Commun. Math. Sci.
Vol. 20, No. 5, pp. 1437–1458

GLOBAL STRONG SOLUTION TO THE CAUCHY PROBLEM OF 2D DENSITY-DEPENDENT BOUSSINESQ EQUATIONS FOR MAGNETOHYDRODYNAMICS CONVECTION WITH THERMAL DIFFUSION*

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Abstract. In this paper, we study the Cauchy problem of density-dependent Boussinesq equations for magnetohydrodynamics convection on the whole 2D space. We first establish global and unique strong solution for the 2D Cauchy problem when the initial density includes vacuum state. Furthermore, we consider that the initial data can be arbitrarily large. We derive a consistent priori estimate by the energy method, and extend the local strong solutions to the global strong solutions. Finally, we obtain the large-time decay rates of the gradients of velocity, temperature field, magnetic field and pressure.

Keywords. MHD-Boussinesq equation; Global strong solution; Density-dependent; Large-time behavior, Vacuum.

AMS subject classifications. 35Q35; 76D03.

1. Introduction

For this paper, we consider the Cauchy problem of 2D nonhomogeneous incompressible Boussinesq equations for magnetohydrodynamics convection which read as follows:

$$\begin{cases} \rho_t+\mathrm{div}(\rho u)=0, \\ (\rho u)_t+\mathrm{div}(\rho u\otimes u)+\nabla P-\mu\triangle u=\rho\theta e_2+b\cdot\triangledown b-\dfrac{1}{2}\triangledown|b|^2, \\ b_t-\nu\triangle b+u\cdot\triangledown b-b\cdot\triangledown u=0, \\ \theta_t+u\cdot\triangledown\theta-\kappa\triangle\theta=0, \\ \mathrm{div}u=0,\quad \mathrm{div}b=0. \end{cases} \tag{1.1}$$

where $x=(x_1,x_2)\in\mathbb{R}^2$ is the spatial coordinate, $t\geq 0$ is time, $\rho=\rho(x,t)$, $u=(u^1,u^2)(x,t)$, $b=(b^1,b^2)(x,t)$ are the density, velocity and the magnetic field, respectively; $\theta=\theta(x,t)$ stands for the temperature of the fluid, and $P=P(x,t)$ denotes the pressure of the fluid; the constant $\mu>0$ is the viscosity coefficient; the constant $\nu>0$ denotes the electrical resistivity, and the constant $\kappa>0$ represent the capillary coefficient. $e^2=(0,1)^T$, where T is the transpose. The initial data and far field conditions are given by

$$\begin{cases} \rho(x,0)=\rho_0(x),\rho u(x,0)=\rho_0 u_0(x),b(x,0)=b_0(x),\theta(x,0)=\theta_0(x),\quad x\in\mathbb{R}^2, \\ (\rho,u,b,\theta)(x,t)\to 0,\quad as\quad |x|\to\infty. \end{cases} \tag{1.2}$$

The Equations (1.1) are a combination of the incompressible Boussinesq equations in fluid dynamics and Maxwell's equations in electromagnetism, where the displacement current is neglected [13, 24]. Specifically, they closely relate to a natural type of the Rayleigh-Bénard convection, which occurs in a horizontal layer of conductive fluid

*Received: November 25, 2020; Accepted (in revised form): December 24, 2021. Communicated by Yaguang Wang.

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heated from below, with the presence of a magnetic field (see [25]). When the fluid is not affected by the temperature, that is $\theta = 0$, the system (1.1) becomes the standard MHD system. Many researchers have studied the MHD equation in the early years. For details, please refer to [8, 10, 21, 23, 27]. Recently, Fan and Li [9] obtained the global existence of strong solutions to the incompressible magnetohydrodynamic equations in a bounded domain by combining the regularity criterion and an abstract bootstrap argument. Under some smallness assumption, Bie-Wang-Yao [4] proved the global well-posedness of solution to the 3D incompressible magnetohydrodynamic equations with variable density in the multiplier space of Besov space. Very recently, Zhong [32] showed the Cauchy problem of the non-barotropic non-resistive magnetohydrodynamic equations with zero heat conduction on the whole 2D space with vacuum as far field density. However, when the initial conditions include vacuum, there are few relevant works. For Equation (1.1) we propose the relationship between magnetic field, fluid temperature and pressure so as to solve the difficulties caused by vacuum.

If the fluid is not affected by the Lorentz force, that is $b = 0$, the system (1.1) becomes the nonhomogeneous Boussinesq system. Boussinesq equation is a simple model widely used in the modeling of atmospheric and oceanic motions, and it plays an important role in the atmospheric sciences (see [24]). Many related research results of the Boussinesq system have emerged. Lorca and Boldrini [19, 20] gained the existence of global weak solutions for Boussinesq equations with small initial values and they also studied the existence of local strong solutions under general initial conditions. Qiu and Yao [26] obtained the local existence and uniqueness of strong solutions of multi-dimensional incompressible density-dependent Boussinesq equations in Besov spaces. For the initial density allowing vacuum states, Liu [17] proved global existence and large-time asymptotic behavior of strong solution to the Cauchy problem of 2D density-dependent Boussinesq equations. Zhong [31] recently showed local existence of strong solutions of the Cauchy problem in $\mathbb{R}^2$ by making use of weighted energy estimate techniques. However, there are still very few people studying the Boussineq equation with a magnetic field in the system (1.1). The temperature field and dissipation term may bring some positive effects to the magnetic fluid. Meanwhile, the strong coupling of velocity field and magnetic field will bring new difficulties.

Recently, much attention has been attracted by the density-dependent viscous MHD-Boussinesq equations. Bian-Gui [1] rigorously justified the stability in a fully nonlinear, dynamical setting from a mathematical point of view in an unbounded domain. Bian-Liu [2] proved the global existence of weak solutions with H^1 initial data. They also obtained a unique global strong solution by imposing a higher regularity assumption on the initial data. Larios and Pei [14] established the local well-posedness of solutions to the fully dissipative 3D Boussinesq-MHD system, and also the fully inviscid, irresistive, non-diffusive Boussinesq-MHD system. Later, Zhao [30] investigated the well-posedness of the Cauchy problem to the Boussinesq-MHD system with partial viscosity and zero magnetic diffusion. Very recently, Liu-Biao [18] studied the global existence and uniqueness of strong and smooth large solutions to the 3D Boussinesq-MHD system with a damping term. In addition, Bian and Pu [3] proved the global axisymmetric smooth solutions for the 3D Boussinesq-MHD equations without magnetic diffusion and heat convection. Up to now, there are few results on MHD-Boussinesq equation when the initial datas can be arbitrarily large. Especially, we have to consider the situation where velocity, temperature and magnetic field have strong coupling and the strong nonlinearity of $b \cdot \nabla b$. More importantly, the particularity of vacuum state should also be considered and temperature also affects the change of density. Therefore, motivated by [23,33], we studied the global existence and large-time asymptotic behavior of strong solution to the Cauchy problem of nonhomogeneous incompressible Boussinesq equations for magnetohydrodynamics convection in $\mathbb{R}^2$.

Now, let's go back to (1.1). We note here the notations and conventions employed throughout the paper. For $R>0$, set

$$B_R \triangleq \{x\in\mathbb{R}^2 \,|\, |x|<R\}, \qquad \int f dx \triangleq \int_{\mathbb{R}^2} f dx.$$

Moreover, for $1\le r\le\infty$, $k\ge 1$, we denote the standard Lebesgue and Sobolev spaces as follows:

$$L^r = L^r(\mathbb{R}^2), \qquad W^{k,r} = W^{k,r}(\mathbb{R}^2), \qquad H^r = W^{k,2}.$$

Then, we define the strong solution to (1.1) as follows:

Definition 1.1. *If all derivatives involved in (1.1) for (ρ,u,P,b,θ) are regular distributions, and Equations (1.1) hold almost everywhere in $\mathbb{R}^2\times(0,T)$, then (ρ,u,P,b,θ) is called a strong solution to (1.1). Without loss of generality, we assume that the initial density ρ_0 satisfies*

$$\int_{\mathbb{R}^2} \rho_0 dx = 1. \tag{1.3}$$

(1.3) implies that there exists a positive constant N_0 such that

$$\int_{B_{N_0}} \rho_0 dx \ge \frac{1}{2}\int \rho_0 dx = \frac{1}{2}. \tag{1.4}$$

Our main result is stated as follows:

Theorem 1.1. *Suppose the initial data (ρ_0,u_0) satisfy that for any given numbers $a>1$ and $q>2$,*

$$\begin{cases} \rho_0\ge 0, \rho_0\bar{x}^a\in L^1\cap H^1\cap W^{1,q}, \nabla u_0\in L^2, \sqrt{\rho_0}u_0\in L^2,\\ b_0\bar{x}^{a/2}\in L^2, \nabla b_0\in L^2, \theta_0\bar{x}^{a/2}\in L^2, \nabla\theta_0\in L^2,\\ divu_0 = divb_0 = 0; \end{cases} \tag{1.5}$$

where

$$\bar{x}\triangleq (e+|x|^2)^{\frac{1}{2}} log^2(e+|x|^2). \tag{1.6}$$

Then the problem (1.1)−(1.2) has a unique global strong solution (ρ,u,P,b,θ) satisfying *that for any* $0<T<\infty$,

$$\begin{cases} 0\le\rho\in C([0,T];L^1\cap H^1\cap W^{1,q}),\\ \rho\bar{x}^a\in L^\infty(0,T;L^2\cap H^1\cap W^{1,q}),\\ \sqrt{\rho}u,\nabla u,\bar{x}^{-1}u,\sqrt{t}\sqrt{\rho}u_t,\sqrt{t}\nabla p,\sqrt{t}\nabla^2 u\in L^\infty(0,T;L^2),\\ b,b\bar{x}^{a/2},\nabla b,\sqrt{t}b_t,\sqrt{t}\nabla^2 b,\sqrt{t}\nabla b\bar{x}^{a/2}\in L^\infty(0,T;L^2),\\ \theta,\theta\bar{x}^{a/2},\nabla\theta,\sqrt{t}\theta_t,\sqrt{t}\nabla^2\theta,\sqrt{t}\nabla\theta\bar{x}^{a/2}\in L^\infty(0,T;L^2),\\ \nabla u\in L^2(0,T;H^1)\cap L^{(q+1)/q}(0,T;W^{1,q}),\\ \nabla P\in L^2(0,T;L^2)\cap L^{(q+1)/q}(0,T;L^q),\\ \nabla b\in L^2(0,T;H^1),b_t,\nabla b\bar{x}^{a/2}\in L^2(0,T;L^2),\\ \nabla\theta\in L^2(0,T;H^1),\theta_t,\nabla\theta\bar{x}^{a/2}\in L^2(0,T;L^2),\\ \sqrt{t}\nabla u\in L^2(0,T;W^{1,q}),\\ \sqrt{\rho}u_t,\sqrt{t}\nabla u_t,\sqrt{t}\nabla b_t,\sqrt{t}\nabla\theta_t,\sqrt{t}\bar{x}^{-1}u_t\in L^2(\mathbb{R}^2\times(0,T)); \end{cases} \tag{1.7}$$

and

$$\inf_{0\le t\le T}\int_{B_{N_1}}\rho(x,t)dx\ge\frac{1}{4}, \tag{1.8}$$

for some positive constant N_1 *depending only* $\|\rho_0\|_{L^1}$, $\|\sqrt{\rho_0}u_0\|_{L^2}$, N_0 *and* T. *Moreover,* (ρ,u,P,b,θ) *has the following decay rates, that is, for* $t\ge1$,

$$\begin{cases} \|\nabla u(\cdot,t)\|_{L^2}+\|\nabla b(\cdot,t)\|_{L^2}+\|\nabla\theta(\cdot,t)\|_{L^2}\le Ct^{-1/2},\\ \|\nabla^2 u(\cdot,t)\|_{L^2}+\|\nabla P(\cdot,t)\|_{L^2}+\||b||\nabla b|\|_{L^2}\le Ct^{-1}; \end{cases} \tag{1.9}$$

where C *depends only on* μ, κ, $\|\rho_0\|_{L^1\cap L^\infty}$, $\|\sqrt{\rho_0}u_0\|_{L^2}$, *and* $\|\nabla u_0\|_{L^2}$.

REMARK 1.1. If the capillarity coefficient is zero, i.e. $\kappa=0$, it should be noted here that, although the Equations (1.1) degenerate near vacuum, Theorem 1.1 is the same as that in [33]. We generalize the main result of [33], and extend the local strong solutions to the global strong solutions.

REMARK 1.2. Note that no compatibility condition (see [5,6]) on the initial data is required in Theorem 1.1 for the local existence and uniqueness of strong solutions.

2. Preliminaries

For the section, we will recall some known lemmas and inequalities. For given initial data, the following lemma assumes that there is a unique local strong solution.

LEMMA 2.1. *Assume that* $(\rho_0,u_0,b_0,\theta_0)$ *satisfies* (1.5). *Then there exists a small time* $T>0$ *and a unique strong solution* (ρ,u,P,b,θ) *to the problem* (1.1)–(1.2) *in* $\mathbb{R}^2\times(0,T)$ *satisfying* (1.7) *and* (1.8).

LEMMA 2.2 (see the Gagliardo-Nirenberg inequality in [12]). *For* $m\in[2,\infty)$, $q\in(1,\infty)$, *and* $r\in(2,\infty)$, *there exists some generic constant* $C>0$ *which may rely on* m, q, *and* r *such that for* $f\in H^1(\mathbb{R}^2)$ *and* $g\in L^q(\mathbb{R}^2)\cap D^{1,r}(\mathbb{R}^2)$, *we have*

$$\|f\|_{L^m(\mathbb{R}^2)}^m\le C\|f\|_{L^2(\mathbb{R}^2)}^2\|\nabla f\|_{L^2(\mathbb{R}^2)}^{m-2}, \tag{2.1}$$

$$\|g\|_{C(\overline{\mathbb{R}^2})} \le C\|g\|_{L^q(\mathbb{R}^2)}^{q(r-2)/(2r+q(r-2))}\|\nabla g\|_{L^r(\mathbb{R}^2)}^{2r/(2r+q(r-2))}. \tag{2.2}$$

The following weighted L^n bounds for elements in $\tilde{D}^{1,2}(\mathbb{R}^2)\triangleq\{v\in H^1_{loc}(\mathbb{R}^2)|\nabla v\in L^2(\mathbb{R}^2)\}$ can be found in [12, Theorem 1.1].

LEMMA 2.3 (see [11]). *For $h\in[2,\infty)$ and $\lambda\in(1+h/2,\infty)$, there exists a positive constant C such that for all $v\in\tilde{D}^{1,2}(\mathbb{R}^2)$,*

$$\Big(\int_{\mathbb{R}^2}\frac{|v|^h}{e+|x|^2}(\log(e+|x|^2))^{-\lambda}dx\Big)^{1/h} \le C\|v\|_{L^2(B_1)}+C\|\nabla v\|_{L^2(\mathbb{R}^2)}. \tag{2.3}$$

The Lemma 2.3 combined with the Poincaré inequality gets the following useful results on weighted bounds, we can also refer to ([11], Lemma 2.4).

LEMMA 2.4 (see [28]). *We can refer to $\bar{x}$ in (1.6), and assume that $\rho\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ is a non-negative function such that*

$$\|\rho\|_{L^1(B_{N_0})}\ge M_0,\quad \|\rho\|_{L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)}\le M_1, \tag{2.4}$$

for positive constants M_0, M_1, and $N_0\ge1$ with $B_{N_0}\subset\mathbb{R}^2$. Then for $\alpha>0$, $\beta>0$, there is a positive constant C depending only on α, β, M_0, M_1, and N_0 such that every $v\in\tilde{D}^{1,2}(\mathbb{R}^2)$ satisfies

$$\|v\bar{x}^{-\beta}\|_{L^{(2+\alpha)/\tilde{\beta}}(\mathbb{R}^2)} \le C\|\rho^{1/2}v\|_{L^2(\mathbb{R}^2)}+C\|\nabla v\|_{L^2(\mathbb{R}^2)}, \tag{2.5}$$

with $\tilde{\beta}=\min\{1,\beta\}$.

Finally, let BMO ($\mathbb{R}^2$) and $\mathcal{H}^1(\mathbb{R}^2)$ represent BMO and Hardy spaces [28, Chapter 4]. In the next section, some facts are more important to prove Lemma 3.2.

LEMMA 2.5 (see [7]).
(i) *There is a positive constant C such that*

$$\|G\cdot M\|_{\mathcal{H}^1(\mathbb{R}^2)}\le C\|G\|_{L^2(\mathbb{R}^2)}\|M\|_{L^2(\mathbb{R}^2)}, \tag{2.6}$$

for all $G\in L^2(\mathbb{R}^2)$ and $M\in L^2(\mathbb{R}^2)$ satisfying

$$divG=0,\quad \nabla^\perp\cdot M=0\quad in\quad D'(\mathbb{R}^2). \tag{2.7}$$

(ii) *There is a positive constant C such that*

$$\|f\|_{BMO(\mathbb{R}^2)}\le C\|\nabla f\|_{L^2(\mathbb{R}^2)}, \tag{2.8}$$

for all $f\in\tilde{D}^{1,2}(\mathbb{R}^2)$.

Proof.
(i) Please refer to [7, Theorem II.1] for a detailed proof.
(ii) It follows, together with the Poincaré inequality, that for any ball $B\subset(\mathbb{R}^2)$

$$\frac{1}{|B|}\int_B\Big|v(x)-\frac{1}{|B|}\int_B v(y)dy\Big|dx\le C\Big(\int_B|\nabla v|^2dx\Big)^{1/2}, \tag{2.9}$$

which directly gives (2.8). ∎

3. Convergence rate of the solution

3.1. Lower order estimates. As $div\, u=0$, we have an estimate on the $L^\infty(0,T;L^r)-norm$ of ρ, as follows:

LEMMA 3.1. *There exists a positive constant C depending only on $\|\rho_0\|_{L^1\cap L^\infty}$ such that*

$$\sup_{t\in[0,T]} \|\rho\|_{L^1\cap L^\infty}\le C. \tag{3.1}$$

Proof. We can see [16, Theorem 2.1]. Then, we will estimate the $L^\infty(0,T;L^2)$-norm of ∇b, $\nabla\theta$ and ∇u. □

LEMMA 3.2. *There exists a positive constant C depending only on μ, ν, κ, $\|\rho_0\|_{L^\infty}$, $\|\nabla u_0\|_{L^2}$, $\|\sqrt{\rho}u_0\|_{L^2}$, $\|b_0\|_{H^1}$ and $\|\theta_0\|_{H^1}$ such that*

$$\sup_{t\in[0,T]} (\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|b\|_{L^4}^4) \\ +\int_0^T (\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|\triangle\theta\|_{L^2}^2+\|\triangle b\|_{L^2}^2+\||\nabla b||b|\|_{L^2}^2)dt\le C. \tag{3.2}$$

Here $\dot{u}\triangleq\partial_t u+u\cdot\nabla u$. One has

$$\sup_{t\in[0,T]} t(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|b\|_{L^4}^4) \\ +\int_0^T t(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|\triangle\theta\|_{L^2}^2+\|\triangle b\|_{L^2}^2+\||\nabla b||b|\|_{L^2}^2)dt\le C. \tag{3.3}$$

Proof. Invoking standard energy estimates, multiplying $(1.1)_2$ by u and integrating the resulting equality over $\mathbb{R}^2$ lead to

$$\int \frac{d}{dt}(\rho u^2)dx+2\mu\int|\nabla u|^2dx \\ =2\int \rho\theta e_2\cdot u dx+2\int b\cdot\nabla b\cdot u dx-\int\nabla|b|^2\cdot u dx. \tag{3.4}$$

Multiplying $(1.1)_3$ by b and integrating the resulting equality over $\mathbb{R}^2$, we get

$$\int\frac{d}{dt}b^2dx+\nu\int|\nabla b|^2dx+\int u\cdot\nabla b\cdot b dx=0. \tag{3.5}$$

Combining (3.3) and (3.4), we obtain from integrating the resulting equality over [0,T] and the Cauchy-Schwarz inequality, that

$$\frac{d}{dt}(\|\sqrt{\rho}u\|_{L^2}^2+\|b\|_{L^2}^2)+2\mu\|\nabla u\|_{L^2}^2+\nu\|\nabla b\|_{L^2}^2=2\int\rho\theta e_2\cdot u dx \\ \le\|\rho\|_{L^\infty}^{1/2}\|\sqrt{\rho}u\|_{L^2}\|\theta\|_{L^2}. \tag{3.6}$$

Multiplying $(1.1)_4$ by θ and integrating over $\mathbb{R}^2$, then integrating the resulting equality over $[0,T]$, we arrive at

$$\sup_{t\in[0,T]}\|\theta\|_{L^2}^2+\kappa\int_0^T\|\nabla\theta\|_{L^2}^2dt\le\|\theta_0\|_{L^2}^2. \tag{3.7}$$

Integrating (3.6) with respect to t and together with (3.1) and (3.7), we have

$$\sup_{t\in[0,T]}(\|\sqrt{\rho}u\|_{L^2}^2+\|b\|_{L^2}^2)+\int_0^T(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)dt\le C. \tag{3.8}$$

Combining (3.7) and (3.8) yields

$$\sup_{t\in[0,T]}(\|\sqrt{\rho}u\|_{L^2}^2+\|b\|_{L^2}^2+\|\theta\|_{L^2}^2)+\int(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)dt\le C. \tag{3.9}$$

Next, multiplying $(1.1)_2$ by $\dot{u}$ and integrating over $\mathbb{R}^2$ give

$$\begin{aligned}\int\rho|\dot{u}|^2dx&=\mu\int\Delta u\cdot\dot{u}dx-\int\nabla P\cdot\dot{u}dx+\int\rho\theta e_2\cdot\dot{u}dx+\int b\cdot\nabla b\cdot\dot{u}dx-\frac{1}{2}\int\nabla|b|^2\cdot\dot{u}dx\\&\triangleq J_1+J_2+J_3+J_4+J_5.\end{aligned} \tag{3.10}$$

In view of the integration by parts and (2.2), we get

$$\begin{aligned}J_1&=\mu\int\triangle u\cdot(u_t+u\cdot\nabla u)dx\\&=-\frac{\mu}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2-\mu\int\partial_iu^j\partial_i(u^k\partial_ku^j)dx\\&\le-\frac{\mu}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^3}^3\\&\le-\frac{\mu}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^2}^2\|\nabla^2u\|_{L^2}.\end{aligned} \tag{3.11}$$

We estimate J_2 by Equation $(1.1)_5$, integration by parts and the duality of $\mathcal{H}^1$ space and BMO (see [7, Chapter IV]). Then, since $div(\partial_ju)=\partial_jdivu=0$, $\nabla^\perp\cdot(\nabla u^j)=0$, and along with (2.7) and (2.9) yields

$$\begin{aligned}J_2&=-\int\nabla P(u_t+u\cdot\nabla u)dx=\int P\partial_ju^i\partial_iu^jdx\\&\le C\|P\|_{BMO}\|\partial_ju^i\partial_iu^j\|_{\mathcal{H}^1}\le C\|\nabla p\|_{L^2}\|\nabla u\|_{L^2}^2.\end{aligned} \tag{3.12}$$

We can deduce the term J_3 by the Cauchy-Schwarz inequality, (3.1), and (3.9), to get

$$J_3=|\int\rho\theta e_2\cdot\dot{u}dx|\le C\|\rho\|_{L^\infty}^{1/2}\|\sqrt{\rho}\dot{u}\|_{L^2}\|\theta\|_{L^2}\le\frac{1}{2}\|\sqrt{\rho}\dot{u}\|_{L^2}^2+C. \tag{3.13}$$

It follows from integration by parts, $(1.1)_3$ and $(1.1)_5$ that

$$\begin{aligned}J_4&=\int b\cdot\nabla b\cdot u_tdx+\int b\cdot\nabla b\cdot(u\cdot\nabla u)dx\\&=-\frac{d}{dt}\int b\cdot\nabla u\cdot bdx+\int b_t\cdot\nabla u\cdot bdx+\int b\cdot\nabla u\cdot b_tdx\\&\quad-\int b^i\partial_iu^j\partial_ju^kb^kdx-\int b^iu^j\partial_i\partial_ju^kb^kdx\\&=-\frac{d}{dt}\int b\cdot\nabla u\cdot bdx+\int(\nu\triangle b-u\cdot\nabla b+b\cdot\nabla u)\cdot\nabla u\cdot bdx\\&\quad+\int b\cdot\nabla u\cdot(\nu\triangle b-u\cdot\nabla b+b\cdot\nabla u)dx-\int b^i\partial_iu^j\partial_ju^kb^kdx\end{aligned}$$

$$
\begin{aligned}
&+\int u^i\partial_j b^i\partial_i u^k b^k dx+\int b^i\partial_i u^k u^j\partial_j b^k dx\\
\le&-\frac{d}{dt}\int b\cdot\nabla u\cdot b dx+\frac{\nu}{2}\|\triangle b\|_{L^2}^2+C\|b\|_{L^2}^2\|\nabla b\|_{L^2}^4+C\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}. \qquad (3.14)
\end{aligned}
$$

The integration by parts together with $(1.1)_5$ and (2.2) leads to

$$
\begin{aligned}
J_5=&\frac{1}{2}\int |b|^2\partial_i u^j\partial_j u^i dx\le C\|b\|_{L^6}^6+C\|\nabla u\|_{L^3}^3\\
\le& C\|b\|_{L^2}^2\|\nabla b\|_{L^2}^4+C\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}. \qquad (3.15)
\end{aligned}
$$

Next, substituting (3.11)-(3.15) into (3.10) gives

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\mu}{2}\|\nabla u\|_{L^2}^2+\int b\cdot\nabla u\cdot b dx\right)+\frac{1}{2}\|\sqrt{\rho}\dot u\|_{L^2}^2\\
\le&\frac{\nu}{2}\|\triangle b\|_{L^2}^2+C\|b\|_{L^2}^2\|\nabla b\|_{L^2}^4+C(\|\nabla^2 u\|_{L^2}+\|\nabla P\|_{L^2})\|\nabla u\|_{L^2}^2. \qquad (3.16)
\end{aligned}
$$

Then, adding $(1.1)_3\times\triangle b$ to $(1.1)_4\times\triangle\theta$ and integrating the resulting equality over $\mathbb{R}^2$, it follows form Hölder's inequality and (2.2) that

$$
\begin{aligned}
&\frac{d}{dt}\int(|\nabla b|^2+|\nabla\theta|^2)dx+2\nu\int|\triangle b|^2dx+2\kappa\int|\triangle\theta|^2dx\\
\le& C\int|\nabla u||\nabla b|^2dx+C\int|\nabla u||b||\triangle b|dx+C\int|\nabla u||\nabla\theta|^2dx\\
\le& C\|\nabla u\|_{L^3}(\|\nabla b\|_{L^2}^{4/3}\|\nabla^2 b\|_{L^2}^{2/3}+\|b\|_{L^6}\|\triangle b\|_{L^2}+\|\nabla\theta\|_{L^2}^{4/3}\|\triangle\theta\|_{L^2}^{2/3})\\
\le& C\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}+C(1+\|b\|_{L^2}^2)\|\nabla b\|_{L^2}^4+\frac{\nu}{2}\|\triangle b\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^2+\frac{\kappa}{2}\|\triangle\theta\|_{L^2}^2, \qquad (3.17)
\end{aligned}
$$

which together with (3.16) and (3.9) estimate

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\mu}{2}\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\int b\cdot\nabla u\cdot b dx\right)+\|\sqrt{\rho}\dot u\|_{L^2}^2+2\nu\|\triangle b\|_{L^2}^2+2\kappa\|\triangle\theta\|_{L^2}^2\\
\le& C(\|\nabla\theta\|_{L^2}^4+\|\nabla b\|_{L^2}^4)+C(\|\nabla^2 u\|_{L^2}+\|\nabla P\|_{L^2})\|\nabla u\|_{L^2}^2. \qquad (3.18)
\end{aligned}
$$

On the other hand, as (ρ,u,P,b,θ) satisfies the following Stokes system

$$
\begin{cases}
-\mu\triangle u+\nabla p=-\rho\dot u+\rho\theta e_2+b\cdot\nabla b-\frac{1}{2}\nabla|b|^2, & x\in\mathbb{R}^2,\\
divu=0, & x\in\mathbb{R}^2,\\
u(x)\to 0, & |x|\to\infty,
\end{cases} \qquad (3.19)
$$

applying the standard L^r-estimates (3.1) to (3.19)(see [29]) that hold for any $s>1$,

$$
\begin{aligned}
\|\nabla^2 u\|_{L^s}+\|\nabla p\|_{L^s}\le& C\|\rho\dot u\|_{L^s}+C\|\rho\theta\|_{L^s}+C\||b||\nabla b|\|_{L^s}\\
\le& C\|\sqrt{\rho}\dot u\|_{L^s}+C\|\rho\theta\|_{L^s}+C\||b||\nabla b|\|_{L^s}. \qquad (3.20)
\end{aligned}
$$

It deduces from (3.18), (3.20), (3.1) and (3.9) that

$$
\begin{aligned}
&\frac{d}{dt}A(t)+\frac{1}{2}\|\sqrt{\rho}\dot u\|_{L^2}^2+2\nu\|\triangle b\|_{L^2}^2+2\kappa\|\triangle\theta\|_{L^2}^2\\
\le& C\|\nabla b\|_{L^2}^4+C\|\nabla\theta\|_{L^2}^4+C\|\nabla u\|_{L^2}^4+\frac{1}{2}\||b||\nabla b|\|_{L^2}^2, \qquad (3.21)
\end{aligned}
$$

where

$$A(t)\triangleq\frac{\mu}{2}\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\int b\cdot\nabla u\cdot b dx, \tag{3.22}$$

and owing to (2.1) and (3.9), one has

$$\frac{\mu}{4}\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2-C_1\|b\|_{L^4}^4\le A(t)\le\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2. \tag{3.23}$$

Then, multiplying $(1.1)_3$ by $b|b|^2$ and integrating the resulting equality by parts over $\mathbb{R}^2$ lead to

$$\begin{aligned}\frac{1}{4}(\||b|^2\|_{L^2}^2)_t+\nu\||b||\nabla b|\|_{L^2}^2+\frac{\nu}{2}\|\nabla|b|^2\|_{L^2}^2\le& C\|\nabla u\|_{L^2}\||b|^2\|_{L^4}^2\\ \le& C\|\nabla u\|_{L^2}\||b|^2\|_{L^2}\|\nabla|b|^2\|_{L^2}\\ \le& C\|\nabla u\|_{L^2}^4+\frac{\nu}{4}\|\nabla|b|^2\|_{L^2}^2+\|\nabla b\|_{L^2}^4.\end{aligned} \tag{3.24}$$

Finally, combining (3.24), (3.21) and (3.23) yields

$$\begin{aligned}&\frac{d}{dt}(A(t)+\|b\|_{L^4}^4)+\frac{1}{2}\|\sqrt{\rho}\dot u\|_{L^2}^2+\nu\|\triangle b\|_{L^2}^2+\kappa\|\triangle\theta\|_{L^2}^2+2\nu\||b||\nabla b|\|_{L^2}^2\\ \le& C(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)(A(t)+\|b\|_{L^4}^4),\end{aligned} \tag{3.25}$$

which together with (3.9), (3.23) and Gronwall's inequality gives (3.2). Then, (3.25) multiplied by t, (3.9) combined with (3.23), (3.2) and Gronwall's inequality gives (3.3). Finally, it finishes the proof of Lemma 3.2. □

LEMMA 3.3. *There is some positive constant C depending only on μ, ν, κ, $\|\rho_0\|_{L^1\cap L^\infty}$, $\|\nabla u_0\|_{L^2}$, $\|b_0\|_{H^1}$ and $\|\rho_0^{1/2}u_0\|_{L^2}$ such that for $i=1,2$,*

$$\sup_{t\in[0,T]} t^i(\|\sqrt{\rho}\dot u\|_{L^2}^2+\||b||\nabla b|\|_{L^2}^2)+\int_0^T t^i(\|\nabla\dot u\|_{L^2}^3+\||b||\triangle b|\|_{L^2}^2)dt\le C, \tag{3.26}$$

and

$$\sup_{t\in[0,T]} t^i(\|\nabla^2 u\|_{L^2}^2+\|\nabla P\|_{L^2}^2)\le C. \tag{3.27}$$

Proof. First, operating $\partial_t+u\cdot\nabla$ to $(1.1)_2^j$, it follows from a few simple calculations that

$$\begin{aligned}\partial_t(\rho\dot u^j)+\mathrm{div}(\rho u\dot u^j)-\mu\triangle\dot u^j=&-\mu\partial_i(\partial_i u\cdot\nabla u^j)-\mu\mathrm{div}(\partial_i u\partial_i u^j)-\partial_j\partial_t P-(u\cdot\nabla)\partial_j P\\ &+\partial_t(\rho\theta e_2^j)+u\cdot\nabla(\rho\theta e_2^j)+\partial_t(b\cdot\nabla b^j)\\ &+u\cdot\nabla(b\cdot\nabla b^j)-\frac{1}{2}\partial_t(\partial_j|b|^2)-\frac{1}{2}u\cdot\nabla(\partial_j|b|^2).\end{aligned} \tag{3.28}$$

Next, (3.28) multiplied by $\dot u^j$, and together with integration by parts and $(1.1)_5$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\dot u\|_{L^2}^2+\mu\|\nabla\dot u\|_{L^2}^2=&-\mu\int\partial_i(\partial_i u\cdot\nabla u^j)\dot u^j dx-\mu\int\mathrm{div}(\partial_i u\partial_i u^j)\dot u^j dx\\ &-\int(\dot u^j\partial_t\partial_j P+\dot u^j(u\cdot\nabla)\partial_j P)dx+\int(\partial_t(\rho\theta e_2^j)+u\cdot\nabla(\rho\theta e_2^j))\cdot\dot u^j dx\end{aligned}$$

$$
\begin{aligned}
&+\int(\partial_t(b\cdot\nabla b^j)+u\cdot\nabla(b\cdot\nabla b^j))\cdot\dot{u}^j\,dx-\frac{1}{2}\int(\partial_t(\partial_j|b|^2)+u\cdot\nabla(\partial_j|b|^2))\cdot\dot{u}^j\,dx\\
&\triangleq\sum_{i=1}^{6}K_i. \qquad (3.29)
\end{aligned}
$$

Following the same argument as [22, Lemma 3.3] we have the estimates of $(K_i(i=1,2,3))$ as

$$
\sum_{i=1}^{3}K_i\le\frac{d}{dt}\int P\partial_j u^i\partial_i u^j\,dx+C(\|p\|_{L^4}^4+\|\nabla u\|_{L^4}^4)+\frac{\mu}{8}\|\nabla\dot{u}\|_{L^2}^2. \qquad (3.30)
$$

Estimating the term K_4 by $(1.1)_1$, $(1.1)_4$, $(1.1)_5$, (3.1), (3.2) and (3.9), one has

$$
\begin{aligned}
K_4&=-\int div(\rho u)\theta e_2^j\cdot\dot{u}^j\,dx+\int\rho e_2^j(\kappa\nabla\theta-u\cdot\nabla\theta)\cdot\dot{u}^j\,dx-\int u^i\rho\theta e_2^j\partial_i\dot{u}^j\,dx\\
&=\int\rho u\theta e_2^j\cdot\nabla\dot{u}^j\,dx+\kappa\int\rho e_2^j\triangle\theta\cdot\dot{u}^j\,dx-\int u^i\rho\theta e_2^j\partial_i\dot{u}^j\,dx\\
&\le C\int\rho|u||\theta||\nabla\dot{u}|dx+C\int\rho|\dot{u}||\triangle\theta|dx\\
&\le C\|\rho\|_{L^\infty}^{1/2}\|\sqrt{\rho}u\|_{L^2}\|\theta\|_{L^\infty}\|\nabla\dot{u}\|_{L^2}+C\|\rho\|_{L^\infty}^{1/2}\|\sqrt{\rho}\dot{u}\|_{L^2}\|\triangle\theta\|_{L^2}\\
&\le\frac{\mu}{8}\|\nabla\dot{u}\|_{L^2}^2+C\|\triangle\theta\|_{L^2}^2+C\|\sqrt{\rho}\dot{u}\|_{L^2}^2+C. \qquad (3.31)
\end{aligned}
$$

We can deduce from integration by parts, $(1.1)_3$, $(1.1)_5$, and (2.2) that

$$
\begin{aligned}
K_5&=-\int b_t^i\partial_i\dot{u}^jb^j\,dx-\int b^i\partial_i\dot{u}^jb_t^j\,dx-\int\partial_iu^ju^ib^k\partial_kb^j\,dx\\
&=\int(-\nu\triangle b^i+u\cdot\nabla b^i-b^i\cdot\nabla u)\partial_i\dot{u}^jb^j\,dx+\int b^i\partial_i\dot{u}^j(-\nu\triangle b^i+u\cdot\nabla b^i-b^i\cdot\nabla u)dx\\
&\quad+\int u^i\partial_i\partial_k\dot{u}^jb^kb^j\,dx+\int\partial_i\dot{u}^j\partial_ku^ib^kb^j\,dx\\
&=-\nu\int\triangle b\cdot\nabla\dot{u}\cdot b dx-\nu\int b\cdot\nabla\dot{u}\cdot\triangle b dx-\int b\cdot\nabla\dot{u}\cdot(b\cdot\nabla u)dx\\
&\le C\||b|^2\|_{L^4}^4+C\|\nabla u\|_{L^4}^4+C\||b||\triangle b|\|_{L^2}^2+\frac{\mu}{8}\|\nabla\dot{u}\|_{L^2}^2. \qquad (3.32)
\end{aligned}
$$

Same as K_5, we also get

$$
\begin{aligned}
K_6&=\int div\dot{u}b\cdot(b\cdot\nabla u+\nu\triangle b)dx-\frac{1}{2}\int\partial_j\dot{u}^ju^i\partial_i|b|^2dx+\frac{1}{2}\int u^i\partial_i\dot{u}^j\partial_j|b|^2dx\\
&\le C\int|\nabla u|^3|b|^2dx+C\int|\nabla u|^2|\triangle b||b|dx-\frac{1}{2}\int\partial_ju^i\partial_i\dot{u}^j|b|^2dx\\
&\le C\||b|^2\|_{L^4}^4+C\|\nabla u\|_{L^4}^4+C\||b||\triangle b|\|_{L^2}^2+\frac{\mu}{8}\|\nabla\dot{u}\|_{L^2}^2. \qquad (3.33)
\end{aligned}
$$

Substituting (3.30)-(3.33) into (3.29) gives

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\frac{\mu}{2}\|\nabla\dot{u}\|_{L^2}^2\le&\frac{d}{dt}\int P\partial_ju^i\partial_iu^j\,dx+C(\|p\|_{L^4}^4+\|\nabla u\|_{L^4}^4)+C\|\sqrt{\rho}\dot{u}\|_{L^2}^2\\
&+C\|\triangle\theta\|_{L^2}^2+C\||b|^2\|_{L^4}^4+C\||b||\triangle b|\|_{L^2}^2+C. \qquad (3.34)
\end{aligned}
$$

Multiplying $(1.1)_3$ by $4\nu^{-1}b\triangle|b|^2$ and integrating the resulting equality over $\mathbb{R}^2$ leads to

$$\begin{aligned}&\nu^{-1}(\|\nabla|b|^2\|_{L^2}^2)_t+2\|\triangle|b|^2\|_{L^2}^2\\&=4\int|\nabla b|^2\triangle|b|^2dx-4\nu^{-1}\int b\cdot\nabla u\cdot b\triangle|b|^2dx+2\nu^{-1}\int u\cdot\nabla|b|^2\triangle|b|^2dx\\&\leq C\|\nabla u\|_{L^4}^4+C\|\nabla b\|_{L^4}^4+C\||b|^2\|_{L^4}^4+\|\triangle|b|^2\|_{L^2}^2.\end{aligned}\tag{3.35}$$

Next, we will adapt some key ideas used in [21] to estimate the term $\||b||\triangle b|\|_{L^2}^2$. Indeed, for $a_1,a_2\in\{-1,0,1\}$, denote

$$\tilde{b}(a_1,a_2)=a_1b^1+a_2b^2,\quad \tilde{u}(a_1,a_2)=a_1u^1+a_2u^2.\tag{3.36}$$

Notice that

$$\begin{aligned}\||b||\triangle b|\|_{L^2}^2\leq&C\|\nabla b\|_{L^4}^4+\|\triangle|\tilde{b}(1,0)|^2\|_{L^2}^2+\|\triangle|\tilde{b}(0,1)|^2\|_{L^2}^2\\&+\|\triangle|\tilde{b}(1,1)|^2\|_{L^2}^2+\|\triangle|\tilde{b}(1,-1)|^2\|_{L^2}^2\end{aligned}\tag{3.37}$$

and

$$\||b||\nabla b|\|_{L^2}^2\leq E(t)\leq C\||b||\nabla b|\|_{L^2}^2\tag{3.38}$$

with

$$E(t)\triangleq\|\nabla|\tilde{b}(1,0)|^2\|_{L^2}^2+\|\nabla|\tilde{b}(0,1)|^2\|_{L^2}^2+\|\nabla|\tilde{b}(1,1)|^2\|_{L^2}^2+\|\nabla|\tilde{b}(1,-1)|^2\|_{L^2}^2,\tag{3.39}$$

which combined with (3.35) and (3.36) implies

$$\begin{aligned}&\frac{d}{dt}(\nu^{-1}C_1E(t))+C_1\||b||\triangle b|\|_{L^2}^2\\&\leq C\|\nabla u\|_{L^4}^4+C\|\nabla b\|_{L^4}^4+C\||b|^2\|_{L^4}^4+C\|\triangle|b|^2\|_{L^2}^2.\end{aligned}\tag{3.40}$$

Putting (3.40) into (3.34), we have

$$\begin{aligned}&\frac{d}{dt}F(t)+\frac{\mu}{2}\|\nabla\dot{u}\|_{L^2}^2+\||b||\triangle b|\|_{L^2}^2\\&\leq C(\|P\|_{L^4}^4+\|\nabla u\|_{L^4}^4)+C\|\sqrt{\rho}\dot{u}\|_{L^2}^2+C\|\triangle\theta\|_{L^2}^2+C\|\nabla b\|_{L^4}^4+C\||b|^2\|_{L^4}^4+C,\end{aligned}\tag{3.41}$$

where

$$F(t)\triangleq\frac{1}{2}\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\nu^{-1}C_1E(t)-\int P\partial_ju^i\partial_iu^jdx\tag{3.42}$$

satisfies

$$\frac{1}{4}\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\frac{\nu^{-1}}{2}C_1E(t)-C\|\nabla u\|_{L^2}^4\leq F(t)\leq C\|\sqrt{\rho}\dot{u}\|_{L^2}^2+CE(t)+C\|\nabla u\|_{L^2}^4.\tag{3.43}$$

We can get the following estimates from (3.12), (3.20), (3.38) and Young's inequality

$$\begin{aligned}|\int P\partial_ju^i\partial_iu^jdx|\leq&C(\|\sqrt{\rho}\dot{u}\|_{L^2}+\|\rho\theta\|_{L^2}+\||b||\nabla b|\|_{L^2}^2)\|\nabla u\|_{L^2}^2\\&\leq\frac{1}{2}\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\frac{\nu^{-1}}{2}C_1E(t)+C\|\nabla u\|_{L^2}^4.\end{aligned}\tag{3.44}$$

Thus, combining with (3.20), (2.1), (3.9) and Sobolev's inequality, we will estimate the term on the right-hand of (3.15) that

$$\begin{aligned}\|P\|_{L^4}^4+\|\nabla u\|_{L^4}^4 &\le C(\|\nabla P\|_{L^{\frac43}}^4+\|\nabla^2 u\|_{L^{\frac43}}^4)\\ &\le C(\|\rho\dot u\|_{L^{\frac43}}^4+\|\rho\theta\|_{L^{\frac43}}^4+\||b||\nabla b|\|_{L^{\frac43}}^4)\\ &\le C(\|\rho\|_{L^2}^2\|\sqrt{\rho}\dot u\|_{L^2}^4+\|\rho\|_{L^4}^4\|\theta\|_{L^2}^4+\|b\|_{L^2}^4\|\nabla b\|_{L^4}^4)\\ &\le C\|\sqrt{\rho}\dot u\|_{L^2}^4+C\|\nabla b\|_{L^2}^2\|\nabla^2 b\|_{L^2}^2+C. \end{aligned}\tag{3.45}$$

We can deduce from (2.1) and (3.9) that

$$\|\nabla b\|_{L^4}^4+\||b|^2\|_{L^4}^4\le C\|\nabla b\|_{L^2}^2\|\nabla^2 b\|_{L^2}^2+C\|\nabla b\|_{L^2}^2\||b||\nabla b|\|_{L^2}^2. \tag{3.46}$$

Then, substituting (3.45) and (3.46) into (3.41), and together with (3.38), (3.43), (3.2) and (3.3), we can obtain that for $i=1,2$,

$$\begin{aligned}&\frac{d}{dt}F(t)+\frac{\mu}{2}\|\nabla\dot u\|_{L^2}^2+\||b||\triangle b|\|_{L^2}^2\\ \le& C(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)(F(t)+\|\nabla u\|_{L^2}^4+1)+C\|\triangle\theta\|_{L^2}^2+C\|\nabla b\|_{L^2}^2\|\nabla^2 b\|_{L^2}^2\\ \le& C(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)F(t)+C\|\triangle\theta\|_{L^2}^2+Ct^{1-i}(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\nabla^2 b\|_{L^2}^2+\|\nabla u\|_{L^2}^4).\end{aligned}\tag{3.47}$$

Next, applying Gronwall's inequality to (3.47), multiplying by $t^i (i=1,2)$, it follows from (3.38), (3.43), (3.2), (3.3) and (3.9) that

$$\begin{aligned}&\sup_{t\in[0,T]}(t^iF(t))+\int_0^T t^i(\|\nabla\dot u\|_{L^2}^2+\||b||\triangle b|\|_{L^2}^2)dt\\ \le&\int_0^T t^{1-i}F(t)dt+C\int_0^T t\|\triangle\theta\|_{L^2}^2dt\\ &+C\int_0^T t(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\nabla^2 b\|_{L^2}^2+\|\nabla u\|_{L^2}^4)dt\\ \le& C(T).\end{aligned}\tag{3.48}$$

Finally, as $i=1,2$, from (3.38), (3.43) and (3.3) we get (3.26). (3.27) is a direct consequence of (3.26) and (3.20). We finish the proof of Lemma 3.3. □

3.2. Lower order estimates. We are concerned with the estimates on the higher-order derivatives of the strong solution (ρ,u,p,b,θ) as follows:

LEMMA 3.4. *For a positive constant C, such that*

$$\sup_{t\in[0,T]}\|\rho\bar x^a\|_{L^1}\le C(T). \tag{3.49}$$

Proof. For $M>1$, let $\varphi_M\in C_0^\infty(B_M)$ satisfy

$$0\le\varphi_M\le1,\qquad \varphi_M(x)=\begin{cases}1, & |x|\le M/2,\\ 0, & |x|\ge M,\end{cases}\qquad |\nabla\varphi_M|\le CM^{-1}. \tag{3.50}$$

We can deduce with $(1.1)_1$ that

$$\frac{d}{dt}\int\rho\varphi_M dx=\int\rho u\cdot\nabla\varphi_M dx\ge -CM^{-1}(\int\rho dx)^{1/2}(\int\rho|u|^2dx)^{1/2}\ge-\tilde CM^{-1}, \tag{3.51}$$

in the last inequality of (3.51), we have applied (3.2) and (3.9). Integrating (3.51) and letting $M = N_1 \triangleq 2N_0 + 4\tilde{C}T$, we obtain after using (1.4) that

$$\begin{aligned}\inf_{t\in[0,T]}\int_{B_{N_1}}\rho dx &\ge \inf_{t\in[0,T]}\int \rho\varphi_{N_1}dx\\ &\ge \int \rho_0\varphi_{N_1}dx - \tilde{C}N_1^{-1}T\\ &\ge \int_{B_{N_0}}\rho_0 dx - \frac{\tilde{C}T}{2N_0+4\tilde{C}T}\\ &\ge \frac{1}{4}.\end{aligned} \tag{3.52}$$

(3.52) along with (3.1), (3.9) and (2.5) for any $\eta\in(0,1]$ and any $s>2$, result in

$$\|u\bar{x}^{-\eta}\|_{L^{s/\eta}} \le C(\|\rho^{1/2}u\|_{L^2} + \|\nabla u\|_{L^2}) \le C. \tag{3.53}$$

Multiplying $(1.1)_1$ by $\bar{x}^a$ and integrating the resulting equality by parts over $\mathbb{R}^2$ we find that

$$\begin{aligned}\frac{d}{dt}\int \rho\bar{x}^a dx &\le C\int \rho|u|\bar{x}^{a-1}\log^2(e+|x|^2)dx\\ &\le C\|\rho\bar{x}^{a-1+\frac{8}{8+a}}\|_{L^{\frac{8+a}{7+a}}}\|u\bar{x}^{-\frac{1}{8+a}}\|_{L^{8+a}}\\ &\le C\int \rho\bar{x}^a dx + C.\end{aligned} \tag{3.54}$$

Using Gronwall's inequality on (3.54) gives (3.49) and it proves Lemma 3.4. ◻

LEMMA 3.5. *There is a positive constant C depending on T such that*

$$\begin{aligned}&\sup_{t\in[0,T]}\|\rho\|_{H^1\cap W^{1,q}} + \int_0^T (\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}} + t\|\nabla^2 u\|_{L^2\cap L^q}^2)dt\\ &\quad + \int_0^T (\|\nabla P\|_{L^2}^2 + \|\nabla P\|_{L^q}^{\frac{q+1}{q}} + t\|\nabla P\|_{L^2\cap L^q}^2)dt \le C(T).\end{aligned} \tag{3.55}$$

Proof. We can follow from $(1.1)_1$ that $\nabla\rho$ holds for any $r\ge 2$,

$$\frac{d}{dt}\|\nabla\rho\|_{L^r} \le C(r)\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^r}. \tag{3.56}$$

Next, employing Lemma 2.2, (3.2) and (3.20), we have for $q>2$,

$$\|\nabla u\|_{L^\infty} \le C\|\nabla u\|_{L^2}^{\frac{q-2}{2(q-1)}}\|\nabla^2 u\|_{L^q}^{\frac{q}{2(q-1)}} \le C\|\rho\dot{u}\|_{L^q}^{\frac{q}{2(q-1)}}. \tag{3.57}$$

It follows from (3.52), (3.1), (2.2) and (3.49) that for any $s>2$,

$$\begin{aligned}\|\rho v\|_{L^s} &\le C\|\rho\bar{x}^{\frac{3a}{4s}}\|_{L^{\frac{4s}{3}}}\|v\bar{x}^{-\frac{3a}{4s}}\|_{L^{4s}}\\ &\le C\|\rho\|_{L^\infty}^{\frac{(4s-3)}{4s}}\|\rho\bar{x}^a\|_{L^1}^{\frac{3}{4s}}(\|\rho^{1/2}v\|_{L^2} + \|\nabla v\|_{L^2})\\ &\le C(\|\rho^{1/2}v\|_{L^2} + \|\nabla v\|_{L^2}),\end{aligned} \tag{3.58}$$

(3.58) combined with the Gagliardo-Nirenberg inequality shows that

$$\begin{aligned}\|\rho\dot{u}\|_{L^q} &\le C\|\rho\dot{u}\|_{L^2}^{\frac{2(q-1)}{q^2-2}}\|\rho\dot{u}\|_{L^{q^2}}^{\frac{q(q-2)}{q^2-2}}\\ &\le C\|\rho\dot{u}\|_{L^2}^{\frac{2(q-1)}{q^2-2}}(\|\sqrt{\rho}\dot{u}\|_{L^2}+\|\nabla\dot{u}\|_{L^2})^{\frac{q(q-2)}{q^2-2}}\\ &\le C(\|\sqrt{\rho}\dot{u}\|_{L^2}+\|\sqrt{\rho}\dot{u}\|_{L^2}^{\frac{2(q-1)}{q^2-2}}\|\nabla\dot{u}\|_{L^2}^{\frac{q(q-2)}{q^2-2}}),\end{aligned} \tag{3.59}$$

which is deformed and when calculated appropriately, leads to

$$\begin{aligned}\int_0^T\|\rho\dot{u}\|_{L^q}^{\frac{q+1}{q}}dt &\le C\int_0^T\|\sqrt{\rho}\dot{u}\|_{L^2}^{\frac{q+1}{q}}dt\\ &\quad+\sup_{t\in[0,T]}(t\|\sqrt{\rho}\dot{u}\|_{L^2}^2)^{\frac{q^2-1}{2q(q^2-2)}}\int_0^T t^{-\frac{q^3+q^2-2q-2}{2q(q^2-2)}}(t\|\nabla\dot{u}\|_{L^2}^2)^{\frac{q(q-2)(q+1)}{2q(q^2-2)}}dt\\ &\le C\int_0^T\|\sqrt{\rho}\dot{u}\|_{L^2}^2dt+C\int_0^T t^{-\frac{q^3+q^2-2q-2}{q^3+q^2-2q}}dt+C\int_0^T t\|\nabla\dot{u}\|_{L^2}^2dt\\ &\le C,\end{aligned} \tag{3.60}$$

$$\int_0^T t\|\rho\dot{u}\|_{L^q}^2dt\le C(\int_0^T t\|\nabla\dot{u}\|_{L^2}^2dt+1)\le C. \tag{3.61}$$

Then, (3.60) and (3.57) imply

$$\int_0^T\|\nabla u\|_{L^\infty}dt\le C(t). \tag{3.62}$$

Next, using Gronwall's inequality on (3.56) shows

$$\sup_{t\in[0,T]}\|\nabla\rho\|_{L^2\cap L^q}\le C. \tag{3.63}$$

Then, letting $s=2$ in (3.20) and integrating the resulting equality over $[0,T]$, we obtain after using (3.1), (3.2) and (3.3) that

$$\int_0^T\|\nabla^2u\|_{L^2}^2dt+\int_0^T\|\nabla P\|_{L^2}^2dt\le C. \tag{3.64}$$

Similarly, setting $r=q$ in (3.20) and integrating the resulting equality over $[0,T]$, we deduce using (3.60), (3.1), (3.2) and (3.3) that

$$\int_0^T\|\nabla^2u\|_{L^q}^{\frac{q+1}{q}}dt+\int_0^T\|\nabla P\|_{L^q}^{\frac{q+1}{q}}dt\le C. \tag{3.65}$$

Multiplying (3.20) by t and integrating the resulting equality over $[0,T]$, we can obtain after using (3.60), (3.1), (3.2) and (3.3) that

$$\int_0^T t\|\nabla^2u\|_{L^2\cap L^q}^2dt+\int_0^T t\|\nabla P\|_{L^2\cap L^q}^2dt\le C. \tag{3.66}$$

Moreover, we can get from (3.64), (3.65) and (3.66) that

$$\int_0^T(\|\nabla^2u\|_{L^2}^2+\|\nabla^2u\|_{L^q}^{\frac{q+1}{q}}+t\|\nabla^2u\|_{L^2\cap L^q}^2)dt$$

$$+\int_0^T (\|\nabla P\|_{L^2}^2+\|\nabla P\|_{L^q}^{\frac{q+1}{q}}+t\|\nabla P\|_{L^2\cap L^q}^2)dt\le C, \tag{3.67}$$

which combined with (3.1) and (3.63) gives (3.55). Lemma 3.5 is proved. ∎

LEMMA 3.6 (see [21]). *There exists a positive constant C depending on T such that for $q>2$,*

$$\sup_{t\in[0,T]}\|\rho\bar{x}^a\|_{L^1\cap H^1\cap W^{1,q}}\le C(T). \tag{3.68}$$

LEMMA 3.7. *There exists a positive constant C such that*

$$\sup_{t\in[0,T]}(\|b\bar{x}^{\frac{a}{2}}\|_{L^2}^2+\|\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2)+\int_0^T(\|\nabla b\bar{x}^{\frac{a}{2}}\|_{L^2}^2+\|\nabla\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2)dt\le C(T), \tag{3.69}$$

and

$$\sup_{t\in[0,T]}t(\|\nabla b\bar{x}^{\frac{a}{2}}\|_{L^2}^2+\|\nabla\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2)+\int_0^T t(\|\triangle b\bar{x}^{\frac{a}{2}}\|_{L^2}^2+\|\triangle\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2)dt\le C(T). \tag{3.70}$$

Proof. First, adding $(1.1)_3\times b\bar{x}^a$ to $(1.1)_4\times\theta\bar{x}^a$ and integrating the resulting equality over $\mathbb{R}^2$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|b\bar{x}^{a/2}\|_{L^2}^2+\|\theta\bar{x}^{a/2}\|_{L^2}^2)+\nu\|\nabla b\bar{x}^{a/2}\|_{L^2}^2+\kappa\|\nabla\theta\bar{x}^{a/2}\|_{L^2}^2\\ \le&C\int(|b|^2+|\theta|^2)\triangle\bar{x}^a dx+\int b\cdot\nabla u\cdot b\bar{x}^a dx+\frac{1}{2}\int(|b|^2+|\theta|^2)u\cdot\nabla\bar{x}^a dx\\ \triangleq&L_1+L_2+L_3,\end{aligned} \tag{3.71}$$

where

$$\begin{aligned}L_1\le&C\int(|b|^2+|\theta|^2)\bar{x}^a\bar{x}^{-2}\log^4(e+|x|^2)dx\\ \le&C\|b\bar{x}^{a/2}\|_{L^2}^2+C\|\theta\bar{x}^{a/2}\|_{L^2}^2,\end{aligned} \tag{3.72}$$

$$\begin{aligned}L_2\le&C\|\nabla u\|_{L^2}\|b\bar{x}^{a/2}\|_{L^4}^2\le C\|b\bar{x}^{a/2}\|_{L^2}(\|\nabla b\bar{x}^{a/2}\|_{L^2}+\|b\nabla\bar{x}^{a/2}\|_{L^2})\\ \le&C\|b\bar{x}^{a/2}\|_{L^2}^2+\frac{\nu}{4}\|\nabla b\bar{x}^{a/2}\|_{L^2}^2,\end{aligned} \tag{3.73}$$

$$\begin{aligned}L_3=&\frac{1}{2}\int(|b|^2+|\theta|^2)u\cdot\nabla\bar{x}^a dx\\ \le&C(\|b\bar{x}^{a/2}\|_{L^4}\|b\bar{x}^{a/2}\|_{L^2}+\|\theta\bar{x}^{a/2}\|_{L^4}\|\theta\bar{x}^{a/2}\|_{L^2})\|u\bar{x}^{-3/4}\|_{L^4}\\ \le&C\|b\bar{x}^{a/2}\|_{L^2}^2+\frac{\nu}{4}\|\nabla b\bar{x}^{a/2}\|_{L^2}^2+\|\theta\bar{x}^{a/2}\|_{L^2}^2+\frac{\kappa}{2}\|\nabla\theta\bar{x}^{a/2}\|_{L^2}^2.\end{aligned} \tag{3.74}$$

Substituting (3.72)-(3.74) into (3.71), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|b\bar{x}^{a/2}\|_{L^2}^2+\|\theta\bar{x}^{a/2}\|_{L^2}^2)+\frac{\nu}{2}\|\nabla b\bar{x}^{a/2}\|_{L^2}^2+\frac{\kappa}{2}\|\nabla\theta\bar{x}^{a/2}\|_{L^2}^2\\ \le&C\|b\bar{x}^{a/2}\|_{L^2}^2+C\|\theta\bar{x}^{a/2}\|_{L^2}^2.\end{aligned} \tag{3.75}$$

Using Gronwall's inequality on (3.75), we obtain (3.69).

Next, multiplying (1.1)$_3$ by $\triangle b\bar{x}^a$ and integrating by parts over $\mathbb{R}^2$, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla b\bar{x}^{a/2}\|_{L^2}^2+\nu\|\triangle b\bar{x}^{a/2}\|_{L^2}^2\\ \le& C\int|\nabla u||\nabla b|^2\bar{x}^a dx+C\int|u||\nabla b|^2|\nabla\bar{x}^a|dx+C\int|\nabla b||\triangle b||\nabla\bar{x}^a|dx\\ &+C\int|b||\nabla u||\triangle b|\bar{x}^a dx+C\int|\nabla b||b|\nabla u||\nabla\bar{x}^a|dx\\ \triangleq&\sum_{i=1}^5\bar{L}_i,\end{aligned}\tag{3.76}$$

we can get the following estimations by the Gagliardo-Nirenberg inequality, (3.2), (3.69) and (3.53)

$$\bar{L}_1\le C\|\nabla u\|_{L^\infty}\|\nabla b\bar{x}^{a/2}\|_{L^2}^2\le C(1+\|\nabla^2u\|_{L^q}^{q+1/q})\|\nabla b\bar{x}^{a/2}\|_{L^2}^2,\tag{3.77}$$

$$\begin{aligned}\bar{L}_2&\le C\||\nabla b|^{2-2/3a}\bar{x}^{a-1/3}\|_{L^{6a/6a-2}}\|u\bar{x}^{-1/3}\|_{L^{6a}}\||\nabla b|^{2/3a}\|_{L^{6a}}\\&\le C\|\nabla b\bar{x}^{a/2}\|_{L^2}^{6a-2/3a}\|\nabla b\|_{L^4}^{2/3a}\le C\|\nabla b\bar{x}^{a/2}\|_{L^2}^2+C\|\nabla b\|_{L^4}^2\\&\le C\|\nabla b\bar{x}^{a/2}\|_{L^2}^2+C\|\triangle b\bar{x}^{a/2}\|_{L^2}^2,\end{aligned}\tag{3.78}$$

$$\begin{aligned}\bar{L}_3+\bar{L}_4&\le\frac{\nu}{4}\|\triangle b\bar{x}^{a/2}\|_{L^2}^2+C\|\nabla b\bar{x}^{a/2}\|_{L^2}^2+C\|b\bar{x}^{a/2}\|_{L^4}^4+C\|\nabla u\|_{L^4}^4\\&\le\frac{\nu}{4}\|\triangle b\bar{x}^{a/2}\|_{L^2}^2+C\|\nabla b\bar{x}^{a/2}\|_{L^2}^2+C\|\nabla^2u\|_{L^2}^2+C,\end{aligned}\tag{3.79}$$

$$\begin{aligned}\bar{L}_5&\le C\|b\bar{x}^{a/2}\|_{L^4}^4+C\|\nabla u\|_{L^4}^4+C\|\nabla b\bar{x}^{a/2}\|_{L^2}^2\\&\le C\|b\bar{x}^{a/2}\|_{L^2}^2(\|\nabla b\bar{x}^{a/2}\|_{L^2}^2+\|b\bar{x}^{a/2}\|_{L^2}^2)+C\|\nabla u\|_{L^4}^4+C\|\nabla b\bar{x}^{a/2}\|_{L^2}^2\\&\le C+C\|\nabla^2u\|_{L^2}^2+C\|\nabla b\bar{x}^{a/2}\|_{L^2}^2.\end{aligned}\tag{3.80}$$

Inserting (3.77)-(3.80) into (3.76) implies that

$$\begin{aligned}&\frac{d}{dt}\|\nabla b\bar{x}^{a/2}\|_{L^2}^2+\nu\|\triangle b\bar{x}^{a/2}\|_{L^2}^2\\ \le& C(1+\|\nabla^2u\|_{L^q}^{q+1/q})\|\nabla b\bar{x}^{a/2}\|_{L^2}^2+C\|\nabla^2u\|_{L^2}^2+C.\end{aligned}\tag{3.81}$$

Multiplying (1.1)$_3$ by $\triangle\theta\bar{x}^a$ and integrating by parts over $\mathbb{R}^2$, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2+\kappa\|\triangle\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2\\ \le& C\int|\nabla u||\nabla\theta|^2\bar{x}^a dx+C\int|u||\nabla\theta|^2|\nabla\bar{x}^a|dx+C\int|\nabla\theta||\triangle\theta||\nabla\bar{x}^a|dx\triangleq\sum_{i=1}^3\tilde{L}_i,\end{aligned}\tag{3.82}$$

where

$$\begin{aligned}\tilde{L}_1&\le C\|\nabla u\|_{L^\infty}\|\nabla\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2\le C\|\nabla u\|_{L^2}^{\frac{q-2}{2(q-1)}}\|\nabla^2u\|_{L^q}^{\frac{q-2}{2(q-1)}}\|\nabla\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2\\&\le C(1+\|\nabla^2u\|_{L^q}^{\frac{q+1}{q}})\|\nabla\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2,\end{aligned}\tag{3.83}$$

$$\begin{aligned}\tilde{L}_2&\le C\||\nabla\theta|^{2-\frac{2}{3a}}\bar{x}^{a-\frac{1}{3}}\|_{L^{\frac{6a}{6a-2}}}\|u\bar{x}^{-\frac{1}{3}}\|_{L^{6a}}\||\nabla\theta|^{\frac{2}{3a}}\|_{L^{6a}}\\&\le C\|\nabla\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^{\frac{6a-2}{3a}}\|\nabla\theta\|_{L^4}^{\frac{2}{3a}}\le C\|\nabla\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2+C\|\nabla\theta\|_{L^4}^2\end{aligned}$$

$$\leq C\|\nabla\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2+\frac{\kappa}{4}\|\triangle\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2, \tag{3.84}$$

$$\tilde{L}_3\leq\frac{\kappa}{4}\|\triangle\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2+C\|\nabla\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2. \tag{3.85}$$

Submitting $\tilde{L}_1$, $\tilde{L}_2$, $\tilde{L}_3$ into (3.82), one has

$$\frac{d}{dt}\|\nabla\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2+\kappa\|\triangle\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2\leq C(1+\|\nabla^2u\|_{L^q}^{\frac{q+1}{q}})\|\nabla\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2. \tag{3.86}$$

Finally, combining (3.81) and (3.86), we get

$$\begin{aligned}&\frac{d}{dt}(\|\nabla b\bar{x}^{a/2}\|_{L^2}^2+\|\nabla\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2)+\nu\|\triangle b\bar{x}^{a/2}\|_{L^2}^2+\kappa\|\triangle\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2\\&\leq C(1+\|\nabla^2u\|_{L^q}^{q+1/q})\|\nabla b\bar{x}^{a/2}\|_{L^2}^2+C\|\nabla^2u\|_{L^2}^2+C(1+\|\nabla^2u\|_{L^q}^{\frac{q+1}{q}})\|\nabla\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2+C.\end{aligned} \tag{3.87}$$

We multiply (3.87) by t, and together with (3.69) and (3.55) , then employing Gronwall's inequlity, one obtains (3.70). This completes the proof of Lemma 3.7. ∎

Lemma 3.8. *There exists a positive constant C such that*

$$\begin{aligned}&\sup_{t\in[0,T]}t(\|\sqrt{\rho}u_t\|_{L^2}^2+\|b_t\|_{L^2}^2+\|\theta_t\|_{L^2}^2+\|\nabla^2b\|_{L^2}^2+\|\nabla^2\theta\|_{L^2}^2)\\&\qquad+\int_0^T t(\|\nabla u_t\|_{L^2}^2+\|\nabla b_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2)dt\leq C(T).\end{aligned} \tag{3.88}$$

Proof. For any $\eta\in(0,1]$ and any $s>2$, we deduce from (3.53), (3.58), (3.9) and (3.2) that

$$\|\rho^\eta u\|_{L^{s/\eta}}+\|u\bar{x}^{-\eta}\|_{L^{s/\eta}}\leq C. \tag{3.89}$$

Next, we will prove that

$$\begin{aligned}&\sup_{t\in[0,T]}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\\&\quad+\int_0^T(\|\sqrt{\rho}u_t\|_{L^2}^2+\|b_t\|_{L^2}^2+\|\theta_t\|_{L^2}^2+\|\triangle b\|_{L^2}^2+\|\triangle\theta\|_{L^2}^2)dt\leq C.\end{aligned} \tag{3.90}$$

With (3.2) at hand, we need to only show

$$\int_0^T(\|\sqrt{\rho}u_t\|_{L^2}^2+\|b_t\|_{L^2}^2+\|\theta_t\|_{L^2}^2)dt\leq C. \tag{3.91}$$

First, owing to (2.2) and (3.89) it is easy to show that

$$\begin{aligned}\|\sqrt{\rho}u_t\|_{L^2}^2&\leq\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|\sqrt{\rho}|u||\nabla u|\|_{L^2}^2\leq\|\sqrt{\rho}\dot{u}\|_{L^2}^2+C\|\sqrt{\rho}u\|_{L^6}^2\|\nabla u\|_{L^3}^2\\&\leq\|\sqrt{\rho}\dot{u}\|_{L^2}^2+C\|\nabla u\|_{L^2}^2+C\|\nabla^2u\|_{L^2}^2.\end{aligned} \tag{3.92}$$

At the same time, $(1.1)_3$, $(1.1)_3$ combined with (2.2) and (3.2) respectively, gives

$$\begin{aligned}\|b_t\|_{L^2}^2+\|\theta_t\|_{L^2}^2&\leq C\|\triangle b\|_{L^2}^2+\||b||\nabla u|\|_{L^2}^2+\||u||\nabla b|\|_{L^2}^2+C\|\triangle\theta\|_{L^2}^2+\||u||\nabla\theta|\|_{L^2}^2\\&\leq C\|\triangle b\|_{L^2}^2+\|b\|_{L^4}^2\|\nabla u\|_{L^4}^2+\|\nabla b\bar{x}^{\frac{a}{2}}\|_{L^2}^2+C\|\triangle\theta\|_{L^2}^2+\|\nabla\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2\end{aligned}$$

$$\begin{aligned}&\le C\|\triangle b\|_{L^2}^2+C\|\nabla u\|_{L^2}^2+C\|\nabla^2 u\|_{L^2}^2+C\|\triangle\theta\|_{L^2}^2\\&\quad+\|\nabla b\bar{x}^{\frac{a}{2}}\|_{L^2}^2+\|\nabla\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2,\end{aligned}\tag{3.93}$$

where

$$\begin{aligned}\||u||\nabla b|\|_{L^2}^2&\le C\|u\bar{x}^{-a/4}\|_{L^8}^4\|\nabla b\|_{L^4}^2+C\|\nabla b\bar{x}^{a/2}\|_{L^2}^2\\&\le\frac{1}{2}\|\nabla^2 b\|_{L^2}^2+C\|\nabla b\bar{x}^{a/2}\|_{L^2}^2\end{aligned}\tag{3.94}$$

and

$$\begin{aligned}\||u||\nabla\theta|\|_{L^2}^2&\le C\|u\bar{x}^{-a/4}\|_{L^8}^4\|\nabla\theta\|_{L^4}^2+C\|\nabla\theta\bar{x}^{a/2}\|_{L^2}^2\\&\le\frac{1}{2}\|\nabla^2\theta\|_{L^2}^2+C\|\nabla\theta\bar{x}^{a/2}\|_{L^2}^2.\end{aligned}\tag{3.95}$$

According to (3.89) and (2.2), we can get (3.91) by the combination of (3.92), (3.93), (3.2), (3.9), (3.9), (3.55) and (3.69).
Next, differentiating $(1.1)_2$ with respect to t gives

$$\begin{aligned}&\rho u_{tt}+\rho u\cdot\nabla u_t-\mu\triangle u_t+\nabla P_t\\&=-\rho_t(u_t+u\cdot\nabla u)-\rho u_t\cdot\nabla u+(\rho\theta e_2)_t+(b\cdot\nabla b-\frac{1}{2}\nabla|b|^2)_t.\end{aligned}\tag{3.96}$$

Multiplying (3.96) by u_t and integrating by parts over $\mathbb{R}^2$, we deduce from $(1.1)_1$ and $(1.1)_4$ that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\rho|u_t|^2dx+\mu\int|\nabla u_t|^2dx\\&\le C\int\rho|u||u_t|(\nabla u_t+|u||\nabla^2 u|+|\nabla u|^2)dx+C\int\rho|u|^2|\nabla u||\nabla u_t|dx\\&\quad+\int\rho|u_t|^2|\nabla u|dx+\int\rho_t\theta e_2\cdot u_tdx+\int\rho\theta_te_2\cdot u_tdx\\&\quad-\int b_t\cdot\nabla u_t\cdot bdx-\int b\cdot\nabla u_t\cdot b_tdx\\&\triangleq\sum_{i=1}^{7}\bar{M}_i,\end{aligned}\tag{3.97}$$

where

$$\begin{aligned}\bar{M}_1&\le C\|\sqrt{\rho}u\|_{L^6}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^6}^{\frac{1}{2}}(\|\nabla u_t\|_{L^2}+\|\nabla u\|_{L^4}^2)\\&\quad+C\|\rho^{\frac{1}{4}}u\|_{L^{12}}^2\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^6}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}\\&\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}(\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla u_t\|_{L^2})^{\frac{1}{2}}(\|\nabla u_t\|_{L^2}+\|\nabla^2 u\|_{L^2})\\&\le\frac{\mu}{8}\|\nabla u_t\|_{L^2}^2+C(1+\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2),\end{aligned}\tag{3.98}$$

$$\begin{aligned}\bar{M}_2+\bar{M}_3&\le C\|\sqrt{\rho}u\|_{L^8}^2\|\nabla u\|_{L^4}\|\nabla u_t\|_{L^2}+\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^6}^{\frac{3}{2}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\\&\le\frac{\mu}{8}\|\nabla u_t\|_{L^2}^2+C(1+\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2).\end{aligned}\tag{3.99}$$

We can obtain the term $\bar{M}_4$ from $(1.1)_1$ together with integration by parts, (3.89), (3.2), (3.9) and (2.2) that

$$\begin{aligned}\bar{M}_4 &= -\int div(\rho u)\theta e_2\cdot u_t dx \le C\int \rho|u||\nabla\theta||u_t|dx + C\int \rho|u||\theta||\nabla u_t|dx\\ &\le \|\sqrt{\rho}u_t\|_{L^2}\|\sqrt{\rho}u\|_{L^4}\|\nabla\theta\|_{L^4} + \|\nabla u_t\|_{L^2}\|\rho u\|_{L^4}\|\theta\|_{L^4}\\ &\le \frac{\mu}{8}\|\nabla u_t\|_{L^2}^2 + C(1+\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2).\end{aligned}\tag{3.100}$$

We can obtain the bound from the Hölder inequality and (3.1) that

$$\bar{M}_5 \le \|\rho\|_{L^\infty}^{1/2}\|\sqrt{\rho}u_t\|_{L^2}\|\theta_t\|_{L^2} \le C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\theta_t\|_{L^2}^2 \tag{3.101}$$

and

$$\begin{aligned}\bar{M}_6 + \bar{M}_7 &\le \frac{\mu}{8}\|\nabla u_t\|_{L^2}^2 + C\||b||b_t|\|_{L^2}^2 \le \frac{\mu}{8}\|\nabla u_t\|_{L^2}^2 + C\|b\|_{L^2}^2\|b_t\|_{L^2}^2\\ &\le \frac{\mu}{8}\|\nabla u_t\|_{L^2}^2 + C\|b_t\|_{L^2}^2 + \frac{\nu}{2}\|\nabla b_t\|_{L^2}^2.\end{aligned}\tag{3.102}$$

Submitting (3.98)-(3.102) into (3.97) gives

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{\rho}u_t\|_{L^2}^2 + \mu\|\nabla u_t\|_{L^2}^2\\ \le& C(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|b_t\|_{L^2}^2 + \|\theta_t\|_{L^2}^2) + C(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2 + 1) + \frac{\mu\nu}{2C_2}\|\nabla b_t\|_{L^2}^2.\end{aligned}\tag{3.103}$$

Next, differentiating $(1.1)_3$ with respect to t shows

$$b_{tt} - b_t\cdot\nabla u - b\cdot\nabla u_t + u_t\cdot\nabla b + u\cdot\nabla b_t - \nu\triangle b_t = 0. \tag{3.104}$$

Now, multiplying (3.104) by b_t and integrating by parts over $\mathbb{R}^2$, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int|b_t|^2dx + \nu\int|\nabla b_t|^2dx\\ =& \int b\cdot\nabla u_t\cdot b_t dx + \int b_t\cdot\nabla u\cdot b_t dx + \int u_t\cdot\nabla b_t\cdot b dx\\ \le& C\|b\|_{L^4}^{1/2a}\|b\bar{x}^{a/2}\|_{L^2}^{(2a-1)/2a}\|u_t\bar{x}^{-(2a-1)/4}\|_{L^{8a}}\|\nabla b_t\|_{L^2}\\ &+ C\|b_t\|_{L^4}\|b\|_{L^4}\|\nabla u_t\|_{L^2} + C\|b_t\|_{L^4}^2\|\nabla u\|_{L^2}\\ \le& C(\|\sqrt{\rho}u_t\|_{L^2} + \|\nabla u_t\|_{L^2})\|\nabla b_t\|_{L^2} + C\|b_t\|_{L^4}^2 + C\|\nabla u_t\|_{L^2}^2\\ \le& C(\|b_t\|_{L^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2) + \frac{\nu}{2}\|\nabla b_t\|_{L^2} + \frac{C_2}{2}\|\nabla u_t\|_{L^2}^2.\end{aligned}\tag{3.105}$$

Adding $(3.103)\times\mu^{-1}C_2$ into (3.105), we get

$$\begin{aligned}&\frac{d}{dt}(\mu^{-1}C_2\|\sqrt{\rho}u_t\|_{L^2}^2 + \|b_t\|_{L^2}^2) + \|\nabla u_t\|_{L^2}^2 + \frac{\nu}{2}\|\nabla b_t\|_{L^2}^2\\ \le& C(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|b_t\|_{L^2}^2) + C(1+\|\nabla^2 u\|_{L^2}^2),\end{aligned}\tag{3.106}$$

which multiplied by t, together with Gronwall's inequality, (3.90), and (3.55) shows

$$\sup_{t\in[0,T]} t(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|b_t\|_{L^2}^2) + \int t(\|\nabla u_t\|_{L^2}^2 + \|\nabla b_t\|_{L^2}^2)dt \le C(T). \tag{3.107}$$

Then, differentiating $(1.1)_4$ with respect to t shows

$$\theta_{tt}-u_t\cdot\nabla\theta+u\cdot\nabla\theta_t-\kappa\triangle\theta_t=0, \tag{3.108}$$

and, multiplying (3.106) by b_t and integrating by parts over $\mathbb{R}^2$, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\theta_t|^2dx+\kappa\int|\nabla\theta_t|^2dx=&\int u_t\theta_t\cdot\theta dx\\ &\le C\|\theta\bar{x}^{a/2}\|_{L^{8a/(4a-1)}}\|u_t\bar{x}^{-(2a-1)/4}\|_{L^{8a}}\|\nabla\theta_t\|_{L^2}\\ &\le C\|\sqrt{\rho}u_t\|_{L^2}^2+C_3\|\nabla u_t\|_{L^2}^2)+\frac{\kappa}{2}\|\nabla\theta_t\|_{L^2}^2.\end{aligned} \tag{3.109}$$

Meanwhile, adding $(3.103)\times\mu^{-1}C_3$ into (3.109), we have

$$\begin{aligned}&\frac{d}{dt}(\mu^{-1}C_3\|\sqrt{\rho}u_t\|_{L^2}^2+\|\theta_t\|_{L^2}^2)+\|\nabla u_t\|_{L^2}^2+\kappa\|\nabla\theta_t\|_{L^2}^2\\ \le& C(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\theta_t\|_{L^2}^2)+C(1+\|\nabla^2u\|_{L^2}^2+\|\nabla^2\theta\|_{L^2}^2),\end{aligned} \tag{3.110}$$

Then, multiplying (3.110) by t and integrating by parts over $[0,T]$, and due to Gronwall's inequality, (3.90), (3.55) and (3.70) we have

$$\sup_{t\in[0,T]}t(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\theta_t\|_{L^2}^2)+\int_0^T t(\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2)dt\le C(T). \tag{3.111}$$

Finally, we deduce from $(1.1)_3$, $(1.1)_4$, (2.2), (3.2), (3.94) and (3.94) that

$$\begin{aligned}&\|\nabla^2b\|_{L^2}^2+\|\nabla^2\theta\|_{L^2}^2\\ \le& C\|b_t\|_{L^2}^2+C\||b||\nabla u|\|_{L^2}^2+C\||u||\nabla b|\|_{L^2}^2+C\|\theta_t\|_{L^2}^2+C\||u||\nabla\theta|\|_{L^2}^2\\ \le& C\|b_t\|_{L^2}^2+C\|b\|_{L^4}^2\|\nabla u\|_{L^2}\|\nabla^2u\|_{L^2}+\frac{1}{2}\|\nabla^2b\|_{L^2}^2\\ &+C\|\nabla b\bar{x}^{a/2}\|_{L^2}^2+C\|\theta_t\|_{L^2}^2+\frac{1}{2}\|\nabla^2\theta\|_{L^2}^2+C\|\nabla\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2\\ \le& C\|b_t\|_{L^2}^2+C\|\nabla^2u\|_{L^2}^2+\frac{1}{2}\|\nabla^2b\|_{L^2}^2+C\|\nabla b\bar{x}^{a/2}\|_{L^2}^2\\ &+C\|\theta_t\|_{L^2}^2+\frac{1}{2}\|\nabla^2\theta\|_{L^2}^2+C\|\nabla\theta\bar{x}^{\frac{a}{2}}\|_{L^2}^2,\end{aligned} \tag{3.112}$$

which combined with (3.111), (3.107), (3.103), (3.70) and (3.27) gains (3.88). Finally, the proof of Lemma 3.8 is finished. □

4. Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1.

Proof. (**Proof of Theorem 1.1.**) According to Lemmas 3.1-3.8, using standard continuity theory of local existence, it is assumed that there is a $T_*>0$ such that systems (1.1) and (1.2) have a local and unique strong solution (ρ,u,P,b,θ) on $\mathbb{R}^2\times(0,T_*]$. Next, we will extend the local solution to all time. Set

$$T^*=\sup\{T|(\rho,u,P,b,\theta)\ \textit{is a strong solution on}\ \mathbb{R}^2\times(0,T]\}. \tag{4.1}$$

We deduce from (3.2), (3.9), (3.27) and (3.88), for any $0<\tau<T<T^*$ with T finite, and any $q\ge 2$ that,

$$\nabla u,\nabla b,\nabla\theta,b,\theta\in C([\tau,T];L^2\cap L^q). \tag{4.2}$$

Then, along with the standard embedding

$$L^\infty(\tau,T;H^1)\cap H^1(\tau,T;L^2)\hookrightarrow C([\tau,T];L^q),\ for\ any\ q\in[2,\infty).$$

And, due to (3.55), (3.68), and [15, Lemma 2.3], we have

$$\rho\in C([0,T];L^1\cap H^1\cap W^{1,q}). \tag{4.3}$$

We declare that

$$T^*=\infty. \tag{4.4}$$

On the contrary, if $T^*<\infty$, we deduce from (4.2), (4.3), (3.2), (3.9), (3.68), and (3.88) that

$$(\rho,u,b,\theta)(x,T^*)=\lim_{t\to T^*}(\rho,u,b,\theta)(x,t)$$

conforms to the initial condition (1.5) at $t=T$. So, we can assume the initial data is $(\rho,u,b,\theta)(x,T^*)$, since the existence and uniqueness of local strong solutions signifies that there is some $T^{**}>T^*$, such that Theorem 1.1 holds for $T=T^{**}$. This is contradictory with the hypothesis of T^* in (4.1), so (4.4) holds. Hence, the previous lemmas and the local existence and uniqueness of strong solutions indicate that (ρ,u,P,b,θ) is actually the unique strong solution on $\mathbb{R}^2\times[0,T]$ for any $0<T<T^*=\infty$. This completes the proof of Theorem 1.1. ∎

Acknowledgments. The author would like to thank the referee's questions and suggestions, and Professor Yong Li for careful reading, helpful suggestions, and valuable comments, which helped a lot to improve the presentation of this manuscript.

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