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ON TWO-DIMENSIONAL STEADY HYPERSONIC-LIMIT EULER FLOWS PASSING RAMPS AND RADON MEASURE SOLUTIONS OF COMPRESSIBLE EULER EQUATIONS*

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Abstract. We proposed rigorous definitions of Radon measure solutions for boundary value problems of steady compressible Euler equations which model hypersonic-limit inviscid flows passing two-dimensional ramps, and their interactions with still gas and pressureless jets. We proved the Newton-Busemann pressure law of drags on a body in hypersonic flow, and constructed various physically interesting measure solutions with density containing Dirac measures supported on curves, also exhibited examples of blow up of certain measure solutions. This established a mathematical foundation for applications in engineering and further studies of measure solutions of compressible Euler equations.

Keywords. Compressible Euler equations; hypersonic; Newton-Busemann pressure law; shock layer; free layer; Dirac measure; measure solution; vacuum; singular Riemann problem.

AMS subject classifications. 35L65; 35L67; 35Q31; 35R06; 35R35; 76K05.

1. Introduction

In gas dynamics, supersonic flow with Mach number greater than five is called hypersonic flow, which bears some peculiar features [1, Section 15.2]. For example, as the Mach number of the flow goes to infinity, it behaves like moving particles without thermal motions (hence pressure approaches zero); when passing a slender body, *shock layer,* i.e., the region bounded by the surface of the body and the shock appearing in front of it, becomes thinner and thinner, and ultimately mass concentrates in an infinite-thin shock layer. There is also a *Mach number independence law* (see [1, Section 15.5] or [15, p.24]), which claims that two flow fields with large but different upstream flow Mach numbers are not different from each other in any fundamental way.

These physical observations imply that there is a limit of hypersonic flow problem, and the limiting flow field cannot be described by Lebesgue measurable functions anymore. Actually a suitable concept of measure solutions of the compressible Euler equations should be introduced to illustrate the above observations and put related physical arguments upon a solid mathematical foundation. However, it is somewhat surprising that there is no such mathematical work before the paper [23]. In [23] the authors showed that for steady Euler flows of polytropic gases, after suitable scaling, the Mach number goes to infinity which means actually that the adiabatic exponent goes to 1. By proposing a definition of measure solutions for supersonic flow passing a two-dimensional straight wedge, the authors verified the above physical observations mathematically, and derived naturally the Newton's sine-squared pressure law. See [1,2,15,21] for the background and physical theory of hypersonic flows, especially [15, Chapter 3] for a detailed introduction to the Newton's theory of hypersonic flow. In [22,24], the authors also studied the related problems of measure solutions of high Mach number limits of piston problems for polytropic gases and Chaplygin gas. These papers demonstrate that the

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concept of measure solutions we proposed works well for these fundamental physical problems.

In this paper we are going to study three typical problems about limiting hypersonic flows passing bodies, with emphasis on explicit solutions derived from rigorous mathematical theory. The first problem deals with hypersonic-limit flow passing a curved ramp and the derivation of the Newton-Busemann pressure law [15, p.133]. The second problem focuses on studying the interactions of hypersonic limit flow and still gas in a "dead gas zone". The third problem involves limiting hypersonic flow interactions with pressureless jets. For the latter two problems, *free layer* (called "delta shock" in the mathematics literature) appears in the flow field that separates gases with different states. We calculate special measure solutions to understand these physical problems, and find some interesting new phenomena, such as blow up of measure solutions in a finite distance, which exhibit the great power of a proper concept of measure solutions to the Euler equations.

In the rest of this section we firstly present the problems, and the concept of measure solutions, as well as the main results. After that, we review briefly some related mathematical works on measure solutions of hyperbolic conservation laws, at the end of the section.

1.1. Formulation of three problems. In the Euclidean plane $\mathbb{R}^2$ with Cartesian coordinates (x,y), the two-dimensional steady non-isentropic compressible Euler equations take the form [1, Section 6.2]

$$\begin{cases} \partial_x(\rho u)+\partial_y(\rho v)=0, \\ \partial_x(\rho u^2+p)+\partial_y(\rho uv)=0, \\ \partial_x(\rho uv)+\partial_y(\rho v^2+p)=0, \\ \partial_x(\rho uE)+\partial_y(\rho vE)=0, \end{cases} \tag{1.1}$$

which can also be written as

$$\partial_x F(U)+\partial_y G(U)=0, \quad U=(\rho,u,v,E)^\top, \tag{1.2}$$

where

$$F(U)=(\rho u,\ \rho u^2+p,\ \rho uv,\ \rho uE)^\top,$$

$$G(U)=(\rho v,\ \rho uv,\ \rho v^2+p,\ \rho vE)^\top.$$

Here ρ, p, (u,v) represent respectively the mass density, pressure, and velocity of the gas; the function E is given by

$$E=\frac{1}{2}(u^2+v^2)+\frac{\gamma}{\gamma-1}\frac{p}{\rho}, \tag{1.3}$$

where $\gamma>1$ is the adiabatic exponent appearing in the state function of polytropic gases:

$$p=\kappa\rho^\gamma \exp(\frac{\hat{S}}{c_v}),$$

with $\hat{S}$ being the entropy, and κ, c_v being positive constants. However, we emphasize that in this work we shall use

$$p=\frac{\gamma-1}{\gamma}\rho(E-\frac{1}{2}(u^2+v^2)), \tag{1.4}$$

which is solved from (1.3), as the state function of the gas. It includes the cases of polytropic gases ($\gamma>1$) and pressureless gas ($\gamma=1$).

It is well-known that to study weak solutions, one shall choose the correct representations of the Euler equations and the state function. Previous works [22–24] have shown that (1.1) and (1.4) are the proper starting point to study physical problems of limiting hypersonic flows and general Radon measure solutions of compressible Euler equations.

1.1.1. Problem 1. Consider the problem of supersonic flow passing an infinite solid ramp $\{(x,y)\in\mathbb{R}^2:\ x\in\mathbb{R},\ y\le b(x)\}$, where $b(x)$ is a given continuous function, satisfying $b(x)=0$ for $x\le 0$, $b(x)\in C^2$ and $b'(x)\ge 0$ for $x>0$ (see Figure 1.1). It is a classic problem in gas dynamics, and has been studied extensively (see, for example, [9, 16, 17] and references therein). Actually, it's Hu and Zhang's work [16, 17] that motivates us to study the hypersonic-limit flow.

To formulate the problem, denote the region filled with gas by

$$\Omega\triangleq\{(x,y)\in\mathbb{R}^2:\ x>0,\ y>b(x)\}.$$

The surface of the ramp is then

$$W\triangleq\{(x,y)\in\mathbb{R}^2:\ x\ge 0,\ y=b(x)\},$$

on which we propose the slip condition

$$v=b'(x)u \quad \text{on } W. \tag{1.5}$$

Without loss of generality (cf. [23] for some non-dimensional scalings), we may assume that the uniform upcoming supersonic flow is

$$U=U_0\triangleq(\rho_0,u_0,v_0,E_0)^\top=(1,1,0,E_0)^\top \quad \text{on } \{x=0,\ y>0\}, \tag{1.6}$$

where $E_0>1/2$ is a given constant. From [23], we know that the Mach number of the upcoming flow $M_0=+\infty$ translates to $\gamma=1$, hence by (1.4), one has $p_0=0$. Therefore limiting hypersonic flow is pressureless Euler flow if there is no physical boundary.

Problem 1: Find a solution to the initial-boundary value problem (1.1), (1.4)-(1.6) in Ω when $\gamma=1$.

1.1.2. Problem 2. For limiting hypersonic flow passing a finite ramp, we consider the case that there is a cliff $W_e\triangleq\{(x,y)\in\mathbb{R}^2:\ x=x_*,\ y<b(x_*)\}$, where $x_*>0$ is given, and there is static uniform gas near the cliff:

$$\underline{U}\triangleq(\underline{\rho},\ 0,\ 0,\ \underline{E})^\top, \tag{1.7}$$

where $\underline{\rho}>0$, $\underline{E}>0$ are given constants. From (1.4) we then solve for the pressure $\underline{p}\ge 0$. In particular for $\underline{p}=0$ we have pressureless static flow. This problem was proposed in [15, p.148] and discussed in a sketchy and qualitative style there.

For limiting hypersonic flow passing the cliff, we assume that there will appear a curve, called *free layer* in [15], to separate the limiting hypersonic flow above it from the static gas below it. Let the free layer be

$$S\triangleq\{(x,y)\in\mathbb{R}^2:\ x\ge x_*,\ y=s(x)\},$$

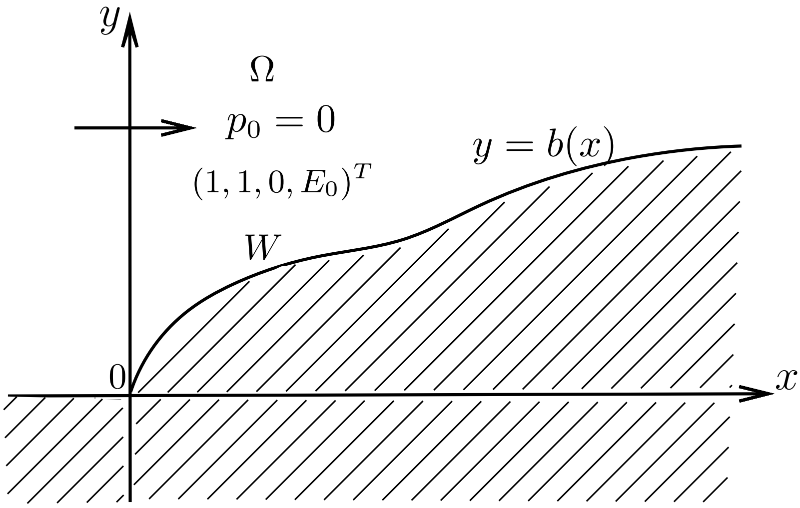


FIG. 1.1. *Limiting hypersonic flow passing an infinite ramp.*

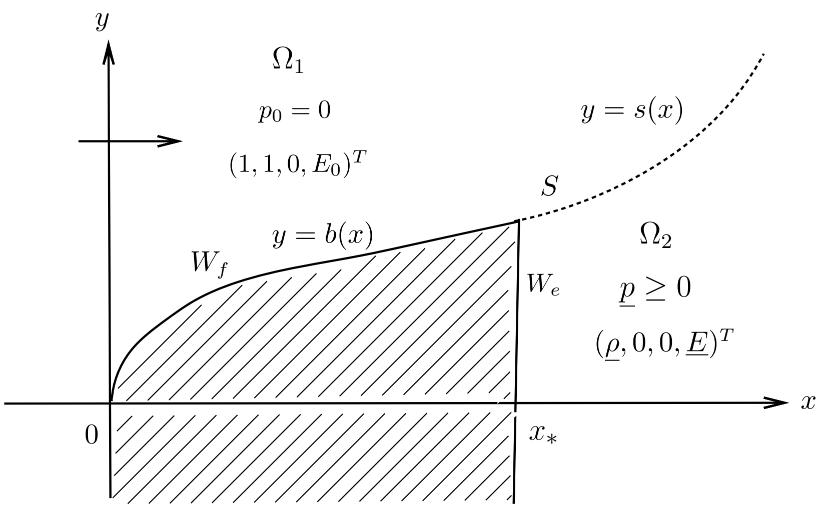


FIG. 1.2. *Interaction of limiting hypersonic flow in* Ω_1 *with uniform static gas in* Ω_2.

where $y = s(x)$ is a function to be solved. Set Ω_2 be the region bounded by W_e and S, which is the region occupied by the uniform static gas (see Figure 1.2). For convenience, we also define Ω_1 to be the region bounded by the positive y-axis and $W_f \cup S$, where

$$W_f \triangleq \{(x,y) \in \mathbb{R}^2 : \ 0 \leq x \leq x_*, \ y = b(x)\}$$

is the surface of the finite ramp.

We still propose the slip condition:

$$v = b'(x)u \quad \text{on } W_f. \tag{1.8}$$

REMARK 1.1. For convenience of later reference, we emphasize that on the free layer, the slip condition also holds

$$v = s'(x)u \quad \text{on } S, \tag{1.9}$$

which can be derived naturally from the definition of measure solutions that will be given below. Note that (u,v) in (1.9) is the velocity of concentrated particles on the free layer, which is usually different from the velocity of the gas that is close to the free layer. See Remark 3.1.

Problem 2: Find a solution to the transonic three-phase flow[1] problem (1.2), (1.4), (1.6)-(1.8).

We remark that although this problem looks quite similar to those studied in [5,6] for supersonic polytropic gas flow passing over a "dead gas zone", the results and methods are quite different. For the latter there is a classical contact discontinuity to separate the moving gas and static gas, and it cannot bear any pressure difference. This is an example that displays the difficulty and fascination of studying the compressible Euler equations: *Similar-looking problems may have drastic differences inside.*

1.1.3. Problems 3. Considering applications to rocket engineering, suppose now that W_f is a wall of a two-dimensional nozzle, and W_e is the exit of the nozzle, where the gas flowing out (i.e., jet) is assumed to be uniform, pressureless and hyperbolic (see Figure 1.3):

$$\underline{U} = (\underline{\rho}, \underline{u}, \underline{v}, \underline{E})^\top \qquad \text{on } W_e, \tag{1.10}$$

where $\underline{\rho} > 0$, $\underline{u} > 0$, $\underline{E} > 0$ and $\underline{v}$ are given constants. As before, we may assume that there is a free layer to separate the limiting hypersonic flow above it and the pressureless jet below it.

Problem 3: Find a solution to the problem (1.2), (1.4), (1.6), (1.8), (1.10).

REMARK 1.2. The steady pressureless Euler system is hyperbolic in the positive x-direction if and only if $u > 0$, with v/u being its eigenvalues, cf. [10, (2.1) in p.325].

1.2. Definition of Radon measure solutions. We now clarify the meaning of solutions to the above problems. As we know from physical observations, there appears concentration of mass along the walls in these problems, and one needs Dirac measures supported on curves to describe such phenomena. So the key point is how to understand the Euler equations when some of the unknowns are measures.

We review some basic notions of measure theory. Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}^2$, and m a Radon measure on $\mathcal{B}$. As a Radon measure, m can be considered as a bounded linear functional on the space $C_0(\mathbb{R}^2)$ which consists of continuous function with compact support: for a test function $\phi \in C_0(\mathbb{R}^2)$, one has

$$\langle m, \phi \rangle = \int_{\mathbb{R}^2} \phi(x,y) m(\mathrm{d}x\mathrm{d}y). \tag{1.11}$$

[1] The three phases are limiting hypersonic flow, static (polytropic) gas, and the free layer itself. For the limiting hypersonic flow, the Euler equations are hyperbolic, while for polytropic static gas, the flow is subsonic and the governing Euler equations are of degenerate elliptic-hyperbolic composite type. Hence the whole flow field is transonic.

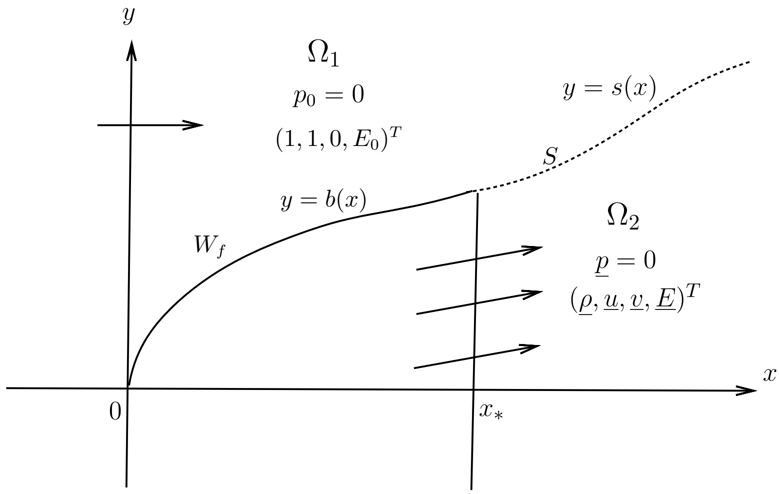


FIG. 1.3. *Interactions of limiting hypersonic flows and pressureless jets.*

For example, $w_L\delta_L$, a Dirac measure supported on a Lipschitz curve L with weight w_L, is defined by

$$\langle w_L\delta_L,\phi\rangle = \int_0^T w_L(t)\phi(x(t),y(t))\sqrt{x'(t)^2+y'(t)^2}\,\mathrm{d}t, \quad \forall\phi\in C_0(\mathbb{R}^2). \tag{1.12}$$

Here $L=\{(x(t),y(t)):\ t\in[0,T)\}$ is a Lipschitz curve given by the parameter t, and $w_L(t)\in L^1_{\mathrm{loc}}(0,T)$. It is singular to the Lebesgue measure $\mathcal{L}^2$ on the plane. The standard notation $\lambda\ll\mu$ means that a measure λ is absolutely continuous with respect to a nonnegative measure μ (cf. [13, p.50]).

We now propose a rigorous definition of Radon measure solutions to Problem 1.

DEFINITION 1.1. *For fixed $\gamma\geq 1$, let m^i, n^i $(i=0,1,2,3), \wp$ be Radon measures on $\overline{\Omega}$, and w_p a nonnegative function belonging to $L^1_{\mathrm{loc}}(\mathbb{R}^+\cup\{0\})$. We call (ϱ,u,v,E) a (Radon) measure solution to Problem 1, provided that the following are valid:*

(i) *For any $\phi\in C^1_0(\mathbb{R}^2)$ (continuously differentiable functions with compact supports), there hold*

$$\langle m^0,\partial_x\phi\rangle+\langle n^0,\partial_y\phi\rangle+\int_0^\infty \rho_0u_0\phi(0,y)\,\mathrm{d}y=0, \tag{1.13}$$

$$\langle m^1,\partial_x\phi\rangle+\langle n^1,\partial_y\phi\rangle+\langle\wp,\partial_x\phi\rangle+\langle w_pn_1\delta_W,\phi\rangle+\int_0^\infty (\rho_0u_0^2+p_0)\phi(0,y)\,\mathrm{d}y=0, \tag{1.14}$$

$$\langle m^2,\partial_x\phi\rangle+\langle n^2,\partial_y\phi\rangle+\langle\wp,\partial_y\phi\rangle+\langle w_pn_2\delta_W,\phi\rangle+\int_0^\infty (\rho_0u_0v_0)\phi(0,y)\,\mathrm{d}y=0, \tag{1.15}$$

$$\langle m^3,\partial_x\phi\rangle+\langle n^3,\partial_y\phi\rangle+\int_0^\infty (\rho_0u_0E_0)\phi(0,y)\,\mathrm{d}y=0, \tag{1.16}$$

where $\mathbf{n}=(n_1,n_2)=(-b'(x),1)/\sqrt{1+b'(x)^2}$ is the inner unit normal vector on W (pointing into Ω);

(ii) *ϱ is a nonnegative Radon measure, such that $\wp \ll \varrho$, $(m^0,n^0) \ll \varrho$, $(m^k,n^k) \ll (m^0,n^0)$ $(k=1,2,3)$, and the corresponding Radon-Nikodym derivatives satisfy $\varrho-$a.e. that*

$$u=\frac{m^0(\mathrm{d}x\mathrm{d}y)}{\varrho(\mathrm{d}x\mathrm{d}y)} \quad \textit{and} \quad v=\frac{n^0(\mathrm{d}x\mathrm{d}y)}{\varrho(\mathrm{d}x\mathrm{d}y)}, \tag{1.17}$$

$$u=\frac{m^1(\mathrm{d}x\mathrm{d}y)}{m^0(\mathrm{d}x\mathrm{d}y)}=\frac{n^1(\mathrm{d}x\mathrm{d}y)}{n^0(\mathrm{d}x\mathrm{d}y)}, \tag{1.18}$$

$$v=\frac{m^2(dxdy)}{m^0(\mathrm{d}x\mathrm{d}y)}=\frac{n^2(\mathrm{d}x\mathrm{d}y)}{n^0(\mathrm{d}x\mathrm{d}y)}, \tag{1.19}$$

and there is a ϱ-a.e. function E, so that

$$E=\frac{m^3(\mathrm{d}x\mathrm{d}y)}{m^0(\mathrm{d}x\mathrm{d}y)}=\frac{n^3(\mathrm{d}x\mathrm{d}y)}{n^0(\mathrm{d}x\mathrm{d}y)}; \tag{1.20}$$

on the (open) null sets of ϱ, we set u,v,E to be zero;

(iii) *If $\varrho \ll \mathcal{L}^2$ in a set A, with ρ being the Radon-Nikodym derivative, and $\wp \ll \mathcal{L}^2$ in A with Radon-Nikodym derivative p, then (1.4) is valid $\mathcal{L}^2$-a.e. in A, and the classical entropy conditions hold across discontinuities of the functions $U=(\rho,u,v,E)^\top$ in A.*

Remark 1.3. It is easy to check that integral weak solutions are measure solutions, cf. [23]. In that paper, it is also shown that for a straight ramp, the well-known piecewise constant integral weak solution containing a shock converges weakly in the sense of measures to the corresponding measure solution with density containing a Dirac measure on the ramp, as the Mach number of the upcoming flow goes to infinity (or equivalently, $\gamma \to 1$). Hence consistency holds for the above definition of measure solutions.

Remark 1.4. The basic idea of our definition of measure solution is firstly to relax the Euler equations to a linear differential system of measures of fluxes of mass and momentum etc., and the nonlinearity of the Euler system is recovered from the algebraic relations of Radon-Nikodym derivatives. The state function (1.4) is no longer required when concentration occurs. So there will not appear difficulties such as products of Dirac measures. The definition makes sense for general multidimensional steady or unsteady compressible Euler equations.

Remark 1.5. $w_p\cdot(-\mathbf{n})$ is the force (lift/drag) acting on the ramp due to the gas flow, hence it is a quantity that received great attention from engineers. We require in this paper $w_p>0$ to guarantee that the mass concentrates actually on the walls.

Remark 1.6. We can define measure solutions to Problems 2 and 3 in a similar way; consult necessary modifications indicated in Sections 3-4.

1.3. Main results and remarks.

The following are the main results we obtained for the above three problems.

Theorem 1.1. *For limiting hypersonic flow passing the infinite ramp W, suppose that*

$$b'(x)\geq 0, \quad b''(x)H(x) > -b'(x)^2\sqrt{1+b'(x)^2}, \tag{1.21}$$

where

$$H(x)\triangleq \int_0^x \frac{b'(t)}{\sqrt{1+b'(t)^2}}\,\mathrm{d}t. \tag{1.22}$$

Then Problem 1 has a measure solution given by (2.22)-(2.21)-(2.23)*, with density containing a weighted Dirac measure supported on W. In particular, we have*

$$w_p(x)=\frac{b''(x)H(x)+b'(x)^2\sqrt{1+b'(x)^2}}{(1+b'(x)^2)^{\frac{3}{2}}}. \tag{1.23}$$

REMARK 1.7. Formula (1.23) is the celebrated *Newton-Busemann pressure law.* As pointed out in [15, p.133]: "This formula is valuable because it is easy to compute and gives a simple basis of comparison." Adolf Busemann (1901–1986) was a German aerospace engineer who also discovered the benefits of the swept wing for aircraft at high speeds.

A derivation of the Newton-Busemann pressure law is given in [2, Sections 3.3, 3.4], and the law is presented as formula (3.29) in [2, p.67]. The formula is derived by a lengthy physical argument (nearly four pages), taking into account the centrifugal force required for a particle to move along the curved ramp. Hence in [15, p.133] Hayes and Probstein wrote "It is not based on any rational theory, however, and its empirical basis should be kept in mind." We believe a significant contribution of this paper is that it establishes a rigorous mathematical foundation for the Newton-Busemann pressure law, as the law can now be proved by short and straightforward computations from the very fundamental compressible Euler equations. This makes the Newton theory a part of modern rational mechanics.

To show equivalence of (1.23) with (3.29) in [2, p.67], note that $b'(x)=\tan\theta$ for θ appearing in (3.29) in [2, p.67], and then rewrite the integration in (1.23) to be integrated for the variable y, using $\mathrm{d}y=b'(x)\,\mathrm{d}x$. The appearance of the factor 2 in (3.29) in [2, p.67] is due to the fact that Anderson employed the scaling (3.16) in [2, p.61] to define the pressure coefficients C_p, where there is a $\frac{1}{2}$ in the denominator; while in our work we just used the scaling $\tilde{p}=\frac{p}{\rho_\infty u_\infty^2}$ without the factor $\frac{1}{2}$ in the denominator (see the scalings below (2.6) in [23]).

REMARK 1.8. Obviously, (1.21) is sufficient to guarantee that $w_p>0$, which means the ramp suffers force from the flow at each point. Such a nontrivial ramp exists. For example, taking $b(x)=\sqrt{x}$, direct computation shows that (1.21) holds for all $x\in[0,+\infty)$.

THEOREM 1.2. *For limiting hypersonic flow passing a finite ramp W_f, suppose* (1.21) *is valid for $0\le x\le x_*$ and $H(x_*)>0$. Then for $\underline{p}\in[0,1]$, Problem 2 has a global measure solution (see* (3.17)-(3.18)*) defined for all $x\ge 0$, with density containing a weighted Dirac measure supported on a curve which coincides with W_f for $0\le x\le x_*$. The curve for $x>x_*$, called "free layer", separates limiting hypersonic flow above it from the static gas below it. The shape of the free layer depends on the pressure of the static gas (cf. Figure 3.1 and Figure 3.2):*

(1) *If $\underline{p}=0$, the free layer (see* (3.24)*) is at most of the order $\sqrt{x}$ as $x\to\infty$.*

(2) *If $0<\underline{p}<1$, the free layer (see* (3.22)*) is of the order $\sqrt{\frac{\underline{p}}{1-\underline{p}}x}$ as $x\to\infty$.*

(3) *If $\underline{p}=1$, the free layer (see* (3.21)*) is of the order x^2 as $x\to\infty$.*

If $\underline{p}>1$, there is a finite point $x_\triangle>x_*$, and Problem 2 has a local measure solution (see (3.17)-(3.18)*) with the above structure defined only on $x\in[0,x_\triangle]$. The solution blows up at $x=x_\triangle$, in the sense that the free layer (see* (3.31)*) rolls up at $x=x_\triangle$ and cannot be prolonged.*

REMARK 1.9. According to [15, p.144], the concept of free layer was introduced by Busemann in 1933, to indicate a shock layer, the pressure behind which is zero. We use the free layer in this paper in a more general sense, which is also called *delta shock* in mathematics literature (see, for example, [7, 10, 25, 27]). There were many discussions and conjectures on free layers in [15, Section 3.3]. The merit of our approach is that we can calculate explicitly the expressions of various free layers based on rigorous mathematics. An application of designing an afterbody that bears no force in limiting hypersonic flow is discussed in Remark 3.4.

REMARK 1.10. To our knowledge, the last conclusion in the above theorem presents the first example of blowing up of measure solutions (delta shocks) for the Euler equations.

THEOREM 1.3. *Under the same assumptions on $b(x)$ as in Theorem 1.2, for Problem 3, i.e., interactions of limiting hypersonic flows and pressureless jets, we have the following results:*

(1) *If $\underline{v}/\underline{u} \geq b'(x_*)$, then Problem 3 has a measure solution (see (4.19)-(4.21)), containing a free layer (see (4.23) and (4.25)) separating the limiting hypersonic flow and pressureless jet, which is of the order $\dfrac{\sqrt{\underline{\rho}}\underline{v}}{1+\sqrt{\underline{\rho}}\underline{u}}x$ as $x \to \infty$.*

(2) *If $\underline{v}/\underline{u} < b'(x_*)$, then Problem 3 has a global measure solution (see (4.44)-(4.45)) containing vacuum. The vacuum starts at $(x_*, b(x_*))$, and is bounded by a free layer (see (4.47)) and a straight contact discontinuity (see (4.48)). Furthermore, if $\underline{v} \leq 0$, the vacuum is unbounded. If $\underline{v} > 0$, the vacuum is bounded.*

REMARK 1.11. To exclude the possibility that particles escape from the free layer and hence leads to obvious non-uniqueness of measure solutions, we use the well-recognized entropy condition of delta shocks (4.5). The free layers obtained for item (1) in the above theorem satisfy this entropy condition.

REMARK 1.12. To prove item (2) in the theorem, we encounter a problem of colliding of free layer and contact discontinuity, and it is reduced to the case studied in (1).

We will prove Theorems 1.1, 1.2 and 1.3 in Sections 2-4 respectively. In a short Section 5, we focus on the role played by singular Riemann problems in the studies of general measure solutions. Here, by singular Riemann problem we mean the initial data is piecewise constant, with density containing a Dirac measure supported on the initial discontinuity point.

Finally we review briefly some mathematical studies on Radon measure solutions of hyperbolic equations. For scalar conservation laws, Liu and Pierre [20] had already found regularizing effects of genuinely nonlinear fluxes — although the initial data could contain Dirac measures, the solutions are always functions. Demengel and Serre [12] studied well-posedness of Cauchy problems of scalar conservation laws with general convex fluxes that grow linearly at infinity, and the initial data being non-negative measures. The main tools are Lax-Oleinik formula and theory of Hamilton-Jacobi equations. See [3] for recent developments. Dal Maso, LeFloch and Murat [11] introduced a product of a measure and certain discontinuous functions, and used it to define and study measure solutions to some 2×2 hyperbolic system [14, 19]. There are also many works studying measure solutions using various flux approximation/regularization, such as vanishing viscosity [26,27] or vanishing pressure [7,8]. Huang and Wang established well-posedness in the class of Radon measures for Cauchy problem of the one-dimensional pressureless Euler equations [18]. See [4] for recent progress on the multidimensional case. We recommend [31] for a rather complete survey of mathematical studies of delta shocks.

Comparing to these established works, the merit of our approach is that our definition of Radon measure solution is rather elementary and flexible, applicable to a large extent of problems, and we can prove from it naturally some physically well-known formulas. However, since the definition employed the special structure of compressible Euler equations, we do not know presently how to extend it to general hyperbolic systems of conservation laws.

2. Limiting hypersonic flow passing an infinite ramp and Newton-Busemann pressure law

In this section we prove Theorem 1.1 by constructing a measure solution to Problem 1.

Let I_A be the characteristic function of a set $A\subset\mathbb{R}^2$, namely $\mathsf{I}_A(x,y)=1$ if $(x,y)\in A$ and $\mathsf{I}_A(x,y)=0$ otherwise. Recall that $\mathcal{L}^2$ is the standard Lebesgue measure on the plane $\mathbb{R}^2$. Now suppose the measures of fluxes are given by

$$\begin{aligned} m^0&=\rho_0u_0\mathsf{I}_\Omega\mathcal{L}^2+w_m^0(x)\delta_W=\mathsf{I}_\Omega\mathcal{L}^2+w_m^0(x)\delta_W,\\ n^0&=\rho_0v_0\mathsf{I}_\Omega\mathcal{L}^2+w_n^0(x)\delta_W=w_n^0(x)\delta_W; \end{aligned} \tag{2.1}$$

$$m^1=\mathsf{I}_\Omega\mathcal{L}^2+w_m^1(x)\delta_W,\quad n^1=w_n^1(x)\delta_W,\quad \wp=0; \tag{2.2}$$

$$m^2=w_m^2(x)\delta_W,\qquad n^2=w_n^2(x)\delta_W; \tag{2.3}$$

$$m^3=E_0\mathsf{I}_\Omega\mathcal{L}^2+w_m^3(x)\delta_W,\quad n^3=w_n^3(x)\delta_W, \tag{2.4}$$

where $w_m^i(x),w_n^i(x)$ $(i=0,1,2,3)$ are functions to be determined.

Substituting (2.1) into (1.13), it follows

$$\begin{aligned}\int_\Omega \partial_x\phi(x,y)\,\mathrm{d}x\mathrm{d}y+\int_0^\infty w_m^0(x)\partial_x\phi(x,b(x))\sqrt{1+b'(x)^2}\,\mathrm{d}x\\ +\int_0^\infty w_n^0(x)\partial_y\phi(x,b(x))\sqrt{1+b'(x)^2}\,\mathrm{d}x+\int_0^\infty \phi(0,y)\,\mathrm{d}y=0.\end{aligned} \tag{2.5}$$

Observing that

$$\begin{aligned}&\int_0^\infty w_m^0(x)\partial_x\phi(x,b(x))\sqrt{1+b'(x)^2}\,\mathrm{d}x\\ &=-w_m^0(0)\sqrt{1+b'(0)^2}\phi(0,0)-\int_0^\infty b'(x)w_m^0(x)\sqrt{1+b'(x)^2}\partial_y\phi(x,b(x))\,\mathrm{d}x\\ &\quad-\int_0^\infty\frac{\mathrm{d}(w_m^0(x)\sqrt{1+b'(x)^2})}{\mathrm{d}x}\phi(x,b(x))\,\mathrm{d}x,\end{aligned}$$

we have

$$\begin{aligned}w_m^0(0)\sqrt{1+b'(0)^2}\phi(0,0)-\int_0^\infty b'(x)\phi(x,b(x))\,\mathrm{d}x+\int_0^\infty\frac{\mathrm{d}(w_m^0(x)\sqrt{1+b'(x)^2})}{\mathrm{d}x}\\ \cdot\phi(x,b(x))\,\mathrm{d}x-\int_0^\infty(w_n^0(x)-b'(x)w_m^0(x))\sqrt{1+b'(x)^2}\partial_y\phi(x,b(x))\,\mathrm{d}x=0.\end{aligned} \tag{2.6}$$

By arbitrariness of ϕ, the above implies

$$w_m^0(0)=0,\quad \frac{\mathrm{d}(w_m^0(x)\sqrt{1+b'(x)^2})}{\mathrm{d}x}=b'(x),\quad w_n^0(x)=b'(x)w_m^0(x), \tag{2.7}$$

and we solve from this

$$w_m^0(x)=\frac{b(x)}{\sqrt{1+b'(x)^2}},\quad w_n^0(x)=\frac{b'(x)b(x)}{\sqrt{1+b'(x)^2}}. \tag{2.8}$$

We may get

$$w_m^3(x)=\frac{E_0 b(x)}{\sqrt{1+b'(x)^2}},\quad w_n^3(x)=\frac{E_0 b'(x)b(x)}{\sqrt{1+b'(x)^2}} \tag{2.9}$$

in the same way.[2]

Similarly, substituting (2.2) into (1.14), we have

$$\begin{aligned}\int_0^\infty [(1-w_p(x))b'(x)-\frac{\mathrm{d}(w_m^1(x)\sqrt{1+b'(x)^2})}{\mathrm{d}x}]\phi(x,b(x))\,\mathrm{d}x-w_m^1(0)\sqrt{1+b'(0)^2}\phi(0,0)\\ +\int_0^\infty (w_n^1(x)-b'(x)w_m^1(x))\sqrt{1+b'(x)^2}\partial_y\phi(x,b(x))\,\mathrm{d}x=0,\end{aligned} \tag{2.10}$$

which yields

$$w_m^1(0)=0,\quad w_n^1(0)=0,\quad w_n^1(x)=b'(x)w_m^1(x), \tag{2.11}$$

$$\frac{\mathrm{d}(w_m^1(x)\sqrt{1+b'(x)^2})}{\mathrm{d}x}=(1-w_p(x))b'(x),\qquad x\ge 0. \tag{2.12}$$

Note the function $w_p(x)$ shall be solved.

Thanks to (2.3), and noticing that $v_0=0$, (1.15) becomes

$$\begin{aligned}-w_m^2(0)\sqrt{1+b'(0)^2}\phi(0,0)+\int_0^\infty [w_p(x)-\frac{\mathrm{d}(w_m^2(x)\sqrt{1+b'(x)^2})}{\mathrm{d}x}]\phi(x,b(x))\mathrm{d}x\\ +\int_0^\infty (w_n^2(x)-b'(x)w_m^2(x))\sqrt{1+b'(x)^2}\partial_y\phi(x,b(x))\mathrm{d}x=0,\end{aligned} \tag{2.13}$$

consequently

$$w_m^2(0)=0,\quad w_n^2(x)=b'(x)w_m^2(x), \tag{2.14}$$

$$\frac{\mathrm{d}(w_m^2(x)\sqrt{1+b'(x)^2})}{\mathrm{d}x}=w_p(x),\quad x\ge 0. \tag{2.15}$$

By requirements (1.5) and (1.18), there holds

$$w_m^2(x)=b'(x)w_m^1(x). \tag{2.16}$$

Then we solve from (2.12), (2.15) and (2.16) that

$$w_m^1(x)=\frac{H(x)}{1+b'(x)^2}, \tag{2.17}$$

$$w_p(x)=\frac{b''(x)H(x)+b'(x)^2\sqrt{1+b'(x)^2}}{(1+b'(x)^2)^{\frac{3}{2}}}, \tag{2.18}$$

[2]Actually for this problem there is a freedom to determine the value of E in these two weights. We choose here E_0 on W, because for supersonic flows without concentration, it is well-known that the quantity E is always constant along flow trajectories. So for this problem we also take it as constant in the whole flow field due to our uniform initial data.

where

$$H(x) \triangleq \int_0^x \frac{b'(t)}{\sqrt{1+b'(t)^2}}\,\mathrm{d}t. \tag{2.19}$$

We thus proved the Newton-Busemann pressure law (1.23).

To write down a measure solution, applying (1.18), one has

$$u|_W = \frac{H(x)}{b(x)\sqrt{1+b'(x)^2}}, \quad v|_W = \frac{b'(x)H(x)}{b(x)\sqrt{1+b'(x)^2}}, \tag{2.20}$$

hence

$$u = \mathsf{I}_\Omega + \frac{H(x)}{b(x)\sqrt{1+b'(x)^2}}\mathsf{I}_W, \quad v = \frac{b'(x)H(x)}{b(x)\sqrt{1+b'(x)^2}}\mathsf{I}_W. \tag{2.21}$$

By (1.17), one gets the measure of mass density:

$$\varrho = \mathsf{I}_\Omega \mathcal{L}^2 + \frac{(b(x))^2}{H(x)}\delta_W. \tag{2.22}$$

Furthermore, recalling that (1.20), (2.8) and (2.9), then

$$E = E_0 \mathsf{I}_\Omega + E_0 \mathsf{I}_W. \tag{2.23}$$

So (2.21), (2.22) and (2.23) constitute a measure solution to Problem 1. This completes the proof of Theorem 1.1.

Remark 2.1. The reason why we require $w_p > 0$ in Theorem 1.1 is to guarantee the assumption lying in (2.1)-(2.4), namely concentration of mass appears just on the surface of the ramp. If at some point $w_p = 0$, then the particles in the shock layer may not feel the ramp and then fly away from the ramp, hence a free layer appears. The region between the free layer and the ramp is vacuum, or with zero pressure, a situation that we will discuss in the next section.

Remark 2.2. One may wonder whether there is a ramp that bears uniform force p per unit area from the upcoming hypersonic limit flow. This means the function $b(x)$ solves the nonlocal ordinary differential equation

$$b''(x)H(x) + b'(x)^2\sqrt{1+b'(x)^2} = p(1+b'(x)^2)^{3/2}, \tag{2.24}$$

where $H(x)$ is defined by (2.19). Some computation reduces this equation to $H''H + (H')^2 = p$, and by $H(0)=0$ we solve that $b(x) = \sqrt{\frac{p}{1-p}}x$, a case studied in [23]. So there is no nontrivial ramp that bears no force in limiting hypersonic flow. It would be interesting to compare this result with Remark 3.4 in the following section.

3. Limiting hypersonic flow passing a finite ramp and interactions with static gas

In this section we prove Theorem 1.2. To define a measure solution, recall that the domain we consider now is

$$\widetilde{\Omega} \triangleq \Omega_1 \cup \Omega_2,$$

where

$$\Omega_1 \triangleq \{(x,y)\in\mathbb{R}^2:\ 0<x\le x_*,\ y>b(x);\ x>x_*, y>s(x)\}$$

and

$$\Omega_2 \triangleq \{(x,y)\in\mathbb{R}^2:\ x>x_*,\ y<s(x)\}$$

represent respectively the region occupied by the limiting hypersonic flow above the free layer S and the region behind the ramp and below S, see Figure 1.2. Comparing to Problem 1, the solid boundary now is $\widehat{W}=W_f\cup W_e$. Therefore, for a definition of measure solutions of Problem 2, we just replace the boundary W appeared in (1.14) and (1.15) in Definition 1.1 by $\widehat{W}$, with $\mathbf{n}=(n_1,n_2)=(-b'(x),1)/\sqrt{1+b'(x)^2}$ being inner unit normal vector on W_f (pointing into Ω_1) and $\mathbf{n}=(n_1,n_2)=(1,0)$ on W_e, pointing into Ω_2.

It turns out that there is a measure solution to Problem 2, which is piecewise constant, connected by a free layer S. We firstly construct such a solution and then study the dependence of the shape of S on the pressure $\underline{p}$ of the static gas.

3.1. Construction of piecewise constant measure solutions. Set $\widetilde{W}=W_f\cup S$, and

$$\begin{aligned} m^0&=\mathsf{I}_{\Omega_1}\mathcal{L}^2+w_m^0(x)\delta_{\widetilde{W}}=\mathsf{I}_{\Omega_1}\mathcal{L}^2+w_m^0(x)\delta_{W_f}+\widetilde{w_m^0}(x)\delta_S,\\ n^0&=w_n^0(x)\delta_{\widetilde{W}}=w_n^0(x)\delta_{W_f}+\widetilde{w_n^0}(x)\delta_S,\end{aligned}\tag{3.1}$$

$$\begin{aligned} m^1&=\mathsf{I}_{\Omega_1}\mathcal{L}^2+w_m^1(x)\delta_{\widetilde{W}}=\mathsf{I}_{\Omega_1}\mathcal{L}^2+w_m^1(x)\delta_{W_f}+\widetilde{w_m^1}(x)\delta_S,\\ n^1&=w_n^1(x)\delta_{\widetilde{W}}=w_n^1(x)\delta_{W_f}+\widetilde{w_n^1}(x)\delta_S,\quad \wp=\underline{p}\mathsf{I}_{\Omega_2}\mathcal{L}^2,\end{aligned}\tag{3.2}$$

$$\begin{aligned} m^2&=w_m^2(x)\delta_{\widetilde{W}}=w_m^2(x)\delta_{W_f}+\widetilde{w_m^2}(x)\delta_S,\\ n^2&=w_n^2(x)\delta_{\widetilde{W}}=w_n^2(x)\delta_{W_f}+\widetilde{w_n^2}(x)\delta_S,\end{aligned}\tag{3.3}$$

$$\begin{aligned} m^3&=E_0\mathsf{I}_{\Omega_1}\mathcal{L}^2+w_m^3(x)\delta_{W_f}+\widetilde{w_m^3}(x)\delta_S,\\ n^3&=w_n^3(x)\delta_{W_f}+\widetilde{w_n^3}(x)\delta_S,\end{aligned}\tag{3.4}$$

where $\widetilde{w_m^i}(x),\widetilde{w_n^i}(x)$ $(i=0,1,2,3)$ are new unknown weights on the free layer.

The calculations to solve the measure solution are quite similar to those presented in Section 2. Substituting (3.1) into (1.13), we find

$$\begin{aligned}&\int_{\Omega_1}\partial_x\phi(x,y)\,\mathrm{d}x\mathrm{d}y+\int_0^{x_*}w_m^0(x)\partial_x\phi(x,b(x))\sqrt{1+b'(x)^2}\,\mathrm{d}x\\ &\quad+\int_{x_*}^{\infty}\widetilde{w_m^0}(x)\partial_x\phi(x,s(x))\sqrt{1+s'(x)^2}\,\mathrm{d}x+\int_0^{x_*}w_n^0(x)\partial_y\phi(x,b(x))\sqrt{1+b'(x)^2}\,\mathrm{d}x\\ &\quad+\int_{x_*}^{\infty}\widetilde{w_n^0}(x)\partial_y\phi(x,s(x))\sqrt{1+s'(x)^2}\,\mathrm{d}x+\int_0^{\infty}\phi(0,y)\,\mathrm{d}y=0.\end{aligned}\tag{3.5}$$

Applying divergence theorem, and noticing that $s(x_*)=b(x_*)$, it follows

$$\begin{aligned}&(w_m^0(x_*)\sqrt{1+b'(x_*)^2}-\widetilde{w_m^0}(x_*)\sqrt{1+s'(x_*)^2})\phi(x_*,b(x_*))\\ &\quad+\int_{x_*}^{\infty}[s'(x)-\frac{\mathrm{d}(\widetilde{w_m^0}(x)\sqrt{1+s'(x)^2})}{\mathrm{d}x}]\phi(x,s(x))\,\mathrm{d}x\\ &\quad+\int_{x_*}^{\infty}(\widetilde{w_n^0}(x)-s'(x)\widetilde{w_m^0}(x))\sqrt{1+s'(x)^2}\partial_y\phi(x,s(x))\,\mathrm{d}x=0.\end{aligned}\tag{3.6}$$

Since ϕ is arbitrary, we get not only (2.7) and (2.8), but also

$$\begin{aligned} &\widetilde{w_m^0}(x_*)\sqrt{1+s'(x_*)^2}=w_m^0(x_*)\sqrt{1+b'(x_*)^2}=b(x_*),\\ &\frac{\mathrm{d}(\widetilde{w_m^0}(x)\sqrt{1+s'(x)^2})}{\mathrm{d}x}=s'(x),\ \widetilde{w_n^0}(x)=s'(x)\widetilde{w_m^0}(x),\quad x>x_*, \end{aligned} \tag{3.7}$$

hence

$$\widetilde{w_m^0}(x)=\frac{s(x)}{\sqrt{1+s'(x)^2}},\quad \widetilde{w_n^0}(x)=\frac{s'(x)s(x)}{\sqrt{1+s'(x)^2}},\quad x>x_*. \tag{3.8}$$

Similar calculation also yields

$$\widetilde{w_m^3}(x)=\frac{E_0 s(x)}{\sqrt{1+s'(x)^2}},\quad \widetilde{w_n^3}(x)=\frac{E_0 s'(x)s(x)}{\sqrt{1+s'(x)^2}}. \tag{3.9}$$

REMARK 3.1. It is important to notice that the two equations in (3.8) imply the slip condition (1.9) on the free layer, namely (1.9) is natural on the free boundary S.

By virtue of

$$w_p n_1\delta_{\widehat{W}}=w_p^f(x)n_1(x)\delta_{W_f}+w_p^e(y)n_1\delta_{W_e}=w_p^f(x)\frac{-b'(x)}{\sqrt{1+b'(x)^2}}\delta_{W_f}+w_p^e(y)\delta_{W_e},$$

from (3.2) and (1.14), removing the test function ϕ, we obtain that

$$\begin{aligned} &w_m^1(0)=0,\ w_n^1(0)=0,\ w_n^1(x)=b'(x)w_m^1(x),\ \frac{\mathrm{d}(w_m^1(x)\sqrt{1+b'(x)^2})}{\mathrm{d}x}=(1-w_p(x))b'(x),\\ &\widetilde{w_m^1}(x_*)\sqrt{1+s'(x_*)^2}=w_m^1(x_*)\sqrt{1+b'(x_*)^2}=\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}},\\ &\widetilde{w_n^1}(x)=s'(x)\widetilde{w_m^1}(x),\quad w_p^e(y)=\underline{p},\quad \frac{\mathrm{d}(\widetilde{w_m^1}(x)\sqrt{1+s'(x)^2})}{\mathrm{d}x}=(1-\underline{p})s'(x), \end{aligned} \tag{3.10}$$

from which we get

$$\widetilde{w_m^1}(x)=\frac{(1-\underline{p})[s(x)-b(x_*)]+\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}}}{\sqrt{1+s'(x)^2}},\qquad x\geq x_*. \tag{3.11}$$

Similarly, noticing

$$w_p n_2\delta_{\widehat{W}}=w_p^f(x)n_2(x)\delta_{W_f}+w_p^e(y)n_2\delta_{W_e}=\frac{w_p^f(x)}{\sqrt{1+b'(x)^2}}\delta_{W_f},$$

thanks to (3.3) and (1.15), we find

$$w_m^2(0)=0,\quad w_n^2(x)=b'(x)w_m^2(x),\quad \frac{\mathrm{d}(w_m^2(x)\sqrt{1+b'(x)^2})}{\mathrm{d}x}=w_p^f(x), \tag{3.12}$$

$$\widetilde{w_m^2}(x_*)\sqrt{1+s'(x_*)^2}=w_m^2(x_*)\sqrt{1+b'(x_*)^2}=\frac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}, \tag{3.13}$$

$$\widetilde{w_n^2}(x)=s'(x)\widetilde{w_m^2}(x),\quad \frac{\mathrm{d}(\widetilde{w_m^2}(x)\sqrt{1+s'(x)^2})}{\mathrm{d}x}=\underline{p}. \tag{3.14}$$

Note that (3.12), as we expected, is the same as (2.14) and (2.15). By (3.13) and (3.14), we discover

$$\widetilde{w_m^2}(x)=\frac{\underline{p}(x-x_*)+\dfrac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}}{\sqrt{1+s'(x)^2}},\qquad x\geq x_*. \tag{3.15}$$

According to (1.18), from (3.8), (3.11) and (3.15), we obtain

$$\begin{aligned} u|_S&=\frac{(1-\underline{p})[s(x)-b(x_*)]+\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}}}{s(x)},\\ v|_S&=\frac{\underline{p}(x-x_*)+\dfrac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}}{s(x)}. \end{aligned} \tag{3.16}$$

Combining with (2.21), we have

$$\begin{aligned} &u=\mathsf{I}_{\Omega_1}+\frac{H(x)}{b(x)\sqrt{1+b'(x)^2}}\mathsf{I}_{W_f}+\frac{(1-\underline{p})[s(x)-b(x_*)]+\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}}}{s(x)}\mathsf{I}_S,\\ &v=\frac{b'(x)H(x)}{b(x)\sqrt{1+b'(x)^2}}\mathsf{I}_{W_f}+\frac{\underline{p}(x-x_*)+\dfrac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}}{s(x)}\mathsf{I}_S,\\ &E=E_0\mathsf{I}_{\Omega_1}+\underline{E}\mathsf{I}_{\Omega_2}+E_0\mathsf{I}_{W_f}+E_0\mathsf{I}_S. \end{aligned} \tag{3.17}$$

Hence the measure of mass density is

$$\varrho=\mathsf{I}_{\Omega_1}\mathcal{L}^2+\underline{\rho}\mathsf{I}_{\Omega_2}\mathcal{L}^2+\frac{(b(x))^2}{H(x)}\delta_{W_f}+\frac{(s(x))^2}{\sqrt{1+s'(x)^2}[(1-\underline{p})(s(x)-b(x_*))+\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}}]}\delta_S. \tag{3.18}$$

From (3.17) and (3.18), if we could solve the free layer $y=s(x)$, then a measure solution to Problem 2 is determined. By the slip condition (1.9), there is an ordinary differential equation for $s(x)$:

$$\begin{cases} \underline{p}(x-x_*)+\dfrac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}=s'(x)[(1-\underline{p})(s(x)-b(x_*))+\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}}],\\ s(x_*)=b(x_*). \end{cases} \tag{3.19}$$

Integrating both sides of (3.19) yields

$$\begin{aligned} &(1-\underline{p})s^2(x)+2[\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}}-(1-\underline{p})b(x_*)]s(x)-\underline{p}(x-x_*)^2\\ &\quad-\frac{2b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}(x-x_*)+b(x_*)[(1-\underline{p})b(x_*)-\frac{2H(x_*)}{\sqrt{1+b'(x_*)^2}}]=0, \end{aligned} \tag{3.20}$$

where $s(x_*)=b(x_*)$. Obviously, solution of $s(x)$ depends on the value of $\underline{p}$. We will discuss this in the next subsection.

3.2. The shape of free layer depending on pressure of downward static gas. We divide the analysis into four cases, namely $\underline{p}=1$, $\underline{p}\in(0,1)$, $\underline{p}=0$, and $\underline{p}>1$.

3.2.1. Case 1: $\underline{p}=1$. For this simple case, from (3.20) we easily see

$$s(x)=\frac{\sqrt{1+b'(x_*)^2}(x-x_*)^2}{2H(x_*)}+b'(x_*)(x-x_*)+b(x_*),\quad \forall x\ge x_*. \tag{3.21}$$

In the following we focus on the more involved situation that $\underline{p}\neq 1$. We solve from (3.20) that

$$s(x)=\frac{\sqrt{\Delta}}{1-\underline{p}}+b(x_*)-\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}(1-\underline{p})}. \tag{3.22}$$

To make sure (3.22) is meaningful, it shall hold

$$\Delta\triangleq(1-\underline{p})\underline{p}(x-x_*)^2+\frac{2(1-\underline{p})H(x_*)b'(x_*)}{\sqrt{1+b'(x_*)^2}}(x-x_*)+(\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}})^2\ge 0,\quad x\ge x_*. \tag{3.23}$$

3.2.2. Case 2: $\underline{p}\in(0,1)$. For this case, $(1-\underline{p})\underline{p}>0$, hence the terms in (3.23) are always nonnegative for $x\ge x_*$, thanks to the assumption that $b'(x)\ge 0$ and $H(x_*)>0$. Therefore for this case the solution of (3.19) is given by (3.22).

Furthermore, from (3.22) we see that $s(x)$ is of the order $\sqrt{\frac{\underline{p}}{1-\underline{p}}}x$ as $x\to\infty$.

REMARK 3.2. As a special case, for $\underline{p}=\frac{(b'(x_*))^2}{1+(b'(x_*))^2}$, the free layer is the straight line $s(x)=b'(x_*)(x-x_*)+b(x_*)$.

3.2.3. Case 3: $\underline{p}=0$. By (3.22), one has

$$s(x)=\sqrt{\frac{2H(x_*)b'(x_*)}{\sqrt{1+b'(x_*)^2}}(x-x_*)+(\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}})^2}+b(x_*)-\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}}. \tag{3.24}$$

Therefore the free layer is of the order $\sqrt{x}$ as $x\to+\infty$. In particular, if $b'(x_*)=0$, then (3.24) implies that $s(x)=b(x_*)$, namely the free layer is a straight line parallel to the upcoming limiting hypersonic flow and no particle impinges on the free layer.

REMARK 3.3. We note that this case includes the situation that the state below the free layer is vacuum.

REMARK 3.4. We have shown in Remark 2.2, the nonexistence of nontrivial ramp that bears no force from the limiting hypersonic flow. However, the above results show that, we could design a forebody with boundary W_f to form a shock layer, and then design an afterbody with boundary (3.24) so that there is no force acting on the afterbody; for this case the free layer becomes a shield which bears all the force from the limiting hypersonic flow. We note that such ideas have already appeared in discussions in [15, p.142].

3.2.4. Case 4: $\underline{p}>1$. For this case, $(1-\underline{p})\underline{p}<0$ and some singularity will appear if we still assume that the free layer is a graph of a function of x. So we use a parametric representation $(x(t),y(t))$ of the free layer, with the parameter t satisfying $x(t_*)=x_*,y(t_*)=b(x_*)$. Substituting this into the definition of measure solutions, analysis like before shows that

$$\begin{cases}\frac{\widetilde{w_n^0}(t_*)\sqrt{x'(t_*)^2+y'(t_*)^2}}{y'(t_*)}=w_m^0(x_*)\sqrt{1+b'(x_*)^2}=b(x_*),\\ \widetilde{w_n^0}(t)x'(t)=y'(t)\widetilde{w_m^0}(t),\quad \frac{\widetilde{w_n^0}(t)\sqrt{x'(t)^2+y'(t)^2}}{y'(t)}=y(t),\qquad t>t_*,\end{cases}\tag{3.25}$$

$$\begin{cases}\frac{\widetilde{w_n^1}(t_*)\sqrt{x'(t_*)^2+y'(t_*)^2}}{y'(t_*)}=\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}},\ \widetilde{w_n^1}(t)x'(t)=y'(t)\widetilde{w_m^1}(t),\\ \frac{\widetilde{w_n^1}(t)\sqrt{x'(t)^2+y'(t)^2}}{y'(t)}=(1-\underline{p})(y(t)-b(x_*))+\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}},\qquad t>t_*,\end{cases}\tag{3.26}$$

$$\begin{cases}\frac{\widetilde{w_n^2}(t_*)\sqrt{x'(t_*)^2+y'(t_*)^2}}{y'(t_*)}=\frac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}},\ \widetilde{w_n^2}(t)x'(t)=y'(t)\widetilde{w_m^2}(t),\\ \frac{\widetilde{w_n^2}(t)\sqrt{x'(t)^2+y'(t)^2}}{y'(t)}=\underline{p}(x(t)-x_*)+\frac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}},\qquad t>t_*.\end{cases}\tag{3.27}$$

Hence we find for $t\geq t_*$,

$$\begin{aligned}u|_S&=\frac{(1-\underline{p})[y(t)-b(x_*)]+\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}}}{y(t)},\\ v|_S&=\frac{\underline{p}(x(t)-x_*)+\frac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}}{y(t)},\end{aligned}\tag{3.28}$$

and

$$\varrho=\mathsf{I}_{\Omega_1}\mathcal{L}^2+\underline{\rho}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_\rho^f(x)\delta_{W_f}+w_\rho^S(t)\delta_S,\tag{3.29}$$

where

$$w_\rho^f(x)=\frac{(b(x))^2}{H(x)},\qquad 0\leq x\leq x_*$$

and

$$w_\rho^S(t)=\frac{y'(t)(y(t))^2}{\sqrt{x'(t)^2+y'(t)^2}[\underline{p}(x(t)-x_*)+\frac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}]},\qquad t\geq t_*,$$

while $(x(t),y(t))$ $(t\geq t_*)$ solve the following ordinary differential equation

$$\begin{cases}[\underline{p}(x(t)-x_*)+\frac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}]x'(t)=y'(t)[(1-\underline{p})(y(t)-b(x_*))+\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}}],\\ y(t_*)=b(x_*).\end{cases}\tag{3.30}$$

Direct integration shows the solution is an ellipse passing $(x_*,b(x_*))$:

$$\begin{aligned}&\underline{p}\left[x(t)-x_*+\frac{1}{\underline{p}}\frac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}\right]^2+(\underline{p}-1)\left[y(t)-b(x_*)-\frac{1}{\underline{p}-1}\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}}\right]^2\\ &=\left[\frac{1}{\underline{p}}b'(x_*)^2+\frac{1}{\underline{p}-1}\right]\frac{H(x_*)^2}{1+b'(x_*)^2}.\end{aligned}\tag{3.31}$$

At the rightmost point $(x_\triangle, y_\triangle)$, where

$$x_\triangle = x(t_\triangle) = x_* + \frac{H(x_*)}{\underline{p}\sqrt{1+b'(x_*)^2}}\left(\sqrt{b'(x_*)^2 + \frac{\underline{p}}{\underline{p}-1}} - b'(x_*)\right), \tag{3.32}$$

$$y_\triangle = y(t_\triangle) = b(x_*) + \frac{H(x_*)}{\sqrt{1+b'(x_*)^2}(\underline{p}-1)}, \tag{3.33}$$

we have

$$u|_{(x_\triangle,\, y_\triangle)} = 0, \quad v|_{(x_\triangle,\, y_\triangle)} = \frac{\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}}\sqrt{b'(x_*)^2 + \dfrac{\underline{p}}{\underline{p}-1}}}{b(x_*) + \dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}(\underline{p}-1)}} > 0, \tag{3.34}$$

and the weight

$$w_\rho^S(t_\triangle) = \frac{y_\triangle^2}{\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}}\sqrt{b'(x_*)^2 + \dfrac{\underline{p}}{\underline{p}-1}}} < +\infty.$$

Since we are treating a hyperbolic problem upside of the free layer, with the positive x-axis being the hyperbolic direction, by causality, the upper branch of the ellipse (3.31) shall no longer be a part of the free layer. Also, noticing that $u(x_\triangle, y) = 0$ holds only at $y = y_\triangle$, the line segment $x = x_\triangle$ with $y \geq y_\triangle$ cannot be a free layer. So we conclude that the measure solution "blows up" (or terminates) at the point $(x_\triangle, y_\triangle)$, in the sense that the free layer satisfies (3.34) and cannot be prolonged anymore.

3.3. Conclusion and examples. In summary, we have the following lemma.

LEMMA 3.1.

(i) *For $\underline{p} = 1$, the free layer takes the form* (3.21).

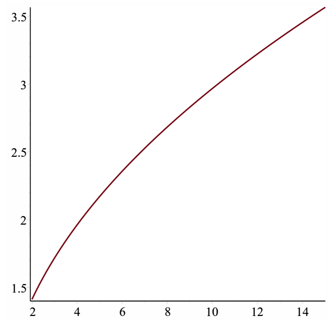
(ii) *For $0 \leq \underline{p} < 1$, the free layer $y = s(x)$ is given by* (3.22), *defined for all $x \geq x_*$.*

(iii) *For $\underline{p} > 1$, the free layer exists only for $x_* \leq x \leq x_\triangle$ and ending at the point $(x_\triangle, y_\triangle)$, where it rolls up and can not be prolonged. So in such a sense the measure solution blows up.*

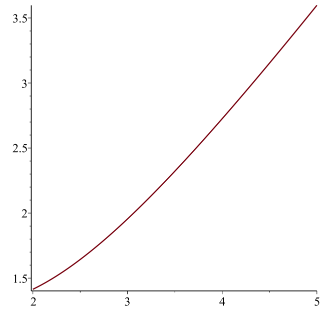
This finishes the proof of Theorem 1.2.

REMARK 3.5. As an example, we take $b(x) = \sqrt{x}$, $x_* = 2$ to draw graphs of free layers, with different pressures of the downward static gas. For $\underline{p} = 0$, we have $s(x) = \sqrt{\frac{2}{3}x - \frac{4}{9}} + \frac{\sqrt{2}}{3}$ (see Figure 3.1(a)). For $\underline{p} = \frac{1}{2}$, $s(x) = 2\sqrt{\frac{1}{4}(x-2)^2 + \frac{1}{3}x + \frac{2}{9}} - \frac{\sqrt{2}}{3}$ (see Figure 3.1(b)). For $\underline{p} = 1$, $s(x) = \frac{3}{4\sqrt{2}}(x-2)^2 + \frac{1}{2\sqrt{2}}(x-2) + \sqrt{2}$ (see Figure 3.1(c)). For $\underline{p} = 2$, $s(x) = -\sqrt{-2(x-2)^2 - \frac{2x}{3} + 20/9} + \frac{5\sqrt{2}}{3}$ and $(x_\triangle, s(x_\triangle)) = (2 + \frac{\sqrt{17}-1}{6}, \frac{5\sqrt{2}}{3})$, where the free layer rolls up (see Figure 3.1(d)).

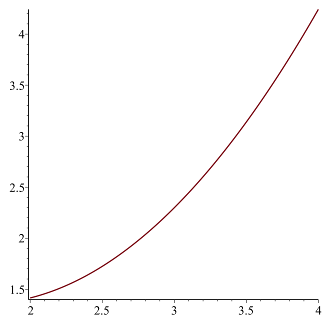
We also draw these graphs in the same frame for comparison, see Figure 3.2.



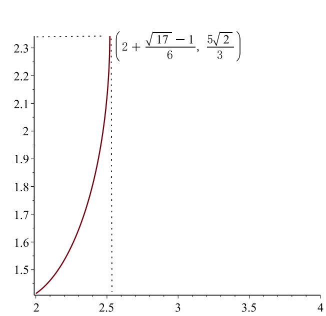
(a) $\underline{p}=0, s(x)=\sqrt{\frac{2}{3}x-\frac{4}{9}}+\frac{\sqrt{2}}{3}$.



(b) $\underline{p}=\frac{1}{2}, s(x)=2\sqrt{\frac{1}{4}x^2-\frac{2}{3}x+\frac{11}{9}}-\frac{\sqrt{2}}{3}$.



(c) $\underline{p}=1, s(x)=\frac{3}{4\sqrt{2}}x^2-\frac{5}{2\sqrt{2}}x+2\sqrt{2}$.



(d) $\underline{p}=2, s(x)=-\sqrt{-2x^2+\frac{22x}{3}-52/9}+\frac{5\sqrt{2}}{3}$.

FIG. 3.1. *(a)(b)(c)(d) demonstrate the free layer for different pressures* $\underline{p}$*, with* $b(x)=\sqrt{x}$ *and* $x_*=2$ *being fixed. So* $(2,\sqrt{2})$ *is the starting point of the free layer. In (d) the point* $(2+\frac{\sqrt{17}-1}{6},\frac{5\sqrt{2}}{3})$ *is the place where the free layer rolls up (terminates).*

4. Interactions of limiting hypersonic flows and pressureless jets

In this section we study Problem 3. We may define its measure solution in the sprit of Definition 1.1. Since there are initial data on $x=x_*$, item (i) in Definition 1.1 shall be replaced by

$$\langle m^0,\partial_x\phi\rangle+\langle n^0,\partial_y\phi\rangle+\int_0^\infty \rho_0u_0\phi(0,y)\,\mathrm{d}y+\int_{-\infty}^{b(x_*)}\underline{\rho}\underline{u}\phi(x_*,y)\,\mathrm{d}y=0, \tag{4.1}$$

$$\begin{aligned}\langle m^1,\partial_x\phi\rangle+\langle n^1,\partial_y\phi\rangle+\langle\wp,\partial_x\phi\rangle+\langle w_pn_1\delta_W,\phi\rangle+\int_0^\infty(\rho_0u_0^2+p_0)\phi(0,y)\,\mathrm{d}y&\\ +\int_{-\infty}^{b(x_*)}(\underline{\rho}\underline{u}^2+\underline{p})\phi(x_*,y)\,\mathrm{d}y=0,&\end{aligned} \tag{4.2}$$

$$\begin{aligned}\langle m^2,\partial_x\phi\rangle+\langle n^2,\partial_y\phi\rangle+\langle\wp,\partial_y\phi\rangle+\langle w_pn_2\delta_W,\phi\rangle+\int_0^\infty\rho_0u_0v_0\phi(0,y)\,\mathrm{d}y&\\ +\int_{-\infty}^{b(x_*)}\underline{\rho}\underline{u}\;\underline{v}\phi(x_*,y)\,\mathrm{d}y=0,&\end{aligned} \tag{4.3}$$

$$\langle m^3,\partial_x\phi\rangle+\langle n^3,\partial_y\phi\rangle+\int_0^\infty \rho_0u_0E_0\phi(0,y)\,\mathrm{d}y+\int_{-\infty}^{b(x_*)}(\underline{\rho}\underline{u}\;\underline{E})\phi(x_*,y)\,\mathrm{d}y=0, \tag{4.4}$$

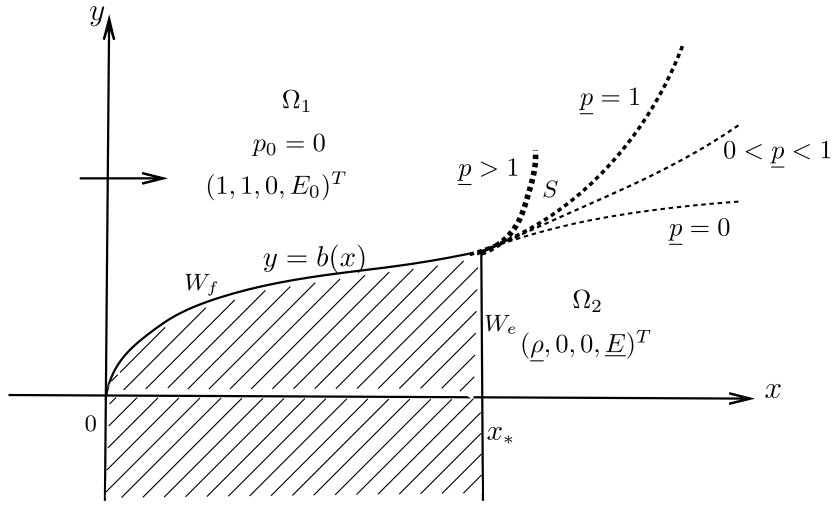


FIG. 3.2. *Different shapes of free layers for different pressures in "dead gas zone".*

where $\phi\in C_0^1(\mathbb{R}^2)$ is an arbitrary test function. The other requirements in Definition 1.1 are unchanged.

To construct a measure solution with physical significance, we need the following *entropy condition*:

$$\frac{\underline{v}}{\underline{u}}\geq\frac{v|_S}{u|_S}=s'(x)\geq\frac{v_0}{u_0}, \tag{4.5}$$

where $S\colon\ y=s(x)$ is a free layer separating constant state U_0 lying above it and constant state $\underline{U}$ below it. It is a generalization of Lax entropy conditions for shocks, and is widely used in the studies of delta shocks, see, for example, [10, p.329, (3.5)], or below (2.5) in [25, p.749] (for the unsteady case). It excludes some anomalous measure solutions such as the case wherein particles escape from the free layer. To fulfill (4.5), the analysis below is separated into two subsections.

4.1. Measure solution consists of two piecewise-constant states without vacuum. For this case we assume that

$$\begin{aligned}&m^0=\mathsf{I}_{\Omega_1}\mathcal{L}^2+\underline{\rho}\underline{u}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_m^0(x)\delta_{\widetilde{W}}=\mathsf{I}_{\Omega_1}\mathcal{L}^2+\underline{\rho}\underline{u}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_m^0(x)\delta_{W_f}+\widetilde{w_m^0}(x)\delta_S,\\&n^0=\underline{\rho}\underline{v}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_n^0(x)\delta_{\widetilde{W}}=\underline{\rho}\underline{v}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_n^0(x)\delta_{W_f}+\widetilde{w_n^0}(x)\delta_S,\end{aligned}\tag{4.6}$$

$$\begin{aligned}&m^1=\mathsf{I}_{\Omega_1}\mathcal{L}^2+\underline{\rho}\underline{u}^2\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_m^1(x)\delta_{\widetilde{W}}=\mathsf{I}_{\Omega_1}\mathcal{L}^2+\underline{\rho}\underline{u}^2\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_m^1(x)\delta_{W_f}+\widetilde{w_m^1}(x)\delta_S,\\&n^1=\underline{\rho}\underline{u}\ \underline{v}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_n^1(x)\delta_{\widetilde{W}}=\underline{\rho}\underline{u}\ \underline{v}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_n^1(x)\delta_{W_f}+\widetilde{w_n^1}(x)\delta_S,\end{aligned}\tag{4.7}$$

$$\begin{aligned}&m^2=\underline{\rho}\underline{u}\underline{v}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_m^2(x)\delta_{\widetilde{W}}=\underline{\rho}\underline{u}\underline{v}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_m^2(x)\delta_{W_f}+\widetilde{w_m^2}(x)\delta_S,\\&n^2=\underline{\rho}\underline{v}^2\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_n^2(x)\delta_{\widetilde{W}}=\underline{\rho}\underline{v}^2\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_n^2(x)\delta_{W_f}+\widetilde{w_n^2}(x)\delta_S,\quad\wp=0,\end{aligned}\tag{4.8}$$

$$\begin{aligned}&m^3=E_0\mathsf{I}_{\Omega_1}\mathcal{L}^2+\underline{\rho}\underline{u}\underline{E}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_m^3(x)\delta_{W_f}+\widetilde{w_m^3}(x)\delta_S,\\&n^3=\underline{\rho}\underline{v}\ \underline{E}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_n^3(x)\delta_{W_f}+\widetilde{w_n^3}(x)\delta_S,\end{aligned}\tag{4.9}$$

where $\widetilde{w^i_m}(x)$, $\widetilde{w^i_n}(x)$ $(i=0,1,2,3)$ are functions to be solved.

The computation mimics previous sections. Substituting (4.6) into (4.1), we get for $x\in[0,x_*]$,

$$w^0_m(0)=0,\quad w^0_n(x)=w^0_m(x)b'(x),\quad \frac{\mathrm{d}(w^0_m(x)\sqrt{1+b'(x)^2})}{\mathrm{d}x}=b'(x), \tag{4.10}$$

and for $x\geq x_*$,

$$\begin{aligned}&\widetilde{w^0_m}(x_*)\sqrt{1+s'(x_*)^2}=w^0_m(x_*)\sqrt{1+b'(x_*)^2}=b(x_*),\\&\frac{\mathrm{d}(\widetilde{w^0_m}(x)\sqrt{1+s'(x)^2})}{\mathrm{d}x}=\underline{\rho}\underline{v}+(1-\underline{\rho}\underline{u})s'(x),\ \widetilde{w^0_n}(x)=s'(x)\widetilde{w^0_m}(x).\end{aligned} \tag{4.11}$$

Hence

$$\begin{aligned}\widetilde{w^0_m}(x)&=\frac{\underline{\rho}\underline{v}(x-x_*)-\underline{\rho}\underline{u}(s(x)-b(x_*))+s(x)}{\sqrt{1+s'(x)^2}},\\\widetilde{w^0_n}(x)&=\frac{\underline{\rho}\underline{v}(x-x_*)-\underline{\rho}\underline{u}(s(x)-b(x_*))+s(x)}{\sqrt{1+s'(x)^2}}s'(x),\end{aligned} \tag{4.12}$$

and similarly

$$\begin{aligned}\widetilde{w^3_m}(x)&=\frac{\underline{\rho}\underline{v}\ \underline{E}(x-x_*)-\underline{\rho}\underline{u}\,\underline{E}(s(x)-b(x_*))+E_0s(x)}{\sqrt{1+s'(x)^2}},\\\widetilde{w^3_n}(x)&=\frac{\underline{\rho}\underline{v}\ \underline{E}(x-x_*)-\underline{\rho}\underline{u}\,\underline{E}(s(x)-b(x_*))+E_0s(x)}{\sqrt{1+s'(x)^2}}s'(x).\end{aligned} \tag{4.13}$$

Note that these equations imply the slip condition (1.9) on the free layer.

By (4.2) and (4.7), we have

$$\begin{aligned}&\widetilde{w^1_m}(x_*)\sqrt{1+s'(x_*)^2}=w^1_m(x_*)\sqrt{1+b'(x_*)^2}=\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}},\\&\widetilde{w^1_n}(x)=s'(x)\widetilde{w^1_m}(x),\quad \frac{\mathrm{d}(\widetilde{w^1_m}(x)\sqrt{1+s'(x)^2})}{\mathrm{d}x}=\underline{\rho}\underline{u}\underline{v}+(1-\underline{\rho}\underline{u}^2)s'(x).\end{aligned} \tag{4.14}$$

Consequently for $x\geq x_*$,

$$\widetilde{w^1_m}(x)=\frac{\underline{\rho}\underline{u}\underline{v}(x-x_*)+(1-\underline{\rho}\underline{u}^2)(s(x)-b(x_*))+\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}}}{\sqrt{1+s'(x)^2}}. \tag{4.15}$$

Moreover, by (4.8), Equation (4.3) is reduced to

$$\begin{aligned}&\widetilde{w^2_m}(x_*)\sqrt{1+s'(x_*)^2}=w^2_m(x_*)\sqrt{1+b'(x_*)^2}=\frac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}},\\&\widetilde{w^2_n}(x)=s'(x)\widetilde{w^2_m}(x),\quad \frac{\mathrm{d}(\widetilde{w^2_m}(x)\sqrt{1+s'(x)^2})}{\mathrm{d}x}=\underline{\rho}\underline{v}^2-\underline{\rho}\underline{u}\underline{v}s'(x).\end{aligned} \tag{4.16}$$

It follows

$$\widetilde{w^2_m}(x)=\frac{\underline{\rho}\underline{v}^2(x-x_*)-\underline{\rho}\underline{u}\underline{v}(s(x)-b(x_*))+\dfrac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}}{\sqrt{1+s'(x)^2}}, \tag{4.17}$$

and by (1.18),

$$
\begin{aligned}
u|_S &= \frac{\underline{\rho}\,\underline{u}\,\underline{v}(x-x_*)+(1-\underline{\rho}\,\underline{u}^2)(s(x)-b(x_*))+\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}}}{\underline{\rho}\,\underline{v}(x-x_*)-\underline{\rho}\,\underline{u}(s(x)-b(x_*))+s(x)},\\
v|_S &= \frac{\underline{\rho}\,\underline{v}^2(x-x_*)-\underline{\rho}\,\underline{u}\,\underline{v}(s(x)-b(x_*))+\dfrac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}}{\underline{\rho}\,\underline{v}(x-x_*)-\underline{\rho}\,\underline{u}(s(x)-b(x_*))+s(x)}.
\end{aligned}
\tag{4.18}
$$

So we discover that

$$
\begin{aligned}
u =& \mathsf{I}_{\Omega_1}+\underline{u}\mathsf{I}_{\Omega_2}+\frac{H(x)}{b(x)\sqrt{1+b'(x)^2}}\mathsf{I}_{W_f}\\
&+\frac{\underline{\rho}\,\underline{u}\,\underline{v}(x-x_*)+(1-\underline{\rho}\,\underline{u}^2)(s(x)-b(x_*))+\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}}}{\underline{\rho}\,\underline{v}(x-x_*)-\underline{\rho}\,\underline{u}(s(x)-b(x_*))+s(x)}\mathsf{I}_S,\\
v =& \underline{v}\mathsf{I}_{\Omega_2}+\frac{b'(x)H(x)}{b(x)\sqrt{1+b'(x)^2}}\mathsf{I}_{W_f}\\
&+\frac{\underline{\rho}\,\underline{v}^2(x-x_*)-\underline{\rho}\,\underline{u}\,\underline{v}(s(x)-b(x_*))+\dfrac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}}{\underline{\rho}\,\underline{v}(x-x_*)-\underline{\rho}\,\underline{u}(s(x)-b(x_*))+s(x)}\mathsf{I}_S,
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
E =& E_0\mathsf{I}_{\Omega_1}+\underline{E}\mathsf{I}_{\Omega_2}+E_0\mathsf{I}_{W_f}\\
&+\frac{\underline{\rho}\,\underline{v}\ \underline{E}(x-x_*)-\underline{\rho}\,\underline{u}\,\underline{E}(s(x)-b(x_*))+E_0 s(x)}{\underline{\rho}\,\underline{v}(x-x_*)-\underline{\rho}\,\underline{u}(s(x)-b(x_*))+s(x)}\mathsf{I}_S,
\end{aligned}
\tag{4.20}
$$

and the measure of mass density is

$$
\begin{aligned}
\varrho =& \mathsf{I}_{\Omega_1}\mathcal{L}^2+\underline{\rho}\mathsf{I}_{\Omega_2}\mathcal{L}^2+\frac{(b(x))^2}{H(x)}\delta_{W_f}\\
&+\frac{[\underline{\rho}\,\underline{v}(x-x_*)-\underline{\rho}\,\underline{u}(s(x)-b(x_*))+s(x)]^2}{\sqrt{1+s'(x)^2}[\underline{\rho}\,\underline{u}\,\underline{v}(x-x_*)+(1-\underline{\rho}\,\underline{u}^2)(s(x)-b(x_*))+\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}}]}\delta_S.
\end{aligned}
\tag{4.21}
$$

So what remains is to determine $s(x)$. Using the slip condition (1.9), from (4.19) we have the following ordinary differential equation:

$$
\begin{cases}
\underline{\rho}\,\underline{v}^2(x-x_*)-\underline{\rho}\,\underline{u}\,\underline{v}(s(x)-b(x_*))+\dfrac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}\\
=s'(x)[\underline{\rho}\,\underline{u}\,\underline{v}(x-x_*)+(1-\underline{\rho}\,\underline{u}^2)(s(x)-b(x_*))+\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}}],\\
s(x_*)=b(x_*).
\end{cases}
\tag{4.22}
$$

LEMMA 4.1. (1) *If $1-\underline{\rho}\,\underline{u}^2=0$ and $\underline{v}/\underline{u}\geq b'(x_*)$, (4.22) has a unique solution $y=s(x)$, which is of the order $\dfrac{\underline{v}}{2\underline{u}}x$ as $x\to+\infty$, and $y=s(x)$ is a free layer satisfying entropy* *condition (4.5), separating the limiting hypersonic flow above it from the pressureless jet below it.*

(2) *For* $1-\underline{\rho}\underline{u}^2\neq 0$ *and* $\underline{v}/\underline{u}\geq b'(x_*)$, (4.22) *has a solution* $y=s(x)$ *which is of the order* $\frac{\sqrt{\underline{\rho}}\underline{v}x}{1+\sqrt{\underline{\rho}}\underline{u}}$ *as* $x\to+\infty$, *and similar conclusion as in (1) also holds.*

Proof. (1) If $1-\underline{\rho}\underline{u}^2=0$, then (4.22) is linear and we get

$$s(x)=\frac{\underline{\rho}\underline{v}^2(x-x_*)^2+2\left[\underline{\rho}\underline{u}\ \underline{v}b(x_*)+\dfrac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}\right](x-x_*)+\dfrac{2b(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}}{2\underline{\rho}\underline{u}\ \underline{v}(x-x_*)+\dfrac{2H(x_*)}{\sqrt{1+b'(x_*)^2}}}. \tag{4.23}$$

Since $\underline{v}/\underline{u}\geq b'(x_*)\geq 0$ and $H(x_*)>0$, the denominator is positive and $s(x)$ is defined for all $x\geq x_*$. By straightforward computations, the denominator

$$d(x)\triangleq \underline{\rho}\underline{v}(x-x_*)-\underline{\rho}\underline{u}(s(x)-b(x_*))+s(x) \tag{4.24}$$

in (4.19) is also positive for all $x\geq x_*$.[3] So is the denominator of the last term in (4.21). Thus the measure solution (4.19)-(4.21) and (4.23) is well-defined.

Considering entropy condition (4.5), since $v_0=0$, $u_0=1$, we need to check that $s'(x)\geq 0$ for all $x\geq x_*$. This is equivalent to the left-hand side of (4.22) being nonnegative on $[x_*,+\infty)$. The verification is also straightforward by substituting (4.23) into it. To show $s'(x)\leq \underline{v}/\underline{u}$, we are led to prove

$$s(x)-b(x_*)\geq \underbrace{\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}}\frac{\underline{u}}{\underline{v}}}_{>0}\underbrace{(b'(x_*)-\frac{\underline{v}}{\underline{u}})}_{\leq 0},\quad \forall x\geq x_*.$$

This is obvious since it holds at $x=x_*$, and recall we have proved that $s'(x)\geq 0$ for $x\geq x_*$.

(2) If $1-\underline{\rho}\underline{u}^2\neq 0$, from (4.22) we have

$$s(x)=\frac{(1-\underline{\rho}\underline{u}^2)b(x_*)-\underline{\rho}\underline{u}\,\underline{v}(x-x_*)-\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}}}{1-\underline{\rho}\underline{u}^2}+\frac{\sqrt{\spadesuit}}{1-\underline{\rho}\underline{u}^2}. \tag{4.25}$$

Since $\underline{v}/\underline{u}\geq b'(x_*)\geq 0$, we get that $\underline{\rho}\underline{u}\,\underline{v}+(1-\underline{\rho}\underline{u}^2)b'(x_*)\geq 0$. This means that for any $x\geq x_*$, we have

$$\begin{aligned}\spadesuit\triangleq\ &\underline{\rho}\underline{v}^2(x-x_*)^2+2[\underline{\rho}\underline{u}\,\underline{v}+(1-\underline{\rho}\underline{u}^2)b'(x_*)]\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}}(x-x_*)\\&+(\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}})^2>0.\end{aligned}$$

[3] For $\underline{\rho}\underline{u}\leq 1$ or equivalently $\underline{u}\geq 1$, this is obvious; for $\underline{u}\in(0,1)$, we are led to show the quadratic function

$$\underline{\rho}\underline{v}^2(1+\underline{u})(x-x_*)^2+\left[2\underline{v}b(x_*)+\frac{2H(x_*)}{\sqrt{1+b'(x_*)^2}}(\frac{\underline{v}}{\underline{u}}-b'(x_*)+\underline{u}b'(x_*))\right](x-x_*)+\frac{2(1+\underline{u})b(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}$$

is positive for all $x\geq x_*$, which is simple.

Therefore (4.25) is defined on $[x_*,\infty)$ and is of the order $\dfrac{\sqrt{\underline{\rho}}\underline{v}x}{1+\sqrt{\underline{\rho}}\underline{u}}$ as $x\to+\infty$.

Next it is easy to see that for all $x\geq x_*$,

$$\underline{\rho}\underline{u}\underline{v}(x-x_*)+(1-\underline{\rho}\underline{u}^2)(s(x)-b(x_*))+\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}}=\sqrt{\spadesuit}>0. \tag{4.26}$$

Supposing that $b'(x_*)>0$, to show

$$\underline{\rho}\underline{v}^2(x-x_*)-\underline{\rho}\underline{u}\underline{v}(s(x)-b(x_*))+\frac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}>0, \tag{4.27}$$

straightforward computation requires

$$(1-\underline{\rho}\underline{u}^2)[\sqrt{\spadesuit}-\alpha(x-x_*)-\beta]<0,\quad \alpha=\frac{\underline{v}}{\underline{u}},\ \ \beta=\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}}\left(1+b'(x_*)\frac{1-\underline{\rho}\underline{u}^2}{\underline{\rho}\underline{u}\underline{v}}\right). \tag{4.28}$$

Since this holds at $x=x_*$, we just need to prove that there is no root greater than x_* to the equation

$$\sqrt{\spadesuit}=\alpha(x-x_*)+\beta.$$

In fact, squaring the equation and after some manipulation, we have

$$\left(\frac{\underline{v}}{\underline{u}}\right)^2(x-x_*)^2+2\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}}\left(\frac{\underline{v}}{\underline{u}}-b'(x_*)+\frac{1}{\underline{\rho}\underline{u}^2}b'(x_*)\right)(x-x_*)$$
$$+\frac{H(x_*)^2}{1+b'(x_*)^2}\frac{b'(x_*)}{\underline{\rho}\underline{u}\underline{v}}\left(\frac{b'(x_*)}{\underline{\rho}\underline{u}\underline{v}}+2-\frac{b'(x_*)\underline{u}}{\underline{v}}\right)=0,$$

and the claim follows by recalling that $\frac{\underline{v}}{\underline{u}}-b'(x_*)\geq 0$. If $b'(x_*)=0$ and $\underline{v}>0$, we easily check that

$$\underline{\rho}\underline{v}^2(x-x_*)-\underline{\rho}\underline{u}\underline{v}(s(x)-b(x_*))+\frac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}\geq 0,\quad \forall x\geq x_*, \tag{4.29}$$

and the equality holds only at $x=x_*$. The equality holds for all $x\geq x_*$ in (4.29) if $\underline{v}=0$ and $b'(x_*)=0$.

From (4.22), (4.26) and (4.27), (4.29), we have shown

$$s'(x)\geq 0 \qquad \forall x\geq x_*. \tag{4.30}$$

Next, to prove

$$s'(x)\leq\frac{\underline{v}}{\underline{u}}, \qquad \forall x\geq x_*, \tag{4.31}$$

by (4.22), we shall verify

$$s(x)-b(x_*)+\frac{H(x_*)}{\sqrt{1+b'(x_*)}}(1-b'(x_*)\frac{\underline{u}}{\underline{v}})\geq 0.$$

It holds at $x=x_*$ and for $x>x_*$ it follows from (4.30). Thus the entropy condition is satisfied by the free layer.

Finally we need to check that $d(x)$ defined in (4.24) is positive on $[x_*,\infty)$. Note that $d(x_*)=b(x_*)>0$, and $d'(x)=\underline{\rho}\underline{v}+(1-\underline{\rho}\underline{u})s'(x)$, which is nonnegative, since if $\underline{\rho}\underline{u}\leq 1$, we use (4.30), and otherwise, by (4.31), $d'(x)\geq\underline{\rho}\underline{v}+(1-\underline{\rho}\underline{u})\frac{\underline{v}}{\underline{u}}=\frac{\underline{v}}{\underline{u}}\geq b'(x_*)\geq 0$. We therefore have a well-defined measure solution of Problem 3 given by (4.19)-(4.21) and (4.25), which also satisfies the entropy condition. ∎

This finishes the proof of (1) in Theorem 1.3.

4.2. Construction of measure solutions containing vacuum. In the previous subsection we considered the case that the jet impinges on the free layer. It may happen that there is vacuum between the free layer and the pressureless jet. In this section we consider this case. It is known that the only classical discontinuity connecting vacuum and pressureless jet is a contact discontinuity [10].

Suppose the free layer is

$$S_u\triangleq\{(x,y)\in\mathbb{R}^2:\ x\geq x_*,\ y=h(x)\},$$

and the contact discontinuity is

$$S_d\triangleq\{(x,y)\in\mathbb{R}^2:\ x\geq x_*,\ y=c(x)\},$$

with $y=h(x)$, $y=c(x)$ being two functions to be determined (see Figure 4.1). The domain we consider is then

$$\widetilde{\Omega}\triangleq\Omega_1\cup\Omega_2\cup\Omega_3,$$

where

$$\begin{aligned}
\Omega_1&\triangleq\{(x,y)\in\mathbb{R}^2:\ 0<x\leq x_*,\ y>b(x)\text{ and }x>x_*,\ y>h(x)\},\\
\Omega_2&\triangleq\{(x,y)\in\mathbb{R}^2:\ x>x_*,\ y<c(x)\},\\
\Omega_3&\triangleq\{(x,y)\in\mathbb{R}^2:\ x>x_*,\ c(x)<y<h(x)\}
\end{aligned}$$

represent the region above S_u, below S_d, and between S_u and S_d, respectively. Above S_u, there is uniform limiting hypersonic flow given by (1.6); the uniform jet below S_d is given by (1.10), with pressure $\underline{p}=0$. The region Ω_3 lying between S_u and S_d is vacuum.

In the following we construct a measure solution to Problem 3 with the above structure. Denote $\widetilde{W}=W_f\cup S_u$. Let

$$\begin{aligned}
m^0&=\mathsf{I}_{\Omega_1}\mathcal{L}^2+\underline{\rho}\underline{u}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_m^0(x)\delta_{\widetilde{W}}=\mathsf{I}_{\Omega_1}\mathcal{L}^2+\underline{\rho}\underline{u}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_m^0(x)\delta_{W_f}+\widetilde{w_m^0}(x)\delta_{S_u},\\
n^0&=\underline{\rho}\underline{v}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_n^0(x)\delta_{\widetilde{W}}=\underline{\rho}\underline{v}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_n^0(x)\delta_{W_f}+\widetilde{w_n^0}(x)\delta_{S_u},
\end{aligned}\tag{4.32}$$

$$\begin{aligned}
m^1&=\mathsf{I}_{\Omega_1}\mathcal{L}^2+\underline{\rho}\underline{u}^2\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_m^1(x)\delta_{\widetilde{W}}=\mathsf{I}_{\Omega_1}\mathcal{L}^2+\underline{\rho}\underline{u}^2\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_m^1(x)\delta_{W_f}+\widetilde{w_m^1}(x)\delta_{S_u},\\
n^1&=\underline{\rho}\underline{u}\ \underline{v}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_n^1(x)\delta_{\widetilde{W}}=\underline{\rho}\underline{u}\ \underline{v}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_n^1(x)\delta_{W_f}+\widetilde{w_n^1}(x)\delta_{S_u},\quad\wp=0,
\end{aligned}\tag{4.33}$$

$$\begin{aligned}
m^2&=\underline{\rho}\underline{u}\underline{v}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_m^2(x)\delta_{\widetilde{W}}=\underline{\rho}\underline{u}\underline{v}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_m^2(x)\delta_{W_f}+\widetilde{w_m^2}(x)\delta_{S_u},\\
n^2&=\underline{\rho}\underline{v}^2\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_n^2(x)\delta_{\widetilde{W}}=\underline{\rho}\underline{v}^2\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_n^2(x)\delta_{W_f}+\widetilde{w_n^2}(x)\delta_{S_u},
\end{aligned}\tag{4.34}$$

$$\begin{aligned}
m^3&=E_0\mathsf{I}_{\Omega_1}\mathcal{L}^2+\underline{\rho}\underline{u}\underline{E}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_m^3(x)\delta_{W_f}+\widetilde{w_m^3}(x)\delta_{S_u},\\
n^3&=\underline{\rho}\underline{v}\ \underline{E}\mathsf{I}_{\Omega_2}\mathcal{L}^2+w_n^3(x)\delta_{W_f}+\widetilde{w_n^3}(x)\delta_{S_u}.
\end{aligned}\tag{4.35}$$

Here $\widetilde{w_m^i}(x)$, $\widetilde{w_n^i}(x)$ $(i=0,1,2,3)$ are unknown functions.

Like before, substituting (4.32) into (4.1), some direct calculation yields

$$\begin{aligned}&\widetilde{w_m^0}(x_*)\sqrt{1+h'(x_*)^2}=w_m^0(x_*)\sqrt{1+b'(x_*)^2}=b(x_*),\\&\frac{\mathrm{d}(\widetilde{w_m^0}(x)\sqrt{1+h'(x)^2})}{\mathrm{d}x}=h'(x),\quad \widetilde{w_n^0}(x)=h'(x)\widetilde{w_m^0}(x),\quad \underline{v}=c'(x)\underline{u}.\end{aligned}\tag{4.36}$$

So we get

$$\widetilde{w_m^0}(x)=\frac{h(x)}{\sqrt{1+h'(x)^2}},\quad \widetilde{w_n^0}(x)=\frac{h'(x)h(x)}{\sqrt{1+h'(x)^2}},\quad \underline{v}=c'(x)\underline{u},\tag{4.37}$$

which imply particularly the slip condition on the free layer and contact discontinuity. Similarly we also have

$$\widetilde{w_m^3}(x)=\frac{E_0h(x)}{\sqrt{1+h'(x)^2}},\quad \widetilde{w_n^3}(x)=\frac{E_0h(x)h'(x)}{\sqrt{1+h'(x)^2}}.\tag{4.38}$$

By (4.33) and (4.2), we find

$$\begin{aligned}&\widetilde{w_m^1}(x_*)\sqrt{1+h'(x_*)^2}=w_m^1(x_*)\sqrt{1+b'(x_*)^2}=\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}},\\&\widetilde{w_n^1}(x)=h'(x)\widetilde{w_m^1}(x),\quad \frac{\mathrm{d}(\widetilde{w_m^1}(x)\sqrt{1+h'(x)^2})}{\mathrm{d}x}=h'(x),\quad \underline{v}=c'(x)\underline{u}.\end{aligned}\tag{4.39}$$

One then solves that

$$\widetilde{w_m^1}(x)=\frac{h(x)-b(x_*)+\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}}}{\sqrt{1+h'(x)^2}},\quad \underline{v}=c'(x)\underline{u}.\tag{4.40}$$

Substituting (4.34) into (4.3), we deduce that

$$\begin{aligned}&\widetilde{w_m^2}(x_*)\sqrt{1+h'(x_*)^2}=w_m^2(x_*)\sqrt{1+b'(x_*)^2}=\frac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}},\\&\widetilde{w_n^2}(x)=h'(x)\widetilde{w_m^2}(x),\quad \frac{\mathrm{d}(\widetilde{w_m^2}(x)\sqrt{1+h'(x)^2})}{\mathrm{d}x}=0,\quad \underline{v}=c'(x)\underline{u}.\end{aligned}\tag{4.41}$$

Therefore

$$\widetilde{w_m^2}(x)=\frac{\dfrac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}}{\sqrt{1+h'(x)^2}},\quad \underline{v}=c'(x)\underline{u}.\tag{4.42}$$

Thanks to (1.18), we see

$$u|_{S_u}=\frac{h(x)-b(x_*)+\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}}}{h(x)},\quad v|_{S_u}=\frac{\dfrac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}}{h(x)},\tag{4.43}$$

hence

$$
\begin{aligned}
u &= \mathsf{I}_{\Omega_1} + \underline{u}\mathsf{I}_{\Omega_2} + \frac{H(x)}{b(x)\sqrt{1+b'(x)^2}}\mathsf{I}_{W_f} + \frac{h(x)-b(x_*)+\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}}}{h(x)}\mathsf{I}_{S_u},\\
v &= \underline{v}\mathsf{I}_{\Omega_2} + \frac{b'(x)H(x)}{b(x)\sqrt{1+b'(x)^2}}\mathsf{I}_{W_f} + \frac{\dfrac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}}{h(x)}\mathsf{I}_{S_u},\\
E &= E_0\mathsf{I}_{\Omega_1} + \underline{E}\mathsf{I}_{\Omega_2} + E_0\mathsf{I}_{W_f} + E_0\mathsf{I}_{S_u},
\end{aligned}
\tag{4.44}
$$

and the measure of mass density is

$$
\varrho = \mathsf{I}_{\Omega_1}\mathcal{L}^2 + \underline{\rho}\mathsf{I}_{\Omega_2}\mathcal{L}^2 + \frac{(b(x))^2}{H(x)}\delta_{W_f} + \frac{(h(x))^2}{\sqrt{1+h'(x)^2}[h(x)-b(x_*)+\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}}]}\delta_{S_u}.
\tag{4.45}
$$

Next we solve $h(x)$. Using the slip condition implied by (4.37) on S_u, we have

$$
\begin{cases}
\dfrac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}} = h'(x)[h(x)-b(x_*)+\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}}],\\
h(x_*) = b(x_*).
\end{cases}
\tag{4.46}
$$

The solution is

$$
h(x) = \sqrt{\frac{2H(x_*)b'(x_*)}{\sqrt{1+b'(x_*)^2}}(x-x_*) + (\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}})^2} + b(x_*) - \frac{H(x_*)}{\sqrt{1+b'(x_*)^2}}.
\tag{4.47}
$$

It is the same as (3.24), while representing the situation that below the free layer is vacuum.

From (4.37) we also infer that $\underline{v} = c'(x)\underline{u}$. So S_d, the contact discontinuity, is a straight line

$$
c(x) = \frac{\underline{v}}{\underline{u}}(x-x_*) + b(x_*).
\tag{4.48}
$$

From (4.47) and (4.48), we know that if the pressureless jet satisfies the requirement that $\underline{v} \le 0$, then for all $x \ge x_*$, $h'(x) \ge 0$ and $c'(x) \le 0$, hence $h(x) \ge c(x)$, and the vacuum between the limiting hypersonic flow and the pressureless jet is unbounded, see Figure 4.1. What happens if $0 < \dfrac{\underline{v}}{\underline{u}} < b'(x_*)$? Simple computation shows that on

$$
x_* \le x \le x^{\Delta} \triangleq x_* + \frac{2(\underline{u}b'(x_*)-\underline{v})\underline{u}}{\underline{v}^2}\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}},
$$

it holds that $h(x) \ge c(x)$. The free layer meets the contact discontinuity at the point $(x^{\Delta}, h(x^{\Delta}))$, with $h(x^{\Delta}) = \dfrac{2(\underline{u}b'(x_*)-\underline{v})}{\underline{v}}\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}} + b(x_*)$. Then from (4.46) we have $h'(x^{\Delta}) = \dfrac{b'(x_*)\underline{v}}{2\underline{u}b'(x_*)-\underline{v}}$. It is easy to show that for $0 < \dfrac{\underline{v}}{\underline{u}} < b'(x_*)$, it holds that $h'(x^{\Delta}) \le \dfrac{\underline{v}}{\underline{u}}$. Therefore considering a problem like Problem 3 with $x = x_*$ replaced by $x = x^{\Delta}$, we have

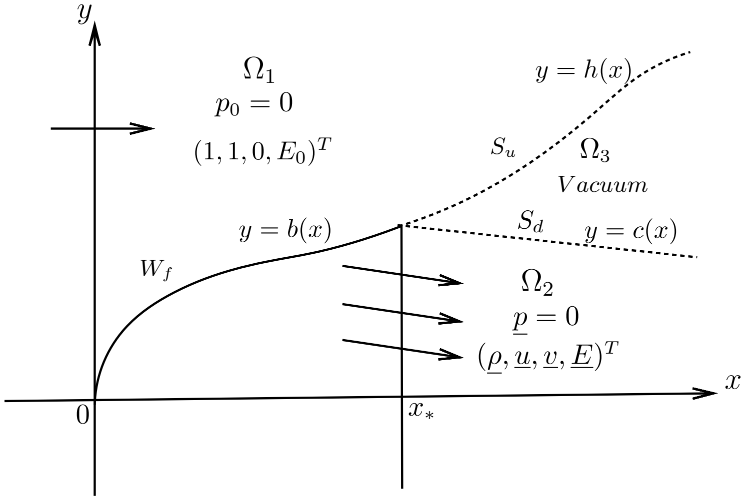


FIG. 4.1. *Unbounded vacuum between free layer and pressureless jet* ($\underline{v}\leq 0$).

a case studied in Section 4.1: *the free layer absorbs the contact discontinuity*. See Figure 4.2. Using the method in Section 4.1, we could determine the free layer starting from the colliding point as follows.

(1) If $\underline{\rho}\underline{u}^2\neq 1$, then for $x\geq x^\triangle$,

$$s(x)=\frac{(1-\underline{\rho}\underline{u}^2)h(x^\triangle)-\underline{\rho}\underline{u}\underline{v}(x-x^\triangle)-\dfrac{2\underline{u}b'(x_*)-\underline{v}}{\underline{v}}\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}}}{1-\underline{\rho}\underline{u}^2}+\frac{\sqrt{(*)}}{1-\underline{\rho}\underline{u}^2},\tag{4.49}$$

where

$$\begin{aligned}(*)=&\underline{\rho}\underline{v}^2(x-x^\triangle)^2+2[\underline{\rho}\underline{u}\underline{v}\cdot\frac{2\underline{u}b'(x_*)-\underline{v}}{\underline{v}}+(1-\underline{\rho}\underline{u}^2)b'(x_*)]\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}}(x-x^\triangle)\\&+(\frac{2\underline{u}b'(x_*)-\underline{v}}{\underline{v}}\cdot\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}})^2.\end{aligned}$$

(2) If $\underline{\rho}\underline{u}^2=1$, then for $x\geq x^\triangle$,

$$s(x)=\frac{\underline{\rho}\underline{v}^2(x-x^\triangle)^2+2[\dfrac{b'(x_*)H(x_*)}{\sqrt{1+b'(x_*)^2}}+\underline{\rho}\underline{u}\underline{v}h(x^\triangle)](x-x^\triangle)+C}{2\underline{\rho}\underline{u}\underline{v}(x-x^\triangle)+2\cdot\dfrac{2\underline{u}b'(x_*)-\underline{v}}{\underline{v}}\dfrac{H(x_*)}{\sqrt{1+b'(x_*)^2}}},\tag{4.50}$$

where

$$C=2h(x^\triangle)\cdot\frac{2\underline{u}b'(x_*)-\underline{v}}{\underline{v}}\frac{H(x_*)}{\sqrt{1+b'(x_*)^2}}.$$

In summary, we proved the following lemma.

LEMMA 4.2. (1) *If* $\underline{v} \le 0$*, then Problem 3 has a measure solution that consists of three piecewise constant states, with the one lying in the middle being an unbounded vacuum.* (2) *If* $0 < \dfrac{\underline{v}}{\underline{u}} < b'(x_*)$*, then Problem 3 has a measure solution that consists of three piecewise constant states, with the one in the middle being a finite vacuum bounded by the curves given by* (4.47)-(4.48)*, lying in* $\{x_* \le x \le x^{\triangle}\}$*. In particular, if* $x \ge x^{\triangle}$*, there is no vacuum and the solution consists of two piecewise constant states.*

This completes the proof of Theorem 1.3.

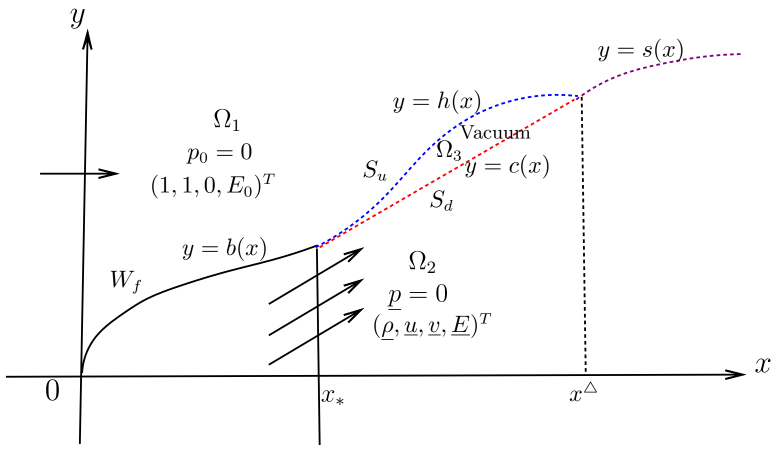


FIG. 4.2. *Bounded vacuum between the limiting hypersonic flow and pressureless jet when* $0 < \dfrac{\underline{v}}{\underline{u}} < b'(x_*)$.

5. Discussions

Considering hypersonic flows, Louie and Ockendon wrote in [21, p.121]: “Although inviscid models have limited practical value, it is important to understand them as well as possible if theoretical progress is to be made with more complicated models for real gases.” This paper maybe considered a part of progress to understand inviscid hypersonic flows. We studied three typical aerodynamical problems on limiting hypersonic flows by using a concept of Radon measure solutions of compressible Euler equations that we proposed. We proved succinctly the well-known Newton-Busemann pressure law that was firstly given by Busemann in 1930s, and constructed various flow fields with free layers, and discovered blow up of some measure solutions. These results demonstrate the power of our concept of measure solutions. However, the uniqueness and stability of the measure solutions we constructed remain interesting open problems. To solve them, some meticulous refinement of our definition might be necessary.

To solve Problem 3, noticing that $\widetilde{\Omega}_1 \cup \widetilde{\Omega}_2 \triangleq \{(x,y) \in \mathbb{R}^2 :\ 0 < x \le x_*,\ y > b(x)\} \cup \{(x,y) \in \mathbb{R}^2 :\ x > x_*,\ y \in \mathbb{R}\}$, using solutions of Problem 1 in $\widetilde{\Omega}_1$, we only need to solve a special boundary value problem of (1.2) in $\widetilde{\Omega}_2$, subjected to boundary conditions given on the line $\{x = x_*\}$:

$$\varrho(x_*, y) = \mathsf{I}_{\{y > b(x_*)\}} \mathcal{L}^1 + \frac{(b(x_*))^2}{H(x_*)} \delta_{\{y = b(x_*)\}} + \underline{\rho} \mathsf{I}_{\{y < b(x_*)\}} \mathcal{L}^1,$$

$$
\begin{aligned}
u(x_*,y) &= \mathsf{I}_{\{y>b(x_*)\}} + \frac{H(x_*)}{b(x_*)\sqrt{1+b'(x_*)^2}}\mathsf{I}_{\{y=b(x_*)\}} + \underline{u}\mathsf{I}_{\{y<b(x_*)\}},\\
v(x_*,y) &= \frac{b'(x_*)H(x_*)}{b(x_*)\sqrt{1+b'(x_*)^2}}\mathsf{I}_{\{y=b(x_*)\}} + \underline{v}\mathsf{I}_{\{y<b(x_*)\}},\\
E(x_*,y) &= E_0\mathsf{I}_{\{y>b(x_*)\}} + E_0\mathsf{I}_{\{y=b(x_*)\}} + \underline{E}\mathsf{I}_{\{y<b(x_*)\}},
\end{aligned}
\tag{5.1}
$$

where $\mathcal{L}^1$ is the Lebesgue measure on the real line $\mathbb{R}$. Unlike the classical Riemann problems, there is a Dirac measure supported on the discontinuity point $(x_*, b(x_*))$ for the density. We call such problems as *Singular Riemann Problems*.

We know that there have been many studies on singular Riemann problems from the mathematical point of view (see, for example, [28–30] and references therein). Our studies of limiting hypersonic flows show that such problems are also models of significant physical problems, hence they require a systematic investigation.

Comparing to classical Riemann problems, solutions to singular Riemann problems may no longer be self-similar and they may contain rather complex wave structures (such as bounded vacuum in Problem 3). If we consider the interactions of limiting hypersonic flows with supersonic polytropic gas jets, the situation is more complicated. We will report the results in another paper.

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