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LOW MACH NUMBER LIMIT OF THE FULL COMPRESSIBLE MHD EQUATIONS WITH CATTANEO'S HEAT TRANSFER LAW*

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Abstract. We study low Mach number limit of the full compressible magnetohydrodynamic (MHD) equations with Cattaneo's heat transfer law in the framework of classical solutions with small density, temperature and heat flux variations. It is rigorously justified that, for well-prepared initial data and a sufficiently small Mach number, the full compressible MHD equations with Cattaneo's heat transfer law admit a smooth solution on the time interval where the smooth solution of the incompressible MHD equations exists, and the solution of the former converges to that of the latter as the Mach number tends to zero. Moreover, we also obtain the convergence rate.

Keywords. Full compressible MHD equations; Cattaneo's heat transfer law; low Mach number limit; incompressible MHD equations.

AMS subject classifications. 76W05; 35B40.

1. Introduction

The three-dimensional full compressible magnetohydrodynamic (MHD) equations with Cattaneo's heat transfer law (hereinafter called the MHD–Cattaneo equations for short) can be written in the following form (see, e.g. [29])

$$\partial_t \rho + \mathrm{div}(\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p - (\mathrm{curl} H) \times H = \mathrm{div} S, \tag{1.2}$$

$$\partial_t \mathcal{E} + \mathrm{div}((\mathcal{E}' + p)u) + \mathrm{div} q = \mathrm{div}((u \times H) \times H - \nu(\mathrm{curl} H) \times H + uS), \tag{1.3}$$

$$\partial_t H - \mathrm{curl}(u \times H) = -\mathrm{curl}(\nu \mathrm{curl} H), \ \ \mathrm{div} H = 0, \tag{1.4}$$

$$\tau \partial_t q + q + \kappa \nabla \theta = 0. \tag{1.5}$$

Here the unknowns ρ, $u = (u_1, u_2, u_3)$, $H = (H_1, H_2, H_3)$, θ and $q = (q_1, q_2, q_3)$ denote the density of the fluid, the fluid velocity, the magnetic field, the thermodynamic temperature and the heat flux, respectively. The stress tensor S is given by

$$S = \mu(\nabla u + \nabla u^\top) + \lambda \mathrm{div} u \mathbb{I}_3,$$

where μ and λ are viscosity coefficients satisfying $\mu > 0$ and $2\mu + 3\lambda \geq 0$, and $\mathbb{I}_3$ is the 3×3 identity matrix. The total energy $\mathcal{E}$ is given by

$$\mathcal{E} = \mathcal{E}' + \frac{|H|^2}{2} \text{ with } \mathcal{E}' = \rho\left(e + \frac{|u|^2}{2}\right).$$

Here e, $\frac{|u|^2}{2}$ and $\frac{|H|^2}{2}$ denote the internal energy, the kinetic energy and the magnetic energy, respectively. The equations of state $p = p(\rho, \theta)$ and $e = e(\rho, \theta)$ relate the pressure p and the internal energy e to the density ρ and the temperature θ. $\nu > 0$ is the magnetic diffusion coefficient of the magnetic field. $\kappa > 0$ denotes the heat conductivity coefficient

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and $\tau > 0$ is the relaxation time. For simplicity, we assume that μ, λ, ν, κ and τ are constants.

For $\tau = 0$, the Cattaneo's heat transfer law (1.5) turns into the Fourier's heat transfer law

$$q + \kappa \nabla \theta = 0,$$

and the MHD–Cattaneo equations (1.1)–(1.5) become the so-called full compressible MHD equations, i.e.,

$$\partial_t \rho + \mathrm{div}(\rho u) = 0, \tag{1.6}$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p - (\mathrm{curl} H) \times H = \mathrm{div} S, \tag{1.7}$$

$$\partial_t \mathcal{E} + \mathrm{div}((\mathcal{E}' + p)u) - \kappa \Delta \theta = \mathrm{div}((u \times H) \times H - \nu (\mathrm{curl} H) \times H + uS), \tag{1.8}$$

$$\partial_t H - \mathrm{curl}(u \times H) = -\mathrm{curl}(\nu \mathrm{curl} H), \ \ \mathrm{div} H = 0. \tag{1.9}$$

There have been a number of studies on this classical system because of the wide applications in the real world and the mathematical challenges involved, see, for example, [3, 4, 8, 9, 12–16, 37] and the references cited therein. One of the important topics on this system is to study its low Mach number limit (also called incompressible limit). Roughly speaking, the basic result is that the solution of slightly compressible model converges to that of the incompressible one as the Mach number tends to zero. Specifically, Jiang, Ju and Li [19] obtained the low Mach number limit of the full compressible MHD system with well-prepared initial data, in which the effect of small entropy or temperature variation is taken into account. Whereafter, together with Z.-P. Xin, they [20] investigated the low Mach number limit of these equations with general initial data and large temperature variations. Cui, Ou and Ren [7] studied the incompressible limits of the full compressible MHD equations for viscous and heat-conductive ideal polytropic flows with magnetic diffusion in a three-dimensional bounded domain. All the above results are about smooth solutions, the incompressible limit in the framework of weak solutions was also established, for example, see [10, 11, 23, 26, 27]. Besides the references mentioned above, there are some other articles on the low Mach number limit of ideal compressible MHD equations. We refer the interested reader to [5, 6, 21, 22, 25, 28, 33, 41] and the references cited therein.

Fourier's heat transfer law implies a parabolic equation with respect to the temperature θ, which predicts that heat waves have an infinite propagation speed and this behavior violates the recognized principle of causality. Compared to Fourier's law, Cattaneo's law leads to the hyperbolic equations with respect to the heat flux q and the temperature θ, which means the heat waves have a finite propagation speed (see [35] for more details). In recent years, Cattaneo's law has been gradually studied by many researchers. For the Navier–Stokes equations with Cattaneo's law, Hu and Racke [17] studied the existence of smooth solutions under different initial data conditions and verified that the solution uniformly converges to that of the full compressible Navier-Stokes equations as the relaxation time τ goes to zero. The low Mach number limit of the Navier–Stokes equations with Cattaneo's law was investigated by the second author [40]. It is justified that, for small density, temperature and heat flux variations, the solution converges to that of the incompressible Navier–Stokes equations as the Mach number tends to zero. For the MHD–Cattaneo Equations (1.1)–(1.5), Liu and Xu [29] showed that the local solution exists for large initial data and the global solution exists for small initial data. They also proved the uniform convergence of the solution to that of the full MHD Equations (1.6)–(1.9) as τ goes to zero. Boukrouche, Boussetouan and Paoli [2] prove an existence result for the incompressible fluid with Tresca's friction at the boundary and heat transfer governed by Cattaneo's law. In addition to fluid dynamics, Cattaneo's law is also used in thermoelasticity which results in the second sound phenomenon, see [1,18,31,32] and the references therein.

Motivated by [19,40], the main purpose of this paper is to investigate the low Mach number limit of the Equations (1.1)–(1.5) in the framework of classical solutions in the whole space $\mathbb{R}^3$ or the three-dimensional torus $\mathbb{T}^3=(\mathbb{R}/(2\pi\mathbb{Z}))^3$, which will be denoted by Ω. We shall focus our study on the fluid obeying the perfect gas relations

$$p=\Re\rho\theta, \qquad e=c_V\theta, \tag{1.10}$$

where the positive constants $\Re$ and c_V are the generic gas constant and the specific heat at constant volume, respectively.

Now, we simplify the Equations (1.1)–(1.5). First of all, for the energy Equation (1.3), we can rewrite it in the form of the temperature θ. Multiplying (1.2) by u and (1.4) by H, and putting them together, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}(\rho|u^2|+|H|^2)+\frac{1}{2}\mathrm{div}(\rho|u^2|u)+\nabla p\cdot u\\ =&(\mathrm{div}S+(\mathrm{curl}H)\times H)\cdot u+(\mathrm{curl}(u\times H)-\mathrm{curl}(\nu\mathrm{curl}H))\cdot H.\end{aligned} \tag{1.11}$$

Subtracting (1.11) from (1.3) and using the following identities

$$\mathrm{div}(H\times\mathrm{curl}H)=|\mathrm{curl}H|^2-(\mathrm{curlcurl}H)\cdot H,$$
$$\mathrm{div}((u\times H)\times H)=(\mathrm{curl}H)\times H\cdot u+\mathrm{curl}(u\times H)\cdot H,$$

we arrive at

$$\rho(\partial_t e+u\cdot\nabla e)+p\mathrm{div}u+\mathrm{div}q=\nu|\mathrm{curl}H|^2+\Psi,$$

where

$$\Psi=\frac{\mu}{2}|\nabla u+\nabla u^\top|^2+\lambda|\mathrm{div}u|^2.$$

Substituting the relations (1.10) into the above equation yields

$$c_V\rho(\partial_t\theta+u\cdot\nabla\theta)+\Re\rho\theta\mathrm{div}u+\mathrm{div}q=\nu|\mathrm{curl}H|^2+\Psi.$$

Secondly, utilizing the mass conservation equation, Equation (1.1), the momentum conservation Equation (1.2) becomes

$$\rho(\partial_t u+u\cdot\nabla u)+\Re\nabla(\rho\theta)-(\mathrm{curl}H)\times H=\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u.$$

Finally, using the identities

$$\begin{aligned}&\mathrm{curl}(u\times H)=u\mathrm{div}H-H\mathrm{div}u+H\cdot\nabla u-u\cdot\nabla H,\\ &\mathrm{curlcurl}H=\nabla\mathrm{div}H-\Delta H,\end{aligned}$$

and the constraint condition $\mathrm{div}H=0$, the magnetic field Equation (1.4) now reads

$$\partial_t H+u\cdot\nabla H-H\cdot\nabla u+H\mathrm{div}u=\nu\Delta H.$$

Thus, we rewrite the MHD–Cattaneo Equations (1.1)–(1.5) in the following form

$$\partial_t\rho+u\cdot\nabla\rho+\rho\mathrm{div}u=0, \tag{1.12}$$

$$\rho(\partial_t u+u\cdot\nabla u)+\Re\nabla(\rho\theta)-(\mathrm{curl}H)\times H=\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u, \tag{1.13}$$

$$c_V\rho(\partial_t\theta+u\cdot\nabla\theta)+\Re\rho\theta\mathrm{div}u+\mathrm{div}q=\nu|\mathrm{curl}H|^2+\Psi, \tag{1.14}$$

$$\partial_t H+u\cdot\nabla H-H\cdot\nabla u+H\mathrm{div}u=\nu\Delta H, \tag{1.15}$$

$$\tau\partial_t q+q+\kappa\nabla\theta=0. \tag{1.16}$$

Dividing each variable in the above system by its reference states and using the same notations for the dimensionless velocity field u, magnetic field H and thermodynamic functions ρ, θ and q, the dimensionless version of the system (1.12)–(1.16), only retaining Mach number (taking all other dimensionless parameters to be one), can be rewritten as follows (see the appendix of [19] for more details)

$$\partial_t\rho+u\cdot\nabla\rho+\rho\mathrm{div}u=0, \tag{1.17}$$

$$\rho(\partial_t u+u\cdot\nabla u)+\frac{\nabla(\rho\theta)}{\epsilon^2}-(\mathrm{curl}H)\times H=\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u, \tag{1.18}$$

$$\rho(\partial_t\theta+u\cdot\nabla\theta)+(\gamma-1)\rho\theta\mathrm{div}u+\mathrm{div}q=\epsilon^2(\nu|\mathrm{curl}H|^2+\Psi), \tag{1.19}$$

$$\partial_t H+u\cdot\nabla H-H\cdot\nabla u+H\mathrm{div}u=\nu\Delta H \tag{1.20}$$

$$\tau\partial_t q+q+\kappa\nabla\theta=0. \tag{1.21}$$

where $\gamma=1+\frac{\Re}{c_V}$ is the ratio of specific heats, and $\epsilon>0$ is the reference Mach number for the slightly compressible fluids, i.e. if $\bar\rho$, $\bar p$ and $\bar u$ represent the reference density, pressure and velocity, then the dimensionless Mach number ϵ can be denoted by the following form

$$\epsilon=\frac{\bar u}{\sqrt{\gamma\bar p/\bar\rho}}.$$

Inspired by [40], we further restrict ourselves to the small density, temperature and heat flux variations. Setting

$$\rho=1+\epsilon\eta^\epsilon,\quad \theta=1+\epsilon\phi^\epsilon,\quad q=\epsilon\psi^\epsilon,$$

then we can rewrite (1.17)–(1.21) as

$$\partial_t\eta^\epsilon+u^\epsilon\cdot\nabla\eta^\epsilon+\frac{1}{\epsilon}(1+\epsilon\eta^\epsilon)\mathrm{div}u^\epsilon=0, \tag{1.22}$$

$$\begin{aligned}&(1+\epsilon\eta^\epsilon)(\partial_t u^\epsilon+u^\epsilon\cdot\nabla u^\epsilon)+\frac{1}{\epsilon}[(1+\epsilon\eta^\epsilon)\nabla\phi^\epsilon+(1+\epsilon\phi^\epsilon)\nabla\eta^\epsilon]-(\mathrm{curl}H^\epsilon)\times H^\epsilon\\&=\mu\Delta u^\epsilon+(\mu+\lambda)\nabla\mathrm{div}u^\epsilon,\end{aligned} \tag{1.23}$$

$$\begin{aligned}&(1+\epsilon\eta^\epsilon)(\partial_t\phi^\epsilon+u^\epsilon\cdot\nabla\phi^\epsilon)+\frac{\gamma-1}{\epsilon}(1+\epsilon\eta^\epsilon)(1+\epsilon\phi^\epsilon)\mathrm{div}u^\epsilon+\mathrm{div}\psi^\epsilon\\&=\epsilon(\nu|\mathrm{curl}H^\epsilon|^2+\Psi^\epsilon),\end{aligned} \tag{1.24}$$

$$\partial_t H^\epsilon+u^\epsilon\cdot\nabla H^\epsilon-H^\epsilon\cdot\nabla u^\epsilon+H^\epsilon\mathrm{div}u^\epsilon=\nu\Delta H^\epsilon, \tag{1.25}$$

$$\tau\partial_t\psi^\epsilon+\kappa\nabla\phi^\epsilon=-\psi^\epsilon. \tag{1.26}$$

Here we have added the superscript ϵ on the unknowns to stress the dependence on the Mach number ϵ. The system (1.22)–(1.26) is equipped with the initial data

$$(\eta^\epsilon,u^\epsilon,\phi^\epsilon,H^\epsilon,\psi^\epsilon)|_{t=0}=(\eta_0^\epsilon(x),u_0^\epsilon(x),\phi_0^\epsilon(x),H_0^\epsilon(x),\psi_0^\epsilon(x)). \tag{1.27}$$

Suppose that the limits $u^\epsilon\to w$ and $H^\epsilon\to B$ exist and denoting the formal limit of $\frac{1}{\epsilon}[(1+\epsilon\eta^\epsilon)\nabla\phi^\epsilon+(1+\epsilon\phi^\epsilon)\nabla\eta^\epsilon]$ by $\nabla\pi$, we can formally obtain the following incompressible MHD equations

$$\partial_t w+w\cdot\nabla w+\nabla\pi-(\mathrm{curl}B)\times B=\mu\Delta w, \tag{1.28}$$

$$\partial_t B+w\cdot\nabla B-B\cdot\nabla w=\nu\Delta B, \tag{1.29}$$

$$\mathrm{div}w=0,\ \mathrm{div}B=0, \tag{1.30}$$

with the initial data

$$(w,B)|_{t=0}=(w_0(x),B_0(x)).$$

In this paper, we shall establish the above limit rigorously in $\Omega=\mathbb{R}^3$ or $\mathbb{T}^3$. Moreover, we shall prove that for sufficiently small ϵ, the MHD–Cattaneo Equations (1.22)–(1.26) admit a smooth solution on the time interval where the smooth solution of the incompressible MHD Equations (1.28)–(1.30) exists. Now we state our main result as follows.

THEOREM 1.1. *Let $s>\frac{3}{2}+2$. Assume that the initial data (1.27) are well-prepared, i.e.,*

$$\|(\eta_0^\epsilon,u_0^\epsilon-w_0,\phi_0^\epsilon,H_0^\epsilon-B_0,\psi_0^\epsilon)\|_s=O(\epsilon), \tag{1.31}$$

for $(w_0,B_0)\in H^{s+3}(\Omega)$ satisfying $\mathrm{div}w_0=\mathrm{div}B_0=0$. Let (w,B,π) be a smooth solution to the incompressible MHD Equations (1.28)–(1.30) with the initial data (w_0,B_0) on $[0,T^*]$ for some $T^*>0$ and satisfy

$$(w,B)\in C([0,T^*];H^{s+2}(\Omega))\cap C^1([0,T^*];H^s(\Omega)),$$
$$\pi\in C([0,T^*];H^{s+2}(\Omega))\cap C^1([0,T^*];H^{s+1}(\Omega)).$$

Then there exists a constant $\epsilon_0>0$, such that for all $\epsilon\le\epsilon_0$, the MHD–Cattaneo Equations (1.22)–(1.26) have a unique smooth solution $(\eta^\epsilon,u^\epsilon,\phi^\epsilon,H^\epsilon,\psi^\epsilon)\in C([0,T^*];H^s(\Omega))$. Moreover, there exists a constant $K>0$ independent of ϵ, such that, for all $\epsilon\le\epsilon_0$,

$$\sup_{t\in[0,T^*]}\left\|\left(\eta^\epsilon-\frac{\epsilon}{2}\pi,u^\epsilon-w,\phi^\epsilon-\frac{\epsilon}{2}\pi,H^\epsilon-B,\psi^\epsilon+\frac{\epsilon\kappa\nabla\pi}{2}\right)\right\|_s\le K\epsilon.$$

REMARK 1.1. The regularity assumptions of (w,B,π) in Theorem 1.1 will be used in error estimates (see more details in Section 4). Moreover, supposing that the initial data $(w_0,B_0)\in H^{s+3}(\Omega)$, we actually obtain from the existence result of the incompressible MHD equations (Theorem 2.2 in Section 2) that $(w,B)\in C([0,T^*];H^{s+3}(\Omega))\cap C^1([0,T^*];H^{s+1}(\Omega))$. If the domain Ω is $\mathbb{T}^3$, then π can be normalized by requesting that $\int_\Omega\pi\mathrm{d}x=0$, and the regularity of π follows from elliptic regularity such that $\pi\in C([0,T^*];H^{s+3}(\Omega))\cap C^1([0,T^*];H^{s+1}(\Omega))$. In the whole space case, to restrict $\pi=0$ in infinity, we can also derive the same regularity of π. Thus, we obtain a better regularity of (w,B,π) than that required in Theorem 1.1.

REMARK 1.2. Theorem 1.1 can be regarded as a generalization of the result in [40]. In fact, taking the magnetic field $H^\epsilon = 0$, the convergence result in Theorem 1.1 also holds for the Navier–Stokes equations with Cattaneo's law. Moreover, compared with the results in [19] for full MHD equations with Fourier's law, we find that for the well-prepared initial data, Cattaneo's law (or hyperbolic heat conduction) and Fourier's law (or parabolic heat conduction) can lead to the same limit equations.

The proof of Theorem 1.1 is based on the method developed in [19]. There are two critical processes. The first one is to construct the approximate system of the MHD–Cattaneo equations by using the solution of the incompressible MHD equations. The second one is to establish the uniform error estimates by energy method. Thus, using the convergence-stability lemma, we can complete the proof of Theorem 1.1.

The outline of this paper is as follows. In the next section, we construct some preliminaries. We first show the existence theory for the MHD–Cattaneo Equations (1.22)–(1.26), then go over the existence result of the incompressible MHD Equations (1.28)–(1.30). Finally, we list some calculus inequalities in Sobolev space and give the nonlinear Gronwall-type inequality, which are used in the error estimates. In Section 3, we construct the approximate system and show our main result. Finally, the uniform error estimates, which are required in Section 3, are obtained in Section 4.

Before ending the introduction, we give some notations used throughout the current paper. We use the letter C to denote various positive constants independent of ϵ, which may vary from line to line. The Greek letter α is used to denote multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, for integers $\alpha_i \geq 0$, $i = 1,2,3$. We denote by $D^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ the partial derivative of order $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. In particular, we use D^k to denote D^α with $|\alpha| = k$. $H^s(\Omega)\,(s \in \mathbb{R})$ denotes the usual Sobolev spaces with norm $\|\cdot\|_s$. For $s = 0$, namely L^2 space, we denote the norm $\|\cdot\|_{L^2}$ by $\|\cdot\|$. $L^\infty(\Omega)$ is the space of bounded measurable functions on Ω with the norm $\|\cdot\|_{L^\infty}$.

2. Preliminaries

In this section, we construct some necessary preliminaries. First, we show the local existence of classical solutions to the Equations (1.22)–(1.26). The method is based on the classical theory of local existence of solutions to quasilinear symmetric hyperbolic–parabolic system. Using the same way, Hu and Racke [17] obtained the local existence of solutions to the Navier–Stokes equations with Cattaneo's law. Here, we give the sketch of the proof on the local existence of solutions to Equations (1.22)–(1.26), for the sake of completeness.

Denoting $U^\epsilon = (\eta^\epsilon, u^\epsilon, \phi^\epsilon, H^\epsilon, \psi^\epsilon)$, we rewrite the system (1.22)–(1.26) in the vector form

$$A_0(U^\epsilon)\partial_t U^\epsilon + \sum_{i=1}^{3} A_i(U^\epsilon)\partial_i U^\epsilon = Q(D^2U^\epsilon) + F(U^\epsilon, DU^\epsilon), \tag{2.1}$$

where

$$Q(D^2U^\epsilon) = (0, \mu\Delta u^\epsilon + (\mu+\lambda)\nabla \mathrm{div} u^\epsilon, 0, \nu\Delta H^\epsilon, 0)^\top,$$

$$F(U^\epsilon, DU^\epsilon) = (0, 0, \epsilon(\nu|\mathrm{curl} H^\epsilon|^2 + \Psi^\epsilon), 0, -\psi^\epsilon)^\top,$$

and the matrices $A_i(U^\epsilon)$, $i = 0,1,2,3$, are given by

$$A_0(U^\epsilon) = \mathrm{diag}(1, 1+\epsilon\eta^\epsilon, 1+\epsilon\eta^\epsilon, 1+\epsilon\eta^\epsilon, 1+\epsilon\eta^\epsilon, 1, 1, 1, \tau, \tau, \tau),$$

$$
A_1(U^\epsilon)=
$$
$$
\begin{pmatrix}
u_1^\epsilon & \frac{1+\epsilon\eta^\epsilon}{\epsilon} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{1+\epsilon\phi^\epsilon}{\epsilon} & u_1^\epsilon(1+\epsilon\eta^\epsilon) & 0 & 0 & \frac{1+\epsilon\eta^\epsilon}{\epsilon} & 0 & H_2^\epsilon & H_3^\epsilon & 0 & 0 & 0\\
0 & 0 & u_1^\epsilon(1+\epsilon\eta^\epsilon) & 0 & 0 & 0 & -H_1^\epsilon & 0 & 0 & 0 & 0\\
0 & 0 & 0 & u_1^\epsilon(1+\epsilon\eta^\epsilon) & 0 & 0 & 0 & -H_1^\epsilon & 0 & 0 & 0\\
0 & \frac{(\gamma-1)(1+\epsilon\eta^\epsilon)(1+\epsilon\phi^\epsilon)}{\epsilon} & 0 & 0 & u_1^\epsilon(1+\epsilon\eta^\epsilon) & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & u_1^\epsilon & 0 & 0 & 0 & 0 & 0\\
0 & H_2^\epsilon & -H_1^\epsilon & 0 & 0 & 0 & u_1^\epsilon & 0 & 0 & 0 & 0\\
0 & H_3^\epsilon & 0 & -H_1^\epsilon & 0 & 0 & 0 & u_1^\epsilon & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \kappa & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
A_2(U^\epsilon)=
$$
$$
\begin{pmatrix}
u_2^\epsilon & 0 & \frac{1+\epsilon\eta^\epsilon}{\epsilon} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & u_2^\epsilon(1+\epsilon\eta^\epsilon) & 0 & 0 & 0 & -H_2^\epsilon & 0 & 0 & 0 & 0 & 0\\
\frac{1+\epsilon\phi^\epsilon}{\epsilon} & 0 & u_2^\epsilon(1+\epsilon\eta^\epsilon) & 0 & \frac{1+\epsilon\eta^\epsilon}{\epsilon} & H_1^\epsilon & 0 & H_3^\epsilon & 0 & 0 & 0\\
0 & 0 & 0 & u_2^\epsilon(1+\epsilon\eta^\epsilon) & 0 & 0 & 0 & -H_2^\epsilon & 0 & 0 & 0\\
0 & 0 & \frac{(\gamma-1)(1+\epsilon\eta^\epsilon)(1+\epsilon\phi^\epsilon)}{\epsilon} & 0 & u_2^\epsilon(1+\epsilon\eta^\epsilon) & 0 & 0 & 0 & 0 & 1 & 0\\
0 & -H_2^\epsilon & H_1^\epsilon & 0 & 0 & u_2^\epsilon & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & u_2^\epsilon & 0 & 0 & 0 & 0\\
0 & 0 & H_3^\epsilon & -H_2^\epsilon & 0 & 0 & 0 & u_2^\epsilon & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \kappa & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
A_3(U^\epsilon)=
$$
$$
\begin{pmatrix}
u_3^\epsilon & 0 & 0 & \frac{1+\epsilon\eta^\epsilon}{\epsilon} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & u_3^\epsilon(1+\epsilon\eta^\epsilon) & 0 & 0 & 0 & -H_3^\epsilon & 0 & 0 & 0 & 0 & 0\\
0 & 0 & u_3^\epsilon(1+\epsilon\eta^\epsilon) & 0 & 0 & 0 & -H_3^\epsilon & 0 & 0 & 0 & 0\\
\frac{1+\epsilon\phi^\epsilon}{\epsilon} & 0 & 0 & u_3^\epsilon(1+\epsilon\eta^\epsilon) & \frac{1+\epsilon\eta^\epsilon}{\epsilon} & H_1^\epsilon & H_2^\epsilon & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{(\gamma-1)(1+\epsilon\eta^\epsilon)(1+\epsilon\phi^\epsilon)}{\epsilon} & u_3^\epsilon(1+\epsilon\eta^\epsilon) & 0 & 0 & 0 & 0 & 0 & 1\\
0 & -H_3^\epsilon & 0 & H_1^\epsilon & 0 & u_3^\epsilon & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -H_3^\epsilon & H_2^\epsilon & 0 & 0 & u_3^\epsilon & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & u_3^\epsilon & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \kappa & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

It is easy to see that the matrices $\hat{A}_0(U^\epsilon)A_i(U^\epsilon)$, $i=1,2,3$, are symmetric matrices by choosing the symmetrizer $\hat{A}_0(U^\epsilon)$ as

$$
\hat{A}_0(U^\epsilon)=\mathrm{diag}\left(\frac{1+\epsilon\phi^\epsilon}{1+\epsilon\eta^\epsilon},\mathbb{I}_3,\frac{1}{(\gamma-1)(1+\epsilon\phi^\epsilon)},\mathbb{I}_3,\frac{1}{\kappa(\gamma-1)(1+\epsilon\phi^\epsilon)}\mathbb{I}_3\right),
$$

where $\mathbb{I}_3$ is the 3×3 identity matrix, and the coefficient matrix of the second-order partial derivative in $Q(D^2U^\epsilon)$ is real symmetric and semi-positive definite. Moreover, denote by G the physical state space of U^ϵ, for each bounded open set G_1 satisfying $G_1 \Subset G$ and $U^\epsilon \in \bar{G}_1$. $\hat{A}_0$ is a positive definite symmetric matrix for sufficiently small ϵ. Following the proof of the local existence theorem for the Cauchy problem of quasilinear symmetric hyperbolic–parabolic system (see [24,36]), we obtain the following result.

THEOREM 2.1. *Let $s \geq \frac{3}{2}+2$. Suppose that $U_0^\epsilon = (\eta_0^\epsilon, u_0^\epsilon, \phi_0^\epsilon, H_0^\epsilon, \psi_0^\epsilon) \in H^s(\Omega)$ and $U_0^\epsilon \in G_0$. Then for each open subset G_1 satisfying $G_0 \Subset G_1 \Subset G$, there exists a $T^\epsilon > 0$ such that the MHD–Cattaneo Equations (1.22)–(1.26) have a unique classical solution $U^\epsilon \in G_1$ and*

$$(\eta^\epsilon, \phi^\epsilon, \psi^\epsilon) \in C([0,T^\epsilon]; H^s(\Omega)) \cap C^1([0,T^\epsilon]; H^{s-1}(\Omega)),$$
$$(u^\epsilon, H^\epsilon) \in C([0,T^\epsilon]; H^s(\Omega)) \cap C^1([0,T^\epsilon]; H^{s-2}(\Omega)).$$

Next, we recall the local existence of solutions to the incompressible MHD equations.

THEOREM 2.2 ([19,34]). *Let $s \geq \frac{3}{2}+2$. Suppose that $(w_0, B_0) \in H^s(\Omega)$ satisfy $\operatorname{div} w_0 = 0$ and $\operatorname{div} B_0 = 0$. Then, there exists a $\bar{T} > 0$ such that the incompressible MHD Equations (1.28)–(1.30) have a unique solution $(w, B) \in L^\infty([0,\bar{T}]; H^s(\Omega))$ satisfying $\operatorname{div} w = 0$ and $\operatorname{div} B = 0$. Moreover, there exists a constant $M > 0$ such that for all $T \in (0, \bar{T})$, it holds that*

$$\sup_{t \in [0,T]} \left\{ \|(w,B)(t)\|_s + \|(\partial_t w, \partial_t B)(t)\|_{s-2} + \|\nabla \pi(t)\|_{s-2} \right\} \leq M.$$

Finally, we list some basic facts on product and commutator estimates in Sobolev space and give the nonlinear Gronwall-type inequality.

LEMMA 2.1 (Moser-type calculus inequalities [25,30]). *Assume that $g, h \in H^s(\Omega) \cap L^\infty(\Omega)$. Then for any α with $1 \leq |\alpha| \leq s$, we have*

$$\|D^\alpha(gh)\| \leq C(\|g\|_{L^\infty} \|D^s h\| + \|h\|_{L^\infty} \|D^s g\|),$$
$$\|[D^\alpha, g]h\| \leq C(\|\nabla g\|_{L^\infty} \|D^{s-1} h\| + \|h\|_{L^\infty} \|D^s g\|),$$

where $[D^\alpha, g]h = D^\alpha(gh) - g D^\alpha h$.

LEMMA 2.2 (Nonlinear Gronwall-type inequality [39]). *Suppose that $\sigma(t)$ is a positive C^1 function of $t \in [0,T)$ with $T \leq \infty$, $m > 1$ and $b_1(t)$, $b_2(t)$ are integrable on $[0,T)$. If*

$$\sigma'(t) \leq b_2(t)\sigma^m(t) + b_1(t)\sigma(t),$$

then there exists a $\delta > 0$, depending only on m, C_{1b} and C_{2b}, such that

$$\sup_{t \in [0,T)} \sigma(t) \leq e^{C_{1b}},$$

whenever $\sigma(0) \in (0, \delta]$. Here

$$C_{1b} = \sup_{t \in [0,T)} \int_0^t b_1(s) \mathrm{d}s \text{ and } C_{2b} = \int_0^T \max\{b_2(t), 0\} \mathrm{d}t.$$

3. Proof of Theorem 1.1

Proof. This section is devoted to proving Theorem 1.1. First of all, according to Theorem 2.1, for any fixed $\epsilon \in (0,1)$, we obtain that there exists a time interval $[0, T^\epsilon)$, such that the Equations (1.22)–(1.26) have a unique classical solution $U^\epsilon \in H^s(\Omega)$ and $U^\epsilon \in G_1$. Now we define

$$T_\epsilon = \sup\{T^\epsilon : U^\epsilon(t,x) \in C([0,T^\epsilon]; H^s), U^\epsilon(t,x) \in G_1, \forall (t,x) \in [0,T^\epsilon] \times \Omega\}.$$

Namely, $[0,T_\epsilon)$ is the maximal time interval of H^s existence. Note that T_ϵ depends on G and may tend to zero as $\epsilon\to 0$. To show that $\liminf_{\varepsilon\to 0} T_\epsilon > 0$, we shall make use of the following convergence-stability lemma, which was first developed for hyperbolic systems by Yong in [38] and is also available for hyperbolic–parabolic systems. We recall it here for the reader's convenience.

LEMMA 3.1 ([19, 38]). *Let $s>\frac{3}{2}+2$. Suppose that $U_0^\epsilon\in G_0$ and the following convergence assumption (A) holds.*

(A) For each ϵ, there exist $T^*>0$ and $U_\epsilon\in L^\infty([0,T^*];H^s)$ for each ϵ, satisfying

$$\bigcup_{x,t,\epsilon}\{U_\epsilon(t,x)\}\Subset G,$$

such that, for $t\in[0,\min\{T^*,T_\epsilon\})$,

$$\sup_{x,t}|U^\epsilon(t,x)-U_\epsilon(t,x)|=o(1),\ \sup_t\|U^\epsilon(t,x)-U_\epsilon(t,x)\|_s=O(1) \text{ as } \epsilon\to 0.$$

Then there exists an $\bar\epsilon>0$ such that, $\forall\epsilon\in(0,\bar\epsilon]$, it holds that

$$T_\epsilon>T^*.$$

To apply Lemma 3.1, we construct the approximation $U_\epsilon=(\eta_\epsilon,u_\epsilon,\phi_\epsilon,H_\epsilon,\psi_\epsilon)$ by

$$\eta_\epsilon=\frac{\epsilon\pi}{2},\ u_\epsilon=w,\ \phi_\epsilon=\frac{\epsilon\pi}{2},\ H_\epsilon=B \text{ and } \psi_\epsilon=-\frac{\epsilon\kappa\nabla\pi}{2},$$

where (w,B,π) is the smooth solution to the incompressible MHD Equations (1.28)–(1.30) obtained in Theorem 2.2. It is easy to verify that U_ϵ satisfies the following approximate system:

$$\partial_t\eta_\epsilon+u_\epsilon\cdot\nabla\eta_\epsilon+\frac{1}{\epsilon}(1+\epsilon\eta_\epsilon)\mathrm{div}u_\epsilon=\frac{\epsilon}{2}(\pi_t+w\cdot\nabla\pi), \tag{3.1}$$

$$\begin{aligned}&(1+\epsilon\eta_\epsilon)(\partial_t u_\epsilon+u_\epsilon\cdot\nabla u_\epsilon)+\frac{1}{\epsilon}[(1+\epsilon\eta_\epsilon)\nabla\phi_\epsilon+(1+\epsilon\phi_\epsilon)\nabla\eta_\epsilon]-(\mathrm{curl}H_\epsilon)\times H_\epsilon\\ =&\mu\Delta u_\epsilon+\frac{\epsilon^2\pi}{2}(w_t+w\cdot\nabla w+\nabla\pi),\end{aligned} \tag{3.2}$$

$$\begin{aligned}&(1+\epsilon\eta_\epsilon)(\partial_t\phi_\epsilon+u_\epsilon\cdot\nabla\phi_\epsilon)+\frac{\gamma-1}{\epsilon}(1+\epsilon\eta_\epsilon)(1+\epsilon\phi_\epsilon)\mathrm{div}u_\varepsilon+\mathrm{div}\psi_\epsilon\\ =&(\frac{\epsilon}{2}+\frac{\epsilon^3\pi}{4})(\pi_t+w\cdot\nabla\pi)-\frac{\epsilon\kappa}{2}\Delta\pi,\end{aligned} \tag{3.3}$$

$$\partial_t H_\epsilon+u_\epsilon\cdot\nabla H_\epsilon-H_\epsilon\cdot\nabla u_\epsilon+H_\epsilon\mathrm{div}u_\epsilon=\nu\Delta H_\epsilon, \tag{3.4}$$

$$\tau\partial_t\psi_\epsilon+\kappa\nabla\phi_\epsilon=-\frac{\epsilon\tau\kappa\nabla\pi_t}{2}+\frac{\epsilon\kappa\nabla\pi}{2}. \tag{3.5}$$

In the next section, we will prove the following error estimates.

LEMMA 3.2. *Suppose that the assumptions in Theorem 1.1 hold. Then there exist constants $K=K(T^*)$ and $\epsilon_0=\epsilon_0(T^*)$ such that for all $\epsilon\in(0,\epsilon_0]$, it holds that*

$$\sup_{t\in[0,\min\{T^*,T_\epsilon\})}\|U^\epsilon(t,x)-U_\epsilon(t,x)\|_s\le K\epsilon. \tag{3.6}$$

With this lemma in hand, we can verify the convergence assumption (A) by using Sobolev's inequality and accomplish the proof of Theorem 1.1. ∎

4. Uniform error estimates

In this section, we are going to establish the uniform error estimates. We rewrite the approximate system (3.1)–(3.5) in the following vector form

$$A_0(U_\epsilon)\partial_t U_\epsilon+\sum_{i=1}^{3}A_i(U_\epsilon)\partial_i U_\epsilon=S(D^2U_\epsilon)+R, \tag{4.1}$$

with $S(D^2U_\epsilon)=(0,\mu\Delta u_\epsilon,0,\nu\Delta H_\epsilon,0)^\top$ and

$$R=\begin{pmatrix}\frac{\epsilon}{2}(\partial_t\pi+w\cdot\nabla\pi)\\ \frac{\epsilon^2\pi}{2}(\partial_t w+w\cdot\nabla w+\nabla\pi)\\ (\frac{\epsilon}{2}+\frac{\epsilon^3\pi}{4})(\partial_t\pi+w\cdot\nabla\pi)-\frac{\epsilon\kappa}{2}\Delta\pi\\ 0\\ -\frac{\epsilon\tau\kappa\nabla\partial_t\pi}{2}+\frac{\epsilon\kappa\nabla\pi}{2}\end{pmatrix}.$$

Due to the regularity assumptions on (w,B,π) in Theorem 1.1, we have

$$\max_{t\in[0,T^*]}\|R(t)\|_s\le C\epsilon. \tag{4.2}$$

Setting $E=U^\epsilon-U_\epsilon=(\eta,u,\phi,H,\psi)$ and utilizing (2.1) and (4.1), we get the error system

$$\begin{aligned}\partial_t E+\sum_{i=1}^{3}\mathcal{A}_i(U^\epsilon)\partial_i E=&\sum_{i=1}^{3}(\mathcal{A}_i(U_\epsilon)-\mathcal{A}_i(U^\epsilon))\partial_i U_\epsilon+A_0^{-1}(U^\epsilon)(Q(D^2U^\epsilon)+F(U^\epsilon,DU^\epsilon))\\ &-A_0^{-1}(U_\epsilon)(S(D^2U_\epsilon)+R),\end{aligned} \tag{4.3}$$

where $\mathcal{A}_i(U)=A_0^{-1}(U)A_i(U)$, $i=1,2,3$.

For any α with $|\alpha|\le s$, we take D^α of (4.3) to obtain that

$$\partial_t D^\alpha E+\sum_{i=1}^{3}\mathcal{A}_i(U^\epsilon)\partial_i D^\alpha E=\mathcal{C}^\alpha+\mathcal{P}^\alpha+\mathcal{Q}^\alpha+\mathcal{R}^\alpha, \tag{4.4}$$

where

$$\begin{aligned}\mathcal{C}^\alpha=&-\sum_{i=1}^{3}[D^\alpha,\mathcal{A}_i(U^\epsilon)]\partial_i E\\ =&\begin{pmatrix}-[D^\alpha,u^\epsilon]\cdot\nabla\eta-[D^\alpha,\frac{1}{\epsilon}(1+\epsilon\eta^\epsilon)]\mathrm{div}u\\ -[D^\alpha,u^\epsilon]\cdot\nabla u-[D^\alpha,\frac{1}{\epsilon}\frac{1+\epsilon\phi^\epsilon}{1+\epsilon\eta^\epsilon}]\nabla\eta-[D^\alpha,\frac{1}{1+\epsilon\eta^\epsilon}H^\epsilon]\times\mathrm{curl}H\\ -[D^\alpha,u^\epsilon]\cdot\nabla\phi-[D^\alpha,\frac{\gamma-1}{\epsilon}(1+\epsilon\phi^\epsilon)]\mathrm{div}u-[D^\alpha,\frac{1}{1+\epsilon\eta^\epsilon}]\mathrm{div}\psi\\ -[D^\alpha,u^\epsilon]\cdot\nabla H+[D^\alpha,H^\epsilon]\cdot\nabla u-[D^\alpha,H^\epsilon]\mathrm{div}u\\ 0\end{pmatrix}:=\begin{pmatrix}\mathcal{C}_1^\alpha\\ \mathcal{C}_2^\alpha\\ \mathcal{C}_3^\alpha\\ \mathcal{C}_4^\alpha\\ 0\end{pmatrix},\end{aligned} \tag{4.5}$$

$$\begin{aligned}\mathcal{P}^\alpha=&\sum_{i=1}^{3}D^\alpha\{(\mathcal{A}_i(U_\epsilon)-\mathcal{A}_i(U^\epsilon))\partial_i U_\epsilon\}\\ =&D^\alpha\begin{pmatrix}-u\cdot\nabla\eta_\epsilon-\eta\mathrm{div}u_\epsilon\\ -u\cdot\nabla u_\epsilon-\frac{1}{\epsilon}(\frac{1+\epsilon\phi^\epsilon}{1+\epsilon\eta^\epsilon}-\frac{1+\epsilon\phi_\epsilon}{1+\epsilon\eta_\epsilon})\nabla\eta_\epsilon-(\frac{H^\epsilon}{1+\epsilon\eta^\epsilon}-\frac{H_\epsilon}{1+\epsilon\eta_\epsilon})\times\mathrm{curl}H_\epsilon\\ -u\cdot\nabla\phi_\epsilon-(\gamma-1)\phi\mathrm{div}u_\epsilon-(\frac{1}{1+\epsilon\eta^\epsilon}-\frac{1}{1+\epsilon\eta_\epsilon})\mathrm{div}\psi_\epsilon\\ -u\cdot\nabla H_\epsilon+H\cdot\nabla u_\epsilon-H\mathrm{div}u_\epsilon\\ 0\end{pmatrix}:=\begin{pmatrix}\mathcal{P}_1^\alpha\\ \mathcal{P}_2^\alpha\\ \mathcal{P}_3^\alpha\\ \mathcal{P}_4^\alpha\\ 0\end{pmatrix},\end{aligned} \tag{4.6}$$

$$\mathcal{Q}^\alpha = D^\alpha\{A_0^{-1}(U^\epsilon)(Q(D^2U^\epsilon)+F(U^\epsilon,DU^\epsilon))-A_0^{-1}(U_\epsilon)S(D^2U_\epsilon)\}$$
$$= D^\alpha\begin{pmatrix} 0 \\ \frac{\mu\Delta u^\epsilon+(\mu+\lambda)\nabla\mathrm{div}u^\epsilon}{1+\epsilon\eta^\epsilon}-\frac{\mu\Delta u_\epsilon}{1+\epsilon\eta_\epsilon} \\ \frac{\epsilon(\nu|\mathrm{curl}H^\epsilon|^2+\Psi^\epsilon)}{1+\epsilon\eta^\epsilon} \\ \nu\Delta H^\epsilon-\nu\Delta H_\epsilon \\ -\frac{\psi^\epsilon}{\tau} \end{pmatrix},$$

and

$$\mathcal{R}^\alpha = D^\alpha\{A_0^{-1}(U_\epsilon)R\}$$
$$= D^\alpha\begin{pmatrix} -\frac{\epsilon}{2}(\pi_t+w\cdot\nabla\pi) \\ -\frac{\epsilon^2\pi}{2(1+\epsilon\eta_\epsilon)}(w_t+w\cdot\nabla w+\nabla\pi) \\ -\frac{2\epsilon+\epsilon^3\pi}{4(1+\epsilon\eta_\epsilon)}(\pi_t+w\cdot\nabla\pi)+\frac{\epsilon\kappa}{2(1+\epsilon\eta_\epsilon)}\Delta\pi \\ 0 \\ \frac{\epsilon\kappa\nabla\pi_t}{2}-\frac{\epsilon\kappa\nabla\pi}{2\tau} \end{pmatrix}.$$

We define the symmetrizer of the error system (4.3) by

$$\tilde{A}_0(U^\epsilon) = \hat{A}_0(U^\epsilon)A_0(U^\epsilon)$$
$$= \mathrm{diag}\left(\frac{1+\epsilon\phi^\epsilon}{1+\epsilon\eta^\epsilon},(1+\epsilon\eta^\epsilon)\mathbb{I}_3,\frac{1+\epsilon\eta^\epsilon}{(\gamma-1)(1+\epsilon\phi^\epsilon)},\mathbb{I}_3,\frac{\tau}{\kappa(\gamma-1)(1+\epsilon\phi^\epsilon)}\mathbb{I}_3\right)$$

and the canonical energy by

$$\|E\|_e^2 := \int\langle\tilde{A}_0(U^\epsilon)E,E\rangle\mathrm{d}x.$$

Multiplying (4.4) by $\tilde{A}_0(U^\epsilon)$, taking the inner product between the resulting equations and $D^\alpha E$ and using integration by parts, we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\|D^\alpha E\|_e^2 = \int\langle\Gamma D^\alpha E,D^\alpha E\rangle\mathrm{d}x+2\int\langle D^\alpha E,\tilde{A}_0(U^\epsilon)(\mathcal{C}^\alpha+\mathcal{P}^\alpha+\mathcal{Q}^\alpha+\mathcal{R}^\alpha)\rangle\mathrm{d}x, \quad (4.7)$$

where

$$\Gamma = (\partial_t,\nabla)\cdot\left(\tilde{A}_0(U^\epsilon),\tilde{A}_0(U^\epsilon)\mathcal{A}_1(U^\epsilon),\tilde{A}_0(U^\epsilon)\mathcal{A}_2(U^\epsilon),\tilde{A}_0(U^\epsilon)\mathcal{A}_3(U^\epsilon)\right).$$

We note that it is only needed to consider $t\in[0,\min\{T^*,T_\epsilon\})$, in which U^ϵ and U_ϵ are regular enough and take values in a bounded subset of G. Thus, we have

$$C^{-1}\|D^\alpha E\|^2 \le \|D^\alpha E\|_e^2 \le C\|D^\alpha E\|^2, \quad (4.8)$$

for some constant $C>0$.

Now, we estimate various terms on the right-hand side of (4.7). Since $\tilde{A}_0(U^\epsilon)$ depends only on $1+\epsilon\eta^\epsilon$ and $1+\epsilon\phi^\epsilon$, we obtain from (1.22) and (1.24) that

$$|\partial_t\tilde{A}_0(U^\epsilon)| \le \left|\left(\frac{1+\epsilon\phi^\epsilon}{(1+\epsilon\eta^\epsilon)^2}+\frac{1}{(\gamma-1)(1+\epsilon\eta^\epsilon)}+1\right)\epsilon\partial_t\eta^\epsilon\right|$$

$$
\begin{aligned}
&+\left|\left(\frac{1+\epsilon\eta^\epsilon}{(\gamma-1)(1+\epsilon\phi^\epsilon)^2}+\frac{\tau}{\kappa(\gamma-1)(1+\epsilon\phi^\epsilon)^2}+\frac{1}{1+\epsilon\eta}\right)\epsilon\partial_t\phi^\epsilon\right| \\
\le& C\big(|\epsilon\partial_t\eta^\epsilon|+|\epsilon\partial_t\phi^\epsilon|\big) \\
\le& C\big(|\epsilon u^\epsilon\cdot\nabla\eta^\epsilon|+|(1+\epsilon\eta^\epsilon)\mathrm{div}u^\epsilon|+|\epsilon u^\epsilon\cdot\nabla\phi^\epsilon|+|(\gamma-1)(1+\epsilon\phi^\epsilon)\mathrm{div}u^\epsilon| \\
&\quad+|\epsilon\mathrm{div}\psi^\epsilon|+|\epsilon^2\Psi^\epsilon|+|\epsilon\nu\mathrm{curl}H^\epsilon|^2\big) \\
\le& C\big(1+\|E\|_s^2\big),
\end{aligned}
$$

where we have used the Cauchy and Sobolev inequalities. Notice that $\tilde{A}_0(U^\epsilon)\mathcal{A}_i(U^\epsilon)$ includes the singular terms $\frac{1+\epsilon\eta^\epsilon}{\epsilon}$ and $\frac{1+\epsilon\phi^\epsilon}{\epsilon}$. Fortunately, we can factor out ϵ in $\partial_i(\frac{1+\epsilon\eta^\epsilon}{\epsilon})$ and $\partial_i(\frac{1+\epsilon\phi^\epsilon}{\epsilon})$ to balance ϵ appearing in the denominator. A direct calculation yields

$$
\begin{aligned}
|\partial_i(\tilde{A}_0(U^\epsilon)\mathcal{A}_i(U^\epsilon))|\le&\left|\partial_i\left(\frac{1+\epsilon\phi^\epsilon}{1+\epsilon\eta^\epsilon}u_i^\epsilon\right)\right|+|\partial_i((1+\epsilon\eta^\epsilon)u_i^\epsilon)| \\
&+\left|\partial_i\left(\frac{1+\epsilon\eta^\epsilon}{\epsilon}\right)\right|+\left|\partial_i\left(\frac{1+\epsilon\phi^\epsilon}{\epsilon}\right)\right| \\
&+\left|\partial_i\left(\frac{(1+\epsilon\eta^\epsilon)u_i^\epsilon}{(\gamma-1)(1+\epsilon\phi^\epsilon)}\right)\right|+\left|\partial_i\left(\frac{1}{(\gamma-1)(1+\epsilon\phi^\epsilon)}\right)\right|+|\partial_iH^\epsilon| \\
\le& C\big(1+\|E\|_s^2\big).
\end{aligned}
$$

Based on the above analysis, we get the following estimates

$$
\begin{aligned}
\int\langle\Gamma D^\alpha E,D^\alpha E\rangle\mathrm{d}x\le& C|\Gamma|\|E\|_s^2 \\
\le& C\left(\left|\partial_t\tilde{A}_0(U^\epsilon)\right|+\sum_{i=1}^3\left|\partial_i\big(\tilde{A}_0(U^\epsilon)\mathcal{A}_i(U^\epsilon)\big)\right|\right)\|E\|_s^2 \\
\le& C\big(1+\|E\|_s^2\big)\|E\|_s^2.
\end{aligned}\tag{4.9}
$$

For the second term on the right-hand side of (4.7), we have

$$
\begin{aligned}
&\int\langle D^\alpha E,\tilde{A}_0(U^\epsilon)(\mathcal{C}^\alpha+\mathcal{P}^\alpha+\mathcal{Q}^\alpha+\mathcal{R}^\alpha)\rangle\mathrm{d}x \\
\le& C\left(\|D^\alpha E\|^2+\|\mathcal{C}^\alpha\|^2+\|\mathcal{P}^\alpha\|^2+\|\mathcal{R}^\alpha\|^2\right)+\int\langle D^\alpha E,\tilde{A}_0(U^\epsilon)\mathcal{Q}^\alpha\rangle\mathrm{d}x.
\end{aligned}\tag{4.10}
$$

First, we shall get the estimates of the commutator $\mathcal{C}^\alpha$ according to the specific form of the expression in (4.5). We get from using Moser-type calculus inequalities that

$$
\begin{aligned}
\|\mathcal{C}_1^\alpha\|\le&\|[D^\alpha,u^\epsilon]\cdot\nabla\eta\|+\left\|\left[D^\alpha,\frac{1+\epsilon\eta^\epsilon}{\epsilon}\right]\mathrm{div}u\right\| \\
\le&\|\nabla u^\epsilon\|_{L^\infty}\|D^{s-1}\nabla\eta\|+\|\nabla\eta\|_{L^\infty}\|D^su^\epsilon\| \\
&+\left\|\nabla\left(\frac{1+\epsilon\eta^\epsilon}{\epsilon}\right)\right\|_{L^\infty}\|D^{s-1}u\|+\|\nabla u\|_{L^\infty}\left\|D^s\left(\frac{1+\epsilon\eta^\epsilon}{\epsilon}\right)\right\| \\
\le&\|u^\epsilon\|_s\|\eta\|_s+\|\eta^\epsilon\|_s^s\|u\|_s \\
\le& C\big(1+\|E\|_s^s\big)\|E\|_s,
\end{aligned}\tag{4.11}
$$

and

$$\begin{aligned}
\|\mathcal{C}_2^\alpha\| \le & \|[D^\alpha,u^\epsilon]\cdot\nabla u\| + \left\|\left[D^\alpha,\frac{1}{\epsilon}\frac{1+\epsilon\phi^\epsilon}{1+\epsilon\eta^\epsilon}\right]\nabla\eta\right\| + \left\|\left[D^\alpha,\frac{1}{1+\epsilon\eta^\epsilon}H^\epsilon\right]\times \mathrm{curl}H\right\| \\
\le & \|\nabla u^\epsilon\|_{L^\infty}\|D^{s-1}\nabla u\| + \|\nabla u\|_{L^\infty}\|D^s u^\epsilon\| \\
& + \left\|\nabla\left(\frac{1}{\epsilon}\frac{1+\epsilon\phi^\epsilon}{1+\epsilon\eta^\epsilon}\right)\right\|_{L^\infty}\|D^{s-1}\nabla\eta\| + \|\nabla\eta\|_{L^\infty}\left\|D^s\left(\frac{1}{\epsilon}\frac{1+\epsilon\phi^\epsilon}{1+\epsilon\eta^\epsilon}\right)\right\| \\
& + \left\|\nabla\left(\frac{H^\epsilon}{1+\epsilon\eta^\epsilon}\right)\right\|_{L^\infty}\|D^{s-1}\nabla H\| + \|\nabla H\|_{L^\infty}\left\|D^s\left(\frac{H^\epsilon}{1+\epsilon\eta^\epsilon}\right)\right\| \\
\le & C\big(\|u^\epsilon\|_s\|u\|_s + \|(\eta^\epsilon,\phi^\epsilon)\|_s^s\|\eta\|_s + \|(\eta^\epsilon,H^\epsilon)\|_s^s\|H\|_s\big) \\
\le & C\big(1+\|E\|_s^s\big)\|E\|_s.
\end{aligned} \tag{4.12}$$

Using a similar fashion, one gets the estimates of $\mathcal{C}_3^\alpha$ and $\mathcal{C}_4^\alpha$ as

$$\|\mathcal{C}_3^\alpha\| + \|\mathcal{C}_4^\alpha\| \le C\big(1+\|E\|_s^s\big)\|E\|_s \tag{4.13}$$

Combining (4.11), (4.12) and (4.13), we obtain that

$$\|\mathcal{C}^\alpha\| \le \|\mathcal{C}_1^\alpha\| + \|\mathcal{C}_2^\alpha\| + \|\mathcal{C}_3^\alpha\| + \|\mathcal{C}_4^\alpha\| \le C\big(1+\|E\|_s^s\big)\|E\|_s. \tag{4.14}$$

Next, we deal with the terms $\mathcal{P}_i^\alpha$, $i=1,2,3,4$, in (4.6). Notice that they have similar structures. The term $\mathcal{P}_2^\alpha$ can be bounded as

$$\begin{aligned}
\|\mathcal{P}_2^\alpha\| \le & \|D^\alpha(u\nabla u_\epsilon)\| + \left\|D^\alpha\left(\frac{1}{\epsilon}\left(\frac{1+\epsilon\phi^\epsilon}{1+\epsilon\eta^\epsilon} - \frac{1+\epsilon\phi_\epsilon}{1+\epsilon\eta_\epsilon}\right)\nabla\eta_\epsilon\right)\right\| \\
& + \left\|D^\alpha\left(\left(\frac{H^\epsilon}{1+\epsilon\eta^\epsilon} - \frac{H_\epsilon}{1+\epsilon\eta_\epsilon}\right)\times\mathrm{curl}H_\epsilon\right)\right\| \\
\le & \|u\|_{L^\infty}\|D^s\nabla u_\epsilon\| + \|\nabla u_\epsilon\|_{L^\infty}\|D^s u\| \\
& + \left\|\frac{1}{\epsilon}\left(\frac{1+\epsilon\phi^\epsilon}{1+\epsilon\eta^\epsilon} - \frac{1+\epsilon\phi_\epsilon}{1+\epsilon\eta_\epsilon}\right)\right\|_{L^\infty}\|D^s\nabla\eta_\epsilon\| + \|\nabla\eta_\epsilon\|_{L^\infty}\left\|\frac{1}{\epsilon}D^s\left(\frac{1+\epsilon\phi^\epsilon}{1+\epsilon\eta^\epsilon} - \frac{1+\epsilon\phi_\epsilon}{1+\epsilon\eta_\epsilon}\right)\right\| \\
& + \left\|\left(\frac{H^\epsilon}{1+\epsilon\eta^\epsilon} - \frac{H_\epsilon}{1+\epsilon\eta_\epsilon}\right)\right\|_{L^\infty}\|D^s\nabla H_\epsilon\| + \|\nabla H_\epsilon\|_{L^\infty}\left\|D^s\left(\frac{H^\epsilon}{1+\epsilon\eta^\epsilon} - \frac{H_\epsilon}{1+\epsilon\eta_\epsilon}\right)\right\| \\
\le & C\big(\|u_\epsilon\|_{s+1}\|u\|_s + \|\eta_\epsilon\|_{s+1}\|(\eta,\phi)\|_s + \|\eta_\epsilon\|_s\|(\eta,\phi)\|_s^s \\
& + \|H_\epsilon\|_{s+1}\|(\eta,H)\|_s + \|H_\epsilon\|_s\|(\eta,H)\|_s^s\big) \\
\le & C\big(1+\|E\|_s^s\big)\|E\|_s,
\end{aligned}$$

where we have used the uniform boundedness of $\|U_\epsilon\|_{s+1}$. The estimates of $\mathcal{P}_1^\alpha$, $\mathcal{P}_3^\alpha$ and $\mathcal{P}_4^\alpha$ can be given in a similar fashion, so we obtain

$$\|\mathcal{P}_1^\alpha\| + \|\mathcal{P}_3^\alpha\| + \|\mathcal{P}_4^\alpha\| \le C\big(1+\|E\|_s^s\big)\|E\|_s.$$

Thus, we arrive at

$$\|\mathcal{P}^\alpha\| \le C\big(1+\|E\|_s^s\big)\|E\|_s. \tag{4.15}$$

For the term $\mathcal{R}^\alpha$, using the fact (4.2), it is easy to see that

$$\|\mathcal{R}^\alpha\| \le C\epsilon. \tag{4.16}$$

Finally, let's concentrate on the estimate of $\int \langle D^\alpha E, \tilde{A}_0(U^\epsilon)\mathcal{Q}^\alpha\rangle \mathrm{d}x$, which is more complex and delicate because of the existence of the highest order derivative terms. We can rewrite it as

$$
\begin{aligned}
&\int \langle D^\alpha E, \tilde{A}_0(U^\epsilon)\mathcal{Q}^\alpha\rangle \mathrm{d}x \\
=& \int (1+\epsilon\eta^\epsilon) D^\alpha u \cdot D^\alpha \left(\frac{\mu\Delta u^\epsilon + (\mu+\lambda)\nabla \mathrm{div} u^\epsilon}{1+\epsilon\eta^\epsilon} - \frac{\mu\Delta u_\epsilon}{1+\epsilon\eta_\epsilon} \right) \mathrm{d}x \\
&+ \int \frac{1+\epsilon\eta^\epsilon}{(\gamma-1)(1+\epsilon\phi^\epsilon)} D^\alpha\phi D^\alpha \left(\frac{\epsilon(\nu|\mathrm{curl} H^\epsilon|^2 + \Psi^\epsilon)}{1+\epsilon\eta^\epsilon} \right) \mathrm{d}x \\
&+ \int D^\alpha H \cdot D^\alpha (\nu\Delta H^\epsilon - \nu\Delta H_\epsilon) \mathrm{d}x - \int \frac{1}{\kappa(\gamma-1)(1+\epsilon\phi^\epsilon)} D^\alpha\psi \cdot D^\alpha\psi^\epsilon \mathrm{d}x \\
=& \int (1+\epsilon\eta^\epsilon) D^\alpha u \cdot D^\alpha \left(\frac{\mu\Delta u + (\mu+\lambda)\nabla \mathrm{div} u}{1+\epsilon\eta^\epsilon} \right) \mathrm{d}x \\
&+ \int (1+\epsilon\eta^\epsilon) D^\alpha u \cdot D^\alpha \left(\frac{\mu\Delta u_\epsilon}{1+\epsilon\eta^\epsilon} - \frac{\mu\Delta u_\epsilon}{1+\epsilon\eta_\epsilon} \right) \mathrm{d}x \\
&+ \int \frac{1+\epsilon\eta^\epsilon}{(\gamma-1)(1+\epsilon\phi^\epsilon)} D^\alpha\phi D^\alpha \left(\frac{\epsilon(\nu|\mathrm{curl} H^\epsilon|^2 + \Psi^\epsilon)}{1+\epsilon\eta^\epsilon} \right) \mathrm{d}x \\
&+ \nu \int D^\alpha H \cdot D^\alpha \Delta H \mathrm{d}x - \int \frac{1}{\kappa(\gamma-1)(1+\epsilon\phi^\epsilon)} D^\alpha\psi \cdot D^\alpha\psi^\epsilon \mathrm{d}x \\
:=& \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5.
\end{aligned}
$$

Using integration by parts, Moser-type calculus inequalities and the regularity of U_ϵ, the estimates of $\mathcal{I}_i$, $i=1,2,3,4,5$, can be obtained as follows.

$$
\begin{aligned}
\mathcal{I}_1 =& \int (1+\epsilon\eta^\epsilon) D^\alpha u \cdot D^\alpha \left(\frac{\mu\Delta u + (\mu+\lambda)\nabla \mathrm{div} u}{1+\epsilon\eta^\epsilon} \right) \mathrm{d}x \\
=& -\int \mu |D^\alpha \nabla u|^2 + (\mu+\lambda)|D^\alpha \mathrm{div} u|^2 \mathrm{d}x \\
&+ \int (1+\epsilon\eta^\epsilon) D^\alpha u \cdot [D^\alpha, (1+\epsilon\eta^\epsilon)^{-1}](\mu\Delta u + (\mu+\lambda)\nabla \mathrm{div} u) \mathrm{d}x \\
\le& -\int \mu |D^\alpha \nabla u|^2 + (\mu+\lambda)|D^\alpha \mathrm{div} u|^2 \mathrm{d}x \\
&+ C\|D^\alpha u\| \|\nabla (1+\epsilon\eta^\epsilon)^{-1}\|_{L^\infty} \|D^{s-1}(\mu\Delta u + (\mu+\lambda)\nabla \mathrm{div} u)\| \\
&+ C\|D^\alpha u\| \|\mu\Delta u + (\mu+\lambda)\nabla \mathrm{div} u\|_{L^\infty} \|D^s (1+\epsilon\eta^\epsilon)^{-1}\| \\
\le& -\int \mu |D^\alpha \nabla u|^2 + (\mu+\lambda)|D^\alpha \mathrm{div} u|^2 \mathrm{d}x + C\epsilon \|u\|_s \|\eta^\epsilon\|_s \|D^{s+1} u\| + C\|u\|_s^2 \|\eta^\epsilon\|_s^s \\
\le& -\int \mu |D^\alpha \nabla u|^2 + (\mu+\lambda)|D^\alpha \mathrm{div} u|^2 \mathrm{d}x + \epsilon^2 \|u\|_{s+1}^2 + C(\|E\|_s^4 + \|E\|_s^{s+2}) + O(\epsilon^2), \\
\mathcal{I}_2 =& \int (1+\epsilon\eta^\epsilon) D^\alpha u \cdot D^\alpha \left(\frac{\mu\Delta u_\epsilon}{1+\epsilon\eta^\epsilon} - \frac{\mu\Delta u_\epsilon}{1+\epsilon\eta_\epsilon} \right) \mathrm{d}x \\
\le& C\|D^\alpha u\| \left\| D^\alpha \left(\frac{\epsilon\eta\Delta u_\epsilon}{(1+\epsilon\eta^\epsilon)(1+\epsilon\eta_\epsilon)} \right) \right\| \\
\le& C\|E\|_s^{s+1}, \\
\mathcal{I}_3 =& \int \frac{1+\epsilon\eta^\epsilon}{(\gamma-1)(1+\epsilon\phi^\epsilon)} D^\alpha\phi D^\alpha \left(\frac{\epsilon(\nu|\mathrm{curl} H^\epsilon|^2 + \Psi^\epsilon)}{1+\epsilon\eta^\epsilon} \right) \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \int \frac{\epsilon}{(\gamma-1)(1+\epsilon\phi^\epsilon)} D^\alpha \phi D^\alpha \left(\nu|\mathrm{curl} H^\epsilon|^2 + \Psi^\epsilon\right) \mathrm{d}x \\
&\quad + \int \frac{1+\epsilon\eta^\epsilon}{(\gamma-1)(1+\epsilon\phi^\epsilon)} D^\alpha \phi [D^\alpha, (1+\epsilon\eta^\epsilon)^{-1}] \left(\nu|\mathrm{curl} H^\epsilon|^2 + \Psi^\epsilon\right) \mathrm{d}x \\
&\le C\epsilon^2 (\|D^{s+1} u\|^2 + \|D^{s+1} H\|^2) + C\|D^\alpha \phi\|^2 \\
&\quad + C\|D^\alpha \phi\| \left\| \nabla \left(\frac{1}{1+\epsilon\eta^\epsilon}\right)\right\|_{L^\infty} \left\| D^{s-1} \left(\nu|\mathrm{curl} H^\epsilon|^2 + \frac{\mu}{2}|\nabla u^\epsilon + (\nabla u^\epsilon)^\top|^2 + \lambda |\mathrm{div} u^\epsilon|^2\right)\right\| \\
&\quad + C\|D^\alpha \phi\| \left\| \left(\nu|\mathrm{curl} H^\epsilon|^2 + \frac{\mu}{2}|\nabla u^\epsilon + (\nabla u^\epsilon)^\top|^2 + \lambda |\mathrm{div} u^\epsilon|^2\right)\right\|_{L^\infty} \left\| D^\alpha \left(\frac{1}{1+\epsilon\eta^\epsilon}\right)\right\| \\
&\le C\epsilon^2 (\|u\|_{s+1}^2 + \|H\|_{s+1}^2) + C(\|E\|_s^2 + \|E\|_s^{s+2}) + O(\epsilon^2),
\end{aligned}
$$

$$
\mathcal{I}_4 = \nu \int D^\alpha H \cdot D^\alpha \Delta H \mathrm{d}x = -\nu \int |D^\alpha \nabla H|^2 \mathrm{d}x,
$$

$$
\mathcal{I}_5 = -\int \frac{1}{\kappa(\gamma-1)(1+\epsilon\phi^\epsilon)} D^\alpha \psi \cdot D^\alpha \psi^\epsilon \mathrm{d}x \le C\|E\|_s^2 + O(\epsilon^2).
$$

Collecting the above estimates, we have

$$
\begin{aligned}
\int \langle D^\alpha F, \tilde{A}_0(U^\epsilon) \mathcal{Q}^\alpha \rangle \mathrm{d}x \le & -\int \mu |D^\alpha \nabla u|^2 + (\mu+\lambda)|D^\alpha \mathrm{div} u|^2 + \nu |D^\alpha \nabla H|^2 \mathrm{d}x \\
& + C\epsilon^2 (\|u\|_{s+1}^2 + \|H\|_{s+1}^2) + C(\|E\|_s^2 + \|E\|_s^{s+2}) + O(\epsilon^2).
\end{aligned} \tag{4.17}
$$

Putting (4.14)–(4.17) into (4.10) gives

$$
\begin{aligned}
&\int \langle D^\alpha E, \tilde{A}_0(U^\epsilon)(\mathcal{C}^\alpha + \mathcal{P}^\alpha + \mathcal{Q}^\alpha + \mathcal{R}^\alpha)\rangle \mathrm{d}x \\
&\le -\int \mu |D^\alpha \nabla u|^2 + (\mu+\lambda)|D^\alpha \mathrm{div} u|^2 + \nu |D^\alpha \nabla H|^2 \mathrm{d}x \\
&\qquad + C\epsilon^2 (\|u\|_{s+1}^2 + \|H\|_{s+1}^2) + C(\|E\|_s^2 + \|E\|_s^{2s+2}) + O(\epsilon^2).
\end{aligned} \tag{4.18}
$$

By substituting (4.9) and (4.18) into (4.7), taking summation for all α with $|\alpha| \le s$, and considering ϵ small enough, we obtain that

$$
\frac{\mathrm{d}}{\mathrm{d}t} \sum_{|\alpha| \le s} \|D^\alpha E\|_e^2 + \xi_1 \|u\|_{s+1}^2 + \xi_2 \|H\|_{s+1}^2 \le C(\|E\|_s^2 + \|E\|_s^{2s+2}) + O(\epsilon^2), \tag{4.19}
$$

where ξ_1 and ξ_2 are positive constants. Thanks to (4.8), we integrate the inequality (4.19) over $(0,t)$ with $t < \min\{T^*, T_\epsilon\}$ to obtain

$$
\|E(t)\|_s^2 \le \|E(0)\|_s^2 + C\int_0^t (\|E(\tau)\|_s^2 + \|E(\tau)\|_s^{2s+2}) + O(\epsilon^2) \mathrm{d}t.
$$

Furthermore, with the help of Gronwall's lemma and the initial data condition (1.31), we obtain that

$$
\|E(t)\|_s^2 \le C\epsilon^2 \exp\left\{ C\int_0^t (1 + \|E(\tau)\|_s^{2s}) \mathrm{d}t \right\}. \tag{4.20}
$$

Denote the right-hand side of (4.20) by $\Phi(t)$ and it is easy to see that $\Phi(t)$ satisfies

$$
\|E(t)\|_s^2 \le \Phi(t),
$$

and

$$\Phi'(t) = C(1+\|E(t)\|_s^{2s})\Phi(t) \le C\Phi(t) + C\Phi^{s+1}(t). \tag{4.21}$$

Applying Lemma 2.2 to (4.21), we get

$$\Phi(t) \le e^{CT^*}$$

for all $t \in [0, \min\{T^*, T_\epsilon\})$, provided that $\Phi(0)$ is suitably small, for example, $\Phi(0) = C\epsilon^2 < \exp\{-CT^*\}$. As a result, we conclude that $\|E(t)\|_s$ is uniformly bounded in $[0, \min\{T^*, T_\epsilon\})$ and the estimate (3.6) holds from (4.20). Thus, the proof of Lemma 3.2 is completed.

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