COMMUN. MATH. SCI.
Vol. 20, No. 6, pp. 1493–1516

GLOBAL SOLVABILITY TO A CANCER INVASION MODEL WITH REMODELING OF ECM AND POROUS MEDIUM DIFFUSION*

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Abstract. In this paper, we deal with a cancer invasion model with remodeling of ECM and slow diffusion. We consider this problem in a bounded domain of $\mathbb{R}^N (N=2,3)$ with zero-flux boundary conditions, and it is shown that for any large initial datum, the problem admits a global 'very' weak solution for any slow diffusion case. It is worth noting that the coexistence of the nonlinear diffusion, haptotaxis and the remodeling of ECM brings essential difficulties. Firstly, unlike the linear diffusion case, the haptotaxis term cannot be merged into the diffusion term, which makes the regularity of ECM less important in the process of making energy estimates. Secondly, the regularity of ECM depends on the worst one of cells density and uPA, therefore, the difficulty caused by the haptotactic term is really highlighted due to the low regularity of ECM. Therefore, it is hard to get the boundedness of cells density because the regularity of ECM is difficult to improve, even for large m.

Keywords. Cancer invasion model; 'Very' weak Solution; Slow diffusion; Remodeling mechanism.

AMS subject classifications. 35K55; 35M10; 92C17.

1. Introduction

It is well known that cancer has always been a major disease that threatens human life and health. Cancer big data shows that nearly 70%-90% of cancers are caused by living habits or environmental factors, and only 10% to 30% of cancers can be attributed to gene mutation [23]. In addition to active prevention, early diagnosis and treatment of cancer is also particularly critical, for example, early screening can reduce the mortality of cervical cancer by 80% and cure rate of early breast cancer is more than 90% [4, 7]. On the other hand, correct understanding and facing cancer is also the key to overcome cancer, and the research on the growth law of cancer cells is helpful for people to understand the formation and development of tumors, and is of great significance for the prevention and control of cancer.

In 1999, Perumpanani and Byrne [14] found the effect of extracellular matrix (ECM) on cancer cells invasion, that is, the invasion of cancer cells is closely related to the degradation of ECM, while matrix degrading enzyme (MDE) is secreted by cancer cells, and in addition to diffusion, cancer cells will also gather towards the direction of higher density of such nondiffusible ECM, which is called haptotaxis. In 2006, Chaplain and Lolas [2] further pointed out that in addition to random diffusion and haptotaxis, cancer cells will also move towards the direction of diffuse MDE, which is called chemotaxis,

*Received: March 16, 2021; Accepted (in revised form): January 03, 2022. Communicated by Min Tang.
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and such a coupled chemotaxis-haptotaxis model is proposed.

$$\begin{cases} u_t = \underbrace{\nabla\cdot(D_u\nabla u)}_{diffusion} - \underbrace{\chi\nabla\cdot(u\nabla v)}_{chemotaxis} - \underbrace{\xi\nabla\cdot(u\nabla w)}_{haptotaxis} + \underbrace{\mu u(1-u-w)}_{proliferation}, \\ \tau v_t = \underbrace{\Delta v}_{diffusion} \underbrace{-v}_{decay} + \underbrace{u}_{production}, \\ w_t = \underbrace{-vw}_{\text{deg}radation} + \underbrace{\eta w(1-u-w)}_{remodeling}, \end{cases} \tag{1.1}$$

where, u,v,w represent the cancer cells density, urokinase Plasminogen Activator (uPA) protease concentration and the extracellular matrix (ECM) density, respectively; D_u is the diffusion coefficient, χ,ξ are the chemotactic and haptotactic coefficients, respectively; $\mu u(1-u-w)$ is the proliferation or death of cancer cells, including competition with the ECM in space; in the second equation, $-v$ denotes the decay of protease, u denotes the protease which is secreted by cancer cells; in the third equation, $-vw$ represents the degradation of ECM, $\eta w(1-u-w)$ represents the remodeling of ECM components.

This model has been widely considered since it was proposed, and a lot of achievements in the research of this kind of models have been obtained. For example, Tao and Winkler [18] proved the global existence of classical solution for $\tau=0$ in two dimensional bounded domain. For the parabolic-parabolic-ODE system, that is $\tau=1$, in dimension 2, Pang and Wang [15] established the global bounded classical solution for large μ, and Jin [9] then removed the largeness restriction on μ. In the 3D case, the global classical solution is obtained respectively for small initial datum [16], or a generalized logistic source [9]. Recently, in 2021, Jin and Xiang [11] studied the following simplified model in dimension 2.

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) - \xi\nabla\cdot(u\nabla w), \quad x\in\Omega, t>0 \\ \tau v_t = \Delta v - v + u, \quad x\in\Omega, t>0 \\ w_t = -vw + \eta w(1-w), \quad x\in\Omega, t>0 \end{cases}$$

with $\tau\in\{0,1\}$, in which, the remodelling term $\eta w(1-u-w)$ is replaced with $\eta w(1-w)$. The authors studied the boundedness, the blow-up phenomenon and the stability of this model, and they established some results which are similar to the standard Keller-Segel model ($w\equiv 0$), these results indicate that the haptotaxis effect is negligible in this model since the regularity of ω is almost equivalent to v. In addition, the case $\eta=0$ has also been studied by many researchers. See for example [17,19–21,25,28], or [3] for a review. Among these, the nonlinear diffusive model, like porous medium diffusion, is also an important research field. Compared with linear diffusion, the nonlinear diffusion model of cells is closer to the actual diffusion behavior in biology, and one of these models of cancer cell is as follows.

$$\begin{cases} u_t = \Delta u^m - \chi\nabla\cdot(u\nabla v) - \xi\nabla\cdot(u\nabla w) + \mu u(1-u-w), \\ v_t = \Delta v - v + u, \\ w_t = -vw. \end{cases} \tag{1.2}$$

For this model, Tao and Winkler [22] obtained the existence of global weak solutions with some $m \geq 1$ fulfilling

$$m > \begin{cases} \dfrac{2n^2+4n-4}{n(n+4)} & if\, n \leq 8, \\ \dfrac{2n^2+3n+2-\sqrt{8n(n+1)}}{n(n+2)} & if\, n \geq 9. \end{cases}$$

Subsequently, the authors of these papers [5,12,24,29] improved the results to $m > \frac{2N}{N+2}$. However, if remodeling effect of ECM is considered in this model (1.2), there is no research. We assume that there is no flux of tumour cells or protease across the boundary of the domain, and this boundary condition is called no-flux boundary condition. This paper is concerned with the following chemotaxis-haptotaxis model with porous medium diffusion and remodeling of ECM with no-flux boundary condition

$$\begin{cases} u_t = \Delta u^m - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1-u-w), \quad in\ Q, \\ v_t = \Delta v - v + u, \quad in\ Q, \\ w_t = -vw + \eta w(1-u-w), \quad in\ Q, \\ \dfrac{\partial u^m}{\partial n} - \chi u \dfrac{\partial v}{\partial n} - \xi u \dfrac{\partial w}{\partial n}\Big|_{\partial\Omega} = 0, \quad \dfrac{\partial v}{\partial n}\Big|_{\partial\Omega} = 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x), \quad x \in \Omega, \end{cases} \tag{1.3}$$

where $m > 1$, $Q = \Omega \times \mathbb{R}^+$, $\Omega \subset \mathbb{R}^N$ $(N = 2,3)$ is a bounded domain with smooth boundary.

Our purpose is to establish the global solvability of a 'very' weak solution in the sense of Definition 2.1. It is worth noting that the coexistence of the nonlinear diffusion and the remodeling of ECM brings essential difficulties. On the one hand, comparing with the linear diffusion case $m = 1$, that is (1.2), for which, one can combine the haptotaxis term and the diffusion term through a transformation, and it makes the regularity of w less important in the process of making energy estimates. On the other hand, comparing with the nonlinear diffusion case without remodeling of ECM, that is (1.2), although the haptotaxis term can not be merged into the diffusion term, it is easy to see that the regularity of w depends entirely on v, so the haptotaxis term and the chemotaxis term have the same difficulty, which also means that the haptotaxis term will not cause additional difficulties compared with the pure-chemotactic model. While for the system (1.3), due to the coexistence of nonlinear diffusion and remodeling effect, on the one hand, the haptotactic term can not be merged into the diffusion term, on the other hand, the regularity of w depends on the worst one of u and v. Therefore, the difficulty caused by the haptotactic term is really highlighted due to the low regularity of w. Thus only a 'very' weak solution in the sense of Definition 2.1 can be obtained in the present paper. To obtain the global 'very' weak solution, we consider an approximation problem by adding $\varepsilon \Delta w$ to the third equation of (1.3). However, this is not enough to ensure the global existence of global bounded solutions in three dimensional domain, thus we also add a damping term εu^3 to the first equation of (1.3). We first establish the global existence of classical solutions to the approximation problem for any $\varepsilon > 0$, then by using some energy estimates and compact discussion, and by letting $\varepsilon \to 0$, we finally establish the global existence of the 'very' weak solutions. Unfortunately, it is hard to get the boundedness of u because the regularity of ECM is really difficult to improve, even for large m.

We give the assumptions of this paper.

$$\begin{cases} u_0 \in L^m(\Omega), v_0 \in H^1(\Omega), w_0 \in L^\infty(\Omega), \text{ and } \nabla\sqrt{w_0} \in L^2(\Omega), \\ u_0, v_0, w_0 \geq 0, u_0 \not\equiv 0, \left.\dfrac{\partial w_0}{\partial n}\right|_{\partial\Omega} = 0. \end{cases} \tag{1.4}$$

On the other hand, by a direct calculation, we see that

$$\begin{aligned} \nabla w_t &= -w\nabla v - v\nabla w + \eta(1-u-w)\nabla w - \eta w\nabla(u+w) \\ &= -w\nabla v + (\eta - v - \eta u - 2\eta w)\nabla w - \eta w\nabla u \\ &= -w\nabla v + (\eta - v - \eta u - 2\eta w)\nabla w - \eta w\nabla u + \frac{\xi\eta}{m} w u^{2-m}\nabla w - \frac{\xi\eta}{m} w u^{2-m}\nabla w \\ &= -w\nabla v + (\eta - v - \eta u - 2\eta w - \frac{\xi\eta}{m} w u^{2-m})\nabla w - \frac{\eta w}{m u^{m-1}}(\nabla u^m - \xi u\nabla w). \end{aligned}$$

From the boundary conditions in (1.3), we infer that

$$\frac{\partial w_t}{\partial n} = (\eta - v - \eta u - 2\eta w - \frac{\xi\eta}{m} w u^{2-m})\frac{\partial w}{\partial n}, \quad \text{on } \partial\Omega.$$

It implies that

$$\left.\frac{\partial w}{\partial n}\right|_{\partial\Omega} = 0$$

since $\frac{\partial w_0}{\partial n}\big|_{\partial\Omega} = 0$. Our main result reads as follows.

THEOREM 1.1. *Assume $N=2$, $m>1$ and (1.4) holds. Then the problem (1.3) admits a global nonnegative 'very' weak solution (u,v,w) in the sense of Definition 2.1 with $(u,v,w) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$, such that for any $T>0$*

$$\sup_{t\in(0,T)} \left(\|w(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty\cap H^1} + \|u(\cdot,t)\|_{L^m} + \|\nabla\sqrt{w}\|_{L^2}\right) \leq M_1, \tag{1.5}$$

$$\begin{aligned} &\left\|D^2 v\right\|_{L^2(Q_T)} + \|v_t\|_{L^2(Q_T)} + \|w_t\|_{L^2(Q_T)} + \left\|u^{m-\frac{3}{2}}\nabla u\right\|_{L^2(Q_T)} \\ &\quad + \left\|\frac{\nabla w}{\sqrt{w}}\cdot\sqrt{u}\right\|_{L^2(Q_T)} + \left\|\frac{\nabla w}{\sqrt{w}}\cdot\sqrt{v}\right\|_{L^2(Q_T)} + \|u\|_{L^{m+1}(Q_T)} \leq M_2, \end{aligned} \tag{1.6}$$

$$\left\|(u^{\frac{m}{2}})_t\right\|_{L^1((0,T);W^{-1,\frac{m+1}{m}}(\Omega))} \leq M_3, \tag{1.7}$$

where

$$\mathcal{X}_1 = \left\{ u \in L^\infty_{loc}(\mathbb{R}^+; L^m(\Omega)) \cap L^{m+1}_{loc}(\mathbb{R}^+; L^{m+1}(\Omega)); u^{m-\frac{1}{2}} \in L^2_{loc}(\mathbb{R}^+; H^1(\Omega)), \right.$$
$$\left. (u^{\frac{m}{2}})_t \in L^1_{loc}\left(\mathbb{R}^+; W^{-1,\frac{m+1}{m}}(\Omega)\right) \right\},$$

$$\mathcal{X}_2 = \left\{ v \in L^\infty_{loc}(\mathbb{R}^+; L^\infty(\Omega) \cap H^1(\Omega)); D^2 v, v_t \in L^2_{loc}(\mathbb{R}^+; L^2(\Omega)) \right\},$$

$$\mathcal{X}_3 = \{ w \in L^\infty_{loc}(\mathbb{R}^+; L^\infty(\Omega)); \nabla\sqrt{w} \in L^\infty_{loc}(\mathbb{R}^+; L^2(\Omega)),$$
$$w_t, \sqrt{u}\nabla\sqrt{w}, \sqrt{v}\nabla\sqrt{w} \in L^2_{loc}(\mathbb{R}^+; L^2(\Omega)) \}$$

$M_i(i=1,2,3)$ depend only on Ω, ξ, χ, μ, η, u_0, v_0, w_0 and T.

THEOREM 1.2. *Assume that $N=3$, $m>1$ and (1.4) holds. Then the problem (1.3) admits a global nonnegative 'very' weak solution (u,v,w) in the sense of Definition 2.1 with $(u,v,w)\in\tilde{\mathcal{D}}_1\times\mathcal{D}_2\times\mathcal{X}_3$ for $m\geq\frac{3}{2}$, and $(u,v,w)\in\mathcal{D}_1\times\mathcal{D}_2\times\mathcal{X}_3$ for $1<m<\frac{3}{2}$, such that for any $T>0$,*

$$\sup_{t\in(0,T)}\|u(\cdot,t)\|_{L^{\tau+1}}+\left\|u^{\frac{m+\tau-2}{2}}\nabla u\right\|_{L^2(Q_T)}+\|u\|_{L^{\tau+2}(Q_T)}\leq M_4,\, for\ 1<m<\frac{3}{2},\tag{1.8}$$

$$\sup_{t\in(0,T)}\|u(\cdot,t)\|_{L^{m}}+\left\|u^{m-\frac{3}{2}}\nabla u\right\|_{L^2(Q_T)}+\|u\|_{L^{m+1}(Q_T)}\leq M_5,\, for\ m\geq\frac{3}{2},\tag{1.9}$$

$$\begin{aligned}&\sup_{t\in(0,T)}\left(\|w(\cdot,t)\|_{L^\infty}+\|v(\cdot,t)\|_{H^1}+\|\nabla\sqrt{w}\|_{L^2}\right)+\|D^2v\|_{L^2(Q_T)}\\&\quad+\|v_t\|_{L^2(Q_T)}+\|w_t\|_{L^2(Q_T)}+\left\|\frac{\nabla w}{\sqrt{w}}\cdot\sqrt{u}\right\|_{L^2(Q_T)}\\&\quad+\left\|\frac{\nabla w}{\sqrt{w}}\cdot\sqrt{u}\right\|_{L^2(Q_T)}+\left\|(u^{\frac{m}{2}})_t\right\|_{L^1((0,T);W^{-1,\frac{\beta+1}{\beta}}(\Omega))}\leq M_6,\ for\ any\ m>1,\end{aligned}\tag{1.10}$$

where $0<\tau<m-1$ is a small constant, $\beta=\max\{m+1,4\}$,

$$\mathcal{D}_1=\Big\{u\in L^\infty_{loc}(\mathbb{R}^+;L^{\tau+1}(\Omega))\cap L^{\tau+2}_{loc}(\mathbb{R}^+;L^{\tau+2}(\Omega));u^{\frac{m+\tau}{2}}\in L^2_{loc}(\mathbb{R}^+;H^1(\Omega)),$$
$$(u^{\frac{m}{2}})_t\in L^1_{loc}\Big(\mathbb{R}^+;W^{-1,\frac{\beta+1}{\beta}}(\Omega)\Big)\Big\},$$

$$\tilde{\mathcal{D}}_1=\Big\{u\in L^\infty_{loc}(\mathbb{R}^+;L^{m}(\Omega))\cap L^{m+1}_{loc}(\mathbb{R}^+;L^{m+1}(\Omega));u^{m-\frac{1}{2}}\in L^2_{loc}(\mathbb{R}^+;H^1(\Omega)),$$
$$(u^{\frac{m}{2}})_t\in L^1_{loc}\Big(\mathbb{R}^+;W^{-1,\frac{\beta+1}{\beta}}(\Omega)\Big)\Big\},$$

$$\mathcal{D}_2=\left\{v\in L^\infty_{loc}(\mathbb{R}^+;H^1(\Omega));D^2v,v_t\in L^2_{loc}(\mathbb{R}^+;L^2(\Omega))\right\},$$

and $M_i(i=4,5,6)$ depend only on Ω, ξ, χ, μ, η, u_0, v_0, w_0 and T.

REMARK 1.1. As we all know, although there is no classical solution in general to porous medium diffusions, the regularity of the solution is still good. However, in the present model (1.3), the remodeling term $\eta\omega(1-u-\omega)$ brings essential difficulties to the estimation of the haptotactic term. Therefore, it is not easy to obtain the boundedness of u, and only a global 'very' weak solution is obtained.

2. Preliminaries

Throughout this paper, we let $\|\cdot\|_{L^p}:=\|\cdot\|_{L^p(\Omega)}$.

Next, we give the definition of 'very' weak solutions.

DEFINITION 2.1. *(u,v,w) is called a 'very' weak solution of (1.3), if $u\geq0,v\geq0,w\geq0$, with $(u,v,w)\in\mathcal{X}_1\times\mathcal{X}_2\times\mathcal{X}_3$ in dimension 2, (in dimension 3, $(u,v,w)\in\tilde{\mathcal{D}}_1\times\mathcal{D}_2\times\mathcal{X}_3$ for $m\geq\frac{3}{2}$; $(u,v,w)\in\mathcal{D}_1\times\mathcal{D}_2\times\mathcal{X}_3$ for $1<m<\frac{3}{2}$) such that for any $T>0$, any $\varphi,\phi,\psi\in C^\infty(\overline{Q_T})$ with $\varphi(x,T)=\phi(x,T)=\psi(x,T)=0$, $\frac{\partial\varphi}{\partial n}\Big|_{\partial\Omega}=0$, the following integral equalities hold*

$$-\iint_{Q_T}u\varphi_t dxdt-\int_\Omega u(x,0)\varphi(x,0)dx+\iint_{Q_T}\nabla u^m\nabla\varphi dxdt-\chi\iint_{Q_T}u\nabla v\nabla\varphi dxdt$$

$$-\xi\iint_{Q_T} u\nabla w\nabla\varphi dxdt=\mu\iint_{Q_T} u(1-u-w)\varphi dxdt,$$

$$-\iint_{Q_T} v\phi_t dxdt-\int_\Omega v(x,0)\phi(x,0)dx+\iint_{Q_T}\nabla v\nabla\phi dxdt$$

$$+\iint_{Q_T} v\phi dxdt-\iint_{Q_T} u\phi dxdt=0,$$

$$-\iint_{Q_T} w\psi_t dxdt-\int_\Omega w(x,0)\psi(x,0)dx$$

$$+\iint_{Q_T} vw\psi dxdt-\eta\iint_{Q_T}\psi w(1-u-w)dxdt=0,$$

where $Q_T=\Omega\times(0,T)$.

For reader's convenience, we give the Gagliardo-Nirenberg interpolation inequality as follows.

LEMMA 2.1. *For functions* $u:\Omega\to\mathbb{R}$ *defined on a bounded Lipschitz domain* $\Omega\in\mathbb{R}^n$, *we have*

$$\left\|D^j u\right\|_{L^p}\le C\left\|D^m u\right\|_{L^r}^{\alpha}\|u\|_{L^q}^{1-\alpha}+C\|u\|_{L^s},$$

where

$$\frac{1}{p}=\frac{j}{N}+(\frac{1}{r}-\frac{m}{N})\alpha+\frac{1-\alpha}{q},\frac{j}{m}\le\alpha\le 1,$$

and $s>0$ *is arbitrary.*

By [13, 26], we have the following two lemmas.

LEMMA 2.2. *Assume that* Ω *is bounded and let* $w\in C^2(\overline{\Omega})$ *satisfy* $\frac{\partial w}{\partial n}\big|_{\partial\Omega}=0$, *where* n *is the outward unit normal vector to the boundary* $\partial\Omega$. *Then we have*

$$\frac{\partial|\nabla w|^2}{\partial n}\le 2\kappa|\nabla w|^2,\quad on\,\partial\Omega,$$

where $\kappa>0$ *is an upper bound for the curvatures of* Ω.

LEMMA 2.3. *Suppose that* $h\in C^2(\mathbb{R})$. *Then for all* $\varphi\in C^2(\overline{\Omega})$ *fulfilling* $\frac{\partial\varphi}{\partial n}=0$ *on* $\partial\Omega$, *we have*

$$\int_\Omega h'(\varphi)|\nabla\varphi|^2\Delta\varphi dx+\frac{2}{3}\int_\Omega h(\varphi)|\Delta\varphi|^2 dx$$

$$=\frac{2}{3}\int_\Omega h(\varphi)\left|D^2\varphi\right|^2 dx-\frac{1}{3}\int_\Omega h''(\varphi)|\nabla\varphi|^4 dx-\frac{1}{3}\int_{\partial\Omega} h(\varphi)\frac{\partial|\nabla\varphi|^2}{\partial n}ds$$

and

$$\int_\Omega\frac{|\nabla\varphi|^4}{\varphi^3}dx\le(2+\sqrt{N})^2\int_\Omega\varphi\left|D^2\ln\varphi\right|^2 dx,$$

where $|D^2\varphi|^2 = \Sigma_{i,j=1}^n |D_{ij}\varphi|^2$.

3. Energy estimation and global solvability to the regularized problems

Consider the approximate problems given as

$$\begin{cases} u_{\varepsilon t} = \Delta\left((u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}} u_\varepsilon\right) - \chi\nabla\cdot(u_\varepsilon \nabla v_\varepsilon) - \xi\nabla\cdot\left(\dfrac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla w_\varepsilon\right) \\ \qquad\qquad +\mu u_\varepsilon(1-u_\varepsilon-w_\varepsilon) - \varepsilon u_\varepsilon^3, \, in\, Q, \\ v_{\varepsilon t} = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon, \, in\, Q, \\ w_{\varepsilon t} = \varepsilon\Delta w_\varepsilon - v_\varepsilon w_\varepsilon + \eta w_\varepsilon\left(1-\dfrac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon} - w_\varepsilon\right), \, in\, Q, \\ \dfrac{\partial u_\varepsilon}{\partial n}\Big|_{\partial\Omega} = 0, \quad \dfrac{\partial v_\varepsilon}{\partial n}\Big|_{\partial\Omega} = 0, \quad \dfrac{\partial w_\varepsilon}{\partial n}\Big|_{\partial\Omega} = 0, \\ u_\varepsilon(x,0) = u_{\varepsilon 0}(x), \quad v_\varepsilon(x,0) = v_{\varepsilon 0}(x), \quad w_\varepsilon(x,0) = w_{\varepsilon 0}(x), x\in\Omega, \end{cases} \tag{3.1}$$

where $\varepsilon\in(0,1)$, $u_{\varepsilon 0}$, $v_{\varepsilon 0}$, $w_{\varepsilon 0}$ satisfy

$$\begin{cases} u_{\varepsilon 0}, v_{\varepsilon 0}, w_{\varepsilon 0} \in C^{2+\alpha}(\overline{\Omega}), \quad \dfrac{\partial u_{\varepsilon 0}}{\partial n}\Big|_{\partial\Omega} = \dfrac{\partial v_{\varepsilon 0}}{\partial n}\Big|_{\partial\Omega} = \dfrac{\partial w_{\varepsilon 0}}{\partial n}\Big|_{\partial\Omega} = 0, u_{\varepsilon 0}, v_{\varepsilon 0}, w_{\varepsilon 0} \geq 0, \\ u_{\varepsilon 0} \rightharpoonup u_0 \text{ in } L^m, \; v_{\varepsilon 0} \rightharpoonup v_0 \text{ in } H^1, \; \sqrt{w_{\varepsilon 0}} \rightharpoonup \sqrt{w_0} \text{ in } H^1, \text{ and } \|w_{\varepsilon 0}\|_{L^\infty} \leq \|w_0\|_{L^\infty}. \end{cases} \tag{3.2}$$

We may assume that $w_{\varepsilon 0} \not\equiv 0$, $u_{\varepsilon 0} \not\equiv 0$, otherwise, we have $w_\varepsilon(x,t) \equiv 0$, and $u_\varepsilon(x,t) \equiv 0$, and the problem becomes very simple and does not need to be studied.

It is not difficult to see that when the approximation term $\varepsilon\Delta w_\varepsilon$ is added to the third equation, the regularity of w_ε will be improved by two orders. Based on this, in order to get the 'very' weak solution of the original problem (1.3), we can use (3.1) to prove some a priori energy estimates which are independent of ε, and the global solution of the original problem (1.3) can then be achieved by letting $\varepsilon\to 0$.

By the third equation of (3.1) , it is easy to obtain that

$$0 \leq w_\varepsilon \leq \max\{1, \|w_{\varepsilon 0}\|_{L^\infty}\} \leq \max\{1, \|w_0\|_{L^\infty}\}. \tag{3.3}$$

Using fixed point theory, or similar to the study of the chemotaxis model [6], it is not difficult to obtain the following local existence result of classical solution to (3.1) for any $\varepsilon > 0$.

LEMMA 3.1. *Assume that* $u_{\varepsilon 0}, v_{\varepsilon 0}, w_{\varepsilon 0}$, *satisfy* (3.2). *Then for any* $\varepsilon\in(0,1)$, *there exists* $T_{\max}\in(0,+\infty]$ *such that the problem* (3.1) *admits a unique classical solution* $(u_\varepsilon, v_\varepsilon, w_\varepsilon) \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega\times(0,T_{\max}))$ *with*

$$u_\varepsilon > 0, v_\varepsilon > 0, w_\varepsilon > 0 \, for\, all\, (x,t)\in\Omega\times(0,T_{\max}),$$

such that either $T_{\max} = \infty$, *or*

$$\lim_{t\to T_{\max}} (\|u_\varepsilon\|_{L^\infty} + \|v_\varepsilon\|_{W^{1,\infty}} + \|w_\varepsilon\|_{W^{1,\infty}}) = +\infty.$$

Next, we give some estimates of $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$.

Lemma 3.2. *Assume that* $N=2,3$, $u_{\varepsilon 0}$, $v_{\varepsilon 0}$, $w_{\varepsilon 0}$ *satisfy* (3.2). *Let* $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ *be a classical solution of* (3.1) *in* $\Omega\times(0,T)$ *for some* $T>0$. *Then*

$$\sup_{t\in(0,T)}\|u_\varepsilon(\cdot,t)\|_{L^1}+\int_0^T(\|u_\varepsilon\|_{L^2}^2+\varepsilon\|u_\varepsilon\|_{L^3}^3)ds\le c_1, \tag{3.4}$$

$$\sup_{t\in(0,T)}\|v_\varepsilon(\cdot,t)\|_{H^1}^2+\int_0^T(\|v_\varepsilon\|_{W^{2,2}}^2+\|v_{\varepsilon t}\|_{L^2}^2)ds\le c_2, \tag{3.5}$$

$$\int_0^T\|w_{\varepsilon t}\|_{L^2}^2ds\le c_3, \tag{3.6}$$

where c_1, c_2 and c_3 depend only on Ω,μ,u_0,v_0,T, and they are independent of ε.

Proof. By a direct integration for the first equation of (3.1), we obtain

$$\frac{d}{dt}\int_\Omega u_\varepsilon dx+\mu\int_\Omega u_\varepsilon^2dx+\mu\int_\Omega u_\varepsilon w_\varepsilon dx+\varepsilon\int_\Omega u_\varepsilon^3dx=\mu\int_\Omega u_\varepsilon dx\le\frac{\mu}{2}\int_\Omega u_\varepsilon^2dx+c_4,$$

then (3.4) is obtained from a direct integration. Using the boundedness of L^2-norm of u_ε, then (3.5) is easily derived by a standard calculation, see for example [9], for simplicity, we omit it.

Multiplying the third equation of (3.1) by $w_{\varepsilon t}$, and integrating it over Ω, we obtain

$$\begin{aligned}\frac{\varepsilon}{2}\frac{d}{dt}\int_\Omega|\nabla w_\varepsilon|^2dx+\int_\Omega|w_{\varepsilon t}|^2dx&=-\int_\Omega v_\varepsilon w_\varepsilon w_{\varepsilon t}dx+\eta\int_\Omega w_\varepsilon w_{\varepsilon t}\left(1-\frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon}-w_\varepsilon\right)dx\\&\le\frac{1}{2}\int_\Omega|w_{\varepsilon t}|^2dx+C\int_\Omega|v_\varepsilon|^2+|u_\varepsilon|^2dx,\end{aligned}$$

and (3.6) is obtained by a direct integration. □

Lemma 3.3. *Assume that* $N=2,3$, $m>1$, $u_{\varepsilon 0}$, $v_{\varepsilon 0}$, $w_{\varepsilon 0}$ *satisfy* (3.2). *Let* $(u_\varepsilon,v_\varepsilon,w_\varepsilon)$ *be a classical solution of* (3.1) *in* $\Omega\times(0,T)$ *for some* $T>0$. *Then*

$$\begin{aligned}&\sup_{t\in(0,T)}\int_\Omega\left(u_\varepsilon\ln u_\varepsilon+\frac{|\nabla w_\varepsilon|^2}{w_\varepsilon}\right)dx+\int_0^T\int_\Omega\frac{(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}}{u_\varepsilon}|\nabla u_\varepsilon|^2dxds\\&+\int_0^T\int_\Omega u_\varepsilon^2(1+\ln u_\varepsilon)dxds+\varepsilon\int_0^T\int_\Omega w_\varepsilon\left|D^2\ln w_\varepsilon\right|^2dxds\\&+\varepsilon\int_0^T\int_\Omega u_\varepsilon^3(1+\ln u_\varepsilon)dxds+\int_0^T\int_\Omega\frac{|\nabla w_\varepsilon|^2}{w_\varepsilon}\left(v_\varepsilon+\eta\frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon}\right)dxds\le C,\end{aligned} \tag{3.7}$$

where C *is independent of* ε, *and it depends only on* $m,|\Omega|,\xi,\chi,\mu,\eta$, u_0, v_0, w_0, T.

Proof. Multiplying the first equation of (3.1) by $1+\ln u_\varepsilon$, and integrating it over Ω, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u_\varepsilon\ln u_\varepsilon dx+\mu\int_\Omega u_\varepsilon^2(1+\ln u_\varepsilon)dx\\&\quad+\int_\Omega\nabla\left((u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon\right)\frac{\nabla u_\varepsilon}{u_\varepsilon}dx+\varepsilon\int_\Omega u_\varepsilon^3(1+\ln u_\varepsilon)dx\end{aligned}$$

$$=\chi\int_\Omega \nabla u_\varepsilon \nabla v_\varepsilon dx+\xi\int_\Omega \frac{\nabla u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla w_\varepsilon dx+\mu\int_\Omega u_\varepsilon(1-w_\varepsilon)(1+\ln u_\varepsilon)dx$$
$$\le -\chi\int_\Omega u_\varepsilon \Delta v_\varepsilon dx+\xi\int_\Omega \frac{\nabla u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla w_\varepsilon dx+\frac{\mu}{2}\int_\Omega u_\varepsilon^2(1+\ln u_\varepsilon)dx+c_5,$$

since w_ε is bounded. From the above inequality we infer that

$$\frac{d}{dt}\int_\Omega u_\varepsilon \ln u_\varepsilon dx+\frac{\mu}{2}\int_\Omega u_\varepsilon^2(1+\ln u_\varepsilon)dx+\int_\Omega \frac{(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}}{u_\varepsilon}|\nabla u_\varepsilon|^2dx+\varepsilon\int_\Omega u_\varepsilon^3(1+\ln u_\varepsilon)dx$$
$$\le -\chi\int_\Omega u_\varepsilon\Delta v_\varepsilon dx+\xi\int_\Omega \frac{\nabla u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla w_\varepsilon dx+c_5. \tag{3.8}$$

We use the third equation of (3.1) to get that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla w_\varepsilon|^2}{w_\varepsilon}dx=\int_\Omega \frac{\nabla w_\varepsilon}{w_\varepsilon}\nabla w_{\varepsilon t}dx-\frac{1}{2}\int_\Omega \frac{|\nabla w_\varepsilon|^2}{w_\varepsilon^2}w_{\varepsilon t}dx$$
$$=-\int_\Omega w_{\varepsilon t}\Big(\frac{\Delta w_\varepsilon}{w_\varepsilon}-\frac{|\nabla w_\varepsilon|^2}{w_\varepsilon^2}\Big)dx-\frac{1}{2}\int_\Omega \frac{|\nabla w_\varepsilon|^2}{w_\varepsilon^2}w_{\varepsilon t}dx$$
$$=\frac{1}{2}\int_\Omega \frac{|\nabla w_\varepsilon|^2}{w_\varepsilon^2}w_{\varepsilon t}dx-\int_\Omega \frac{\Delta w_\varepsilon}{w_\varepsilon}w_{\varepsilon t}dx$$
$$=\frac{1}{2}\int_\Omega \Big(\frac{|\nabla w_\varepsilon|^2}{w_\varepsilon^2}-2\frac{\Delta w_\varepsilon}{w_\varepsilon}\Big)\left(\varepsilon\Delta w_\varepsilon-v_\varepsilon w_\varepsilon+\eta w_\varepsilon\Big(1-\frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon}-w_\varepsilon\Big)\right)dx$$
$$=-\varepsilon\int_\Omega \frac{|\Delta w_\varepsilon|^2}{w_\varepsilon}dx+\frac{\varepsilon}{2}\int_\Omega \frac{|\nabla w_\varepsilon|^2}{w_\varepsilon^2}\Delta w_\varepsilon dx+\frac{1}{2}\eta\int_\Omega \Big(\frac{1}{w_\varepsilon}-3\Big)|\nabla w_\varepsilon|^2dx+\int_\Omega \Delta w_\varepsilon v_\varepsilon dx$$
$$+\eta\int_\Omega \Delta w_\varepsilon\frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon}dx-\frac{1}{2}\int_\Omega \frac{|\nabla w_\varepsilon|^2}{w_\varepsilon}\left(v_\varepsilon+\eta\frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon}\right)dx$$
$$=-\varepsilon\int_\Omega \frac{|\Delta w_\varepsilon|^2}{w_\varepsilon}dx+\frac{\varepsilon}{2}\int_\Omega \frac{|\nabla w_\varepsilon|^2}{w_\varepsilon^2}\Delta w_\varepsilon dx+\frac{1}{2}\eta\int_\Omega |\nabla w_\varepsilon|^2\Big(\frac{1}{w_\varepsilon}-3\Big)dx$$
$$-\int_\Omega \nabla w_\varepsilon\nabla v_\varepsilon dx-\eta\int_\Omega \frac{\nabla u_\varepsilon \nabla w_\varepsilon}{1+\varepsilon u_\varepsilon}dx-\frac{1}{2}\int_\Omega \frac{|\nabla w_\varepsilon|^2}{w_\varepsilon}\left(v_\varepsilon+\eta\frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon}\right)dx. \tag{3.9}$$

Noticing that

$$\int_\Omega w_\varepsilon|D^2\ln w_\varepsilon|^2dx=\int_\Omega\left(\frac{|D^2w_\varepsilon|^2}{w_\varepsilon}+\frac{|\nabla w_\varepsilon|^4}{w_\varepsilon^3}-2\frac{\nabla w_\varepsilon D^2w_\varepsilon\nabla w_\varepsilon}{w_\varepsilon^2}\right)dx$$
$$=\int_\Omega\left(\frac{|D^2w_\varepsilon|^2}{w_\varepsilon}+\frac{|\nabla w_\varepsilon|^4}{w_\varepsilon^3}-\frac{\nabla w_\varepsilon\nabla|\nabla w_\varepsilon|^2}{w_\varepsilon^2}\right)dx$$
$$=\int_\Omega\left(\frac{|D^2w_\varepsilon|^2}{w_\varepsilon}+\frac{|\nabla w_\varepsilon|^4}{w_\varepsilon^3}+\frac{|\nabla w_\varepsilon|^2}{w_\varepsilon^2}\Delta w_\varepsilon-2\frac{|\nabla w_\varepsilon|^4}{w_\varepsilon^3}\right)dx$$
$$=\int_\Omega\left(\frac{|D^2w_\varepsilon|^2}{w_\varepsilon}+\frac{|\nabla w_\varepsilon|^2}{w_\varepsilon^2}\Delta w_\varepsilon-\frac{|\nabla w_\varepsilon|^4}{w_\varepsilon^3}\right)dx, \tag{3.10}$$

then by Lemma 2.3 with $h(w)=-\frac{3\varepsilon}{2w}$ and combining with the above equality, we see that

$$-\varepsilon\int_\Omega \frac{|\Delta w_\varepsilon|^2}{w_\varepsilon}dx+\frac{\varepsilon}{2}\int_\Omega \frac{|\nabla w_\varepsilon|^2}{w_\varepsilon^2}\Delta w_\varepsilon dx$$

$$
\begin{aligned}
&= -\varepsilon\int_\Omega \frac{|\nabla w_\varepsilon|^2}{w_\varepsilon^2}\Delta w_\varepsilon dx - \varepsilon\int_\Omega \frac{\left|D^2 w_\varepsilon\right|^2}{w_\varepsilon}dx + \varepsilon\int_\Omega \frac{|\nabla w_\varepsilon|^4}{w_\varepsilon^3}dx + \frac{\varepsilon}{2}\int_{\partial\Omega}\frac{1}{w_\varepsilon}\frac{\partial}{\partial n}|\nabla w_\varepsilon|^2 ds\\
&= -\varepsilon\int_\Omega w_\varepsilon\left|D^2\ln w_\varepsilon\right|^2 dx + \frac{\varepsilon}{2}\int_{\partial\Omega}\frac{1}{w_\varepsilon}\frac{\partial}{\partial n}|\nabla w_\varepsilon|^2 ds.
\end{aligned}
$$

Substituting the above equality into (3.9) to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla w_\varepsilon|^2}{w_\varepsilon}dx + \varepsilon\int_\Omega w_\varepsilon\left|D^2\ln w_\varepsilon\right|^2 dx + \int_\Omega \frac{|\nabla w_\varepsilon|^2}{w_\varepsilon}\left(v_\varepsilon + \eta\frac{\ln(1+\varepsilon u)}{\varepsilon} + 3\eta w_\varepsilon\right)dx\\
\le& \frac{\varepsilon}{2}\int_{\partial\Omega}\frac{1}{w_\varepsilon}\frac{\partial}{\partial n}|\nabla w_\varepsilon|^2 ds - \eta\int_\Omega \frac{\nabla u_\varepsilon\nabla w_\varepsilon}{1+\varepsilon u_\varepsilon}dx - \int_\Omega \nabla v_\varepsilon\nabla w_\varepsilon dx + \frac{1}{2}\eta\int_\Omega\frac{|\nabla w_\varepsilon|^2}{w_\varepsilon}dx. \qquad (3.11)
\end{aligned}
$$

Using Lemma 2.3, we see that

$$
\int_\Omega \frac{|\nabla w_\varepsilon|^4}{w_\varepsilon^3}dx \le (2+\sqrt{N})^2\int_\Omega w_\varepsilon\left|D^2\ln w_\varepsilon\right|^2 dx, \qquad (3.12)
$$

Using the boundary trace embedding theorem [1,8], that is

$$
\|u\|_{L^2(\partial\Omega)} \le \delta\|\nabla u\|_{L^2(\Omega)} + C_\delta\|u\|_{L^2(\Omega)}, \quad \text{for any } \delta>0,
$$

and combining with (3.12) with δ sufficiently small, we see that

$$
\begin{aligned}
\frac{\varepsilon}{2}\int_{\partial\Omega}\frac{1}{w_\varepsilon}\frac{\partial}{\partial n}|\nabla w_\varepsilon|^2 ds &\le \kappa\varepsilon\int_{\partial\Omega}\frac{1}{w_\varepsilon}|\nabla w_\varepsilon|^2 ds = \kappa\varepsilon\int_{\partial\Omega}\left|w_\varepsilon^{\frac{1}{2}}\nabla\ln w_\varepsilon\right|^2 ds\\
&\le \delta\varepsilon\int_\Omega\left|\nabla\left(w_\varepsilon^{\frac{1}{2}}\nabla\ln w_\varepsilon\right)\right|^2 dx + \varepsilon C_\delta\int_\Omega w_\varepsilon|\nabla\ln w_\varepsilon|^2 dx\\
&\le \delta\varepsilon\int_\Omega\left|\frac{1}{2}w_\varepsilon^{-\frac{1}{2}}\nabla w_\varepsilon\nabla\ln w_\varepsilon + w_\varepsilon^{\frac{1}{2}}D^2\ln w_\varepsilon\right|^2 dx + \varepsilon C_\delta\int_\Omega\frac{|\nabla w_\varepsilon|^2}{w_\varepsilon}dx\\
&\le 2\delta\varepsilon\int_\Omega\left(w_\varepsilon\left|D^2\ln w_\varepsilon\right|^2 + \frac{1}{4}\frac{|\nabla w_\varepsilon|^4}{w_\varepsilon^3}\right)dx + \varepsilon C_\delta\int_\Omega\frac{|\nabla w_\varepsilon|^2}{w_\varepsilon}dx\\
&\le \frac{\varepsilon}{4}\int_\Omega w_\varepsilon\left|D^2\ln w_\varepsilon\right|^2 dx + \varepsilon C_\delta\int_\Omega\frac{|\nabla w_\varepsilon|^2}{w_\varepsilon}dx.
\end{aligned}
$$

Noticing that w_ε is bounded, then

$$
C\varepsilon\int_\Omega\frac{|\nabla w_\varepsilon|^2}{w_\varepsilon}dx \le \frac{\varepsilon}{4}\frac{1}{(2+\sqrt{N})^2}\int_\Omega\frac{|\nabla w_\varepsilon|^4}{w_\varepsilon^3}dx + c_6\varepsilon \le \frac{\varepsilon}{4}\int_\Omega w_\varepsilon\left|D^2\ln w_\varepsilon\right|^2 dx + c_6\varepsilon.
$$

Combining the above two inequalities, we obtain

$$
\frac{\varepsilon}{2}\int_{\partial\Omega}\frac{1}{w_\varepsilon}\frac{\partial}{\partial n}|\nabla w_\varepsilon|^2 ds \le \frac{\varepsilon}{2}\int_\Omega w_\varepsilon\left|D^2\ln w_\varepsilon\right|^2 dx + c_6\varepsilon. \qquad (3.13)
$$

Substituting (3.13) into (3.11) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\frac{|\nabla w_\varepsilon|^2}{w_\varepsilon}dx + \frac{\varepsilon}{2}\int_\Omega w_\varepsilon\left|D^2\ln w_\varepsilon\right|^2 dx + \frac{1}{2}\int_\Omega\frac{|\nabla w_\varepsilon|^2}{w_\varepsilon}\left(v_\varepsilon + \eta\frac{\ln(1+\varepsilon u)}{\varepsilon} + 3\eta w_\varepsilon\right)dx\\
\le& -\eta\int_\Omega\frac{\nabla u_\varepsilon\nabla w_\varepsilon}{1+\varepsilon u_\varepsilon}dx + \int_\Omega \Delta v_\varepsilon w_\varepsilon dx + \frac{1}{2}\eta\int_\Omega\frac{|\nabla w_\varepsilon|^2}{w_\varepsilon}dx + c_6\varepsilon. \qquad (3.14)
\end{aligned}
$$

Combining (3.8) with (3.14), we arrive at

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u_\varepsilon \ln u_\varepsilon dx+\int_\Omega \frac{(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}}{u_\varepsilon}|\nabla u_\varepsilon|^2dx+\frac{\mu}{2}\int_\Omega u_\varepsilon^2(1+u_\varepsilon)dx+\varepsilon\int_\Omega u_\varepsilon^3(1+u_\varepsilon)dx\\&\quad+\frac{\xi}{2\eta}\frac{d}{dt}\int_\Omega\frac{|\nabla w_\varepsilon|^2}{w_\varepsilon}dx+\frac{\varepsilon\xi}{2\eta}\int_\Omega w_\varepsilon|D^2\ln w_\varepsilon|^2dx\\&\quad+\frac{\xi}{2\eta}\int_\Omega\frac{|\nabla w_\varepsilon|^2}{w_\varepsilon}\left(v_\varepsilon+\eta\frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon}+3\eta w_\varepsilon\right)dx\\&\le-\chi\int_\Omega u_\varepsilon\Delta v_\varepsilon dx+\frac{\xi}{\eta}\int_\Omega\Delta v_\varepsilon w_\varepsilon dx+\frac{1}{2}\xi\int_\Omega\frac{|\nabla w_\varepsilon|^2}{w_\varepsilon}dx+c_7(1+\varepsilon)\\&\le\chi\int_\Omega|u_\varepsilon|^2dx+\chi\int_\Omega|\Delta v_\varepsilon|^2dx+\frac{1}{2}\xi\int_\Omega\frac{|\nabla w_\varepsilon|^2}{w_\varepsilon}dx+c_8(1+\varepsilon).\end{aligned}$$

Using (3.4), (3.5), we can easily obtain (3.7). □

LEMMA 3.4. *Assume that (3.2) holds, $N=2$ with $m>1$, or $N=3$ with $m\geq\frac{3}{2}$. Let $(u_\varepsilon,v_\varepsilon,w_\varepsilon)$ be a classical solution of (3.1) in $\Omega\times(0,T)$ for some $T>0$. Then*

$$\sup_{t\in(0,T)}\int_\Omega u_\varepsilon^m dx+\int_0^T\int_\Omega(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon^{m-2}|\nabla u_\varepsilon|^2dxds+\int_0^T\int_\Omega(u_\varepsilon^{m+1}+\varepsilon u_\varepsilon^{m+2})dxds\leq C, \tag{3.15}$$

where C is independent of ε, and it depends only on $m,\Omega,\xi,\chi,\mu,\eta$, u_0, v_0, w_0, T.

Proof. From Lemma 2.1, we see that

$$\begin{aligned}&\|\nabla v_\varepsilon\|_{L^4}^4\le c_9\|\nabla v_\varepsilon\|_{L^2}^2\|\Delta v_\varepsilon\|_{L^2}^2+c_{10}\|\nabla v_\varepsilon\|_{L^2}^4,\quad \text{when } N=2,\\&\|\nabla v_\varepsilon\|_{L^{\frac{10}{3}}}^{\frac{10}{3}}\le c_{11}\|\nabla v_\varepsilon\|_{L^2}^{\frac{4}{3}}\|\Delta v_\varepsilon\|_{L^2}^2+c_{12}\|\nabla v_\varepsilon\|_{L^2}^{\frac{10}{3}},\quad \text{when } N=3,\end{aligned}$$

recalling (3.5), we obtain

$$\int_0^T\|\nabla v_\varepsilon\|_{L^4}^4ds\le c_{13}\quad\text{when } N=2, \tag{3.16}$$

$$\int_0^T\|\nabla v_\varepsilon\|_{L^{\frac{10}{3}}}^{\frac{10}{3}}ds\le c_{14}\quad\text{when } N=3. \tag{3.17}$$

Multiplying the first equation of (3.1) by mu_ε^{m-1}, integrating it over Ω, recalling (3.3), and using Young's inequality yields

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u_\varepsilon^m dx+\mu m\int_\Omega u_\varepsilon^{m+1}dx+m(m-1)\int_\Omega(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon^{m-2}|\nabla u_\varepsilon|^2dx+m\varepsilon\int_\Omega u_\varepsilon^{m+2}dx\\&\le\chi m(m-1)\int_\Omega u_\varepsilon^{m-1}\nabla u_\varepsilon\nabla v_\varepsilon dx+\xi m(m-1)\int_\Omega\frac{u_\varepsilon^{m-1}}{1+\varepsilon u_\varepsilon}\nabla u_\varepsilon\nabla w_\varepsilon dx\\&\quad+\frac{\mu m}{4}\int_\Omega u_\varepsilon^{m+1}dx+c_{15}\\&\le\frac{m(m-1)}{2}\int_\Omega u_\varepsilon^{2m-3}|\nabla u_\varepsilon|^2dx+c_{16}\int_\Omega u_\varepsilon|\nabla v_\varepsilon|^2dx+c_{17}\int_\Omega\frac{u_\varepsilon}{(1+\varepsilon u_\varepsilon)^2}|\nabla w_\varepsilon|^2dx\\&\quad+\frac{\mu m}{4}\int_\Omega u_\varepsilon^{m+1}dx+c_{15}\end{aligned}$$

$$
\begin{aligned}
\leq & \frac{m(m-1)}{2}\int_\Omega (u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon^{m-2}|\nabla u_\varepsilon|^2dx + c_{18}\int_\Omega |\nabla v_\varepsilon|^{\frac{2(m+1)}{m}}dx \\
& + c_{17}\int_\Omega \frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon}|\nabla w_\varepsilon|^2dx + \frac{m\mu}{2}\int_\Omega u_\varepsilon^{m+1}dx + c_{15},
\end{aligned}
$$

since

$$
\frac{u_\varepsilon}{(1+\varepsilon u_\varepsilon)^2} \leq \frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon}
$$

for any $\varepsilon u_\varepsilon \geq 0$. Noticing that $\frac{2(m+1)}{m} < 4$ for any $m > 1$, and $\frac{2(m+1)}{m} < \frac{10}{3}$ for $m \geq \frac{3}{2}$, recalling (3.7), (3.16), (3.17), and by a direct integration, we obtain (3.15). ∎

REMARK 3.1. In dimension 2 with $m > 1$, or in dimension 3 with $m \geq \frac{3}{2}$, we have

$$
\sup_{t\in(0,T)} \|u(\cdot,t)\|_{L^m} \leq C,
$$

while for the three dimensional case with $1 < m < \frac{3}{2}$, it is hard to get the above estimation. Nevertheless, we can get the following estimate instead of the above result,

$$
\sup_{t\in(0,T)} \|u(\cdot,t)\|_{L^{\tau+1}} \leq C.
$$

This is also sufficient for the later proof.

LEMMA 3.5. *When $N = 3$, $1 < m < \frac{3}{2}$, assume that (3.2) holds. Let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be a classical solution of (3.1) in $\Omega \times (0,T)$ for some $T > 0$. Then there exists a small positive constant $\tau < m-1$, such that*

$$
\begin{aligned}
& \sup_{0<t<T}\int_\Omega u_\varepsilon^{\tau+1}dx + \int_0^T\int_\Omega \left((u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon^{\tau-1}|\nabla u_\varepsilon|^2 + u_\varepsilon^{\tau+2}\right)dxdt \\
& + \varepsilon\int_0^T\int_\Omega u_\varepsilon^{3+\tau}dxdt \leq C, \qquad (3.18)
\end{aligned}
$$

where C is independent of ε, and it depends only on $\tau, m, \Omega, \xi, \chi, \mu, \eta$, u_0, v_0, w_0 and T.

Proof. Multiplying the first equation of (3.1) by u_ε^r for $0 < r \leq m-1$, similar to the proof above, we obtain

$$
\begin{aligned}
& \frac{1}{r+1}\frac{d}{dt}\int_\Omega u_\varepsilon^{r+1}dx + r\int_\Omega (u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon^{r-1}|\nabla u_\varepsilon|^2dx + \frac{\mu}{2}\int_\Omega u_\varepsilon^{r+2}dx + \varepsilon\int_\Omega u_\varepsilon^{3+r}dx \\
\leq & -\frac{r\chi}{r+1}\int_\Omega u_\varepsilon^{r+1}\Delta v_\varepsilon dx + r\xi\int_\Omega \frac{u_\varepsilon^r}{1+\varepsilon u_\varepsilon}\nabla u_\varepsilon \nabla w_\varepsilon dx + c_{19} \\
\leq & \frac{r\chi}{r+1}\int_\Omega (u_\varepsilon^{r+2} + |\Delta v_\varepsilon|^{r+2})dx + \frac{r}{2}\int_\Omega u_\varepsilon^{m+r-2}|\nabla u_\varepsilon|^2dx + \frac{r\xi^2}{2}\int_\Omega \frac{u_\varepsilon^{r-m+2}}{(1+\varepsilon u_\varepsilon)^2}|\nabla w_\varepsilon|^2dx + c_{19}.
\end{aligned}
$$

Noticing that $\frac{1}{2} < r-m+2 \leq 1$ since $m < \frac{3}{2}$ and $0 < r \leq m-1$, then

$$
\frac{u_\varepsilon^{r-m+2}}{(1+\varepsilon u_\varepsilon)^2} \leq \frac{u_\varepsilon}{(1+\varepsilon u_\varepsilon)^2} + 1 \leq \frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon} + 1.
$$

Thus, we arrive at

$$
\frac{1}{r+1}\frac{d}{dt}\int_\Omega u_\varepsilon^{r+1}dx + \frac{r}{2}\int_\Omega (u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon^{r-1}|\nabla u_\varepsilon|^2dx + \frac{\mu}{2}\int_\Omega u_\varepsilon^{r+2}dx + \varepsilon\int_\Omega u_\varepsilon^{3+r}dx
$$

$$\leq \frac{r\chi}{r+1}\int_\Omega (u_\varepsilon^{r+2}+|\Delta v_\varepsilon|^{r+2})dx+\frac{r\xi^2}{2}\int_\Omega \left(\frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon}+1\right)|\nabla w_\varepsilon|^2dx+c_{19}.$$

By a direct integration and the L^p theory of linear parabolic equations, we get that

$$\begin{aligned}
&\frac{1}{r+1}\sup_{0<t<T}\int_\Omega u_\varepsilon^{r+1}dx+\frac{r}{2}\int_0^T\int_\Omega (u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon^{r-1}|\nabla u_\varepsilon|^2dxdt\\
&\quad+\frac{\mu}{2}\int_0^T\int_\Omega u_\varepsilon^{r+2}dxdt+\varepsilon\int_0^T\int_\Omega u_\varepsilon^{3+r}dxdt\\
&\leq \frac{2r\chi}{r+1}\int_0^T\int_\Omega (u_\varepsilon^{r+2}+|\Delta v_\varepsilon|^{r+2})dxdt+r\xi^2\int_0^T\int_\Omega\left(\frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon}+1\right)|\nabla w_\varepsilon|^2dxdt+2Tc_{19}.\\
&\leq c_{20}r\chi\int_0^T\int_\Omega u_\varepsilon^{r+2}dxdt+r\xi^2\int_0^T\int_\Omega\left(\frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon}+1\right)|\nabla w_\varepsilon|^2dxdt+c_{21},
\end{aligned}$$

where c_{20} is independent of T, r and μ, c_{21} depends on T. Taking $r=\tau$ with $0<\tau<\min\{m-1,\frac{\mu}{4\chi c_{20}}\}$ in the above inequality, and using (3.7), we complete the proof. □

The above estimates are not enough to ensure the global existence of the approximation problem (3.1). In what follows, these estimates may depend on ε.

LEMMA 3.6. *Assume (3.2) holds, $N=2,3$, $m>1$. Let $(u_\varepsilon,v_\varepsilon,w_\varepsilon)$ be a classical solution of (3.1) in $\Omega\times(0,T)$ for some $T>0$. Then*

$$\sup_{t\in(0,T)}\int_\Omega u_\varepsilon^{m+2}dx+\int_0^T\int_\Omega u_\varepsilon^{2m-1}|\nabla u_\varepsilon|^2dxds+\int_0^T\int_\Omega u_\varepsilon^{m+4}dxds\leq C_\varepsilon, \tag{3.19}$$

where C_ε depends on $m,\Omega,\xi,\chi,\mu,\eta,u_{\varepsilon0},v_{\varepsilon0},w_{\varepsilon0},T$ and ε.

Proof. Multiplying the first equation of (3.1) by $(m+2)u_\varepsilon^{m+1}$ and integrating it over Ω, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^{m+2}dx + \mu(m+2)\int_\Omega u_\varepsilon^{m+3}dx+\varepsilon(m+2)\int_\Omega u_\varepsilon^{m+4}dx\\
&=-(m+2)\int_\Omega \nabla\left((u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon\right)\nabla u_\varepsilon^{m+1}dx+\chi(m+2)\int_\Omega u_\varepsilon\nabla v_\varepsilon\nabla u_\varepsilon^{m+1}dx\\
&\quad+\xi(m+2)\int_\Omega\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla w_\varepsilon\nabla u_\varepsilon^{m+1}dx+\mu(m+2)\int_\Omega u_\varepsilon^{m+2}(1-w_\varepsilon)dx.
\end{aligned}$$

Similar with the proof of Lemma 3.4, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^{m+2}dx + \frac{(m+1)(m+2)}{2}\int_\Omega u_\varepsilon^{2m-1}|\nabla u_\varepsilon|^2dx\\
&\quad+\mu(m+2)\int_\Omega u_\varepsilon^{m+3}dx+\varepsilon(m+2)\int_\Omega u_\varepsilon^{m+4}dx\\
&\leq c_{22}\int_\Omega u_\varepsilon^3|\nabla v_\varepsilon|^2dx+c_{23}\int_\Omega\frac{u_\varepsilon^3}{(1+\varepsilon u_\varepsilon)^2}|\nabla w_\varepsilon|^2dx+\frac{\varepsilon(m+2)}{4}\int_\Omega u_\varepsilon^{m+4}dx+c_{24}\\
&\leq\frac{(m+2)\varepsilon}{2}\int_\Omega u_\varepsilon^{m+4}dx+c_{25}\int_\Omega|\nabla v_\varepsilon|^{\frac{2(m+4)}{m+1}}dx+c_{26}\int_\Omega\frac{|\nabla w_\varepsilon|^4}{|w_\varepsilon|^3}+c_{23}\int_\Omega\frac{u_\varepsilon^6}{(1+\varepsilon u_\varepsilon)^4}dx+c_{24}.
\end{aligned}$$

By Lemma 2.1, we obtain

$$\|\nabla v_\varepsilon\|_{L^{\frac{2(m+4)}{m+1}}}^{\frac{2(m+4)}{m+1}}\leq c_{41}\|\nabla v_\varepsilon\|_{L^2}^{\frac{(2m+3)(m+4)}{(m+1)(m+3)}}\|\nabla^2 v_\varepsilon\|_{L^{m+4}}^{\frac{3(m+4)}{(m+3)(m+1)}}+c_{42}\|\nabla v_\varepsilon\|_{L^2}^{\frac{2(m+4)}{m+1}},\ \text{when } N=2,$$

$$\|\nabla v_\varepsilon\|_{L^{\frac{2(m+4)}{m+1}}}^{\frac{2(m+4)}{m+1}} \le c_{41}\|\nabla v_\varepsilon\|_{L^2}^{\frac{10(m+4)}{5m+14}} \|\nabla^2 v_\varepsilon\|_{L^{m+4}}^{\frac{18(m+4)}{(5m+14)(m+1)}} + c_{42}\|\nabla v_\varepsilon\|_{L^2}^{\frac{2(m+4)}{m+1}}, \text{ when } N=3.$$

Combining the above three inequalities, using (3.4), (3.5) and (3.12), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^{m+2}dx + \frac{(m+1)(m+2)}{2}\int_\Omega u_\varepsilon^{2m-1}|\nabla u_\varepsilon|^2 dx\\
&\quad + \mu(m+2)\int_\Omega u_\varepsilon^{m+3}dx + \frac{(m+2)\varepsilon}{2}\int_\Omega u_\varepsilon^{m+4}dx\\
&\le c_{27}\int_\Omega \left|\nabla^2 v_\varepsilon\right|^{\beta(m+4)}dx + c_{28}\int_\Omega w_\varepsilon|D^2 w_\varepsilon|^2 dx + c_{23}\int_\Omega \frac{u_\varepsilon^6}{(1+\varepsilon u_\varepsilon)^4}dx + c_{24},
\end{aligned} \tag{3.20}$$

where

$$\beta = \begin{cases} \dfrac{3}{(m+3)(m+1)}, & \text{when } N=2,\\ \dfrac{18}{(5m+14)(m+1)}, & \text{when } N=3, \end{cases}$$

it is easy to see that $0<\beta<1$. By (3.20) and using the L^p theory of linear parabolic equations, we get that

$$\begin{aligned}
&\sup_{0<t<T}\int_\Omega u_\varepsilon^{m+2}dx + \frac{(m+1)(m+2)}{2}\int_0^T\int_\Omega u_\varepsilon^{2m-1}|\nabla u_\varepsilon|^2 dxdt\\
&\quad + \frac{(m+2)\varepsilon}{2}\int_0^T\int_\Omega u_\varepsilon^{m+4}dxdt\\
&\le 2c_{27}\int_0^T\int_\Omega \left|\nabla^2 v_\varepsilon\right|^{\beta(m+4)}dxdt + 2c_{28}\int_0^T\int_\Omega w_\varepsilon|D^2 w_\varepsilon|^2 dxdt\\
&\quad + 2c_{23}\int_0^T\int_\Omega \frac{u_\varepsilon^6}{(1+\varepsilon u_\varepsilon)^4}dxdt + 2c_{24}\\
&\le c_{29}\int_0^T\int_\Omega |u_\varepsilon|^{\beta(m+4)}dxdt + 2c_{28}\int_0^T\int_\Omega w_\varepsilon|D^2 w_\varepsilon|^2 dxdt + c_{30}\int_0^T\int_\Omega u_\varepsilon^2 dxdt + c_{31}\\
&\le \frac{(m+2)\varepsilon}{4}\int_0^T\int_\Omega |u_\varepsilon|^{m+4}dxdt + 2c_{28}\int_0^T\int_\Omega w_\varepsilon|D^2 w_\varepsilon|^2 dxdt + c_{30}\int_0^T\int_\Omega u_\varepsilon^2 dxdt + c_{32}.
\end{aligned}$$

By (3.4) and (3.7), we complete the proof. ◻

LEMMA 3.7. *Assume $N=2,3$, $m>1$, and (3.2) holds. Let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be a classical solution in $\Omega\times(0,T)$. Then for any $\varepsilon>0$, we have*

$$\sup_{t\in(0,T)}\|v_\varepsilon\|_{L^\infty} \le C, \quad \text{when } N=2 \tag{3.21}$$

$$\sup_{t\in(0,T)}\|v_\varepsilon\|_{W^{1,\infty}} + \sup_{t\in(0,T)}\|w_\varepsilon\|_{W^{1,\infty}} \le C_\varepsilon, \tag{3.22}$$

where C is independent of ε in dimension 2, C_ε depends on ε, both of them depend on m, Ω, ξ , χ, μ, η, $u_{\varepsilon 0}$, $v_{\varepsilon 0}$, $w_{\varepsilon 0}$, T.

Proof. By Duhamel's principle, we see that the solution $v_\varepsilon, w_\varepsilon$ can be expressed as follows,

$$v_\varepsilon = e^{-t}e^{t\Delta}v_{\varepsilon 0} + \int_0^t e^{-(t-s)}e^{(t-s)\Delta}u(s)ds,$$

$$w_\varepsilon = e^{-t}e^{\varepsilon t\Delta}w_{\varepsilon 0} + \int_0^t e^{-(t-s)}e^{\varepsilon(t-s)\Delta}\left((1-v_\varepsilon(s))w_\varepsilon(s)+\eta w_\varepsilon(s)\left(1-\frac{\ln(1+\varepsilon u_\varepsilon(s))}{\varepsilon}-w_\varepsilon(s)\right)\right)ds,$$

where $\{e^{t\Delta}\}_{t\geq 0}$ is the Neumann heat semigroup in Ω; for more details of Neumann heat semigroup, please refer to [27]. Here, we replace $e^{t\Delta}$ with $e^{-t}e^{t\Delta}$, so for this case, the condition $\int_\Omega u dx = 0$ in [27] is unnecessary. When $N=2$, using (3.15), for any $t\in(0,T)$, we arrive at

$$\begin{aligned}\|v_\varepsilon(\cdot,t)\|_{L^\infty} &\leq e^{-t}\|v_{\varepsilon 0}\|_{L^\infty} + \int_0^t e^{-(t-s)}(1+(t-s)^{-\frac{1}{m}})\|u_\varepsilon(s)\|_{L^m}ds\\ &\leq e^{-t}\|v_{\varepsilon 0}\|_{L^\infty} + \sup_{s\in(0,T)}\|u_\varepsilon(s)\|_{L^m}\int_0^t e^{-(t-s)}(1+(t-s)^{-\frac{1}{m}})ds\\ &\leq e^{-t}\|v_{\varepsilon 0}\|_{L^\infty} + \sup_{s\in(0,T)}\|u_\varepsilon(s)\|_{L^m}\int_0^\infty e^{-s}(1+s^{-\frac{1}{m}})ds \leq c_{33},\end{aligned}$$

since $\frac{1}{m}<1$. It is easy to see that c_{33} is independent of ε. In dimension 3, similar to the proof above, and using (3.18), we also obtain the L^∞ estimate of v_ε. Furthermore, we also have

$$\begin{aligned}\|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty} &\leq e^{-t}\|\nabla v_{\varepsilon 0}\|_{L^\infty} + \int_0^t e^{-(t-s)}(1+(t-s)^{-\frac{1}{2}-\frac{N}{2(m+2)}})\|u_\varepsilon(s)\|_{L^{m+2}}ds\\ &\leq e^{-t}\|\nabla v_{\varepsilon 0}\|_{L^\infty} + \sup_{s\in(0,T)}\|u_\varepsilon(s)\|_{L^{m+2}}\int_0^t e^{-(t-s)}(1+(t-s)^{-\frac{1}{2}-\frac{N}{2(m+2)}})ds\\ &\leq C_\varepsilon \quad for\ any\ t\in(0,T),\end{aligned}$$

since $\frac{1}{2}+\frac{N}{2(m+2)}<1$ for $N=2,3$, $m>1$.

$$\begin{aligned}&\|\nabla w_\varepsilon(\cdot,t)\|_{L^\infty} \leq e^{-t}\|\nabla w_{\varepsilon 0}\|_{L^\infty}\\ &\quad + \int_0^t e^{-(t-s)}(\varepsilon(t-s))^{-\frac{1}{2}-\frac{N}{2(m+2)}}\left\|(1-v_\varepsilon)w_\varepsilon+\eta w_\varepsilon\left(1-\frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon}-w_\varepsilon\right)(s)\right\|_{L^{m+2}}ds\\ &\leq e^{-t}\|\nabla w_{\varepsilon 0}\|_{L^\infty}\\ &\quad + c_{34}\sup_{s\in(0,T)}(\|u_\varepsilon\|_{L^{m+2}}+\|v_\varepsilon(s)\|_{L^{m+2}}+1)\int_0^t e^{-(t-s)}(1+(\varepsilon(t-s))^{-\frac{1}{2}-\frac{N}{2(m+2)}})ds\\ &\leq \tilde{C}_\varepsilon, \quad \text{for any } t\in(0,T),\end{aligned}$$

since $\frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon}\leq u_\varepsilon$ when $u_\varepsilon\geq 0$, and C_ε, $\tilde{C}_\varepsilon$ depend on ε. □

Using (3.21), (3.22), we adapt the classical Moser's iterative technique to prove the L^∞ estimate of u_ε.

LEMMA 3.8. *Assume $N=2,3$, $m>1$, and (3.2) holds. Let $(u_\varepsilon,v_\varepsilon,w_\varepsilon)$ be a classical solution in $\Omega\times(0,T)$. Then for any $\varepsilon>0$, we have*

$$\sup_{t\in(0,T)}\|u_\varepsilon\|_{L^\infty}\leq C_\varepsilon, \tag{3.23}$$

where C_ε depends on ε, m, Ω, ξ , χ, μ, η, $u_{\varepsilon 0}$, $v_{\varepsilon 0}$, $w_{\varepsilon 0}$, T.

Proof. Multiplying the first equation of (3.1) by pu_ε^{p-1} for $p>m$, using (3.22) yields

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^p dx+p(p-1)\int_\Omega u_\varepsilon^{p+m-3}|\nabla u_\varepsilon|^2dx+\mu p\int_\Omega(u_\varepsilon^{p+1}+u_\varepsilon^p w_\varepsilon)dx+\int_\Omega u_\varepsilon^p dx\\
\le&\chi p(p-1)\int_\Omega u_\varepsilon^{p-1}\nabla u_\varepsilon\nabla v_\varepsilon dx+\xi p(p-1)\int_\Omega \frac{u_\varepsilon^{p-1}\nabla u_\varepsilon\nabla w_\varepsilon}{1+\varepsilon u_\varepsilon}dx+(\mu p+1)\int_\Omega u_\varepsilon^p dx\\
\le&\frac{1}{2}p(p-1)\int_\Omega u_\varepsilon^{p+m-3}|\nabla u_\varepsilon|^2dx+c_{35}p^2\int_\Omega u_\varepsilon^{p+1-m}dx+c_{36}p\int_\Omega u_\varepsilon^p dx,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^p dx+\frac{2p(p-1)}{(p+m-1)^2}\int_\Omega\left|\nabla u_\varepsilon^{\frac{m+p-1}{2}}\right|^2dx+\int_\Omega u_\varepsilon^p dx\\
\le&c_{35}p^2\int_\Omega u_\varepsilon^{p+1-m}dx+c_{36}p\int_\Omega u_\varepsilon^p dx.
\end{aligned}\tag{3.24}$$

Then completely similar to the proof of Lemma 3.5 in [10], we complete the proof. ∎

LEMMA 3.9. *Assume $N=2,3$, $m>1$, and (3.2) holds. Then for any $\varepsilon\in(0,1)$, the problem (3.1) admits a unique global classical solution $(u_\varepsilon,v_\varepsilon,w_\varepsilon)\in C^{2+\alpha,1+\alpha/2}(\Omega\times(0,+\infty))$.*

Proof. Recalling Lemma 3.1. We only need to prove that $T_{\max}=\infty$. Suppose the contrary, that is $T_{\max}<+\infty$. We take $T=T_{\max}$ in Lemma 3.7 and Lemma 3.8. It is a contradiction. The proof is completed. ∎

4. Global existence of 'very' weak solutions

From Section 3, we see that the problem (3.1) admits a unique global classical solution, and these estimates in Lemma 3.2–Lemma 3.5 are independent of ε. To show the global existence of 'very' weak solutions of the problem (1.3), we also need some estimates for $u_{\varepsilon t}$, for this purpose, we prove the following lemma.

LEMMA 4.1. *Assume $N=2,3$, $1<m<2$. Let $(u_\varepsilon,v_\varepsilon,w_\varepsilon)$ be the global classical solution for any $\varepsilon\in(0,1)$. Then we have*

$$\int_0^T\int_\Omega(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon^{m-3}|\nabla u_\varepsilon|^2dxdt\le C,\tag{4.1}$$

where C is independent of ε, and it depends only on m, Ω, ξ, χ, μ, η, u_0, v_0, w_0, T.

Proof. When $1<m<2$, multiplying the first equation of (3.1) by $-u_\varepsilon^{m-2}$, then integrating the resultant equation over Ω, and using (3.3), (3.4) yields

$$\begin{aligned}
&-\frac{1}{m-1}\frac{d}{dt}\int_\Omega u_\varepsilon^{m-1}dx+(2-m)\int_\Omega(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon^{m-3}|\nabla u_\varepsilon|^2dx\\
=&(2-m)\int_\Omega\left(\chi u_\varepsilon^{m-2}\nabla u_\varepsilon\nabla v_\varepsilon+\xi\frac{u_\varepsilon^{m-2}}{1+\varepsilon u_\varepsilon}\nabla u_\varepsilon\nabla w_\varepsilon\right)dx\\
&-\mu\int_\Omega u_\varepsilon^{m-1}(1-u_\varepsilon-w_\varepsilon)dx+\varepsilon\int_\Omega u_\varepsilon^{m+1}dx\\
\le&\frac{2-m}{2}\int_\Omega u_\varepsilon^{2m-4}|\nabla u_\varepsilon|^2dx+(2-m)\chi^2\int_\Omega|\nabla v_\varepsilon|^2dx+(2-m)\xi^2\int_\Omega|\nabla w_\varepsilon|^2dx
\end{aligned}$$

$$+\mu\int_\Omega u^m dx+\varepsilon\int_\Omega u_\varepsilon^{m+1}dx+C$$

since $0<m-1<1$. Noticing that $m<2$, integrating the above inequality from 0 to T, and using (3.5), (3.7), (3.15), (3.18), we complete the proof. ∎

LEMMA 4.2. *Let $(u_\varepsilon,v_\varepsilon,w_\varepsilon)$ be the global classical solution for any $\varepsilon\in(0,1)$. Then $(u_\varepsilon^{\frac{m}{2}})_t\in L^1_{loc}\Big(\mathbb{R}^+;W^{-1,\frac{m+1}{m}}(\Omega)\Big)$ for $N=2$, $(u_\varepsilon^{\frac{m}{2}})_t\in L^1_{loc}\Big(\mathbb{R}^+;W^{-1,\frac{\beta+1}{\beta}}(\Omega)\Big)$ for $N=3$, that is, for any $T>0$,*

$$\int_0^T\Big\|(u_\varepsilon^{\frac{m}{2}})_t\Big\|_{W^{-1,\frac{m+1}{m}}(\Omega)}dt\le C_T,\quad \text{when } N=2, \tag{4.2}$$

$$\int_0^T\Big\|(u_\varepsilon^{\frac{m}{2}})_t\Big\|_{W^{-1,\frac{\beta+1}{\beta}}(\Omega)}dt\le \tilde{C}_T,\quad \text{when } N=3, \tag{4.3}$$

where $\beta=\max\{m+1,4\}$, C_T and $\tilde{C}_T$ are independent of ε, and they depend only on m, Ω, ξ , χ, μ, η, u_0, v_0, w_0, T.

Proof. For any $\varphi\in C_0^\infty(Q_T)$, $m>1$, we see that

$$\begin{aligned}\int_0^T\int_\Omega(u_\varepsilon^{\frac{m}{2}})_t\varphi dxds=&-\frac{m}{2}\int_0^T\int_\Omega\nabla\Big((u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon\Big)\nabla(u_\varepsilon^{\frac{m}{2}-1}\varphi)dxds\\&+\frac{m\chi}{2}\int_0^T\int_\Omega u_\varepsilon\nabla v_\varepsilon\nabla(u_\varepsilon^{\frac{m}{2}-1}\varphi)dxds+\frac{m\xi}{2}\int_0^T\int_\Omega\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla w_\varepsilon\nabla(u_\varepsilon^{\frac{m}{2}-1}\varphi)dxds\\&+\frac{m\mu}{2}\int_0^T\int_\Omega u_\varepsilon^{\frac{m}{2}}(1-u_\varepsilon-w_\varepsilon)\varphi dxds-\frac{m\varepsilon}{2}\int_0^T\int_\Omega u_\varepsilon^{\frac{m}{2}+2}\varphi dxds=J_1+J_2+J_3+J_4+J_5.\end{aligned}$$

(i) We first consider the case $N=2$, $m>1$; or $N=3$, $m\ge\frac{3}{2}$.

For J_1, using (3.15), we see that

$$\begin{aligned}|J_1|\le& C\int_0^T\int_\Omega(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon^{\frac{m}{2}-2}|\nabla u_\varepsilon|^2|\varphi|dxds\\&+C\int_0^T\int_\Omega(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon^{\frac{m}{2}-1}|\nabla u_\varepsilon||\nabla\varphi|dxds\\\le& C\|\varphi\|_{L^\infty(Q_T)}\int_0^T\int_\Omega(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon^{\frac{m}{2}-2}|\nabla u_\varepsilon|^2dxds\\&+C\Big\|(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{4}}u_\varepsilon^{\frac{m}{2}-1}\nabla u_\varepsilon\Big\|_{L^2(Q_T)}\big\|(u_\varepsilon^2+\varepsilon)\big\|^{\frac{m-1}{4}}_{L^{\frac{m+1}{2}}(Q_T)}\|\nabla\varphi\|_{L^{m+1}(Q_T)}\\\le& C_1\|\nabla\varphi\|_{L^{m+1}(Q_T)}+C\|\varphi\|_{L^\infty(Q_T)}\int_0^T\int_\Omega(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon^{\frac{m}{2}-2}|\nabla u_\varepsilon|^2dxds.\end{aligned}$$

When $m\ge2$, noticing that $-1\le\frac{m}{2}-2<m-2$, then from (3.7) and (3.15), we infer that

$$\begin{aligned}&C\int_0^T\int_\Omega(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon^{\frac{m}{2}-2}|\nabla u_\varepsilon|^2dxds\\\le& C\int_0^T\int_\Omega(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}(u_\varepsilon^{m-2}+u_\varepsilon^{-1})|\nabla u_\varepsilon|^2dxds\le\tilde{C};\end{aligned}$$

when $1<m<2$, noticing that $m-3<\frac{m}{2}-2<m-2$, and using (3.15) and (4.1), we get that

$$\begin{aligned}&C\int_0^T\int_\Omega (u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon^{\frac{m}{2}-2}|\nabla u_\varepsilon|^2\,dxds\\ \le &C\int_0^T\int_\Omega (u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}(u_\varepsilon^{m-2}+u_\varepsilon^{m-3})|\nabla u_\varepsilon|^2 dxds\le \hat{C}.\end{aligned}$$

Summing up, we obtain

$$|J_1|\le C_1\|\nabla\varphi\|_{L^{m+1}(Q_T)}+C_2\|\varphi\|_{L^\infty(Q_T)}.$$

Next, we consider J_2, it is easy to see that

$$\begin{aligned}|J_2|\le& C\int_0^T\int_\Omega \left|u_\varepsilon^{\frac{m}{2}-1}\varphi\nabla u_\varepsilon\nabla v_\varepsilon+u_\varepsilon^{\frac{m}{2}}\nabla v_\varepsilon\nabla\varphi\right|dxds\\ \le& C\left\|u_\varepsilon^{\frac{m}{2}-1}\nabla u_\varepsilon\right\|_{L^2(Q_T)}\|\nabla v_\varepsilon\|_{L^2(Q_T)}\|\varphi\|_{L^\infty(Q_T)}\\ &+C\|u_\varepsilon\|_{L^{m+1}(Q_T)}^{\frac{m}{2}}\|\nabla v_\varepsilon\|_{L^{2+\frac{2}{m}}(Q_T)}\|\nabla\varphi\|_{L^{m+1}(Q_T)}\\ \le& C\left\|(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{4}}u_\varepsilon^{-1/2}\nabla u_\varepsilon\right\|_{L^2(Q_T)}\|\nabla v_\varepsilon\|_{L^2(Q_T)}\|\varphi\|_{L^\infty(Q_T)}\\ &+C\|u_\varepsilon\|_{L^{m+1}(Q_T)}^{\frac{m}{2}}\|\nabla v_\varepsilon\|_{L^{2+\frac{2}{m}}(Q_T)}\|\nabla\varphi\|_{L^{m+1}(Q_T)}.\end{aligned}$$

Noticing that $2+\frac{2}{m}<4$ for $m>1$, $2+\frac{2}{m}<\frac{10}{3}$ for $m\ge\frac{3}{2}$, and using (3.16), (3.17) yields

$$\|\nabla v_\varepsilon\|_{L^{2+\frac{2}{m}}(Q_T)}\le C.$$

Thus, combining (3.7) and (3.15), we finally arrive at

$$|J_2|\le C_3\|\nabla\varphi\|_{L^{m+1}(Q_T)}+C_4\|\varphi\|_{L^\infty(Q_T)}.$$

For J_3, noticing that $\frac{u_\varepsilon}{(1+\varepsilon u_\varepsilon)^2}\le\frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon}$, and using (3.7), (3.15), we derive that

$$\begin{aligned}|J_3|\le& C\|u_\varepsilon^{\frac{m}{2}-1}\nabla u_\varepsilon\|_{L^2(Q_T)}\|\nabla w_\varepsilon\|_{L^2(Q_T)}\|\varphi\|_{L^\infty(Q_T)}\\ &+C\left\|\nabla w_\varepsilon\frac{u_\varepsilon^{\frac{1}{2}}}{1+\varepsilon u_\varepsilon}\right\|_{L^2(Q_T)}\|u_\varepsilon\|_{L^{m+1}(Q_T)}^{\frac{m-1}{2}}\|\nabla\varphi\|_{L^{m+1}(Q_T)}\\ \le& C\|u_\varepsilon^{\frac{m}{2}-1}\nabla u_\varepsilon\|_{L^2(Q_T)}\|\nabla w_\varepsilon\|_{L^2(Q_T)}\|\varphi\|_{L^\infty(Q_T)}\\ &+C\left\|\frac{\nabla w_\varepsilon}{\sqrt{w_\varepsilon}}\frac{\sqrt{\ln(1+\varepsilon u_\varepsilon)}}{\sqrt{\varepsilon}}\right\|_{L^2(Q_T)}\|u_\varepsilon\|_{L^{m+1}(Q_T)}^{\frac{m-1}{2}}\|\nabla\varphi\|_{L^{m+1}(Q_T)}\\ \le& C_5\|\varphi\|_{L^\infty(Q_T)}+C_6\|\nabla\varphi\|_{L^{m+1}(Q_T)}.\end{aligned}$$

For J_4, J_5, from (3.15), we infer that

$$|J_4|\le C\left(\|u_\varepsilon\|_{L^{\frac{m}{2}+1}(Q_T)}^{\frac{m}{2}+1}+\|u_\varepsilon\|_{L^{\frac{m}{2}}(Q_T)}^{\frac{m}{2}}\right)\|\varphi\|_{L^\infty(Q_T)}\le C_7\|\varphi\|_{L^\infty(Q_T)},$$

and

$$|J_5|\leq C\varepsilon\|u_\varepsilon\|_{L^{\frac{m}{2}+2}(Q_T)}^{\frac{m}{2}+2}\|\varphi\|_{L^\infty(Q_T)}\leq C_8\|\varphi\|_{L^\infty(Q_T)}.$$

Summing up, we conclude that when $N=2$, $m>1$; or $N=3$, $m\geq\frac{3}{2}$,

$$\left|\int_0^T\int_\Omega(u_\varepsilon^{\frac{m}{2}})_t\varphi dxds\right|\leq C_9\|\varphi\|_{L^\infty(Q_T)}+C_{10}\|\nabla\varphi\|_{L^{m+1}(Q_T)}. \tag{4.4}$$

(ii) Next, we turn our attention to the case $N=3$, $1<m<\frac{3}{2}$. Recalling (3.7), (3.18) and (4.1), noticing that $m-3<\frac{m}{2}-2<\tau-1$, we obtain that

$$\begin{aligned}
|J_1|\leq& C\int_0^T\int_\Omega(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon^{\frac{m}{2}-2}|\nabla u_\varepsilon|^2|\varphi|dxds\\
&+C\int_0^T\int_\Omega(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon^{\frac{m}{2}-1}|\nabla u_\varepsilon||\nabla\varphi|dxds\\
\leq& C\|\varphi\|_{L^\infty(Q_T)}\int_0^T\int_\Omega(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon^{\frac{m}{2}-2}|\nabla u_\varepsilon|^2dxds\\
&+C\left\|(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{4}}u_\varepsilon^{-\frac{1}{2}}\nabla u_\varepsilon\right\|_{L^2(Q_T)}\left\|(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{4}}u_\varepsilon^{\frac{m-1}{2}}\nabla\varphi\right\|_{L^2(Q_T)}\\
\leq& C\|\varphi\|_{L^\infty(Q_T)}\int_0^T\int_\Omega(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}\left(u_\varepsilon^{m-3}+u_\varepsilon^{\tau-1}\right)|\nabla u_\varepsilon|^2dxds\\
&+C\left\|(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{4}}u_\varepsilon^{-\frac{1}{2}}\nabla u_\varepsilon\right\|_{L^2(Q_T)}\left\|(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{4}}u_\varepsilon^{\frac{m-1}{2}}\right\|_{L^{\frac{2}{m-1}}(Q_T)}\|\nabla\varphi\|_{L^{\frac{2}{2-m}}(Q_T)}\\
\leq& \hat{C}\|\varphi\|_{L^\infty(Q_T)}+C\left\|(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{4}}u_\varepsilon^{-\frac{1}{2}}\nabla u_\varepsilon\right\|_{L^2(Q_T)}\left\|u_\varepsilon^2+\varepsilon\right\|_{L^1(Q_T)}\|\nabla\varphi\|_{L^{\frac{2}{2-m}}(Q_T)}\\
\leq& \tilde{C}_1\|\varphi\|_{L^\infty(Q_T)}+\tilde{C}_2\|\nabla\varphi\|_{L^{\frac{2}{2-m}}(Q_T)}.
\end{aligned}$$

For J_2, using (3.4), (3.5) and (3.7) yields

$$\begin{aligned}
|J_2|=&\frac{m\chi}{2}\left|\int_0^T\int_\Omega\left(u_\varepsilon^{\frac{m}{2}-1}\varphi\nabla u_\varepsilon\nabla v_\varepsilon+u_\varepsilon^{\frac{m}{2}}\varphi\Delta v_\varepsilon\right)dxds\right|\\
\leq&\frac{m\chi}{2}\left\|u_\varepsilon^{\frac{m}{2}-1}\nabla u_\varepsilon\right\|_{L^2(Q_T)}\|\nabla v_\varepsilon\|_{L^2(Q_T)}\|\varphi\|_{L^\infty(Q_T)}\\
&+C\|u_\varepsilon\|_{L^m(Q_T)}^{\frac{m}{2}}\|\Delta v_\varepsilon\|_{L^2}\|\varphi\|_{L^\infty(Q_T)}\\
\leq&\tilde{C}_3\|\varphi\|_{L^\infty(Q_T)}.
\end{aligned}$$

As for J_3, noticing that $\frac{u_\varepsilon}{(1+\varepsilon u_\varepsilon)^2}\leq\frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon}$, from (3.7), we infer that

$$\begin{aligned}
|J_3|\leq& C\|u_\varepsilon^{\frac{m}{2}-1}\nabla u_\varepsilon\|_{L^2(Q_T)}\|\nabla w_\varepsilon\|_{L^2(Q_T)}\|\varphi\|_{L^\infty(Q_T)}\\
&+C\left\|\nabla w_\varepsilon\frac{u_\varepsilon^{\frac{1}{2}}}{1+\varepsilon u_\varepsilon}\right\|_{L^2(Q_T)}\|u_\varepsilon\|_{L^1(Q_T)}^{\frac{m-1}{2}}\|\nabla\varphi\|_{L^{\frac{2}{2-m}}(Q_T)}\\
\leq&\tilde{C}_4\|\varphi\|_{L^\infty(Q_T)}+\tilde{C}_5\|\nabla\varphi\|_{L^{\frac{2}{2-m}}(Q_T)}.
\end{aligned}$$

Similarly, for J_4, J_5, noticing that $\frac{m}{2}+1<2$, then using (3.4), we get that

$$|J_4|\le C\left(\|u_\varepsilon\|^{\frac{m}{2}+1}_{L^{\frac{m}{2}+1}(Q_T)}+\|u_\varepsilon\|^{\frac{m}{2}}_{L^{\frac{m}{2}}(Q_T)}\right)\|\varphi\|_{L^\infty(Q_T)}\le \tilde{C}_6\|\varphi\|_{L^\infty(Q_T)},$$

and

$$|J_5|\le C\varepsilon\|u_\varepsilon\|^{\frac{m}{2}+2}_{L^{\frac{m}{2}+2}(Q_T)}\|\varphi\|_{L^\infty(Q_T)}\le \tilde{C}_7\|\varphi\|_{L^\infty(Q_T)}.$$

Summing up, we conclude that when $N=3$, $1<m<\frac{3}{2}$,

$$\left|\int_0^T\int_\Omega (u_\varepsilon^{\frac{m}{2}})_t\varphi dxds\right|\le \tilde{C}_8\|\varphi\|_{L^\infty(Q_T)}+\tilde{C}_9\|\nabla\varphi\|_{L^{\frac{2}{2-m}}(Q_T)}. \tag{4.5}$$

From the above discussion, we see that when $N=2$, using (4.4),

$$\left|\int_0^T\int_\Omega (u_\varepsilon^{\frac{m}{2}})_t\varphi dxds\right|\le C_{11}\sup_{0<t<T}\|\varphi\|_{W^{1,m+1}} \tag{4.6}$$

since $W^{1,m+1}(\Omega)\hookrightarrow L^\infty(\Omega)$; when $N=3$, note that $\frac{2}{2-m}<4$ for $m<\frac{3}{2}$, we let $\beta=\max\{4,m+1\}$. Then by (4.4) and (4.5),

$$\left|\int_0^T\int_\Omega (u_\varepsilon^{\frac{m}{2}})_t\varphi dxds\right|\le C_{12}\sup_{0<t<T}\|\varphi\|_{W^{1,\beta}}. \tag{4.7}$$

This lemma is proved. □

Proof. **(Proof of Theorem 1.1 and Theorem 1.2.)** By Lemma 3.9, for any $\varepsilon\in(0,1)$, there exists a global classical solution $(u_\varepsilon,v_\varepsilon,w_\varepsilon)$. We let $\varepsilon\to 0^+$ (passing to subsequences if necessary) to obtain the global 'very' weak solution. In what follows, we let '$\to$' denote the strong convergence, and '$\rightharpoonup$' denote the weak convergence.

The following proofs are based on the uniform energy estimations in Lemma 3.2-Lemma 3.5, Lemma 4.1 and (4.2). For simplicity, we only prove the case $N=2$, $m>1$, and the case $N=3$, $m\ge\frac{3}{2}$. The proof for $N=3$, $1<m<\frac{3}{2}$, is similar, so we omit it.

Recalling (3.7), (3.15), (4.2) and (4.3), and using Aubin-Lions lemma,

$$u_\varepsilon^{\frac{m}{2}}\to u^{\frac{m}{2}},\quad \text{in } L^q(Q_T) \text{ for any } q<\frac{2(m+1)}{m}.$$

It implies

$$u_\varepsilon\to u,\quad \text{in } L^q(Q_T) \text{ for any } q<m+1.$$

By (3.15), we also have

$$u_\varepsilon\rightharpoonup u,\quad \text{in } L^{m+1}(Q_T),$$

$$\nabla u_\varepsilon^{m-\frac{1}{2}}\rightharpoonup\nabla u^{m-\frac{1}{2}}, \text{in } L^2(Q_T)\ ,$$

and

$$\varepsilon u_\varepsilon^3\to 0,\quad \text{in } L^{\frac{m+2}{3}}(Q_T),$$

since

$$\iint_{Q_T} |\varepsilon u_\varepsilon^3|^{\frac{m+2}{3}} dxdt = \varepsilon^{\frac{m-1}{3}} \varepsilon \iint_{Q_T} u_\varepsilon^{m+2} dxdt \le C\varepsilon^{\frac{m-1}{3}}.$$

Noticing that $(a+b)^\alpha \le a^\alpha + b^\alpha$ for any $a,b>0$, $\alpha\in(0,1)$, then when $m\le 3$,

$$\left|(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}} u_\varepsilon - u^m\right| \le \left|(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}} u_\varepsilon - u_\varepsilon^m\right| + |u_\varepsilon^m - u^m| \le \varepsilon^{\frac{m-1}{2}} u_\varepsilon + |u_\varepsilon^m - u^m|,$$

when $m>3$,

$$\begin{aligned}\left|(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}} u_\varepsilon - u^m\right| &\le \left|(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}} u_\varepsilon - u_\varepsilon^m\right| + |u_\varepsilon^m - u^m| \\ &\le \frac{m-1}{2}\varepsilon(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}-1} u_\varepsilon + |u_\varepsilon^m - u^m| \\ &\le C\varepsilon u_\varepsilon^{m-2} + C\varepsilon^{\frac{m-1}{2}} u_\varepsilon + |u_\varepsilon^m - u^m|,\end{aligned}$$

which implies that

$$(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}} u_\varepsilon \to u^m \, in \, L^p(Q_T), \, for \, any \, p < \frac{m+1}{m}.$$

Employing (3.5), it gives

$$\begin{aligned} &v_\varepsilon \rightharpoonup v, \quad \text{in } W_2^{2,1}(Q_T), \\ &v_\varepsilon \to v, \quad \nabla v_\varepsilon \to \nabla v, \quad \text{in } L^2(Q_T).\end{aligned}$$

Recalling (3.7), (3.10) and (3.12), it yields

$$\varepsilon \iint_{Q_T} |D^2 w_\varepsilon|^2 dxdt \le C\varepsilon \iint_{Q_T} \frac{|D^2 w_\varepsilon|^2}{w_\varepsilon} dxdt \le \tilde{C}\varepsilon \iint_{Q_T} w_\varepsilon |D^2 \ln w_\varepsilon|^2 dxdt \le C_T.$$

Using (3.3), (3.6) and (3.7), we also have

$$\begin{aligned} &w_\varepsilon \to w, \quad \text{in } L^p(Q_T) \text{ for any } p>1, \\ &w_\varepsilon \overset{*}{\rightharpoonup} w, \quad \text{in } L^\infty(Q_T), \\ &\varepsilon \Delta w_\varepsilon \to 0, \quad \text{in } L^2(Q_T), \\ &\nabla w_\varepsilon \rightharpoonup \nabla w, \nabla\sqrt{w_\varepsilon} \rightharpoonup \nabla\sqrt{w}, w_{\varepsilon t} \rightharpoonup w_t \quad \text{in } L^2(Q_T).\end{aligned}$$

Noting that when $\varepsilon\to 0$, using (3.7) and (3.15),

$$\begin{aligned} &\left\|\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla w_\varepsilon - u_\varepsilon \nabla w_\varepsilon\right\|_{L^{\frac{2(m+2)}{m+6}}} \le \left\|\frac{\varepsilon u_\varepsilon^2}{1+\varepsilon u_\varepsilon}\nabla w_\varepsilon\right\|_{L^{\frac{2(m+2)}{m+6}}} \\ \le &\|\varepsilon u_\varepsilon^2\|_{L^{\frac{m+2}{2}}(Q_T)} \|\nabla w_\varepsilon\|_{L^2(Q_T)} \to 0.\end{aligned}$$

On the other hand, we also note that

$$\begin{aligned} &\|u_\varepsilon \nabla w_\varepsilon\|_{L^{\frac{2m+2}{m+3}}(Q_T)} \le \|u_\varepsilon\|_{L^{m+1}(Q_T)} \|\nabla w_\varepsilon\|_{L^2(Q_T)}, \\ &\left\|\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla w_\varepsilon\right\|_{L^{\frac{2m+2}{m+3}}(Q_T)} \le \|u_\varepsilon\|_{L^{m+1}(Q_T)} \|\nabla w_\varepsilon\|_{L^2(Q_T)},\end{aligned}$$

which implies that

$$\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla w_\varepsilon \rightharpoonup u\nabla w \, in \, L^{\frac{2m+2}{m+3}}(Q_T).$$

Since

$$\begin{aligned}&\left|\frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon}-u_\varepsilon\right|=\left|(\frac{1}{1+\theta\varepsilon u_\varepsilon}-1)u_\varepsilon\right|\\=&\left|\frac{-\theta\varepsilon u_\varepsilon}{1+\theta\varepsilon u_\varepsilon}\cdot u_\varepsilon\right|\le \varepsilon u_\varepsilon^2\to 0, \text{ in } L^{\frac{m+2}{2}}(Q_T) \text{ with } \theta\in(0,1),\end{aligned}$$

then

$$\frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon}w_\varepsilon \rightharpoonup uw, \quad \text{in} \quad L^{m+1}(Q_T).$$

Similarly, using (3.16) and (3.18), we also have

$$\begin{aligned}&u_\varepsilon\nabla v_\varepsilon \rightharpoonup u\nabla v, \quad \text{in } L^{\frac{4(m+1)}{m+5}}(Q_T) \text{ for } N=2,\\&u_\varepsilon\nabla v_\varepsilon \rightharpoonup u\nabla v, \quad \text{in } L^{\frac{10(m+1)}{3m+13}}(Q_T) \text{ for } N=3.\end{aligned}$$

Recalling that for any $\varphi,\phi,\psi\in C^\infty(\overline{Q_T})$ with $\frac{\partial\varphi}{\partial n}=0$ and $\varphi(x,T)=\phi(x,T)=\psi(x,T)=0$,

$$\begin{aligned}&-\iint_{Q_T}u_\varepsilon\varphi_t dxdt-\int_\Omega u_{\varepsilon 0}\varphi(x,0)dx-\iint_{Q_T}(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon\Delta\varphi dxdt\\&-\chi\iint_{Q_T}u_\varepsilon\nabla v_\varepsilon\nabla\varphi dxdt-\xi\iint_{Q_T}\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla w_\varepsilon\nabla\varphi dxdt\\=&\mu\iint_{Q_T}u_\varepsilon(1-u_\varepsilon-w_\varepsilon)\varphi dxdt-\varepsilon\iint_{Q_T}u_\varepsilon^3\varphi dxds,\\&-\iint_{Q_T}v_\varepsilon\phi_t dxdt-\int_\Omega v_{\varepsilon 0}\phi(x,0)dx+\iint_{Q_T}\nabla v_\varepsilon\nabla\phi dxdt\\&+\iint_{Q_T}v_\varepsilon\phi dxdt-\iint_{Q_T}u_\varepsilon\phi dxdt=0,\\&-\iint_{Q_T}w_\varepsilon\psi_t dxdt-\int_\Omega w_{\varepsilon 0}\psi(x,0)dx+\iint_{Q_T}v_\varepsilon w_\varepsilon\psi dxdt\\=&\varepsilon\iint_{Q_T}\nabla w_\varepsilon\nabla\psi dxdt+\eta\iint_{Q_T}w_\varepsilon(1-\frac{\ln(1+\varepsilon u_\varepsilon)}{\varepsilon}-w_\varepsilon)\psi dxdt.\end{aligned}$$

Letting $\varepsilon\to 0$, we conclude that

$$\begin{aligned}&-\iint_{Q_T}u\varphi_t dxdt-\int_\Omega u_0\varphi(x,0)dx+\iint_{Q_T}u^m\Delta\varphi dxdt-\chi\iint_{Q_T}u\nabla v\nabla\varphi dxdt\\&-\xi\iint_{Q_T}u\nabla w\nabla\varphi dxdt=\mu\iint_{Q_T}u(1-u-w)\varphi dxdt,\\&-\iint_{Q_T}v\phi_t dxdt-\int_\Omega v_0\phi(x,0)dx+\iint_{Q_T}\nabla v\nabla\phi dxdt+\iint_{Q_T}v\phi dxdt-\iint_{Q_T}u\phi dxdt=0,\\&-\iint_{Q_T}w\psi_t dxdt-\int_\Omega w_0\psi(x,0)dx+\iint_{Q_T}vw\psi dxdt=\eta\iint_{Q_T}w(1-u-w)\psi dxdt.\end{aligned}$$

Noting that

$$\|u^m\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(Q_T)} = \left\|mu^{m-\frac{3}{2}}u^{\frac{1}{2}}\nabla u\right\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(Q_T)} \le \left\|mu^{m-\frac{3}{2}}\nabla u\right\|^{\frac{4}{3}}_{L^2(Q_T)}\left\|u^{\frac{1}{2}}\right\|^{\frac{4}{3}}_{L^4(Q_T)} \le C,$$

which implies $\nabla u^m \in L^{\frac{4}{3}}(Q_T)$. Then we also have

$$-\iint_{Q_T} u\varphi_t dxdt - \int_\Omega u_0\varphi(x,0)dx + \iint_{Q_T} \nabla u^m \nabla\varphi dxdt - \chi\iint_{Q_T} u\nabla v\nabla\varphi dxdt$$
$$-\xi\iint_{Q_T} u\nabla w\nabla\varphi dxdt = \mu\iint_{Q_T} u(1-u-w)\varphi dxdt.$$

Hence, (u,v,w) is a 'very' weak solution of (1.3) such that these regularity estimates in (1.5)-(1.10) hold. □

Acknowledgement. The authors would like to thank the anonymous referees for their valuable comments and suggestions, which made some significant changes in this revision. The research of C. Jin was supported in part by NSFC Grant No. 11871230, Guangdong Basic and Applied Basic Research Foundation Grant No. 2021A1515010336.

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