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ZERO DISSIPATION LIMIT PROBLEM OF 1-D NAVIER-STOKES EQUATIONS*

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Abstract. The zero dissipation limit problem of the Navier-Stokes equations with zero viscosity in the case of the superposition of two rarefaction waves and a contact discontinuity is considered in this paper. It is proved that when the heat conductivity coefficient tends to zero, there exists a unique global solution of the compressible Navier-Stokes equations which converges uniformly to the Riemann solution of the corresponding Euler equations away from the initial time and the contact discontinuity. In addition, the uniform convergence rate in terms of the heat conductivity coefficient is obtained. This result is proved by a combination of the energy method from [F.M. Huang, Y. Wang, and T. Yang, Kinet. Relat. Models, 3:685–728, 2010] and [S.X. Ma, J. Math. Anal. Appl., 387:1033–1043, 2012].

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1. Introduction

For the 1-D conservation laws of viscous flows

$$u_t + f(u)_x = \varepsilon (B(u)u_x)_x, u \in \mathbb{R}^n, x \in \mathbb{R}^1, t \in \mathbb{R}^+, \varepsilon > 0, \tag{1.1}$$

where $f(u) \in \mathbb{R}^n$ satisfies some assumptions to ensure the hyperbolic nature of the corresponding inviscid system (1.2),

$$u_t + f(u)_x = 0, \tag{1.2}$$

ε is the viscosity constant and $B(u)$ is the viscosity matrix. When $B(u)$ is positive definite, Goodman-Xin [3] verified that solutions of (1.1) converge to the piecewise smooth solutions of (1.2) with finitely many noninteracting shocks, by using a matched asymptotic analysis. Later, Yu [17] revealed a rich structure of nonlinear wave interactions due to the presence of shocks and initial layers for hyperbolic conservation laws. Bianchini and Bressan [1] considered

$$u_t + f(u)_x = \varepsilon u_{xx}, \tag{1.3}$$

the initial data of which has small total variation, and justified that the vanishing viscosity limits of solutions of (1.3) are that of (1.2), by uniform BV estimates.

The compressible Navier-Stokes equations in Lagrangian coordinates takes the form

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \varepsilon (\frac{u_x}{v})_x, \\ (e + \frac{u^2}{2})_t + (pu)_x = \kappa (\frac{\theta_x}{v})_x + \varepsilon (\frac{u u_x}{v})_x, \end{cases} \tag{1.4}$$

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where v,u,θ,p,e represent the specific volume, the velocity, the temperature, the pressure and the internal energy, ε,κ are the viscosity and heat-conductivity coefficients, respectively. Formally, as the coefficients ε,κ tend to zero, the limiting system of (1.4) is the compressible Euler equations

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ (e+\frac{u^2}{2})_t + (pu)_x = 0. \end{cases} \tag{1.5}$$

Since the viscosity matrix of (1.4) is only semi-positive definite, the method in [1] cannot be applied to the Navier-Stokes equations.

The isentropic compressible Navier-Stokes equations in Euler coordinates can be written as:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = \varepsilon u_{xx}, \end{cases} \tag{1.6}$$

where ρ denotes the density. Recently, by using the compensated compactness method, Chen-Perepelitsa [2] proved that the solution of (1.6) converges to that of the compressible Euler equations. Note that this result allows the initial data containing vacuum in the interior domain. However, the framework of compensated compactness is basically limited to 2×2 systems so far, so that this result could not be applied to the full compressible Navier-Stokes Equations (1.4). But for the compressible Navier-Stokes equations, there are many investigations on the limits to the Euler system with basic wave patterns.

When the Euler flow contains a single shock, for the isentropic compressible Navier-Stokes equations in Lagrangian coordinates

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \varepsilon(\frac{u_x}{v})_x, \end{cases} \tag{1.7}$$

Hoff and Liu [4] firstly showed that the solutions to the isentropic Navier-Stokes equations with shock data exist and converge to the inviscid shocks as the viscosity vanishes, uniformly away from the shocks. By closely following the framework in [3] but without the smallness condition on the shock strength, Wang [14] verified that solutions of (1.7) converge to that of the corresponding inviscid problem. Then, also motivated by [3], Wang [16] generalized the result to the full compressible Navier-Stokes equations.

In the case that the corresponding Euler equations have rarefaction wave solutions, Xin [15] proved that the solutions of (1.7) exist for all time, and converge to the centered rarefaction waves as the viscosity vanishes, uniformly away from the initial discontinuities. Later, Jiang, Ni, and Sun [11] improved the first part with weak centered rarefaction waves data and Zeng [18] improved the other results, respectively, in Xin [15] to the full compressible Navier-Stokes Equations (1.4), provided that the viscosity and heat-conductivity coefficients are of the same order.

When it comes to the contact discontinuity, since the contact discontinuity is linearly degenerate, there are few results for this case. Ma [12] proved that, when the solution of the inviscid Euler equations is piecewise constant with a contact discontinuity, smooth solutions to the Navier-Stokes equations exist and converge to the inviscid solution away from the contact discontinuity at a rate of $\kappa^{\frac{1}{4}}$ as the heat-conductivity coefficient

κ tends to zero, provided that the viscosity coefficient is of a higher order than the heat-conductivity coefficient κ. This result doesn't require the smallness of the strength of the contact discontinuity. It's worthy to note that this is the first paper to deal with the zero dissipation limit result for contact discontinuity. Then, Ma [13] constructed a new ansatz to improve the convergence rate to $\kappa^{\frac{3}{4}}$, provided that the viscosity coefficient is of a higher order than or the same order as the heat-conductivity coefficient.

For composite wave, Huang, Wang, and Yang [8] considered the case when the solution of the Euler equations is a Riemann solution consisting of two rarefaction waves and a contact discontinuity and proved the fluid dynamic limit to compressible Euler equations from compressible Navier-Stokes equations with smooth initial values. Additionally, the uniform convergence rate $\varepsilon^{\frac{1}{5}}$ was obtained in terms of viscosity ε. This is the first rigorous proof of this limit for a Riemann solution with superposition of three waves. Then, Huang, Wang, and Yang [9] studied the case that the Riemann solution of the Euler equations consists of the superposition of a shock wave and a rarefaction wave, and showed that there exists a family of smooth solutions to the compressible Navier-Stokes equations that converges to the Riemann solution away from the initial time and shock layers. This is also the first mathematical justification for the case that Riemann solution contains these two typical nonlinear hyperbolic waves. Compared to the result in Huang, Wang, and Yang [8], Huang, Jiang, and Wang [5] used weighted energy estimates motivated by Huang, Li, and Matsumura [6] to circumvent the difficulties occuring due to the discontinuity of the initial data.

In this paper, we study the zero dissipation limit problem of the Navier-Stokes equations with zero viscosity in the case when the solution of the Euler equations is a Riemann solution consisting of two rarafaction waves and a contact discontinuity:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ (e + \frac{u^2}{2})_t + (pu)_x = \kappa(\frac{\theta_x}{v})_x. \end{cases} \tag{1.8}$$

We consider the polytropic gas where

$$p = \frac{R\theta}{v}, e = \frac{R\theta}{\gamma - 1}, \tag{1.9}$$

with $R > 0, \gamma > 1$ being the gas parameters.

2. Main results

For later use, let us firstly recall the wave curves for the two types of basic waves studied in this paper. Given the right end state (v_+, u_+, θ_+), the following wave curves in the phase space $\{(v,u,\theta) | v > 0, \theta > 0\}$ are defined for the Euler equations.

- Contact discontinuity wave curve:

$$CD(v_+, u_+, \theta_+) = \{(v,u,\theta) | u = u_+, p = p_+, v \neq v_+\}. \tag{2.1}$$

- i-Rarefaction wave curve ($i = 1, 3$):

$$R_i(v_+, u_+, \theta_+) := \{(v,u,\theta) | v < v_+, u = u_+ - \int_{v_+}^{v} \lambda_i(\eta, s_+) d\eta, s(v,\theta) = s_+\}, \tag{2.2}$$

  where $s_+ = s(v_+, \theta_+)$ and $\lambda_i = \lambda_i(v,s)$ is the i-th characteristic speed of the Euler system (1.5).

As in [10] and [15], for compressible Navier-Stokes equations, the corresponding wave profiles can be defined approximately as follows.

2.1. Contact discontinuity. If $(v_-,u_-,\theta_-)\in CD(v_+,u_+,\theta_+)$, i.e.

$$u_-=u_+,p_-=p_+,v_-\neq v_+,$$

then the following Riemann problem of the Euler system (1.5) with Riemann initial data

$$(v,u,\theta)(t=0,x)=\begin{cases}(v_-,u_-,\theta_-), & x<0,\\ (v_+,u_+,\theta_+), & x>0,\end{cases}$$

admits a single contact discontinuity solution

$$(v^{cd},u^{cd},\theta^{cd})(t,x)=\begin{cases}(v_-,u_-,\theta_-), & x<0,t>0,\\ (v_+,u_+,\theta_+), & x>0,t>0.\end{cases} \tag{2.3}$$

As in [7], the viscous contact wave $(V^{CD},U^{CD},\Theta^{CD})(t,x)$ can be defined as follows. Since we expect that

$$P^{CD}\approx p_+=p_-, \quad and \quad |U^{CD}|\ll 1, \tag{2.4}$$

the leading order of the energy equation $(1.8)_3$ is

$$\frac{R}{\gamma-1}\Theta_t+p_+U_x=\kappa(\frac{\Theta_x}{V})_x. \tag{2.5}$$

Substituting (2.4) into (2.5) and using $(1.8)_1$ yields a nonlinear diffusion equation

$$\Theta_t=a\kappa(\frac{\Theta_x}{\Theta})_x, \quad \Theta(t,\pm\infty)=\theta_\pm, \quad a=\frac{p_+(\gamma-1)}{\gamma R^2}>0, \tag{2.6}$$

which admits a unique self-similar solution $\hat{\Theta}(x,t)=\hat{\Theta}(\xi)$, $\xi=\frac{x}{\sqrt{1+t}}$, $\delta^{CD}=|\theta_+-\theta_-|$, then Θ satisfies

$$|(\kappa(1+t))^{\frac{l}{2}}\partial_x^l\hat{\Theta}|+|\hat{\Theta}-\theta_\pm|\le O(1)\delta^{CD}e^{-\frac{c_0x^2}{\kappa(1+t)}} \quad as\ |x|\to+\infty \tag{2.7}$$

Now the viscous contact wave $(V^{CD},U^{CD},\Theta^{CD})$ can be defined as:

$$\begin{cases}V^{CD}(t,x)=R\frac{\hat{\Theta}(t,x)}{p_+},\\ U^{CD}(t,x)=u_++\frac{\kappa(\gamma-1)}{\gamma R}\frac{\hat{\Theta}_x(t,x)}{\hat{\Theta}(t,x)},\\ \Theta^{CD}(t,x)=\hat{\Theta}(t,x)-\frac{\kappa(\gamma-1)}{\gamma R p_+}\hat{\Theta}_t(t,x).\end{cases} \tag{2.8}$$

Then $(V^{CD},U^{CD},\Theta^{CD})$ satisfies

$$\begin{cases}V_t^{CD}-U_x^{CD}=0,\\ U_t^{CD}+P_x^{CD}=0,\\ \frac{R}{\gamma-1}\Theta_t^{CD}+P^{CD}U_x^{CD}=\kappa(\frac{\Theta_x^{CD}}{V^{CD}})_x+Q^{CD},\end{cases} \tag{2.9}$$

where $P^{CD}=R\frac{\Theta^{CD}}{V^{CD}}$ and the error term

$$\begin{aligned}Q^{CD}&=-\frac{\kappa}{\gamma p_+}\hat{\Theta}_{tt}+\frac{\kappa}{\gamma p_+}\frac{\hat{\Theta}_t^2}{\hat{\Theta}}+\frac{\kappa^2(\gamma-1)}{\gamma R^2}(\frac{\hat{\Theta}_t}{\hat{\Theta}})_{xx}+\frac{\kappa^2(\gamma-1)}{\gamma R^2}(\frac{\hat{\Theta}_t}{\hat{\Theta}})_x\frac{\hat{\Theta}_x}{\hat{\Theta}}\\ &=O(1)\delta^{CD}\kappa(1+t)^{-2}e^{-\frac{c_0x^2}{\kappa(1+t)}}, \quad as\ |x|\to+\infty.\end{aligned} \tag{2.10}$$

2.2. Rarefaction waves. The following approximate rarefaction wave profile satisfying the Euler equations was motivated by [15]. For the completeness of the presentation, we introduce its definition and properties.

If $(v_-,u_-,\theta_-)\in R_i(v_+,u_+,\theta_+)$ $(i=1,3)$, then there exists an i-rarefaction wave $(v^{r_i},u^{r_i},\theta^{r_i})(x/t)$ which is a global solution of the following Riemann problem

$$\begin{cases} v_t-u_x=0,\\ u_t+p_x(v,\theta)=0,\\ \frac{R}{\gamma-1}\theta_t+p(v,\theta)u_x=0,\\ (v,u,\theta)(t=0,x)=\begin{cases}(v_-,u_-,\theta_-),\ x<0,\\ (v_+,u_+,\theta_+),\ x>0.\end{cases}\end{cases} \tag{2.11}$$

Consider the following inviscid Burgers equation with Riemann data

$$\begin{cases} w_t+ww_x=0,\\ w(t=0,x)=\begin{cases} w_-,\ x<0,\\ w_+,\ x>0.\end{cases}\end{cases} \tag{2.12}$$

If $w_-<w_+$, then the above Riemann problem admits a rarefaction wave solution

$$w^r(t,x)=w^r(\frac{x}{t})=\begin{cases} w_-,\ \frac{x}{t}\le w_-,\\ \frac{x}{t},\ w_-\le\frac{x}{t}\le w_+,\\ w_+,\ \frac{x}{t}\ge w_+.\end{cases} \tag{2.13}$$

The following lemma from [8] is obvious.

LEMMA 2.1.
For any shift $t_0>0$ in the time variable, we have

$$|w^r(t+t_0,x)-w^r(t,x)|\le\frac{C}{t}t_0,$$

where C is a positive constant depending only on $w_\pm$.

We should note that Lemma 2.1 plays an important role in the wave interaction estimates for the rarefaction waves.

As in [15], the approximate rarefaction wave $(V^R,U^R,\Theta^R)(t,x)$ to (1.8) can be constructed by the solution of the Burgers equation

$$\begin{cases} w_t+ww_x=0,\\ w(0,x)=w_\sigma(x)=w(\frac{x}{\sigma})=\frac{w_++w_-}{2}+\frac{w_+-w_-}{2}\tanh\frac{x}{\sigma},\end{cases} \tag{2.14}$$

where $\sigma>0$ is a small parameter to be determined. The solution $w_\sigma^r(t,x)$ of (2.14) is given by

$$w_\sigma^r(t,x)=w_\sigma(x_0(t,x)),\quad x=x_0(t,x)+w_\sigma(x_0(t,x))t.$$

The solution $w_\sigma^r(t,x)$ has the following properties.

LEMMA 2.2 ([15]). *Let $w_-<w_+$, (2.14) has a unique smooth solution $w_\sigma^r(t,x)$ satisfying*

(1) $w_- < w^r_\sigma(t,x) < w_+,\ (w^r_\sigma)_x(t,x) \geq 0$;

(2) *For any* p $(1\leq p\leq+\infty)$, *there exists a constant* C *such that*

$$\|\frac{\partial}{\partial x}w^r_\sigma(t,\cdot)\|_{L^p(\mathbb{R})}\leq C\ \min\ \{(w_+-w_-)\sigma^{-1+1/p},\ (w_+-w_-)^{1/p}t^{-1+1/p}\},$$
$$\|\frac{\partial^2}{\partial x^2}w^r_\sigma(t,\cdot)\|_{L^p(\mathbb{R})}\leq C\ \min\ \{(w_+-w_-)\sigma^{-2+1/p},\ \sigma^{-1+1/p}t^{-1}\};$$

(3) *If* $x-w_-t<0$ *and* $w_->0$, *then*

$$|w^r_\sigma(t,x)-w_-|\leq(w_+-w_-)e^{-\frac{2|x-w_-t|}{\sigma}},$$
$$|\frac{\partial}{\partial x}w^r_\sigma(t,\cdot)|\leq\frac{2(w_+-w_-)}{\sigma}e^{-\frac{2|x-w_-t|}{\sigma}},$$
$$|\frac{\partial^2}{\partial x^2}w^r_\sigma(t,\cdot)|\leq\frac{4(w_+-w_-)}{\sigma^2}e^{-\frac{2|x-w_-t|}{\sigma}},$$
$$|\frac{\partial^3}{\partial x^3}w^r_\sigma(t,\cdot)|\leq\frac{8(w_+-w_-)}{\sigma^3}e^{-\frac{2|x-w_-t|}{\sigma}};$$

If $x-w_+t>0$ *and* $w_+<0$, *then*

$$|w^r_\sigma(t,x)-w_+|\leq(w_+-w_-)e^{-\frac{2|x-w_+t|}{\sigma}},$$
$$|\frac{\partial}{\partial x}w^r_\sigma(t,\cdot)|\leq\frac{2(w_+-w_-)}{\sigma}e^{-\frac{2|x-w_+t|}{\sigma}},$$
$$|\frac{\partial^2}{\partial x^2}w^r_\sigma(t,\cdot)|\leq\frac{4(w_+-w_-)}{\sigma^2}e^{-\frac{2|x-w_+t|}{\sigma}},$$
$$|\frac{\partial^3}{\partial x^3}w^r_\sigma(t,\cdot)|\leq\frac{8(w_+-w_-)}{\sigma^3}e^{-\frac{2|x-w_+t|}{\sigma}};$$

(4) $\sup\limits_{x\in\mathbb{R}}|w^r_\sigma(t,x)-w^r(\frac{x}{t})|\leq\min\{(w_+-w_-),\ \frac{\sigma}{t}[\ln(1+t)+|\ln\sigma|]\}$.

Then the smooth approximate rarefaction wave profile denoted by $(V^{R_i},U^{R_i},\Theta^{R_i})(t,x)(i=1,3)$ can be defined by

$$\begin{cases} S^{R_i}(t,x)=s(V^{R_i}(t,x),\Theta^{R_i}(t,x))=s_+,\\ w_\pm=\lambda_{i\pm}:=\lambda_i(v_\pm,\theta_\pm),\\ w^r_\sigma(t+t_0,x)=\lambda_i(V^{R_i}(t,x),s_+),\\ U^{R_i}(t,x)=u_+-\int_{v_+}^{V^{R_i}(t,x)}\ \lambda_i(v,s_+)\ dv, \end{cases} \tag{2.15}$$

where t_0 is the shift used to control the interaction between waves in different families with the property that $t_0\to0$ as $\kappa\to0$. In the following, motivated by [8], we choose

$$t_0=\kappa^{\frac{1}{5}},\quad and\quad \sigma=\kappa^{\frac{2}{5}}. \tag{2.16}$$

Note that $(V^{R_i},U^{R_i},\Theta^{R_i})(t,x)(i=1,3)$ defined above satisfies

$$\begin{cases} V^{R_i}_t-U^{R_i}_x=0,\\ U^{R_i}_t+P^{R_i}_x=0,\\ \frac{R}{\gamma-1}\Theta^{R_i}_t+P^{R_i}U^{R_i}_x=0, \end{cases} \tag{2.17}$$

where $P^{R_i}=p(V^{R_i},\Theta^{R_i})(t,x)$, and its properties can be shown as follows by Lemma 2.1 and Lemma 2.2.

LEMMA 2.3. *The approximate rarefaction waves* $(V^{R_i},U^{R_i},\Theta^{R_i})(t,x)(i=1,3)$ *constructed in (2.15) have the following properties:*

(1) $U_x^{R_i}(t,x)>0$ *for* $x\in\mathbb{R}$, $t>0$;

(2) *For any* $1\le p\le+\infty$, *the following estimates hold*

$$\begin{aligned}
&\|(V^{R_i},U^{R_i},\Theta^{R_i})_x\|_{L^p(dx)}\le C(t+t_0)^{-1+\frac{1}{p}},\\
&\|(V^{R_i},U^{R_i},\Theta^{R_i})_{xx}\|_{L^p(dx)}\le C\sigma^{-1+\frac{1}{p}}(t+t_0)^{-1},\\
&\|(V^{R_i},U^{R_i},\Theta^{R_i})_{xxx}\|_{L^p(dx)}\le C\sigma^{-2+\frac{1}{p}}(t+t_0)^{-1},
\end{aligned}$$

where the positive constant C *only depends on* p *and the wave strength;*

(3) *If* $x\ge\lambda_{1+}(t+t_0)$, *then*

$$\begin{aligned}
&|(V^{R_1},U^{R_1},\Theta^{R_1})(t,x)-(v_-,u_-,\theta_-)|\le Ce^{-\frac{2|x-\lambda_{1+}(t+t_0)|}{\sigma}},\\
&|(V^{R_1},U^{R_1},\Theta^{R_1})_x(t,x)|\le\frac{C}{\sigma}e^{-\frac{2|x-\lambda_{1+}(t+t_0)|}{\sigma}},\\
&|(V^{R_1},U^{R_1},\Theta^{R_1})_{xx}(t,x)|\le\frac{C}{\sigma^2}e^{-\frac{2|x-\lambda_{1+}(t+t_0)|}{\sigma}},\\
&|(V^{R_1},U^{R_1},\Theta^{R_1})_{xxx}(t,x)|\le\frac{C}{\sigma^3}e^{-\frac{2|x-\lambda_{1+}(t+t_0)|}{\sigma}};
\end{aligned}$$

If $x\le\lambda_{3-}(t+t_0)$, *then*

$$\begin{aligned}
&|(V^{R_3},U^{R_3},\Theta^{R_3})(t,x)-(v_+,u_+,\theta_+)|\le Ce^{-\frac{2|x-\lambda_{3-}(t+t_0)|}{\sigma}},\\
&|(V^{R_3},U^{R_3},\Theta^{R_3})_x(t,x)|\le\frac{C}{\sigma}e^{-\frac{2|x-\lambda_{3-}(t+t_0)|}{\sigma}},\\
&|(V^{R_3},U^{R_3},\Theta^{R_3})_{xx}(t,x)|\le\frac{C}{\sigma^2}e^{-\frac{2|x-\lambda_{3-}(t+t_0)|}{\sigma}},\\
&|(V^{R_3},U^{R_3},\Theta^{R_3})_{xxx}(t,x)|\le\frac{C}{\sigma^3}e^{-\frac{2|x-\lambda_{3-}(t+t_0)|}{\sigma}};
\end{aligned}$$

(4) *There exist positive constants* C *and* σ_0 *such that for* $\sigma\in(0,\sigma_0)$ *and* $t,t_0>0$,

$$\sup_{x\in\mathbb{R}}|(V^{R_i},U^{R_i},\Theta^{R_i})(t,x)-(v^{r_i},u^{r_i},\theta^{r_i})(\frac{x}{t})|\le\frac{C}{t}[\sigma\ln(1+t+t_0)+\sigma|\ln\sigma|+t_0].$$

2.3. Superposition of rarefaction waves and contact discontinuity. Now we define the solution profile that consists of the superposition of two rarefaction waves and a contact discontinuity. Let $(v_-,u_-,\theta_-)\in R_1-CD-R_3(v_+,u_+,\theta_+)$. Then there exist uniquely two intermediate states (v_*,u_*,θ_*) and (v^*,u^*,θ^*) such that $(v_-,u_-,\theta_-)\in R_1(v_*,u_*,\theta_*)$, $(v_*,u_*,\theta_*)\in CD(v^*,u^*,\theta^*)$ and $(v^*,u^*,\theta^*)\in R_3(v_+,u_+,\theta_+)$.

Thus the wave pattern $(\bar V,\bar U,\bar\Theta)(t,x)$ consisting of 1-rarefaction wave, 2-contact discontinuity and 3-rarefaction wave that solves the corresponding Riemann problem of the Euler system (1.5) can be defined by

$$\begin{pmatrix}\bar V\\ \bar U\\ \bar\Theta\end{pmatrix}(t,x)=\begin{pmatrix}v^{r_1}+v^{cd}+v^{r_3}\\ u^{r_1}+u^{cd}+u^{r_3}\\ \theta^{r_1}+\theta^{cd}+\theta^{r_3}\end{pmatrix}(t,x)-\begin{pmatrix}v_*+v^*\\ u_*+u^*\\ \theta_*+\theta^*\end{pmatrix},\tag{2.18}$$

where $(v^{r_1},u^{r_1},\theta^{r_1})(t,x)$ is the 1-rarefaction wave defined in (2.2) with the right state (v_+,u_+,θ_+) replaced by (v_*,u_*,θ_*), $(v^{cd},u^{cd},\theta^{cd})(t,x)$ is the contact discontinuity defined in (2.3) with the states (v_-,u_-,θ_-) and (v_+,u_+,θ_+) replaced by (v_*,u_*,θ_*) and (v^*,u^*,θ^*) respectively, and $(v^{r_3},u^{r_3},\theta^{r_3})(t,x)$ is the 3-rarefaction wave defined in (2.2) with the left state (v_-,u_-,θ_-) replaced by (v^*,u^*,θ^*).

Corresponding to (2.18), the approximate wave pattern $(V,U,\Theta)(t,x)$ of the compressible Navier-Stokes equations can be defined by

$$\begin{pmatrix} V\\ U\\ \Theta\end{pmatrix}(t,x)=\begin{pmatrix} V^{R_1}+V^{CD}+V^{R_3}\\ U^{R_1}+U^{CD}+U^{R_3}\\ \Theta^{R_1}+\Theta^{CD}+\Theta^{R_3}\end{pmatrix}(t,x)-\begin{pmatrix} v_*+v^*\\ u_*+u^*\\ \theta_*+\theta^*\end{pmatrix}, \tag{2.19}$$

where $(V^{R_1},U^{R_1},\Theta^{R_1})(t,x)$ is the approximate 1-rarefaction wave defined in (2.15) with the right state (v_+,u_+,θ_+) replaced by (v_*,u_*,θ_*), $(V^{CD},U^{CD},\Theta^{CD})(t,x)$ is the viscous contact wave defined in (2.8) with the states (v_-,u_-,θ_-) and (v_+,u_+,θ_+) replaced by (v_*,u_*,θ_*) and (v^*,u^*,θ^*) respectively, and $(V^{R_3},U^{R_3},\Theta^{R_3})(t,x)$ is the approximate 3-rarefaction wave defined in (2.15) with the left state (v_-,u_-,θ_-) replaced by (v^*,u^*,θ^*).

Thus, from the properties of the viscous contact wave in (2.7) and the approximate rarefaction wave in Lemma 2.3, we have the following relation between the approximate wave pattern $(V,U,\Theta)(t,x)$ of the compressible Navier-Stokes equations and the exact inviscid wave pattern $(\bar V,\bar U,\bar\Theta)(t,x)$ to the Euler equations

$$|(V,U,\Theta)(t,x)-(\bar V,\bar U,\bar\Theta)(t,x)|\le\frac{C}{t}[\sigma\ln(1+t+t_0)+\sigma|\ln\sigma|+t_0]+C\delta^{CD}e^{-\frac{cx^2}{\kappa(1+t)}}, \tag{2.20}$$

with t_0, σ in (2.16).

By (2.9) and (2.17), the superposition wave profile $(V,U,\Theta)(t,x)$ defined in (2.19) satisfies the following system

$$\begin{cases} V_t-U_x=0,\\ U_t+P_x=Q_1,\\ \frac{R}{\gamma-1}\Theta_t+PU_x=\kappa(\frac{\Theta_x}{V})_x+Q_2. \end{cases} \tag{2.21}$$

where $P=p(V,\Theta)$, and

$$\begin{aligned} Q_1 &= (P-P^{R_1}-P^{CD}-P^{R_3})_x,\\ Q_2 &= PU_x-P^{CD}U^{CD}_x-P^{R_1}U^{R_1}_x-P^{R_3}U^{R_3}_x-\kappa(\frac{\Theta_x}{V})_x+\kappa(\frac{\Theta^{CD}_x}{V^{CD}})_x+Q^{CD}. \end{aligned}$$

Direct calculation shows that

$$\begin{aligned} Q_1=&O(1)[|(V^{R_1}_x,\Theta^{R_1}_x)||(V^{CD}-v_*,\Theta^{CD}-\theta_*,V^{R_3}-v^*,\Theta^{R_3}-\theta^*)|\\ &+|(V^{R_3}_x,\Theta^{R_3}_x)||(V^{R_1}-v_*,\Theta^{R_1}-\theta_*,V^{CD}-v^*,\Theta^{CD}-\theta^*)|\\ &+|(V^{CD}_x,\Theta^{CD}_x)||(V^{R_1}-v_*,\Theta^{R_1}-\theta_*,V^{R_3}-v^*,\Theta^{R_3}-\theta^*)|]\\ :=&\ Q_{11}, \end{aligned} \tag{2.22}$$

$$\begin{aligned} Q_{1x}=&O(1)[|(\Theta^{R_1}_{xx},\Theta^{R_1}_xV^{R_1}_x)||(V^{CD}-v_*,V^{R_3}-v^*)|\\ &+|(\Theta^{R_3}_{xx},\Theta^{R_3}_xV^{R_3}_x)||(V^{R_1}-v_*,V^{CD}-v^*)|\\ &+|(\Theta^{CD}_{xx},\Theta^{CD}_xV^{CD}_x)||(V^{R_1}-v_*,V^{R_3}-v^*)| \end{aligned}$$

$$
\begin{aligned}
&+|(V_{xx}^{R_1},V_x^{R_1 2})||(V^{CD}-v_*,\Theta^{CD}-\theta_*,V^{R_3}-v^*,\Theta^{R_3}-\theta^*)|\\
&+|(V_{xx}^{CD},V_x^{CD2})||(V^{R_1}-v_*,\Theta^{R_1}-\theta_*,V^{R_3}-v^*,\Theta^{R_3}-\theta^*)|\\
&+|(V_{xx}^{R_3},V_x^{R_3 2})||(V^{R_1}-v_*,\Theta^{R_1}-\theta_*,V^{CD}-v^*,\Theta^{CD}-\theta^*)|\\
&+|(\Theta_x^{R_1},V_x^{R_1})||(V_x^{CD},V_x^{R_3})|+|(\Theta_x^{CD},V_x^{CD})||(V_x^{R_1},V_x^{R_3})|\\
&+|(\Theta_x^{R_3},V_x^{R_3})||(V_x^{CD},V_x^{R_1})|]\\
:=&Q_{1x1}, \qquad (2.23)
\end{aligned}
$$

$$
\begin{aligned}
Q_{1xx}=&O(1)[|(V_{xxx}^{R_1},V_{xx}^{R_1}V_x^{R_1},V_x^{R_1 3},V_{xx}^{R_1}\Theta_x^{R_1},V_x^{R_1 2}\Theta_x^{R_1},V_x^{R_1}\Theta_{xx}^{R_1},\\
&\Theta_{xxx}^{R_1})||(V^{CD}-v_*,V^{R_3}-v^*)|\\
&+|(V_{xxx}^{R_3},V_{xx}^{R_3}V_x^{R_3},V_x^{R_3 3},V_{xx}^{R_3}\Theta_x^{R_3},V_x^{R_3 2}\Theta_x^{R_3},V_x^{R_3}\Theta_{xx}^{R_3},\\
&\Theta_{xxx}^{R_3})||(V^{R_1}-v_*,V^{CD}-v^*)|\\
&+|(V_{xxx}^{CD},V_{xx}^{CD}V_x^{CD},V_x^{CD3},V_{xx}^{CD}\Theta_x^{CD},V_x^{CD2}\Theta_x^{CD},V_x^{CD}\Theta_{xx}^{CD},\\
&\Theta_{xxx}^{CD})||(V^{R_1}-v_*,V^{R_3}-v^*)|\\
&+|(V_{xxx}^{R_1},V_{xx}^{R_1}V_x^{R_1},V_x^{R_1 3})||(\Theta^{CD}-\theta_*,\Theta^{R_3}-\theta^*)|\\
&+|(V_{xxx}^{R_3},V_{xx}^{R_3}V_x^{R_3},V_x^{R_3 3})||(\Theta^{R_1}-\theta_*,\Theta^{CD}-\theta^*)|\\
&+|(V_{xxx}^{CD},V_{xx}^{CD}V_x^{CD},V_x^{CD3})||(\Theta^{R_1}-\theta_*,\Theta^{R_3}-\theta^*)|\\
&+|(V_{xx}^{R_1},V_x^{R_1 2})||(V_x^{CD},V_x^{R_3},\Theta_x^{CD},\Theta_x^{R_3})|\\
&+|(V_{xx}^{R_3},V_x^{R_3 2})||(V_x^{CD},V_x^{R_1},\Theta_x^{CD},\Theta_x^{R_1})|\\
&+|(V_{xx}^{CD},V_x^{CD2})||(V_x^{R_1},V_x^{R_3},\Theta_x^{R_1},\Theta_x^{R_3})|\\
&+|V_x^{R_1}||(\Theta_{xx}^{CD},\Theta_{xx}^{R_3})|+|V_x^{CD}||(\Theta_{xx}^{R_1},\Theta_{xx}^{R_3})|\\
&+|V_x^{R_3}||(\Theta_{xx}^{CD},\Theta_{xx}^{R_1})|+|V_x^{R_1}V_x^{CD}V_x^{R_3}|+|V_x^{R_1}V_x^{R_3}|\\
&+|V_x^{R_1}V_x^{CD}|+|V_x^{CD}V_x^{R_3}|]\\
:=&Q_{1xx1}. \qquad (2.24)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
Q_2=&O(1)\{|U_x^{R_1}||(V^{CD}-v_*,\Theta^{CD}-\theta_*,V^{R_3}-v^*,\Theta^{R_3}-\theta^*)|\\
&+|U_x^{R_3}||(V^{R_1}-v_*,\Theta^{R_1}-\theta_*,V^{CD}-v^*,\Theta^{CD}-\theta^*)|\\
&+|U_x^{CD}||(V^{R_1}-v_*,\Theta^{R_1}-\theta_*,V^{R_3}-v^*,\Theta^{R_3}-\theta^*)|\\
&+\kappa|(\Theta_{xx}^{CD},\Theta_x^{CD}V_x^{CD})||(V^{R_1}-v_*,V^{R_3}-v^*)|\\
&+\kappa[|(\Theta_x^{CD},V_x^{CD})(V_x^{R_1},V_x^{R_3},\Theta_x^{R_1},\Theta_x^{R_3})|+|(\Theta_x^{R_1},\Theta_x^{R_3})(V_x^{R_1},V_x^{R_3})|]\}\\
&+O(1)\kappa(|\Theta_{xx}^{R_1}|+|\Theta_{xx}^{R_3}|)+|Q^{CD}|\\
:=&\ Q_{21}+Q_{22}+|Q^{CD}|. \qquad (2.25)
\end{aligned}
$$

$$
\begin{aligned}
Q_{2x}=&O(1)\{|U_{xx}^{R_1}||(V^{CD}-v_*,\Theta^{CD}-\theta_*,V^{R_3}-v^*,\Theta^{R_3}-\theta^*)|\\
&+|U_{xx}^{CD}||(V^{R_1}-v_*,\Theta^{R_1}-\theta_*,V^{R_3}-v^*,\Theta^{R_3}-\theta^*)|\\
&+|U_{xx}^{R_3}||(V^{R_1}-v_*,\Theta^{R_1}-\theta_*,V^{CD}-v^*,\Theta^{CD}-\theta^*)|\\
&+|(\Theta_x^{R_1},V_x^{R_1})||U_x^{R_1}||(V^{CD}-v_*,V^{R_3}-v^*)|
\end{aligned}
$$

$$
\begin{aligned}
&+|(\Theta_x^{CD},V_x^{CD})||U_x^{CD}||(V^{R_1}-v_*,V^{R_3}-v^*)|\\
&+|(\Theta_x^{R_3},V_x^{R_3})||U_x^{R_3}||(V^{R_1}-v_*,V^{CD}-v^*)|\\
&+\kappa[|\Theta_{xxx}^{CD}|+|\Theta_{xx}^{CD}||V_x^{CD}|+|\Theta_x^{CD}||V_{xx}^{CD}|+|\Theta_x^{CD}||V_x^{CD}|^2]|(V^{R_1}-v_*,V^{R_3}-v^*)|\\
&+|V_x^{R_1}U_x||(\Theta^{CD}-\theta_*,\Theta^{R_3}-\theta^*)|+|V_x^{CD}U_x||(\Theta^{R_1}-\theta_*,\Theta^{R_3}-\theta^*)|\\
&+|V_x^{R_3}U_x||(\Theta^{R_1}-\theta_*,\Theta^{CD}-\theta^*)|+|(\Theta_x^{R_1},V_x^{R_1})||(U_x^{CD},U_x^{R_3})|\\
&+|(\Theta_x^{CD},V_x^{CD})||(U_x^{R_1},U_x^{R_3})|+|(\Theta_x^{R_3},V_x^{R_3})||(U_x^{CD},U_x^{R_1})|\\
&+\kappa[|\Theta_{xx}^{CD}||(V_x^{R_1},V_x^{R_3})|+|(\Theta_{xx}^{R_1},\Theta_{xx}^{R_3})||(V_x^{R_1},V_x^{CD},V_x^{R_3})|\\
&+|\Theta_x^{CD}||(V_{xx}^{R_1},V_{xx}^{R_3})|+|(\Theta_x^{R_1},\Theta_x^{R_3})||(V_{xx}^{R_1},V_{xx}^{CD},V_{xx}^{R_3})|\\
&+|\Theta_x^{CD}||V_x^{CD}||(V_x^{R_1},V_x^{R_3})|+|\Theta_x^{CD}||(V_x^{R_1},V_x^{R_3})|^2\\
&+|(\Theta_x^{R_1},\Theta_x^{R_3})||(V_x^{R_1},V_x^{CD},V_x^{R_3})|^2]\}\\
&+O(1)\kappa|(\Theta_{xxx}^{R_1},\Theta_{xxx}^{R_3})|+|Q_x^{CD}|\\
:=\;& Q_{2x1}+Q_{2x2}+|Q_x^{CD}|. \qquad (2.26)
\end{aligned}
$$

Here Q_{11},Q_{1x1},Q_{1xx1} and Q_{21},Q_{2x1} represent the interactions coming from different wave patterns, Q_{22},Q_{2x2} represents the error terms coming from the approximate rarefaction wave profiles, and Q^{CD},Q_x^{CD} is the error term defined in (2.10) due to the viscous contact wave.

Then, as in [8], we estimate the interaction terms $Q_{11},Q_{1x1},Q_{1xx1},Q_{21},Q_{2x1}$ by dividing the whole domain $\Omega=\{(t,x)|(t,x)\in\mathbb{R}^+\times\mathbb{R}\}$ into three regions:

$$
\begin{aligned}
\Omega_-&=\{(t,x)|2x\le\lambda_{1*}(t+t_0)\},\\
\Omega_{CD}&=\{(t,x)|\lambda_{1*}(t+t_0)<2x<\lambda_3^*(t+t_0)\},\\
\Omega_+&=\{(t,x)|2x\ge\lambda_3^*(t+t_0)\},
\end{aligned}
$$

where $\lambda_{1*}=\lambda_1(v_*,\theta_*)$, $\lambda_3^*=\lambda_3(v^*,\theta^*)$.

Now from Lemma 2.3, we have the following estimates in each region:

- In Ω_-,

$$
\begin{aligned}
|V^{R_3}-v^*|&=O(1)e^{-\frac{2|x|+2\lambda_3^*(t+t_0)}{\sigma}}\\
&=O(1)e^{-\lambda_3^*\kappa^{-1/5}}e^{-\frac{2|x|+\lambda_3^*(t+t_0)}{\kappa^{2/5}}},\\
|(V^{CD}-v_*,V^{CD}-v^*)|&=O(1)\delta^{CD}e^{-\frac{C[\lambda_{1*}(t+t_0)]^2}{4\kappa(1+t)}}\\
&=O(1)e^{-\frac{Ct_0(t+t_0)}{\kappa}}\\
&=O(1)e^{-\frac{Ct_0(|x|+t+t_0)}{\kappa}}\\
&=O(1)e^{-C\kappa^{-3/5}}e^{-\frac{C(|x|+t+t_0)}{\kappa^{4/5}}};
\end{aligned}
$$

- In Ω_{CD},

$$
\begin{aligned}
|V^{R_1}-v_*|&=O(1)e^{-\frac{2|x|+2|\lambda_{1*}|(t+t_0)}{\sigma}}\\
&=O(1)e^{-|\lambda_{1*}|\kappa^{-1/5}}e^{-\frac{2|x|+|\lambda_{1*}|(t+t_0)}{\kappa^{2/5}}},\\
|V^{R_3}-v^*|&=O(1)e^{-\frac{2|x|+2\lambda_3^*(t+t_0)}{\sigma}}\\
&=O(1)e^{-\lambda_3^*\kappa^{-1/5}}e^{-\frac{2|x|+\lambda_3^*(t+t_0)}{\kappa^{2/5}}};
\end{aligned}
$$

- In Ω_+,

$$\begin{aligned}|V^{R_1}-v_*|&=O(1)e^{-\frac{2|x|+2|\lambda_1^*|(t+t_0)}{\sigma}}\\&=O(1)e^{-|\lambda_{1*}|\kappa^{-1/5}}e^{-\frac{2|x|+2|\lambda_{1*}|(t+t_0)}{\kappa^{2/5}}},\\|(V^{CD}-v_*,V^{CD}-v^*)|&=O(1)\delta^{CD}e^{-\frac{C[\lambda_3^*(t+t_0)]^2}{4\kappa(1+t)}}\\&=O(1)e^{-\frac{Ct_0(t+t_0)}{\kappa}}\\&=O(1)e^{-\frac{Ct_0(|x|+t+t_0)}{\kappa}}\\&=O(1)e^{-C\kappa^{-3/5}}e^{-\frac{C(|x|+t+t_0)}{\kappa^{4/5}}}.\end{aligned}$$

Hence, in summary, it follows from (2.22)-(2.26) that

$$|(Q_{11},Q_{1x1},Q_{1xx1},Q_{21},Q_{2x1})|=O(1)e^{-C\kappa^{-1/5}}e^{-\frac{C(|x|+t+t_0)}{\kappa^{2/5}}},\tag{2.27}$$

for some positive constant C.

Now, we consider (1.8) with the initial values

$$(v,u,\theta)(t=0,x)=(V,U,\Theta)(t=0,x).\tag{2.28}$$

Introduce the following scaled variables

$$y=\frac{x}{\kappa},\ \tau=\frac{t}{\kappa},$$

then the unknown functions and the approximate wave profiles will be denoted by $(v,u,\theta)(\tau,y)$ and $(V,U,\Theta)(\tau,y)$. Set the perturbation around the composite wave pattern $(V,U,\Theta)(\tau,y)$ by

$$(\phi,\psi,\zeta)(\tau,y)=(v-V,u-U,\theta-\Theta)(\tau,y).$$

Then the perturbation $(\phi,\psi,\zeta)(\tau,y)$ satisfies the system

$$\begin{cases}\phi_\tau-\psi_y=0,\\\psi_\tau+(p-P)_y=-\kappa Q_1,\\\frac{R}{\gamma-1}\zeta_\tau+pu_y-PU_y=(\frac{\theta_y}{v})_y-(\frac{\Theta_y}{V})_y-\kappa Q_2,\\(\phi,\psi,\zeta)(\tau=0,y)=0.\end{cases}\tag{2.29}$$

2.4. Main result. The main results of this paper are as follows:

THEOREM 2.1. *Given a Riemann solution $(\bar V,\bar U,\bar\Theta)(t,x)$ defined in (2.18), which is a superposition of two rarefaction waves and a contact discontinuity for the Euler system (1.5), there exist small positive constants δ_0 and κ_0 such that if the contact wave strength $\delta^{CD}\le\delta_0$ and the heat conductivity coefficient $\kappa\le\kappa_0$, then the compressible Navier-Stokes Equations (1.8)-(1.9) and (2.28) admit a unique global solution $(v,u,\theta)(t,x)$ satisfying*

$$\sup_{(t,x)\in\Sigma_h}|(v,u,\theta)(t,x)-(\bar V,\bar U,\bar\Theta)(t,x)|\le C_h\kappa^{\frac15},\forall h>0,\tag{2.30}$$

where $\Sigma_h=\{(t,x)|t\ge h,\frac{x}{\sqrt{1+t}}\ge h\kappa^\alpha,0<\alpha<\frac12\}$, the positive constant C_h depends only on h but is independent of κ.

REMARK 2.1. Theorem 2.1 shows that, we can obtain the same result as that in [8] when the viscosity coefficient equals to zero or the viscosity coefficient is of a higher order than that of the coefficient of heat conductivity κ. Compared with the result in [8], the main difficulty in the proof here is that, since the viscosity coefficient equals zero, the first two equations of the Navier-Stokes system (1.8) are hyperbolic and only the last one is parabolic. We can not directly lead to the a priori estimates for $\int\|\psi_y(\tau,\cdot)\|^2 d\tau$ (see (2.29)). To circumvent such difficulty, we use the method in [13] to control the term $\int\|(\phi_y,\psi_y)(\tau,\cdot)\|^2 d\tau$ by $\int\|\zeta_y(\tau,\cdot)\|^2 d\tau$, with the help of the symmetry of matrix S (see Section 3.2).

NOTATION 2.1. *In this paper,* $|a| = (\sum_{i=1}^n a_i^2)^{\frac{1}{2}}$ *if* $a = (a_i)$ *is a vector in* $\mathbb{R}^n$ *and* $|A| = (\sum_{i=1}^n \sum_{j=1}^n A_{ij}^2)^{\frac{1}{2}}$ *if* $A = (A_{ij})_{n\times n}$ *is a matrix. For function spaces,* $H^l(\mathbb{R})$ $(l \geq 1)$ *denotes the* l*-th order Sobolev space with its norm* $\|f\|_l = (\sum_{j=0}^l \|\partial_y^j f\|^2)^{\frac{1}{2}}$ *and* $\|\cdot\| := \|\cdot\|_{L^2(dy)}$*, where* $L^2(dz)$ *means the* L^2 *integral over* $\mathbb{R}$ *with respect to the Lebesgue measure* dz*, and* $z = x$ *or* y*.*

3. Proof of Theorem 2.1

We will prove Theorem 2.1 in this section. Due to the estimate (2.20), to prove the main theorem, it suffices to verify the following theorem.

THEOREM 3.1. *There exist small positive constants* δ_1 *and* κ_1 *such that if the initial values and the contact wave strength* δ^{CD} *satisfy*

$$\mathcal{N}(\tau)|_{\tau=0} + \delta^{CD} \leq \delta_1, \tag{3.1}$$

and $\kappa \leq \kappa_1$*, then (2.29) admits a unique global solution* $(v,u,\theta)(\tau,y)$ *satisfying*

$$\sup_{\tau,y} |(v,u,\theta)(\tau,y) - (V,U,\Theta)(\tau,y)| \leq C\kappa^{\frac{1}{5}}. \tag{3.2}$$

Here $\mathcal{N}(\tau)$ *is defined by (3.3) below.*

We will focus on the reformulated system (2.29). By the standard local existence theory to (2.29) and the continuity argument, to prove the global existence, we only need to close the following a priori estimate

$$\mathcal{N}(\tau) = \sup_{0\leq\tau'\leq\tau} \|(\phi,\psi,\zeta)(\tau',\cdot)\|_2^2 \leq \chi^2, \tag{3.3}$$

where χ is a small positive constant depending only on the initial values and the strength of the contact wave. This is a consequence of a series of lemmas. We start with the lower order estimate.

3.1. Lower order estimate.

LEMMA 3.1. *Under the assumptions of Theorem 3.1, there exists a constant* $C > 0$ *and a sufficiently small constant* $\beta_1 > 0$*, such that*

$$\begin{aligned}
&\|(\phi,\psi,\zeta)(\tau,\cdot)\|^2 + \int_0^\tau \|\zeta_y\|^2\ d\tau + \int_0^\tau \|\sqrt{(U_y^{R_1}, U_y^{R_3})}(\phi,\zeta)\|^2\ d\tau \\
&\leq C_{\beta_1} \int_0^\tau (\tau+\tau_0)^{-2} \|(\phi,\zeta)\|^2\ d\tau + C_{\beta_1}\kappa^{\frac{2}{5}} \\
&\quad + \beta_1 \int_0^\tau \|(\phi_y,\psi_y)\|^2 d\tau + C\int_0^\tau \int_{\mathbb{R}} \kappa(1+\kappa\tau)^{-1}\delta^{CD} e^{-\frac{C_0\kappa y^2}{1+\kappa\tau}} (\phi^2+\zeta^2)\ dy d\tau.
\end{aligned} \tag{3.4}$$

Proof. Similar to [8], we multiply $(2.29)_2$ by ψ firstly,

$$(\frac{\psi^2}{2})_\tau-(p-P)\psi_y=-\kappa\psi Q_1-[\psi(p-P)]_y. \tag{3.5}$$

Since $p-P=\frac{R\zeta}{v}+R\Theta(\frac{1}{v}-\frac{1}{V})$ and $\phi_\tau=\psi_y$, we have

$$(\frac{\psi^2}{2})_\tau-R\frac{\zeta}{v}\psi_y-R\Theta(\frac{1}{v}-\frac{1}{V})\phi_\tau=-\kappa\psi Q_1-[\psi(p-P)]_y. \tag{3.6}$$

Set $\Phi(z)=z-1-\ln z$, then it's easy to note that $\Phi(1)=\Phi'(1)=0$ and $\Phi(z)$ is strictly convex around $z=1$. Direct calculation yields that

$$[R\Theta\Phi(\frac{v}{V})]_\tau=R\Theta_\tau\Phi(\frac{v}{V})-R\Theta(\frac{1}{v}-\frac{1}{V})\phi_\tau-\frac{PV_\tau}{vV}\phi^2, \tag{3.7}$$

$$[\frac{R}{\gamma-1}\Theta\Phi(\frac{\theta}{\Theta})]_\tau=\frac{R}{\gamma-1}(1-\frac{\Theta}{\theta})\zeta_\tau+\frac{R}{\gamma-1}\Phi(\frac{\theta}{\Theta})\Theta_\tau-\frac{R}{\gamma-1}\frac{\Theta_\tau\zeta^2}{\theta\Theta}. \tag{3.8}$$

By $(2.29)_3$,

$$\begin{aligned}&\frac{R}{\gamma-1}(1-\frac{\Theta}{\theta})\zeta_\tau\\ =&(1-\frac{\Theta}{\theta})[-pu_y+PU_y+(\frac{\theta_y}{v})_y-(\frac{\Theta_y}{V})_y-\kappa Q_2]\\ =&-\frac{R\zeta}{v}\psi_y-\frac{\zeta}{\theta}(p-P)U_y-\frac{\zeta_y^2}{\theta v}-\frac{\zeta_y}{\theta}(\frac{1}{v}-\frac{1}{V})\Theta_y+\frac{\zeta\theta_y}{\theta^2}(\frac{\theta_y}{v}-\frac{\Theta_y}{V})-\kappa\frac{\zeta}{\theta}Q_2+[\frac{\zeta}{\theta}(\frac{\theta_y}{v}-\frac{\Theta_y}{V})]_y.\end{aligned} \tag{3.9}$$

Substituting (3.7)-(3.9) into (3.6) gives

$$\begin{aligned}&[\frac{\psi^2}{2}+R\Theta\Phi(\frac{v}{V})+\frac{R}{\gamma-1}\Theta\Phi(\frac{\theta}{\Theta})]_\tau+\frac{\zeta_y^2}{\theta v}+J_1\\ =&-\frac{\zeta_y}{\theta}(\frac{1}{v}-\frac{1}{V})\Theta_y+\frac{\zeta\theta_y}{\theta^2}(\frac{\theta_y}{v}-\frac{\Theta_y}{V})-\kappa\frac{\zeta}{\theta}Q_2-\kappa\psi Q_1+[\frac{\zeta}{\theta}(\frac{\theta_y}{v}-\frac{\Theta_y}{V})-\psi(p-P)]_y,\end{aligned} \tag{3.10}$$

where

$$J_1=\frac{\zeta}{\theta}(p-P)U_y-R\Theta_\tau\Phi(\frac{v}{V})-\frac{R}{\gamma-1}\Phi(\frac{\theta}{\Theta})\Theta_\tau+\frac{PV_\tau}{vV}\phi^2+\frac{R}{\gamma-1}\frac{\Theta_\tau\zeta^2}{\theta\Theta}. \tag{3.11}$$

Direct calculation shows that

$$J_1=PU_y[\gamma\Phi(\frac{v}{V})+\Phi(\frac{\theta V}{v\Theta})]-[(\frac{\Theta_y}{V})_y+\kappa Q_2][(\gamma-1)\Phi(\frac{v}{V})-\Phi(\frac{\Theta}{\theta})]. \tag{3.12}$$

Substituting (3.12) into (3.10) gives

$$\begin{aligned}&[\frac{\psi^2}{2}+R\Theta\Phi(\frac{v}{V})+\frac{R}{\gamma-1}\Theta\Phi(\frac{\theta}{\Theta})]_\tau+\frac{\zeta_y^2}{\theta v}+P(U_y^{R_1}+U_y^{R_3})[\gamma\Phi(\frac{v}{V})+\Phi(\frac{\theta V}{v\Theta})]\\ =&J_2-\kappa\frac{\zeta}{\theta}Q_2-\kappa\psi Q_1+(\cdots)_y,\end{aligned} \tag{3.13}$$

where

$$
\begin{aligned}
J_2 = & -PU_y^{CD}[\gamma\Phi(\frac{v}{V})+\Phi(\frac{\theta V}{v\Theta})]+[(\frac{\Theta_y}{V})_y+\kappa Q_2][(\gamma-1)\Phi(\frac{v}{V})-\Phi(\frac{\Theta}{\theta})] \\
& -\frac{\zeta_y}{\theta}(\frac{1}{v}-\frac{1}{V})\Theta_y+\frac{\zeta\theta_y}{\theta^2}(\frac{\theta_y}{v}-\frac{\Theta_y}{V}).
\end{aligned} \tag{3.14}
$$

Here, $(\cdots)_y$ represents the conservative terms which vanish after integrating in y over $\mathbb{R}$.

Since $\Phi(z)$ is strictly convex around $z=1$, by the a priori assumption (3.3) with sufficiently small $\chi>0$, there exist positive constants c_1, c_2 such that

$$
\begin{aligned}
& c_1\phi^2\le\Phi(\frac{v}{V})\le c_2\phi^2,\ c_1\zeta^2\le\Phi(\frac{\Theta}{\theta}),\Phi(\frac{\theta}{\Theta})\le c_2\zeta^2, \\
& c_1(\phi^2+\zeta^2)\le\Phi(\frac{\theta V}{v\Theta})\le c_2(\phi^2+\zeta^2).
\end{aligned} \tag{3.15}
$$

Hence, we have

$$
\begin{aligned}
\int_{\mathbb{R}}|J_2|dy\le & \int_{\mathbb{R}}\frac{1}{4}\frac{\zeta_y^2}{\theta v}dy+C\int_{\mathbb{R}}\kappa(1+\kappa\tau)^{-1}\delta^{CD}e^{-\frac{C_0\kappa y^2}{1+\kappa\tau}}(\phi^2+\zeta^2)dy \\
& +C_\beta(\tau+\tau_0)^{-2}\|(\phi,\zeta)\|^2+\beta\int_{\mathbb{R}}(\phi_y^2+\zeta_y^2)dy+\int_{\mathbb{R}}\kappa|Q_2|(\phi^2+\zeta^2)dy,
\end{aligned} \tag{3.16}
$$

where $\beta>0$ is a constant to be determined later. We now estimate the terms $\kappa\frac{\zeta}{\theta}Q_2$, $\kappa\psi Q_1$ in (3.13) and $\kappa|Q_2|(\phi^2+\zeta^2)$ in (3.16).

Firstly, from the estimate in (2.27), we have

$$
\begin{aligned}
\int_0^\tau\int_{\mathbb{R}}\kappa|Q_1||\psi|dyd\tau = & \int_0^\tau\int_{\mathbb{R}}\kappa|Q_{11}||\psi|dyd\tau \\
\le & \int_0^\tau\|\psi\|_{L_y^\infty}\int_{\mathbb{R}}|Q_{11}|dxd\tau \\
\le & C\int_0^\tau\|\psi\|^{\frac{1}{2}}\|\psi_y\|^{\frac{1}{2}}e^{-C\kappa^{-1/5}}e^{-\frac{C(t+t_0)}{\kappa^{2/5}}}d\tau \\
\le & \beta\int_0^\tau\|\psi_y\|^2d\tau+C_\beta e^{-C\kappa^{-1/5}}\sup_{[0,\tau]}\|\psi(\tau)\|^{\frac{2}{3}} \\
\le & \beta\int_0^\tau\|\psi_y\|^2d\tau+\beta\sup_{[0,\tau]}\|\psi(\tau)\|^2+C_\beta e^{-C\kappa^{-1/5}}.
\end{aligned} \tag{3.17}
$$

For $\kappa\frac{\zeta}{\theta}Q_2$, (2.25) yields that

$$
\int_0^\tau\int_{\mathbb{R}}\kappa|\frac{\zeta}{\theta}||Q_2|dyd\tau=\int_0^\tau\int_{\mathbb{R}}\kappa|\frac{\zeta}{\theta}|(|Q_{21}|+|Q_{22}|+|Q^{CD}|)dyd\tau.
$$

We estimate these three terms separately below. By (2.27), similar to (3.17), we get

$$
\begin{aligned}
& \int_0^\tau\int_{\mathbb{R}}\kappa|\frac{\zeta}{\theta}||Q_{21}|dyd\tau \\
\le & \beta\int_0^\tau\|\zeta_y\|^2d\tau+\beta\sup_{[0,\tau]}\|\zeta(\tau)\|^2+C_\beta e^{-C\kappa^{-1/5}},
\end{aligned} \tag{3.18}
$$

From Lemma 2.3, we have

$$\begin{aligned}\int_0^\tau\int_{\mathbb{R}}\kappa\left|\frac{\zeta}{\theta}\right||Q_{22}|dyd\tau &\leq C\kappa\int_0^\tau\|\zeta\|_{L^\infty_y}(\|\Theta^{R_1}_{xx}\|_{L^1}+\|\Theta^{R_3}_{xx}\|_{L^1})d\tau\\ &\leq C\int_0^\tau\|\zeta\|^{\frac12}\|\zeta_y\|^{\frac12}(\tau+\tau_0)^{-1}d\tau\\ &\leq \beta\int_0^\tau\|\zeta_y\|^2d\tau+C_\beta\int_0^\tau(\tau+\tau_0)^{-\frac43}\|\zeta\|^{\frac23}d\tau\\ &\leq \beta\int_0^\tau\|\zeta_y\|^2d\tau+C_\beta\sup_{[0,\tau]}\|\zeta(\tau)\|^{\frac23}3\tau_0^{-\frac13}\\ &\leq \beta\int_0^\tau\|\zeta_y\|^2d\tau+\beta\sup_{[0,\tau]}\|\zeta(\tau)\|^2+C_\beta\kappa^{\frac25}. \end{aligned}\tag{3.19}$$

For the last term, using the estimate in (2.10), we obtain

$$\begin{aligned}\int_0^\tau\int_{\mathbb{R}}\kappa\left|\frac{\zeta}{\theta}\right||Q^{CD}|dyd\tau &\leq C\kappa^2\int_0^\tau\|\zeta\|_{L^\infty_y}\int_{\mathbb{R}}(1+\kappa\tau)^{-2}e^{-\frac{C_0\kappa y^2}{1+\kappa\tau}}dyd\tau\\ &\leq C\kappa^{\frac32}\int_0^\tau\|\zeta\|^{\frac12}\|\zeta_y\|^{\frac12}(1+\kappa\tau)^{-\frac32}d\tau\\ &\leq \beta\int_0^\tau\|\zeta_y\|^2d\tau+C_\beta\kappa^2\sup_{[0,\tau]}\|\zeta(\tau)\|^{\frac23}\int_0^\tau(1+\kappa\tau)^{-2}d\tau\\ &\leq \beta\int_0^\tau\|\zeta_y\|^2d\tau+\beta\sup_{[0,\tau]}\|\zeta(\tau)\|^2+C_\beta\kappa^{\frac32}. \end{aligned}\tag{3.20}$$

Finally, we can estimate the term $\kappa|Q_2|(\phi^2+\zeta^2)$ similarly as the terms $\kappa\frac{\zeta}{\theta}Q_2$, $\kappa\psi Q_1$ and get that

$$\begin{aligned}&\int_0^\tau\int_{\mathbb{R}}\kappa|Q_2|(\phi^2+\zeta^2)dyd\tau\\ \leq&\beta\int_0^\tau(\|\phi_y\|^2+\|\zeta_y\|^2)d\tau+C_\beta\int_0^\tau(\tau+\tau_0)^{-2}\|(\phi,\zeta)\|^2(\tau)d\tau\\ &+C_\beta e^{-C\kappa^{-\frac15}}\sup_{[0,\tau]}\|(\phi,\zeta)(\tau)\|^2+\kappa\int_0^\tau\int_{\mathbb{R}}\delta^{CD}\kappa(1+\kappa\tau)^{-2}e^{-\frac{C_0\kappa y^2}{1+\kappa\tau}}(\phi^2+\zeta^2)dyd\tau. \end{aligned}\tag{3.21}$$

By substituting (3.16)-(3.21) into (3.13) and choosing β,κ suitably small, we can obtain (3.4). □

3.2. Higher order estimate.

LEMMA 3.2. *Under the assumptions of Theorem 3.1, there exists a sufficiently small constant $\beta_4>0$, such that*

$$\begin{aligned}&\|(\phi,\psi,\zeta)(\tau,\cdot)\|_1^2+\int_0^\tau\|(\phi_y,\psi_y,\zeta_y)\|^2\ d\tau+\int_0^\tau\|\zeta_{yy}\|^2d\tau\\ \leq&\beta_4\int_0^\tau\|(\phi_{yy},\psi_{yy})\|^2d\tau+C_{\beta_4}\int_0^\tau(\tau+\tau_0)^{-2}\|(\phi,\psi,\zeta)\|^2\ d\tau+C_{\beta_4}\kappa^{\frac25}\\ &+C_{\beta_4}\int_0^\tau\int_{\mathbb{R}}\delta^{CD}\kappa(1+\kappa\tau)^{-1}e^{-\frac{C_0\kappa y^2}{1+\kappa\tau}}(\phi^2+\psi^2+\zeta^2)\ dyd\tau. \end{aligned}\tag{3.22}$$

Proof.

Step 1. Multiplying $(2.29)_2$ by $-(\Theta\psi_y)_y$ and $(2.29)_3$ by $-\zeta_{yy}$, and then integrating the resulting equation with respect to y,τ yields that

$$
\begin{aligned}
&\int_{\mathbb{R}}(\Theta\frac{\psi_y^2}{2}+\frac{p\Theta}{v}\frac{\phi_y^2}{2}+\frac{R}{\gamma-1}\frac{\zeta_y^2}{2})dy+\int_0^\tau\int_{\mathbb{R}}\frac{1}{v}\zeta_{yy}^2dyd\tau\\
&\quad+\int_0^\tau\int_{\mathbb{R}}(\frac{4p}{v}+\frac{\gamma-1}{v^2}p)(U_y^{R_1}+U_y^{R_3})\Theta\frac{\phi_y^2}{2}dyd\tau\\
=&\int_0^\tau\int_{\mathbb{R}}[-(\frac{4p}{v}+\frac{\gamma-1}{v^2}p)(U_y^{CD}+\psi_y)+\frac{\gamma-1}{v^2}(\frac{\theta_y}{v})_y]\Theta\frac{\phi_y^2}{2}dyd\tau\\
&\quad-\int_0^\tau\int_{\mathbb{R}}[R(-2\frac{V_y^2}{V^3}+\frac{V_{yy}}{V^2})\Theta+2R\frac{V_y}{V^2}\Theta_y+\kappa Q_{1y}+2R\frac{v_y}{v^3}\theta V_y-\frac{R}{v^2}\theta_yV_y\\
&\quad-\frac{R\theta}{v^2}V_{yy}-\frac{R}{v^2}\theta_yv_y]\Theta\psi_ydyd\tau\\
&\quad+\int_0^\tau\int_{\mathbb{R}}(R\Theta U_y\frac{-\phi}{vV}+\frac{R\zeta}{v}U_y+\frac{v_y\theta_y}{v^2}+\frac{\phi}{vV}\Theta_{yy}-\frac{V_y\Theta_y}{V^2}+\kappa Q_2)\zeta_{yy}\ dyd\tau\\
&\quad+\int_0^\tau\int_{\mathbb{R}}\{\frac{-p}{v}\phi_y\Theta_y\psi_y+(\frac{\psi_y^2}{2}+\frac{p}{v}\frac{\phi_y^2}{2})\frac{\gamma-1}{R}[-PU_y+(\frac{\Theta_y}{V})_y+\kappa Q_2]\\
&\quad+\frac{R\Theta}{vV}\phi\Theta_{yy}\psi_y-\frac{R\Theta}{v}\psi_y\zeta_{yy}+\frac{R\theta}{v}\psi_y\zeta_{yy}\}dyd\tau\\
=&I_1+I_2+I_3+I_4.
\end{aligned}\tag{3.23}
$$

We now estimate I_1,I_2,I_3,I_4 separately below. By (2.7), (2.8) and (3.3), we get

$$
\begin{aligned}
I_1=&\int_0^\tau\int_{\mathbb{R}}[-(\frac{4p}{v}+\frac{\gamma-1}{v^2}p)(U_y^{CD}+\psi_y)+\frac{\gamma-1}{v^2}(\frac{\theta_y}{v})_y]\Theta\frac{\phi_y^2}{2}dyd\tau\\
\le& C\int_0^\tau\int_{\mathbb{R}}\kappa|U_x^{CD}|\phi_y^2dyd\tau+C\int_0^\tau\int_{\mathbb{R}}|\psi_y|\phi_y^2dyd\tau\\
&+C\int_0^\tau\int_{\mathbb{R}}|(\frac{1}{v})_y\theta_y|\frac{\phi_y^2}{2}dyd\tau+C\int_0^\tau\int_{\mathbb{R}}|\frac{1}{v}\theta_{yy}|\frac{\phi_y^2}{2}dyd\tau\\
\le& C\int_0^\tau\|\kappa U_x^{CD}\|_{L_x^\infty}\|\phi_y\|^2d\tau+C\int_0^\tau\|\phi_y\|_{L_y^\infty}\int_{\mathbb{R}}|\phi_y||\psi_y|dyd\tau\\
&+C\int_0^\tau\int_{\mathbb{R}}|-\frac{v_y}{v^2}\theta_y|\frac{\phi_y^2}{2}dyd\tau+C\int_0^\tau\int_{\mathbb{R}}|\frac{1}{v}\theta_{yy}|\frac{\phi_y^2}{2}dyd\tau\\
\le& C\int_0^\tau\kappa^2[\kappa(1+t)]^{-1}\|\phi_y\|^2d\tau+C\int_0^\tau\|\phi_y\|^{\frac12}\|\phi_{yy}\|^{\frac12}\|\psi_y\|\|\phi_y\|d\tau\\
&+C\int_0^\tau\int_{\mathbb{R}}|-\frac{V_y+\phi_y}{v^2}||(\Theta_y+\zeta_y)|\frac{\phi_y^2}{2}dyd\tau+C\int_0^\tau\int_{\mathbb{R}}|\frac{\Theta_{yy}+\zeta_{yy}}{v}|\frac{\phi_y^2}{2}dyd\tau\\
\le& C\kappa\int_0^\tau\|\phi_y\|^2d\tau+C\chi\int_0^\tau\|(\phi_y,\psi_y)\|^2d\tau+C\int_0^\tau\int_{\mathbb{R}}|V_y||\Theta_y|\phi_y^2dyd\tau\\
&+C\int_0^\tau\int_{\mathbb{R}}(|V_y||\zeta_y|+|\Theta_y||\phi_y|)\phi_y^2dyd\tau+C\int_0^\tau\int_{\mathbb{R}}|\phi_y|^3|\zeta_y|dyd\tau\\
&+C\int_0^\tau\int_{\mathbb{R}}|\Theta_{yy}+\zeta_{yy}|\phi_y^2dyd\tau\\
=& C\kappa\int_0^\tau\|\phi_y\|^2d\tau+C\chi\int_0^\tau\|(\phi_y,\psi_y)\|^2d\tau+I_{11}+I_{12}+I_{13}+I_{14}.
\end{aligned}\tag{3.24}
$$

By (2.7), (2.8) and Lemma 2.3, we have

$$
\begin{aligned}
I_{11} &= C\int_0^\tau\int_{\mathbb{R}}|V_y||\Theta_y|\phi_y^2dyd\tau\\
&\le C\int_0^\tau\|\kappa V_x\|_{L_x^\infty}\|\kappa\Theta_x\|_{L_x^\infty}\int_{\mathbb{R}}\phi_y^2dyd\tau\\
&\le C\int_0^\tau[(\tau+\tau_0)^{-2}+\kappa(1+\kappa\tau)^{-1}]\|\phi_y\|^2d\tau\\
&\le C\kappa\int_0^\tau\|\phi_y\|^2d\tau.
\end{aligned}
$$

Similar to I_{11} and using (3.3) additionally,

$$
\begin{aligned}
I_{12} &= C\int_0^\tau\int_{\mathbb{R}}(|V_y||\zeta_y|+|\Theta_y||\phi_y|)\phi_y^2dyd\tau\\
&\le C\int_0^\tau\int_{\mathbb{R}}(V_y^2+\Theta_y^2)\phi_y^2dyd\tau+C\int_0^\tau\int_{\mathbb{R}}(\zeta_y^2+\phi_y^2)\phi_y^2dyd\tau\\
&\le C\int_0^\tau(\|\kappa V_x\|_{L_x^\infty}^2+\|\kappa\Theta_x\|_{L_x^\infty}^2)\|\phi_y\|^2d\tau+C\int_0^\tau(\|\zeta_y\|_{L_x^\infty}^2+\|\phi_y\|_{L_x^\infty}^2)\|\phi_y\|^2d\tau\\
&\le C\kappa\int_0^\tau\|\phi_y\|^2d\tau+C\int_0^\tau(\|\zeta_y\|\|\zeta_{yy}\|+\|\phi_y\|\|\phi_{yy}\|)\|\phi_y\|^2d\tau\\
&\le C(\kappa+\chi^2)\int_0^\tau\|\phi_y\|^2d\tau.
\end{aligned}
$$

Similarly, we can easily obtain that

$$
\begin{aligned}
I_{13} &= C\int_0^\tau\int_{\mathbb{R}}|\phi_y|^3|\zeta_y|dyd\tau\\
&\le C\chi^2\int_0^\tau\|\phi_y\|^2d\tau.
\end{aligned}
$$

$$
\begin{aligned}
I_{14} &= C\int_0^\tau\int_{\mathbb{R}}|\Theta_{yy}+\zeta_{yy}|\phi_y^2dyd\tau\\
&\le C\int_0^\tau\|\Theta_{yy}\|_{L^\infty}\|\phi_y\|^2d\tau+C\int_0^\tau\|\phi_y\|_{L^\infty}\int_{\mathbb{R}}|\phi_y||\zeta_{yy}|dyd\tau\\
&\le C\kappa\int_0^\tau\|\phi_y\|^2d\tau+C\chi\int_0^\tau\|(\phi_y,\zeta_{yy}\|^2d\tau.
\end{aligned}
$$

Hence, substituting $I_{11}-I_{14}$ into (3.24) gives

$$
I_1\le C(\chi+\kappa)\int_0^\tau\|(\phi_y,\psi_y)\|^2d\tau+C(\chi+\kappa)\int_0^\tau\|\zeta_{yy}\|^2d\tau.
$$

By (2.7), (2.8) and Lemma 2.3 again, we get

$$
\begin{aligned}
I_2 = &-\int_0^\tau\int_{\mathbb{R}}[R(-2\frac{V_y^2}{V^3}+\frac{V_{yy}}{V^2})\Theta+2R\frac{V_y}{V^2}\Theta_y+\kappa Q_{1y}+2R\frac{v_y}{v^3}\theta V_y-\frac{R}{v^2}\theta_yV_y\\
&-\frac{R\theta}{v^2}V_{yy}-\frac{R}{v^2}\theta_yv_y]\Theta\psi_ydyd\tau
\end{aligned}
$$

$$
\begin{aligned}
&\le C\int_0^\tau\int_{\mathbb{R}}[|V_y|^2+|V_{yy}|+|V_y||\Theta_y|+|V_y|(|\phi_y|+|\zeta_y|)+|\Theta_y||\phi_y| \\
&\quad +|\phi_y||\zeta_y|+\kappa|Q_{1y}|]|\psi_y|dyd\tau \\
&\le C_\beta\int_0^\tau\int_{\mathbb{R}}(V_y^4+\Theta_y^4+V_{yy}^2)dyd\tau+\beta\int_0^\tau\|\psi_y\|^2d\tau \\
&\quad C\int_0^\tau\|(V_y,\Theta_y)\|_{L^\infty}\|(\phi_y,\psi_y,\zeta_y)\|^2d\tau \\
&\quad +C\int_0^\tau\|\zeta_y\|_{L^\infty}\|(\phi_y,\psi_y)\|^2d\tau+C_\beta\int_0^\tau\int_{\mathbb{R}}\kappa^2|Q_{1y}|^2dyd\tau \\
&\le C(\beta+\kappa^{\frac{1}{2}}+\chi)\int_0^\tau\|(\phi_y,\psi_y,\zeta_y)\|^2d\tau+C_\beta\kappa^{\frac{1}{2}},
\end{aligned}
\tag{3.25}
$$

where, for $\int_0^\tau\int_{\mathbb{R}}\kappa^2|Q_{1y}|^2dyd\tau$, we use (2.27).

For I_3, we split it into three terms and then estimate them separately

$$
\begin{aligned}
I_3=&\int_0^\tau\int_{\mathbb{R}}(R\Theta U_y\frac{-\phi}{vV}+\frac{R\zeta}{v}U_y+\frac{v_y\theta_y}{v^2}+\frac{\phi}{vV}\Theta_{yy}-\frac{V_y\Theta_y}{V^2}+\kappa Q_2)\zeta_{yy}\ dyd\tau \\
\le& C\int_0^\tau\int_{\mathbb{R}}[|U_y|(|\phi|+|\zeta|)]|\zeta_{yy}|dyd\tau+\int_0^\tau\int_{\mathbb{R}}\kappa|Q_2||\zeta_{yy}|\ dyd\tau \\
&+C\int_0^\tau\int_{\mathbb{R}}(|V_y||\Theta_y|+|V_y||\zeta_y|+|\Theta_y||\phi_y|+|\phi_y||\zeta_y|+|\Theta_{yy}||\phi|)|\zeta_{yy}|dyd\tau \\
=&I_{31}+I_{32}+I_{33}.
\end{aligned}
\tag{3.26}
$$

By (2.7), (2.8) and Lemma 2.3, we get

$$
\begin{aligned}
I_{31}=&C\int_0^\tau\int_{\mathbb{R}}[|U_y|(|\phi|+|\zeta|)]|\zeta_{yy}|dyd\tau \\
\le&\beta\int_0^\tau\|\zeta_{yy}\|^2d\tau+C_\beta\int_0^\tau\int_{\mathbb{R}}|U_y|^2|(\phi,\zeta)|^2dyd\tau \\
\le&\beta\int_0^\tau\|\zeta_{yy}\|^2d\tau+C_\beta\int_0^\tau(\tau+\tau_0)^{-2}\|(\phi,\zeta)\|^2d\tau \\
&+C_\beta\int_0^\tau\int_{\mathbb{R}}\delta^{CD}\kappa(1+\kappa\tau)^{-1}e^{-\frac{C_0\kappa y^2}{1+\kappa\tau}}|(\phi,\zeta)|^2dyd\tau.
\end{aligned}
$$

$$
\begin{aligned}
I_{32}=&\int_0^\tau\int_{\mathbb{R}}\kappa|Q_2||\zeta_{yy}|\ dyd\tau \\
\le&\beta\int_0^\tau\|\zeta_{yy}\|^2d\tau+C_\beta\int_0^\tau\int_{\mathbb{R}}\kappa^2|Q_2|^2dyd\tau \\
\le&\beta\int_0^\tau\|\zeta_{yy}\|^2d\tau+C_\beta\int_0^t\int_{\mathbb{R}}(|Q_{21}|^2+|Q_{22}|^2+|Q^{CD}|^2)dxdt \\
\le&\beta\int_0^\tau\|\zeta_{yy}\|^2d\tau+C_\beta\kappa^{\frac{7}{5}}.
\end{aligned}
$$

$$
\begin{aligned}
I_{33}=&C\int_0^\tau\int_{\mathbb{R}}(|V_y||\Theta_y|+|V_y||\zeta_y|+|\Theta_y||\phi_y|+|\phi_y||\zeta_y|+|\Theta_{yy}||\phi|)|\zeta_{yy}|dyd\tau \\
\le&\beta\int_0^\tau\|\zeta_{yy}\|^2d\tau+C_\beta\int_0^\tau\int_{\mathbb{R}}(V_y^4+\Theta_y^4)dyd\tau+C\int_0^\tau\|\phi_y\|_{L^\infty}\|(\zeta_y,\zeta_{yy})\|^2d\tau
\end{aligned}
$$

$$+C\int_0^\tau \|(V_y,\Theta_y)\|_{L^\infty}\|(\zeta_y,\phi_y,\zeta_{yy})\|^2 d\tau + C\int_0^\tau \|\phi\|_{L^\infty}\|(\Theta_{yy},\zeta_{yy})\|^2 d\tau$$
$$\leq (\beta + C\kappa^{\frac{1}{2}} + C\chi)\int_0^\tau \|\zeta_{yy}\|^2 d\tau + C(\kappa^{\frac{1}{2}}+\chi)\int_0^\tau \|(\phi_y,\zeta_y)\|^2 d\tau + (C_\beta + C\chi)\kappa^{\frac{1}{2}}.$$

Hence, substituting $I_{31}-I_{33}$ into (3.26) gives

$$I_3 \leq (\beta + C\kappa^{\frac{1}{2}} + C\chi)\int_0^\tau \|\zeta_{yy}\|^2 d\tau + C_\beta\int_0^\tau (\tau+\tau_0)^{-2}\|(\phi,\zeta)\|^2 d\tau$$
$$+ C_\beta \int_0^\tau\int_{\mathbb{R}} \delta^{CD}\kappa(1+\kappa\tau)^{-1} e^{-\frac{C_0\kappa y^2}{1+\kappa\tau}}|(\phi,\zeta)|^2 dy d\tau$$
$$+ C(\kappa^{\frac{1}{2}}+\chi)\int_0^\tau \|(\phi_y,\zeta_y)\|^2 d\tau + (C_\beta + C\chi)\kappa^{\frac{1}{2}}.$$

By (2.7), (2.8), (3.3) and Lemma 2.3, we obtain

$$I_4 = \int_0^\tau\int_{\mathbb{R}} \{\frac{-p}{v}\phi_y\Theta_y\psi_y + (\frac{\psi_y^2}{2} + \frac{p}{v}\frac{\phi_y^2}{2})\frac{\gamma-1}{R}[-PU_y + (\frac{\Theta_y}{V})_y + \kappa Q_2]$$
$$+ \frac{R\Theta}{vV}\phi\Theta_{yy}\psi_y - \frac{R\Theta}{v}\psi_y\zeta_{yy} + \frac{R\theta}{v}\psi_y\zeta_{yy}\} dy d\tau$$
$$\leq C\int_0^\tau\int_{\mathbb{R}} |\Theta_y||\phi_y||\psi_y| + (|U_y| + |\Theta_{yy}| + |\Theta_y||V_y|)(|\phi_y|^2 + |\psi_y|^2)$$
$$+ \kappa|Q_2|(\phi_y^2+\psi_y^2) + |\phi||\Theta_{yy}||\psi_y| + |\zeta||\psi_y||\zeta_{yy}| dy d\tau$$
$$\leq C\int_0^\tau \|(\Theta_y, U_y, \Theta_{yy}\|_{L^\infty} + \|\Theta_y\|_{L^\infty}\|V_y\|_{L^\infty})\|(\phi_y,\psi_y)\|^2 d\tau$$
$$+ \int_0^\tau \kappa\|Q_2\|_{L^\infty}\|(\phi_y,\psi_y)\|^2 d\tau$$
$$+ C\int_0^\tau \|\phi\|_{L^\infty}\int_{\mathbb{R}}(\Theta_{yy}^2 + \psi_y^2) dy d\tau + C\int_0^\tau \|\zeta\|_{L^\infty}\|(\psi_y,\zeta_{yy})\|^2 d\tau$$
$$\leq C(\chi + \kappa^{\frac{1}{2}})\int_0^\tau \|(\phi_y,\psi_y)\|^2 d\tau + C\chi\int_0^\tau \|\zeta_{yy}\|^2 d\tau + C\chi\kappa^{\frac{1}{2}}.$$

Then combining I_1-I_4 and taking β,κ,χ sufficiently small yields

$$\|(\phi_y,\psi_y,\zeta_y)\|^2(\tau) + \int_0^\tau \|\zeta_{yy}\|^2 d\tau + \int_0^\tau \|\sqrt{(U_y^{R_1},U_y^{R_3})}\phi_y\|^2 d\tau$$
$$\leq C(\chi + \kappa^{\frac{1}{2}} + \beta_2)\int_0^\tau \|(\phi_y,\psi_y,\zeta_y)\|^2 d\tau + C_{\beta_2}\int_0^\tau (\tau+\tau_0)^2\|(\phi,\zeta)\|^2 d\tau$$
$$+ C_{\beta_2}\int_0^\tau\int_{\mathbb{R}} \delta^{CD}\kappa(1+\kappa\tau)^{-1}e^{-\frac{C_0\kappa y^2}{1+\kappa\tau}}|(\phi,\zeta)|^2 dy d\tau + (C_{\beta_2} + C\chi)\kappa^{\frac{1}{2}}. \tag{3.27}$$

Step 2. In this step, we will estimate $\int_0^\tau \|(\phi_y,\psi_y)\|^2 d\tau$ as in [13]. Denote $J = (v,u,\theta)(\tau,y)$, $L = (V,U,\Theta)(\tau,y)$, $W = J - L = (\phi,\psi,\zeta)(\tau,y)$. Make a scaling in (1.8) with $y = \frac{x}{\kappa}$, $\tau = \frac{t}{\kappa}$ and rewrite the resulting equations in the following symmetric form

$$A^0(J)J_\tau + A(J)J_y = B(J)J_{yy} + g(J,J_y), \tag{3.28}$$

where $g(J,J_y) = (0,0,(\frac{1}{v})_y\theta_y)^T$ and

$$A^0(J) = \begin{pmatrix} -\theta p_v & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \frac{R}{\gamma-1} \end{pmatrix},\ A(J) = \begin{pmatrix} 0 & \theta p_v & 0 \\ \theta p_v & 0 & p \\ 0 & p & 0 \end{pmatrix},\ B(J) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{v} \end{pmatrix}.$$

Similarly, make a scaling in (2.21) and rewrite the resulting equations in the following form

$$A^0(L)L_\tau + A(L)L_y = B(L)L_{yy} + g(L,L_y) + F, \tag{3.29}$$

where $F = (0,\kappa Q_1,\kappa Q_2)^T$. Linearizing (3.28) at L and then subtracting (3.29) from the resulting system, one gets that

$$A^0(L)W_\tau + A(L)W_y = B(L)W_{yy} + H, \tag{3.30}$$

where

$$\begin{aligned} H =& A^0(L)\{[A^0(L)^{-1}A(L) - A^0(J)^{-1}A(J)]J_y + [A^0(J)^{-1}B(J) - A^0(L)^{-1}B(L)]J_{yy} \\ &+ A^0(J)^{-1}g(J,J_y) - A^0(L)^{-1}g(L,L_y)\} - F. \end{aligned} \tag{3.31}$$

Rewriting (3.29) in the following form

$$\begin{aligned} A^0(J)L_\tau + A(J)L_y =& B(J)L_{yy} + g(L,L_y) + F + [A^0(J) - A^0(L)]L_\tau + [A(J) - A(L)]L_y \\ &+ [B(L) - B(J)]L_{yy}, \end{aligned}$$

and then subtracting the above equation from (3.28) gives

$$A^0(J)W_\tau + A(J)W_y = B(J)W_{yy} + G, \tag{3.32}$$

where

$$G = g(J,J_y) - g(L,L_y) - F + [A^0(L) - A^0(J)]L_\tau + [A(L) - A(J)]L_y + [B(J) - B(L)]L_{yy}.$$

We set

$$S(J) = \begin{pmatrix} 0 & -p & 0 \\ v & 0 & 2\theta \\ 0 & -\frac{2R}{\gamma-1} & 0 \end{pmatrix},$$

multiply (3.30) with $S(L)$ and then with W_y^T on the left, finally integrate with respect to y on $\mathbb{R}$ to obtain

$$\begin{aligned} &\int_{\mathbb{R}} \langle S(L)A^0(L)W_\tau, W_y\rangle\, dy + \int_{\mathbb{R}} \langle S(L)A(L)W_y, W_y\rangle\, dy \\ &= \int_{\mathbb{R}} \langle S(L)B(L)W_{yy}, W_y\rangle\, dy + \int_{\mathbb{R}} \langle S(L)H, W_y\rangle\, dy, \end{aligned} \tag{3.33}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $\mathbb{R}^3$.

Since

$$SA^0(J) = \begin{pmatrix} 0 & -\theta p & 0 \\ \theta p & 0 & \frac{2R\theta}{\gamma-1} \\ 0 & -\frac{2R\theta}{\gamma-1} & 0 \end{pmatrix}$$

is skew-symmetric, integrating by parts and using (3.32), we can write the first term on the left-hand side of (3.33) as

$$
\begin{aligned}
&\int_{\mathbb{R}} \langle S(L)A^0(L)W_\tau, W_y\rangle dy \\
=&\frac{1}{2}\{\frac{d}{d\tau}\int_{\mathbb{R}} \langle S(L)A^0(L)W, W_y\rangle dy - \int_{\mathbb{R}} \langle (S(L)A^0(L))_\tau W, W_y\rangle dy \\
&+\int_{\mathbb{R}} \langle (S(L)A^0(L))_y W, W_\tau\rangle dy\} \\
=&\frac{1}{2}\{\frac{d}{d\tau}\int_{\mathbb{R}} \langle S(L)A^0(L)W, W_y\rangle dy - \int_{\mathbb{R}} \langle (S(L)A^0(L))_\tau W, W_y\rangle dy \\
&-\int_{\mathbb{R}} \langle (S(L)A^0(L))_y W, A^0(J)^{-1}A(J)W_y\rangle dy \\
&+\int_{\mathbb{R}} \langle (S(L)A^0(L))_y W, A^0(J)^{-1}B(J)W_{yy}\rangle dy \\
&+\int_{\mathbb{R}} \langle (S(L)A^0(L))_y W, A^0(J)^{-1}G\rangle dy\}.
\end{aligned}
$$

Substitute this into (3.33) to get

$$
\begin{aligned}
&\int_{\mathbb{R}} \langle S(L)A(L)W_y, W_y\rangle dy \\
=&-\frac{1}{2}\frac{d}{d\tau}\int_{\mathbb{R}} \langle S(L)A^0(L)W, W_y\rangle dy + \frac{1}{2}\{\int_{\mathbb{R}} \langle (S(L)A^0(L))_\tau W, W_y\rangle dy \\
&+\int_{\mathbb{R}} \langle (S(L)A^0(L))_y W, A^0(J)^{-1}A(J)W_y\rangle dy\} \\
&-\frac{1}{2}\int_{\mathbb{R}} \langle (S(L)A^0(L))_y W, A^0(J)^{-1}B(J)W_{yy}\rangle dy \\
&-\frac{1}{2}\int_{\mathbb{R}} \langle (S(L)A^0(L))_y W, A^0(J)^{-1}G\rangle dy \\
&+\int_{\mathbb{R}} \langle S(L)B(L)W_{yy}, W_y\rangle dy + \int_{\mathbb{R}} \langle S(L)H, W_y\rangle dy. \qquad (3.34)
\end{aligned}
$$

Now, we estimate all of the terms above separately. Firstly,

$$
\begin{aligned}
&\int_{\mathbb{R}} \langle S(L)A(L)W_y, W_y\rangle dy \\
=&\int_{\mathbb{R}} (\frac{P^3}{R}\phi_y^2 + \Theta P\psi_y^2 + \frac{3-\gamma}{\gamma-1}P^2\phi_y\zeta_y - \frac{2R}{\gamma-1}P\zeta_y^2)dy \\
\geq&\int_{\mathbb{R}} (\frac{P^3}{R}\phi_y^2 + \Theta P\psi_y^2)dy - C\int_{\mathbb{R}} \zeta_y^2 dy. \qquad (3.35)
\end{aligned}
$$

By Young's inequality, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}} \langle (S(L)A^0(L))_\tau W, W_y\rangle dy + \int_{\mathbb{R}} \langle (S(L)A^0(L))_y W, A^0(J)^{-1}A(J)W_y\rangle dy \\
\leq& C\int_{\mathbb{R}} (|\Theta_\tau| + |V_\tau| + |\Theta_y| + |V_y|)|W||W_y| dy \\
\leq& \beta\int_{\mathbb{R}} W_y^2 dy + C_\beta\int_{\mathbb{R}} \delta^{CD}\kappa(1+\kappa\tau)^{-1}e^{-\frac{C_0\kappa y^2}{1+\kappa\tau}}W^2 dy + C_\beta(\tau+\tau_0)^{-2}\int_{\mathbb{R}} W^2 dy. \qquad (3.36)
\end{aligned}
$$

Direct calculations and Young's inequality lead to

$$\begin{aligned}
&-\frac{1}{2}\int_{\mathbb{R}}\langle (S(L)A^0(L))_y W, A^0(J)^{-1}B(J)W_{yy}\rangle dy+\int_{\mathbb{R}}\langle S(L)B(L)W_{yy},W_y\rangle dy\\
=&\int_{\mathbb{R}}\frac{\psi\Theta_y\zeta_{yy}}{v}dy+\int_{\mathbb{R}}\frac{2\Theta}{V}\zeta_{yy}\psi_y dy\\
\le&\beta\int_{\mathbb{R}}\Theta_y^2\psi^2dy+C_\beta\int_{\mathbb{R}}\zeta_{yy}^2dy+\beta\int_{\mathbb{R}}\psi_y^2dy\\
\le&\beta(\tau+\tau_0)^{-2}\int_{\mathbb{R}}\psi^2dy+C_\beta\int_{\mathbb{R}}\zeta_{yy}^2dy+\beta\int_{\mathbb{R}}\psi_y^2dy\\
&+\beta\int_{\mathbb{R}}\delta^{CD}\kappa(1+\kappa\tau)^{-1}e^{-\frac{C_0\kappa y^2}{1+\kappa\tau}}\psi^2dy.
\end{aligned}\tag{3.37}$$

By the definition of G, we get

$$\begin{aligned}
&-\int_0^\tau\int_{\mathbb{R}}\langle (S(L)A^0(L))_y W, A^0(J)^{-1}G\rangle dyd\tau\\
\le&C\int_0^\tau\int_{\mathbb{R}}|\Theta_y||\phi+\zeta|[|\Theta_y||\phi+\zeta|+\kappa|Q_1|]+|\Theta_y||\psi|(|\phi||\Theta_{yy}|\\
&+|\zeta+\phi||U_y|+|V_y||\zeta_y|+|\phi_y||\zeta_y|+|V_y||\Theta_y||\phi|+|\Theta_y||\phi_y|+\kappa|Q_2|)dyd\tau\\
\le&C\int_0^\tau\int_{\mathbb{R}}(|L_y|^2+\Theta_y^4+\Theta_{yy}^2)W^2dyd\tau+\beta\int_0^\tau\int_{\mathbb{R}}(\zeta_y^2+\phi_y^2)dyd\tau\\
&+C_\beta\int_0^\tau\int_{\mathbb{R}}(\Theta_y^4+V_y^4)\psi^2dyd\tau+C\int_0^\tau\int_{\mathbb{R}}|\Theta_y||\psi||\phi_y||\zeta_y|dyd\tau\\
&+C\int_0^\tau\int_{\mathbb{R}}(|\Theta_y||\phi+\zeta|\kappa|Q_1|+|\Theta_y||\psi|\kappa|Q_2|)dyd\tau\\
\le&(C+C_\beta)\int_0^\tau\int_{\mathbb{R}}(\tau+\tau_0)^{-2}W^2dyd\tau+\beta\int_0^\tau\int_{\mathbb{R}}(\zeta_y^2+\phi_y^2)dyd\tau\\
&+C\int_0^\tau\kappa\|\Theta_x\|_{L^\infty}\|\psi\|_{L^\infty}\int_{\mathbb{R}}|\phi_y||\zeta_y|dyd\tau+C\kappa^{\frac{1}{2}}\int_0^\tau\int_{\mathbb{R}}(|\phi+\zeta|\kappa|Q_1|+|\psi|\kappa|Q_2|)dyd\tau\\
&+(C+C_\beta)\int_0^\tau\int_{\mathbb{R}}\delta^{CD}\kappa(1+\kappa\tau)^{-1}e^{-\frac{C_0\kappa y^2}{1+\kappa\tau}}W^2dyd\tau\\
\le&(C+C_\beta)\int_0^\tau\int_{\mathbb{R}}(\tau+\tau_0)^{-2}W^2dyd\tau+\beta\int_0^\tau\|(\phi_y,\psi_y,\zeta_y)\|^2d\tau+C\kappa\int_0^\tau\int_{\mathbb{R}}\phi_y^2dyd\tau\\
&+C\chi^2\int_0^\tau\int_{\mathbb{R}}\zeta_y^2dyd\tau+\beta\sup_{[0,\tau]}\|(\phi,\psi,\zeta)(\tau)\|^2+C_\beta\kappa^{\frac{9}{10}}\\
&+(C+C_\beta)\int_0^\tau\int_{\mathbb{R}}\delta^{CD}\kappa(1+\kappa\tau)^{-1}e^{-\frac{C_0\kappa y^2}{1+\kappa\tau}}W^2dyd\tau,
\end{aligned}\tag{3.38}$$

where in the last inequality, we have used the estimates similar to (3.17)-(3.20). By the definition of H, we have

$$\begin{aligned}
&\int_0^\tau\int_{\mathbb{R}}\langle S(L)H,W_y\rangle dyd\tau\\
\le&C\int_0^\tau\int_{\mathbb{R}}(|\Theta_y|+|V_y|)|W||W_y|+|\psi||\Theta_{yy}||\psi_y|+|V_y||\Theta_y||\phi||\psi_y|+|W||W_y|^2dyd\tau\\
&+C\int_0^\tau\int_{\mathbb{R}}(|\Theta_y|+|V_y|)|W_y|^2+|\psi||\psi_y||\zeta_{yy}|+|\phi_y||\psi_y||\zeta_y|dyd\tau
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\tau\int_{\mathbb{R}}\langle -S(L)F,W_y\rangle\,dyd\tau\\
\leq&(\beta+C\chi+C\kappa^{\frac{1}{2}})\int_0^\tau\|W_y\|^2d\tau+C_\beta\int_0^\tau\int_{\mathbb{R}}\delta^{CD}\kappa(1+\kappa\tau)^{-1}e^{-\frac{C_0\kappa y^2}{1+\kappa\tau}}W^2dyd\tau\\
&+C_\beta\int_0^\tau(\tau+\tau_0)^{-2}\|W\|^2d\tau+C\chi\int_0^\tau\|(\phi_y,\psi_y)\|^2d\tau+C\chi\int_0^\tau\|\zeta_{yy}\|^2d\tau\\
&+\int_0^\tau\int_{\mathbb{R}}\langle -S(L)F,W_y\rangle\,dyd\tau\\
\leq&(\beta+C\chi+C\kappa^{\frac{1}{2}})\int_0^\tau\|W_y\|^2d\tau+\beta\int_0^\tau\|(\phi_{yy},\psi_{yy})\|^2d\tau+(\beta+C\chi)\int_0^\tau\|\zeta_{yy}\|^2d\tau\\
&+\beta\sup_{[0,\tau]}\|(\phi_y,\psi_y,\zeta_y)(\tau)\|^2+C_\beta\int_0^\tau(\tau+\tau_0)^{-2}\|W\|^2d\tau+C_\beta\kappa^{\frac{3}{2}}\\
&+C_\beta\int_0^\tau\int_{\mathbb{R}}\delta^{CD}\kappa(1+\kappa\tau)^{-1}e^{-\frac{C_0\kappa y^2}{1+\kappa\tau}}W^2dyd\tau, \qquad (3.39)
\end{aligned}
$$

where for $\int_0^\tau\int_{\mathbb{R}}\langle -S(L)F,W_y\rangle\,dyd\tau$, we have used the estimates similar to (3.17)-(3.20) again.

Since

$$
\begin{aligned}
&-\int_0^\tau\frac{1}{2}\frac{d}{d\tau}\int_{\mathbb{R}}\langle S(L)A^0(L)W,W_y\rangle\,dyd\tau\\
\leq&C_\beta\|(\phi,\psi,\zeta)(\tau)\|^2+\beta\|(\phi_y,\psi_y,\zeta_y)(\tau)\|^2,
\end{aligned}
$$

combining this estimate and (3.34)-(3.39), by choosing β,χ,κ sufficiently small, we get

$$
\begin{aligned}
\int_0^\tau\|(\phi_y,\psi_y)\|^2d\tau\leq&(C_{\beta_3}+\beta_3)\sup_{[0,\tau]}\|(\phi,\psi,\zeta)\|^2+\beta_3\sup_{[0,\tau]}\|(\phi_y,\psi_y,\zeta_y)\|^2\\
&+C_{\beta_3}\int_0^\tau\int_{\mathbb{R}}\delta^{CD}\kappa(1+\kappa\tau)^{-1}e^{-\frac{C_0\kappa y^2}{1+\kappa\tau}}W^2dyd\tau+C_{\beta_3}\int_0^\tau(\tau+\tau_0)^{-2}\|W\|^2d\tau\\
&+C_{\beta_3}\int_0^\tau\|\zeta_{yy}\|^2d\tau+(C+\beta_3+\chi+\kappa^{\frac{1}{2}})\int_0^\tau\|\zeta_y\|^2d\tau+\beta_3\int_0^\tau\|(\phi_{yy},\psi_{yy})\|^2d\tau\\
&+(\beta_3+\chi)\int_0^\tau\|\zeta_{yy}\|^2d\tau+C_{\beta_3}\kappa^{\frac{3}{2}}. \qquad (3.40)
\end{aligned}
$$

Combining (3.4), (3.27) and (3.40), and then choosing β_1, κ, χ sufficiently small, we can obtain (3.22), which completes the proof of Lemma 3.2. □

For the second order derivatives, we have the following result:

LEMMA 3.3. *Under the assumptions of Theorem 3.1, there exists a constant $C>0$ such that*

$$
\begin{aligned}
&\|(\phi,\psi,\zeta)(\tau,\cdot)\|_2^2+\int_0^\tau\|(\phi_y,\psi_y,\zeta_y)\|_1^2\,d\tau+\int_0^\tau\|\zeta_{yyy}\|^2d\tau\\
\leq&C\int_0^\tau(\tau+\tau_0)^{-2}\|(\phi,\psi,\zeta)\|^2\,d\tau+C\kappa^{\frac{2}{5}}\\
&\quad+C\delta^{CD}\int_0^\tau\int_{\mathbb{R}}\kappa(1+\kappa\tau)^{-1}e^{-\frac{C_0\kappa y^2}{1+\kappa\tau}}(\phi^2+\psi^2+\zeta^2)\,dyd\tau. \qquad (3.41)
\end{aligned}
$$

The proof is similar to that of Lemma 3.2. Hence we omit it for simplicity.

In order to close the estimate, we only need to control the last term in (3.41). Here, we will use the following technique by using the heat kernel motivated by [6].

LEMMA 3.4. *Suppose that $h(\tau,y)$ satisfies*

$$h\in L^\infty(0,+\infty:L^2(\mathbb{R})),\ h_y\in L^2(0,+\infty;L^2(\mathbb{R})),\ h_\tau\in L^2(0,+\infty;H^{-1}(\mathbb{R})),$$

Then

$$\begin{aligned}&\int_0^\tau\int_{\mathbb{R}}\kappa(1+\kappa\tau)^{-1}e^{-\frac{2a\kappa y^2}{1+\kappa\tau}}h^2(\tau,y)dyd\tau\\ \le& C_a[\|h(0,y)\|^2+\int_0^\tau\|h_y\|^2d\tau+\int_0^\tau\langle h_\tau,hg_a^2\rangle_{H^{-1}\times H^1}d\tau]\end{aligned}$$

where

$$g_a(\tau,y)=\kappa^{\frac{1}{2}}(1+\kappa\tau)^{-\frac{1}{2}}\int_{-\infty}^y e^{-\frac{a\kappa\eta^2}{1+\kappa\tau}}d\eta,$$

and $a>0$ is the constant to be determined later.

Similar to the one given in [6], we can prove Lemma 3.4 by carefully dealing with the parameter κ. Therefore, we omit it.

Based on Lemma 3.4, we can obtain

LEMMA 3.5. *There exists a constant $C>0$ such that if δ^{CD} and κ_0 are small enough, then we have*

$$\begin{aligned}&\int_0^\tau\int_{\mathbb{R}}\kappa(1+\kappa\tau)^{-1}e^{-\frac{C_0\kappa y^2}{1+\kappa\tau}}|(\phi,\psi,\zeta)|^2dyd\tau\\ \le& C\|(\phi,\psi,\zeta)(\tau,\cdot)\|^2+C\int_0^\tau\|(\phi_y,\psi_y,\zeta_y)\|^2d\tau+C\int_0^\tau(\tau+\tau_0)^{-\frac{3}{2}}\|(\phi,\psi,\zeta)\|^2d\tau+C\kappa^{\frac{2}{5}}.\end{aligned}$$

The proof is similar to that in [8], the main difference here is that we deal with the coefficient of the heat conductivity κ but not the viscosity coefficient. Hence we omit the proof too.

Now from (3.41) and Lemma 3.5, if the strength of the contact wave δ^{CD} and the parameter χ in the a priori estimate (3.3) are sufficiently small, we can obtain

$$\begin{aligned}&\|(\phi,\psi,\zeta)(\tau,\cdot)\|_2^2+\int_0^\tau\|(\phi_y,\psi_y,\zeta_y)\|_1^2\,d\tau+\int_0^\tau\|\zeta_{yyy}\|^2d\tau\\ \le& C\int_0^\tau(\tau+\tau_0)^{-\frac{3}{2}}\|(\phi,\psi,\zeta)\|^2\,d\tau+C\kappa^{\frac{2}{5}}.\end{aligned}$$

Then by the Gronwall inequality, we have

$$\|(\phi,\psi,\zeta)(\tau,\cdot)\|_2^2+\int_0^\tau\|(\phi_y,\psi_y,\zeta_y)\|_1^2\,d\tau+\int_0^\tau\|\zeta_{yyy}\|^2d\tau\le C\kappa^{\frac{2}{5}}.$$

Finally, by the Sobolev imbedding,

$$\|(\phi,\psi,\zeta)\|_{L^\infty}\le C\|(\phi,\psi,\zeta)\|^{\frac{1}{2}}\|(\phi_y,\psi_y,\zeta_y)\|^{\frac{1}{2}}\le C\kappa^{\frac{1}{5}},$$

which completes the proof of Theorem 2.1.

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