u, \qquad \text{weakly in } L^2\big((0,T); D\big(\mathcal{L}_\mathcal{B}^{(1-s)/2}\big)\big),$$

where $u\in L^2\big((0,T);D\big(\mathcal{L}_{\mathcal{B}}^{(1-s)/2}\big)\big)$.

Proof. From Proposition 4.2, we have

$$\iint_{\Omega_T} |\nabla\mathcal{K}_s u_\delta|^2\,dxdt \le \frac{C}{\mu},$$

where C is a positive constant which does not depend on δ. Therefore, the right-hand side is (uniformly) bounded in $\mathbf{L}^2(\Omega_T)$ w.r.t. δ. Thus we obtain (along a suitable subsequence) that, $\nabla\mathcal{K}_s u_\delta$ converges weakly to $\mathbf{v}$ in $\mathbf{L}^2(\Omega_T)$.

The next step is to show that $\mathbf{v}=\nabla\mathcal{K}_s u$ in $\mathbf{L}^2(\Omega_T)$. First, we prove the regularity of u. From the equivalent norm (2.8) we deduce that

$$\iint_{\Omega_T} \left|\mathcal{L}_{\mathcal{B}}^{(1-s)/2} u_\delta(t,x)\right|^2 dxdt \le \Lambda_2 \iint_{\Omega_T} |\nabla\mathcal{H}_s u_\delta(t,x)|^2 dxdt.$$

On the other hand, from Corollary 4.1, we obtain that $\nabla\mathcal{H}_s u_\delta$ is (uniformly) bounded in $\mathbf{L}^2(\Omega_T)$ w.r.t. δ. Thus $\{u_\delta\}$ is (uniformly) bounded in $L^2\big((0,T);D\big(\mathcal{L}_{\mathcal{B}}^{(1-s)/2}\big)\big)$. Consequently, it is possible to select a subsequence, still denoted by $\{u_\delta\}$, converging weakly to u in $L^2\big((0,T);D\big(\mathcal{L}_{\mathcal{B}}^{(1-s)/2}\big)\big)$, where we have used the uniqueness of the limit. Therefore, using again (2.8) and the Poincare-type inequality (Corollary 2.1), it follows that

$$\iint_{\Omega_T} |\nabla\mathcal{K}_s u(t,x)|^2 dxdt \le \Lambda_1^{-1}\lambda_1^{-s} \iint_{\Omega_T} |\mathcal{L}_{\mathcal{B}}^{(1-s)/2} u(t,x)|^2 dxdt,$$

where λ_1 is the first eigenvalue of $\mathcal{L}$. Thus, we obtain that $\nabla\mathcal{K}_s u\in\mathbf{L}^2(\Omega_T)$, and hence $\nabla\mathcal{K}_s u_\delta$ converges weakly to $\nabla\mathcal{K}_s u$ in $\mathbf{L}^2(\Omega_T)$. ∎

LEMMA 4.3. *Under the hypothesis of Theorem 4.2, there exists a subsequence of* $\{u_\delta\}_{\delta>0}$ *such that,*

$$u_\delta \to u \qquad \textit{strongly in } L^2(\Omega_T),$$

where $u\in L^2\big((0,T);D\big(\mathcal{L}_{\mathcal{B}}^{(1-s)/2}\big)\big)$.

Proof. Here we apply the Aubin-Lions compactness theorem. First, from Lemma 4.2 we have

$$u_\delta \to u, \text{ weakly in } L^2\big((0,T);D\big(\mathcal{L}_{\mathcal{B}}^{(1-s)/2}\big)\big).$$

On the other hand, from Propositions 4.1, 4.2 and 4.3, together with the (uniform) boundedness of $\nabla\mathcal{K}_s u_\delta$ in $\mathbf{L}^2(\Omega_T)$, we have

$$\int_0^T \|\partial_t u_\delta\|^2_{H^{-1}(\Omega)}\,dt \le C\ (\|u_0\|_\infty+\mu). \tag{4.14}$$

One observes that, at this point $\mu>0$ is fixed. Thus, the right-hand side of (4.14) is bounded in $L^2((0,T);H^{-1}(\Omega))$ w.r.t. δ. Therefore, there exists a subsequence such that $\partial_t u_\delta$ converges weakly to $\partial_t u$ in $L^2(0,T;H^{-1}(\Omega))$. Then, applying the Aubin-Lions compactness theorem (see [17], Lemma 2.48) it follows that, u_δ converges to u (along a suitable subsequence) strongly in $L^2(\Omega_T)$ as δ goes to zero. ∎

Proof. **(Proof of Proposition 4.4.)** The idea of the proof of (4.13) is to pass to the limit in (4.5) as $\delta\to 0^+$. From Lemma 4.1 it is enough to pass to the limit in the first integral in the left-hand side of (4.5). We can proceed in a similar way as before for the sequence $u_{0,\delta}$.

On the other hand, by Corollary 4.1 and the Hölder inequality, we have that the second integral in the left-hand side of (4.5) is zero, given that $A\in L^\infty(\Omega)$ and $\phi\in L^2(\Omega)$.

Now, we study the convergence of the integral in right-hand side of (4.5). First, since $A(x)$ is symmetric, it is sufficient to show $(\mu+u_\delta)\nabla\mathcal{K}_s u_\delta$ converges weakly in $\mathbf{L}^2(\Omega_T)$. Indeed, by Lemmas 4.2 and 4.3, we obtain that $(\mu+u_\delta)\nabla\mathcal{K}_s u_\delta$ converges weakly to $(\mu+u)\nabla\mathcal{K}_s u$ as $\delta\to 0^+$. Hence, the equality (4.13) follows. ∎

COROLLARY 4.2. *Let u be the function given by Proposition 4.4, then it satisfies:*

(1) *For almost all $(t,x)\in\Omega_T$*

$$0\le u(t)+\mu\le\|u_0\|_\infty,\quad \textit{and} \tag{4.15}$$

$$\int_\Omega u(x,t)dx=\int_\Omega u_0(x)dx. \tag{4.16}$$

(2) *First energy estimate: For $\eta(\lambda):=(\lambda+\mu)\log(1+(\lambda/\mu))-\lambda$, $(\lambda\ge 0)$, and almost all $t\in(0,T)$,*

$$\int_\Omega \eta(u(t))dx+\Lambda_1\int_0^t\int_\Omega|\nabla\mathcal{H}_s u|^2\,dxdt'\le\int_\Omega\eta(u_0)\,dx. \tag{4.17}$$

(3) *Second energy estimate: For almost all $0<t_1<t_2<T$,*

$$\frac{1}{2}\int_\Omega|\mathcal{H}_s u(t_2)|^2dx+\Lambda_1\int_{t_1}^{t_2}\int_\Omega(\mu+u)|\nabla\mathcal{K}_s u|^2\,dx\,dt\le\frac{1}{2}\int_\Omega|\mathcal{H}_s u(t_1)|^2dx. \tag{4.18}$$

(4) *For each $v\in H^1_{\Gamma_0}(\Omega)$,*

$$\int_0^T\langle\partial_t u,v\rangle dt=\iint_{\Omega_T}(\mu+u)A(x)\nabla\mathcal{K}_s u\cdot\nabla v\,dxdt, \tag{4.19}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between $\left(H^1_{\Gamma_0}(\Omega)\right)^*$ and $H^1_{\Gamma_0}(\Omega)$.

Proof. (1) To show (4.15), recall that u_δ converges strongly to u in $L^2(\Omega_T)$ and therefore (for a subsequence) u_δ converges a.e. to u in Ω_T, then passing the limit in (4.6) as $\delta\to 0^+$, we obtain the (4.15). Assertion (4.16) is obtained by (4.7) together with the Dominated Convergence Theorem.

(2) To prove the first energy estimate (4.17), we pass to the limit in (4.9) as $\delta\to 0^+$. As u_δ converges almost everywhere to u in Ω_T, and η is a continuous function, it follows that $\eta(u_\delta)$ converges almost everywhere to $\eta(u)$ in Ω_T. Moreover, u_δ is bounded in $L^\infty(\Omega_T)$ w.r.t. δ, then for almost all $t\in(0,T)$

$$\lim_{\delta\to 0^+}\int_\Omega\eta(u_\delta(t))\,dx=\int_\Omega\eta(u(t))\,dx,$$

where we have used the Dominated Convergence Theorem. We can proceed in a similar way as before for the sequence $u_{0,\delta}$.

On the other hand, using the idea of the proof of Lemma 4.2 it is possible to show that (for a subsequence) $\nabla\mathcal{H}_s u_\delta$ converges weakly to $\nabla\mathcal{H}_s u$ in $\mathbf{L}^2(\Omega_T)$. Then, we have

$$\int_0^t\int_\Omega |\nabla\mathcal{H}_s u|^2\,dxdt' \leq \liminf_{\delta\to 0^+}\int_0^t\int_\Omega |\nabla\mathcal{H}_s u_\delta|^2\,dxdt'$$

for almost all $t\in(0,T)$. Also observe that the second integral in the left-hand side of (4.9) is positive, hence we throw it out. Therefore passing to the limit in (4.9) as δ tends to zero, we obtain the assertion.

(3) To show the second energy estimate (4.18), we pass to the limit in (4.11) as δ goes to zero. First, we have to study the convergence of each integral in (4.2). One notes that, due to the continuity in $L^2(\Omega_T)$ and Lemma 4.3, it follows that $\mathcal{H}_s u_\delta$ strongly converges to $\mathcal{H}_s u$ in $L^2(\Omega_T)$. Consequently, it is possible to select a subsequence, still denoted by $\mathcal{H}_s u_\delta(t)$ such that, for almost all $t\in(0,T)$

$$\lim_{\delta\to 0^+}\int_\Omega |\mathcal{H}_s u_\delta(t,x)|^2\,dx = \int_\Omega |\mathcal{H}_s u(t,x)|^2\,dx.$$

On the other hand, since second integral in the left-hand side of (4.11) is positive for all $\delta>0$, we throw it out. Finally, the convergence of the third integral follows from Lemmas 4.2 and 4.3. Then, passing to the limit in (4.11) as $\delta\to 0^+$, we obtain (4.18).

(4) Assertion (4.19) follows by similar ideas, so we pass to the limit in (4.12) as $\delta\to 0^+$, and the proof is concluded. $\square$

REMARK 4.1. The function u obtained above depends on the fixed parameter μ. For each $\mu>0$, we write from now on u_μ instead of u.

4.2.2. Limit transition $\mu\to 0^+$. Here, we prove the existence of weak solutions for the initial mixed-boundary value problem (1.1). To show this, we consider the sequence $\{u_\mu\}_{\mu>0}$, obtained in Proposition 4.4, which satisfies Corollary 4.2 for each $\mu>0$, (4.13)–(4.19).

Proof. **(Proof of Theorem 4.1.)** To show the existence of solutions we pass to the limit in (4.13) as $\mu\to 0^+$. From (4.15) and $\mu\in(0,1)$, we see that $\{u_\mu\}_{\mu>0}$ is (uniformly) bounded in $L^\infty(\Omega_T)$ w.r.t μ. Hence, it is possible to select a subsequence, still denoted by $\{u_\mu\}$, converging weakly-$\star$ to u in $L^\infty(\Omega_T)$, which is enough to pass to the limit in the first integral in the left-hand side of (4.13).

Now, we study the convergence of the integral in right-hand side of (4.13). First, since $A(x)$ is symmetric, it is sufficient to show $(\mu+u_\mu)\nabla\mathcal{K}_s u_\mu$ converges weakly in $\mathbf{L}^2(\Omega_T)$. On the other hand, we recall that, for each $\lambda\geq 0$,

$$\begin{aligned}\eta(\lambda) &= (\lambda+\mu)\log(1+\lambda/\mu)-\lambda,\\ &= (\lambda+\mu)\log(\lambda+\mu)-(\lambda+\mu)\log\mu-\lambda.\end{aligned}$$

Then, from (4.16) and (4.17) we obtain for almost all $t\in(0,T)$

$$\begin{aligned}&\Lambda_1\int_0^t\int_\Omega |\nabla\mathcal{H}_s u_\mu|^2\ dxdt + \int_\Omega (u_\mu(t)+\mu)\log(u_\mu(t)+\mu)\ dx\\ &\leq \int_\Omega (u_0+\mu)\log(u_0+\mu)\ dx.\end{aligned}\tag{4.20}$$

Since $f=f^+-f^-$, where $f^\pm=\max\{\pm f,0\}$, it follows from (4.20) that

$$\Lambda_1\int_0^t\int_\Omega|\nabla\mathcal{H}_s u_\mu|^2\,dxdt+\int_\Omega(u_\mu(t)+\mu)\log^+(u_\mu(t)+\mu)\,dx$$
$$\le\int_\Omega(u_0+\mu)\log(u_0+\mu)\,dx+\int_\Omega(u_\mu(t)+\mu)\log^-(u_\mu(t)+\mu)dx.$$

We observe that the right-hand side of the above inequality is bounded w.r.t. μ (small enough), because u_μ is bounded in $L^\infty(\Omega_T)$ w.r.t. μ, and

$$\int_\Omega(u_\mu(t)+\mu)\log^-(u_\mu(t)+\mu)dx,$$

is bounded w.r.t. μ (small enough). Consequently, we have that $\nabla\mathcal{H}_s u_\mu$ is (uniformly) bounded in $L^2(\Omega_T)$.

On the other hand, using (2.8) and the Poincaré inequality (Corollary 2.1), we obtain that

$$\begin{aligned}\iint_{\Omega_T}|\nabla\mathcal{K}_s u_\mu(t,x)|^2dxdt&\le\Lambda_1^{-1}\iint_{\Omega_T}\left|\mathcal{L}_\mathcal{B}^{1/2-s}u_\mu(t,x)\right|^2dxdt\\&\le\Lambda_1^{-1}\lambda_1^{-s}\iint_{\Omega_T}|\mathcal{L}_\mathcal{B}^{1/2-s/2}u_\mu(t,x)|^2dxdt\\&\le\Lambda_1^{-1}\lambda_1^{-s}\Lambda_2\iint_{\Omega_T}|\nabla\mathcal{H}_s u_\mu(t,x)|^2dxdt.\end{aligned}$$

Therefore, $\nabla\mathcal{K}_s u_\mu$ is (uniformly) bounded in $L^2(\Omega_T)$ w.r.t. $\mu>0$, and thus we obtain (along a suitable subsequence) that $\nabla\mathcal{K}_s u_\mu$ converges weakly to $\mathbf{v}$ in $\mathbf{L}^2(\Omega_T)$. It remains to show that $\mathbf{v}=\nabla\mathcal{K}_s u$. Moreover, applying the same ideas as in the proof of the Proposition 4.4, it is possible to select a subsequence, still denoted by $\{u_\mu\}$, converging weakly to u in $L^2\left(0,T;D\left(\mathcal{L}_\mathcal{B}^{(1-s)/2}\right)\right)$, such that

$$\mathbf{v}=\nabla\mathcal{K}_s u\text{ in }\mathbf{L}^2(\Omega_T).$$

Hence $\nabla\mathcal{K}_s u_\delta$ converges weakly to $\nabla\mathcal{K}_s u$ in $\mathbf{L}^2(\Omega_T)$.

Now, we prove strong convergence for $\{u_\mu\}_{\mu>0}$ in $L^2(\Omega_T)$. To show this, we apply again the Aubin-Lions compactness theorem. Since the coefficients of the matrix $A(x)$ are in $C(\overline{\Omega})\cap C^{0,1}_{\text{loc}}(\Omega)$, together with the boundedness of $\nabla\mathcal{K}_s u_\mu$ in $\mathbf{L}^2(\Omega_T)$, and the uniform limitation of u_μ, we have from (4.19) that

$$\int_0^T\|\partial_t u_\mu\|^2_{H^{-1}(\Omega)}dt\le C,\tag{4.21}$$

where C is a positive constant which does not depend on μ. Then, passing to a subsequence (still denoted by $\{u_\mu\}$), we obtain that

$$\partial_t u_\mu\text{ converges weakly to }\partial_t u\text{ in }L^2(0,T;H^{-1}(\Omega)).$$

Applying the Aubin-Lions compactness theorem, it follows that u_μ converges strongly to u (along a suitable sequence) in $L^2(\Omega_T)$. Consequently, we obtain that $(\mu+u_\mu)\nabla\mathcal{K}_s u_\mu$

converges weakly to $u\ \nabla\mathcal{K}_s u$ as $\mu\to 0^+$. Then, we are ready to pass to the limit in (4.13) as $\mu\to 0^+$ to get

$$\iint_{\Omega_T} u(t,x)\big(\partial_t\varphi(t,x) - A(x)\nabla\mathcal{K}_s(u(t,x))\cdot\nabla\varphi(t,x)\big)\ dxdt + \int_\Omega u_0(x)\varphi(0,x)dx = 0,$$

for all $\varphi\in C_c^\infty([0,T);C_{\Gamma_0}^\infty(\bar\Omega))$. ☐

Corollary 4.3. *The solution u of the initial mixed-boundary value problem (1.1) given by Theorem 4.1, satisfies:*

(1) *For almost all $t\in(0,T)$, we have*

$$\|u(t)\|_\infty \le \|u_0\|_\infty, \quad \textit{and} \tag{4.22}$$

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)\ dx. \tag{4.23}$$

(2) *First energy estimate: For almost all $t\in(0,T)$,*

$$\Lambda_1\int_0^t\int_\Omega |\nabla\mathcal{H}_s u|^2\ dxdt' + \int_\Omega u(t)\log(u(t))\ dx \le \int_\Omega u_0\log(u_0)\ dx. \tag{4.24}$$

(3) *Second energy estimate: For almost all $0<t_1<t_2<T$,*

$$\frac{1}{2}\int_\Omega |\mathcal{H}_s u(t_2)|^2\ dx + \Lambda_1\int_{t_1}^{t_2}\int_\Omega u|\nabla\mathcal{K}_s u|^2\,dx\,dt \le \frac{1}{2}\int_\Omega |\mathcal{H}_s u(t_1)|^2 dx. \tag{4.25}$$

Proof. In order to show (4.22)-(4.25), we may follow similar lines as in the proof of Corollary 4.2. Therefore, we omit them here. ☐

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Appendix. Let us fix here some notation and background used in this paper, we first consider the notion of C^1-(admissible) deformations, which is used to give the correct notion of traces. One can refer to [18].

Definition A.1. *Let $\Omega\subset\mathbb{R}^n$ be an open set. A C^1-map $\Psi:[0,1]\times\Gamma\to\overline{\Omega}$ is said to be a C^1 admissible deformation, when it satisfies the following conditions:*

(1) *For all $r\in\Gamma$, $\Psi(0,r)=r$.*
(2) *The derivative of the map $[0,1]\ni\tau\mapsto\Psi(\tau,r)$ at $\tau=0$ is not orthogonal to $\nu(r)$, for each $r\in\Gamma$.*

Moreover, for each $\tau\in[0,1]$, we denote: Ψ_τ the mapping from Γ to $\overline{\Omega}$, given by $\Psi_\tau(r):=\Psi(\tau,r)$; ν_τ the unit outward normal field in $\Psi_\tau(\Gamma)$. In particular, $\nu_0(x)=\nu(x)$ is the unit outward normal field in Γ.

It must be recognized that domains with C^2 boundaries always have C^1 admissible deformations. Indeed, it is enough to take $\Psi(\tau,r)=r-\epsilon\tau\nu(r)$ for sufficiently small $\epsilon>0$.

Now, we define a level set function h associated with the deformation Ψ_τ. For $\delta > 0$ sufficiently small we define

$$h(x) := \begin{cases} \min\{\tau,\delta\}, & \text{if } x \in \Omega, \\ -\min\{\tau,\delta\}, & \text{if } x \in \mathbb{R}^n \setminus \Omega, \end{cases}$$

which is Lipschitz continuous in $\mathbb{R}^n$, and C^1 on the closure of $\{x \in \mathbb{R}^n : |h(x)| < \delta\}$, moreover

$$|\nabla h(x)| = \begin{cases} 1 & \text{for } 0 \leq h(x) < \delta, \\ 0 & \text{for } h(x) = \delta. \end{cases} \tag{A.1}$$

LEMMA A.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with C^2 boundary. For each $k \in \mathbb{N}$, and all $x \in \mathbb{R}^n$, consider*

$$\xi_k(x) := 1 - \exp(-k\ h(x)). \tag{A.2}$$

Then, the sequence $\{\xi_k\}$ satisfies

$$\lim_{k\to+\infty} \int_\Omega |1-\xi_k|^2 dx = 0, \quad \textit{and} \quad \lim_{k\to+\infty} \int_\Omega |\nabla \xi_k|^2 dx = 0. \tag{A.3}$$

Proof. For more details see Málek, Necas, Rokyta and Ruzicka [17], p. 129. ∎

Last but not least, let us consider the following approximating sequences. Choose a non-negative function $\gamma \in C_c^1(\mathbb{R})$, with support contained in $[0,1]$, such that, $\int \gamma(t)dt = 1$. Then, we consider the sequences $\{\delta_j\}_{j\in\mathbb{N}}$, and $\{H_j\}_{j\in\mathbb{N}}$, defined by

$$\delta_j(t) := j\ \gamma(jt), \quad H_j(t) := \int_0^t \delta_j(s)\ ds.$$

Thus, $H_j'(t) = \delta_j(t)$, and clearly the sequence $\delta_j(t)$ converges, as $j \to \infty$, to the Dirac δ-measure in $\mathcal{D}'(\mathbb{R})$, while the sequence $H_j(t)$ converges pointwise to the Heaviside function

$$H(t) = \begin{cases} 1, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

To finish this section, we show the existence and uniqueness of $u_{\mu,\delta}$ for the approximate parabolic problem (4.1)–(4.4). To this end, we first apply the Banach Fixed Point Theorem to prove the local-in-time existence of the solution, and thus applying a contradiction argument we extend it to be global in time. Since (4.1) is a fractional non-standard parabolic equation, we present the important details and omit the usual ones.

THEOREM A.1. *Let $u_{0\delta}$ be a non-negative regularized initial data. Then the problem (4.1)–(4.4) admits a unique strong solution*

$$u_{\mu,\delta} \in C([0,T); H^1_{\Gamma_0}(\Omega)) \cap L^2((0,T); H^2(\Omega')) \cap L^\infty(\Omega_T),$$
$$\partial_t u_{\mu,\delta} \in L^2(\Omega_T),$$

for each Ω' compactly contained in Ω.

Proof. The proof will be divided into four steps.

(1) First, for each $\tilde{u} \in L^\infty(\Omega_T) \cap L^2(0,T;D(\mathcal{L}_{\mathcal{B}}^{1-s}))$, the following problem

$$\begin{cases} \partial_t u_{\mu,\delta} - \delta \operatorname{div}(A(x)\nabla u_{\mu,\delta}) = \operatorname{div}(\mathbf{q}_\mu(x,\tilde{u})) & \text{in}\, \Omega_T, \\ u_{\mu,\delta} = u_{0\delta} & \text{in}\{t=0\}\times\Omega, \\ u_{\mu,\delta} = 0 & \text{on}\,(0,T)\times\Gamma_0, \\ \delta A\nabla u_{\mu,\delta}\cdot\nu = -\mathbf{q}_\mu(x,\tilde{u})\cdot\nu & \text{on}\,(0,T)\times\Gamma_1, \end{cases} \tag{A.4}$$

has a unique weak solution

$$u_{\mu,\delta} \in L^2\big((0,T);H^1_{\Gamma_0}(\Omega)\big)\cap C\big([0,T);L^2(\Omega)\big)\cap L^\infty(\Omega_T).$$

Indeed, since $\tilde{u} \in L^\infty(\Omega_T) \cap L^2(0,T;D(\mathcal{L}_{\mathcal{B}}^{1-s}))$, it follows that

$$\mathbf{q}_\mu(x,\tilde{u}) \in L^2((0,T);H^1_{\Gamma_0}(\Omega)).$$

Then applying the parabolic theory, see Theorem 11.8 in Chipot [8], (also Chipot, Rougirel [9]), there exists a unique weak solution

$$u_{\mu,\delta} \in L^2\big((0,T);H^1_{\Gamma_0}(\Omega)\big)\cap C\big([0,T);L^2(\Omega)\big)\cap L^\infty(\Omega_T)$$

of the problem (A.4).

(2) Now, we show the local-in-time existence of the solution to (4.1)–(4.4). To prove that, we define the following map

$$\begin{aligned} u_{\mu,\delta}(t,x) = \mathcal{T}(\tilde{u})(t,x) := &\int_\Omega K(t,x,y)\; u_{0,\delta}(y)dy \\ &+ \int_0^t\int_\Omega (\tilde{u}(t',y)+\mu)\nabla_y K(t-t',x,y)\cdot\nabla\mathcal{K}_s\tilde{u}(t',y)dydt, \end{aligned}$$

where $K(t,x,y)$, $(x,y \in \Omega)$, is the heat kernel of the operator $\mathcal{L}u = -\operatorname{div}(A(\cdot)\nabla u)$ with mixed Dirichlet-Neumann boundary data, see [10]. Moreover, for $t>0$ sufficiently small, it is not difficult to show that $\mathcal{T}$ is a contraction. Then, applying the Banach Fixed Point Theorem, there exists a unique local-in-time weak solution

$$u_{\mu,\delta} \in L^2\big((0,T_M);H^1_{\Gamma_0}(\Omega)\big)\cap C\big([0,T_M);L^2(\Omega)\big)\cap L^\infty(\Omega_{T_M}),$$

where T_M denotes the maximal time of existence.

(3) We claim that the local solution $u_{\mu,\delta}$ satisfies

$$\begin{aligned} &u_{\mu,\delta} \in C\left([0,T_M);H^1_{\Gamma_0}(\Omega)\right)\cap L^2\left((0,T_M);H^2(\Omega')\right)\cap L^\infty(\Omega_{T_M}), \\ &\partial_t u_{\mu,\delta} \in L^2\left(\Omega_{T_M}\right). \end{aligned} \tag{A.5}$$

Indeed, since $u_{\mu,\delta} \in L^2\big((0,T_M);H^1_{\Gamma_0}(\Omega)\big)\cap C\big([0,T_M);L^2(\Omega)\big)\cap L^\infty(\Omega_{T_M})$, we have

$$\operatorname{div}\left((u_{\mu,\delta}+\mu)A(x)\nabla\mathcal{K}_s u_{\mu,\delta}\right) \in L^2\left((0,T_M);L^2(\Omega)\right).$$

Therefore, from Equation (4.1) and the standard parabolic regularity theory (see [1]), we obtain (A.5). Consequently, $u_{\mu,\delta}$ satisfies the partial differential Equation (4.1) in the strong sense, that is, for almost all $(t,x) \in (0,T_M)\times\Omega'$.

(4) Finally, we claim that $T_M = T$, for any $T > 0$. Conversely, let us suppose that, $T_M < T$. Then, there exists an increasing sequence $\{t_j\}_{j=1}^{\infty}$, such that, $t_j \to T_M^-$ as $j \to \infty$ and

$$\lim_{j\to\infty} \|u_{\mu,\delta}(t_j,\cdot)\|_{L^\infty(\Omega)} = +\infty. \tag{A.6}$$

Although, due to a similar proof given to (4.6), we may show that

$$0 \le u_{\mu,\delta}(t,x) + \mu \le \|u_{0\delta}\|_{L^\infty(\Omega)},$$

for each $t \in (0, T_M)$ and almost all $x \in \Omega$, which contradicts (A.6). ☐

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