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ON GLOBAL REGULARITY FOR A MODEL OF THE REGULARIZED BOUSSINESQ EQUATIONS WITH ZERO DIFFUSION*

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Abstract. In this paper, we consider the n-dimensional regularized incompressible Boussinesq equations with a Leray-regularization through a smoothing kernel of order α in the quadratic term and a β-fractional Laplacian in the velocity equation. Attention is focused on the case that the temperature equation is a pure transport equation without regularizing the velocity in the nonlinear term. We establish the global regularity for the regularized Boussinesq equations with zero diffusion in the critical case $\alpha+\beta=\frac{1}{2}+\frac{n}{4}$ and $\beta\geq\frac{1}{2}$. In addition, a regularity criterion via the temperature is also established for the critical case $\alpha+\beta=\frac{1}{2}+\frac{n}{4}$ and $0<\beta<\frac{1}{2}$.

Keywords. Boussinesq equations; Fractional dissipation; Global regularity.

AMS subject classifications. 35Q35; 35B65; 76D03.

1. Introduction and main results

The standard incompressible Boussinesq equations with zero diffusion read as follows

$$\begin{cases}\partial_t v+(v\cdot\nabla)v-\Delta v+\nabla p=\theta e_n, \quad x\in\mathbb{R}^n, \quad t>0,\\ \partial_t\theta+(v\cdot\nabla)\theta=0,\\ \nabla\cdot v=0,\\ v(x,0)=v_0(x), \quad \theta(x,0)=\theta_0(x),\end{cases} \tag{1.1}$$

where $v(x,t)=(v_1(x,t),v_2(x,t),\cdots,v_n(x,t))$ is a vector field denoting the velocity, $\theta=\theta(x,t)$ is a scalar function denoting the temperature, p is the scalar pressure and e_n is the unit vector $(0,0,\cdots,1)$. v_0 and θ_0 are the given initial data satisfying $\nabla\cdot v_0=0$. The Boussinesq equations model geophysical fluids such as atmospheric fronts and oceanic currents as well as fluids in our daily life such as the Rayleigh-Benard convection (see [5, 13, 15] for more details). Moreover, from the mathematical point of view, the full inviscid case is analogous to the incompressible axi-symmetric swirling three dimensional Euler equations (see e.g. [13]).

The incompressible Boussinesq equations not only have many applications in modeling fluids and geophysical fluids but also are mathematically important. The global well-posedness problems on the Boussinesq equations have recently attracted considerable interest. For the case $n=2$, Chae [4] and Hou-Li [7] established the global well-posedness of the problem (1.1), independently. Later, Hmidi-Keraani-Rousset [8] successfully established the global well-posedness for the system (1.1) when $-\Delta$ was weakened to half Laplacian, namely, $\sqrt{-\Delta}$. However, when $n\geq3$, the global regularity problem of (1.1) is a very challenging open problem in fluid mechanics. Remarkably, some interesting models have been proposed to guarantee the global regularity in the

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higher dimensions. One natural generalization is to replace $-\Delta$ by $\Lambda^{2\beta}$, namely,

$$\begin{cases} \partial_t v+(v\cdot\nabla)v+\Lambda^{2\beta}v+\nabla p=\theta e_n, \quad x\in\mathbb{R}^n, \quad t>0, \\ \partial_t\theta+(v\cdot\nabla)\theta=0, \\ \nabla\cdot v=0, \end{cases} \tag{1.2}$$

where the fractional Laplacian operator Λ^γ denotes the Zygmund operator defined through the Fourier transform, namely

$$\widehat{\Lambda^\gamma f}(\xi)=|\xi|^\gamma\hat{f}(\xi).$$

In fact, the global existence and regularity result holds true for the system (1.2) with $\beta\geq\frac{1}{2}+\frac{n}{4}$ (see [10,17–19] for details). On the other hand, a weaker nonlinearity and a strong viscous dissipation could work together to imply the regularity (see [14]). Recently, inspired by the idea of Olson and Titi [14], Bessaih and Ferrario [2] proposed (in fact $n=3$) the following regularized n-dimensional incompressible Boussinesq equations with zero diffusion

$$\begin{cases} \partial_t v+(u\cdot\nabla)v+\Lambda^{2\beta}v+\nabla p=\theta e_n, \quad x\in\mathbb{R}^n, \quad t>0, \\ \partial_t\theta+(u\cdot\nabla)\theta=0, \\ v=u+\Lambda^{2\alpha}u, \\ \nabla\cdot u=\nabla\cdot v=0. \end{cases} \tag{1.3}$$

Moreover, they established the global regularity result for the system (1.3) in the case $n=3$ provided that $\alpha+\beta\geq\frac{5}{4}$ and $\frac{1}{2}<\beta<\frac{5}{4}$. In our recent paper [22], the unnatural restriction $\frac{1}{2}<\beta<\frac{5}{4}$ was removed and the global results were also extended to arbitrary spatial dimensions. More precisely, the system (1.3) admits a unique global regular solution as long as $\alpha\geq 0$ and $\beta\geq 0$ satisfy $\alpha+\beta\geq\frac{1}{2}+\frac{n}{4}$. These results are also true even for some logarithmically supercritical cases (see [22] for details).

It should be noted that the work of Olson and Titi [14] showed that the lack of viscous diffusion strength may be compensated by regularizing the velocity within the nonlinear term. This drives us to believe that regularizing the nonlinear term within the velocity equation may be enough. Consequently, this paper aims at the global regularity of the following regularized n-dimensional incompressible Boussinesq equations with zero diffusion [1]

$$\begin{cases} \partial_t v+(u\cdot\nabla)v+\Lambda^{2\beta}v+\nabla p=\theta e_n, \quad x\in\mathbb{R}^n, \quad t>0, \\ \partial_t\theta+(v\cdot\nabla)\theta=0, \\ v=u+\Lambda^{2\alpha}u, \\ \nabla\cdot u=\nabla\cdot v=0, \\ v(x,0)=v_0(x), \quad \theta(x,0)=\theta_0(x). \end{cases} \tag{1.4}$$

The physical motivation of this regularization defined in terms of smoothing kernels is very related to a sub-grid length scale in the model and these kernels work as a kind of filter with certain widths (see [14] for more details). When $\alpha=0$, (1.4) reduces to

[1]The author would like to express his special thanks to one of the anonymous referees of our paper [22] for providing us with the interesting model (1.4) and helping us to derive some results for this model.

(1.2). Compared with the system (1.3), we only regularize the nonlinear term in the velocity equation. More precisely, the system (1.4) is only regularized $(v\cdot\nabla)v$ to $(u\cdot\nabla)v$, without $(v\cdot\nabla)\theta$ to $(u\cdot\nabla)\theta$. As a matter of fact, the global regularity problem for the system (1.4) is more interesting and challenging to some extent. Now let us state our result of this paper as follows.

THEOREM 1.1. *Assume that $n\geq 3$ and $v_0\in H^s(\mathbb{R}^n), \theta_0\in H^{s-\beta}(\mathbb{R}^n)$ with $s>1+\frac{n}{2}$. If $\alpha\geq 0$ and $\beta\geq\frac{1}{2}$ satisfy*

$$\alpha+\beta\geq\frac{1}{2}+\frac{n}{4},$$

then the Boussinesq system (1.4) admits a unique global regular solution (v,θ) such that for any given $T>0$,

$$v\in L^\infty([0,T];H^s(\mathbb{R}^n))\cap L^2([0,T];H^{s+\beta}(\mathbb{R}^n)),\quad \theta\in L^\infty([0,T];H^{s-\beta}(\mathbb{R}^n)).$$

REMARK 1.1. The proof of Theorem 1.1 is divided into two cases, namely,

$$\textbf{Case 1}: \alpha+\beta\geq\frac{1}{2}+\frac{n}{4}\ \text{ with }\ \beta>\frac{1}{2};$$

$$\textbf{Case 2}: \alpha\geq\frac{n}{4}\ \text{ and }\ \beta=\frac{1}{2}.$$

The proof of Theorem 1.1 is not trivial and involves the combination of an array of tools and new techniques. The core of the proof is to establish a global a priori bound. This is obtained by consecutively proving more and more regular global bounds. Let us now explain the main difficulty and our arguments. When $\alpha+\beta\geq\frac{1}{2}+\frac{n}{4}$, the best regularity estimate is

$$\|u(\tau)\|_{L_T^2H^{1+\frac{n}{2}}}\leq C(T,v_0,\theta_0), \tag{1.5}$$

which can not help us get any regularity for θ because θ satisfies a pure transport equation with the convective term $v\cdot\nabla\theta$. On the one hand, for **Case 1**, we fully exploit the space-time estimates (see Lemma 2.3) to derive the following key estimates (see Lemma 2.4)

$$\|\nabla v(t)\|_{L_t^1L_x^\infty}+\|\nabla\theta(t)\|_{L_t^1L_x^\infty}\leq C(t,v_0,\theta_0).$$

Therefore, the global H^s-estimate for **Case 1** follows from the above estimates immediately. On the other hand, the proof of **Case 2** is much more difficult which needs several techniques, such as Littlewood-Paley technique, maximal regularity type estimate. To bypass the above mentioned difficulty, we first establish two new commutators (2.16) and (2.17). Then, combining the localized maximum principle, (1.5) and the two commutators (2.16) and (2.17) altogether, we derive the crucial estimate $\|v(t)\|_{L_T^\infty B_{\infty,\infty}^1}\leq C(T,v_0,\theta_0)$, where $B_{p,r}^s$ denotes the nonhomogeneous Besov space (see Appendix A for details). Finally, with this estimate at our disposal, the desired global H^s-estimate follows immediately.

REMARK 1.2. We remark that our main efforts are devoted to the proof of the critical case $\alpha+\beta=\frac{1}{2}+\frac{n}{4}$ as the subcritical case $\alpha+\beta>\frac{1}{2}+\frac{n}{4}$ is more easier and can be handled in the same manner with only some suitable modifications.

Remark 1.3. Unfortunately, at present we are not able to show that Theorem 1.1 holds true under the condition $\alpha+\beta\geq\frac{1}{2}+\frac{n}{4}$ with $\beta<\frac{1}{2}$. The key reason is that the temperature equation is a pure transport equation without regularizing the velocity in the nonlinear term. Thus, the best information of θ is the boundedness of $\|\theta(t)\|_{L_t^\infty L^\infty}$, without any regularity. Therefore, it would be interesting and challenging to show the global regularity result for this remainder case. This is left for the future. However, if one adds some certain regularity on θ, then the global regularity of solution (v,θ) actually holds true. More precisely, as a by-product of the proof of Theorem 1.1, we have the following regularity criterion result.

Theorem 1.2. *Assume that $n\geq3$ and $v_0\in H^s(\mathbb{R}^n),\theta_0\in H^{s-\beta}(\mathbb{R}^n)$ with $s>1+\frac{n}{2}$. Let (v,θ) be the local (in time) smooth solution of the Boussinesq system (1.4) with $\alpha+\beta\geq\frac{1}{2}+\frac{n}{4}$ and $0<\beta<\frac{1}{2}$ corresponding to the initial condition (v_0,θ_0). If the following condition holds true*

$$\|\theta(t)\|_{\widetilde{L}_T^p B_{\infty,\infty}^{1-2\beta+\frac{2\beta}{p}}}<\infty \quad \text{for } 1\leq p\leq\infty, \tag{1.6}$$

then the solution (v,θ) can be extended beyond time T, where $\widetilde{L}_T^p B_{q,r}^s$ denotes the mixed space-time Besov spaces (see Appendix A for its definition).

The rest of the paper unfolds as follows. In Section 2 we carry out the proof of Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.2. In Appendix A, we present the Besov spaces and some useful lemmas. For the convenience of the reader, we present the proof of Lemma 2.1 and Lemma 2.2 in Appendix B.

2. The proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We first state that the existence and uniqueness of local smooth solutions in the functional spaces H^s with $s>1+\frac{n}{2}$ can be performed through the standard approach (see for example [6,13]). Thus, in order to complete the proof of Theorem 1.1, it is sufficient to establish *a priori* estimates that hold for any fixed $T>0$. In this paper, we shall use the convention that C denotes a generic constant, whose value may change from line to line. We shall write $C(\lambda_1,\lambda_2,\cdots,\lambda_k)$ as the constant C depends on the quantities $\lambda_1,\lambda_2,\cdots,\lambda_k$. We also denote $\Psi\approx\Upsilon$ if there exist two constants $C_1\leq C_2$ such that $C_1\Upsilon\leq\Psi\leq C_2\Upsilon$. For a quasi-Banach space X and for any $0<T\leq\infty$, we use standard notation $L^p(0,T;X)$ or $L_T^p(X)$ for the quasi-Banach space of Bochner measurable functions f from $(0,T)$ to X endowed with the norm

$$\|f\|_{L_T^p(X)}:=\begin{cases}\left(\int_0^T\|f(.,t)\|_X^p\,dt\right)^{\frac{1}{p}}, & 1\leq p<\infty,\\ \sup\limits_{0\leq t\leq T}\|f(.,t)\|_X, & p=\infty.\end{cases}$$

We now recall the following two lemmas (Lemma 2.1 and Lemma 2.2), which can be proved in the same way as Lemma 2.5 and Lemma 2.6 of [22]. For the sake of convenience, we present the proof in Appendix B.

Lemma 2.1. *Assume (v_0,θ_0) satisfies the assumptions stated in Theorem 1.1. Then the corresponding smooth solution (v,θ) of (1.4) admits the following bounds for any $t>0$*

$$\|\theta(t)\|_{L^p}\leq\|\theta_0\|_{L^p}, \quad p\in[1,\infty], \tag{2.1}$$

$$\|v(t)\|_{L^2}^2 + \int_0^t \|v(\tau)\|_{H^\beta}^2 \, d\tau \le C(t, v_0, \theta_0), \tag{2.2}$$

$$\|u(t)\|_{H^{2\alpha}}^2 + \int_0^t \|u(\tau)\|_{H^{2\alpha+\beta}}^2 \, d\tau \le C(t, v_0, \theta_0). \tag{2.3}$$

LEMMA 2.2. *Assume (v_0, θ_0) satisfies the assumptions stated in Theorem 1.1. If $\alpha \ge 0$ and $\beta \ge 0$ satisfy $\alpha + \beta \ge \frac{1}{2} + \frac{n}{4}$, then the corresponding smooth solution (v, θ) of (1.4) admits the following bounds for any $t > 0$*

$$\|v(t)\|_{H^\beta}^2 + \int_0^t \|v(\tau)\|_{H^{2\beta}}^2 \, d\tau \le C(t, v_0, \theta_0), \tag{2.4}$$

$$\|u(t)\|_{H^{2\alpha+\beta}}^2 + \int_0^t \|u(\tau)\|_{H^{2\alpha+2\beta}}^2 \, d\tau \le C(t, v_0, \theta_0). \tag{2.5}$$

We note that θ satisfies a pure transport equation with the convective term $v \cdot \nabla\theta$, which is a big obstacle to derive the regularity of θ. Actually, in order to obtain the regularity of θ, we need to control the norm $\|\nabla v\|_{L^\infty}$. Consequently, our next main goal is to derive this crucial estimate. To this end, we divided the proof into two cases.

Case 1: $\alpha + \beta \ge \frac{1}{2} + \frac{n}{4}$ with $\beta > \frac{1}{2}$.

In this case, we will make use of the following space-time estimates (see [20, Lemma 3.1]), which play a key role in proving our main result.

LEMMA 2.3. *Consider the following fractional dissipation equation with $\gamma > 0$*

$$\partial_t f + \Lambda^\gamma f = g, \qquad f(x,0) = f_0(x),$$

then for any $0 < \varepsilon \le \gamma$ and for any $1 \le p, q \le \infty$, we have

$$\|\Lambda^{\gamma-\varepsilon} f\|_{L_t^q L_x^p} \le C(t, f_0) + C\|g\|_{L_t^q L_x^p}. \tag{2.6}$$

Now we are in a position to derive the key estimate $\|\nabla v\|_{L_t^1 L_x^\infty}$.

LEMMA 2.4. *Assume (v_0, θ_0) satisfy the assumptions of Theorem 1.1. Let (v, θ) be the corresponding smooth solution of the system (1.4). If $\alpha + \beta = \frac{1}{2} + \frac{n}{4}$ with $\beta > \frac{1}{2}$, then the following estimate holds*

$$\|\Lambda^{2\beta-\varepsilon} v(t)\|_{L_t^1 L_x^q} \le C(t, v_0, \theta_0) \tag{2.7}$$

for any $0 < \varepsilon < 2\beta - 1$ and for any $\frac{n}{2\beta-1} < q < \infty$. In particular, we have

$$\|\nabla v(t)\|_{L_t^1 L_x^\infty} \le C(t, v_0, \theta_0). \tag{2.8}$$

This further implies

$$\|\nabla \theta(t)\|_{L_t^1 L_x^\infty} \le C(t, v_0, \theta_0). \tag{2.9}$$

Proof. Due to $\nabla \cdot v = 0$, we rewrite the first equation of (1.4) as

$$\partial_t v + \Lambda^{2\beta} v = \mathcal{P}(\theta e_n - u \cdot \nabla v),$$

where $\mathcal{P}$ denotes the Leray projector over divergence-free vector-fields, namely,

$$\mathcal{P} := \left(\mathbb{I} - \frac{\nabla \nabla \cdot}{\Delta} \right).$$

Applying the estimate (2.6) to the above equation yields

$$\begin{aligned}\|\Lambda^{2\beta-\varepsilon}v(t)\|_{L^1_tL^q_x} &\le C(t,v_0,\theta_0)+\|\mathcal{P}(\theta e_n-u\cdot\nabla v)\|_{L^1_tL^q_x}\\ &\le C(t,v_0,\theta_0)+C\|\theta e_n-u\cdot\nabla v\|_{L^1_tL^q_x}\\ &\le C(t,v_0,\theta_0)+C\|\theta\|_{L^1_tL^q_x}+C\|u\cdot\nabla v\|_{L^1_tL^q_x}\\ &\le C(t,v_0,\theta_0)+C\|u\cdot\nabla v\|_{L^1_tL^q_x},\end{aligned} \tag{2.10}$$

where we have used the fact that the Calderon-Zygmund type operators are bounded on L^r for $1<r<\infty$. Thanks to the Hölder inequality and the Gagliardo-Nirenberg inequality (see Lemma A.5), it ensures for any $0<\varepsilon<2\beta-1$

$$\begin{aligned}C\|u\cdot\nabla v\|_{L^1_tL^q_x} &\le C\|u\|_{L^\infty_tL^{q_1}_x}\|\nabla v\|_{L^1_tL^{q_2}_x}\\ &\le C\|u\|_{L^\infty_tL^{q_1}_x}\|\nabla v\|_{L^1_tL^2_x}^{1-\frac{nq(q_2-2)}{q_2[2(2\beta-1-\varepsilon)q+(q-2)n]}}\|\Lambda^{2\beta-\varepsilon}v\|_{L^1_tL^q_x}^{\frac{nq(q_2-2)}{q_2[2(2\beta-1-\varepsilon)q+(q-2)n]}}\\ &\le \frac{1}{2}\|\Lambda^{2\beta-\varepsilon}v(t)\|_{L^1_tL^q_x}+C\|u\|_{L^\infty_tL^{q_1}_x}^{\frac{q_2[2(2\beta-1-\varepsilon)q+(q-2)n]}{q_2[2(2\beta-1-\varepsilon)q+(q-2)n]-nq(q_2-2)}}\times\|\nabla v\|_{L^1_tL^2_x},\end{aligned}$$

where q_1 and q_2 should satisfy

$$\frac{1}{q}-\frac{1}{q_2}=\frac{1}{q_1}<\frac{2\beta-1-\varepsilon}{n}.$$

If we further take q_1 satisfying

$$2\alpha+\beta-\frac{n}{2}\ge-\frac{n}{q_1},$$

then it follows from (2.4) and (2.5) that

$$\|u\|_{L^\infty_tL^{q_1}_x}+\|\nabla v\|_{L^1_tL^2_x}\le C\|u\|_{L^\infty_tH^{2\alpha+\beta}_x}+C\|v\|_{L^1_tH^{2\beta}_x}\le C(t,v_0,\theta_0). \tag{2.11}$$

It should be pointed out that the above q_1 and q_2 would work as long as $\beta>\frac{1}{2}$. Putting (2.10)-(2.11) together leads to

$$\|\Lambda^{2\beta-\varepsilon}v(t)\|_{L^1_tL^q_x}\le C(t,v_0,\theta_0),$$

which is (2.7). Taking $q>\frac{n}{2\beta-1-\varepsilon}$, we have that

$$\|\nabla v\|_{L^\infty}\le C\|\nabla v\|_{L^2}+C\|\Lambda^{2\beta-\varepsilon}v\|_{L^q}.$$

As a result, the desired estimate (2.8) follows directly. We now apply gradient operator to $(1.4)_2$ to get

$$\partial_t\nabla\theta+(v\cdot\nabla)\nabla\theta=-(\nabla v\cdot\nabla)\theta. \tag{2.12}$$

Multiplying the Equation (2.12) by $|\nabla\theta|^{p-2}\nabla\theta$, integrating by parts and using $\nabla\cdot v=0$, it yields

$$\frac{1}{p}\frac{d}{dt}\|\nabla\theta(t)\|_{L^p}^p\le\|\nabla v\|_{L^\infty}\|\nabla\theta\|_{L^p}^p,$$

which further gives

$$\frac{d}{dt}\|\nabla\theta(t)\|_{L^p}\le\|\nabla v\|_{L^\infty}\|\nabla\theta\|_{L^p}.$$

Letting $p\to\infty$, we have

$$\frac{d}{dt}\|\nabla\theta(t)\|_{L^\infty}\le\|\nabla v\|_{L^\infty}\|\nabla\theta\|_{L^\infty}. \tag{2.13}$$

The Gronwall inequality and (2.8) allow us to deduce

$$\|\nabla\theta(t)\|_{L_t^1L_x^\infty}\le C(t,v_0,\theta_0).$$

This ends the proof of Lemma 2.4. □

With the above estimates at our disposal, we are now ready to deduce the global H^s-estimate for **Case 1**.

Proof. **(The global H^s-estimate for Case 1.)** Applying Λ^s to $(1.4)_1$ and $\Lambda^{s-\beta}$ to $(1.4)_2$, taking the L^2 inner product with $\Lambda^s v$ and $\Lambda^{s-\beta}\theta$ respectively, then adding them up, we can get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\Lambda^s v(t)\|_{L^2}^2+\|\Lambda^{s-\beta}\theta(t)\|_{L^2}^2)+\|\Lambda^{s+\beta}v\|_{L^2}^2\\&=-\int_{\mathbb{R}^n}[\Lambda^s,u\cdot\nabla]v\cdot\Lambda^s v\,dx-\int_{\mathbb{R}^n}[\Lambda^{s-\beta},v\cdot\nabla]\theta\Lambda^{s-\beta}\theta\,dx\\&\quad+\int_{\mathbb{R}^n}\Lambda^s\theta e_n\cdot\Lambda^s v\,dx.\end{aligned} \tag{2.14}$$

By using the commutator (A.3), it is obvious to see

$$\begin{aligned}-\int_{\mathbb{R}^n}[\Lambda^s,u\cdot\nabla]v\cdot\Lambda^s v\,dx&\le C\|[\Lambda^s,u\cdot\nabla]v\|_{L^2}\|\Lambda^s v\|_{L^2}\\&\le C(\|\nabla u\|_{L^\infty}\|\Lambda^s v\|_{L^2}+\|\nabla v\|_{L^\infty}\|\Lambda^s u\|_{L^2})\|\Lambda^s v\|_{L^2}\\&\le C(\|\nabla u\|_{L^\infty}+\|\nabla v\|_{L^\infty})(\|u\|_{L^2}^2+\|\Lambda^s v\|_{L^2}^2)\\&\le C(1+\|\nabla v\|_{L^\infty})(1+\|\Lambda^s v\|_{L^2}^2),\end{aligned}$$

$$\begin{aligned}-\int_{\mathbb{R}^n}[\Lambda^{s-\beta},v\cdot\nabla]\theta\Lambda^{s-\beta}\theta\,dx&\le C\|[\Lambda^{s-\beta},v\cdot\nabla]\theta\|_{L^2}\|\Lambda^{s-\beta}\theta\|_{L^2}\\&\le C(\|\nabla v\|_{L^\infty}\|\Lambda^{s-\beta}\theta\|_{L^2}+\|\nabla\theta\|_{L^\infty}\|\Lambda^{s-\beta}v\|_{L^2})\|\Lambda^{s-\beta}\theta\|_{L^2}\\&\le C(\|\nabla v\|_{L^\infty}+\|\nabla\theta\|_{L^\infty})(1+\|\Lambda^{s-\beta}v\|_{L^2}^2+\|\Lambda^{s-\beta}\theta\|_{L^2}^2)\\&\le C(\|\nabla v\|_{L^\infty}+\|\nabla\theta\|_{L^\infty})(1+\|\Lambda^s v\|_{L^2}^2+\|\Lambda^{s-\beta}\theta\|_{L^2}^2).\end{aligned}$$

According to the Young inequality, it yields

$$\int_{\mathbb{R}^n}\Lambda^s\theta e_n\cdot\Lambda^s v\,dx\le C\|\Lambda^{s+\beta}v\|_{L^2}\|\Lambda^{s-\beta}\theta\|_{L^2}\le\frac{1}{2}\|\Lambda^{s+\beta}v\|_{L^2}^2+C\|\Lambda^{s-\beta}\theta\|_{L^2}^2.$$

Putting all the above estimates together, it is easy to show that

$$\frac{d}{dt}X(t)+\|\Lambda^{s+\beta}v\|_{L^2}^2\le C(1+\|\nabla v\|_{L^\infty}+\|\nabla\theta\|_{L^\infty})X(t)$$

where

$$X(t):=1+\|\Lambda^s v(t)\|_{L^2}^2+\|\Lambda^{s-\beta}\theta(t)\|_{L^2}^2.$$

Thanks to (2.8) and (2.9), we obtain by using the Gronwall inequality

$$\|\Lambda^s v(t)\|_{L^2}^2 + \|\Lambda^{s-\beta}\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{s+\beta} v(\tau)\|_{L^2}^2 \, d\tau \le C(t, v_0, \theta_0).$$

This completes the proof of Theorem 1.1 for **Case 1**. ∎

Case 2: $\alpha \ge \frac{n}{4}$ and $\beta = \frac{1}{2}$.

In this case, we first establish the following crucial estimate.

LEMMA 2.5. *Assume (v_0, θ_0) satisfies the assumptions stated in Theorem 1.1. Let $\alpha \ge \frac{n}{4}$ and $\beta = \frac{1}{2}$, then the following estimate holds*

$$\|v(t)\|_{L_T^\infty B^1_{\infty,\infty}} \le C(T, v_0, \theta_0), \tag{2.15}$$

where $C(T, v_0, \theta_0)$ is a constant depending on T and the initial data.

To prove Lemma 2.5, we first establish the following commutator estimates, which play an important role.

LEMMA 2.6. *Let u be a divergence-free vector field and $1 \le r \le \infty$, then for any $k \ge 0$ and $0 \le \delta \le 1$, we have*

$$\|[\Delta_k \mathcal{R}_l \mathcal{R}_i, u \cdot \nabla] v\|_{L_T^r L^\infty} \le C(k+1) 2^{-(1-\delta)k} \|u\|_{\widetilde{L}_T^r B^1_{\infty,\infty}} \|v\|_{\widetilde{L}_T^\infty B^{1-\delta}_{\infty,\infty}}, \tag{2.16}$$

$$\|[\Delta_k, u \cdot \nabla] v\|_{L_T^r L^\infty} \le C(k+1) 2^{-(1-\delta)k} \|u\|_{\widetilde{L}_T^r B^1_{\infty,\infty}} \|v\|_{\widetilde{L}_T^\infty B^{1-\delta}_{\infty,\infty}}, \tag{2.17}$$

where $\mathcal{R}_m := \frac{\partial_m}{\sqrt{-\Delta}}$ $(m = 1, 2, \cdots, n)$ are the classical Riesz transforms.

Proof. First, we make use of Bony's decomposition to show that

$$\begin{aligned}
[\Delta_k \mathcal{R}_l \mathcal{R}_i, u \cdot \nabla] v = & \sum_{|j-k| \le 4} [\Delta_k \mathcal{R}_l \mathcal{R}_i, S_{j-1} u \cdot \nabla] \Delta_j v + \sum_{|j-k| \le 4} [\Delta_k \mathcal{R}_l \mathcal{R}_i, \Delta_j u \cdot \nabla] S_{j-1} v \\
& + \sum_{j-k \ge -4} [\Delta_k \mathcal{R}_l \mathcal{R}_i, \widetilde{\Delta}_j u \cdot \nabla] \Delta_j v \\
:= & N_1 + N_2 + N_3.
\end{aligned}$$

Now we recall the following fact. Let $\mathcal{A}$ be an annulus centered at the origin. Then for every F with spectrum supported on $2^j \mathcal{A}$, there exists $\eta \in \mathcal{S}(\mathbb{R}^n)$ whose Fourier transform supported away from the origin, such that

$$\mathcal{R}_l \mathcal{R}_i F = 2^{jn} \eta(2^j .) \star F.$$

For fixed k, the summation over $|j-k| \le 4$ involves only a finite number of $j's$. For the sake of brevity, we shall replace the summations by their representative term with $j = k$ in N_1 and N_2. By Lemma A.1 and Lemma A.2, we have

$$\begin{aligned}
\|N_1\|_{L_T^r L^\infty} \le & C \|x 2^{kn} \eta(2^k x)\|_{L_T^\infty L^1} \|\nabla S_{k-1} u\|_{L_T^r L^\infty} \|\Delta_k \nabla v\|_{L_T^\infty L^\infty} \\
\le & C 2^{-k} \sum_{k' \le k-2} \|\Delta_{k'} \nabla u\|_{L_T^r L^\infty} \|\Delta_k \nabla v\|_{L_T^\infty L^\infty} \\
\le & C 2^{-(1-\delta)k} \sum_{k' \le k-2} \|\Delta_{k'} \nabla u\|_{L_T^r L^\infty} \|\Delta_k \Lambda^{1-\delta} v\|_{L_T^\infty L^\infty}
\end{aligned}$$

$$\leq C(k+1)2^{-(1-\delta)k}\|u\|_{\widetilde{L}_T^r B^1_{\infty,\infty}}\|v\|_{\widetilde{L}_T^\infty B^{1-\delta}_{\infty,\infty}},$$

$$\begin{aligned}\|N_2\|_{L_T^r L^\infty} \leq & C\|x 2^{kn}\eta(2^k x)\|_{L_T^\infty L^1}\|\Delta_k \nabla u\|_{L_T^r L^\infty}\|S_{k-1}\nabla v\|_{L_T^\infty L^\infty}\\ \leq & C2^{-k}\|\Delta_k \nabla u\|_{L_T^r L^\infty}\sum_{k'\leq k-2}\|\Delta_{k'}\nabla v\|_{L_T^\infty L^\infty}\\ \leq & C2^{-k}\|\Delta_k \nabla u\|_{L_T^r L^\infty}\sum_{k'\leq k-2}2^{\delta k'}\|\Delta_{k'}\Lambda^{1-\delta} v\|_{L_T^\infty L^\infty}\\ \leq & C2^{-(1-\delta)k}\|u\|_{\widetilde{L}_T^r B^1_{\infty,\infty}}\|v\|_{\widetilde{L}_T^\infty B^{1-\delta}_{\infty,\infty}}.\end{aligned}$$

The last term N_3 can be rewritten as

$$N_3 = \sum_{j-k\geq -4}\Delta_k \mathcal{R}_l \mathcal{R}_i(\widetilde{\Delta}_j u\cdot\nabla\Delta_j v) - \sum_{j-k\geq -4}\widetilde{\Delta}_j u\cdot\nabla\Delta_k\mathcal{R}_l\mathcal{R}_i\Delta_j v := N_3^1 + N_3^2.$$

Due to $\nabla\cdot u = 0$ and $k\geq 0$, we conclude by Lemma A.1 that

$$\begin{aligned}\|N_3^1\|_{L_T^r L^\infty} = & \left\|\sum_{j-k\geq -4}\Delta_k\mathcal{R}_l\mathcal{R}_i\nabla\cdot(\widetilde{\Delta}_j u\otimes\Delta_j v)\right\|_{L_T^r L^\infty}\\ \leq & C\sum_{j-k\geq -4}2^k\|\widetilde{\Delta}_j u\Delta_j v\|_{L_T^r L^\infty}\\ \leq & C\sum_{j-k\geq -4}2^k\|\widetilde{\Delta}_j u\|_{L_T^r L^\infty}\|\Delta_j v\|_{L_T^\infty L^\infty}\\ = & C\sum_{j-k\geq -4,\ j\geq 0}2^k\|\widetilde{\Delta}_j u\|_{L_T^r L^\infty}\|\Delta_j v\|_{L_T^\infty L^\infty}\\ & + C\sum_{j-k\geq -4,\ j=-1}2^k\|\widetilde{\Delta}_j u\|_{L_T^r L^\infty}\|\Delta_j v\|_{L_T^\infty L^\infty}\\ \leq & C2^{-(1-\delta)k}\sum_{j-k\geq -4,\ j\geq 0}2^{(2-\delta)(k-j)}\|\widetilde{\Delta}_j\nabla u\|_{L_T^r L^\infty}\|\Delta_j\Lambda^{1-\delta}v\|_{L_T^\infty L^\infty}\\ & + C\sum_{j-k\geq -4,\ j=-1}2^k\|\widetilde{\Delta}_j u\|_{L_T^r L^\infty}\|\Delta_j v\|_{L_T^\infty L^\infty}\\ \leq & C2^{-(1-\delta)k}\|u\|_{\widetilde{L}_T^r B^1_{\infty,\infty}}\|v\|_{\widetilde{L}_T^\infty B^{1-\delta}_{\infty,\infty}} + C2^{-(1-\delta)k}\|u\|_{\widetilde{L}_T^r B^0_{\infty,\infty}}\|v\|_{\widetilde{L}_T^\infty B^0_{\infty,\infty}}\\ \leq & C2^{-(1-\delta)k}\|u\|_{\widetilde{L}_T^r B^1_{\infty,\infty}}\|v\|_{\widetilde{L}_T^\infty B^{1-\delta}_{\infty,\infty}}.\end{aligned}$$

With the same argument, it yields

$$\begin{aligned}\|N_3^2\|_{L_T^r L^\infty} \leq & C\sum_{j-k\geq -4}\|\widetilde{\Delta}_j u\|_{L_T^r L^\infty}\|\nabla\Delta_k\mathcal{R}_l\mathcal{R}_i\Delta_j v\|_{L_T^\infty L^\infty}\\ \leq & C\sum_{j-k\geq -4}2^k\|\widetilde{\Delta}_j u\|_{L_T^r L^\infty}\|\Delta_j v\|_{L_T^\infty L^\infty}\\ \leq & C2^{-(1-\delta)k}\|u\|_{\widetilde{L}_T^r B^1_{\infty,\infty}}\|v\|_{\widetilde{L}_T^\infty B^{1-\delta}_{\infty,\infty}}.\end{aligned}$$

This allows us to get

$$\|N_3\|_{L_T^r L^\infty} \leq C(k+1)2^{-(1-\delta)k}\|u\|_{\widetilde{L}_T^r B^1_{\infty,\infty}}\|v\|_{\widetilde{L}_T^\infty B^{1-\delta}_{\infty,\infty}}.$$

Summing up all the above estimates, we obtain

$$\|[\Delta_k\mathcal{R}_l\mathcal{R}_i,u\cdot\nabla]v\|_{L^r_TL^\infty}\leq C(k+1)2^{-(1-\delta)k}\|u\|_{\widetilde{L}^r_TB^1_{\infty,\infty}}\|v\|_{\widetilde{L}^\infty_TB^{1-\delta}_{\infty,\infty}},$$

which is the desired estimate (2.16). Finally, (2.17) follows from the proof of (2.16). This completes the proof of Lemma 2.6. ∎

With (2.16) and (2.17) at our disposal, we are in a position to prove Lemma 2.5.

Proof. **(Proof of Lemma 2.5.)** Keeping in mind, it suffices to consider $\beta=\frac{1}{2}$. To this end, we apply Δ_k with $k\geq 0$ to the first equation of (1.4) to get

$$\partial_t\Delta_kv+\Delta_k\{(u\cdot\nabla)v\}+\Lambda\Delta_kv+\nabla\Delta_kp=\Delta_k\theta e_n. \tag{2.18}$$

By $\nabla\cdot u=\nabla\cdot v=0$, we can rewrite (2.18) as

$$\begin{aligned}\partial_t\Delta_kv+(u\cdot\nabla)\Delta_kv+\Lambda\Delta_kv&=-[\Delta_k,u\cdot\nabla]v+\Delta_k\theta e_n+\Delta_k\frac{\nabla\nabla\cdot}{-\Delta}(\theta e_n)\\&\quad-\Delta_k\frac{\nabla\partial_m}{-\Delta}(u_i\partial_iv_m)\\&=-[\Delta_k,u\cdot\nabla]v+\Delta_k\theta e_n+\Delta_k\frac{\nabla\nabla\cdot}{-\Delta}(\theta e_n)\\&\quad-\left[\Delta_k\frac{\nabla\partial_m}{-\Delta},u_i\right]\partial_iv_m,\end{aligned}$$

where we have used the fact

$$u_i\Delta_k\frac{\nabla\partial_m}{-\Delta}\partial_iv_m=u_i\Delta_k\frac{\nabla\partial_i}{-\Delta}\partial_mv_m=0.$$

This implies

$$\begin{aligned}\partial_t\Delta_kv+(u\cdot\nabla)\Delta_kv+\Lambda\Delta_kv&=-[\Delta_k,u\cdot\nabla]v+\Delta_k(\mathbb{I}+\mathcal{R}\mathcal{R}_n)(\theta e_n)\\&\quad-[\Delta_k\mathcal{R}\mathcal{R}_m,u\cdot\nabla]v_m,\end{aligned}\tag{2.19}$$

where $\mathbb{I}$ is the $n\times n$ identity matrix and $\mathcal{R}:=\frac{\nabla}{\sqrt{-\Delta}}$ is the classical Riesz transform. According to the localized maximum principle (see [16, Theorem 1.1]), we deduce from (2.19) that

$$\begin{aligned}\frac{d}{dt}\|\Delta_kv\|_{L^\infty}+c2^k\|\Delta_kv\|_{L^\infty}&\leq\|[\Delta_k,u\cdot\nabla]v\|_{L^\infty}+\|\Delta_k(\mathbb{I}+\mathcal{R}\mathcal{R}_n)(\theta e_n)\|_{L^\infty}\\&\quad+\|[\Delta_k\mathcal{R}\mathcal{R}_m,u\cdot\nabla]v_m\|_{L^\infty},\end{aligned}$$

where $c>0$ is an absolute constant. We therefore obtain

$$\begin{aligned}\|\Delta_kv(t)\|_{L^\infty}&\leq\|\Delta_kv_0\|_{L^\infty}e^{-c2^kt}+\int_0^te^{-c2^k(t-\tau)}\|[\Delta_k,u\cdot\nabla]v(\tau)\|_{L^\infty}\,d\tau\\&\quad+\int_0^te^{-c2^k(t-\tau)}\|\Delta_k(\mathbb{I}+\mathcal{R}\mathcal{R}_n)(\theta e_n)(\tau)\|_{L^\infty}\,d\tau\\&\quad+\int_0^te^{-c2^k(t-\tau)}\|[\Delta_k\mathcal{R}\mathcal{R}_m,u\cdot\nabla]v_m(\tau)\|_{L^\infty}\,d\tau.\end{aligned}\tag{2.20}$$

Recalling (2.16) and (2.17), a straightforward computation gives

$$\begin{aligned}&\left\|\int_0^t e^{-c2^k(t-\tau)}\|[\Delta_k,u\cdot\nabla]v(\tau)\|_{L^\infty}\,d\tau\right\|_{L_T^\infty}\\ \le& C\|e^{-c2^kt}\|_{L_T^2}\,\|[\Delta_k,u\cdot\nabla]v(\tau)\|_{L_T^2L^\infty}\\ \le& C(k+1)2^{-(\frac{3}{2}-\delta)k}\|u\|_{\widetilde{L}_T^2B^1_{\infty,\infty}}\|v\|_{\widetilde{L}_T^\infty B^{1-\delta}_{\infty,\infty}},\end{aligned}$$

$$\begin{aligned}&\left\|\int_0^t e^{-c2^k(t-\tau)}\|[\Delta_k\mathcal{R}\mathcal{R}_m,u\cdot\nabla]v_m(\tau)\|_{L^\infty}\,d\tau\right\|_{L_T^\infty}\\ \le& C\|e^{-c2^kt}\|_{L_T^2}\,\|[\Delta_k\mathcal{R}\mathcal{R}_m,u\cdot\nabla]v_m(\tau)\|_{L_T^2L^\infty}\\ \le& C(k+1)2^{-(\frac{3}{2}-\delta)k}\|u\|_{\widetilde{L}_T^2B^1_{\infty,\infty}}\|v\|_{\widetilde{L}_T^\infty B^{1-\delta}_{\infty,\infty}}.\end{aligned}$$

Due to $k\ge 0$, we may deduce

$$\begin{aligned}&\left\|\int_0^t e^{-c2^k(t-\tau)}\|\Delta_k(\mathbb{I}+\mathcal{R}\mathcal{R}_n)(\theta e_n)(\tau)\|_{L^\infty}\,d\tau\right\|_{L_T^\infty}\\ \le& C\left\|\int_0^t e^{-c2^k(t-\tau)}\|\theta(\tau)\|_{L^\infty}\,d\tau\right\|_{L_T^\infty}\\ \le& C\left(\int_0^t e^{-c2^kt}\,dt\right)\|\theta(\tau)\|_{L_T^\infty L^\infty}\\ \le& C2^{-k}\|\theta_0\|_{L^\infty}.\end{aligned}$$

Inserting the above three estimates into (2.20) yields for any $k\ge 0$

$$\begin{aligned}\|\Delta_k v(t)\|_{L_T^\infty L^\infty}\le&\|\Delta_k v_0\|_{L^\infty}e^{-c2^kt}+C(k+1)2^{-(\frac{3}{2}-\delta)k}\|u\|_{\widetilde{L}_T^2B^1_{\infty,\infty}}\|v\|_{\widetilde{L}_T^\infty B^{1-\delta}_{\infty,\infty}}\\ &+C2^{-k}\|\theta_0\|_{L^\infty}.\end{aligned}\tag{2.21}$$

Moreover, it is obvious to check that

$$\|\Delta_{-1}v(t)\|_{L_T^\infty L^\infty}\le C\|\Delta_{-1}v(t)\|_{L_T^\infty L^2}\le C(T,v_0,\theta_0).\tag{2.22}$$

Fixing $\delta\in(0,\frac{1}{2})$, it follows from (2.21) and (2.22) that

$$\begin{aligned}\|v(t)\|_{\widetilde{L}_T^\infty B^1_{\infty,\infty}}\le& C(T,v_0,\theta_0)+C\sup_{k\ge 0}\{(k+1)2^{-(\frac{1}{2}-\delta)k}\}\|u\|_{\widetilde{L}_T^2B^1_{\infty,\infty}}\|v\|_{\widetilde{L}_T^\infty B^{1-\delta}_{\infty,\infty}}\\ \le& C(T,v_0,\theta_0)+C\|u\|_{\widetilde{L}_T^2B^1_{\infty,\infty}}\|v\|_{\widetilde{L}_T^\infty B^{1-\delta}_{\infty,\infty}}.\end{aligned}\tag{2.23}$$

Invoking the facts of (A.1), we thus deduce from (2.23) that

$$\|v(t)\|_{L_T^\infty B^1_{\infty,\infty}}\le C(T,v_0,\theta_0)+C\|u(t)\|_{L_T^2B^1_{\infty,\infty}}\|v(t)\|_{L_T^\infty B^{1-\delta}_{\infty,\infty}}.$$

Noticing the following interpolation inequality

$$\|v(t)\|_{L_T^\infty B^{1-\delta}_{\infty,\infty}}\le C\|v(t)\|_{L_T^\infty L^2}^{\frac{2\delta}{n+2}}\|v(t)\|_{L_T^\infty B^1_{\infty,\infty}}^{\frac{n+2-2\delta}{n+2}},$$

it yields

$$\begin{aligned}\|v(t)\|_{L^\infty_T B^1_{\infty,\infty}} \le & C(T,v_0,\theta_0)+C\|u(t)\|_{L^2_T B^1_{\infty,\infty}}\|v(t)\|^{\frac{2\delta}{n+2}}_{L^\infty_T L^2}\|v(t)\|^{\frac{n+2-2\delta}{n+2}}_{L^\infty_T B^1_{\infty,\infty}}\\ \le & \frac{1}{2}\|v(t)\|_{L^\infty_T B^1_{\infty,\infty}}+C(T,v_0,\theta_0)+C\|u(t)\|^{\frac{n+2}{2\delta}}_{L^2_T B^1_{\infty,\infty}}\|v(t)\|_{L^\infty_T L^2},\end{aligned}$$

which implies

$$\|v(t)\|_{L^\infty_T B^1_{\infty,\infty}} \le C(T,v_0,\theta_0)+C\|u(t)\|^{\frac{n+2}{2\delta}}_{L^2_T B^1_{\infty,\infty}}\|v(t)\|_{L^\infty_T L^2}. \tag{2.24}$$

It follows from the simple embedding and (2.5) that

$$\int_0^T \|u(\tau)\|^2_{B^1_{\infty,\infty}}\,d\tau \le C\int_0^T \|u(\tau)\|^2_{H^{1+\frac{n}{2}}}\,d\tau \le C(T,v_0,\theta_0).$$

This along with (2.24) yields

$$\|v(t)\|_{L^\infty_T B^1_{\infty,\infty}} \le C(T,v_0,\theta_0).$$

We therefore conclude the proof of Lemma 2.5. □

With the above estimates at our disposal, we are now ready to prove Theorem 1.1 for **Case 2**.

Proof. **(The global H^s-estimate for Case 2.)** Recalling (2.14), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\Lambda^s v(t)\|^2_{L^2}+\|\Lambda^{s-\beta}\theta(t)\|^2_{L^2})+\|\Lambda^{s+\beta}v\|^2_{L^2}\\ &=-\int_{\mathbb{R}^n}[\Lambda^s,u\cdot\nabla]v\cdot\Lambda^s v\,dx-\int_{\mathbb{R}^n}[\Lambda^{s-\beta},v\cdot\nabla]\theta\Lambda^{s-\beta}\theta\,dx+\int_{\mathbb{R}^n}\Lambda^s\theta e_n\cdot\Lambda^s v\,dx.\end{aligned}$$

It follows from the commutator (A.3) that

$$\begin{aligned}\left|\int_{\mathbb{R}^n}[\Lambda^s,u\cdot\nabla]v\cdot\Lambda^s v\,dx\right| \le & C\|[\Lambda^s,u\cdot\nabla]v\|_{L^2}\|\Lambda^s v\|_{L^2}\\ \le & C(\|\nabla u\|_{L^\infty}\|\Lambda^s v\|_{L^2}+\|\nabla v\|_{L^\infty}\|\Lambda^s u\|_{L^2})\|\Lambda^s v\|_{L^2}\\ \le & C(\|\nabla u\|_{L^\infty}+\|\nabla v\|_{L^\infty})(\|\Lambda^s u\|^2_{L^2}+\|\Lambda^s v\|^2_{L^2})\\ \le & C(\|\nabla u\|_{L^\infty}+\|\nabla v\|_{L^\infty})\|\Lambda^s v\|^2_{L^2}.\end{aligned} \tag{2.25}$$

Thanks to $\nabla\cdot v=0$ and (A.4), it yields for $\beta\ge\frac{1}{2}$ that

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}[\Lambda^{s-\beta},v\cdot\nabla]\theta\Lambda^{s-\beta}\theta\,dx\right|=\left|\int_{\mathbb{R}^n}[\Lambda^{s-\beta}\partial_i,v_i]\theta\Lambda^{s-\beta}\theta\,dx\right|\\ &\le C\|[\Lambda^{s-\beta}\partial_i,v_i]\theta\|_{L^2}\|\Lambda^{s-\beta}\theta\|_{L^2}\\ &\le C(\|\nabla v\|_{L^\infty}\|\Lambda^{s-\beta}\theta\|_{L^2}+\|\theta\|_{L^\infty}\|\Lambda^{s+1-\beta}v\|_{L^2})\|\Lambda^{s-\beta}\theta\|_{L^2}\\ &\le C\|\nabla v\|_{L^\infty}\|\Lambda^{s-\beta}\theta\|^2_{L^2}+C\|\theta\|_{L^\infty}\|v\|^{\frac{2\beta-1}{s+\beta}}_{L^2}\|\Lambda^{s+\beta}v\|^{\frac{s+1-\beta}{s+\beta}}_{L^2}\|\Lambda^{s-\beta}\theta\|_{L^2}\\ &\le\frac{1}{4}\|\Lambda^{s+\beta}v\|^2_{L^2}+C\|\nabla v\|_{L^\infty}\|\Lambda^{s-\beta}\theta\|^2_{L^2}+C\|\theta\|^{\frac{2(s+\beta)}{s+3\beta-1}}_{L^\infty}\|v\|^{\frac{2(2\beta-1)}{s+3\beta-1}}_{L^2}\|\Lambda^{s-\beta}\theta\|^{\frac{2(s+\beta)}{s+3\beta-1}}_{L^2}.\end{aligned} \tag{2.26}$$

The Young inequality yields

$$\int_{\mathbb{R}^n} \Lambda^s\theta e_n\cdot\Lambda^s v\,dx \le C\|\Lambda^{s+\beta}v\|_{L^2}\|\Lambda^{s-\beta}\theta\|_{L^2} \le \frac{1}{4}\|\Lambda^{s+\beta}v\|_{L^2}^2 + C\|\Lambda^{s-\beta}\theta\|_{L^2}^2. \quad (2.27)$$

Putting all the above estimates together, it is easy to show that

$$\frac{d}{dt}Z(t)+\|\Lambda^{s+\beta}v\|_{L^2}^2 \le C\left(1+\|\nabla u\|_{L^\infty}+\|\nabla v\|_{L^\infty}+\|\theta\|_{L^\infty}^{\frac{2(s+\beta)}{s+3\beta-1}}\|v\|_{L^2}^{\frac{2(2\beta-1)}{s+3\beta-1}}\right)Z(t)$$

where

$$Z(t):=1+\|\Lambda^s v(t)\|_{L^2}^2+\|\Lambda^{s-\beta}\theta(t)\|_{L^2}^2.$$

Recall the logarithmic Sobolev inequalities (see [3, 12] for instance)

$$\|\nabla f\|_{L^\infty} \le C\Big(1+\|f\|_{L^2}+\|\Lambda^{1+\frac{n}{2}}f\|_{L^2}\log\big(1+\|f\|_{\dot H^s}\big)\Big),\quad s>1+\frac{n}{2}, \quad (2.28)$$

$$\|\nabla f\|_{L^\infty} \le C\Big(1+\|f\|_{L^2}+\|f\|_{B^1_{\infty,\infty}}\log\big(1+\|f\|_{\dot H^s}\big)\Big),\quad s>1+\frac{n}{2}. \quad (2.29)$$

Now we thus deduce

$$\frac{d}{dt}Z(t)+\|\Lambda^{s+\beta}v\|_{L^2}^2 \le C\left(1+\|\Lambda^{1+\frac{n}{2}}u\|_{L^2}+\|v\|_{B^1_{\infty,\infty}}+\|\theta\|_{L^\infty}^{\frac{2(s+\beta)}{s+3\beta-1}}\|v\|_{L^2}^{\frac{2(2\beta-1)}{s+3\beta-1}}\right)\times Z(t)\ln Z(t).$$

According to (2.1)–(2.5) and (2.15), we obtain

$$\int_0^T \left(1+\|\Lambda^{1+\frac{n}{2}}u(t)\|_{L^2}+\|v(t)\|_{B^1_{\infty,\infty}}+\|\theta(t)\|_{L^\infty}^{\frac{2(s+\beta)}{s+3\beta-1}}\|v(t)\|_{L^2}^{\frac{2(2\beta-1)}{s+3\beta-1}}\right)dt \le C(T,v_0,\theta_0),$$

which together with the Gronwall inequality yields for any $t\le T$

$$\|\Lambda^s v(t)\|_{L^2}^2+\|\Lambda^{s-\beta}\theta(t)\|_{L^2}^2+\int_0^T\|\Lambda^{s+\beta}v(\tau)\|_{L^2}^2\,d\tau \le C(T,v_0,\theta_0).$$

This ends the proof of Theorem 1.1 for **Case 2**. □

3. The proof of Theorem 1.2

The proof of Theorem 1.2 can be performed in the same fashion as that of Theorem 1.1 with some certain modifications. Now we present the details as follows. We first point out that due to $\alpha+\beta\ge\frac{1}{2}+\frac{n}{4}$, Lemma 2.1 and Lemma 2.2 are still true. For simplicity, we denote

$$M:=\|\theta(t)\|_{\widetilde{L}^p_T B^{1-2\beta+\frac{2\beta}{p}}_{\infty,\infty}}<\infty.$$

Now we will show the following crucial estimate under the assumption (1.6).

LEMMA 3.1. *Assume (v_0,θ_0) satisfies the assumptions stated in Theorem 1.2. Let $\alpha+\beta\ge\frac{1}{2}+\frac{n}{4}$ with $0<\beta<\frac{1}{2}$, then it holds under the assumption (1.6)*

$$\|v(t)\|_{L^\infty_T B^1_{\infty,\infty}} \le C(T,M,v_0,\theta_0), \quad (3.1)$$

where $C(T,M,v_0,\theta_0)$ is a constant depending on T, M and the initial data.

Proof. We first deduce from (2.19) that

$$\partial_t\Delta_k v+(u\cdot\nabla)\Delta_k v+\Lambda^{2\beta}\Delta_k v=-[\Delta_k,u\cdot\nabla]v+\Delta_k(\mathbb{I}+\mathcal{R}\mathcal{R}_n)(\theta e_n)-[\Delta_k\mathcal{R}\mathcal{R}_m,u\cdot\nabla]v_m.$$

Again, thanks to the localized maximum principle (see [16, Theorem 1.1]), it yields

$$\frac{d}{dt}\|\Delta_k v\|_{L^\infty}+c2^{2\beta k}\|\Delta_k v\|_{L^\infty}\le\|[\Delta_k,u\cdot\nabla]v\|_{L^\infty}+\|\Delta_k(\mathbb{I}+\mathcal{R}\mathcal{R}_n)(\theta e_n)\|_{L^\infty}+\|[\Delta_k\mathcal{R}\mathcal{R}_m,u\cdot\nabla]v_m\|_{L^\infty},\tag{3.2}$$

where $c>0$ is an absolute constant. Consequently, we have

$$\begin{aligned}\|\Delta_k v(t)\|_{L^\infty}\le&\|\Delta_k v_0\|_{L^\infty}e^{-c2^{2\beta k}t}+\int_0^t e^{-c2^k(t-\tau)}\|[\Delta_k,u\cdot\nabla]v(\tau)\|_{L^\infty}\,d\tau\\&+\int_0^t e^{-c2^{2\beta k}(t-\tau)}\|\Delta_k(\mathbb{I}+\mathcal{R}\mathcal{R}_n)(\theta e_n)(\tau)\|_{L^\infty}\,d\tau\\&+\int_0^t e^{-c2^{2\beta k}(t-\tau)}\|[\Delta_k\mathcal{R}\mathcal{R}_m,u\cdot\nabla]v_m(\tau)\|_{L^\infty}\,d\tau.\end{aligned}$$

By (2.16) and (2.17), one thus obtains

$$\begin{aligned}&\left\|\int_0^t e^{-c2^{2\beta k}(t-\tau)}\|[\Delta_k,u\cdot\nabla]v(\tau)\|_{L^\infty}\,d\tau\right\|_{L^\infty_T}\\\le&C\|e^{-c2^{2\beta k}t}\|_{L^2_T}\,\|[\Delta_k,u\cdot\nabla]v(\tau)\|_{L^2_TL^\infty}\\\le&C(k+1)2^{-(1+\beta-\delta)k}\|u\|_{\widetilde{L}^2_TB^1_{\infty,\infty}}\|v\|_{\widetilde{L}^\infty_TB^{1-\delta}_{\infty,\infty}},\end{aligned}$$

$$\begin{aligned}&\left\|\int_0^t e^{-c2^{2\beta k}(t-\tau)}\|[\Delta_k\mathcal{R}\mathcal{R}_m,u\cdot\nabla]v_m(\tau)\|_{L^\infty}\,d\tau\right\|_{L^\infty_T}\\\le&C\|e^{-c2^{2\beta k}t}\|_{L^2_T}\,\|[\Delta_k\mathcal{R}\mathcal{R}_m,u\cdot\nabla]v_m(\tau)\|_{L^2_TL^\infty}\\\le&C(k+1)2^{-(1+\beta-\delta)k}\|u\|_{\widetilde{L}^2_TB^1_{\infty,\infty}}\|v\|_{\widetilde{L}^\infty_TB^{1-\delta}_{\infty,\infty}}.\end{aligned}$$

By means of $k\ge0$, one can bound it as follows

$$\begin{aligned}&\left\|\int_0^t e^{-c2^{2\beta k}(t-\tau)}\|\Delta_k(\mathbb{I}+\mathcal{R}\mathcal{R}_n)(\theta e_n)(\tau)\|_{L^\infty}\,d\tau\right\|_{L^\infty_T}\\\le&C\left\|\int_0^t e^{-c2^{2\beta k}(t-\tau)}\|\Delta_k\theta(\tau)\|_{L^\infty}\,d\tau\right\|_{L^\infty_T}\\\le&C\|e^{-c2^{2\beta k}t}\|_{L^{p'}_T}\|\Delta_k\theta(\tau)\|_{L^p_TL^\infty}\\\le&C2^{-\frac{2\beta(p-1)k}{p}}\|\Delta_k\theta(\tau)\|_{L^p_TL^\infty}.\end{aligned}$$

Plugging the above estimates into (3.2) gives that for any $k\ge0$

$$\|\Delta_k v(t)\|_{L^\infty_TL^\infty}\le\|\Delta_k v_0\|_{L^\infty}e^{-c2^{2\beta k}t}+C(k+1)2^{-(1+\beta-\delta)k}\|u\|_{\widetilde{L}^2_TB^1_{\infty,\infty}}\|v\|_{\widetilde{L}^\infty_TB^{1-\delta}_{\infty,\infty}}+C2^{-\frac{2\beta(p-1)k}{p}}\|\Delta_k\theta(\tau)\|_{L^p_TL^\infty}.\tag{3.3}$$

Moreover, one has

$$\|\Delta_{-1}v(t)\|_{L_T^\infty L^\infty}\leq C\|\Delta_{-1}v(t)\|_{L_T^\infty L^2}\leq C(T,v_0,\theta_0). \tag{3.4}$$

Taking $\delta\in(0,\beta)$, we deduce from (3.3) and (3.4) that

$$\begin{aligned}\|v(t)\|_{\widetilde{L}_T^\infty B_{\infty,\infty}^1}\leq& C(T,v_0,\theta_0)+C\sup_{k\geq 0}\{(k+1)2^{-(\beta-\delta)k}\}\|u\|_{\widetilde{L}_T^2 B_{\infty,\infty}^1}\|v\|_{\widetilde{L}_T^\infty B_{\infty,\infty}^{1-\delta}}\\&+C\|\theta(t)\|_{\widetilde{L}_T^p B_{\infty,\infty}^{1-2\beta+\frac{2\beta}{p}}}\\ \leq& C(T,M,v_0,\theta_0)+C\|u\|_{\widetilde{L}_T^2 B_{\infty,\infty}^1}\|v\|_{\widetilde{L}_T^\infty B_{\infty,\infty}^{1-\delta}}.\end{aligned}$$

Invoking the following facts of (A.1), it implies

$$\|v(t)\|_{L_T^\infty B_{\infty,\infty}^1}\leq C(T,M,v_0,\theta_0)+C\|u(t)\|_{L_T^2 B_{\infty,\infty}^1}\|v(t)\|_{L_T^\infty B_{\infty,\infty}^{1-\delta}}.$$

With the same argument in dealing with (2.24), we derive

$$\|v(t)\|_{L_T^\infty B_{\infty,\infty}^1}\leq C(T,M,v_0,\theta_0)+C\|u(t)\|_{L_T^2 B_{\infty,\infty}^1}^{\frac{n+2}{2\delta}}\|v(t)\|_{L_T^\infty L^2},$$

which further gives

$$\|v(t)\|_{L_T^\infty B_{\infty,\infty}^1}\leq C(T,M,v_0,\theta_0).$$

We therefore conclude the proof of Lemma 3.1. □

With the help of (3.1), we are now in a position to prove Theorem 1.2. Keeping in mind (2.14), we get that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\Lambda^s v(t)\|_{L^2}^2+\|\Lambda^{s-\beta}\theta(t)\|_{L^2}^2)+\|\Lambda^{s+\beta}v\|_{L^2}^2\\ =&-\int_{\mathbb{R}^n}[\Lambda^s,u\cdot\nabla]v\cdot\Lambda^s v\,dx-\int_{\mathbb{R}^n}[\Lambda^{s-\beta},v\cdot\nabla]\theta\Lambda^{s-\beta}\theta\,dx+\int_{\mathbb{R}^n}\Lambda^s\theta e_n\cdot\Lambda^s v\,dx.\end{aligned}$$

By (2.25) and (2.27), one arrives at

$$\left|\int_{\mathbb{R}^n}[\Lambda^s,u\cdot\nabla]v\cdot\Lambda^s v\,dx\right|\leq C(\|\nabla u\|_{L^\infty}+\|\nabla v\|_{L^\infty})\|\Lambda^s v\|_{L^2}^2,$$

$$\int_{\mathbb{R}^n}\Lambda^s\theta e_n\cdot\Lambda^s v\,dx\leq C\|\Lambda^{s+\beta}v\|_{L^2}\|\Lambda^{s-\beta}\theta\|_{L^2}\leq\frac{1}{4}\|\Lambda^{s+\beta}v\|_{L^2}^2+C\|\Lambda^{s-\beta}\theta\|_{L^2}^2.$$

However, the estimate (2.26) depends heavily on the requirement $\beta\geq\frac{1}{2}$. However, our case $0<\beta<\frac{1}{2}$ fails. Thus, we should handle this case differently. The following crude estimate is an easy consequence of (A.3)

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}[\Lambda^{s-\beta},v\cdot\nabla]\theta\Lambda^{s-\beta}\theta\,dx\right|\\ \leq& C\|[\Lambda^{s-\beta},v\cdot\nabla]\theta\|_{L^2}\|\Lambda^{s-\beta}\theta\|_{L^2}\\ \leq& C(\|\nabla v\|_{L^\infty}\|\Lambda^{s-\beta}\theta\|_{L^2}+\|\nabla\theta\|_{L^\infty}\|\Lambda^{s-\beta}v\|_{L^2})\|\Lambda^{s-\beta}\theta\|_{L^2}\\ \leq& C\|\nabla v\|_{L^\infty}\|\Lambda^{s-\beta}\theta\|_{L^2}^2+C\|\nabla\theta\|_{L^\infty}\|v\|_{L^2}^{\frac{2\beta}{s+\beta}}\|\Lambda^{s+\beta}v\|_{L^2}^{\frac{s-\beta}{s+\beta}}\|\Lambda^{s-\beta}\theta\|_{L^2}\end{aligned}$$

$$\leq \frac{1}{4}\|\Lambda^{s+\beta}v\|_{L^2}^2 + C\|\nabla v\|_{L^\infty}\|\Lambda^{s-\beta}\theta\|_{L^2}^2 + C\|\nabla\theta\|_{L^\infty}^{\frac{2(s+\beta)}{s+3\beta}}\|v\|_{L^2}^{\frac{4\beta}{s+3\beta}}\|\Lambda^{s-\beta}\theta\|_{L^2}^{\frac{2(s+\beta)}{s+3\beta}}.$$

Putting all the above estimates together, it is easy to show that

$$\begin{aligned}&\frac{d}{dt}(\|\Lambda^s v(t)\|_{L^2}^2 + \|\Lambda^{s-\beta}\theta(t)\|_{L^2}^2) + \|\Lambda^{s+\beta}v\|_{L^2}^2\\ \leq& C(1+\|\nabla u\|_{L^\infty} + \|\nabla v\|_{L^\infty})(\|\Lambda^s v\|_{L^2}^2 + \|\Lambda^{s-\beta}\theta\|_{L^2}^2) + C\|\nabla\theta\|_{L^\infty}^{\frac{2(s+\beta)}{s+3\beta}}\|v\|_{L^2}^{\frac{4\beta}{s+3\beta}}\|\Lambda^{s-\beta}\theta\|_{L^2}^{\frac{2(s+\beta)}{s+3\beta}}. \quad (3.5)\end{aligned}$$

However, at this moment we have no estimate for $\|\nabla\theta\|_{L^\infty}$. To this end, some special techniques are required. To begin with, according to (3.1), it is not difficult to see that for any small constant $\epsilon > 0$ to be fixed hereafter, there exists $T_0 = T_0(\epsilon) \in (0,T)$ such that

$$\int_{T_0}^{T}\|v(\tau)\|_{B^1_{\infty,\infty}}\,d\tau \leq \epsilon.$$

For any $T_0 \leq t \leq T$, we denote

$$\Gamma(t) := \max_{\tau\in[T_0,t]}(\|\Lambda^s v(\tau)\|_{L^2}^2 + \|\Lambda^{s-\beta}\theta(\tau)\|_{L^2}^2), \quad s > 1 + \frac{n}{2}.$$

Consequently, one may verify that $\Gamma(t)$ is a monotonically nondecreasing function. Then, the next objective is to show

$$\lim_{t\to T^-}\Gamma(t) \leq C < \infty.$$

Recalling (2.13), (2.29) and noticing the monotonicity of $\Gamma(t)$, it is not hard to check for any $T_0 \leq t < T$ that

$$\begin{aligned}\|\nabla\theta(t)\|_{L^\infty} \leq& \|\nabla\theta(T_0)\|_{L^\infty}\exp\Big[\int_{T_0}^t\|\nabla v(\tau)\|_{L^\infty}\,d\tau\Big]\\ \leq& C\exp\Big[C\int_{T_0}^t\Big(1 + \|v(\tau)\|_{L^2} + \|v(\tau)\|_{B^1_{\infty,\infty}}\ln(e + \|\Lambda^s v(\tau)\|_{L^2}\Big)\,d\tau\Big]\\ \leq& C\exp\Big[\int_{T_0}^t C(1+\|v\|_{L^2})\,d\tau\Big]\exp\Big[C_0\Big(\int_{T_0}^t\|v(\tau)\|_{B^1_{\infty,\infty}}\,d\tau\Big)\ln\big(e+\Gamma(t)\big)\Big]\\ \leq& C\exp\Big[C_0\Big(\int_{T_0}^t\|v(\tau)\|_{B^1_{\infty,\infty}}\,d\tau\Big)\ln\big(e+\Gamma(t)\big)\Big]\\ \leq& C(e+\Gamma(t))^{C_0\epsilon},\end{aligned}$$

where $C_0 > 0$ is an absolute constant whose value is independent of ϵ, T or T_0. Consequently, it implies

$$\|\nabla\theta(t)\|_{L^\infty} \leq C(e+\Gamma(t))^{C_0\epsilon} \quad \text{for any } T_0 \leq t < T. \quad (3.6)$$

Integrating (3.5) over the interval (T_0,t) and making use of (2.28)-(2.29), it leads to

$$\begin{aligned}&\|\Lambda^s v(t)\|_{L^2}^2 + \|\Lambda^{s-\beta}\theta(t)\|_{L^2}^2 - \|\Lambda^s v(T_0)\|_{L^2}^2 - \|\Lambda^{s-\beta}\theta(T_0)\|_{L^2}^2\\ \leq& C\int_{T_0}^t(1 + \|u(\tau)\|_{H^{1+\frac{n}{2}}} + \|v(\tau)\|_{B^1_{\infty,\infty}})\ln\big(e + \|\Lambda^s v(\tau)\|_{L^2}^2 + \|\Lambda^{s-\beta}\theta(\tau)\|_{L^2}^2\big)\\ &\times(\|\Lambda^s v(\tau)\|_{L^2}^2 + \|\Lambda^{s-\beta}\theta(\tau)\|_{L^2}^2)\,d\tau + C\int_{T_0}^t\|\nabla\theta(\tau)\|_{L^\infty}^{\frac{2(s+\beta)}{s+3\beta}}\|\Lambda^{s-\beta}\theta(\tau)\|_{L^2}^{\frac{2(s+\beta)}{s+3\beta}}\,d\tau,\end{aligned}$$

which along with (3.6) yields

$$\begin{aligned} e+\Gamma(t) \leq& C+\Gamma(T_0)+C\int_{T_0}^{t}(1+\|u(\tau)\|_{H^{1+\frac{n}{2}}}+\|v(\tau)\|_{B^1_{\infty,\infty}})\ln\big(e+\Gamma(\tau)\big)\Gamma(\tau)\,d\tau \\ &+C\int_{T_0}^{t}\big(e+\Gamma(\tau)\big)^{\frac{2C_0\epsilon(s+\beta)}{s+3\beta}}\big(e+\Gamma(\tau)\big)^{\frac{s+\beta}{s+3\beta}}\,d\tau \\ \leq& C+\Gamma(T_0)+C\int_{T_0}^{t}(1+\|u(\tau)\|_{H^{1+\frac{n}{2}}}+\|v(\tau)\|_{B^1_{\infty,\infty}})\times\ln\big(e+\Gamma(\tau)\big)\big(e+\Gamma(\tau)\big)\,d\tau, \end{aligned}$$

where in the last line we have chosen $0<\epsilon\leq\frac{2\beta}{2C_0(s+\beta)}$. This tells us that

$$e+\Gamma(t)\leq C+\Gamma(T_0)+C\int_{T_0}^{t}A(\tau)\ln\big(e+\Gamma(\tau)\big)\big(e+\Gamma(\tau)\big)\,d\tau, \tag{3.7}$$

where

$$A(\tau):=1+\|u(\tau)\|_{H^{1+\frac{n}{2}}}+\|v(\tau)\|_{B^1_{\infty,\infty}}.$$

Thanks to (2.5) and (3.1), we observe

$$\int_0^T A(\tau)\,d\tau<\infty.$$

Applying the Log-Gronwall inequality to (3.7), we end up with

$$\Gamma(t)\leq C<\infty,\quad T_0\leq t\leq T.$$

By the local well-posedness results, the solution (v,θ) can be extended beyond time T. Consequently, we complete the proof of Theorem 1.2.

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Appendix A. Besov spaces and some useful facts. This Appendix includes several parts. It recalls the Littlewood-Paley theory, introduces the Besov spaces, provides Bernstein inequalities as well as several facts used in the proof of our main result. We start with the Littlewood-Paley theory. We choose some smooth radial nonincreasing function χ with values in $[0,1]$ such that $\chi\in C_0^\infty(\mathbb{R}^n)$ is supported in the ball $\mathcal{B}:=\{\xi\in\mathbb{R}^n,|\xi|\leq\frac{4}{3}\}$ and with value 1 on $\{\xi\in\mathbb{R}^n,|\xi|\leq\frac{3}{4}\}$, then we set $\varphi(\xi)=\chi\big(\frac{\xi}{2}\big)-\chi(\xi)$. One easily verifies that $\varphi\in C_0^\infty(\mathbb{R}^n)$ is supported in the annulus $\mathcal{C}:=\{\xi\in\mathbb{R}^n,\frac{3}{4}\leq|\xi|\leq\frac{8}{3}\}$ and satisfies

$$\chi(\xi)+\sum_{j\geq 0}\varphi(2^{-j}\xi)=1,\quad\forall\xi\in\mathbb{R}^n.$$

Let $h=\mathcal{F}^{-1}(\varphi)$ and $\widetilde{h}=\mathcal{F}^{-1}(\chi)$, then we introduce the dyadic blocks Δ_j of our decomposition by setting

$$\Delta_j u=0,\ \ j\leq -2;\qquad \Delta_{-1}u=\chi(D)u=\int_{\mathbb{R}^n}\widetilde{h}(y)u(x-y)\,dy;$$

$$\Delta_j u = \varphi(2^{-j}D)u = 2^{jn}\int_{\mathbb{R}^n} h(2^j y)u(x-y)\,dy, \quad \forall j\in\mathbb{N}.$$

We shall also denote

$$S_j u := \sum_{-1\le k\le j-1} \Delta_k u, \qquad \widetilde{\Delta}_j u := \Delta_{j-1}u + \Delta_j u + \Delta_{j+1}u.$$

The nonhomogeneous Besov spaces are defined through the dyadic decomposition.

DEFINITION A.1. *Let $s\in\mathbb{R}, (p,r)\in[1,+\infty]^2$. The nonhomogeneous Besov space $B^s_{p,r}$ is defined as a space of $f\in S'(\mathbb{R}^n)$ such that*

$$B^s_{p,r} = \{f\in S'(\mathbb{R}^n); \|f\|_{B^s_{p,r}} < \infty\},$$

where

$$\|f\|_{B^s_{p,r}} = \begin{cases} \Big(\sum_{j\ge -1} 2^{jrs}\|\Delta_j f\|^r_{L^p}\Big)^{\frac{1}{r}}, & r<\infty,\\ \sup_{j\ge -1} 2^{js}\|\Delta_j f\|_{L^p}, & r=\infty. \end{cases}$$

We shall also need the mixed space-time spaces

$$\|f\|_{L^\rho_T B^s_{p,r}} := \Big\|(2^{js}\|\Delta_j f\|_{L^p})_{l^r_j}\Big\|_{L^\rho_T}$$

and

$$\|f\|_{\widetilde{L}^\rho_T B^s_{p,r}} := (2^{js}\|\Delta_j f\|_{L^\rho_T L^p})_{l^r_j}.$$

The following links are a direct consequence of the Minkowski inequality

$$L^\rho_T B^s_{p,r} \hookrightarrow \widetilde{L}^\rho_T B^s_{p,r}, \qquad \text{if } r\ge\rho, \quad \text{and} \quad \widetilde{L}^\rho_T B^s_{p,r} \hookrightarrow L^\rho_T B^s_{p,r}, \qquad \text{if } \rho\ge r. \tag{A.1}$$

In particular,

$$\widetilde{L}^r_T B^s_{p,r} \approx L^r_T B^s_{p,r}.$$

The following lemma provides Bernstein-type inequalities for fractional derivatives.

LEMMA A.1 (see [1]). *Assume $1\le a\le b\le\infty$. If the integer $j\ge -1$, then it holds*

$$\|\Lambda^k \Delta_j f\|_{L^b} \le C_1 2^{jk+jn(\frac{1}{a}-\frac{1}{b})}\|\Delta_j f\|_{L^a}, \quad k\ge 0.$$

If the integer $j\ge 0$, then we have

$$C_2 2^{jk}\|\Delta_j f\|_{L^b} \le \|\Lambda^k\Delta_j f\|_{L^b} \le C_3 2^{jk+jn(\frac{1}{a}-\frac{1}{b})}\|\Delta_j f\|_{L^a}, \quad k\in\mathbb{R},$$

where C_1, C_2 and C_3 are constants depending on k, a and b only.

Let us recall the following commutator estimate (see [9, Lemma 3.2]).

LEMMA A.2. *Let f, g and h be three functions such that $\nabla f\in L^\infty$, $g\in L^\infty$ and $xh\in L^1$. Then it holds*

$$\|h\star(fg) - f(h\star g)\|_{L^\infty} \le \|xh\|_{L^1}\|\nabla f\|_{L^\infty}\|g\|_{L^\infty},$$

where $\star$ stands for the convolution symbol.

We recall the following commutators estimate (see [21, Lemma 2.6]).

LEMMA A.3. *Let f be a divergence-free vector field and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ with $p\in[2,\infty)$, $p_1,p_2\in[2,\infty]$, $r\in[1,\infty]$ as well as $s\in(-1,1-\delta)$ for $\delta\in(0,2)$, then it holds*

$$\|[\Lambda^{\delta},f\cdot\nabla]g\|_{B^{s}_{p,r}}\leq C(p,r,\delta,s)\big(\|\nabla f\|_{L^{p_1}}\|g\|_{B^{s+\delta}_{p_2,r}}+\|f\|_{L^2}\|g\|_{L^2}\big). \tag{A.2}$$

We also need the classical Kato-Ponce type commutator estimate (see [11]).

LEMMA A.4. *Let $s>0$. Let $p,p_1,p_3\in(1,\infty)$ and $p_2,p_4\in[1,\infty]$ satisfy*

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}.$$

Then there exist some constants C such that

$$\|[\Lambda^{s},f]g\|_{L^p}\leq C\left(\|\Lambda^{s}f\|_{L^{p_1}}\|g\|_{L^{p_2}}+\|\Lambda^{s-1}g\|_{L^{p_3}}\|\nabla f\|_{L^{p_4}}\right). \tag{A.3}$$

In some context, we also need the following variant version of (A.3), whose proof is the same as that for (A.3)

$$\|[\Lambda^{s-1}\partial_i,f]g\|_{L^r}\leq C\left(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{q_1}}+\|\Lambda^{s}f\|_{L^{p_2}}\|g\|_{L^{q_2}}\right). \tag{A.4}$$

The following lemma is the fractional version of the Gagliardo-Nirenberg inequality.

LEMMA A.5. *Let $1<p,q,r<\infty, 0\leq\theta\leq 1$ and $s,s_1,s_2\in\mathbb{R}$, then the following fractional Gagliardo-Nirenberg inequality*

$$\|\Lambda^{s}u\|_{L^p(\mathbb{R}^n)}\leq C\|\Lambda^{s_1}u\|_{L^q(\mathbb{R}^n)}^{1-\theta}\|\Lambda^{s_2}u\|_{L^r(\mathbb{R}^n)}^{\theta},$$

holds if and only if

$$\frac{1}{p}-\frac{s}{n}=(1-\theta)(\frac{1}{q}-\frac{s_1}{n})+\theta(\frac{1}{r}-\frac{s_2}{n}),\quad s\leq(1-\theta)s_1+\theta s_2.$$

Appendix B. The proof of Lemma 2.1 and Lemma 2.2.

Proof. **(Proof of Lemma 2.1.)** Multiplying the Equation $(1.4)_2$ by $|\theta|^{p-2}\theta$, integrating by parts and using $\nabla\cdot v=0$, we get

$$\frac{d}{dt}\|\theta(t)\|_{L^p}=0.$$

The desired estimate (2.1) follows by integrating it in time. Multiplying the Equation $(1.4)_1$ by v and integrating by parts, it yields by using (2.1) with $p=2$ that

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2+\|\Lambda^{\beta}v\|_{L^2}^2\leq\int_{\mathbb{R}^n}|v|\,|\theta|\,dx\ \leq\|v\|_{L^2}\|\theta\|_{L^2}\leq\|v\|_{L^2}\|\theta_0\|_{L^2}. \tag{B.1}$$

It follows that

$$\frac{d}{dt}\|v(t)\|_{L^2}\leq\|\theta_0\|_{L^2}.$$

Integrating in time yields

$$\|v(t)\|_{L^2} \leq \|v_0\|_{L^2} + t\|\theta_0\|_{L^2}.$$

Recalling (B.1) and integrating in time imply

$$\begin{aligned}\|v(t)\|_{L^2}^2 + 2\int_0^t \|\Lambda^\beta v(\tau)\|_{L^2}^2 \, d\tau \leq & \|v_0\|_{L^2}^2 + 2\int_0^t \|v(\tau)\|_{L^2}\|\theta_0\|_{L^2} \, d\tau \\ \leq & \|v_0\|_{L^2}^2 + 2\int_0^t (\|v_0\|_{L^2} + \tau\|\theta_0\|_{L^2})\|\theta_0\|_{L^2} \, d\tau \\ = & (\|v_0\|_{L^2} + t\|\theta_0\|_{L^2})^2.\end{aligned}$$

Notice the simple fact

$$\int_0^t \|v(\tau)\|_{H^\beta}^2 \, d\tau \approx \int_0^t (\|v(\tau)\|_{L^2}^2 + \|\Lambda^\beta v(\tau)\|_{L^2}^2) \, d\tau \leq C(t, v_0, \theta_0).$$

This leads to (2.2). Noticing the following facts

$$\widehat{v}(\xi) = \widehat{u}(\xi) + |\xi|^{2\alpha}\widehat{u}(\xi), \qquad \frac{1}{2} \leq \frac{(1+|\xi|^2)^\alpha}{1+|\xi|^{2\alpha}} \leq 2^\alpha,$$

we have

$$\begin{aligned}\|u\|_{H^\sigma} = & \|(1+|\xi|^2)^{\frac{\sigma}{2}}\widehat{u}(\xi)\|_{L^2} = \left\| \frac{(1+|\xi|^2)^{\frac{\sigma}{2}}}{1+|\xi|^{2\alpha}}\widehat{v}(\xi) \right\|_{L^2} \\ = & \left\| \frac{(1+|\xi|^2)^\alpha}{1+|\xi|^{2\alpha}}(1+|\xi|^2)^{\frac{\sigma-2\alpha}{2}}\widehat{v}(\xi) \right\|_{L^2} \\ \approx & \|(1+|\xi|^2)^{\frac{\sigma-2\alpha}{2}}\widehat{v}(\xi)\|_{L^2} \approx \|v\|_{H^{\sigma-2\alpha}},\end{aligned}$$

which gives

$$\|u\|_{H^\sigma} \approx \|v\|_{H^{\sigma-2\alpha}}. \tag{B.2}$$

An easy consequence of (2.2) and (B.2) is that

$$\begin{aligned}\|u(t)\|_{H^{2\alpha}}^2 + \int_0^t \|u(\tau)\|_{H^{2\alpha+\beta}}^2 \, d\tau \approx & \|v(t)\|_{L^2}^2 + \int_0^t \|v(\tau)\|_{H^\beta}^2 \, d\tau \\ \leq & C(t, v_0, \theta_0).\end{aligned}$$

This completes the proof of Lemma 2.1. ∎

Proof. **(Proof of Lemma 2.2.)** Applying Λ^β to Equation $(1.4)_1$ and taking the inner product with $\Lambda^\beta v$ yield

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^\beta v(t)\|_{L^2}^2 + \|\Lambda^{2\beta} v\|_{L^2}^2 = -\int_{\mathbb{R}^n} \Lambda^\beta(u\cdot\nabla v)\Lambda^\beta v \, dx + \int_{\mathbb{R}^n} \Lambda^\beta \theta e_n \Lambda^\beta v \, dx. \tag{B.3}$$

To bound the first term at the right-hand side of (B.3), we split it into two cases, namely, the case $\beta < 1$ and the case $\beta \geq 1$. For the case $\beta < 1$, we have by using commutator estimate (A.2) that

$$\left| \int_{\mathbb{R}^n} \Lambda^\beta(u\cdot\nabla v)\Lambda^\beta v \, dx \right| = \left| \int_{\mathbb{R}^n} [\Lambda^\beta, u\cdot\nabla] v \, \Lambda^\beta v \, dx \right|$$

$$
\begin{aligned}
&\le C\|[\Lambda^{\beta},u\cdot\nabla]v\|_{H^{-\beta}}\|v\|_{H^{2\beta}}\\
&\le C(\|\nabla u\|_{L^{\frac{n}{\beta}}}\|v\|_{B^{0}_{\frac{2n}{n-2\beta},2}}+\|u\|_{L^2}\|v\|_{L^2})\|v\|_{H^{2\beta}}\\
&\le C(\|u\|_{H^{2\alpha+\beta}}\|v\|_{H^{\beta}}+\|u\|_{L^2}\|v\|_{L^2})\|v\|_{H^{2\beta}}\\
&\le \frac{1}{4}\|\Lambda^{2\beta}v\|_{L^2}^2+C(1+\|u\|_{H^{2\alpha+\beta}}^2)\|\Lambda^{\beta}v\|_{L^2}^2+C,
\end{aligned}
$$

where we have used the following facts

$$
\|\nabla u\|_{L^{\frac{n}{\beta}}}\le C\|u\|_{H^{2\alpha+\beta}},\quad \alpha+\beta=\frac{1}{2}+\frac{n}{4},
$$

and

$$
\|v\|_{B^{0}_{\frac{2n}{n-2\beta},2}}\le C\|v\|_{H^{\beta}}.
$$

For the case $\beta\ge 1$, the commutator estimate (A.3) would suffice our purpose. In fact, it implies

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^n}\Lambda^{\beta}(u\cdot\nabla v)\Lambda^{\beta}v\,dx\Big|&=\Big|\int_{\mathbb{R}^n}[\Lambda^{\beta},u\cdot\nabla]v\,\Lambda^{\beta}v\,dx\Big|\\
&\le C\|[\Lambda^{\beta},u\cdot\nabla]v\|_{L^2}\|\Lambda^{\beta}v\|_{L^2}\\
&\le C(\|\nabla u\|_{L^{\frac{n}{\beta}}}\|\Lambda^{\beta}v\|_{L^{\frac{2n}{n-2\beta}}}+\|\nabla v\|_{L^{\frac{2n}{n-4\beta+2}}}\|\Lambda^{\beta}u\|_{L^{\frac{n}{2\beta-1}}})\|\Lambda^{\beta}v\|_{L^2}\\
&\le C(\|u\|_{H^{2\alpha+\beta}}\|\Lambda^{2\beta}v\|_{L^2}+\|\Lambda^{2\beta}v\|_{L^2}\|u\|_{H^{2\alpha+\beta}})\|\Lambda^{\beta}v\|_{L^2}\\
&\le \frac{1}{4}\|\Lambda^{2\beta}v\|_{L^2}^2+C(1+\|u\|_{H^{2\alpha+\beta}}^2)\|\Lambda^{\beta}v\|_{L^2}^2.
\end{aligned}
$$

The following is a direct consequence of the Young inequality

$$
\int_{\mathbb{R}^n}\Lambda^{\beta}\theta e_n\Lambda^{\beta}v\,dx\le\|\theta\|_{L^2}\|\Lambda^{2\beta}v\|_{L^2}\le\frac{1}{4}\|\Lambda^{2\beta}v\|_{L^2}^2+C.
$$

Summing up the above estimates gives

$$
\frac{d}{dt}\|\Lambda^{\beta}v\|_{L^2}^2+\|\Lambda^{2\beta}v\|_{L^2}^2\le C(1+\|u\|_{H^{2\alpha+\beta}}^2)\|\Lambda^{\beta}v\|_{L^2}^2+C.
$$

The classical Gronwall inequality ensures

$$
\|\Lambda^{\beta}v(t)\|_{L^2}^2+\int_0^t\|\Lambda^{2\beta}v(\tau)\|_{L^2}^2\,d\tau\le C,
$$

which along with (2.2) implies (2.4). It is not hard to see that (2.5) is an easy consequence of (2.3) and the relation $v=u+\Lambda^{2\alpha}u$. This completes the proof of Lemma 2.2.

□

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