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STABILITY OF LARGE AMPLITUDE VISCOUS SHOCK WAVE FOR 1-D ISENTROPIC NAVIER-STOKES SYSTEM IN THE HALF SPACE*

LIN CHANG†

Abstract. In this paper, the asymptotic-time behavior of solutions to an initial boundary value problem in the half space for 1-D isentropic Navier-Stokes system is investigated. It is shown that the viscous shock wave is stable for an impermeable wall problem where the velocity is zero on the boundary provided that the shock wave is initially far away from the boundary. Moreover, the strength of the shock wave could be arbitrarily large. This work essentially improves the result of [A. Matsumura and M. Mei, Arch. Ration. Mech. Anal., 146(1):1–22, 1999], where the strength of the shock wave is sufficiently small.

Keywords. Impermeable wall problem; large amplitude shock; asymptotic stability.

AMS subject classifications. 35Q30; 76N10.

1. Introduction

We consider a 1-D isentropic Navier-Stokes system for general viscous gas, which reads in the Lagrangian coordinate as,

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = (\mu(v)\frac{u_x}{v})_x, \end{cases} \tag{1.1}$$

where $t>0, x\in\mathbb{R}_+$, and $v(x,t)=\frac{1}{\rho(x,t)}$ is the specific volume, $u(x,t)$ the fluid velocity, $p=av^{-\gamma}$ the pressure with constant $a>0$, $\gamma>1$ the adiabatic constant, and $\mu(v)=\mu_0 v^{-\alpha}$ the viscosity coefficient with $\alpha\geq 0$. When the viscosity $\mu(v)\equiv 0$, the system (1.1) becomes the famous Euler system

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \end{cases} \tag{1.2}$$

that has rich wave phenomena such as shock and rarefaction waves. When $\mu(v)>0$, the shock wave is mollified as the so-called viscous shock wave. Without loss of generality, we assume $\mu_0=1$ in what follows.

Since the system (1.1) is regular than the Euler system (1.2), it is very interesting and important to study the stability of the viscous version of the shock wave, i.e., the viscous shock wave, for the viscous conservation laws such as the NS system (1.1) with the initial data:

$$(v,u)(x,0)=(v_0,u_0)(x)\longrightarrow(v_\pm,u_\pm), \quad \text{as} \quad x\to\infty. \tag{1.3}$$

The stability of viscous shock wave for the Cauchy problem (1.1), (1.3) has been extensively studied in a large amount of literature since the pioneering works of [2,13], see the other interesting works [1,4–9,11,14,17]. It is noted that most of the above works require the strength of the shock wave to be suitably small, that is, the shock must be weak. The stability of a large amplitude shock (strong shock) is more interesting and challenging in both mathematics and physics, see the works [3,6,10,13,16,18,19].

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†School of Mathematics Science, Beihang University, Beijing, China (changlin23@buaa.edu.cn).

It is shown by Matsumura-Nishihara [13] that the viscous shock wave is stable if $|v_+ - v_-| < C(\gamma-1)^{-1}$, that is, when $\gamma \to 1$, the strength of the shock wave could be large. This condition is relaxed in [6] to the condition that $|v_+ - v_-| < C(\gamma-1)^{-2}$ later. Recently, the restriction on the strength of the shock was removed in [16] by an elegant weighted energy method as $\alpha > \frac{\gamma-1}{2}$. Vasseur-Yao [18] removed the condition $\alpha > \frac{\gamma-1}{2}$ by introducing a beautiful variable transformation. Moreover, He-Huang [3] extended the result of [18] to general pressure $p(v)$ and general viscosity $\mu(v)$, where $\mu(v)$ could be any positive smooth function.

On the other hand, it is also interesting to investigate the stability of viscous shock wave under the effect of a boundary. In 1999, Matsumura-Mei [12] considered an impermeable wall problem of (1.1) in the half space $x \geq 0$, i.e.,

$$\begin{cases} (v,u)(x,0) = (v_0,u_0)(x) \longrightarrow (v_+,u_+), \ x \to +\infty, \\ u(0,t) = 0, \quad t \in \mathbb{R}_+, \end{cases} \tag{1.4}$$

where $v_+ > 0, u_+ < 0$. The impermeable wall means that there is no flow across the boundary so that the velocity at the boundary $x = 0$ has to be zero. It was proved in [12] that the solution of (1.1), (1.4), with $\alpha = 0$, time-asymptotically tends to an outgoing shock wave (2-shock) connecting the left state $(v_-, 0)$ and the right one (v_+, u_+) if $|v_+ - v_-| < C(\gamma-1)^{-2}$, and the outgoing shock is initially far away from the boundary so that the interaction between the shock and the boundary is weak, where v_- is determined by the RH condition, i.e.,

$$\begin{cases} -s(v_+ - v_-) - (u_+ - u_-) = 0, \\ -s(u_+ - u_-) + (p(v_+) - p(v_-)) = 0, \end{cases} \tag{1.5}$$

with $u_- = 0$. Matsumura-Nishihara [15] removed the condition that the shock is initially far away from the boundary by extending the half space to the whole space, with the price that the shock wave has to be weak even for $\gamma = 1$ case.

In this paper, we aim to prove that the large-amplitude shock wave is still stable for the impermeable wall problem (1.1)-(1.4). Roughly speaking, there exists a 2-viscous shock wave (outgoing shock) (V_2, U_2) connecting $(v_-, 0)$ and (v_+, u_+) with v_- determined by the RH condition (1.5), and (V_2, U_2) is asymptotically stable if it is initially far away from the boundary. The precise statement of the main result is given in Theorem 2.1.

We outline the strategy as follows. Motivated by [18] and [3], we introduce a new variable $h = u - v^{(-\alpha+1)} v_x$ and formulate a new equation $(4.2)_2$ in which the viscous term is moved to the mass equation $(4.2)_1$ so that the two nonlinear terms p_x and $(\frac{v_x}{v^{\alpha+1}})_x$ are decoupled and the interaction between nonlinear terms is weakened. Since the strength of outgoing shock is arbitrarily large, the interaction between the 2-shock and the boundary $x = 0$ is strong. We have to assume that the outgoing shock is initially far away from the boundary so that the interaction is weak. Since the boundary terms with first-order derivatives are controlled, we can obtain the low order estimates through careful analysis. But the idea using the new system (4.2) does not work in the higher order estimation since it is very difficult to control the second-order derivatives of boundary terms for the new system. Note that the second derivatives of u on the boundary can be controlled, we then turn to original system (1.1) to obtain the higher order energy estimates, and finally complete the a priori estimates.

The rest of the paper will be arranged as follows. In Section 2, the outgoing shock wave is formulated and the main result is stated. In Section 3, the problem is reformulated by the anti-derivatives of the perturbations around the viscous shock wave.

In Section 4, the a priori estimates are established. In Section 5, the main theorem is proved.

Notation. The functional $\|\cdot\|_{L^p(\Omega)}$ is defined by $\|f\|_{L^p(\Omega)}=(\int_\Omega |f|^p(\xi)\,\mathrm{d}\xi)^{\frac{1}{p}}$. The symbol Ω is often omitted, when $\Omega=(0,\infty)$. As $p=2$, for simplicity we denote,

$$\|f\|=\left(\int_0^\infty f^2(\xi)\,\mathrm{d}\xi\right)^{\frac{1}{2}}.$$

In addition, H^m denotes the m-th order Sobolev space of functions defined by

$$\|f\|_m=\left(\sum_{k=0}^m\|\partial_\xi^k f\|^2\right)^{\frac{1}{2}}.$$

2. Preliminaries and main theorem

2.1. Viscous shock profile and location of the shift. As pointed out by [12], the solution of the impermeable wall problem (1.1)-(1.4) is expected to tend toward the outgoing viscous shock $(V,U)(\xi)$ satisfying

$$\begin{cases} -sV'-U'=0,\\ -sU'+p(V)'=\left(\frac{U'}{V^{\alpha+1}}\right)',\\ (V,U)(-\infty)=(v_-,0),\quad (V,U)(+\infty)=(v_+,u_+), \end{cases} \tag{2.1}$$

where $'=\mathrm{d}/\mathrm{d}\xi$, $\xi=x-st$, s is the shock speed determined by the RH condition (1.5) and $v_\pm>0, u_+<0$ are given constants. From $(2.1)_1$ and $(2.1)_2$, one gets

$$s^2V'+p(V)'=-\left(\frac{sV'}{V^{\alpha+1}}\right)'. \tag{2.2}$$

Integrating (2.2) over $(\pm\infty,\xi)$ gives

$$\frac{sV'}{V^{\alpha+1}}=-s^2V-p(V)-b=:h(V), V(\pm\infty)=v_\pm, \tag{2.3}$$

$$U=-s(V-v_-)=-s(V-v_+)+u_+, \tag{2.4}$$

where $b=-s^2v_\pm-p(v_\pm)$.

Proposition 2.1 ([12]). *There exists a unique viscous shock profile $(V,U)(\xi)$ up to a shift satisfying*

$$0<v_-<V(\xi)<v_+,\quad h(V)>0,\quad U'<0, \tag{2.5}$$

$$|V(\xi)-v_\pm|=O(1)|v_+-v_-|e^{-C_\pm|\xi|}, \tag{2.6}$$

as $\xi\to\pm\infty$, where $C_\pm=\frac{v_\pm^{\alpha+1}}{s}|p'(v_\pm)+s^2|$, $s=\frac{-u_+}{v_+-v_-}$.

We expect $\int_0^\infty[v(x,t)-V(x-st+\beta_0-\beta)]\,\mathrm{d}x\to 0$ as $t\to\infty$. As in [12], the shift of the viscous shock profile is given by

$$\beta_0=\frac{1}{v_+-v_-}\left\{\int_0^\infty[v_0(x)-V(x-\beta)]\,\mathrm{d}x+\int_0^\infty U(-st-\beta)\,\mathrm{d}t\right\}. \tag{2.7}$$

2.2. Main theorem. We assume that for $\beta>0$, the initial data satisfies

$$v_0(x)-V(x-\beta)\in H^1\cap L^1 \quad u_0(x)-U(x-\beta)\in H^1\cap L^1, \tag{2.8}$$

and

$$u_0(0)=0 \tag{2.9}$$

as the compatibility condition. Set

$$(A_0,B_0)(x):=-\int_x^\infty (v_0(y)-V(y-\beta),u_0(y)-U(y-\beta))\,\mathrm{d}y.$$

We further assume that

$$(A_0,B_0)\in L^2. \tag{2.10}$$

The shift β_0 has the following properties.

LEMMA 2.1 ([12]). *Under the assumptions (2.8)-(2.10), the shift β_0 defined by (2.7) satisfies*

$$\beta_0\to 0 \quad as \quad \|A_0,B_0\|_2\to 0 \quad and \quad \beta\to+\infty.$$

The main theorem is stated as follows.

THEOREM 2.1. *For any $u_+<0$ and $v_+>0$, suppose that (2.8)-(2.10) hold. Then there exists a positive constant δ_0 such that if*

$$\|(A_0,B_0)\|_2+\beta^{-1}\le\delta_0,$$

then the initial-boundary value problem (1.1), (1.4) has a unique global solution $(v,u)(x,t)$, satisfying

$$\begin{aligned} &v(x,t)-V(x-st+\beta_0-\beta)\in C^0([0,+\infty);H^1)\cap L^2([0,+\infty);H^1),\\ &u(x,t)-U(x-st+\beta_0-\beta)\in C^0([0,+\infty);H^1)\cap L^2([0,+\infty);H^2), \end{aligned} \tag{2.11}$$

where $s>0$ is defined by (1.5), and

$$\sup_{x\in\mathbb{R}_+}|(v,u)(x,t)-(V,U)(x-st+\beta_0-\beta)|\to 0,\ as\ t\to+\infty. \tag{2.12}$$

REMARK 2.1. The condition $v_+-v_-<C(\gamma-1)^{-2}$ in [12] is removed.

3. Reformulation of the original problem

Set

$$\begin{aligned} \phi(x,t)&=-\int_x^\infty v(y,t)-V(y-st+\beta_0-\beta)\,\mathrm{d}y,\\ \psi(x,t)&=-\int_x^\infty u(y,t)-U(y-st+\beta_0-\beta)\,\mathrm{d}y, \end{aligned} \tag{3.1}$$

which means that we look for the solution $(v,u)(x,t)$ in the form

$$v(x,t)=\phi_x(x,t)+V(x-st+\beta_0-\beta),$$

$$u(x,t) = \psi_x(x,t) + U(x - st + \beta_0 - \beta). \tag{3.2}$$

The initial perturbations ϕ and ψ satisfy

LEMMA 3.1 ([12]). *Under the assumptions (2.8)-(2.10), the initial perturbation* $(\phi,\psi)(x,0) := (\phi_0,\psi_0)(x) \in H^2$ *and satisfies*

$$\|(\phi_0,\psi_0)\|_2 \to 0 \quad \text{as} \quad \|(A_0,B_0)\|_2 \to 0 \text{ and } \beta \to +\infty.$$

Motivated by [12], substituting (3.2) into (1.1) and integrating the resulting system with respect to x, we have

$$\begin{cases} \phi_t - \psi_x = 0, \\ \psi_t - f(V)\phi_x - \frac{\psi_{xx}}{V^{\alpha+1}} = F, \end{cases} \tag{3.3}$$

with the initial conditions and Neumann boundary condition:

$$\begin{aligned} &(\phi_0,\psi_0)(x) \in H^2, \quad x \geq 0, \\ &\psi_x|_{x=0} = \phi_t|_{x=0} = -U(st + \beta_0 - \beta), \quad t \geq 0, \end{aligned} \tag{3.4}$$

where

$$f(V) = -p'(V) + (\alpha+1)\frac{sV_x}{V^{\alpha+2}} = -p'(V) + (\alpha+1)\frac{h(V)}{V} > 0, \tag{3.5}$$

$$\begin{aligned} F &= \frac{u_x}{v^{\alpha+1}} - \frac{U_x}{V^{\alpha+1}} - \frac{\psi_{xx}}{V^{\alpha+1}} + (\alpha+1)\frac{U_x\phi_x}{V^{\alpha+2}} - [p(v) - p(V) - p'(V)\phi_x] \\ &= O(1)(|\phi_x|^2 + |\phi_x\psi_{xx}|). \end{aligned} \tag{3.6}$$

We will seek the solution in the functional space $X_\delta(0,T)$ for any $0 \leq T < +\infty$,

$$\begin{aligned} X_\delta(0,T) := \{ &(\phi,\psi) \in C([0,T];H^2) | \phi_x \in L^2(0,T;H^1), \psi_x \in L^2(0,T;H^2) \\ &\sup_{0\leq t\leq T} \|(\phi,\psi)(t)\|_2 \leq \delta \}, \end{aligned}$$

where $\delta \ll 1$ is small.

PROPOSITION 3.1 (A priori estimate). *Suppose that* $(\phi,\psi) \in X_\delta(0,T)$ *is the solution of (3.3), (3.4) for some time* $T > 0$. *There exists a positive constant* δ_0 *independent of* T, *such that if*

$$\sup_{0\leq t\leq T} \|(\phi,\psi)(t)\|_2 \leq \delta \leq \delta_0,$$

for $t \in [0,T]$, *then*

$$\|(\phi,\psi)(t)\|_2^2 + \int_0^t (\|\phi_x(t)\|_1^2 + \|\psi_x(t)\|_2^2)\,\mathrm{d}t \leq C_0(\|(\phi_0,\psi_0)\|_2^2 + e^{-C_-\beta}),$$

where $C_0 > 1$ *and* C_- *are positive constants independent of* T.

As long as Proposition 3.1 holds, the local solution (ϕ,ψ) can be extended to $T = +\infty$. We have the following lemma.

LEMMA 3.2. *If $(\phi_0,\psi_0)\in H^2$, there exists a positive constant $\delta_1=\frac{\delta_0}{\sqrt{C_0}}$, such that if*

$$\|(\phi_0,\psi_0)\|_2^2+e^{-C_-\beta}\le\delta_1^2,$$

then the initial-boundary problem (3.3), (3.4) has a unique global solution $(\phi,\psi)\in X_{\delta_0}(0,\infty)$ satisfying

$$\sup_{t\ge0}\|(\phi,\psi)(t)\|_2^2+\int_0^\infty(\|\phi_x(t)\|_1^2+\|\psi_x(t)\|_2^2)\mathrm{d}t\le C_0(\|(\phi_0,\psi_0)\|_2^2+e^{-C_-\beta}).$$

4. A priori estimate

Throughout this section, we assume that the problem (3.3),(3.4) has a solution $(\phi,\psi)\in X_\delta(0,T)$, for some $T>0$,

$$\sup_{0\le t\le T}\|(\phi,\psi)(t)\|_2\le\delta. \tag{4.1}$$

It follows from the Sobolev inequality that $\frac{1}{2}v_+\le v\le\frac{3}{2}v_-$, and

$$\sup_{0\le t\le T}\{\|(\phi,\psi)(t)\|_{L^\infty}+\|(\phi_x,\psi_x)(t)\|_{L^\infty}\}\le\delta.$$

4.1. Low order estimate. In order to remove the condition $v_+-v_-<C(\gamma-1)^{-2}$ in [12], we introduce a new perturbation (ϕ,Ψ) instead of (ϕ,ψ), where Ψ will be defined below.

Inspired by [18] and [3], we introduce a new variable h which depends on v and u, i.e., $h=u-v^{-(\alpha+1)}v_x$. Through a direct calculation, v and h satisfy the following system

$$\begin{cases} v_t-h_x=(\frac{v_x}{v^{\alpha+1}})_x,\\ h_t+p_x=0.\end{cases} \tag{4.2}$$

Then the initial-boundary conditions given in (1.4) are changed into

$$\begin{cases} (v,h)(x,0)=(v_0,u_0-{v_0}^{-(\alpha+1)}v_{0x})(x)\longrightarrow(v_+,u_+),\ x\to+\infty,\\ h(0,t)=u(0,t)-v(0,t)^{-(\alpha+1)}v_x(0,t)=-v(0,t)^{-(\alpha+1)}v_x(0,t),t\in\mathbb{R}_+.\end{cases}$$

Let $H=U-V^{-(\alpha+1)}V_x$. Then (2.1) is equivalent to

$$\begin{cases} V_t-H_x=\left(\frac{V_x}{V^{\alpha+1}}\right)_x,\\ H_t+p(V)_x=0,\\ (V,H)(-\infty)=(v_-,0),\quad (V,H)(+\infty)=(v_+,u_+).\end{cases} \tag{4.3}$$

We define

$$-\int_x^\infty(h-H)\mathrm{d}x=\Psi. \tag{4.4}$$

Substituting (4.3) from (4.2) and integrating the resulting system with respect to x, we have from (4.4), $(3.1)_1$ that

$$\begin{cases} \phi_t-\Psi_x-\frac{\phi_{xx}}{V^{\alpha+1}}+(\alpha+1)\frac{V_x\phi_x}{V^{\alpha+2}}=G,\\ \Psi_t+p'(V)\phi_x=-p(v|V),\end{cases} \tag{4.5}$$

where

$$G=\frac{v_x}{v^{\alpha+1}}-\frac{V_x}{V^{\alpha+1}}-\frac{\phi_{xx}}{V^{\alpha+1}}+(\alpha+1)\frac{V_x\phi_x}{V^{\alpha+2}},$$

$$p(v|V)=(p(v)-p(V))-p'(V)\phi_x,$$

with the initial data

$$\phi(x,0)\in H^2,\quad \Psi(x,0)\in H^1,$$

and boundary data

$$\Phi(0,t)=-\int_x^\infty [u(y,0)-U(y+\beta_0-\beta)]\,\mathrm{d}y+\left(V^{-(\alpha+1)}-v^{-(\alpha+1)}\right)(x,0).$$

Lemma 4.1 ([3]). *Under the assumption of (4.1), it holds that*

$$\begin{aligned}&p(v|V)\le C\phi_x^2,\\ &|p(v|V)_x|\le C(|\phi_{xx}\phi_x|+|V_x|\phi_x^2),\\ &|G|\le C(|\phi_{xx}\phi_x|+|V_x|\phi_x^2).\end{aligned}$$

In addition, some boundary estimates are given as follows.

Lemma 4.2. *Under the same assumptions of Proposition 3.1, for $0\le t\le T$, it holds that:*

$$\left|\int_0^t(\phi\Psi)|_{x=0}\,\mathrm{d}t\right|\le Ce^{-C_-\beta},\qquad \left|\int_0^t(\phi\phi_x)|_{x=0}\,\mathrm{d}t\right|\le Ce^{-C_-\beta},\tag{4.6}$$

$$\left|\int_0^t(\phi_x\phi_t)|_{x=0}\,\mathrm{d}t\right|\le Ce^{-C_-\beta},\qquad \left|\int_0^t(\psi_x\psi_t)|_{x=0}\,\mathrm{d}t\right|\le Ce^{-C_-\beta},\tag{4.7}$$

$$\left|\int_0^t(\psi_x\psi_{xx})|_{x=0}\,\mathrm{d}t\right|\le Ce^{-C_-\beta},\qquad \left|\int_0^t(\psi_{xt}\psi_{xx})|_{x=0}\,\mathrm{d}t\right|\le Ce^{-C_-\beta},\tag{4.8}$$

and

$$\begin{aligned}&\|\Psi_0\|_1^2\le\|\psi_0\|_1^2+C\|\phi_0\|_2^2,\quad \|\psi\|^2\le\|\Psi\|^2+C\|\phi\|_1^2,\\ &\|\psi_x\|^2\le\|\Psi_x\|^2+C\|\phi_x\|_1^2,\end{aligned}\tag{4.9}$$

where $C_-=\frac{v_-^{\alpha+1}}{s}|p'(v_-)+s^2|>0$.

Proof. Note that

$$\begin{aligned}\Psi(x,t)&=-\int_x^\infty[u(y,t)-U(y-st+\beta_0-\beta)]\,\mathrm{d}y\\ &\qquad+(g(v(x,t))-g(V(x-st+\beta_0-\beta)))\\ &:=\psi(x,t)+p(x,t)\le\psi(x,t)+C|\phi_x(x,t)|,\\ \psi(x,t)&=\Psi(x,t)-p(x,t)\le\psi(x,t)+C|\phi_x(x,t)|,\end{aligned}\tag{4.10}$$

where $g(v)=\frac{1}{\alpha}v^{-\alpha}$, if $\alpha\ne0$; $g(v)=-\ln v$, if $\alpha\ne0$, one has (4.9) from (4.10) immediately. Motivated by [12], we have

$$|\psi(0,t)|\le C,\quad |\phi_x(0,t)|\le C,\quad |\phi(0,t)|\le Ce^{-C_-\beta}e^{-C_-st}.\tag{4.11}$$

Combining (4.10) and (4.11), we have (4.6). The estimates (4.7) and (4.8) can be found in [12]. Thus the proof is completed. ∎

LEMMA 4.3. *Under the same assumptions of Proposition 3.1, it holds that*

$$\|(\phi,\Psi)\|^2(t)+\int_0^t\int_0^\infty\left(\frac{1}{p'(V)}\right)_t\Psi^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^t\|\phi_x\|^2\,\mathrm{d}t$$
$$\le C\|(\phi_0,\Psi_0)\|^2+C\delta\int_0^t\|\phi_{xx}\|^2\,\mathrm{d}t+Ce^{-C_-\beta}.$$

Proof. Multiply $(4.5)_1$ and $(4.5)_2$ by ϕ and $\frac{\Psi}{-p'(V)}$ respectively, sum them up, and integrate the result with respect to t and x over $[0,t]\times[0,\infty)$. We have

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty\left(\phi^2-\frac{\Psi^2}{p'(V)}\right)\mathrm{d}x+\int_0^t\int_0^\infty\left\{\frac{1}{2}\left(\frac{1}{p'(V)}\right)_t\Psi^2+\frac{\phi_x^2}{V^{\alpha+1}}\right\}\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^t\int_0^\infty G\phi\,\mathrm{d}x\,\mathrm{d}t+\int_0^t\int_0^\infty\frac{p(v|V)\Psi}{p'(V)}\mathrm{d}x\,\mathrm{d}t\\
&\quad-\int_0^t(\phi\Psi+(V^{-(\alpha+1)})\phi\phi_x)|_{x=0}\,\mathrm{d}t+\frac{1}{2}\int_0^\infty\left(\phi^2-\frac{\Psi^2}{p'(V)}\right)\Big|_{t=0}\mathrm{d}x\\
&=:\sum_{i=1}^4 A_i.
\end{aligned}\tag{4.12}$$

Utilizing Lemma 4.1, we can get

$$\begin{aligned}
&|A_1+A_2|\\
&\le C\left(\int_0^t\int_0^\infty|\phi_x\phi_{xx}\phi|+\left|V_x\phi_x^2\phi\right|+\left|\Psi\phi_x^2\right|\mathrm{d}x\,\mathrm{d}t\right)\\
&\le C\int_0^t\|\phi\|_{L^\infty}\int_0^\infty|\phi_x\phi_{xx}|\,\mathrm{d}x\,\mathrm{d}t+C\int_0^t(\|\phi\|_{L^\infty}+\|\Psi\|_{L^\infty})\int_0^\infty\phi_x^2\,\mathrm{d}x\,\mathrm{d}t\\
&\le C(\|\phi\|_2+\|\psi\|_1)\int_0^t\|\phi_x\|^2+\|\phi_{xx}\|^2\,\mathrm{d}t\\
&\le C\delta\int_0^t\|\phi_x\|^2+\|\phi_{xx}\|^2\,\mathrm{d}t.
\end{aligned}\tag{4.13}$$

With the help of Lemma 4.2, one has

$$|A_3|\le Ce^{-C_-\beta}.\tag{4.14}$$

Taking δ sufficiently small, using (4.12)–(4.14), we get Lemma 4.3. ∎

LEMMA 4.4. *Under the same assumptions of Proposition 3.1, it holds that*

$$\|(\phi,\Psi)(t)\|_1^2+\int_0^t\|\phi_x\|_1^2\,\mathrm{d}t\le C\|(\phi_0,\Psi_0)\|_1^2+Ce^{-C_-\beta}.$$

Proof. Multiply $(4.5)_1$ and $(4.5)_2$ by $-\phi_{xx}$ and $\frac{\Psi_{xx}}{p'(V)}$ respectively, sum over the result, integrate the result with respect to t and x over $[0,t]\times[0,\infty)$. We have

$$\frac{1}{2}\int_0^\infty\left(\phi_x^2-\frac{\Psi_x^2}{p'(V)}\right)\mathrm{d}x+\int_0^t\int_0^\infty\left\{\frac{1}{2}\left(\frac{1}{p'(V)}\right)_t\Psi_x^2+\frac{\phi_{xx}^2}{V^{\alpha+1}}\right\}\mathrm{d}x\,\mathrm{d}t$$

$$
\begin{aligned}
=&\frac{1}{2}\int_0^\infty\left(\phi_x^2-\frac{\Psi_x^2}{p'(V)}\right)\Big|_{t=0}\mathrm{d}x-\int_0^t\int_0^\infty\left[G-(\alpha+1)\frac{V_x}{V^{\alpha+2}}\phi_x\right]\phi_{xx}\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^t\int_0^\infty\left(\frac{1}{p'(V)}\right)_x p'(V)\Psi_x\phi_x\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^t\left(\phi_t\phi_x-\phi_x\Psi_x-\frac{\Psi_t\Psi_x}{p'(V)}-\frac{p(v|V)}{p'(V)}\Psi_x\right)\Big|_{x=0}\mathrm{d}t\\
&+\int_0^t\int_0^\infty\frac{1}{p'(V)}p(v|V)_x\Psi_x\,\mathrm{d}x\,\mathrm{d}t\\
=:&\frac{1}{2}\int_0^\infty\left(\phi_x^2-\frac{\Psi_x^2}{p'(V)}\right)\Big|_{t=0}\mathrm{d}x+\sum_{i=1}^4 B_i.
\end{aligned}
\tag{4.15}
$$

Now we estimate B_i term by term. The Cauchy inequality indicates that

$$
\begin{aligned}
|B_1|\le& C\int_0^t\int_0^\infty(|\phi_{xx}\phi_x|+|V_x\phi_x^2|)|\phi_{xx}|+|\phi_x\phi_{xx}|\,\mathrm{d}x\,\mathrm{d}t\\
\le&(C\delta+\varepsilon)\int_0^t\|\phi_{xx}\|^2\,\mathrm{d}t+C_\varepsilon\int_0^t\|\phi_x\|^2\,\mathrm{d}t,
\end{aligned}
\tag{4.16}
$$

and

$$
\begin{aligned}
|B_2|\le&\int_0^t\int_0^\infty\left|p'(V)\Psi_x\phi_x\left(\frac{1}{p'(V)}\right)_x\right|\mathrm{d}x\,\mathrm{d}t\\
\le&\frac{1}{4}\int_0^t\int_0^\infty\left(\frac{1}{p'(V)}\right)_t\Psi_x^2\,\mathrm{d}x\,\mathrm{d}t+C\int_0^t\|\phi_x\|^2\,\mathrm{d}t.
\end{aligned}
\tag{4.17}
$$

Making use of the estimate (4.7) for the boundary, we have

$$
\begin{aligned}
B_3=&-\int_0^t\left(\phi_t\phi_x-\phi_x\Psi_x-\frac{\Psi_t\Psi_x}{p'(V)}-\frac{p(v|V)}{p'(V)}\Psi_x\right)\Big|_{x=0}\mathrm{d}t\\
=&-\int_0^t(\phi_t\phi_x)|_{x=0}\le Ce^{-C_-\beta}.
\end{aligned}
\tag{4.18}
$$

By (4.9) and the Sobolev inequality, we obtain

$$
\begin{aligned}
|B_4|\le&\int_0^t\int_0^\infty\left|\frac{1}{p'(V)}p(v|V)_x\Psi_x\right|\mathrm{d}x\,\mathrm{d}t\\
\le&C\int_0^t\int_0^\infty\left|(\phi_x\phi_{xx}+V_x\phi_x^2)\Psi_x\right|\mathrm{d}x\,\mathrm{d}t\\
\le&C\int_0^t\int_0^\infty\left\{\left|(\phi_x\phi_{xx}+V_x\phi_x^2)\psi_x\right|+|(\phi_{xx}\phi_{xx}+V_x\phi_x\phi_{xx})\phi_x|\right\}\mathrm{d}x\,\mathrm{d}t\\
\le&C(\|\phi\|_2+\|\psi\|_2)\int_0^t\|\phi_x\|^2+\|\phi_{xx}\|^2\,\mathrm{d}t\\
\le&C\delta\int_0^t(\|\phi_{xx}\|^2+\|\phi_x\|^2)\,\mathrm{d}t.
\end{aligned}
\tag{4.19}
$$

From (4.15)–(4.19), we get

$$\begin{aligned}&\frac{1}{2}\int_0^\infty\left(\phi_x^2-\frac{\Psi_x^2}{p'(V)}\right)\mathrm{d}x+\frac{1}{4}\int_0^t\int_0^\infty\left[\left(\frac{1}{p'(V)}\right)_t\Psi_x^2+\frac{\phi_{xx}^2}{V^{\alpha+1}}\right]\mathrm{d}x\mathrm{d}t\\ \le&(C+C\delta+C_\varepsilon)\int_0^t\|\phi_x\|^2\mathrm{d}t+(C\delta+\varepsilon)\int_0^t\|\phi_{xx}\|^2\mathrm{d}t\\ &+Ce^{-C_-\beta}+C\left(\|\phi_{0x}\|^2+\|\Psi_{0x}\|^2\right).\end{aligned}$$

Choosing ε sufficiently small, together with Lemma 4.3, we complete the proof of Lemma 4.4. ∎

LEMMA 4.5. *Under the same assumptions of Proposition 3.1, it holds that*

$$\int_0^t\|\Psi_x(t)\|^2\mathrm{d}t\le C\|(\phi_0,\Psi_0)\|_1^2+Ce^{-C_-\beta}.$$

Proof. Multiply $(4.5)_1$ by Ψ_x and make use of $(4.5)_2$. We get

$$\begin{aligned}\Psi_x^2=&(\phi\Psi_x)_t+[\phi(p(v)-p(V))]_x-\phi_x(p(v)-p(V))\\ &-\frac{\Psi_x\phi_{xx}}{V^{\alpha+1}}-\Psi_x\left[G-(\alpha+1)\frac{V_x\phi_x}{V^{\alpha+2}}\right].\end{aligned}\tag{4.20}$$

Integrate (4.20) with respect to t and x over $[0,t]\times[0,\infty)$. We have

$$\begin{aligned}&\int_0^t\|\Psi_x\|^2\mathrm{d}t\\ =&-\int_0^\infty\phi\Psi_x|_{t=0}\mathrm{d}x-\int_0^t\int_0^\infty\Psi_x\left[G-(\alpha+1)\frac{V_x\phi_x}{V^{\alpha+2}}\right]\mathrm{d}x\mathrm{d}t\\ &+\int_0^\infty\phi\Psi_x\mathrm{d}x-\int_0^t\int_0^\infty\frac{\Psi_x\phi_{xx}}{V^{\alpha+1}}\mathrm{d}x\mathrm{d}t\\ &-\int_0^t\int_0^\infty\phi_x(p(v)-p(V))\mathrm{d}x\mathrm{d}t-\int_0^t\phi(p(v)-p(V))|_{x=0}\mathrm{d}t\\ =:&-\int_0^\infty\phi\Psi_x|_{t=0}\mathrm{d}x+\sum_{i=1}^5H_i.\end{aligned}\tag{4.21}$$

We estimate H_i term by term. By the Cauchy inequality, it holds that

$$\begin{aligned}H_1\le&C\int_0^t\int_0^\infty\Psi_x(|\phi_x\phi_{xx}|+|V_x\phi_x|)\mathrm{d}x\mathrm{d}t\\ \le&\varepsilon\int_0^t\|\Psi_x\|^2\mathrm{d}t+C_\varepsilon\int_0^t(\|\phi_{xx}\|^2+\|\phi_x\|^2)\mathrm{d}t.\end{aligned}\tag{4.22}$$

In addition, it is straightforward to imply that

$$\begin{aligned}&H_2+H_3+H_4\\ \le&\|(\phi,\Psi_x)\|^2+\varepsilon\int_0^t\|\Psi_x\|^2\mathrm{d}t+C_\varepsilon\int_0^t\|\phi_{xx}\|^2\mathrm{d}t+C\int_0^t\|\phi_x\|^2\mathrm{d}t.\end{aligned}\tag{4.23}$$

Making use of the estimate (4.6) for the boundary, we have

$$H_5 = -\int_0^t \phi(p(v)-p(V))|_{x=0}\,\mathrm{d}t \le C\int_0^t \phi\phi_x|_{x=0}\,\mathrm{d}t \le Ce^{-C_-\beta}. \tag{4.24}$$

Collecting (4.21)-(4.24) and using Lemma 4.4, we complete the proof of Lemma 4.5. □

Combining Lemmas 4.3-4.5, we obtain the following low order estimates

$$\|(\phi,\Psi)\|_1^2(t) + \int_0^t \|\Psi_x\|^2\,\mathrm{d}t + \int_0^t \|\phi_x\|_1^2\,\mathrm{d}t \le C\|(\phi_0,\Psi_0)\|_1^2 + Ce^{-C_-\beta}. \tag{4.25}$$

If we continue to get the estimates of second-order derivatives ϕ_{xx},Ψ_{xx}, new difficulties arise from the boundary. In fact, multiplying the result $\{(4.5)_1\}_{xx}$ by ϕ_{xx} and $\{(4.5)_2\}_{xx}\times(-p'(V))^{-1}\Psi_{xx}$, we obtain

$$\begin{aligned}
&\left(\frac{\phi_{xx}^2}{2} - \frac{\Psi_{xx}^2}{2p'(V)}\right)_t + \left(\frac{1}{V^{\alpha+1}}\right)\phi_{xxx}^2 - \frac{s}{2}\left(\frac{1}{2p'(V)}\right)_x \Psi_{xx}^2 \\
&= \Bigg\{ G_{xx}\phi_{xx} + \Big(\frac{1}{V^{\alpha+1}}\Big)_{xxx}\phi_x\phi_{xx} + 2\Big(\frac{1}{V^{\alpha+1}}\Big)_{xx}\phi_{xx}^2 + \frac{p(v|V)_{xx}}{p'(V)}\Psi_{xx} \\
&\quad -2\left(\frac{1}{p'(V)}\right)_x (p'(V)\phi_x)_x\Psi_{xx} - \left(\frac{1}{p'(V)}\right)_{xx} p'(V)\phi_x\Psi_{xx} \Bigg\} \\
&\quad + \left\{ \left(\left(\frac{\phi_{xx}}{V^{\alpha+1}}\right)_x \phi_{xx} + \phi_{xx}\Psi_{xx} \right)_x \right\} := I_1 + I_2.
\end{aligned}$$

The last term I_2 vanishes after integration for the Cauchy problem. However, both ϕ_{xx} and Ψ_{xx} are unknown on the boundary and appear after integration for the initial-boundary problem. In particular, Ψ_{xx} contains ϕ_{xxx} and it is very difficult to estimate $\phi_{xx}\phi_{xxx}$ on the boundary. Thus, we turn to the original equation (3.3) to study the higher order estimates. Inequality (4.25) can be rewritten by the variables ϕ and ψ as

LEMMA 4.6. *Under the same assumptions of Proposition 3.1, it holds that*

$$(\|\phi\|_1^2 + \|\psi\|^2)(t) + \int_0^t \|\psi_x\|^2\,\mathrm{d}t + \int_0^t \|\phi_x\|_1^2\,\mathrm{d}t \le C\|\phi_0\|_2^2 + C\|\psi_0\|_1^2 + Ce^{-C_-\beta}.$$

Proof. From (4.10), $\psi(x,t) = \Psi(x,t) - p(x,t)$, which gives that

$$|\psi(x,t)| \le |\Psi(x,t)| + C|\phi_x|, \tag{4.26}$$

and

$$\|\psi(x,t)\|^2 \le \|\Psi(x,t)\|^2 + C\|\phi_x\|^2. \tag{4.27}$$

Then it follows from (4.25) that

$$\|\psi\|^2 \le \|\Psi\|^2 + C\|\phi\|_1^2 \le C\|(\phi_0,\Psi_0)\|_1^2 + Ce^{-C_-\beta}. \tag{4.28}$$

Similarly, it holds that

$$\int_0^t \|\psi_x\|^2\,\mathrm{d}t \le \int_0^t \|\Psi_x\|^2 + C\|\phi_x\|_1^2\,\mathrm{d}t \le C\|(\phi_0,\Psi_0)\|_1^2 + Ce^{-C_-\beta}. \tag{4.29}$$

Therefore, Lemma 4.6 is obtained by (4.25), (4.28), (4.29) and $(4.9)_1$. □

4.2. High order estimate.

LEMMA 4.7. *Under the same assumptions of Proposition 3.1, it holds that*

$$\|\psi_x\|^2(t)+\int_0^t \|\psi_{xx}\|^2 \mathrm{d}t \le C\|\phi_0\|_2^2+C\|\psi_0\|_1^2+Ce^{-C_-\beta}. \tag{4.30}$$

Proof. Multiplying $(3.3)_2$ by $-\psi_{xx}$, integrating the result with respect to t and x over $[0,t]\times[0,\infty)$ gives

$$\begin{aligned}
&\frac{1}{2}\|\psi_x\|^2(t)+\int_0^t\int_0^\infty \frac{\psi_{xx}^2}{V^{\alpha+1}}\mathrm{d}x\mathrm{d}t\\
=&\frac{1}{2}\|\psi_{0x}\|^2-\int_0^t \{\psi_x\psi_t\}|_{x=0}\mathrm{d}t-\int_0^t\int_0^\infty f(V)\phi_x\psi_{xx}\mathrm{d}x\mathrm{d}t-\int_0^t\int_0^\infty F\psi_{xx}\mathrm{d}x\mathrm{d}t\\
=:&\frac{1}{2}\|\psi_{0x}\|^2+\sum_{i=1}^3 M_i.
\end{aligned} \tag{4.31}$$

Making use of the estimate (4.7) for the boundary, we have

$$M_1\le Ce^{-C_-\beta}. \tag{4.32}$$

The Cauchy inequality implies that

$$M_2\le \varepsilon\int_0^t \|\psi_{xx}\|^2\mathrm{d}t+C_\varepsilon\int_0^t\|\phi_x\|^2\mathrm{d}t. \tag{4.33}$$

By (3.6) and the Sobolev inequality, we have

$$\begin{aligned}
M_3&\le C\int_0^t\int_0^\infty \left(|\phi_x|^2+|\phi_x||\psi_{xx}|\right)|\psi_{xx}|\mathrm{d}x\mathrm{d}t\\
&\le C\int_0^t\int_0^\infty |\phi_x|\left(|\phi_x|^2+|\psi_{xx}|^2\right)\mathrm{d}x\mathrm{d}t\\
&\le C\delta\int_0^t\left(\|\phi_x\|^2+\|\psi_{xx}\|^2\right)\mathrm{d}t.
\end{aligned} \tag{4.34}$$

Substituting (4.32)-(4.34) into (4.31) and using Lemma 4.6, we obtain (4.30). □

LEMMA 4.8. *Under the same assumptions of Proposition 3.1, it holds that*

$$\|\phi_{xx}\|^2+\int_0^t\|\phi_{xx}\|^2\mathrm{d}t\le C\|\phi_0\|_2^2+C\|\psi_0\|_1^2+Ce^{-C_-\beta}+C\delta\int_0^t\|\psi_{xxx}\|^2\mathrm{d}t. \tag{4.35}$$

Proof. Differentiating $(3.3)_1$ with respect to x, using $(3.3)_2$, we have

$$\frac{\phi_{xt}}{V^{\alpha+1}}+f(V)\phi_x=\psi_t-F. \tag{4.36}$$

Differentiating (4.36) with respect to x and multiplying the derivative by ϕ_{xx}, integrating the result with respect to t and x over $[0,t]\times[0,\infty)$, using (2.3), one has

$$\frac{1}{2}\int_0^\infty\frac{\phi_{xx}^2}{V^{\alpha+1}}\mathrm{d}x+\int_0^t\int_0^\infty\left(f(V)-\frac{(\alpha+1)h(V)}{2V}\right)\phi_{xx}^2\mathrm{d}x\mathrm{d}t$$

$$
\begin{aligned}
=&\frac{1}{2}\int_0^\infty \frac{\phi_{xx}^2}{V^{\alpha+1}}\Big|_{t=0}\mathrm{d}x-\int_0^\infty \psi_x\phi_{xx}\Big|_{t=0}\mathrm{d}x+\int_0^\infty \psi_x\phi_{xx}\,\mathrm{d}x\\
&+\int_0^t \psi_x\psi_{xx}\Big|_{x=0}\mathrm{d}t+\int_0^t \|\psi_{xx}\|^2\,\mathrm{d}t-\int_0^t\int_0^\infty F_x\phi_{xx}\,\mathrm{d}x\,\mathrm{d}t\\
&+(\alpha+1)\int_0^t\int_0^\infty \frac{V_x}{V^{\alpha+2}}\phi_{xt}\phi_{xx}\,\mathrm{d}x\,\mathrm{d}t-\int_0^t\int_0^\infty f(V)_x\phi_x\phi_{xx}\,\mathrm{d}x\,\mathrm{d}t\\
=:&\frac{1}{2}\int_0^\infty \frac{\phi_{xx}^2}{V^{\alpha+1}}\Big|_{t=0}\mathrm{d}x-\int_0^\infty \psi_x\phi_{xx}\Big|_{t=0}\mathrm{d}x+\sum_{i=1}^{6}N_i.
\end{aligned}\tag{4.37}
$$

By (2.5) and (3.5), one has

$$
f(V)-\frac{(\alpha+1)h(V)}{2V}\geq -p'(v_+)>0. \tag{4.38}
$$

The Cauchy inequality yields

$$
N_1\leq \varepsilon\|\phi_{xx}\|^2+C_\varepsilon\|\psi_x\|^2. \tag{4.39}
$$

Making use of the estimate (4.8) for the boundary, it follows that

$$
N_2\leq Ce^{-C_-\beta}. \tag{4.40}
$$

N_3 can be controlled by (4.30). By the Cauchy inequality, we have

$$
|N_4|\leq \varepsilon\int_0^t \|\phi_{xx}\|^2\,\mathrm{d}t+C_\varepsilon\int_0^t \|F_x\|^2\,\mathrm{d}t.
$$

Using

$$
\begin{aligned}
\|F_x\|^2&\leq C\int_0^\infty (\phi_x^4+\phi_x^2\phi_{xx}^2+\psi_{xx}^2\phi_{xx}^2+\psi_{xxx}^2\phi_x^2+\phi_x^2\psi_{xx}^2)\,\mathrm{d}x\\
&\leq C\delta\left(\|\phi_x\|_1^2+\|\psi_x\|_2^2\right),
\end{aligned}
$$

we have the estimate of N_4

$$
|N_4|\leq \varepsilon\int_0^t \|\phi_{xx}\|^2\,\mathrm{d}t+C_\varepsilon\delta\int_0^t \left(\|\phi_x\|_1^2+\|\psi_x\|_2^2\right)\mathrm{d}t. \tag{4.41}
$$

The Cauchy inequality yields

$$
|N_5|\leq C\int_0^t\int_0^\infty \left|\frac{V_x}{V^{\alpha+2}}\psi_{xx}\phi_{xx}\right|\mathrm{d}x\,\mathrm{d}t\leq \varepsilon\int_0^t \|\phi_{xx}\|^2\,\mathrm{d}t+C_\varepsilon\int_0^t \|\psi_{xx}\|^2\,\mathrm{d}t, \tag{4.42}
$$

$$
|N_6|\leq \varepsilon\int_0^t \|\phi_{xx}\|^2\,\mathrm{d}t+C_\varepsilon\int_0^t \|\phi_x\|^2\,\mathrm{d}t. \tag{4.43}
$$

Choosing ε small, substituting (4.38)-(4.43) into (4.37) and using Lemma 4.6, Lemma 4.7, we have (4.35). □

On the other hand, differentiating the second equation of (3.3) with respect to x, multiplying the derivative by $-\psi_{xxx}$, integrating the resulting equality over $[0,\infty)\times[0,t]$, using Lemmas 4.6-4.8, we can get the highest order estimate in the same way, which is listed as follows and the proof is omitted.

LEMMA 4.9. *Under the same assumptions of Proposition 3.1, it holds that*

$$\|\psi_{xx}(t)\|^2+\int_0^t \|\psi_{xxx}\|^2 \mathrm{d}t \leq C\|(\phi_0,\psi_0)\|_2^2+Ce^{-C_-\beta}. \tag{4.44}$$

Finally, Proposition 3.1 is obtained by Lemmas 4.5-4.9.

5. Proof of Theorem 2.1

Now we turn to the proof of the main theorem, i.e., Theorem 2.1. It is straightforward to imply (2.11) from Lemma 3.2. It remains to show (2.12). We will use the following useful lemma.

LEMMA 5.1 ([13]). *Assume that the function $f(t)\geq 0 \in L^1(0,+\infty)\cap BV(0,+\infty)$. Then it holds that $f(t)\to 0$ as $t\to\infty$.*

Proof. **(Proof of Theorem 2.1.)** Differentiating the first equation of (3.3) with respect to x, multiplying the resulting equation by ϕ_x, and integrating on $(0,\infty)$, we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}(\|\phi_x\|^2)\right| \leq C(\|\phi_x\|^2+\|\psi_{xx}\|^2).$$

Using Lemma 3.2, we have

$$\int_0^\infty \left|\frac{\mathrm{d}}{\mathrm{d}t}(\|\phi_x\|^2)\right| \mathrm{d}t \leq C\left\{\|(\phi_0,\psi_0)\|_2^2+e^{-c_-\beta}\right\}\leq C,$$

which implies $\|\phi_x\|^2 \in L^1(0,+\infty)\cap BV(0,+\infty)$. By Lemma 5.1, we have

$$\|\phi_x\|\to 0 \quad \text{as} \quad t\to+\infty.$$

Since $\|\phi_{xx}\|$ is bounded, the Sobolev inequality implies that

$$\|v-V\|_\infty^2=\|\phi_x\|_\infty^2 \leq 2\|\phi_x(t)\|\,\|\phi_{xx}(t)\|\to 0.$$

Similarly, we have

$$\|u-U\|_\infty^2=\|\psi_x\|_\infty^2 \leq 2\|\psi_x(t)\|\,\|\psi_{xx}(t)\|\to 0.$$

Therefore, the proof of Theorem 2.1 is completed. □

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