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GLOBAL WELL-POSEDNESS AND DECAY OF THE LOW-REGULARITY SOLUTION TO THE 3D DENSITY-DEPENDENT MAGNETOHYDRODYNAMIC EQUATIONS WITH VACUUM*

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Abstract. In this paper, we consider the initial-boundary value problem of the 3D density-dependent magnetohydrodynamic equations with a low-regularity initial data. Assume that the initial density $\rho_0 \geq 0$ is bounded, and the scaling invariant quantity

$$\left(\|\rho_0^{1/2}\mathrm{u}_0\|_{L^2}^2+\|\mathrm{H}_0\|_{L^2}^2\right)\left(\|\nabla \mathrm{u}_0\|_{L^2}^2+\|\nabla \mathrm{H}_0\|_{L^2}^2\right)$$

is sufficiently small, then we prove that this system admits a unique global low-regularity solution. Here, no compatibility conditions are imposed on the initial data, and the initial density is allowed to vanish. In particular, we also obtain the exponential decay of the solution by introducing a delicate time-weighted estimate.

Keywords. Magnetohydrodynamic equations; global existence and uniqueness; exponential decay; vacuum.

AMS subject classifications. 76W05; 76N10; 35B40.

1. Introduction

Magnetohydrodynamics (MHD) is concerned with the macroscopic interaction of electrically conducting fluids with a magnetic field, which has a very broad range of applications, such as, the intensely heated and ionized fluids in an electromagnetic field in astrophysics, geophysics, and plasma physics. In this paper, we consider the 3D density-dependent MHD equations in $\Omega\times\mathbb{R}_+$ as

$$\begin{cases}\rho_t+\mathrm{div}(\rho\mathrm{u})=0,\\ (\rho\mathrm{u})_t+\mathrm{div}(\rho\mathrm{u}\otimes\mathrm{u})-\mu\Delta\mathrm{u}+\nabla P=\nabla\times\mathrm{H}\times\mathrm{H},\\ \mathrm{H}_t+\nu\nabla\times\nabla\times\mathrm{H}=\nabla\times(\mathrm{u}\times\mathrm{H}),\\ \mathrm{div}\mathrm{H}=0,\ \mathrm{div}\mathrm{u}=0,\end{cases}\tag{1.1}$$

where $\rho(x,t)\geq 0$ denotes the density, $\mathrm{u}=(\mathrm{u}^1(x,t),\mathrm{u}^2(x,t),\mathrm{u}^3(x,t))$ the velocity, $P(x,t)$ the pressure and $\mathrm{H}=(\mathrm{H}^1(x,t),\mathrm{H}^2(x,t),\mathrm{H}^3(x,t))$ the magnetic field, respectively. The positive constants μ and ν denote the viscosity of fluid and the magnetic diffusivity coefficient. $\Omega\subset\mathbb{R}^3$ is a given bounded domain with C^2 boundary.

We shall consider the initial-boundary value problem of (1.1) with the initial conditions:

$$(\rho,\mathrm{u},\mathrm{H})(x,0)=(\rho_0,\mathrm{u}_0,\mathrm{H}_0)\quad\text{in }\Omega,\tag{1.2}$$

and the boundary conditions

$$\begin{cases}\mathrm{u}|_{\partial\Omega}=0,\\ (\mathrm{H}\cdot n)|_{\partial\Omega}=\nabla\times\mathrm{H}\times n|_{\partial\Omega}=0,\end{cases}\tag{1.3}$$

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where n is the unit outward normal to $\partial\Omega$. Here, $(1.3)_2$ is a more physical boundary condition on the magnetic field, known as the perfectly conducting boundary, which can be used to describe a class of containers made of perfectly conductive materials.

System (1.1) describes in particular the motion of several conducting incompressible immiscible fluids (without surface tension) in presence of a magnetic field. Due to its important physical background and mathematical importance, there is a lot of literature devoted to its mathematical theory. Let us firstly give some historical reviews on the homogeneous incompressible MHD (i.e., $\rho\equiv const$ in (1.1)). Duvaut-Lions [13] first proved the global existence of Leray-Hopf weak solutions, and local existence and uniqueness of strong solution in the classical Sobolev space $H^s(\mathbb{R}^N)$ with $s\geq N$. Later, Sermange-Temam [28] generalized these results in [13], and they also proved the uniqueness of the global weak solution in dimension two. However, whether the weak solution is regular or the unique strong solution can exist globally is still an open problem for the spatial dimension $N\geq 3$. So a lot of works were devoted to finding various regularity criteria in terms of the velocity field only (see, e.g., [7, 18]). Also the global regularity problem on the MHD equations with partial dissipation has been extensively studied (see, e.g., [5, 24]). Recently, He-Huang-Wang [19] proved the global existence and uniqueness of the strong solution provided that the difference between the magnetic field and the velocity is small initially.

The inhomogeneous case (1.1) has been also studied by a lot of authors. The global existence of weak solutions with finite energy was establishend by Gerbeau-Le Bris [15] and Desjardins-Le Bris [12] in the whole space $\mathbb{R}^3$ and in the torus $\mathbb{T}^3$, respectively. However, the question of uniqueness of such a weak solution remains open, even in two dimensions. Abidi-Hmidi [1, 2] established the global (with small initial data) existence and uniqueness of a strong solution in some Besov spaces with positive initial density (no vacuum). If the initial density vanishes in some sets, the analysis becomes more subtle, since the system degenerates in the vacuum region. So there exist a lot of works devoted to this more physical and interesting case. Among these related results, Chen-Tan-Wang [8] first showed a local unique strong solution to (1.1), where the initial data satisfies

$$0\leqslant\rho_0\in H^2,\ (\mathrm{u}_0,\mathrm{H}_0)\in H^2. \tag{1.4}$$

In particular, they need the following compatibility condition

$$-\mu\Delta\mathrm{u}_0+\nabla P_0-(\mathrm{H}_0\cdot\nabla)\mathrm{H}_0=\sqrt{\rho_0}g, \tag{1.5}$$

with some $(P_0,g)\in H^1\times L^2$. Based on the local existence result in [8], Huang-Wang [20] established the global existence of strong solutions to the 2D initial boundary value problem (1.1)–(1.3) with a general large data in a bounded domain, and Gong-Li [17] considered the 3D case for a small initial data. Recently, Lü-Xu-Zhong [26] considered the 2D Cauchy problem (1.1), where they obtained the local and global well-posedness of the unique strong solution with a general large data and $\rho_0\in W^{1,p}(\mathbb{R}^2)(p>2)$.

From the results mentioned above, it reveals that the uniqueness of solutions to the MHD system (1.1) is linked to its regularity, that is, if the initial data is smooth enough, the uniqueness of the corresponding solution can be proved. A natural question is: When the initial data is discontinuous, does system (1.1)–(1.3) admit a unique solution? And whether the corresponding solution could admit some decay properties? The main motivation of this paper is to provide the positive answers on these two topics. Indeed, there were some results on these two topics of the nonhomogeneous incompressible Navier–Stokes equations (i.e., $\mathrm{H}\equiv 0$ in (1.1)):

$$\begin{cases} \rho_t+\operatorname{div}(\rho \mathrm{u})=0, \\ (\rho \mathrm{u})_t+\operatorname{div}(\rho \mathrm{u}\otimes \mathrm{u})-\mu\Delta \mathrm{u}+\nabla P=0, \\ \operatorname{div}\mathrm{u}=0. \end{cases} \tag{1.6}$$

Here, we mention some interesting works in [9, 10, 22, 27] on these topics. Ladyženskaja-Solonnikov [22] first addressed the question of unique resolvability of (1.6) with less regularity on the initial data. Under the assumption that $u_0\in W^{2-\frac{2}{p},p}(\Omega)(p>n)$ is divergence free and vanishes on $\partial\Omega$ and that $\rho_0\in C^1(\Omega)$ is bounded away from zero ($\Omega\subset\mathbb{R}^3$ is a bounded domain), they then proved global well-posedness with $\|u_0\|_{W^{2-\frac{2}{p},p}}\le\varepsilon$ for a sufficiently small ε. To weaken their regularity condition on the initial data, Danchin-Mucha [9, 10] first introduced the Lagrangian coordinates to prove the uniqueness of the solution of the system (1.6) in a critical functional framework, where the discontinuous density is allowed. Abidi-Gui-Zhang [3] also discussed the well-posedness results for (1.6) in various Besov spaces. Later, Paicu-Zhang-Zhang [27] proved the global existence and uniqueness of solution in Sobolev space, in which the initial data $(\rho_0,\mathrm{u}_0)\in L^\infty\left(\mathbb{R}^3\right)\times H^1\left(\mathbb{R}^3\right)$ satisfies

$$0<c_0\le\rho_0\le C_0<+\infty,\quad \|u_0\|_{\dot H^1}\le\varepsilon,$$

for some small $\varepsilon>0$ depending only on the given constants c_0, C_0. Chen-Zhang-Zhao [6] refined the result in [27] with a smallness condition on $\|u_0\|_{\dot H^{1/2}}$, and they also showed that if $u_0\in L^p\left(\mathbb{R}^3\right)$ with $p\in\left[\frac{6}{5},2\right]$, the velocity admits a decay estimate for $t>1$ and $k=0,1$ as

$$\|\nabla^k u(t)\|_{L^2}\le C(1+t)^{-\frac{k}{2}-\alpha(p)},$$

with $\alpha(p)=\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)$. We should point out that the initial vacuum has been eliminated in these works mentioned above. Recently, Danchin-Mucha [11] proved the global unique solvability of system (1.6), where the initial density is allowed provided that $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}\le\varepsilon$ with ε sufficiently small. Here $\Omega\subset\mathbb{R}^d$ is a bounded domain or the torus $\mathbb{T}^d$ with $d=2,\ 3$.

In this paper, we generalize the global well-posedness result in [11] to the 3D initial boundary value problem (1.1)–(1.3), and improve their result by obtaining an exponential decay of the solution.

Before stating the main results of the paper, let us introduce a few notations. Set

$$\int f\mathrm{d}x:=\int_\Omega f\mathrm{d}x.$$

For $1\le r\le\infty$ and $k\in\mathbb{N}$, we denote the Lebesgue space and the Sobolev space by

$$L^r:=L^r(\Omega),\quad W^{k,r}:=\{f\in L^r: D^\alpha f\in L^r, |\alpha|\le k\},\ H^k:=W^{k,2},$$

$$H_0^1:=\left\{\mathrm{u}\in H^1,\ \mathrm{u}=0 \text{ on } \partial\Omega\right\},\ H_n^1:=\left\{\mathrm{u}\in H^1,\ \mathrm{u}\cdot n=0 \text{ on } \partial\Omega\right\}.$$

Let us now state our main result in this paper as follows.

Theorem 1.1. *Let Ω be a bounded domain with C^2 boundary in $\mathbb{R}^3$. Assume that the initial data $(\rho_0, \mathrm{u}_0, \mathrm{H}_0)$ satisfy for a given constant $\bar{\rho} > 0$ that*

$$0 \le \rho_0 \le \bar{\rho}, \ (\rho_0, \mathrm{u}_0, \mathrm{H}_0) \in L^\infty \times H_0^1 \times H_n^1, \tag{1.7}$$

and the divergence-free condition

$$\operatorname{div} \mathrm{u}_0 = 0, \ \operatorname{div} \mathrm{H}_0 = 0 \text{ in } \Omega. \tag{1.8}$$

There exists a positive constant ϵ, depending only on Ω, μ, ν and $\bar{\rho}$, such that if

$$E_0 := \left(\|\rho_0^{1/2} \mathrm{u}_0\|_{L^2}^2 + \|\mathrm{H}_0\|_{L^2}^2 \right) \left(\|\nabla \mathrm{H}_0\|_{L^2}^2 + \|\nabla \mathrm{u}_0\|_{L^2}^2 \right) \le \epsilon, \tag{1.9}$$

then the initial-boundary value problem (1.1)–(1.3) admits a unique global solution satisfying for any $0 < \tau < +\infty$:

$$\begin{cases} 0 \le \rho \in L^\infty(0,+\infty; L^\infty) \cap C([0,+\infty); L^q), \ \rho^{1/2} \mathrm{u} \in C\left([0,+\infty); L^2\right), \\ \mathrm{u} \in L^\infty\left(0,+\infty; H_0^1\right) \cap L^2\left(0,+\infty; H^2\right) \cap L^2\left(\tau,\infty; W^{2,6}\right) \cap C\left([\tau,+\infty); W^{1,p}\right), \\ \nabla P \in L^2\left(\tau,\infty; L^6\right) \cap L^2\left(0,+\infty; L^2\right), \\ \mathrm{H} \in L^\infty\left(0,+\infty; H_n^1\right) \cap L^2\left(0,+\infty; H^2\right) \cap L^2\left(\tau,+\infty; W^{2,6}\right) \cap C\left([\tau,+\infty); W^{1,p}\right), \\ \sqrt{\rho} \mathrm{u}_t \in L^2\left(0,+\infty; L^2\right) \cap L^\infty\left(\tau,+\infty; L^2\right), \ \mathrm{u}_t \in L^2\left(\tau,+\infty; H_0^1\right), \\ \mathrm{H}_t \in L^2\left(0,+\infty; L^2\right) \cap L^\infty\left(\tau,+\infty; L^2\right) \cap L^2\left(\tau,+\infty; H_n^1\right), \end{cases} \tag{1.10}$$

where $q \in [1,+\infty)$ and $p \in [2,6)$. In particular, (u,H) has the following decay rates:

$$\left\| \rho^{1/2} \mathrm{u}(\cdot,t) \right\|_{L^2}^2 \le C e^{-\sigma t}, \text{ for all } t > 0, \tag{1.11}$$

and

$$\|\mathrm{u}(\cdot,t)\|_{W^{1,p}}^2 + \|\mathrm{H}(\cdot,t)\|_{W^{1,p}}^2 \le C e^{-\sigma t}, \text{ for all } t > 1, \tag{1.12}$$

for some $\sigma > 0$ defined in Lemma 3.1.

Remark 1.1.

(1) It is easy to check that Theorem 1.1 still holds for the inhomogeneous incompressible Navier–Stokes equation (1.6). Compared with the result in [11], the exponential decay (1.11)–(1.12) is new. The key idea to this improvement is that some new a priori estimates are obtained by introducing a dedicated initial layer analysis. So our results also can be viewed as a generalization and improvement of those in [11].

(2) In this paper, we prove the global well-posedness of the system (1.1)–(1.3) with much lower regularity on the initial data, and in particular, the initial density is allowed to have a discontinuity. Our result improves the ones in [17,20], where they all need continuous initial data to guarantee the uniqueness.

(3) We would like to point out that our result can be extended to Navier-slip boundary condition for velocity as follows:

$$\mathrm{u} \cdot n = 0, \quad \operatorname{curl} \mathrm{u} \times n = 0 \quad \text{on } \partial\Omega.$$

For more detailed derivation of the boundary condition, please refer to [4].

Let us now make some comments on the analysis of Theorem 1.1. Compared with the previous works [17, 20], we prove the global existence of the solution with a low-regularity initial data and without compatibility condition on the initial data. Because $\rho_0 \in L^\infty$ only, it gives rise to more difficulty to prove the uniqueness. On the other hand, from the classical weak-strong uniqueness results given by Lions [25] and Germain [16] for the inhomogeneous incompressible Navier–Stokes, we know that $\nabla\rho \in L^\infty L^3$ is a sufficient condition to prove the uniqueness. We can not expect more regularity on ρ than ρ_0 due to the hyperbolicity of the density equation $(1.1)_1$. As a consequence, it seems impossible to establish the uniqueness with only the regularity assumption (1.7). To overcome this difficulty, the technique of Lagrangian coordinates is borrowed from Danchin-Mucha [10,11], in which they considered the density-dependent incompressible Navier-Stokes equations. We would like to point out that it is not a trivial extension from Navier–Stokes equations to MHD system (1.1), because we have to make more considerable effort to deal with the new difficulties caused by the strong coupling between velocity and magnetic field. For example, we find that we must introduce an additional term $\|\mathrm{H}\|_{L^4}^4$ to establish the one order energy inequality due to the inclusion of the magnetic field (see Lemma 3.2). In addition, some new estimates for magnetic fields H are needed to be developed in the Lagrangian coordinates (see Section 4.2). After removing the compatibility condition (1.5), we have to introduce a time-weighted energy method to obtain higher regularity of the solution. To obtain the decay rate, we make the initial layer analysis by virtue of the time-weighted function $\eta(t)=\min\{t,1\}$ instead of $\eta(t)=t$ with $t\geq 0$ in contrast with [11]. The main advantage of choosing such a time-weighted function is that it allows us to get the uniform estimate of $\|e^{\sigma t}\eta^{1/2}\rho^{1/2}\mathrm{u}_t\|_{L^\infty L^2}^2$ and $\|e^{\sigma t}\eta^{1/2}\mathrm{H}_t\|_{L^\infty L^2}^2$ with respect to time $t>0$ (see Lemma 3.3). Consequently, the exponential decay of the solution follows.

The rest of the paper is organized as follows. In Section 2, we shall recall some well-known inequalities, which will be used frequently in the sequel. In Section 3, we first deduce some uniform estimates and the time-weighted estimate of the lower-order derivative of the smooth approximated solutions, and then the estimates of the higher-order derivative. Finally, we complete the proof of Theorem 1.1 in Section 4.

2. Auxiliary lemmas

In this section, we firstly recall some well-known Sobolev embedding inequalities (see [14, 21]), which will be frequently used in this paper.

LEMMA 2.1. *The following inequalities hold for all $2\leq p\leq 6$:*

$$(1)\ \|f\|_{L^p}\leq C\|f\|_{H^1},\ \forall\ f\in H^1, \tag{2.1}$$

$$(2)\ \|f\|_{L^\infty}\leq C\|f\|_{W^{1,r}},\ \forall\ f\in W^{1,r}\text{ with } r\in(3,\infty), \tag{2.2}$$

$$(3)\ \|f\|_{L^p}\leq C\|\nabla f\|_{L^2},\ \forall\ f\in H_0^1\text{ or } H_n^1, \tag{2.3}$$

$$(4)\ \|f\|_{L^p}^p\leq C\|f\|_{L^2}^{(6-p)/2}\|\nabla f\|_{L^2}^{(3p-6)/2},\ \forall\ f\in H_0^1\text{ or } H_n^1. \tag{2.4}$$

To deal with the estimate of the magnetic field H, we also need to state the following lemma (see [29]), which reveals that $\|\nabla\mathrm{H}\|_{L^2}$ is equivalent to $\|\mathrm{curl}\mathrm{H}\|_{L^2}$ as $\mathrm{div}\,\mathrm{H}=0$. Therefore, we shall not distinguish these two quantities with no confusion in the sequel.

LEMMA 2.2. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with C^2 boundary and $v\in H_0^1$ or H_n^1 be a vector-valued function. Then, it holds*

$$\|\nabla v\|_{L^2}\leq C\left(\|\mathrm{div}v\|_{L^2}+\|\mathrm{curl}v\|_{L^2}\right). \tag{2.5}$$

In particular, if $\operatorname{div} v = 0$*, it holds that*

$$\|\nabla v\|_{L^2} \leq C\|\operatorname{curl} v\|_{L^2}. \tag{2.6}$$

Let us give the last lemma (see [11, Lemma 3.4]), which plays a key role in improving the time-regularity for the velocity field by virtue of its time-weighted estimate.

LEMMA 2.3. *Let* $p \in [1,\infty]$, $v \in L^2(0,T;L^p)$ *and* $e^{\sigma t}\eta^{1/2}(t)v_t \in L^2(0,T;L^p)$ *with* $\eta(t) = \min\{t,1\}$, $t \geq 0$. *Then,* v *is in* $H^{\frac{1}{2}-\alpha}(0,T;L^p)$ *for all* $\alpha \in \left(0,\frac{1}{2}\right)$. *In particular, we have*

$$\|v\|^2_{H^{1/2-\alpha}(0,T;L^p)} \leq \|v\|^2_{L^2(0,T;L^p)} + C(\alpha,T)\left\|\sqrt{t}v_t\right\|^2_{L^2(0,T;L^p)}. \tag{2.7}$$

3. Some a priori estimates

This section is devoted to the proof of a priori estimates for a solution $(\rho,\mathrm{u},\mathrm{H})$ to the initial-boundary value problem (1.1)–(1.3). In particular, all of the estimates hold for its approximate solutions, which will be constructed in Section 4. In this section, the letter C stands for a generic constant, depending on the up bound $\bar{\rho}$ of the initial density, ν, μ and Ω, but independent of T, and may change its value even in a single string of estimates.

We begin with the following standard energy estimate for the solution $(\rho,\mathrm{u},\mathrm{H})$ to (1.1)–(1.3) and its time-weighted estimate. Although the time-weighted estimate in the following lemma is simple, it is crucial to obtain the uniform estimate of higher order derivative of the solution.

LEMMA 3.1. *For any given* $T > 0$, *let* $(\rho,\mathrm{u},\mathrm{H})$ *be a smooth solution to* (1.1)–(1.3) *on* $\Omega \times (0,T)$. *Then, it holds that*

$$0 \leq \rho(x,t) \leq \bar{\rho}, \tag{3.1}$$

$$\begin{aligned}&\sup_{0\leq t\leq T}\int\left(\rho|\mathrm{u}|^2 + |\mathrm{H}|^2\right)\mathrm{d}x + \int_0^T\left(2\mu\|\nabla\mathrm{u}\|^2_{L^2} + 2\nu\|\nabla\mathrm{H}\|^2_{L^2}\right)\mathrm{d}t\\ &\leq \int\left(\rho_0|\mathrm{u}_0|^2 + |\mathrm{H}_0|^2\right)\mathrm{d}x,\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}&\sup_{0\leq t\leq T} e^{\sigma t}\int\left(\rho|\mathrm{u}|^2 + |\mathrm{H}|^2\right)\mathrm{d}x + \int_0^T e^{\sigma t}\left(\mu\|\nabla\mathrm{u}\|^2_{L^2} + \nu\|\nabla\mathrm{H}\|^2_{L^2}\right)\mathrm{d}t\\ &\leq \int\left(\rho_0|\mathrm{u}_0|^2 + |\mathrm{H}_0|^2\right)\mathrm{d}x,\end{aligned} \tag{3.3}$$

where σ *is a given positive constant depending on* Ω, μ, ν *and* $\bar{\rho}$.

Proof. Equation (3.1) is a direct consequence of the transport Equation $(1.1)_1$. (3.2) is the basic energy inequality for system (1.1). To do this, multiplying $(1.1)_2$ by u and then integrating by parts over Ω lead to[1]

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|\mathrm{u}|^2\mathrm{d}x + \mu\int|\nabla\mathrm{u}|^2\mathrm{d}x = -\int\mathrm{H}\cdot\nabla\mathrm{u}\cdot\mathrm{H}\mathrm{d}x. \tag{3.4}$$

[1]Hereafter we shall make frequent use of the following identities: for all vector functions Φ and Ψ, we have

$$\nabla\times\Phi\times\Phi = -\frac{1}{2}\nabla|\nabla\Phi|^2 + \Phi\cdot\nabla\Phi,\ \Delta\Phi = \nabla\operatorname{div}\Phi - \nabla\times(\nabla\times\Phi),$$

Similarly, from $(1.1)_3$ it holds that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathrm{H}|^2 \mathrm{d}x + \nu\int |\nabla\times \mathrm{H}|^2 \mathrm{d}x = \int \mathrm{H}\cdot\nabla \mathrm{u}\cdot H \mathrm{d}x, \tag{3.5}$$

where we use the following calculation

$$\begin{aligned}\int \nabla\times(\nabla\times \mathrm{H})\cdot \mathrm{H}\mathrm{d}x &= -\int_{\partial\Omega}(\nabla\times \mathrm{H}\times n)\cdot \mathrm{H}\mathrm{d}S + \int |\nabla\times \mathrm{H}|^2\mathrm{d}x\\ &= \int |\nabla\times \mathrm{H}|^2 dx.\end{aligned}$$

Adding (3.4) to (3.5), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u|^2 + |\mathrm{H}|^2\mathrm{d}x + 2\int \mu|\nabla \mathrm{u}|^2 + \nu|\nabla\times\mathrm{H}|^2\mathrm{d}x = 0, \tag{3.6}$$

and then (3.2) follows immediately after integrating (3.6) over $(0,T)$.

It is easy to prove that there exists a constant $\sigma>0$, depending on $\bar\rho$, μ, ν, Ω and the constants in Lemma 2.2, such that

$$\sigma\left(\|\rho^{1/2}\mathrm{u}\|_{L^2}^2 + \|\mathrm{H}\|_{L^2}^2\right) \le \mu\|\nabla \mathrm{u}\|_{L^2}^2 + \nu\|\nabla\times\mathrm{H}\|_{L^2}^2,$$

which, together with (3.6), leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\mathrm{u}|^2 + |\mathrm{H}|^2\mathrm{d}x + \sigma\int \rho|\mathrm{u}|^2 + |\mathrm{H}|^2\mathrm{d}x + \int \mu|\nabla\mathrm{u}|^2 + \nu|\nabla\times\mathrm{H}|^2\mathrm{d}x \le 0.$$

Then, we obtain (3.3) by multiplying the above inequality by $e^{\sigma t}$ and integrating the result over $(0,T)$. ∎

Next, we shall obtain a key estimate $\|\nabla\mathrm{u},\nabla\mathrm{H}\|_{L^\infty(0,T;L^2)}$ provided that E_0 is small enough.

Lemma 3.2. *For any given $T>0$, let $(\rho,\mathrm{u},\mathrm{H})$ be a smooth solution to (1.1)–(1.3) on $\Omega\times(0,T)$. Then, there exists a constant C such that*

$$\sup_{0\le t\le T}\left(\|\mathrm{u}\|_{H^1}^2 + \|\mathrm{H}\|_{L^4}^4 + \|\mathrm{H}\|_{H^1}^2\right) + \int_0^T \left(\|\rho^{1/2}\mathrm{u}_t\|_{L^2}^2 + \|\mathrm{H}_t\|_{L^2}^2 + \|\mathrm{H}\|_{H^2}^2 + \|\mathrm{u}\|_{H^2}^2\right)\mathrm{d}t \le C, \tag{3.7}$$

provided that (1.9) holds. Moreover, it holds that

$$\begin{aligned}&\sup_{0\le t\le T} e^{\sigma t}\left(\|\nabla\mathrm{u}\|_{L^2}^2 + \|\nabla\times\mathrm{H}\|_{L^2}^2 + \|\mathrm{H}\|_{L^4}^4\right)\\ &\quad + \int_0^T e^{\sigma t}\left(\|\rho^{1/2}\mathrm{u}_t\|_{L^2}^2 + \|\mathrm{H}_t\|_{L^2}^2 + \|\mathrm{H}\|_{H^2}^2 + \|\mathrm{u}\|_{H^2}^2\right)\mathrm{d}t \le C.\end{aligned} \tag{3.8}$$

$$\nabla\times(\Phi\times\Psi) = (\Psi\cdot\nabla)\Phi - (\Phi\cdot\nabla)\Psi + (\mathrm{div}\,\Psi)\Phi - (\mathrm{div}\,\Phi)\Psi,$$

and

$$\int \nabla\times\Phi\cdot\Psi\mathrm{d}x = \int \Phi\cdot\nabla\times\Psi\mathrm{d}x + \int_{\partial\Omega} n\times\Phi\cdot\Psi\mathrm{d}S.$$

Proof. Firstly, multiplying $(1.1)_2$ by u_t and integrating by parts over Ω yield

$$\begin{aligned}
&\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \mathrm{u}\|_{L^2}^2+\|\rho^{1/2}\mathrm{u}_t\|_{L^2}^2\\
&=-\int(\mathrm{H}\cdot\nabla\mathrm{u}_t\cdot\mathrm{H}+\rho(\mathrm{u}\cdot\nabla)\mathrm{u}\cdot\mathrm{u}_t)\mathrm{d}x\\
&=-\frac{\mathrm{d}}{\mathrm{d}t}\int\mathrm{H}\cdot\nabla\mathrm{u}\cdot\mathrm{H}\mathrm{d}x+\int\mathrm{H}\cdot\nabla\mathrm{u}\cdot\mathrm{H}_t+\mathrm{H}_t\cdot\nabla\mathrm{u}\cdot\mathrm{H}\mathrm{d}x-\int\rho(\mathrm{u}\cdot\nabla)\mathrm{u}\cdot\mathrm{u}_t\mathrm{d}x.
\end{aligned}$$

On the other hand, it follows from $(1.1)_3$ that

$$\nu\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\times\mathrm{H}\|_{L^2}^2+\left(\|\mathrm{H}_t\|_{L^2}^2+\nu^2\|\nabla\times\nabla\times\mathrm{H}\|_{L^2}\right)=\int|\mathrm{H}\cdot\nabla\mathrm{u}-\mathrm{u}\cdot\nabla\mathrm{H}|^2\mathrm{d}x.$$

After summing up the last two identities, we obtain that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\mu|\nabla\mathrm{u}|^2+2\mathrm{H}\cdot\nabla\mathrm{u}\cdot\mathrm{H}+\nu|\nabla\times\mathrm{H}|^2\mathrm{d}x+2\|\rho^{1/2}\mathrm{u}_t\|_{L^2}^2+\|\mathrm{H}_t\|_{L^2}^2+\nu^2\|\nabla\times\nabla\times\mathrm{H}\|_{L^2}^2\\
&=2\int\mathrm{H}\cdot\nabla\mathrm{u}\cdot\mathrm{H}_t+\mathrm{H}_t\cdot\nabla\mathrm{u}\cdot\mathrm{H}\mathrm{d}x-2\int\rho(\mathrm{u}\cdot\nabla)\mathrm{u}\cdot\mathrm{u}_t\mathrm{d}x+\int|\mathrm{H}\cdot\nabla\mathrm{u}-\mathrm{u}\cdot\nabla\mathrm{H}|^2\mathrm{d}x\\
&:=\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3.
\end{aligned}\tag{3.9}$$

Let us now estimate the terms on the right-hand side of the above identity term by term. By the Hölder inequality, Young inequality and Gagliardo-Nirenberg inequality, it yields

$$\begin{aligned}
\mathrm{I}_1&=2\int\mathrm{H}_t\cdot\nabla\mathrm{u}\cdot\mathrm{H}+\mathrm{H}\cdot\nabla\mathrm{u}\cdot\mathrm{H}_t\mathrm{d}x\\
&\le\frac{1}{2}\|\mathrm{H}_t\|_{L^2}^2+C\|\mathrm{H}\|_{L^6}^2\|\nabla\mathrm{u}\|_{L^3}^2\\
&\le\frac{1}{2}\|\mathrm{H}_t\|_{L^2}^2+C\|\nabla\mathrm{H}\|_{L^2}^2\|\nabla\mathrm{u}\|_{L^2}\|\nabla\mathrm{u}\|_{H^1},\\
\mathrm{I}_2&=-2\int\rho(\mathrm{u}\cdot\nabla)\mathrm{u}\cdot\mathrm{u}_t\mathrm{d}x\\
&\le\frac{3}{2}\|\rho^{1/2}\mathrm{u}_t\|_{L^2}^2+C\|\mathrm{u}\|_{L^6}^2\|\nabla\mathrm{u}\|_{L^3}^2\\
&\le\frac{3}{2}\|\rho^{1/2}\mathrm{u}_t\|_{L^2}^2+C\|\nabla\mathrm{u}\|_{L^2}^3\|\nabla\mathrm{u}\|_{H^1},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{I}_3&=\int|\mathrm{H}\cdot\nabla\mathrm{u}-\mathrm{u}\cdot\nabla\mathrm{H}|^2\mathrm{d}x\\
&\le C\|\mathrm{H}\|_{L^6}^2\|\nabla\mathrm{u}\|_{L^3}^2+C\|\mathrm{u}\|_{L^6}^2\|\nabla\mathrm{H}\|_{L^3}^2\\
&\le C\|\nabla\mathrm{H}\|_{L^2}^2\|\nabla\mathrm{u}\|_{L^2}\|\nabla\mathrm{u}\|_{H^1}+C\|\nabla\mathrm{u}\|_{L^2}^2\|\nabla\mathrm{H}\|_{L^2}\|\nabla\mathrm{H}\|_{H^1}.
\end{aligned}$$

Substituting I_1–I_3 into (3.9) and using the Young inequality, it holds that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\mu|\nabla\mathrm{u}|^2+2\mathrm{H}\cdot\nabla\mathrm{u}\cdot\mathrm{H}+\nu|\nabla\times\mathrm{H}|^2\mathrm{d}x+\frac{1}{2}\|\rho^{1/2}\mathrm{u}_t\|_{L^2}^2+\frac{1}{2}\|\mathrm{H}_t\|_{L^2}^2+\nu^2\|\nabla\times\nabla\times\mathrm{H}\|_{L^2}^2\\
&\le C\left(\|\nabla\mathrm{H}\|_{L^2}^6+\|\nabla\mathrm{u}\|_{L^2}^6\right)+\varepsilon(\|\nabla\mathrm{u}\|_{H^1}^2+\|\nabla\mathrm{H}\|_{H^1}^2).
\end{aligned}\tag{3.10}$$

To close the estimates, we have to get the estimate of $\|\nabla \mathrm{u}\|_{H^1}$ and $\|\nabla \mathrm{H}\|_{H^1}$ on the right-hand side of the above inequality. To this end, we need to rewrite $(1.1)_2$ and $(1.1)_3$ into the following form, that is, u and H satisfy, respectively, the following Stokes equations

$$\begin{cases} -\Delta \mathrm{u}+\nabla P=-\rho \mathrm{u}_t-\rho \mathrm{u}\cdot\nabla \mathrm{u}+\frac{1}{2}\nabla|\mathrm{H}|^2-\mathrm{H}\cdot\nabla \mathrm{H}, \text{ in } \Omega,\\ \mathrm{div}\mathrm{u}=0, \text{ in } \Omega,\\ \mathrm{u}=0, \text{ on } \partial\Omega, \end{cases}$$

and elliptic equations

$$\begin{cases} -\nu\Delta \mathrm{H}=-\mathrm{H}_t-\mathrm{u}\cdot\nabla \mathrm{H}+\mathrm{H}\cdot\nabla \mathrm{u}, \text{ in } \Omega,\\ \mathrm{div}\mathrm{H}=0, \text{ in } \Omega,\\ \mathrm{H}\cdot n=\nabla\times \mathrm{H}=0, \text{ on } \partial\Omega. \end{cases}$$

By the well-known regularity theory on Stokes equations (see [14]), we have

$$\begin{aligned} \|\mathrm{u}\|_{H^2}+\|\nabla P\|_{H^1} &\le C(\|\rho \mathrm{u}_t\|_{L^2}+\|\rho \mathrm{u}\cdot\nabla \mathrm{u}\|_{L^2}+\|\nabla|\mathrm{H}|^2\|_{L^2}+\|\mathrm{H}\cdot\nabla \mathrm{H}\|_{L^2})\\ &\le C(\|\rho^{1/2}\mathrm{u}_t\|_{L^2}+\|\mathrm{u}\|_{L^6}\|\nabla \mathrm{u}\|_{L^3}+\|\mathrm{H}\|_{L^6}\|\nabla \mathrm{H}\|_{L^3})\\ &\le C(\|\rho^{1/2}\mathrm{u}_t\|_{L^2}+\|\nabla \mathrm{u}\|_{L^2}^{3/2}\|\mathrm{u}\|_{H^2}^{1/2}+\|\nabla \mathrm{H}\|_{L^2}^{3/2}\|\mathrm{H}\|_{H^2}^{1/2}), \end{aligned}$$

and

$$\begin{aligned} \|\mathrm{H}\|_{H^2} &\le C\|\mathrm{H}_t\|_{L^2}+C\|\mathrm{u}\cdot\nabla \mathrm{H}\|_{L^2}+C\|\mathrm{H}\cdot\nabla \mathrm{u}\|_{L^2}\\ &\le C\|\mathrm{H}_t\|_{L^2}+C\|\mathrm{u}\|_{L^6}\|\nabla \mathrm{H}\|_{L^3}+C\|\mathrm{H}\|_{L^6}\|\nabla \mathrm{u}\|_{L^3}\\ &\le C\|\mathrm{H}_t\|_{L^2}+C\|\nabla \mathrm{u}\|_{L^2}\|\nabla \mathrm{H}\|_{L^2}^{1/2}\|\mathrm{H}\|_{H^2}^{1/2}+C\|\nabla \mathrm{H}\|_{L^2}\|\nabla \mathrm{u}\|_{L^2}^{1/2}\|\mathrm{u}\|_{H^2}^{1/2}, \end{aligned}$$

which, together with the Young inequality, immediately leads to

$$\|\mathrm{u}\|_{H^2}+\|\mathrm{H}\|_{H^2}+\|\nabla P\|_{L^2}\le C(\|\rho^{1/2}\mathrm{u}_t\|_{L^2}+\|\mathrm{H}_t\|_{L^2}+\|\nabla \mathrm{H}\|_{L^2}^3+\|\nabla \mathrm{u}\|_{L^2}^3). \tag{3.11}$$

On the other hand, multiplying $(1.1)_3$ by $\mathrm{H}|\mathrm{H}|^2$ and integrating the resulting equality by parts over Ω lead to

$$\begin{aligned} &\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int|\mathrm{H}|^4\mathrm{d}x+\nu\||\mathrm{H}||\nabla\times \mathrm{H}|\|_{L^2}^2+\frac{\nu}{2}\|\nabla|\mathrm{H}|^2|\|_{L^2}^2\\ &\le C\|\nabla \mathrm{u}\|_{L^3}\|\mathrm{H}\|_{L^6}^4+C\int|\mathrm{H}||\nabla|\mathrm{H}|^2||\nabla \mathrm{H}|\mathrm{d}x\\ &\le C\|\nabla \mathrm{u}\|_{L^2}^{1/2}\|\nabla \mathrm{u}\|_{H^1}^{1/2}\|\nabla \mathrm{H}\|_{L^2}^4+\frac{\nu}{4}\|\nabla|\mathrm{H}|^2|\|_{L^2}^2+C\|\mathrm{H}\|_{L^6}^2\|\nabla \mathrm{H}\|_{L^3}^2\\ &\le C\|\nabla \mathrm{u}\|_{L^2}^{1/2}\|\nabla \mathrm{u}\|_{H^1}^{1/2}\|\nabla \mathrm{H}\|_{L^2}^4+\frac{\nu}{4}\|\nabla|\mathrm{H}|^2|\|_{L^2}^2+C\|\nabla \mathrm{H}\|_{L^2}^3\|\nabla \mathrm{H}\|_{H^1}\\ &\le C\|\nabla \mathrm{u}\|_{L^2}^6+C\|\nabla \mathrm{H}\|_{L^2}^6+\frac{\nu}{4}\|\nabla|\mathrm{H}|^2\|_{L^2}^2+\varepsilon\|\nabla \mathrm{H}\|_{H^1}^2+\varepsilon\|\nabla \mathrm{u}\|_{H^1}^2, \end{aligned} \tag{3.12}$$

where we use the following calculation

$$\begin{aligned} &\int\nabla\times(\nabla\times)\mathrm{H}\cdot|\mathrm{H}|^2\mathrm{H}\mathrm{d}x\\ &=\int|\mathrm{H}|^2\nabla\times \mathrm{H}\cdot\nabla\times \mathrm{H}\mathrm{d}x+\int\nabla\times \mathrm{H}\cdot(\nabla|\mathrm{H}|^2\times \mathrm{H})\mathrm{d}x-\int|\mathrm{H}|^2\nabla\times \mathrm{H}\times n\cdot \mathrm{H}\mathrm{d}S \end{aligned}$$

$$= \int ||\mathrm{H}||\nabla\times\mathrm{H}||^2 \mathrm{d}x + \frac{1}{2}\int |\nabla|\mathrm{H}|^2|^2 \mathrm{d}x - \int \mathrm{H}\cdot\nabla\mathrm{H}\cdot\nabla|\mathrm{H}|^2 \mathrm{d}x.$$

It is easy to prove that there exist two constants $C_1>0$ and $C_2>0$ such that

$$\begin{aligned}
&2\mu\|\nabla \mathrm{u}\|_{L^2}^2+\nu\|\nabla\times\mathrm{H}\|_{L^2}^2+C_2\|\mathrm{H}\|_{L^4}^4\\
\geq &\mathrm{Y}_0(t):=\mu\|\nabla\mathrm{u}\|_{L^2}^2+\nu\|\nabla\times\mathrm{H}\|_{L^2}^2+C_1\|\mathrm{H}\|_{L^4}^4+2\int \mathrm{H}\cdot\nabla\mathrm{u}\cdot\mathrm{H}dx\\
\geq &\frac{\mu}{2}\|\nabla\mathrm{u}\|_{L^2}^2+\nu\|\nabla\times\mathrm{H}\|_{L^2}^2+\frac{C_1}{2}\|\mathrm{H}\|_{L^4}^4\geq 0.
\end{aligned}$$

Multiplying (3.12) by $4C_1$ and summing the result and (3.10) together, and then taking ε suitably small lead to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Y}(t)\leq C\mathrm{Y}^2(t)(\|\nabla\mathrm{u}\|_{L^2}^2+\|\nabla\mathrm{H}\|_{L^2}^2), \tag{3.13}$$

where $\mathrm{Y}(t)$ is defined as

$$\mathrm{Y}(t):=\mathrm{Y}_0(t)+\frac{1}{4}\int_0^t \|\rho^{1/2}\mathrm{u}_s\|_{L^2}^2+\|\mathrm{H}_s\|_{L^2}^2+\nu^2\|\nabla\times\nabla\times\mathrm{H}\|_{L^2}^2+\frac{\nu}{2}\|\nabla|\mathrm{H}|^2\|_{L^2}^2\mathrm{d}s.$$

For any $t\in[0,T)$, a direct calculation immediately yields

$$\mathrm{Y}(t)\leq\frac{\mathrm{Y}(0)}{1-C\mathrm{Y}(0)\int_0^t\|\nabla\mathrm{u}\|_{L^2}^2+\|\nabla\mathrm{H}\|_{L^2}^2\mathrm{d}s}. \tag{3.14}$$

On the other hand, using the energy inequality (3.2), it holds that

$$\begin{aligned}
&\mathrm{Y}(0)\int_0^t\|\nabla\mathrm{u}\|_{L^2}^2+\|\nabla\mathrm{H}\|_{L^2}^2\mathrm{d}s\\
\leq &C_3\left(\|\rho_0^{1/2}\mathrm{u}_0\|_{L^2}^2+\|\mathrm{H}_0\|_{L^2}^2\right)\left(\|\nabla\mathrm{H}_0\|_{L^2}^2+\|\nabla\mathrm{u}_0\|_{L^2}^2+\|\mathrm{H}_0\|_{L^4}^4\right)\\
\leq &C_3E_0(1+E_0^{\frac{1}{2}}).
\end{aligned}$$

If we choose a E_0 such that

$$E_0\leq\epsilon=:\min\left\{1,\frac{1}{4C_3}\right\},$$

then we deduce from (3.14) for all $t\in[0,T)$

$$\mathrm{Y}(t)\leq 2\mathrm{Y}(0),$$

which immediately leads to (3.7).

To obtain (3.8), we deduce from (3.13) and (3.7) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Y}_0(t)+(\|\rho^{1/2}\mathrm{u}_t\|_{L^2}^2+\|\mathrm{H}_t\|_{L^2}^2)\leq C\mathrm{Y}_0(t). \tag{3.15}$$

We multiply (3.15) by $e^{\sigma t}$ to obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{\sigma t}\mathrm{Y}_0(t)+e^{\sigma t}(\|\rho^{1/2}\mathrm{u}_t\|_{L^2}^2+\|\mathrm{H}_t\|_{L^2}^2)\leq Ce^{\sigma t}\mathrm{Y}_0(t). \tag{3.16}$$

Once $e^{\sigma t}\mathrm{Y}_0(t)\in L^2(\mathbb{R}^+)$, it is easy to prove (3.8) by integrating (3.16) over $(0,T)$. Indeed, it holds

$$\int_0^T e^{\sigma t}\mathrm{Y}_0(t)\mathrm{d}t\le\int_0^T e^{\sigma t}\left(\frac{\mu}{2}\|\nabla \mathrm{u}\|_{L^2}^2+\nu\|\nabla\times\mathrm{H}\|_{L^2}^2+C_2\|\mathrm{H}\|_{L^4}^4\right)\mathrm{d}t$$
$$\le C+C\int_0^T e^{\sigma t}\|\mathrm{H}\|_{L^2}\|\nabla\times\mathrm{H}\|_{L^2}^3\mathrm{d}t\le C,$$

due to (3.3) and (3.7). The proof of this lemma is completed. ∎

The following lemma is concerned with the weighted L^2-estimate of $\rho^{1/2}\mathrm{u}_t$ and H_t.

Lemma 3.3. *For any given $T>0$, let $(\rho,\mathrm{u},\mathrm{H})$ be a smooth solution to (1.1)–(1.3) on $\Omega\times[0,T)$. Then, there exists a constant C such that*

$$\sup_{0\le t\le T} e^{\sigma t}\left(\|\eta^{1/2}\rho^{1/2}\mathrm{u}_t\|_{L^2}^2+\|\eta^{1/2}\mathrm{H}_t\|_{L^2}^2+\|\eta^{1/2}\mathrm{u}\|_{H^2}^2+\|\eta^{1/2}\mathrm{H}\|_{H^2}^2\right)$$
$$+\int_0^T e^{\sigma t}\|\eta^{1/2}\nabla\mathrm{u}_t\|_{L^2}^2+e^{\sigma t}\|\eta^{1/2}\nabla\mathrm{H}_t\|_{L^2}^2\mathrm{d}t\le C, \tag{3.17}$$

where $\eta(t):=\min\{t,1\}$.

Proof. Applying ∂_t to the momentum Equations $(1.1)_2$ yields

$$\rho\mathrm{u}_{tt}+\rho\mathrm{u}\cdot\nabla\mathrm{u}_t-\mu\triangle\mathrm{u}_t=-\rho\mathrm{u}_t\cdot\nabla\mathrm{u}-\rho_t(\mathrm{u}_t+\mathrm{u}\cdot\nabla\mathrm{u})-\nabla P_t+(\mathrm{H}\cdot\nabla\mathrm{H}-\frac{1}{2}\nabla|\mathrm{H}|^2)_t.$$

Multiplying the above equation by $\eta(t)\mathrm{u}_t$ and integrating by parts over Ω, one gets

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta(t)^{1/2}\rho^{1/2}\mathrm{u}_t\|_{L^2}^2+\|\eta(t)^{1/2}\nabla\mathrm{u}_t\|_{L^2}^2$$
$$=\frac{1}{2}\eta'(t)\|\rho^{1/2}\mathrm{u}_t\|_{L^2}^2-\int\eta(t)\rho_t|\mathrm{u}_t|^2\mathrm{d}x-\int\rho(\mathrm{u}_t\cdot\nabla\mathrm{u})\cdot\eta(t)\mathrm{u}_t\mathrm{d}x$$
$$-\int\rho_t(\mathrm{u}\cdot\nabla\mathrm{u})\cdot\eta(t)\mathrm{u}_t\mathrm{d}x+\int(\mathrm{H}_t\cdot\nabla\mathrm{H}+\mathrm{H}\cdot\nabla\mathrm{H}_t)\cdot\eta(t)\mathrm{u}_t\mathrm{d}x. \tag{3.18}$$

Similarly, differentiating $(1.1)_3$ with respect to t and multiplying the resulting equation by $\eta(t)\mathrm{H}_t$, we obtain after integrating by parts that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta(t)^{1/2}\mathrm{H}_t\|_{L^2}^2+\|\eta(t)^{1/2}\nabla\times\mathrm{H}_t\|_{L^2}^2$$
$$=\frac{1}{2}\eta'(t)\|\mathrm{H}_t\|_{L^2}^2-\int\mathrm{u}_t\cdot\nabla\mathrm{H}\cdot\eta(t)\mathrm{H}_t\mathrm{d}x+\int\mathrm{H}_t\cdot\nabla\mathrm{u}\cdot\eta(t)\mathrm{H}_t-\mathrm{H}\cdot\nabla\mathrm{H}_t\cdot\eta(t)\mathrm{u}_t\mathrm{d}x. \tag{3.19}$$

Combining (3.18) with (3.19), one gets that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\eta(t)^{1/2}\mathrm{H}_t\|_{L^2}^2+\|\eta(t)^{1/2}\rho^{1/2}\mathrm{u}_t\|_{L^2}^2\right)+\|\eta(t)^{1/2}\nabla\mathrm{u}_t\|_{L^2}^2+\|\eta(t)^{1/2}\nabla\times\mathrm{H}_t\|_{L^2}^2$$
$$=\frac{1}{2}\eta'(t)\int\rho|u_t|^2+|\mathrm{H}_t|^2\mathrm{d}x-\int\eta(t)\rho_t|\mathrm{u}_t|^2\mathrm{d}x-\int\eta(t)\rho_t(\mathrm{u}\cdot\nabla\mathrm{u})\cdot\mathrm{u}_t\mathrm{d}x$$
$$-\int\eta(t)\rho(\mathrm{u}_t\cdot\nabla\mathrm{u})\cdot\mathrm{u}_t\mathrm{d}x+\int\eta(t)(\mathrm{H}_t\cdot\nabla\mathrm{H}\cdot\mathrm{u}_t-\mathrm{u}_t\cdot\nabla\mathrm{H}\cdot\mathrm{H}_t)\mathrm{d}x+\int\eta(t)\mathrm{H}_t\cdot\nabla\mathrm{u}\cdot\mathrm{H}_t\mathrm{d}x$$
$$:=\frac{1}{2}\eta'(t)\int\rho|u_t|^2+|\mathrm{H}_t|^2\mathrm{d}x+\sum_{i=1}^5\mathrm{R}_i. \tag{3.20}$$

Let us estimate every term on the right-hand side of (3.20). First, using the Hölder inequality, Young inequality and $(1.1)_1$, we obtain

$$
\begin{aligned}
|\,\mathrm{R}_1| &= \left|\int \eta(t)\rho\mathrm{u}\cdot\nabla|\mathrm{u}_t|^2\mathrm{d}x\right| \\
&\leq C\|\eta(t)^{1/2}\nabla\mathrm{u}_t\|_{L^2}\|\eta(t)^{1/2}\rho^{1/2}\mathrm{u}_t\|_{L^3}\|\rho^{1/2}\mathrm{u}\|_{L^6} \\
&\leq C\|\nabla\mathrm{u}\|_{L^2}\|\eta(t)^{1/2}\rho^{1/2}\mathrm{u}_t\|_{L^2}^{1/2}\|\eta(t)^{1/2}\nabla\mathrm{u}_t\|_{L^2}^{3/2} \\
&\leq \varepsilon\|\eta(t)^{1/2}\nabla\mathrm{u}_t\|_{L^2}^2 + C(\varepsilon)\|\nabla\mathrm{u}\|_{L^2}^4\|\eta(t)^{1/2}\rho^{1/2}\mathrm{u}_t\|_{L^2}^2.
\end{aligned}
$$

Using $(1.1)_1$ once again, it holds

$$
\begin{aligned}
|\mathrm{R}_2| &= \left|\int \eta(t)\rho\mathrm{u}\cdot\nabla(\mathrm{u}\cdot\nabla\mathrm{u}\cdot\mathrm{u}_t)\mathrm{d}x\right| \\
&\leq C\int \eta(t)\rho|\mathrm{u}||\nabla\mathrm{u}|^2|\mathrm{u}_t| + \eta(t)\rho|\mathrm{u}|^2|\nabla^2\mathrm{u}||\mathrm{u}_t| + \eta(t)\rho|\mathrm{u}|^2|\nabla\mathrm{u}||\nabla\mathrm{u}_t|\mathrm{d}x \\
&:= \sum_{i=1}^{3}\mathrm{R}_{2i}.
\end{aligned}
$$

By the Gagliardo-Nirenberg inequality, one obtains that

$$
\begin{aligned}
|\mathrm{R}_{21}| &\leq C\int \eta(t)\rho|\mathrm{u}||\nabla\mathrm{u}|^2|\mathrm{u}_t|\mathrm{d}x \\
&\leq C\|\eta(t)^{1/2}\rho^{1/2}\mathrm{u}_t\|_{L^2}\|\nabla\mathrm{u}\|_{L^6}^2\eta(t)^{1/2}\|\rho^{1/2}\mathrm{u}\|_{L^6} \\
&\leq C\|\eta(t)^{1/2}\rho^{1/2}\mathrm{u}_t\|_{L^2}^2\|\nabla\mathrm{u}\|_{H^1}^2 + C\|\nabla\mathrm{u}\|_{H^1}^2,
\end{aligned}
$$

$$
\begin{aligned}
|\mathrm{R}_{22}| &= \int \eta(t)\rho|\mathrm{u}|^2|\nabla^2\mathrm{u}||\mathrm{u}_t|\mathrm{d}x \\
&\leq C\eta(t)^{1/2}\|\nabla^2\mathrm{u}\|_{L^2}\|\mathrm{u}\|_{L^6}^2\|\eta(t)^{1/2}\mathrm{u}_t\|_{L^6} \\
&\leq C\|\nabla^2\mathrm{u}\|_{L^2}\|\eta(t)^{1/2}\nabla\mathrm{u}_t\|_{L^2}\eta(t)^{1/2}\|\nabla\mathrm{u}\|_{L^2}^2 \\
&\leq \varepsilon\|\eta(t)^{1/2}\nabla\mathrm{u}_t\|_{L^2}^2 + C(\varepsilon)\|\nabla^2\mathrm{u}\|_{L^2}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathrm{R}_{23}| &= \left|\int \eta(t)\rho|\mathrm{u}|^2|\nabla\mathrm{u}||\nabla\mathrm{u}_t|\mathrm{d}x\right| \\
&\leq C\|\nabla\mathrm{u}\|_{L^6}\eta(t)^{1/2}\|\mathrm{u}\|_{L^6}^2\|\eta(t)^{1/2}\nabla\mathrm{u}_t\|_{L^2} \\
&\leq C\|\nabla\mathrm{u}\|_{L^6}\eta(t)^{1/2}\|\nabla\mathrm{u}\|_{L^2}^2\|\eta(t)^{1/2}\nabla\mathrm{u}_t\|_{L^2} \\
&\leq \varepsilon\|\eta(t)^{1/2}\nabla\mathrm{u}_t\|_{L^2}^2 + C(\varepsilon)\|\nabla\mathrm{u}\|_{H^1}^2,
\end{aligned}
$$

due to (3.2) and (3.7). Inserting all the estimates of R_{21}–R_{23} into R_2, one gets

$$
|\mathrm{R}_2| \leq 2\varepsilon\|\eta(t)^{1/2}\nabla\mathrm{u}_t\|_{L^2}^2 + C\|\eta(t)^{1/2}\rho^{1/2}\mathrm{u}_t\|_{L^2}^2\|\nabla\mathrm{u}\|_{H^1}^2 + C\|\nabla\mathrm{u}\|_{H^1}^2.
$$

Using the Gagliardo-Nirenberg inequality again, we also get

$$
|\mathrm{R}_3| \leq C\int \eta(t)\rho|\mathrm{u}_t|^2|\nabla\mathrm{u}|\mathrm{d}x \leq C\|\nabla\mathrm{u}\|_{L^2}\|\eta(t)^{1/2}\rho^{1/2}\mathrm{u}_t\|_{L^4}^2
$$

$$\le C\|\nabla \mathrm{u}\|_{L^2}\|\eta(t)^{1/2}\rho^{1/2}\mathrm{u}_t\|_{L^2}^{1/2}\|\eta(t)^{1/2}\nabla \mathrm{u}_t\|_{L^2}^{3/2}$$
$$\le \varepsilon\|\eta(t)^{1/2}\nabla \mathrm{u}_t\|_{L^2}^2 + C(\varepsilon)\|\nabla \mathrm{u}\|_{L^2}^4\|\eta(t)^{1/2}\rho^{1/2}\mathrm{u}_t\|_{L^2}^2,$$

$$|\mathrm{R}_4| \le C\int \eta(t)|\mathrm{H}_t||\nabla \mathrm{H}||\mathrm{u}_t|\mathrm{d}x \le C\|\nabla \mathrm{H}\|_{L^3}\|\eta(t)^{1/2}\mathrm{H}_t\|_{L^2}\|\eta(t)^{1/2}\mathrm{u}_t\|_{L^6}$$
$$\le C\|\nabla \mathrm{H}\|_{H^1}^{1/2}\|\eta(t)^{1/2}\mathrm{H}_t\|_{L^2}\|\eta(t)^{1/2}\nabla \mathrm{u}_t\|_{L^2}$$
$$\le \varepsilon\|\eta(t)^{1/2}\nabla \mathrm{u}_t\|_{L^2}^2 + C(\varepsilon)\|\nabla \mathrm{H}\|_{H^1}^2\|\eta(t)^{1/2}\mathrm{H}_t\|_{L^2}^2,$$

and

$$|\mathrm{R}_5| \le C\int \eta(t)|\mathrm{H}_t|^2|\nabla \mathrm{u}|\mathrm{d}x \le C\|\eta(t)^{1/2}\mathrm{H}_t\|_{L^4}^2\|\nabla \mathrm{u}\|_{L^2}$$
$$\le \varepsilon\|\eta(t)^{1/2}\nabla \mathrm{H}_t\|_{L^2}^2 + C\|\nabla \mathrm{u}\|_{L^2}^4\|\eta(t)^{1/2}\mathrm{H}_t\|_{L^2}^2.$$

Substituting all estimates of R_1–R_5 into (3.20), and then taking ε small enough, one obtains

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\eta(t)^{1/2}\rho^{1/2}\mathrm{u}_t\|_{L^2}^2 + \|\eta(t)^{1/2}\mathrm{H}_t\|_{L^2}^2\right) + \left(\|\eta(t)^{1/2}\nabla \mathrm{u}_t\|_{L^2}^2 + \|\eta(t)^{1/2}\nabla \mathrm{H}_t\|_{L^2}^2\right)\\ &\le C(\|\eta(t)^{1/2}\rho^{1/2}\mathrm{u}_t\|_{L^2}^2 + \|\eta(t)^{1/2}\mathrm{H}_t\|_{L^2}^2)(\|\nabla \mathrm{u}\|_{H^1}^2 + \|\nabla \mathrm{H}\|_{H^1}^2 + \|\nabla \mathrm{u}\|_{L^2}^4)\\ &\quad + C(\|\nabla \mathrm{u}\|_{H^1}^2 + \|\rho^{1/2}u_t\|_{L^2}^2 + \|\mathrm{H}_t\|_{L^2}^2).\end{aligned} \tag{3.21}$$

Next, we multiply (3.21) by $e^{\sigma t}$ to obtain that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}e^{\sigma t}\left(\|\eta^{1/2}\rho^{1/2}\mathrm{u}_t\|_{L^2}^2 + \|\eta^{1/2}\mathrm{H}_t\|_{L^2}^2\right) + e^{\sigma t}\left(\|\eta^{1/2}\nabla \mathrm{u}_t\|_{L^2}^2 + \|\eta^{1/2}\nabla \mathrm{H}_t\|_{L^2}^2\right)\\ &\le Ce^{\sigma t}(\|\eta^{1/2}\rho^{1/2}\mathrm{u}_t\|_{L^2}^2 + \|\eta^{1/2}\mathrm{H}_t\|_{L^2}^2)(\|\nabla^2 \mathrm{u}\|_{L^2}^2 + \|\nabla \mathrm{H}\|_{H^1}^2 + \|\nabla \mathrm{u}\|_{L^2}^4)\\ &\quad + Ce^{\sigma t}\left(\|\nabla \mathrm{u}\|_{H^1}^2 + \|\rho^{1/2}\mathrm{u}_t\|_{L^2}^2 + \|\mathrm{H}_t\|_{L^2}^2\right).\end{aligned}$$

Using (3.8), we obtain the following inequality by integrating the above inequality over $(0,t)$

$$\begin{aligned}&e^{\sigma t}\left(\|\eta^{1/2}\rho^{1/2}\mathrm{u}_t\|_{L^2}^2 + \|\eta^{1/2}\mathrm{H}_t\|_{L^2}^2\right) + \int_0^t e^{\sigma t}\|\eta^{1/2}\nabla \mathrm{u}_t\|_{L^2}^2 + e^{\sigma t}\|\eta^{1/2}\nabla \mathrm{H}_t\|_{L^2}^2\mathrm{d}s\\ &\le C + C\int_0^t e^{\sigma t}(\|\eta^{1/2}\rho^{1/2}\mathrm{u}_t\|_{L^2}^2 + \|\eta^{1/2}\mathrm{H}_t\|_{L^2}^2)(\|\nabla \mathrm{u}\|_{H^1}^2 + \|\nabla \mathrm{H}\|_{H^1}^2 + \|\nabla \mathrm{u}\|_{L^2}^4)\mathrm{d}s.\end{aligned}$$

This, together with (3.7), (3.11) and the Gronwall inequality, leads to (3.17). This lemma is completed. ∎

Next, we shall apply the classical $W^{2,p}$-estimate of elliptic equations to improve the integrability of the solution.

Lemma 3.4. *For any given $T>0$, let $(\rho,\mathrm{u},\mathrm{H},P)$ be a smooth solution to (1.1)–(1.3) on $\Omega\times(0,T)$. Then, it holds that*

$$\|e^{\sigma t}\eta^{1/2}\mathrm{u}, e^{\sigma t}\eta^{1/2}\mathrm{H}\|_{L^2(0,T;W^{2,6})} + \|e^{\sigma t}\eta^{1/2}\nabla P\|_{L^2(0,T;L^6)} \le C. \tag{3.22}$$

Proof. We deduce from $(1.1)_2$ that $e^{\sigma t}\eta^{1/2}\mathrm{u}$ satisfies the following Stokes systems

$$\begin{cases} -\triangle e^{\sigma t}\eta^{1/2}\mathrm{u}+\nabla e^{\sigma t}\eta^{1/2}P=e^{\sigma t}\eta^{1/2}\left(-\rho\mathrm{u}_t-\rho\mathrm{u}\cdot\nabla\mathrm{u}+\frac{1}{2}\nabla|\mathrm{H}|^2-\mathrm{H}\cdot\nabla\mathrm{H}\right), \text{ in } \Omega,\\ \mathrm{div}e^{\sigma t}\eta^{1/2}\mathrm{u}=0, \text{ in } \Omega,\\ e^{\sigma t}\eta^{1/2}\mathrm{u}=0, \text{ on } \partial\Omega. \end{cases}$$

Applying L^p-estimates to the above Stokes equations and using Sobolev inequality, it holds that

$$\begin{aligned}
&\|e^{\sigma t}\eta^{1/2}\nabla\mathrm{u}\|_{W^{1,6}}+\|e^{\sigma t}\eta^{1/2}\nabla P\|_{L^6}\\
&\le C\left(\left\|e^{\sigma t}\eta^{1/2}\rho\mathrm{u}_t\right\|_{L^6}+\|e^{\sigma t}\eta^{1/2}\rho\mathrm{u}\cdot\nabla\mathrm{u}\|_{L^6}+\|e^{\sigma t}\eta^{1/2}\mathrm{H}\cdot\nabla\mathrm{H}\|_{L^6}\right)\\
&\le C\left(\left\|e^{\sigma t}\eta^{1/2}\nabla\mathrm{u}_t\right\|_{L^2}+\|\rho\mathrm{u}\|_{L^\infty}\|e^{\sigma t}\eta^{1/2}\nabla\mathrm{u}\|_{L^6}+\|\mathrm{H}\|_{L^\infty}\|e^{\sigma t}\eta^{1/2}\nabla\mathrm{H}\|_{L^6}\right)\\
&\le C(\left\|e^{\sigma t}\eta^{1/2}\nabla\mathrm{u}_t\right\|_{L^2}+\|\nabla\mathrm{u}\|_{H^1}\|e^{\sigma t}\eta^{1/2}\nabla\mathrm{u}\|_{H^1}+\|\nabla\mathrm{H}\|_{H^1}\|e^{\sigma t}\eta^{1/2}\nabla\mathrm{H}\|_{H^1})\\
&\le C\left(\left\|e^{\sigma t}\eta^{1/2}\nabla\mathrm{u}_t\right\|_{L^2}+\|\nabla\mathrm{u}\|_{H^1}+\|\nabla\mathrm{H}\|_{H^1}\right),
\end{aligned}$$

due to (3.17). This, together with (3.7) and (3.17), immediately leads to

$$\|e^{\sigma t}\eta^{1/2}\mathrm{u}\|_{L^2(0,T;W^{2,6})}+\|e^{\sigma t}\eta^{1/2}\nabla P\|_{L^2(0,T;W^{1,6})}\le C.$$

Applying the same method to $(1.1)_3$, we obtain

$$\|e^{\sigma t}\eta^{1/2}\mathrm{H}\|_{L^2(0,T;W^{2,6})}\le C.$$

This completes the proof of the lemma. □

Finally, let us conclude this section by presenting more information on the integrability of the solution, in particular, the estimate of $\|\nabla\mathrm{u}\|_{L^1L^\infty}$, which will be used in the proof of uniqueness.

LEMMA 3.5. *For any given $T>0$, let $(\rho,\mathrm{u},\mathrm{H})$ be a smooth solution to (1.1)–(1.3) on $\Omega\times(0,T)$. Then, it holds that*

$$\|e^{\sigma t}\eta^{1/2}\mathrm{u}\|_{L^p(0,T;W^{2,r})}+\|e^{\sigma t}\eta^{1/2}\nabla P\|_{L^p(0,T;W^{1,r})}\le C, \tag{3.23}$$

with $r\in[2,6]$ and $2\le p\le\frac{4r}{3r-6}$. In particular, we have

$$\int_0^T\|\nabla\mathrm{u}\|_{L^\infty}\mathrm{d}t\le C. \tag{3.24}$$

Proof. Lemmas 3.3 and 3.4 are indeed two special cases of this lemma. Taking $r=6$ and $p=2$, (3.23) is reduced to (3.22), and taking $r=2$ and $p=+\infty$, this case corresponds to (3.17). So that we only need to prove that (3.23) holds with $r\in(2,6)$ and $p\in(2,\frac{4r}{3r-6})$. To this end, we shall apply the interpolation inequality between $L^2(0,T;W^{2,6})$ and $L^\infty(0,T;H^2)$ ((3.17) and (3.22)):

$$\int_0^T\|e^{\sigma t}\eta^{1/2}\nabla\mathrm{u}\|_{W^{1,r}}^p\mathrm{d}t\le C\int_0^T\|e^{\sigma t}\eta^{1/2}\nabla\mathrm{u}\|_{H^1}^{\frac{p(6-r)}{2r}}\|e^{\sigma t}\eta^{1/2}\nabla\mathrm{u}\|_{W^{1,6}}^{\frac{3p(r-2)}{2r}}\mathrm{d}t$$

$$\leq C\left(\int_0^T \|e^{\sigma t}\eta^{1/2}\nabla \mathrm{u}\|_{W^{1,6}}^2 \mathrm{d}t\right)^{\frac{3p(r-2)}{4r}} \left(\int_0^T \|e^{\sigma t}\eta^{1/2}\nabla \mathrm{u}\|_{H^1}^{\frac{2p(6-r)}{4r-3p(r-2)}} \mathrm{d}t\right)^{\frac{4r-3p(r-2)}{4r}}$$

$$\leq C\left(\int_0^T \|e^{\sigma t}\eta^{1/2}\nabla \mathrm{u}\|_{H^1}^2 \mathrm{d}t\right)^{\frac{4r-3p(r-2)}{4r}} \left(\sup_{0\leq t\leq T} \|e^{\sigma t}\sup \eta^{1/2}\nabla \mathrm{u}\|_{H^1}\right)^{p-2} \leq C,$$

due to the fact $\frac{3p(r-2)}{2r} < 2 < \frac{2p(6-r)}{4r-3p(r-2)}$ and (3.8).

Finally, taking $r=4$ and $p=8/3$ in (3.23), we obtain the desired result (3.24) as follows

$$\int_0^T \|\nabla \mathrm{u}\|_{L^\infty} dt \leq \int_0^T \|e^{\sigma t}\eta^{1/2}\nabla \mathrm{u}\|_{W^{1,4}} e^{-\sigma t}\eta^{-1/2} \mathrm{d}t$$
$$\leq \left(\int_0^T e^{-8\sigma t/5}\eta^{-4/5} dt\right)^{5/8} \|\eta^{1/2}e^{\sigma t}\nabla \mathrm{u}\|_{L^{8/3}(0,T;W^{1,4})} \leq C.$$

The proof of Lemma 3.5 is completed. ∎

4. Proof of Theorem 1.1

In this section, we shall divide the proof of Theorem 1.1 into two parts: the existence and uniqueness.

4.1. Proof of the existence. In this subsection, we shall apply the a priori estimates in Section 3 to complete the proof of existence in Theorem 1.1. We firstly construct smooth approximated solutions $(\rho^\delta, \mathrm{u}^\delta, \mathrm{H}^\delta)$ to (1.1)–(1.3) by mollifying the initial data, then establish some estimates on the approximated solutions, which are uniform with respect to the mollifying parameter δ and time $T>0$. Finally, we obtain the existence of the solution to the original problem by compactness theorem, which allow us to justify the passing to the limit as $\delta \to 0^+$.

To this end, let $(\rho_0, \mathrm{u}_0, \mathrm{H}_0)$ be the initial data satisfying the conditions (1.7)–(1.8) in Theorem 1.1 and define

$$\rho_0^\delta = j_\delta * \rho_0 + \delta, \quad \mathrm{u}_0^\delta = \mathrm{u}_0, \quad \mathrm{H}_0^\delta = \mathrm{H}_0,$$

where $j_\delta = j_\delta(x)$ is the standard Friedrich's mollifier of width δ. It thus holds

$$\begin{cases} 0 < \delta \leq \rho_0^\delta \leq \bar{\rho} + \delta < \infty, \\ \lim\limits_{\delta\to 0^+} \left(\|\rho_0^\delta - \rho_0\|_{L^p} + \|\mathrm{u}_0^\delta - \mathrm{u}_0\|_{H^1} + \|\mathrm{H}_0^\delta - \mathrm{H}_0\|_{H^1}\right) = 0, \text{ for any } p \in (1, +\infty), \end{cases} \tag{4.1}$$

and the initial energy of the mollified data now becomes

$$E_0^\delta := (\|\sqrt{\rho_0^\delta}\mathrm{u}_0^\delta\|_{L^2}^2 + \|\mathrm{H}_0^\delta\|_{L^2}^2)(\|\nabla \mathrm{u}_0^\delta\|_{L^2}^2 + \|\nabla \mathrm{H}_0^\delta\|_{L^2}^2). \tag{4.2}$$

Thanks to (4.1), it is easy to get that

$$\lim_{\delta\to 0^+} E_0^\delta = E_0.$$

The short-time existence of approximate solutions $(\rho^\delta, \mathrm{u}^\delta, \mathrm{H}^\delta)$, defined up to a positive time $T_* > 0$, to the MHD Equations (1.1) with initial data $(\rho_0^\delta, \mathrm{u}_0^\delta, \mathrm{H}_0^\delta)$ can be obtained through the same method as [23].

Indeed, the approximated solutions satisfy all the a priori estimates (3.1)–(3.3), (3.7)-(3.8), (3.17) and (3.22) in Lemmas 3.1–3.4, independent of δ and T. By these bounds of the approximated solution, it suffices to pass to the limit to obtain a solution of the original problem (1.1)–(1.3). Firstly, from the estimates in Lemmas 3.1–3.4, we find that the sequence $(\rho^\delta, \mathrm{u}^\delta, \mathrm{H}^\delta)$ converges, up to a subsequence, to some limit $(\rho, \mathrm{u}, \mathrm{H})$, that is, as $\delta \to 0^+$, we have

$$
\begin{gathered}
\rho^\delta \rightharpoonup \rho, \text{ weakly} - * \text{ in } L^\infty(\mathbb{R}^+; L^\infty),\\
\mathrm{u}^\delta \rightharpoonup \mathrm{u}, \text{weakly} - * \text{ in } L^\infty(\mathbb{R}^+; H_0^1), \quad \mathrm{H}^\delta \rightharpoonup \mathrm{H}, \text{ weakly} - * \text{ in } L^\infty(\mathbb{R}^+; H_n^1),\\
\mathrm{u}^\delta \rightharpoonup \mathrm{u}, \text{weakly in } L^2(\mathbb{R}^+; H^2), \quad \mathrm{H}^\delta \rightharpoonup \mathrm{H}, \text{ weakly in } L^2(\mathbb{R}^+; H^2),\\
e^{\sigma t}\eta^{1/2}\mathrm{u}^\delta \rightharpoonup e^{\sigma t}\eta^{1/2}\mathrm{u}, \text{ weakly} - * \text{ in } L^\infty(\mathbb{R}^+; H^2),\\
e^{\sigma t}\eta^{1/2}\mathrm{H}^\delta \rightharpoonup e^{\sigma t}\eta^{1/2}\mathrm{H}, \text{ weakly} - * \text{ in } L^\infty(\mathbb{R}^+; H^2),\\
e^{\sigma t}\eta^{1/2}\mathrm{u}^\delta \rightharpoonup e^{\sigma t}\eta^{1/2}\mathrm{u}, \ e^{\sigma t}\eta^{1/2}\mathrm{H}^\delta \rightharpoonup e^{\sigma t}\eta^{1/2}\mathrm{H}, \text{ weakly in } L^2(\mathbb{R}^+; W^{2,6}),\\
\sqrt{\rho^\delta}\mathrm{u}_t^\delta \rightharpoonup \sqrt{\rho}\mathrm{u}_t, \ \mathrm{H}_t^\delta \rightharpoonup \mathrm{H}_t, \text{ weakly in } L^2(\mathbb{R}^+; L^2),\\
\eta^{1/2}\sqrt{\rho^\delta}\mathrm{u}_t^\delta \rightharpoonup \eta^{1/2}\sqrt{\rho}\mathrm{u}_t, \ \eta^{1/2}\mathrm{H}_t^\delta \rightharpoonup \eta^{1/2}\mathrm{H}_t, \text{ weakly} - * \text{ in } L^\infty(\mathbb{R}^+; L^2),\\
e^{\sigma t}\eta^{1/2}\mathrm{u}_t^\delta \rightharpoonup e^{\sigma t}\eta^{1/2}\mathrm{u}_t, \text{ weakly in } L^2(\mathbb{R}^+; H_0^1),\\
e^{\sigma t}\eta^{1/2}\mathrm{H}_t^\delta \rightharpoonup e^{\sigma t}\eta^{1/2}\mathrm{H}_t, \text{ weakly in } L^2(\mathbb{R}^+; H_n^1).
\end{gathered} \tag{4.3}
$$

On the other hand, to guarantee the convergence of the nonlinear term in the definition of the weak solution, we need to deduce some strong convergence. From (3.7) and (3.17), we firstly apply the Lemma 2.3 to obtain for any $\alpha \in (0,1/2)$ and $0 < T < \infty$ that

$$\|\mathrm{u}^\delta(t)\|_{H^{1/2-\alpha}(0,T;L^6)} \le C.$$

Applying the interpolation theorem between the above norm and $\|\mathrm{u}^\delta\|_{L^\infty(\mathbb{R}_+;H^1)}$, it is easy to obtain that for $0 < \beta \le \frac{1}{6\alpha+1}$ and $T > 0$

$$\|\mathrm{u}^\delta(t)\|_{\mathrm{H}^\beta((0,T)\times\Omega)} \le C. \tag{4.4}$$

This implies for any $p \in (1, 2 + \frac{2}{12\alpha+1})$ that

$$\mathrm{u}^\delta \to \mathrm{u} \text{ in } L^p((0,T)\times\Omega), \tag{4.5}$$

due to the standard compact embedding theorem. Then, by the above strong convergence result and boundedness of u^δ in (3.7), it (at least) suffices to yield that

$$\mathrm{u}^\delta \to \mathrm{u} \text{ in } L^2_{loc}(\mathbb{R}_+; H^1). \tag{4.6}$$

From (3.7) and (3.17), we can deduce that H^δ is bounded in $\{v | v \in L^\infty(\mathbb{R}_+; H^1), v_t \in L^2(\mathbb{R}_+; L^2)\}$. Thus, by the Aubin-Lions-Simon theorem, we obtain for any $p \in [2,6)$ that

$$\mathrm{H}^\delta \to \mathrm{H} \text{ in } C_{loc}(\mathbb{R}_+; L^p). \tag{4.7}$$

All this information obtained is more than enough to justify that $(\rho, \mathrm{u}, \mathrm{H})$ is a weak solution to (1.1)–(1.3).

To make decay rate more clear, in what follows, we need to justify the continuity of $\rho^{1/2}\mathrm{u}(t)$ and $(\mathrm{u},\mathrm{H})(t)$ in the strong topology in $[0,+\infty)$, which indeed can be rigorous by appropriate regularization. Firstly, for any $0<\tau<\infty$, it holds

$$\mathrm{u}^{\delta},\ \mathrm{H}^{\delta}\in L^{\infty}(\tau,+\infty;H^2),\ \mathrm{u}_t^{\delta}\in L^2(\tau,+\infty;H_0^1),\ \mathrm{H}_t^{\delta}\in L^2(\tau,+\infty;H_n^1). \tag{4.8}$$

Thus, one can deduce the strong continuity of the solution by Aubin-Lions-Simon theorem that

$$\mathrm{u},\ \mathrm{H}\in C([\tau,\infty);W^{1,p}). \tag{4.9}$$

for any $p\in[2,6)$.

On the other hand, using the method of renormalized solutions, arguing as in [25], one eventually proves that for all $p\geq 1$

$$\rho^{\delta}\to\rho \text{ in } C([0,+\infty);L^p),$$

and

$$\sqrt{\rho^{\delta}}\to\sqrt{\rho} \text{ in } C([0,+\infty);L^p).$$

Together with (4.9), we conclude that

$$\sqrt{\rho}\mathrm{u}\in C\left(0,+\infty;L^2\right). \tag{4.10}$$

Next, we turn our attention to prove the continuity of $\|\sqrt{\rho}\mathrm{u}(\cdot,t)\|_{L^2}$ at $t=0$. By the standard method, we can also prove that $\sqrt{\rho}$ is a solution of

$$\partial_t\sqrt{\rho}+\operatorname{div}(\sqrt{\rho}\mathrm{u})=0,\quad \sqrt{\rho}_{|t=0}=\sqrt{\rho_0}. \tag{4.11}$$

This, together with the fact $\sqrt{\rho}\mathrm{u}\in L^{\infty}(0,T;L^2)$, yields

$$\sqrt{\rho}_t\in L^{\infty}(0,+\infty;H^{-1}).$$

Due to $\mathrm{u}\in L^{\infty}(0,+\infty;H^1)$, we obtain

$$\sqrt{\rho}_t\mathrm{u}\in L^{\infty}(0,+\infty;W^{-1,3/2}).$$

Note that

$$(\sqrt{\rho}\mathrm{u})_t=\sqrt{\rho}_t\mathrm{u}+\sqrt{\rho}\mathrm{u}_t,$$

consequently it leads to

$$(\sqrt{\rho}\mathrm{u})_t\in L^2(0,+\infty;W^{-1,3/2}),$$

due to $\sqrt{\rho}\mathrm{u}_t\in L^2(0,+\infty;L^2)$. On other hand, noticing $\sqrt{\rho}\mathrm{u}\in L^{\infty}(0,T;L^2)$, it immediately leads to

$$\sqrt{\rho}\mathrm{u}\in C([0,+\infty);L^2-w). \tag{4.12}$$

By the above weak continuity and energy inequality, we obtain that

$$\operatorname{ess}\lim\sup_{t\to 0^+}\int_{\Omega}|\sqrt{\rho}\mathbf{u}-\sqrt{\rho_0}\mathbf{u}_0|^2+|\mathrm{H}-\mathrm{H}_0|^2\,dx$$

$$\leq \operatorname{ess}\limsup_{t\to 0^+}\left(\int_\Omega (\rho|\mathbf{u}|^2+|\mathrm{H}|^2)\,\mathrm{d}x-\int_\Omega\left(\rho_0|\mathrm{u}_0|^2+|\mathrm{H}_0|^2\right)\mathrm{d}x\right)$$
$$+\operatorname{ess}\lim_{t\to 0^+}\left(2\int_\Omega \sqrt{\rho_0}\mathbf{u}_0(\sqrt{\rho_0}\mathbf{u}_0-\sqrt{\rho}\mathbf{u})\,\mathrm{d}x+\int_\Omega 2\mathrm{H}_0(\mathrm{H}-\mathrm{H}_0)\,\mathrm{d}x\right)=0.$$

This, together with (4.10), immediately leads to the strong continuity $\sqrt{\rho}\mathrm{u}\in C([0,+\infty);L^2)$. The proof of the existence and temporal decay of Theorem 1.1 is completed.

4.2. Proof of the uniqueness. Finally, we complete the proof of the uniqueness. Due to the lack of regularity of the density, similar to the studies of the Navier–Stokes equations, there is no hope to prove uniqueness of solutions to system (1.1) at the level of the Eulerian coordinates. Therefore, we shall prove it for the solutions written in the Lagrangian coordinates. In the sequel, we shall pay more attention to the terms involving the magnetic fields.

To proceed, we introduce a flow $X:\mathbb{R}_+\times\Omega\to\Omega$ of u, which is a solution to the following integral equations

$$X(t,y)=y+\int_0^t \mathrm{u}(\tau,X(\tau,y))\,\mathrm{d}\tau. \tag{4.13}$$

Because of $\mathrm{u}|_{\partial\Omega}=0$, it holds that $\partial\Omega=X(t,\partial\Omega)$. We denote the Eulerian coordinates by (t,X) with $X=X(t,y)$, where the fixed $(t,y)\in\mathbb{R}^+\times\Omega$ stand for the Lagrangian coordinates. In Lagrangian coordinates (t,y), we define the Lagrangian unknowns $(\eta,\mathbf{v},\mathrm{d},Q)$ as

$$\begin{cases}\eta(t,y):=\rho(t,X(t,y)) \quad\text{and}\quad \mathbf{v}(t,y):=\mathrm{u}(t,X(t,y)),\\ \mathrm{d}(t,y):=\mathrm{H}(t,X(t,y)) \quad\text{and}\quad Q(t,y):=(P+\frac{1}{2}|\mathrm{H}|^2)(t,X(t,y)).\end{cases} \tag{4.14}$$

Notice that from (4.13), it holds that

$$\nabla_y X(t,y)=\mathbb{I}d+\int_0^t \nabla_y\mathbf{v}(\tau,y)\mathrm{d}\tau.$$

Setting $A(t):=(\nabla_y X(t,\cdot))^{-1}$, we get that, in the (t,y)-coordinates, operators ∇, div, curl and Δ translate into

$$\begin{aligned}&\nabla_{\mathbf{v}}:={}^TA\nabla_y,\quad \mathrm{curl}_{\mathbf{v}}:={}^TA\nabla_y\times,\\ &\mathrm{div}_{\mathbf{v}}:={}^TA:\nabla_y=\mathrm{div}_y(A\cdot)\ \text{ and }\ \Delta_{\mathbf{v}}:=\mathrm{div}_y(A^TA\nabla_y\cdot),\end{aligned} \tag{4.15}$$

and then $(\eta,\mathbf{v},\mathrm{d},Q)$ satisfies

$$\begin{cases}\eta_t=0 & \text{in}\quad (0,T)\times\Omega,\\ \eta\mathbf{v}_t-\Delta_{\mathbf{v}}\mathbf{v}+\nabla_{\mathbf{v}}Q=\mathrm{d}\cdot\nabla_{\mathbf{v}}\mathrm{d} & \text{in}\quad (0,T)\times\Omega,\\ \mathrm{d}_t-\Delta_{\mathbf{v}}\mathrm{d}-\mathrm{d}\cdot\nabla_{\mathbf{v}}\mathbf{v}=0 & \text{in}\quad (0,T)\times\Omega,\\ \mathrm{div}_{\mathbf{v}}\mathbf{v}=0,\quad \mathrm{div}_{\mathbf{v}}\mathrm{d}=0 & \text{in}\quad (0,T)\times\Omega.\end{cases} \tag{4.16}$$

From the result of [10,11] on the Navier–Stokes equations, one can conclude that the system (4.16) is equivalent to (1.1) whenever

$$\int_0^T \|\nabla_y\mathbf{v}\|_{L^\infty}\mathrm{d}\tau\leq\frac{1}{2}. \tag{4.17}$$

In fact, under the condition (4.17), one gets that$\nabla_y X - \mathbb{I}d$ is small (in an appropriate Sobolev space), and then the mapping X is a diffeomorphism. This is due to the fact that if condition (4.17) is fulfilled then one may write that

$$A = (\mathbb{I}\mathrm{d} + (\nabla_y X - \mathbb{I}\mathrm{d}))^{-1} = \sum_{k=0}^{+\infty} (-1)^k \left(\int_0^t \nabla_y \mathbf{v}(\tau,\cdot)\mathrm{d}\tau \right)^k, \tag{4.18}$$

and thus

$$\|A - \mathbb{I}\mathrm{d}\|_{L^\infty((0,T)\times\Omega)} \le \frac{1}{3}.$$

In what follows, let $(\rho^1,\mathrm{u}^1,\mathrm{H}^1,P^1)$ and $(\rho^2,\mathrm{u}^2,\mathrm{H}^2,P^2)$ be two solutions to system (1.1) with the same initial data (1.2) and boundary condition (1.3), and satisfy the properties of Theorem 1.1. We now define the corresponding Lagrangian unknowns by

$$(\eta^1,\mathbf{v}^1,\mathrm{d}^1,Q^1) \text{ and } (\eta^2,\mathbf{v}^2,\mathrm{d}^2,Q^2).$$

Note that the continuity equation becomes

$$\eta^1 = \eta^2 = \rho_0,$$

the density can be regarded as a parameter function in Lagrangian coordinates. Moreover, from Lemmas 3.1–3.5, we have for $i = 1,2$, the following regularity

$$\begin{cases} \nabla \mathbf{v}^i, \nabla \mathrm{d}^i \in L^1(0,T;L^\infty) \cap L^4(0,T;L^3) \cap L^2(0,T;L^6), \\ t^{1/2}\nabla \mathbf{v}^i, t^{1/2}\nabla \mathrm{d}^i \in L^2(0,T;L^\infty) \cap L^2(0,T;W^{1,6}), \\ \mathbf{v}^i,\ \mathrm{d}^i \in L^4(0,T;L^\infty),\ \mathrm{d}^i \in L^\infty(0,T;L^3), \\ t^{1/2}\nabla Q^i \in L^2(0,T;L^4),\ t^{1/2}\mathbf{v}^i_t,\ t^{1/2}\mathrm{d}^i_t \in L^{4/3}(0,T;L^6). \end{cases} \tag{4.19}$$

Denoting $\delta\mathbf{v} := \mathbf{v}^2 - \mathbf{v}^1$, $\delta\mathrm{d} := \mathrm{d}^2 - \mathrm{d}^1$ and $\delta Q := Q^2 - Q^1$, and subtracting the system of $(\eta^1,\mathbf{v}^1,\mathrm{d}^1,Q^1)$ from the one satisfied by $(\eta^2,\mathbf{v}^2,\mathrm{d}^2,Q^2)$, we get

$$\begin{cases} \rho_0 \delta\mathbf{v}_t - \Delta_{\mathbf{v}^1}\delta\mathbf{v} + \nabla_{\mathbf{v}^1}\delta Q = (\Delta_{\mathbf{v}^1} - \Delta_{\mathbf{v}^2})\mathbf{v}^2 - (\nabla_{\mathbf{v}^1} - \nabla_{\mathbf{v}^2})Q^2 \\ \qquad\qquad -\mathrm{d}^2\cdot\nabla_{\mathbf{v}^2}\delta\mathrm{d} + \mathrm{d}^2\cdot(\nabla_{\mathbf{v}^2} - \nabla_{\mathbf{v}^1})\mathrm{d}^1 + \delta\mathrm{d}\cdot\nabla_{\mathbf{v}^1}\mathrm{d}^1, \\ \delta\mathrm{d}_t - \Delta_{\mathbf{v}^1}\delta\mathrm{d} = (\Delta_{\mathbf{v}^2} - \Delta_{\mathbf{v}^1})\mathrm{d}^2 - \mathrm{d}^2\cdot\nabla_{\mathbf{v}^2}\delta\mathbf{v} + \mathrm{d}^2\cdot(\nabla_{\mathbf{v}^2} - \nabla_{\mathbf{v}^1})\mathbf{v}^1 + \delta\mathrm{d}\cdot\nabla_{\mathbf{v}^1}\mathbf{v}^1, \\ \mathrm{div}_{\mathbf{v}^1}\delta\mathbf{v} = (\mathrm{div}_{\mathbf{v}^1} - \mathrm{div}_{\mathbf{v}^2})\mathbf{v}^2, \quad \mathrm{div}_{\mathbf{v}^1}\delta\mathrm{d} = (\mathrm{div}_{\mathbf{v}^1} - \mathrm{div}_{\mathbf{v}^2})\mathrm{d}^2, \\ \delta\mathbf{v}|_{\partial\Omega} = 0, \quad \nabla\times\delta\mathrm{d}\times n|_{\partial\Omega} = 0. \\ \delta\mathbf{v}|_{t=0} = 0, \quad \delta\mathrm{d}|_{t=0} = 0. \end{cases} \tag{4.20}$$

To complete this uniqueness, it is a key step to prove the following claim.

Claim: *There exists a sufficiently small* $T_0 > 0$, *such that*

$$\int_0^{T_0}\int_\Omega \left(|\nabla\delta\mathbf{v}|^2 + |\nabla\delta\mathrm{d}|^2\right)(t,y)\mathrm{d}y\mathrm{d}t = 0. \tag{4.21}$$

Proof. **(Proof of claim (4.21).)** To this end, we decompose $\delta\mathrm{u}$ and $\delta\mathrm{d}$ into two parts

$$\delta\mathbf{v} = \tilde{\mathbf{v}} + \hat{\mathbf{v}} \text{ and } \delta\mathrm{d} = \tilde{\mathrm{d}} + \hat{\mathrm{d}},$$

respectively, where $\tilde{\mathbf{v}}$ is the solution to the problem:

$$\operatorname{div}_{\mathbf{v}^1}\tilde{\mathbf{v}} = (\operatorname{div}_{\mathbf{v}^1} - \operatorname{div}_{\mathbf{v}^2})\mathbf{v}^2 = \operatorname{div}(\delta A\,\mathbf{v}^2) = {}^T\delta A : \nabla \mathbf{v}^2, \tag{4.22}$$

and $\tilde{\mathrm{d}}$ is the solution to the problem:

$$\operatorname{div}_{\mathbf{v}^1}\tilde{\mathrm{d}} = (\operatorname{div}_{\mathbf{v}^1} - \operatorname{div}_{\mathbf{v}^2})\mathrm{d}^2 = \operatorname{div}(\delta A\,\mathrm{d}^2) = {}^T\delta A : \nabla \mathrm{d}^2, \tag{4.23}$$

with $\delta A := A^2 - A^1$ and $A^i := A(u^i)$. Similar to the paper of Danchin-Mucha [11] (see Lemma A 4.3 of [11] for details), one has for $0 < T_0 \leq T$

$$\|\tilde{\mathbf{v}}\|_{L^4(0,T_0;L^2)} + \|\nabla\tilde{\mathbf{v}}\|_{L^2((0,T_0)\times\Omega)} + \|\tilde{\mathbf{v}}_t\|_{L^{4/3}(0,T_0;L^{3/2})} \leq c(T_0)\|\nabla\delta\mathbf{v}\|_{L^2((0,T_0)\times\Omega)} \tag{4.24}$$

and

$$\|\tilde{\mathrm{d}}\|_{L^4(0,T_0;L^2)} + \|\nabla\tilde{\mathrm{d}}\|_{L^2((0,T_0)\times\Omega)} + \|\tilde{\mathrm{d}}_t\|_{L^{4/3}(0,T_0;L^{3/2})} \leq c(T_0)\|\nabla\delta\mathrm{d}\|_{L^2((0,T_0)\times\Omega)} \tag{4.25}$$

with $c(T_0)$ going to 0 as $T_0 \to 0$.

Next, we rewrite the equations satisfied by $(\delta\mathbf{v}, \delta\mathrm{d}, \delta Q)$ as the following system

$$\begin{cases} \rho_0\hat{\mathbf{v}}_t - \Delta_{\mathbf{v}^1}\hat{\mathbf{v}} + \nabla_{\mathbf{v}^1}\delta Q = (\Delta_{\mathbf{v}^2} - \Delta_{\mathbf{v}^1})\mathbf{v}^2 - (\nabla_{\mathbf{v}^2} - \nabla_{\mathbf{v}^1})Q^2 + \mathrm{d}^2\cdot\nabla_{\mathbf{v}^2}\delta\mathrm{d} \\ \qquad\qquad + \mathrm{d}^2\cdot(\nabla_{\mathbf{v}^2} - \nabla_{\mathbf{v}^1})\mathrm{d}^1 + \delta\mathrm{d}\cdot\nabla_{\mathbf{v}^1}\mathrm{d}^1 - \rho_0\tilde{\mathbf{v}}_t + \Delta_{\mathbf{v}^1}\tilde{\mathbf{v}}, \\ \hat{\mathrm{d}}_t - \Delta_{\mathbf{v}^1}\hat{\mathrm{d}} = (\Delta_{\mathbf{v}^2} - \Delta_{\mathbf{v}^1})\mathrm{d}^2 + \mathrm{d}^2\cdot\nabla_{\mathbf{v}^2}\delta\mathbf{v} + \mathrm{d}^2\cdot(\nabla_{\mathbf{v}^2} - \nabla_{\mathbf{v}^1})\mathbf{v}^1 \\ \qquad\qquad + \delta\mathrm{d}\cdot\nabla_{\mathbf{v}^1}\mathbf{v}^1 - \tilde{\mathrm{d}}_t + \Delta_{\mathbf{v}^1}\tilde{\mathrm{d}}, \\ \operatorname{div}_{\mathbf{v}^1}\hat{\mathbf{v}} = 0, \quad \operatorname{div}_{\mathbf{v}^1}\hat{\mathrm{d}} = 0, \\ \hat{\mathbf{v}}|_{t=0} = 0, \quad \hat{\mathrm{d}}|_{t=0} = 0. \end{cases} \tag{4.26}$$

Multiplying the first equation and the second equation of (4.26) by $\hat{\mathbf{v}}$ and $\hat{\mathrm{d}}$, respectively, and after integrating by parts, one gets

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int(\rho_0\hat{\mathbf{v}} + |\hat{\mathrm{d}}|^2)\mathrm{d}x + \int(|\nabla_{\mathbf{v}_1}\hat{\mathbf{v}}|^2 + |\operatorname{curl}_{\mathbf{v}_1}\hat{\mathrm{d}}|^2)\mathrm{d}y \\ &= \int(\Delta_{\mathbf{v}^2} - \Delta_{\mathbf{v}^1})\mathbf{v}^2\cdot\hat{\mathbf{v}}\mathrm{d}y + \int(\nabla_{\mathbf{v}_1} - \nabla_{\mathbf{v}_2})Q^2\cdot\hat{\mathbf{v}}\mathrm{d}y - \int\rho_0\tilde{\mathbf{v}}_t\cdot\hat{\mathbf{v}}\mathrm{d}y + \int\Delta_{\mathbf{v}^1}\tilde{\mathbf{v}}\cdot\hat{\mathbf{v}}\mathrm{d}y \\ &\quad + \int\mathrm{d}^2\cdot\nabla_{\mathbf{v}^2}\delta\mathrm{d}\cdot\hat{\mathbf{v}}\mathrm{d}y + \int\mathrm{d}^2\cdot(\nabla_{\mathbf{v}_2} - \nabla_{\mathbf{v}_1})\mathrm{d}^1\cdot\hat{\mathbf{v}}\mathrm{d}y + \int\delta\mathrm{d}\cdot\nabla_{\mathbf{v}^1}\mathrm{d}^1\cdot\hat{\mathbf{v}}\mathrm{d}y \\ &\quad + \int(\Delta_{\mathbf{v}^2} - \Delta_{\mathbf{v}^1})\mathrm{d}^2\cdot\hat{\mathrm{d}}\mathrm{d}x - \int\tilde{\mathrm{d}}_t\cdot\hat{\mathrm{d}}\mathrm{d}y + \int\Delta_{\mathbf{v}^1}\tilde{\mathrm{d}}\cdot\hat{\mathrm{d}}\mathrm{d}y \\ &\quad + \int\mathrm{d}^2\cdot\nabla_{\mathbf{v}^2}\delta\mathbf{v}\cdot\hat{\mathrm{d}}\mathrm{d}y + \int\mathrm{d}^2\cdot(\nabla_{\mathbf{v}^2} - \nabla_{\mathbf{v}^1})\mathbf{v}^1\cdot\hat{\mathrm{d}}\mathrm{d}y + \int\delta\mathrm{d}\cdot\nabla_{\mathbf{v}^1}\mathbf{v}^1\cdot\hat{\mathrm{d}}\mathrm{d}y \\ &= \sum_{i=1}^{13} I_i, \end{aligned} \tag{4.27}$$

where we have used the identities

$$\int\nabla_{\mathbf{v}^1}\delta Q\cdot\hat{\mathbf{v}}\mathrm{d}y = -\int\operatorname{div}_{\mathbf{v}^1}\hat{\mathbf{v}}\,\delta Q\mathrm{d}y = 0 \quad\text{and}\quad -\int\Delta\hat{\mathrm{d}}_{\mathbf{v}^1}\cdot\hat{\mathrm{d}}\mathrm{d}y = \int|\operatorname{curl}_{\mathbf{v}^1}\hat{\mathrm{d}}|^2\mathrm{d}y.$$

Exactly along the same lines as the paper of Danchin-Mucha [11] (see pages 26–27 of [11] for more details), one has

$$\int_0^{T_0} I_1(t)\mathrm{d}t \le C(T_0)\|\nabla\delta\mathbf{v}\|_{L^2(0,T_0;L^2)}\|\nabla\hat{\mathbf{v}}\|_{L^2(0,T_0;L^2)};$$

$$\int_0^{T_0} I_2(t)\mathrm{d}t \le C(T_0)(\|\sqrt{\rho_0}\hat{\mathbf{v}}\|_{L^\infty(0,T_0;L^2)}+\|\nabla\hat{\mathbf{v}}\|_{L^2(0,T_0;L^2)})\|\nabla\delta\mathbf{v}\|_{L^2(0,T_0;L^2)};$$

$$\int_0^{T_0} I_3(t)\mathrm{d}t \le C(T_0)(\|\sqrt{\rho_0}\hat{\mathbf{v}}\|_{L^\infty(0,T_0;L^2)}+\|\nabla\hat{\mathbf{v}}\|_{L^2(0,T_0;L^2)})^{\frac{1}{2}} \times\|\sqrt{\rho_0}\hat{\mathbf{v}}\|^{\frac{1}{2}}_{L^\infty(0,T_0;L^2)}\|\nabla\delta\mathbf{v}\|_{L^2(0,T_0;L^2)};$$

$$\int_0^{T_0} I_4(t)\mathrm{d}t \le \frac{1}{4}\int_0^{T_0}\|\nabla_{\mathbf{v}^1}\hat{\mathbf{v}}\|^2_{L^2}\mathrm{d}t + C(T_0)\|\nabla\delta\mathbf{v}\|^2_{L^2(0,T_0;L^2)};$$

$$\int_0^{T_0} I_8(t)\mathrm{d}t \le C(T_0)\|\nabla\delta\mathbf{v}\|_{L^2(0,T_0;L^2)}\|\nabla\hat{\mathrm{d}}\|_{L^2(0,T_0;L^2)};$$

$$\int_0^{T_0} I_9(t)\mathrm{d}t \le C(T_0)(\|\hat{\mathrm{d}}\|_{L^\infty(0,T_0;L^2)}+\|\nabla\hat{\mathrm{d}}\|_{L^2(0,T_0;L^2)})^{\frac{1}{2}}\|\hat{\mathrm{d}}\|^{\frac{1}{2}}_{L^\infty(0,T_0;L^2)}\|\nabla\delta\mathrm{d}\|_{L^2(0,T_0;L^2)};$$

$$\int_0^{T_0} I_{10}(t)\mathrm{d}t \le \frac{1}{4}\int_0^{T_0}\|\nabla_{\mathbf{v}^1}\hat{\mathrm{d}}\|^2_{L^2}\mathrm{d}t + C(T_0)\|\nabla\delta\mathrm{d}\|^2_{L^2(0,T_0;L^2)},$$

where $C(T_0)$ is positive constant depending on T_0 and going to 0 when T_0 tends to 0. For terms I_5 and I_{11}, one has

$$\begin{aligned}
&\int_0^{T_0} I_5(t)+I_{11}(t)\mathrm{d}t = \int_0^{T_0}\int \mathrm{d}^2\cdot(\nabla_{\mathbf{v}^2}\hat{\mathbf{v}}\cdot\delta\mathrm{d} - \nabla_{\mathbf{v}^2}\delta\mathbf{v}\cdot\hat{\mathrm{d}})\mathrm{d}y\mathrm{d}t\\
&= \int_0^{T_0}\int \mathrm{d}^2\cdot(\nabla_{\mathbf{v}^2}\hat{\mathbf{v}}\cdot\tilde{\mathrm{d}} - \nabla_{\mathbf{v}^2}\tilde{\mathbf{v}}\cdot\hat{\mathrm{d}})\mathrm{d}y\mathrm{d}t = \int_0^{T_0}\int \mathrm{d}^2\cdot(\nabla_{\mathbf{v}^2}\hat{\mathbf{v}}\cdot\tilde{\mathrm{d}} + \nabla_{\mathbf{v}^2}\hat{\mathrm{d}}\cdot\tilde{\mathbf{v}})\mathrm{d}y\mathrm{d}t\\
&= \int_0^{T_0}\int \mathrm{d}^2\cdot({}^TA_2\nabla\hat{\mathbf{v}}\cdot\tilde{\mathrm{d}} + {}^TA_2\nabla\hat{\mathrm{d}}\cdot\tilde{\mathbf{v}})\mathrm{d}y\mathrm{d}t\\
&\le \int_0^{T_0}\|\mathrm{d}^2\|_{L^\infty}(\|\nabla\hat{\mathbf{v}}\|_{L^2}\|\tilde{\mathrm{d}}\|_{L^2}+\|\nabla\hat{\mathrm{d}}\|_{L^2}\|\tilde{\mathbf{v}}\|_{L^2})\mathrm{d}t\\
&\le \|\mathrm{d}^2\|_{L^4(0,T_0;L^\infty)}(\|\nabla\hat{\mathbf{v}}\|_{L^2(0,T_0;L^2)}\|\tilde{\mathrm{d}}\|_{L^4(0,T_0;L^2)}+\|\nabla\hat{\mathrm{d}}\|_{L^2(0,T_0;L^2}\|\tilde{\mathbf{v}}\|_{L^4(0,T_0;L^2)})\\
&\le C(T_0)(\|\nabla\hat{\mathbf{v}}\|_{L^2(0,T_0;L^2)}\|\nabla\delta\mathrm{d}\|_{L^2(0,T_0;L^2)}+\|\nabla\hat{\mathrm{d}}\|_{L^2(0,T_0;L^2)}\|\nabla\delta\mathbf{v}\|_{L^2(0,T_0;L^2)}),
\end{aligned}$$

where the fact $\mathrm{div}_{\mathbf{v}^2}\mathrm{d}^2=0$ is used. Using (4.24) and (4.25), one can estimate $I_7(t)$ as follows

$$\begin{aligned}
\int_0^{T_0} I_7(t)\mathrm{d}t &= \int_0^{T_0}\int \delta\mathrm{d}\cdot\nabla_{\mathbf{v}^1}\mathrm{d}^1\cdot\hat{\mathbf{v}}\mathrm{d}y\mathrm{d}t \le \int_0^{T_0}\|\nabla\mathrm{d}^1\|_{L^3}\|\delta\mathrm{d}\|_{L^2}\|\hat{\mathbf{v}}\|_{L^6}\mathrm{d}t\\
&\le C\|\delta\mathrm{d}\|_{L^4(0,T_0;L^2)}\|\nabla\mathrm{d}^1\|_{L^4(0,T_0;L^3)}\|\hat{\mathbf{v}}\|_{L^2(0,T_0;L^6)}\\
&\le C(T_0)\|\nabla\hat{\mathbf{v}}\|_{L^2(0,T_0;L^2)}(\|\hat{\mathrm{d}}\|_{L^\infty(0,T_0;L^2)}+\|\tilde{\mathrm{d}}\|_{L^4(0,T_0;L^2)})\\
&\le C(T_0)\|\nabla\hat{\mathbf{v}}\|_{L^2(0,T_0;L^2)}(\|\hat{\mathrm{d}}\|_{L^\infty(0,T_0;L^2)}+\|\nabla\delta\mathrm{d}\|_{L^2(0,T_0;L^2)}),
\end{aligned}$$

and similarly I_{13} with the following estimate

$$\begin{aligned}\int_0^{T_0} I_{13}(t)\mathrm{d}t = & -\int_0^{T_0}\int \delta\mathrm{d}\cdot\nabla_{\mathbf{v}^1}\mathbf{v}^1\cdot\hat{\mathrm{d}}\mathrm{d}y\mathrm{d}t \le \int_0^{T_0}\|\nabla\mathbf{v}^1\|_{L^3}\|\delta\mathrm{d}\|_{L^2}\|\hat{\mathrm{d}}\|_{L^6}\mathrm{d}t\\ &\le C\|\delta\mathrm{d}\|_{L^4(0,T_0;L^2)}\|\nabla\mathbf{v}^1\|_{L^4(0,T_0;L^3)}\|\nabla\hat{\mathrm{d}}\|_{L^2(0,T_0;L^2)}\\ &\le C(T_0)\|\nabla\hat{\mathrm{d}}\|_{L^2(0,T_0;L^2)}(\|\hat{\mathrm{d}}\|_{L^\infty(0,T_0;L^2)}+\|\nabla\delta\mathrm{d}\|_{L^2(0,T_0;L^2)}),\end{aligned}$$

where the Poincaré inequality is used. To deal with I_6, it holds that

$$I_6(t) = \int(\mathrm{d}^2\cdot{}^T\delta A\nabla)\mathrm{d}^1\cdot\hat{\mathbf{v}}\mathrm{d}y \le C\|t^{-\frac{1}{2}}\delta A\|_{L^2}\|t^{\frac{1}{2}}\nabla\mathrm{d}^1\|_{L^\infty}\|\mathrm{d}^2\|_{L^3}\|\hat{\mathbf{v}}\|_{L^6}.$$

Notice that using the fact that if both $\mathbf{v}^1$ and $\mathbf{v}^2$ fulfill (4.19), then one has (see (4.12) of Danchin-Mucha [11] for more details)

$$\sup_{t\in[0,T_0]}\|t^{-\frac{1}{2}}\delta A\|_{L^2} \le C\sup_{t\in[0,T_0]}\|t^{-\frac{1}{2}}\int_0^t\nabla\delta\mathbf{v}\mathrm{d}\tau\|_{L^2} \le C\|\nabla\delta\mathbf{v}\|_{L^2(0,T_0;L^2)}.$$

Thus, using the Poincaré inequality, one gets

$$\begin{aligned}\int_0^{T_0} I_6(t)\mathrm{d}t \le & C\|t^{-\frac{1}{2}}\delta A\|_{L^\infty(0,T_0;L^2)}\|t^{\frac{1}{2}}\nabla\mathrm{d}^1\|_{L^2(0,T_0;L^\infty)}\|\mathrm{d}^2\|_{L^\infty(0,T_0;L^3)}\|\nabla\hat{\mathbf{v}}\|_{L^2(0,T_0;L^2)}\\ \le & C(T_0)\|\nabla\delta\mathbf{v}\|_{L^2(0,T_0;L^2)}\|\nabla\hat{\mathbf{v}}\|_{L^2(0,T_0;L^2)}.\end{aligned}$$

Similarly, it holds

$$I_{12}(t) = \left|\int \mathrm{d}^2\delta A\nabla\mathbf{v}^1\cdot\hat{\mathrm{d}}\mathrm{d}x\right| \le C\|t^{-1/2}\delta A\|_{L^2}\|\mathrm{d}^2\|_{L^3}\|t^{1/2}\nabla\mathbf{v}^1\|_{L^\infty}\|\hat{\mathrm{d}}\|_{L^6}, \tag{4.28}$$

and then

$$\int_0^{T_0} I_{12}(t)\mathrm{d}t \le C(T_0)\|\nabla\hat{\mathrm{d}}\|_{L^2(0,T_0;L^2)}\|\nabla\delta\mathbf{v}\|_{L^2(0,T_0;L^2)}. \tag{4.29}$$

So altogether, substituting the estimates $I_1 - I_{13}$ into (4.27), using the Young inequality, and then integrating over $(0,T_0)$ immediately lead to, for a small enough $T_0 > 0$, that

$$\begin{aligned}&\sup_{t\in[0,T_0]}(\|\sqrt{\rho_0}\hat{\mathbf{v}}(t)\|_{L^2}^2+\|\hat{\mathrm{d}}(t)\|_{L^2}^2)+\int_0^{T_0}\|\nabla\hat{\mathbf{v}}\|_{L^2}^2+\|\nabla\hat{\mathrm{d}}\|_{L^2}^2\mathrm{d}t\\ &\le C(T_0)\int_0^{T_0}\|\nabla\delta\mathbf{v}\|_{L^2}^2+\|\nabla\delta\mathrm{d}\|_{L^2}^2\mathrm{d}t,\end{aligned} \tag{4.30}$$

where $C(T_0)$ is a positive constant depending on T_0 and goes to 0 when T_0 tends to 0. This, together with (4.24) and (4.25), we conclude that

$$\int_0^{T_0}\|\nabla\delta\mathbf{v}\|_{L^2}^2+\|\nabla\delta\mathrm{d}\|_{L^2}^2\mathrm{d}t \le C(T_0)\int_0^{T_0}\|\nabla\delta\mathbf{v}\|_{L^2}^2+\|\nabla\delta\mathrm{d}\|_{L^2}^2\mathrm{d}t. \tag{4.31}$$

Hence, **claim** (4.21) follows from taking a small enough T_0 in (4.31).

Now, plugging information of the **claim** (4.21) in (4.30) yields

$$\|\sqrt{\rho_0}\hat{\mathbf{v}}\|_{L_\infty(0,T_0;L_2)}+\|\hat{\mathrm{d}}\|_{L^\infty(0,T_0;L^2)}+\|\nabla\hat{\mathbf{v}}\|_{L^2(0,T_0;L^2)}+\|\nabla\hat{\mathrm{d}}\|_{L^2(0,T_0;L^2)}=0,$$

which, together with the Poincaré inequality, implies that

$$(\hat{\mathbf{v}},\hat{\mathrm{d}}) \equiv 0 \text{ on } [0,T_0]\times\Omega.$$

This, together with (4.24) and (4.25), clearly yields

$$(\tilde{\mathbf{v}},\tilde{\mathrm{d}}) \equiv 0 \text{ on } [0,T_0]\times\Omega.$$

Therefore we proved that for a small enough $T_0>0$ it holds

$$\mathbf{v}^1 \equiv \mathbf{v}^2,\ \mathrm{d}^1 \equiv \mathrm{d}^2 \quad \text{on } [0,T_0]\times\Omega.$$

Reverting to Eulerian coordinates, we conclude that the two solutions $(\mathbf{v}^1,\mathrm{d}^1)$ and $(\mathbf{v}^2,\mathrm{d}^2)$ coincide on $[0,T_0]\times\Omega$. By virtue of the standard connectivity arguments, we finally prove the uniqueness on the whole $\mathbb{R}_+$. ∎

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