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NONEQUILIBRIUM-DIFFUSION LIMIT OF THE COMPRESSIBLE EULER-P1 APPROXIMATION MODEL ARISING FROM RADIATION HYDRODYNAMICS*

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Abstract. We prove rigorously the nonequilibrium-diffusion limit of the compressible Euler-P1 approximation model arising in radiation hydrodynamics. For sufficiently well-prepared initial data, we obtain the uniform estimates of smooth solutions and establish the convergence of the model to the Euler system coupled with a nonlinear diffusion equation.

Keywords. Diffusion limit; nonequilibrium regime; radiation hydrodynamics; Euler-P1 approximation.

AMS subject classifications. 35D35; 35Q31; 35Q35; 76D30.

1. Introduction

As pointed out in [3,14,20], the motion of a compressible inviscid radiative flow can be described by the Euler system coupled with a radiative transfer equation

$$S_r\partial_t\rho+\mathrm{div}_x(\rho\mathbf{u})=0, \tag{1.1}$$

$$S_r\partial_t(\rho\mathbf{u})+\mathrm{div}_x(\rho\mathbf{u}\otimes\mathbf{u})+\frac{1}{Ma^2}\nabla_x P=-\frac{\mathcal{P}}{Ma^2}S_F, \tag{1.2}$$

$$S_r\partial_t(\rho e)+\mathrm{div}_x(\rho e\mathbf{u})+P\mathrm{div}_x\mathbf{u}=-\mathcal{P}\mathcal{C}S_E+\mathcal{P}S_F\cdot\mathbf{u}, \tag{1.3}$$

$$\frac{S_r}{\mathcal{C}}\partial_t I+\omega\cdot\nabla_x I=S. \tag{1.4}$$

Here the dimensionless unknowns $\rho(t,x)$, $\mathbf{u}(t,x)$, and $e(t,x)$ represent the density, velocity, and specific internal energy, respectively, as functions of the time $t\geq 0$ and the spatial variable $x\in\Omega:=\mathbb{T}^3=(\mathbb{R}\backslash(2\pi\mathbb{Z}))^3$. $I(x,t,\nu,\omega)$ is the dimensionless radiative intensity depending on the frequency $\nu\geq 0$ and direction $\omega\in\mathcal{S}^2$ of photons with $\mathcal{S}^2\subset\mathbb{R}^3$ being the unit sphere. The pressure $P=P(\rho,\theta)$ and the internal energy $e=e(\rho,\theta)$ are smooth functions of ρ and the temperature θ, and satisfy the Gibbs relation $\theta\mathrm{d}s=\mathrm{d}e+\rho\mathrm{d}(1/\rho)$ for some smooth entropy function $s=s(\rho,\theta)$. The radiative source term S in (1.4) is given by

$$S=\mathcal{L}\sigma_a(B(\nu,\theta)-I)+\mathcal{L}\mathcal{L}_s\sigma_s\left(\widetilde{I}-I\right), \tag{1.5}$$

where

$$B(\nu,\theta)=\frac{\nu^3}{e^{\frac{\nu}{\theta}}-1},\quad \widetilde{I}=\frac{1}{4\pi}\int_{\mathcal{S}^2}I\mathrm{d}\omega, \tag{1.6}$$

and $\sigma_a=\sigma_a(\nu,\theta)\geq 0$ and $\sigma_s=\sigma_s(\nu,\theta)\geq 0$ represent the absorption coefficient and scattering coefficient, respectively. Moreover, the radiative flux S_F and the radiative energy

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source S_E are given by

$$S_F = \frac{1}{c}\int_0^\infty \int_{\mathcal{S}^2} \omega S \mathrm{d}\omega \mathrm{d}\nu, \quad S_E = \int_0^\infty \int_{\mathcal{S}^2} S \mathrm{d}\omega \mathrm{d}\nu. \tag{1.7}$$

Furthermore, we denote by Sr, Ma, and $\mathcal{C}$ the Strouhal, Mach, and "infrarelativistic" numbers corresponding to hydrodynamics, and by $\mathcal{L}$, $\mathcal{L}_s$, and $\mathcal{P}$ the various dimensionless numbers corresponding to radiation. For the sake of convenience, the energy Equation (1.3) will be rewritten in the following form

$$S_r \partial_t(\rho s) + \mathrm{div}_x(\rho s \mathbf{u}) = -\frac{\mathcal{P}\mathcal{C}S_E}{\theta} + \frac{\mathcal{P}S_F \cdot \mathbf{u}}{\theta}. \tag{1.8}$$

For simplification, we assume that σ_a and σ_s are two positive constants in the following derivation. It should be noted that the general case $\sigma_a = \sigma_a(\rho,\theta)$ and $\sigma_s = \sigma_s(\rho,\theta)$ can be also dealt with similarly.

When the distribution of photons is almost isotropic, the well-known $P1$ approximation is frequently used (see [4,5,20]). The main advantage of considering $P1$ approximation is that the radiative transfer equation is transformed to equations independent of the angular directions which are easy to solve numerically. In fact, the equation of radiative transfer is very complicated while the $P1$ model can approach the full radiative heat transfer with very low computational cost. By using the famous $P1$ hypothesis $\int_0^\infty I d\nu = I_0 + \mathbf{I}_1 \cdot \omega$ where I_0 and $\mathbf{I}_1$ are independent of ω and ν (see [10, 11]), we can obtain the following Euler-P1 approximation radiation model

$$S_r \partial_t \rho + \mathrm{div}_x(\rho \mathbf{u}) = 0, \tag{1.9}$$

$$S_r \partial_t(\rho \mathbf{u}) + \mathrm{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{Ma^2}\nabla_x P = \frac{4\pi \mathcal{P}\mathcal{L}}{Ma^2}(\sigma_a + \mathcal{L}_s \sigma_s)\mathbf{I}_1, \tag{1.10}$$

$$S_r \partial_t(\rho s) + \mathrm{div}_x(\rho s \mathbf{u}) = \frac{4\pi \mathcal{P}\mathcal{C}\mathcal{L}\sigma_a}{\theta}\left(I_0 - C_1\theta^4\right) - \frac{4\pi \mathcal{P}\mathcal{C}\mathcal{L}}{\theta}(\sigma_a + \sigma_s \mathcal{L}_s)\mathbf{I}_1 \cdot \mathbf{u}, \tag{1.11}$$

$$\frac{S_r}{\mathcal{C}}\partial_t I_0 + \mathrm{div}_x \mathbf{I}_1 = \mathcal{L}\sigma_a\left(C_1\theta^4 - I_0\right), \tag{1.12}$$

$$\frac{S_r}{\mathcal{C}}\partial_t \mathbf{I}_1 + \nabla_x I_0 = -\mathcal{L}(\sigma_a + \mathcal{L}_s\sigma_s)\mathbf{I}_1, \tag{1.13}$$

where C_1 is some positive constant, and I_0 and $\mathbf{I}_1$ denote the leading term of the radiation distribution and the small correction terms with respect to the travelling angle, respectively.

The system (1.1)-(1.4) can be considered as a simplified model in radiation hydrodynamics, the physical foundations of which were described by Pomraning [20] and Mihalas and Weibel-Mihalas [17] in the framework of special relativity. The asymptotic regimes of the system (1.1)-(1.4) have been investigated formally and numerically by Lowrie, Morel and Hittinger [14] and the well-known equilibrium and nonequilibrium limits of the system (1.1)-(1.4) were formally given by Buet and Despres [3] through the Chapman-Enskog expansion. For the case that viscosity and heat-conduction are included (the so-called Navier-Stokes-Fourier-Radiation model (NSFR)), Ducomet and Nečasová [7] studied both equilibrium and nonequilibrium diffusion limits for NSFR system with the Dirichlet boundary condition for the velocity field in the framework of relative entropy method. For the P1-approximation of the radiative transfer equation, Danchin and Ducomet [5] justified the diffusive limits for the barotropic model of viscous radiative flow in the critical functional framework with the small data. To the best of our knowledge, there are very few rigorous results of the asymptotic limits for the inviscid system (1.1)-(1.4). Recently, Jiang, Ju and Liao [10] rigorously proved the nonequilibrium-diffusion limit of the system (1.9)-(1.13) at low Mach number with the large temperature variation. We mention that Zhong and Jiang [25] proved the local existence and finite-time blow-up of smooth solutions to the system (1.1)-(1.4).

This paper aims to provide the rigorous justification of the nonequilibrium-diffusion limit of the system (1.9)-(1.13) when the Mach number is fixed. In order to analyze the nonequilibrium diffusion regime, we use $\mathcal{L}_s=\epsilon^{-2}$ and $\mathcal{L}=\epsilon$. We also assume that the flow is strongly under-relativistic, i.e., $\mathcal{C}=\epsilon^{-1}$ and that a moderate amount of radiation is present, i.e., $\mathcal{P}=1$. On the other hand, since we are interested in the case that the Mach number is fixed, we put $Ma=1$. Moreover, we set $Sr=1$ in the previous system. In addition, we focus on the fluids obeying the ideal polytropic gas relations

$$e^\epsilon=C_V\theta^\epsilon,\quad P^\epsilon=R\rho^\epsilon\theta^\epsilon=A(\rho^\epsilon)^\gamma\exp\left(\frac{\gamma-1}{R}s^\epsilon\right),\tag{1.14}$$

where the specific gas constants A, R, and the specific heat at constant volume C_V are positive constants and $\gamma>1$ is the adiabatic constant. In what follows, for simplicity of presentation, we set the physical constants C_V, R, and A to be one. Besides, if we further ignore the influence of other constants, then we can deduce from the above assumptions and (1.9)-(1.14) that

$$\frac{1}{P^\epsilon}(\partial_t P^\epsilon+\mathbf{u}^\epsilon\cdot\nabla_x P^\epsilon)+\gamma\mathrm{div}_x\mathbf{u}^\epsilon=\frac{(\gamma-1)\left(I_0^\epsilon-(\theta^\epsilon)^4\right)}{P^\epsilon}-(\gamma-1)\left(\epsilon+\frac{1}{\epsilon}\right)\frac{\mathbf{I}_1^\epsilon\cdot\mathbf{u}^\epsilon}{P^\epsilon},\tag{1.15}$$

$$\rho^\epsilon(\partial_t\mathbf{u}^\epsilon+\mathbf{u}^\epsilon\cdot\nabla_x\mathbf{u}^\epsilon)+\nabla_x P^\epsilon=\left(\epsilon+\frac{1}{\epsilon}\right)\mathbf{I}_1^\epsilon,\tag{1.16}$$

$$\partial_t s^\epsilon+\mathbf{u}^\epsilon\cdot\nabla_x s^\epsilon=\frac{I_0^\epsilon-(\theta^\epsilon)^4}{P^\epsilon}-\left(\epsilon+\frac{1}{\epsilon}\right)\frac{\mathbf{I}_1^\epsilon\cdot\mathbf{u}^\epsilon}{P^\epsilon},\tag{1.17}$$

$$\partial_t I_0^\epsilon+\frac{\mathrm{div}_x\mathbf{I}_1^\epsilon}{\epsilon}=(\theta^\epsilon)^4-I_0^\epsilon,\tag{1.18}$$

$$\partial_t\mathbf{I}_1^\epsilon+\frac{\nabla_x I_0^\epsilon}{\epsilon}=-\left(1+\frac{1}{\epsilon^2}\right)\mathbf{I}_1^\epsilon.\tag{1.19}$$

The Equations (1.15)-(1.19) are supplemented with initial data

$$(P^\epsilon,\mathbf{u}^\epsilon,s^\epsilon,I_0^\epsilon,\mathbf{I}_1^\epsilon)(t,x)|_{t=0}=(P_0^\epsilon,\mathbf{u}_0^\epsilon,s_0^\epsilon,I_{00}^\epsilon,\mathbf{I}_{10}^\epsilon)(x),\tag{1.20}$$

where $x\in\Omega:=\mathbb{T}^3$.

A local existence result for (1.15)-(1.20) in the following sense can be shown in a similar way as that in [13, 15, 21]. Thus we omit the details of the proof.

THEOREM 1.1 (Local existence). *Let $\epsilon\in(0,1]$, and suppose that the initial data $(P_0^\epsilon,\mathbf{u}_0^\epsilon,s_0^\epsilon,I_{00}^\epsilon,\mathbf{I}_{10}^\epsilon)\in H^3(\Omega)$ and there are positive constants $\underline{\rho}$ and $\underline{\theta}$ independent of ϵ, such that $\rho_0(x)\geq\underline{\rho}$, $\theta_0(x)\geq\underline{\theta}$. Then there exists a positive constant $T^\epsilon=T^\epsilon(P_0^\epsilon,\mathbf{u}_0^\epsilon,s_0^\epsilon,I_{00}^\epsilon,\mathbf{I}_{10}^\epsilon,\epsilon)$, such that the initial value problem (1.15)–(1.20) admits a unique solution $(P^\epsilon,\mathbf{u}^\epsilon,s^\epsilon,I_0^\epsilon,\mathbf{I}_1^\epsilon)(t,x)$, and for any $\delta>0$, it holds that*

$$(P^\epsilon,\mathbf{u}^\epsilon,s^\epsilon,I_0^\epsilon,\mathbf{I}_1^\epsilon)(t,x)\in C^1([0,T^\epsilon]\times\Omega)\bigcap\left(\bigcap_{j=0}^{2}C^j\left([0,T^\epsilon];H^{3-j-\delta}(\Omega)\right)\right),$$
$$\partial_t^j(P^\epsilon,\mathbf{u}^\epsilon,s^\epsilon,I_0^\epsilon,\mathbf{I}_1^\epsilon)(t,x)\in L^\infty\left([0,T^\epsilon];H^{3-j}(\Omega)\right),\quad j=0,1,2,3.$$

If we introduce

$$
\begin{aligned}
M^\epsilon(t) &= \|(P^\epsilon,\mathbf{u}^\epsilon,s^\epsilon,I_0^\epsilon,\mathbf{I}_1^\epsilon)(t)\|_E \\
&= \sup_{0\le\tau\le t}\sum_{i=0}^{3}\left\|\partial_t^i(P^\epsilon,\mathbf{u}^\epsilon,s^\epsilon,I_0^\epsilon,\mathbf{I}_1^\epsilon)(\tau)\right\|_{H^{3-i}(\Omega)}^2,
\end{aligned}
\tag{1.21}
$$

then the main result of this paper is stated as follows, which shows the uniform estimates of strong solutions to (1.15)–(1.20), and the corresponding nonequilibrium-diffusion limit.

THEOREM 1.2. *In addition to the hypotheses of Theorem 1.1, if we further assume that the initial datum $(P_0^\epsilon(x),\mathbf{u}_0^\epsilon(x),s_0^\epsilon(x),I_{00}^\epsilon(x),\mathbf{I}_{10}^\epsilon(x))$ satisfies*

(i) $\|(P_0^\epsilon,\mathbf{u}_0^\epsilon,s_0^\epsilon,I_{00}^\epsilon,\mathbf{I}_{10}^\epsilon)\|_E\le C_2$;

(ii) *there exists $(P_0(x),\mathbf{u}_0(x),s_0(x),I_{00}(x),\mathbf{I}_{10}(x))\in H^3(\Omega)$ such that*

$$
\lim_{\epsilon\to 0}\|(P_0^\epsilon-P_0,\mathbf{u}_0^\epsilon-\mathbf{u}_0,s_0^\epsilon-s_0,I_{00}^\epsilon-I_{00},\mathbf{I}_{10}^\epsilon-\mathbf{I}_{10})\|_{H^3(\Omega)}=0.
$$

Then we can conclude that

(a) *the solution $(P^\epsilon,\mathbf{u}^\epsilon,s^\epsilon,I_0^\epsilon,\mathbf{I}_1^\epsilon)(t,x)$ obtained in Theorem 1.1 exists on a time interval $[0,T]$ for some $T>0$ independent of $\epsilon>0$ and satisfies*

$$
\|(P^\epsilon,\mathbf{u}^\epsilon,s^\epsilon,I_0^\epsilon,\mathbf{I}_1^\epsilon)(T)\|_E\le C_3.
$$

(b) *There exists $(\rho,u,\theta,s,I_0,\mathbf{I}_1)(t,x)\in C\left([0,T];H^{3-\delta}(\Omega)\right)\cap C^1\left([0,T];H^{2-\delta}(\Omega)\right)\cap C^1([0,T]\times\Omega)$ for any $\delta>0$, such that as $\epsilon\to 0$,*

$$
(\rho^\epsilon(P^\epsilon,s^\epsilon),u^\epsilon,\theta^\epsilon(P^\epsilon,s^\epsilon),s^\epsilon,I_0^\epsilon,\mathbf{I}_1^\epsilon)(t,x)\longrightarrow(\rho,,u,\theta,s,I_0,\mathbf{I}_1)(t,x)
$$

in $C\left([0,T];H^{3-\delta}(\Omega)\right)$.

(c) *$(\rho,u,\theta,s,I_0,\mathbf{I}_1)(t,x)$ satisfy the following equations*

$$
\begin{aligned}
&\partial_t\rho+\mathbf{u}\cdot\nabla_x\rho+\rho\, div_x\mathbf{u}=0,\\
&\rho(\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla_x\mathbf{u})+\nabla_x(\rho\theta+I_0)=0,\\
&\rho(\partial_t\theta+\mathbf{u}\cdot\nabla_x\theta)+\rho\theta\, div_x\mathbf{u}-\nabla_x I_0\cdot\mathbf{u}=I_0-\theta^4,\\
&\partial_t I_0-\Delta_x I_0+I_0=\theta^4,\\
&\mathbf{I}_1=0,\\
&P=\rho\theta=\rho^\gamma\exp((\gamma-1)s),
\end{aligned}
$$

with initial data

$$
\begin{aligned}
&(\rho(P(t,x),s(t,x)),\mathbf{u}(t,x),\theta(P(t,x),s(t,x)),s(t,x),I_0(t,x),\mathbf{I}_1(t,x))|_{t=0}\\
&=(\rho(P_0(x),s_0(x)),\mathbf{u}_0(x),\theta(P_0(x),s_0(x)),s_0(x),I_{00}(x),\mathbf{I}_{10}(x)).
\end{aligned}
$$

Here C_2 and C_3 are positive constants which are independent of $\epsilon>0$.

REMARK 1.1. To simplify the statement, we shall use "$\partial_t\mathbf{I}_{10}^\epsilon$" to signify the quantity $\partial_t\mathbf{I}_1^\epsilon|_{t=0}:=-\frac{1}{\epsilon}\nabla_x I_{00}-(1+\frac{1}{\epsilon^2})\mathbf{I}_{10}$ through the Equation (1.19). Other time derivatives of the corresponding variables are defined in similar ways.

REMARK 1.2. The equilibrium diffusion regime is defined by setting $\mathcal{L}=\epsilon^{-1}$ and $\mathcal{L}_s=\epsilon^2$ (see [3,7,14]). To study the equilibrium-diffusion limit of the system (1.9)-(1.13), new tricks should be developed to deal with the singular terms caused by radiation effect.

REMARK 1.3. In [10], the authors consider the case of a small amount of radiation i.e, $\mathcal{P}=\epsilon$, while we consider in this paper the more realistic case , i.e., $\mathcal{P}=1$.

REMARK 1.4. It is more interesting to give a rigorous justification of the diffusion limit of the original system (1.1)-(1.4) for which the techniques used in the present paper are not sufficient since the additional direction variable is involved in the kinetic equation. It will be our purpose in the future.

Similar as in [1, 10, 16, 19], it suffices to show the following theorem to deduce the main result Theorem 1.2.

THEOREM 1.3. *Let T^ϵ be the maximal time of existence of the solution to the system (1.15)-(1.19) in the sense of Theorem 1.1. Then for any $t\in[0,T^\epsilon)$, we have*

$$M^\epsilon(t)\le C_0(M^\epsilon(0))+C(M^\epsilon(t))t,$$

for some given non-decreasing continuous functions $C_0(\cdot)$ and $C(\cdot)$.

Now we outline the main difficulties of the problem and our strategy to prove the main theorem. Indeed, the key point in our analysis lies in deducing the uniform energy estimates (independent of ϵ) on the solution $(P^\epsilon(t,x),\mathbf{u}^\epsilon(t,x),s^\epsilon(t,x),I_0^\epsilon(t,x),\mathbf{I}_1^\epsilon(t,x))$. Due to the singular terms caused by radiation effect in the system (1.15)-(1.19), the classical theory of singular limits for quasi-linear hyperbolic equations [1, 12, 16, 21] is invalid for our problem. Besides, compared with the problem considered in [10]($Ma=\epsilon$), we cannot derive energy estimates on $\mathrm{div}_x\mathbf{u}^\epsilon$ and $\nabla_x P^\epsilon$ through (1.15) and (1.16) since the Mach number is fixed in our case. For our purpose, we shall introduce the norm $\|(P^\epsilon,\mathbf{u}^\epsilon,s^\epsilon,I_0^\epsilon,\mathbf{I}_1^\epsilon)(t)\|_E$ (see (1.21)) to deduce the uniform estimates on the solution. Our norm includes the derivatives of solutions with respect to time and space, and in particular the higher order time derivatives are involved since the coefficients of the time derivatives of the velocity and the pressure in (1.15)-(1.19) are variable. Such kind of norm has been used by Browning and Kreiss in [2] to study the singular limit of nonlinear partial differential equations. However, the problem considered here is quite distinct from the system in [2]. To prove the main theorem, we make full use of the special structure of the system to deduce the desired energy estimates on the singular radiative term $\mathbf{I}_1^\epsilon$. In particular, we shall deduce the uniform estimate on $\int_0^t\left(1+\frac{1}{\epsilon^2}\right)\sum_{i=0}^{3}\left\|\partial_t^i\mathbf{I}_1^\epsilon(\tau)\right\|_{H^{3-i}(\Omega)}^2\mathrm{d}\tau$ (see Lemma 3.1) by using the energy method, which helps us control the singular radiative terms on the right-hand side of (1.15), (1.16), and (1.17).

Before concluding this section, we mention other results concerning the $P1$ approximation radiation model, please refer to [4, 9, 11, 22, 24]. Moreover, interested readers can also refer to [3, 6, 8, 14, 23, 25] for more references therein.

The rest of the paper is arranged as follows. Some elementary facts and useful inequalities will be given in Section 2. Then we will deduce uniform estimates for the norms $\|(I_0^\epsilon,\mathbf{I}_1^\epsilon)(t)\|_E$ and $\|(P^\epsilon,\mathbf{u}^\epsilon,s^\epsilon)(t)\|_E$ in Section 3 and Section 4, respectively. Finally, the proof of Theorem 1.2 will be given in Section 5.

Notations. Throughout this paper, $\epsilon<1$ represents some small positive constant. $C\ge 1$ or $C_i\ge 1$ $(i=1,2,...)$ is used to denote a generic positive constant which is independent of ϵ. Note that these constants may vary from line to line. By $C_0(\cdot,)$, $C(\cdot,)$ and $F(\cdot,)$, we denote positive non-decreasing continuous functions independent of ϵ. $A \lesssim B$ means that $A \le CB$ holds for some positive constant C independent of ϵ.

$L^q(\Omega)$ (or L^q) ($1 \le q \le \infty$) denotes the usual Lebesgue space on Ω with norm $\|\cdot\|_{L^q(\Omega)}$, while $H^q(\Omega)$ denotes the usual Sobolev space with norm $\|\cdot\|_{H^q(\Omega)}$. On the other hand, we denote by $C(I;H^q(\Omega))$ the space of continuous functions on the interval I with values in $H^q(\Omega)$. For simplicity, we use $\|\cdot\|$ and $\|\cdot\|_q$ to denote the norms $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{H^q(\Omega)}$, respectively.

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, we denote $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ and $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$. For an integer m, the symbol D_x^m denotes the summation of all terms ∂_x^α with the multi-index α satisfying $|\alpha| = m$.

In the estimates that follow, Hölder's and Sobolev's inequalities as well as the interpolation inequality will be used without further claim. We shall drop the superscript "ϵ" in $\rho^\epsilon, q^\epsilon, \mathbf{u}^\epsilon, s^\epsilon, \theta^\epsilon, I_0^\epsilon$, and $\mathbf{I}_1^\epsilon$ for the sake of brevity. Moreover, we write $M^\epsilon(t)$, $M^\epsilon(0)$, div_x, and ∇_x as M, M_0, div, and ∇, respectively, for short.

2. Some useful inequalities

In this section, some useful inequalities are given since they will be frequently used later on. The first lemma focuses on some Moser-type calculus inequalities in Sobolev spaces, whose proof can be found in [12, 18] for example.

LEMMA 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$.*

(i) *For $f, g \in H^m(\Omega) \cap L^\infty(\Omega)$ and $|\alpha| \le m$, $m > n/2$, it holds that*

$$\|\partial_x^\alpha (fg)\| \lesssim \|f\|_{L^\infty} \|D_x^m g\| + \|g\|_{L^\infty} \|D_x^m f\|; \tag{2.1}$$

(ii) *For $f \in H^m(\Omega)$, $D_x^1 f \in L^\infty(\Omega)$, $g \in H^{m-1}(\Omega) \cap L^\infty(\Omega)$, and $|\alpha| \le m$, $m > n/2+1$, it holds that*

$$\|\partial_x^\alpha (fg) - f \partial_x^\alpha g\| \lesssim \left\|D_x^1 f\right\|_{L^\infty} \left\|D_x^{m-1} g\right\| + \|g\|_{L^\infty} \|D_x^m f\|; \tag{2.2}$$

(iii) *Assume $g(u)$ is a smooth vector-valued function on Ω, $k(x)$ is a continuous function with $k \in \Omega_1$, $\overline{\Omega}_1 \subset\subset \Omega$, and $k \in H^m(\Omega) \cap L^\infty(\Omega)$. Then for $m \ge 1$,*

$$\|D_x^m g(k)\| \lesssim \left|\frac{\partial g}{\partial k}\right|_{m-1, \overline{\Omega}_1} \|k\|_{L^\infty}^{m-1} \|D_x^m k\|. \tag{2.3}$$

Here $|\cdot|_{r, \overline{\Omega}_1}$ is the C^r-norm on the set $\overline{\Omega}_1$;

(iv) *If $f, g \in H^m(\Omega)$, $m > n/2$, then $f \cdot g \in H^m(\Omega)$, and*

$$\|fg\|_m \lesssim \|f\|_m \|g\|_m. \tag{2.4}$$

The following lemma is a consequence of (1.14) and the definition of the norm $\|\cdot\|_E$.

LEMMA 2.2. *Assume that the conditions in Theorem 1.2 hold, we have for all $0 \le t \le T^\epsilon$ that*

$$\|\rho\|_E + \|\theta\|_E \le F(\|P\|_E, \|s\|_E) \tag{2.5}$$

for some non-decreasing continuous ϵ-independent function $F(\cdot,\cdot)$.

3. Uniform estimates on $\|(I_0,\mathbf{I}_1)\|_E$

In this section, we shall first deduce uniform estimates on $\|(I_0,\mathbf{I}_1)\|_E$. At the same time, we also obtain the estimates on $\int_0^t \left(1+\frac{1}{\epsilon^2}\right)\sum_{i=0}^{3}\left\|\partial_t^i\mathbf{I}_1(\tau)\right\|_{3-i}^2 \mathrm{d}\tau$, which will play an important role in dealing with the singular radiative terms on the right-hand side of (1.15)-(1.17). In fact, we have

LEMMA 3.1. *Assume that the conditions in Theorem 1.2 hold, then we have for all* $0\le t\le T^\epsilon$ *that*

$$\sum_{i=0}^{3}\left\|\partial_t^i(I_0,\mathbf{I}_1)(t)\right\|_{3-i}^2+\int_0^t\left[\sum_{i=0}^{3}\left\|\partial_t^i I_0(\tau)\right\|_{3-i}^2+\left(1+\frac{1}{\epsilon^2}\right)\sum_{i=0}^{3}\left\|\partial_t^i\mathbf{I}_1(\tau)\right\|_{3-i}^2\right]d\tau$$
$$\le C_0(M_0)+C(M)t. \tag{3.1}$$

Proof. We apply the operator ∂_x^α $(0\le|\alpha|\le 3)$ to (1.18) and (1.19), multiply the resulting equations by $\partial_x^\alpha I_0$ and $\partial_x^\alpha\mathbf{I}_1$, respectively, and integrate over $[0,t]\times\Omega$. A simple summation shows that

$$\begin{aligned}
&\frac{1}{2}\|(\partial_x^\alpha I_0,\partial_x^\alpha\mathbf{I}_1)(t)\|^2+\int_0^t\left[\|\partial_x^\alpha I_0(\tau)\|^2+\left(1+\frac{1}{\epsilon^2}\right)\|\partial_x^\alpha\mathbf{I}_1(\tau)\|^2\right]\mathrm{d}\tau\\
&\le C_0(M_0)+\frac{1}{2}\int_0^t\|\partial_x^\alpha I_0(\tau)\|^2\mathrm{d}\tau+C\int_0^t\left\|\partial_x^\alpha(\theta^4)\right\|^2\mathrm{d}\tau\\
&\le C_0(M_0)+\frac{1}{2}\int_0^t\|\partial_x^\alpha I_0(\tau)\|^2\mathrm{d}\tau+C\int_0^t\left(\|\theta\|_3^8+\|\theta\|_3^{12}\right)\tau\\
&\le C_0(M_0)+C(M)t+\frac{1}{2}\int_0^t\|\partial_x^\alpha I_0(\tau)\|^2\mathrm{d}\tau.
\end{aligned}\tag{3.2}$$

Here (2.3) has been used in the penultimate inequality of (3.2).

Second, applying the operator $\partial_x^\beta\partial_t$ $(0\le|\beta|\le 2)$ to (1.18) and (1.19), multiplying the resulting equations by $\partial_x^\beta\partial_t I_0$ and $\partial_x^\beta\partial_t\mathbf{I}_1$ in $L^2(\Omega)$, respectively, and adding up the resulting equalities, we arrive at

$$\begin{aligned}
&\frac{1}{2}\left\|\left(\partial_x^\beta\partial_t I_0,\partial_x^\beta\partial_t\mathbf{I}_1\right)(t)\right\|^2+\int_0^t\left[\left\|\partial_x^\beta\partial_t I_0(\tau)\right\|^2+\left(1+\frac{1}{\epsilon^2}\right)\left\|\partial_x^\beta\partial_t\mathbf{I}_1(\tau)\right\|^2\right]\mathrm{d}\tau\\
&\le C_0(M_0)+\frac{1}{2}\int_0^t\left\|\partial_x^\beta\partial_t I_0(\tau)\right\|^2\mathrm{d}\tau+C\int_0^t\left\|\partial_x^\beta(\theta^3\partial_t\theta)\right\|^2\mathrm{d}\tau\\
&\le C_0(M_0)+\frac{1}{2}\int_0^t\left\|\partial_x^\beta\partial_t I_0(\tau)\right\|^2\mathrm{d}\tau+C\int_0^t\left(\|\theta\|_{L^\infty}^3\|D_x^2\partial_t\theta\|+\|\partial_t\theta\|_{L^\infty}\|D_x^2(\theta^3)\|\right)\mathrm{d}\tau\\
&\le C_0(M_0)+\frac{1}{2}\int_0^t\left\|\partial_x^\beta\partial_t I_0(\tau)\right\|^2\mathrm{d}\tau+C\int_0^t\|\partial_t\theta\|_2^2\left(\|\theta\|_2^8+\|\theta\|_2^6\right)\mathrm{d}\tau\\
&\le C_0(M_0)+C(M)t+\frac{1}{2}\int_0^t\left\|\partial_x^\beta\partial_t I_0(\tau)\right\|^2\mathrm{d}\tau.
\end{aligned}\tag{3.3}$$

We point out that (2.1), (2.3), and (2.5) have been employed in deriving the above inequality.

Third, we differentiate (1.18) and (1.19) with respect to t twice, multiply the resulting equations by $\partial_{tt}I_0$ and $\partial_{tt}\mathbf{I}_1$ in $L^2(\Omega)$, respectively, then add up the resulting identities to deduce that

$$\frac{1}{2}\|(\partial_{tt}I_0,\partial_{tt}\mathbf{I}_1)(t)\|^2+\int_0^t\left[\|\partial_{tt}I_0(\tau)\|^2+\left(1+\frac{1}{\epsilon^2}\right)\|\partial_{tt}\mathbf{I}_1(\tau)\|^2\right]\mathrm{d}\tau$$

$$
\begin{aligned}
&\le C_0(M_0)+\int_0^t\int_\Omega |\partial_{tt}I_0|\left(\theta^2|\partial_t\theta|^2+\theta^3|\partial_{tt}\theta|\right)\mathrm{d}x\mathrm{d}\tau\\
&\le C_0(M_0)+\int_0^t \|\partial_{tt}I_0\|\,\|\theta\|_2^2\left(\|\partial_t\theta\|_2^2+\|\theta\|_2\|\partial_{tt}\theta\|\right)\mathrm{d}\tau\\
&\le C_0(M_0)+C(M)t. \qquad (3.4)
\end{aligned}
$$

Moreover, we apply the operator $\partial_{tt}D_x$ to (1.18) and (1.19), multiply the resulting equations by $\partial_{tt}D_xI_0$ and $\partial_{tt}D_x\mathbf{I}_1$ in $L^2(\Omega)$, and sum up the resulting equalities to conclude that

$$
\begin{aligned}
&\frac{1}{2}\|(\partial_{tt}D_xI_0,\partial_{tt}D_x\mathbf{I}_1)(t)\|^2+\int_0^t\left[\|\partial_{tt}D_xI_0(\tau)\|^2+\left(1+\frac{1}{\epsilon^2}\right)\|\partial_{tt}D_x\mathbf{I}_1(\tau)\|^2\right]\mathrm{d}\tau\\
&\le C_0(M_0)+\int_0^t\int_\Omega|\partial_{tt}D_xI_0|\left(\theta|D_x\theta||\partial_t\theta|^2+\theta^2|\partial_t\theta\partial_tD_x\theta|+\theta^2|D_x\theta\partial_{tt}\theta|+\theta^3|\partial_{tt}D_x\theta|\right)\mathrm{d}x\mathrm{d}\tau\\
&\le C_0(M_0)+\int_0^t\|\partial_{tt}I_0\|_1\left(\|\theta\|_2^2\|\partial_t\theta\|_2^2+\|\theta\|_3^4\|\partial_{tt}\theta\|+\|\theta\|_2^3\|\partial_{tt}\theta\|_1\right)\mathrm{d}\tau\\
&\le C_0(M_0)+C(M)t. \qquad (3.5)
\end{aligned}
$$

Finally, differentiating (1.18) and (1.19) with respect to time thrice, multiplying the resulting equations by $\partial_{ttt}I_0$ and $\partial_{ttt}\mathbf{I}_1$ in $L^2(\Omega)$, then summing up, one has

$$
\begin{aligned}
&\frac{1}{2}\|(\partial_{ttt}I_0,\partial_{ttt}\mathbf{I}_1)(t)\|^2+\int_0^t\left[\|\partial_{ttt}I_0(\tau)\|^2+\left(1+\frac{1}{\epsilon^2}\right)\|\partial_{ttt}\mathbf{I}_1(\tau)\|^2\right]\mathrm{d}\tau\\
&\le C_0(M_0)+\int_0^t\int_\Omega|\partial_{ttt}I_0|\left(\theta|\partial_t\theta|^3+\theta^2|\partial_t\theta\partial_{tt}\theta|+\theta^3|\partial_{ttt}\theta|\right)\mathrm{d}x\mathrm{d}\tau\\
&\le C_0(M_0)+\int_0^t\|\partial_{ttt}I_0\|\left(\|\theta\|\,\|\partial_t\theta\|_2^3+\|\theta\|_2^2\|\partial_t\theta\|_2\|\partial_{tt}\theta\|+\|\theta\|_2^3\|\partial_{ttt}\theta\|\right)\mathrm{d}\tau\\
&\le C_0(M_0)+C(M)t. \qquad (3.6)
\end{aligned}
$$

Then (3.1) is a consequence of (3.2)-(3.6). ◻

4. Uniform estimates on $\|(P,\mathbf{u},s)\|_E$

The main task of this section is to derive uniform estimates on $\|(P,\mathbf{u},s)\|_E$. For this purpose, we will utilize the antisymmetric structure of the system (1.15)-(1.19) to eliminate the high-order space derivative terms caused by div$\mathbf{u}$ and ∇P on the left-hand side of (1.15) and (1.16), respectively. On the other hand, the singular radiative terms appearing on the right-hand side of (1.15)-(1.17) will be bounded by the term $\int_0^t\left(1+\frac{1}{\epsilon^2}\right)\sum_{i=0}^3\left\|\partial_t^i\mathbf{I}_1(\tau)\right\|_{3-i}^2\mathrm{d}\tau$ as pointed out before. To begin with, we first deduce the estimate on $\|(P,\mathbf{u})\|_3^2$. In fact, we have

LEMMA 4.1. *Assume that the conditions in Theorem 1.2 hold, then we have for all $0\le t\le T^\epsilon$ that*

$$
\|(P,\mathbf{u})\|_3^2\le C_0(M_0)+C(M)t. \qquad (4.1)
$$

Proof. Multiplying ∂_x^α(1.18) and ∂_x^α(1.19) $(0\le|\alpha|\le3)$ by $\partial_x^\alpha P$ and $\partial_x^\alpha\mathbf{u}$, respectively, summing up the resulting identities and integrating over $[0,t]\times\Omega$, we obtain that

$$
\frac{1}{2}\int_\Omega\left(P^{-1}|\partial_x^\alpha P|^2+\gamma\rho|\partial_x^\alpha\mathbf{u}|^2\right)\mathrm{d}x=\frac{1}{2}\int_\Omega\left(P^{-1}|\partial_x^\alpha P|^2+\gamma\rho|\partial_x^\alpha\mathbf{u}|^2\right)(0)\mathrm{d}x+\sum_{j=1}^4K_j, \quad (4.2)
$$

where

$$
\begin{aligned}
K_1 &= \int_0^t \int_\Omega \left\{ \frac{\gamma}{2}\partial_t\rho|\partial_x^\alpha \mathbf{u}|^2 - \frac{1}{2}P^{-2}\partial_t P|\partial_x^\alpha P|^2 + (\gamma-1)\partial_x^\alpha\left[P^{-1}(I_0-\theta^4)\right]\partial_x^\alpha P \right\} \mathrm{d}x\mathrm{d}\tau,\\
K_2 &= -\int_0^t \int_\Omega \Big\{ \left[\partial_x^\alpha(P^{-1}\partial_t P) - P^{-1}\partial_t\partial_x^\alpha P + \partial_x^\alpha(P^{-1}\nabla P\cdot\mathbf{u}) - \partial_x^\alpha \nabla P\cdot P^{-1}\mathbf{u}\right]\partial_x^\alpha P\\
&\quad + \gamma\left[\partial_x^\alpha(\rho\partial_t\mathbf{u}) - \rho\partial_t\partial_x^\alpha\mathbf{u} + \partial_x^\alpha(\rho\mathbf{u}\cdot\nabla\mathbf{u}) - \rho\mathbf{u}\cdot\nabla\partial_x^\alpha\mathbf{u}\right]\cdot\partial_x^\alpha\mathbf{u}\Big\}\mathrm{d}x\mathrm{d}\tau,\\
K_3 &= -\int_0^t\int_\Omega \left(P^{-1}\partial_x^\alpha P\mathbf{u}\cdot\nabla\partial_x^\alpha P + \gamma\rho\mathbf{u}\cdot\nabla\partial_x^\alpha\mathbf{u}\cdot\partial_x^\alpha\mathbf{u}\right)\mathrm{d}x\mathrm{d}\tau,\\
K_4 &= \left(\epsilon + \frac{1}{\epsilon}\right)\int_0^t\int_\Omega \left[\gamma\partial_x^\alpha\mathbf{I}_1\cdot\partial_x^\alpha\mathbf{u} - (\gamma-1)\partial_x^\alpha\left(P^{-1}\mathbf{I}_1\cdot\mathbf{u}\right)\partial_x^\alpha P\right]\mathrm{d}x\mathrm{d}\tau.
\end{aligned}
$$

For the term K_1, we employ (2.1), (2.3), and (2.5) to infer that

$$
\begin{aligned}
K_1 &\lesssim \int_0^t \left[\|\partial_t P\|_2\left(\|\mathbf{u}\|_3^2 + \|P\|_3^2\right) + \left(\|\partial_x^\alpha(P^{-1}I_0)\| + \|\partial_x^\alpha(P^{-1}\theta^4)\|\right)\|P\|_3\right]\mathrm{d}\tau\\
&\lesssim C(M)t + \int_0^t \Big[\|P^{-1}\|_{L^\infty}\left(\|D_x^3 I_0\| + \|D_x^3(\theta^4)\|\right)\\
&\quad + \left(\|I_0\|_{L^\infty} + \|\theta\|_{L^\infty}^4\right)\|D_x^3(P^{-1})\|\Big]\|P\|_3\mathrm{d}\tau\\
&\lesssim C(M)t + \int_0^t \left[\|I_0\|_3 + \|I_0\|_2\|P\|_3^3 + \left(\|\theta\|_2^3 + \|\theta\|_2\right)\|\theta\|_3^3 + \|\theta\|_2^4\|P\|_3^3\right]\|P\|_3\mathrm{d}\tau\\
&\le C(M)t.
\end{aligned}
$$

Moreover, it follows from (2.2) and (2.3) that

$$
\begin{aligned}
K_2 &\lesssim \int_0^t \Big[\|P\|_3\left(\|\partial_x^\alpha(P^{-1}\partial_t P) - P^{-1}\partial_t\partial_x^\alpha P\| + \|\partial_x^\alpha(P^{-1}\nabla P\cdot\mathbf{u}) - \partial_x^\alpha\nabla P\cdot P^{-1}\mathbf{u}\|\right)\\
&\quad + \|\mathbf{u}\|_3\left(\|\partial_x^\alpha(\rho\partial_t\mathbf{u}) - \rho\partial_t\partial_x^\alpha\mathbf{u}\| + \|\partial_x^\alpha(\rho\mathbf{u}\cdot\nabla\mathbf{u}) - \rho\mathbf{u}\cdot\nabla\partial_x^\alpha\mathbf{u}\|\right)\Big]\mathrm{d}\tau\\
&\lesssim \int_0^t \Big[\Big(\|D_x^1(P^{-1})\|_{L^\infty}\|D_x^2\partial_t P\| + \|\partial_t P\|_{L^\infty}\|D_x^3(P^{-1})\|\\
&\quad + \|D_x^1(P^{-1}\mathbf{u})\|_{L^\infty}\|D_x^2\nabla P\| + \|\nabla P\|_{L^\infty}\|D_x^3(P^{-1}\mathbf{u})\|\Big)\|P\|_3\\
&\quad + \Big(\|D_x^1\rho\|_{L^\infty}\|D_x^2\partial_t\mathbf{u}\| + \|\partial_t\mathbf{u}\|_{L^\infty}\|D_x^3\rho\| + \|D_x^1(\rho\mathbf{u})\|_{L^\infty}\|D_x^2\nabla\mathbf{u}\|\\
&\quad + \|\nabla\mathbf{u}\|_{L^\infty}\|D_x^3(\rho\mathbf{u})\|\Big)\|\mathbf{u}\|_3\Big]\mathrm{d}\tau\\
&\lesssim \int_0^t \Big[\left(\|\partial_t P\|_2 + \|\partial_t P\|_2\|P\|_3^2 + \|P\|_3\|\mathbf{u}\|_2 + \|\mathbf{u}\|_3 + \|P\|_3^3\|\mathbf{u}\|_2\right)\|P\|_3^2\\
&\quad + \left(\|\partial_t\mathbf{u}\|_2 + \|\mathbf{u}\|_3^2\right)\|\rho\|_3\|\mathbf{u}\|_3\Big]\mathrm{d}\tau \le C(M)t.
\end{aligned}
$$

On the other hand, we use integration by parts to conclude that

$$\begin{aligned}
K_3 &= \frac{1}{2}\int_0^t\int_\Omega \left[|\partial_x^\alpha P|^2 \mathrm{div}\left(P^{-1}\mathbf{u}\right) + \gamma|\partial_x^\alpha \mathbf{u}|^2 \mathrm{div}\left(\rho\mathbf{u}\right)\right]\mathrm{d}\tau\\
&\lesssim \int_0^t\int_\Omega \left[|\partial_x^\alpha P|^2\left(P^{-1}|\mathrm{div}\mathbf{u}| + P^{-2}|\nabla P||\mathbf{u}|\right) + |\partial_x^\alpha \mathbf{u}|^2\left(|\nabla\rho||\mathbf{u}| + \rho|\mathrm{div}\mathbf{u}|\right)\right]\mathrm{d}\tau\\
&\lesssim \int_0^t \left(\|P\|_3^2 + \|P\|_3^3 + \|\rho\|_3\|\mathbf{u}\|_3^2\right)\|\mathbf{u}\|_3\,\mathrm{d}\tau \le C(M)t.
\end{aligned}$$

Furthermore, we utilize (2.3), (2.4), and (3.1) to bound the singular term K_4 as follows

$$\begin{aligned}
K_4 &= \left(\epsilon + \frac{1}{\epsilon}\right)\int_0^t \|\mathbf{u}\|_3\|\mathbf{I}_1\|_3\left(1 + \|P\|_3\left\|P^{-1}\right\|_3\right)\mathrm{d}\tau\\
&\lesssim \int_0^t \left(\epsilon^2 + \frac{1}{\epsilon^2}\right)\|\mathbf{I}_1\|_3^2\mathrm{d}\tau + \int_0^t \|\mathbf{u}\|_3^2\left(1 + \|P\|_3^8\right)\mathrm{d}\tau\\
&\le C_0(M_0) + C(M)t.
\end{aligned}$$

Plugging all the above estimates into (4.2), we can derive (4.1). ∎

Our next lemma focuses on estimates of $\|(\partial_t P, \partial_t \mathbf{u})\|_2^2$.

LEMMA 4.2. *Assume that the conditions in Theorem 1.2 hold, then we have for all $0 \le t \le T^\epsilon$ that*

$$\|(\partial_t P, \partial_t \mathbf{u})\|_2^2 \le C_0(M_0) + C(M)t. \tag{4.3}$$

Proof. Applying the operator $\partial_x^\beta\partial_t$ $(0 \le |\beta| \le 2)$ to (1.18) and (1.19), multiplying the resulting equations by $\partial_x^\beta\partial_t P$ and $\gamma\partial_x^\beta\partial_t\mathbf{u}$ in $L^2(\Omega)$, respectively, and adding up the resulting equalities, one has

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \left(P^{-1}|\partial_x^\beta\partial_t P|^2 + \gamma\rho|\partial_x^\beta\partial_t\mathbf{u}|^2\right)\mathrm{d}x\\
&= \frac{1}{2}\int_\Omega \left(P^{-1}|\partial_x^\beta\partial_t P|^2 + \gamma\rho|\partial_x^\beta\partial_t\mathbf{u}|^2\right)(0)\mathrm{d}x + \sum_{j=5}^{8} K_j,
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
K_5 = \int_0^t\int_\Omega \Bigg[&\partial_x^\beta\partial_t\mathbf{u}\cdot\left(\frac{\gamma}{2}\partial_t\rho\partial_x^\beta\partial_t\mathbf{u} - \gamma\partial_x^\beta\left(\partial_t\rho\partial_t\mathbf{u} + \partial_t\rho\mathbf{u}\cdot\nabla\mathbf{u} + \rho\partial_t\mathbf{u}\cdot\nabla\mathbf{u}\right)\right)\\
&+ \partial_x^\beta\partial_t P\Bigg(\partial_x^\beta\left(P^{-2}|\partial_t P|^2 + P^{-2}\partial_t P\cdot\mathbf{u} - P^{-1}\nabla P\cdot\partial_t\mathbf{u}\right) - \frac{1}{2}P^{-2}\partial_t P\partial_x^\beta\partial_t P\\
&+ (\gamma-1)\partial_x^\beta\left(P^{-1}\partial_t I_0 - P^{-2}\partial_t P I_0 + P^{-2}\partial_t P\theta^4 - 4P^{-1}\theta^3\partial_t\theta\right)\Bigg)\Bigg]\mathrm{d}x\mathrm{d}\tau,\\
K_6 = -\int_0^t\int_\Omega \Big\{&\partial_x^\beta\partial_t P\left[\partial_x^\beta\left(P^{-1}\partial_{tt}P\right) - P^{-1}\partial_{tt}\partial_x^\beta P + \partial_x^\beta\left(P^{-1}\nabla\partial_t P\cdot\mathbf{u}\right) - P^{-1}\mathbf{u}\cdot\partial_t\nabla\partial_x^\beta P\right]\\
&+ \gamma\partial_x^\beta\partial_t\mathbf{u}\cdot\left[\partial_x^\beta\left(\rho\partial_{tt}\mathbf{u}\right) - \rho\partial_x^\beta\partial_{tt}\mathbf{u} + \partial_x^\beta\left(\rho\mathbf{u}\cdot\nabla\partial_t\mathbf{u}\right) - \rho\mathbf{u}\cdot\nabla\partial_x^\beta\partial_t\mathbf{u}\right]\Big\}\mathrm{d}x\mathrm{d}\tau,\\
K_7 = -\int_0^t\int_\Omega &\left(P^{-1}\partial_x^\beta\partial_t P\mathbf{u}\cdot\nabla\partial_t\partial_x^\beta P + \gamma\rho\mathbf{u}\cdot\nabla\partial_x^\beta\partial_t\mathbf{u}\cdot\partial_x^\beta\partial_t\mathbf{u}\right)\mathrm{d}x\mathrm{d}\tau,
\end{aligned}$$

$$K_8=\left(\epsilon+\frac{1}{\epsilon}\right)\int_0^t\int_\Omega\Big[\gamma\partial_x^\beta\partial_t\mathbf{I}_1\cdot\partial_x^\beta\partial_t\mathbf{u}-(\gamma-1)\partial_x^\beta\Big(P^{-1}\partial_t\mathbf{I}_1\cdot\mathbf{u}-P^{-2}\partial_tP\mathbf{I}_1\cdot\mathbf{u}$$
$$+P^{-1}\mathbf{I}_1\cdot\partial_t\mathbf{u}\Big)\partial_x^\beta\partial_tP\Big]\mathrm{d}x\mathrm{d}\tau.$$

We apply (2.3)-(2.4) to bound K_5 and K_6 as follows

$$\begin{aligned}K_5\lesssim&\int_0^t\Big[\|\partial_t\rho\|_2\|\partial_t\mathbf{u}\|_2^2+\|\partial_tP\|_3^2+\big\|P^{-2}|\partial_tP|^2\big\|_2\|\partial_tP\|_2+\big\|P^{-2}\partial_tP\nabla P\cdot\mathbf{u}\big\|_2\|\partial_tP\|_2\\&+\big\|P^{-1}\nabla P\cdot\partial_t\mathbf{u}\big\|_2\|\partial_tP\|_2+\|\partial_t\rho\partial_t\mathbf{u}\|_2\|\partial_t\mathbf{u}\|_2+\|\partial_t\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_2\|\partial_t\mathbf{u}\|_2\\&+\|\rho\partial_t\mathbf{u}\cdot\nabla\mathbf{u}\|_2\|\partial_t\mathbf{u}\|_2+\Big(\big\|P^{-1}\partial_tI_0\big\|_2+\big\|P^{-2}\partial_tPI_0\big\|_2\\&+\big\|P^{-2}\partial_tP\theta^4\big\|_2+\big\|P^{-1}\theta^3\partial_t\theta\big\|_2\Big)\|\partial_tP\|_2\Big]\mathrm{d}\tau\\\lesssim&C(M)t+\int_0^t\Big[\|\partial_tP\|_2^2\left(\|P\|_2+\|P\|_2^2\right)^2\left(\|\partial_tP\|_2+\|P\|_3\|\mathbf{u}\|_2+\|I_0\|_2+\|\theta\|_2^4\right)\\&+\|\partial_tP\|_2\left(\|P\|_2+\|P\|_2^2\right)\left(\|P\|_3\|\partial_t\mathbf{u}\|_2+\|\partial_tI_0\|_2+\|\theta\|_2^3\|\partial_t\theta\|_2\right)\\&+\|\partial_t\mathbf{u}\|_2\left(\|\partial_t\rho\|_2\|\partial_t\mathbf{u}\|_2+\|\partial_t\rho\|_2\|\mathbf{u}\|_3^2+\|\rho\|_2\|\partial_t\mathbf{u}\|_2\|\mathbf{u}\|_3\right)\Big]\mathrm{d}\tau\le C(M)t,\end{aligned}$$

$$\begin{aligned}K_6\lesssim&\textstyle\int_0^t\Big[\left(\left\|\partial_x^\beta\left(P^{-1}\partial_{tt}P\right)-P^{-1}\partial_{tt}\partial_x^\beta P\right\|+\left\|\partial_x^\beta\left(P^{-1}\nabla\partial_tP\cdot\mathbf{u}\right)-P^{-1}\mathbf{u}\cdot\partial_t\nabla\partial_x^\beta P\right\|\right)\|\partial_tP\|_2\\&+\left(\left\|\partial_x^\beta\left(\rho\partial_{tt}\mathbf{u}\right)-\rho\partial_x^\beta\partial_{tt}\mathbf{u}\right\|+\left\|\partial_x^\beta\left(\rho\mathbf{u}\cdot\nabla\partial_t\mathbf{u}\right)-\rho\mathbf{u}\cdot\nabla\partial_x^\beta\partial_t\mathbf{u}\right\|\right)\|\partial_t\mathbf{u}\|_2\Big]\mathrm{d}\tau\\\lesssim&\int_0^t\Big[\|\partial_tP\|_2\|P\|_3\left(\|P\|_3\|\partial_{tt}P\|+\|\partial_{tt}P\|_1\right)+\|\partial_tP\|_2^2\|\mathbf{u}\|_3\left(1+\|P\|_3^3\right)\\&+\|\rho\|_3\|\partial_t\mathbf{u}\|_2\left(\|\partial_{tt}\mathbf{u}\|_1+\|\mathbf{u}\|_3\|\partial_t\mathbf{u}\|_2\right)\Big]\mathrm{d}\tau\\\le&C(M)t.\end{aligned}$$

After integration by parts, K_7 can be bounded as follows

$$\begin{aligned}K_7=&\frac{1}{2}\int_0^t\int_\Omega\left[|\partial_x^\beta\partial_tP|^2\mathrm{div}\left(P^{-1}\mathbf{u}\right)+\gamma|\partial_x^\beta\partial_t\mathbf{u}|^2\mathrm{div}\left(\rho\mathbf{u}\right)\right]\mathrm{d}\tau\\\lesssim&\int_0^t\left[\|\partial_tP\|_2^2\left(\|\mathbf{u}\|_3+\|P\|_3\|\mathbf{u}\|_2\right)+\|\rho\|_3\|\mathbf{u}\|_3\|\partial_t\mathbf{u}\|_2^2\right]\mathrm{d}\tau\le C(M)t.\end{aligned}$$

Moreover, we utilize (2.3), (2.4), and (3.1) to achieve

$$\begin{aligned}K_8=&\left(\epsilon+\frac{1}{\epsilon}\right)\int_0^t\Big[\left(\big\|P^{-1}\partial_t\mathbf{I}_1\cdot\mathbf{u}\big\|_2+\big\|P^{-2}\partial_tP\mathbf{I}_1\cdot\mathbf{u}\big\|_2+\big\|P^{-1}\mathbf{I}_1\cdot\partial_t\mathbf{u}\big\|_2\right)\|\partial_tP\|_2\\&+\|\partial_t\mathbf{I}_1\|_2\|\partial_t\mathbf{u}\|_2\Big]\mathrm{d}\tau\\\lesssim&\int_0^t\left(\epsilon^2+\frac{1}{\epsilon^2}\right)\|\partial_t\mathbf{I}_1\|_2^2\mathrm{d}\tau+\int_0^t\Big[\left(\|P\|_2+\|P\|_2^2\right)^2\left(\|\mathbf{u}\|_2^2+\|\partial_t\mathbf{u}\|_2^2\right)\|\partial_tP\|_2^2\end{aligned}$$

$$+\left(\|P\|_2+\|P\|_2^2\right)^4\|\partial_t P\|_2^4\|\mathbf{u}\|_2^2+\|\partial_t\mathbf{u}\|_2^2\Big]\mathrm{d}\tau$$
$$\le C_0(M_0)+C(M)t.$$

Inserting all the above estimates into (4.4), we can conclude (4.3). □

Now we turn to bound terms $\|(\partial_{tt}P,\partial_{tt}\mathbf{u})\|_1^2$. In fact, we have

LEMMA 4.3. *Assume that the conditions in Theorem 1.2 hold, then we have for all $0\le t\le T^\epsilon$ that*

$$\|(\partial_{tt}P,\partial_{tt}\mathbf{u})\|_1^2\le C_0(M_0)+C(M)t. \tag{4.5}$$

Proof. We apply the operator $\partial_x^\eta\partial_{tt}$ $(0\le|\eta|\le1)$ to (1.18) and (1.19), multiply by $\partial_x^\eta\partial_{tt}P$ and $\gamma\partial_x^\eta\partial_{tt}\mathbf{u}$ in $L^2(\Omega)$, respectively, and add up the resulting equalities to reach

$$\begin{aligned}&\frac12\int_\Omega\left(P^{-1}|\partial_x^\eta\partial_{tt}P|^2+\gamma\rho|\partial_x^\eta\partial_{tt}\mathbf{u}|^2\right)\mathrm{d}x\\&=\frac12\int_\Omega\left(P^{-1}|\partial_x^\eta\partial_{tt}P|^2+\gamma\rho|\partial_x^\eta\partial_{tt}\mathbf{u}|^2\right)(0)\mathrm{d}x+\sum_{j=9}^{12}K_j,\end{aligned}\tag{4.6}$$

where

$$\begin{aligned}K_9=&\int_0^t\int_\Omega\Big\{\gamma\partial_x^\eta\partial_{tt}\mathbf{u}\cdot\Big[\frac12\partial_t\rho\partial_{tt}\partial_x^\eta\mathbf{u}-\partial_x^\eta\Big(2\partial_t\rho\partial_{tt}\mathbf{u}+\partial_{tt}\rho\partial_t\mathbf{u}+\partial_{tt}\rho\mathbf{u}\cdot\nabla\mathbf{u}\\&+2\partial_t\rho\partial_t\mathbf{u}\cdot\nabla\mathbf{u}+2\partial_t\rho\mathbf{u}\cdot\nabla\partial_t\mathbf{u}+\rho\partial_{tt}\mathbf{u}\cdot\nabla\mathbf{u}+2\rho\partial_t\mathbf{u}\cdot\nabla\partial_t\mathbf{u}\Big)\Big]+\partial_x^\eta\partial_{tt}P\cdot\\&\Big[\partial_x^\eta\Big(3P^{-2}\partial_tP\partial_{tt}P-\frac12P^{-2}\partial_tP\partial_{tt}\partial_x^\eta P-2P^{-3}|\partial_tP|^2\partial_tP+2P^{-2}\partial_tP\nabla\partial_tP\cdot\mathbf{u}\\&-2P^{-1}\nabla\partial_tP\cdot\partial_t\mathbf{u}-2P^{-3}|\partial_tP|^2\nabla P\cdot\mathbf{u}+P^{-2}\partial_{tt}P\nabla P\cdot\mathbf{u}+2P^{-2}\partial_tP\nabla P\cdot\partial_t\mathbf{u}\\&-P^{-1}\nabla P\cdot\partial_{tt}\mathbf{u}\Big)+(\gamma-1)\partial_x^\eta\Big(P^{-1}\partial_{tt}I_0-2P^{-2}\partial_tP\partial_tI_0+2P^{-3}|\partial_tP|^2I_0-P^{-2}\partial_{tt}PI_0\\&-2P^{-3}|\partial_tP|^2\theta^4+P^{-2}\partial_{tt}P\theta^4+8P^{-2}\partial_tP\theta^3\partial_t\theta\\&\qquad-12P^{-1}\theta^2|\partial_t\theta|^2-4P^{-1}\theta^3\partial_{tt}\theta\Big)\Big]\Big\}\mathrm{d}x\mathrm{d}\tau,\end{aligned}$$

$$\begin{aligned}K_{10}=&-\int_0^t\int_\Omega\Big\{\left[\partial_x^\eta(P^{-1}\partial_{ttt}P)-P^{-1}\partial_{ttt}\partial_x^\eta P+\partial_x^\eta(P^{-1}\nabla\partial_{tt}P\cdot\mathbf{u})-\nabla\partial_x^\eta\partial_{tt}P\cdot P^{-1}\mathbf{u}\right]\partial_x^\eta\partial_{tt}P\\&+\gamma\left[\partial_x^\eta(\rho\partial_{ttt}\mathbf{u})-\rho\partial_{ttt}\partial_x^\eta\mathbf{u}+\partial_x^\eta(\rho\mathbf{u}\cdot\nabla\partial_{tt}\mathbf{u})-\rho\mathbf{u}\cdot\nabla\partial_x^\eta\partial_{tt}\mathbf{u}\right]\cdot\partial_x^\eta\partial_{tt}\mathbf{u}\Big\}\mathrm{d}x\mathrm{d}\tau,\end{aligned}$$

$$K_{11}=-\int_0^t\int_\Omega\left(P^{-1}\mathbf{u}\cdot\nabla\partial_{tt}\partial_x^\eta P\partial_x^\eta\partial_{tt}P+\gamma\rho\mathbf{u}\cdot\nabla\partial_x^\eta\partial_{tt}\mathbf{u}\cdot\partial_x^\eta\partial_{tt}\mathbf{u}\right)\mathrm{d}x\mathrm{d}\tau,$$

$$\begin{aligned}K_{12}=&-(\gamma-1)\left(\epsilon+\tfrac1\epsilon\right)\int_0^t\int_\Omega\Big[\partial_x^\eta\partial_{tt}P\partial_x^\eta\Big(P^{-1}\partial_{tt}\mathbf{I}_1\cdot\mathbf{u}+2P^{-1}\partial_t\mathbf{I}_1\cdot\partial_t\mathbf{u}+2P^{-3}|\partial_tP|^2\mathbf{I}_1\cdot\mathbf{u}\\&-P^{-1}\partial_{tt}P\mathbf{I}_1\cdot\mathbf{u}-2P^{-2}\partial_tP\mathbf{I}_1\cdot\partial_t\mathbf{u}+P^{-1}\mathbf{I}_1\cdot\partial_{tt}\mathbf{u}\Big)-\frac{\gamma}{\gamma-1}\partial_x^\eta\partial_{tt}\mathbf{I}_1\cdot\partial_x^\eta\partial_{tt}\mathbf{u}\Big]\mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

In view of (2.3)-(2.5), we have

$$K_9\lesssim\int_0^t\Big[\|\partial_{tt}\mathbf{u}\|_1\Big(\|\partial_t\rho\|_2\|\partial_{tt}\mathbf{u}\|_1+\|\partial_t\rho\partial_{tt}\mathbf{u}\|_1+\|\partial_{tt}\rho\partial_t\mathbf{u}\|_1+\|\partial_{tt}\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_1$$

$$
\begin{aligned}
&+\|\partial_t\rho\partial_t\mathbf{u}\cdot\nabla\mathbf{u}\|_1+\|\partial_t\rho\mathbf{u}\cdot\nabla\partial_t\mathbf{u}\|_1+\|\rho\partial_{tt}\mathbf{u}\cdot\nabla\mathbf{u}\|_1+\|\rho\partial_t\mathbf{u}\cdot\nabla\partial_t\mathbf{u}\|_1\Big)\\
&+\|\partial_{tt}P\|_1\Big(\|\partial_tP\|_2\|\partial_{tt}P\|_1+\left\|P^{-3}|\partial_tP|^2\partial_tP\right\|_1+\left\|P^{-2}\partial_tP\partial_{tt}P\right\|_1\\
&+\left\|P^{-2}\partial_tP\nabla\partial_tP\cdot\mathbf{u}\right\|_1+\left\|P^{-1}\nabla\partial_tP\cdot\partial_t\mathbf{u}\right\|_1+\left\|P^{-3}|\partial_tP|^2\nabla P\cdot\mathbf{u}\right\|_1\\
&+\left\|P^{-2}\partial_{tt}P\nabla P\cdot\mathbf{u}\right\|_1+\left\|P^{-2}\partial_tP\nabla P\cdot\partial_t\mathbf{u}\right\|_1+\left\|P^{-1}\nabla P\cdot\partial_{tt}\mathbf{u}\right\|_1\\
&+\left\|P^{-1}\partial_{tt}I_0\right\|_1+\left\|P^{-2}\partial_tP\partial_tI_0\right\|_1+\left\|P^{-3}|\partial_tP|^2I_0\right\|_1+\left\|P^{-2}\partial_{tt}PI_0\right\|_1\\
&+\left\|P^{-3}|\partial_tP|^2\theta^4\right\|_1+\left\|P^{-2}\partial_{tt}P\theta^4\right\|_1+\left\|P^{-2}\partial_tP\theta^3\partial_t\theta\right\|_1+\left\|P^{-1}\theta^2|\partial_t\theta|^2\right\|_1\\
&+\left\|P^{-1}\theta^3\partial_{tt}\theta\right\|_1\Big)\mathrm{d}\tau\\
&\lesssim\int_0^t\|\partial_{tt}\mathbf{u}\|_1\Big(\|\partial_t\rho\|_2\|\partial_{tt}\mathbf{u}\|_1+\|\partial_{tt}\rho\|_1\|\partial_t\mathbf{u}\|_2+\|\partial_{tt}\rho\|_1\|\mathbf{u}\|_3^2\\
&+\|\partial_t\rho\|_2\|\partial_t\mathbf{u}\|_2\|\mathbf{u}\|_3+\|\rho\|_3\|\partial_{tt}\mathbf{u}\|_1\|\mathbf{u}\|_3+\|\rho\|_3\|\partial_t\mathbf{u}\|_2^2\Big)\\
&+\|\partial_{tt}P\|_1\Big[\left(\|P\|_2+\|P\|_2^2\right)\left(\|\partial_tP\|_2\|\partial_tI_0\|_2+\|\theta\|_2^2\|\partial_t\theta\|_2^2\right)\\
&+\left(\|P\|_2+\|P\|_2^2\right)^2\|\partial_tP\|_2\left(\|P\|_3\|\partial_t\mathbf{u}\|_2+\|\theta\|_2^3\|\partial_t\theta\|_2\right)\\
&+\left(\|P\|_2+\|P\|_2^2\right)^3\|\partial_tP\|_2^2\left(\|\partial_tP\|_2+\|P\|_3\|\mathbf{u}\|_2+\|I_0\|_2+\|\theta\|_2^4\right)\\
&+\|P\|_3(\|P\|_3\|\partial_{tt}P\|\|\mathbf{u}\|_2+\|\partial_{tt}P\|_1\|\mathbf{u}\|_3+\|P\|_3\|\partial_{tt}\mathbf{u}\|+\|\partial_{tt}\mathbf{u}\|_1)\\
&+\|P\|_3\|\partial_{tt}P\|\left(\|I_0\|_2+\|\theta\|_2^4\right)+\|\partial_{tt}P\|_1\left(\|I_0\|_3+\|\theta\|_3^4\right)\\
&+(1+\|P\|_3)\left(\|\partial_{tt}I_0\|_1+\|\partial_{tt}\theta\|_1\|\theta\|_3^3\right)\\
&+\|\partial_tP\|_2\Big(\|\partial_{tt}P\|_1+\|P\|_3\|\partial_{tt}P\|+\|P\|_3\|\partial_tP\|_2\|\mathbf{u}\|_3+\|\partial_tP\|_2\|\mathbf{u}\|_3\\
&+\|P\|_3\|\partial_t\mathbf{u}\|_2+\|\partial_t\mathbf{u}\|_2\Big)\Big]\le C(M)t,
\end{aligned}
$$

and

$$
\begin{aligned}
K_{10}&\lesssim\textstyle\int_0^t\Big[(\|\partial_x^\eta(P^{-1}\partial_{ttt}P)-P^{-1}\partial_{ttt}\partial_x^\eta P\|+\|\partial_x^\eta(P^{-1}\nabla\partial_{tt}P\cdot\mathbf{u})-P^{-1}\mathbf{u}\cdot\nabla\partial_{tt}\partial_x^\eta P\|)\|\partial_{tt}P\|_1\\
&\quad+(\|\partial_x^\eta(\rho\partial_{ttt}\mathbf{u})-\rho\partial_x^\eta\partial_{ttt}\mathbf{u}\|+\|\partial_x^\eta(\rho\mathbf{u}\cdot\nabla\partial_{tt}\mathbf{u})-\rho\mathbf{u}\cdot\nabla\partial_x^\eta\partial_{tt}\mathbf{u}\|)\|\partial_{tt}\mathbf{u}\|_1\Big]\mathrm{d}\tau\\
&\lesssim\int_0^t\Big\{\|\partial_{tt}P\|_1\|P\|_3\Big[\|\partial_{ttt}P\|+\left(1+\|P\|_3^2\right)\|\mathbf{u}\|_3\|\partial_{tt}P\|_1\Big]\\
&\quad+\|\rho\|_3\|\partial_{tt}\mathbf{u}\|_1(\|\mathbf{u}\|_3+\|\partial_{ttt}\mathbf{u}\|)\Big\}\mathrm{d}\tau\le C(M)t.
\end{aligned}
$$

On the other hand, we employ (2.4) and integration by parts to obtain that

$$
\begin{aligned}
K_{11}&=\frac{1}{2}\int_0^t\int_\Omega\left[|\partial_x^\eta\partial_{tt}P|^2\mathrm{div}\left(P^{-1}\mathbf{u}\right)+\gamma|\partial_x^\eta\partial_{tt}\mathbf{u}|^2\mathrm{div}(\rho\mathbf{u})\right]\mathrm{d}\tau\\
&\lesssim\int_0^t\Big[\left(\|P\|_3+\|P\|_3^3\right)\|\mathbf{u}\|_3\|\partial_{tt}P\|_1^2+\|P\|_3\|\mathbf{u}\|_3\|\partial_{tt}\mathbf{u}\|_1\Big]\mathrm{d}\tau\le C(M)t.
\end{aligned}
$$

Finally, using (2.3), (2.4), and (3.1), we can infer that

$$
\begin{aligned}
K_{12} \lesssim & \left(\epsilon+\tfrac{1}{\epsilon}\right) \int_0^t \Big[(1+\|P\|_3)\|\partial_{tt}P\|_1 (\|\partial_{tt}\mathbf{I}_1\|_1\|\mathbf{u}\|_2+\|\partial_{tt}P\|_1\|\mathbf{I}_1\|_3\|\mathbf{u}\|_3+\|\mathbf{I}_1\|_3\|\partial_{tt}\mathbf{u}\|_1) \\
& +\|\mathbf{I}_1\|_2\left(\|P\|_2+\|P\|_2^2\right)^2\|\partial_t P\|_2\|\partial_{tt}P\|_1 \\
& \qquad \cdot\left(\|\partial_t\mathbf{u}\|_2+\|P\|_2\|\partial_t P\|_2\|\mathbf{u}\|_2+\|P\|_2^2\|\partial_t P\|_2\|\mathbf{u}\|_2\right) \\
& +\|\partial_t\mathbf{I}_1\|_2\left(\|P\|_2+\|P\|_2^2\right)\|\partial_t\mathbf{u}\|_2\|\partial_{tt}P\|_1+\|\partial_{tt}\mathbf{I}_1\|_1\|\partial_{tt}\mathbf{u}\|_1\Big]\mathrm{d}\tau \\
\lesssim & \int_0^t\left(\epsilon^2+\frac{1}{\epsilon^2}\right)\sum_{i=0}^3\left\|\partial_t^i\mathbf{I}_1(\tau)\right\|_{3-i}^2\mathrm{d}\tau+C(M)t \\
\le & \, C_0(M_0)+C(M)t.
\end{aligned}
$$

We insert all the above estimates into (4.6) to deduce (4.5). ∎

The next lemma aims to bound terms $\|(\partial_{ttt}P,\partial_{ttt}\mathbf{u})\|^2$.

LEMMA 4.4. *Assume that the conditions in Theorem 1.2 hold, then we have for all $0\le t\le T^\epsilon$ that*

$$
\|(\partial_{ttt}P,\partial_{ttt}\mathbf{u})\|^2\le C_0(M_0)+C(M)t. \tag{4.7}
$$

Proof. Differentiating (1.18) and (1.19) with respect to time thrice, multiplying by $\partial_{ttt}P$ and $\gamma\partial_{ttt}\mathbf{u}$ in $L^2(\Omega)$, respectively, then adding up the resulting equalities, we arrive at

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega\left(P^{-1}|\partial_{ttt}P|^2+\gamma\rho|\partial_{ttt}\mathbf{u}|^2\right)\mathrm{d}x \\
&=\frac{1}{2}\int_\Omega\left(P^{-1}|\partial_{ttt}P|^2+\gamma\rho|\partial_{ttt}\mathbf{u}|^2\right)(0)\mathrm{d}x+\sum_{j=13}^{16}K_j,
\end{aligned} \tag{4.8}
$$

where

$$
\begin{aligned}
K_{13}=&\int_0^t\int_\Omega\partial_{ttt}P\Big[\frac{7}{2}P^{-2}\partial_t P\partial_{ttt}P+3P^{-2}\partial_t P\nabla\partial_{tt}P\cdot\mathbf{u}-3P^{-1}\nabla\partial_{tt}P\cdot\partial_t\mathbf{u} \\
&+6P^{-4}|\partial_t P|^4-12P^{-3}|\partial_t P|^2\partial_{tt}P+3P^{-2}|\partial_{tt}P|^2-6P^{-3}|\partial_t P|^2\nabla\partial_t P\cdot\mathbf{u} \\
&+3P^{-2}\partial_{tt}P\nabla\partial_t P\cdot\mathbf{u}+6P^{-2}\partial_t P\nabla\partial_t P\cdot\partial_t\mathbf{u}-3P^{-1}\nabla\partial_t P\cdot\partial_{tt}\mathbf{u}+6P^{-4}|\partial_t P|^2\nabla P\cdot\mathbf{u} \\
&-6P^{-3}\partial_t P\partial_{tt}P\nabla P\cdot\mathbf{u}-6P^{-3}|\partial_t P|^2\nabla P\cdot\partial_t\mathbf{u}+P^{-2}\partial_{ttt}P\nabla P\cdot\mathbf{u} \\
&+3P^{-2}\partial_{tt}P\nabla P\cdot\partial_t\mathbf{u}+3P^{-2}\partial_t P\nabla P\cdot\partial_{tt}\mathbf{u}-P^{-1}\nabla P\cdot\partial_{ttt}\mathbf{u} \\
&+(\gamma-1)\Big(P^{-1}\partial_{ttt}I_0-3P^{-2}\partial_t P\partial_{tt}I_0+6P^{-3}|\partial_t P|^2\partial_t I_0-3P^{-2}\partial_{tt}P\partial_t I_0 \\
&-6P^{-4}|\partial_t P|^2\partial_t P I_0+6P^{-3}\partial_t P\partial_{tt}P I_0-P^{-2}\partial_{ttt}P I_0+6P^{-4}|\partial_t P|^2\partial_t P\theta^4 \\
&-6P^{-3}\partial_t P\partial_{tt}P\theta^4-24P^{-3}|\partial_t P|^2\theta^3\partial_t\theta+P^{-2}\theta^4\partial_{ttt}P+12P^{-2}\partial_{tt}P\theta^3\partial_t\theta \\
&+12P^{-2}\partial_t P\theta^3\partial_{tt}\theta+36P^{-2}\partial_t P\theta^2|\partial_t\theta|^2-24P^{-1}\theta|\partial_t\theta|^2\partial_t\theta \\
&-36P^{-1}\theta^2\partial_t\theta\partial_{tt}\theta-4P^{-1}\theta^3\partial_{ttt}\theta\Big)\Big]\mathrm{d}x\mathrm{d}\tau,
\end{aligned}
$$

$$
\begin{aligned}
K_{14}=&-\gamma\int_0^t\int_\Omega \partial_{ttt}\mathbf{u}\cdot\bigg(\frac{5}{2}\partial_t P\partial_{ttt}\mathbf{u}+3\partial_{tt}\rho\partial_{tt}\mathbf{u}+\partial_{tt}\rho\partial_t\mathbf{u}+\partial_{ttt}\rho\mathbf{u}\cdot\nabla\mathbf{u}\\
&+3\partial_{tt}\rho\partial_t\mathbf{u}\cdot\nabla\mathbf{u}+3\partial_{tt}\rho\mathbf{u}\cdot\nabla\partial_t\mathbf{u}+3\partial_t\rho\partial_{tt}\mathbf{u}\cdot\nabla\mathbf{u}+6\partial_t\rho\partial_t\mathbf{u}\cdot\nabla\partial_t\mathbf{u}\\
&+3\partial_t\rho\mathbf{u}\cdot\nabla\partial_{tt}\mathbf{u}+\rho\partial_{ttt}\mathbf{u}\cdot\nabla\mathbf{u}+3\rho\partial_{tt}\mathbf{u}\cdot\nabla\partial_t\mathbf{u}+3\rho\partial_t\mathbf{u}\cdot\nabla\partial_{tt}\mathbf{u}\bigg)\mathrm{d}x\mathrm{d}\tau,
\end{aligned}
$$

$$
K_{15}=-\int_0^t\int_\Omega\left(P^{-1}\mathbf{u}\cdot\nabla\partial_{ttt}P\partial_{ttt}P+\gamma\rho\mathbf{u}\cdot\nabla\partial_{ttt}\mathbf{u}\cdot\partial_{ttt}\mathbf{u}\right)\mathrm{d}x\mathrm{d}\tau,
$$

$$
\begin{aligned}
K_{16}=&(\gamma-1)\left(\epsilon+\frac{1}{\epsilon}\right)\int_0^t\int_\Omega\bigg[\partial_{ttt}P\Big(P^{-2}\partial_t P\partial_{tt}\mathbf{I}_1\cdot\mathbf{u}-P^{-1}\partial_{ttt}\mathbf{I}_1\cdot\mathbf{u}-3P^{-1}\partial_{tt}\mathbf{I}_1\cdot\partial_t\mathbf{u}\\
&+4P^{-2}\partial_t P\partial_t\mathbf{I}_1\cdot\partial_t\mathbf{u}-3P^{-1}\partial_t\mathbf{I}_1\cdot\partial_{tt}\mathbf{u}+6P^{-4}|\partial_t P|^2\partial_t P\mathbf{I}_1\cdot\mathbf{u}-6P^{-3}\partial_t P\partial_{tt}P\mathbf{I}_1\cdot\mathbf{u}\\
&-2P^{-3}|\partial_t P|^2\partial_t\mathbf{I}_1\cdot\mathbf{u}-6P^{-3}|\partial_t P|^2\mathbf{I}_1\cdot\partial_t\mathbf{u}+P^{-2}\mathbf{I}_1\cdot\mathbf{u}\partial_{ttt}P+P^{-2}\partial_{tt}P\partial_t\mathbf{I}_1\cdot\mathbf{u}\\
&+3P^{-2}\partial_{tt}P\mathbf{I}_1\cdot\partial_t\mathbf{u}+3P^{-2}\partial_t P\mathbf{I}_1\cdot\partial_{tt}\mathbf{u}-P^{-1}\mathbf{I}_1\cdot\partial_{tt}\mathbf{u}\Big)+\frac{\gamma}{\gamma-1}\partial_{ttt}\mathbf{I}_1\cdot\partial_{ttt}\mathbf{u}\bigg]\mathrm{d}x\mathrm{d}\tau.
\end{aligned}
$$

With the help of interpolations and Hölder's inequalities, K_{13} and K_{14} can be bounded as follows

$$
\begin{aligned}
K_{13}\lesssim&\int_0^t\bigg[\|\partial_t P\|_2\Big(\|\partial_{ttt}P\|+\|\mathbf{u}\|_2\|\partial_{tt}P\|_1+\|\partial_{tt}\mathbf{u}\|_1+\|P\|_3\|\mathbf{u}\|_2\|\partial_{tt}P\|+\|P\|_3\|\partial_{tt}\mathbf{u}\|\\
&+\|\partial_{tt}I_0\|+\|I_0\|_2\|\partial_{tt}P\|+\|\partial_{tt}P\|\|\theta\|_2^4+\|\theta\|_2^3\|\partial_{tt}\theta\|+\|\theta\|_2^2\|\partial_t\theta\|_2^2\Big)\\
&+\|\partial_t P\|_2^2\left(\|\partial_{tt}P\|+\|\partial_t\mathbf{u}\|_2+\|P\|_3\|\partial_t\mathbf{u}\|+\|\partial_t I_0\|+\|\theta\|_2^3\|\partial_t\theta\|\right)\\
&+\|\partial_t P\|_2^3\left(\|\partial_t P\|_2+\|\mathbf{u}\|_2+\|\mathbf{u}\|_2\|P\|_1+\|I_0\|+\|\theta\|_2^4\right)\\
&+\|\partial_{tt}P\|_1\left(\|\partial_t\mathbf{u}\|_2+\|\partial_{tt}P\|_1+\|P\|_3\|\partial_t\mathbf{u}\|_1+\|\partial_t I_0\|_2+\|\theta\|_2^3\|\partial_t\theta\|_2\right)\\
&+\|\partial_{ttt}P\|\left(\|P\|_3\|\mathbf{u}\|_2+\|I_0\|_2+\|\theta\|_2^4\right)+\|P\|_3\|\partial_{ttt}\mathbf{u}\|+\|\partial_{ttt}I_0\|\\
&+\|\theta\|_2\left(\|\partial_t\theta\|_2^3+\|\theta\|_2\|\partial_t\theta\|_2\|\partial_{tt}\theta\|+\|\theta\|_2^2\|\partial_{ttt}\theta\|\right)\bigg]\|\partial_{ttt}P\|\mathrm{d}\tau\\
\le& C(M)t,
\end{aligned}
$$

and

$$
\begin{aligned}
K_{14}\lesssim&\int_0^t\|\partial_{ttt}\mathbf{u}\|\bigg[\|\rho\|_2\left(\|\mathbf{u}\|_3\|\partial_{ttt}\mathbf{u}\|+\|\partial_t\mathbf{u}\|_2\|\partial_{tt}\mathbf{u}\|_1\right)\\
&+\|\partial_t\rho\|_2\left(\|\mathbf{u}\|_3\|\partial_{tt}\mathbf{u}\|+\|\partial_t\mathbf{u}\|_2^2+\|\mathbf{u}\|_2\|\partial_{tt}\mathbf{u}\|_1\right)\\
&+\|\partial_{tt}\rho\|_1\left(\|\partial_{tt}\mathbf{u}\|_1+\|\partial_t\mathbf{u}\|_2\|\mathbf{u}\|_3+\|\mathbf{u}\|_2\|\partial_t\mathbf{u}\|_2\right)\\
&+\|\partial_{ttt}\rho\|\left(\|\partial_t\mathbf{u}\|_2+\|\mathbf{u}\|_3^2\right)+\|\partial_t P\|_2\|\partial_{ttt}\mathbf{u}\|\bigg]\mathrm{d}\tau\\
\le& C(M)t.
\end{aligned}
$$

Moreover, we use (2.3)-(2.5), and integration by parts to show that

$$
K_{15}=\frac{1}{2}\int_0^t\int_\Omega\left[|\partial_{ttt}P|^2\mathrm{div}\left(P^{-1}\mathbf{u}\right)+\gamma|\partial_{ttt}\mathbf{u}|^2\mathrm{div}\left(\rho\mathbf{u}\right)\right]\mathrm{d}\tau
$$

$$\lesssim \int_0^t \left[\left(\|P\|_3+\|P\|_3^3\right)\|\mathbf{u}\|_3\|\partial_{ttt}P\|^2+\|\rho\|_3\|\mathbf{u}\|_3\|\partial_{ttt}\mathbf{u}\|^2\right]\mathrm{d}\tau \le C(M)t.$$

Finally, it follows from (3.1) that

$$\begin{aligned}
K_{16} \lesssim &\left(\epsilon+\frac{1}{\epsilon}\right)\int_0^t \|\partial_{ttt}P\|\Big[\|\mathbf{I}_1\|_2\Big(\|\partial_t P\|_2^3\|\mathbf{u}\|+\|\partial_t P\|_2\|\mathbf{u}\|_2\|\partial_{tt}P\|+\|\partial_t P\|_2^2\|\partial_t\mathbf{u}\| \\
&+\|\mathbf{u}\|_2\|\partial_{ttt}P\|+\|\partial_{tt}P\|\|\partial_t\mathbf{u}\|_2+\|\partial_t P\|_2\|\partial_{tt}\mathbf{u}\|+\|\partial_{tt}\mathbf{u}\|\Big) \\
&+\|\partial_t\mathbf{I}_1\|_2\left(\|\partial_t P\|_2\|\partial_t\mathbf{u}\|+\|\partial_{tt}\mathbf{u}\|+\|\partial_t P\|_2^2\|\mathbf{u}\|_2+\|\partial_{tt}P\|\|\mathbf{u}\|_2\right) \\
&+\|\partial_{tt}\mathbf{I}_1\|\left(\|\partial_t P\|_2\|\mathbf{u}\|_2+\|\partial_t\mathbf{u}\|_2\right)+\|\partial_{ttt}\mathbf{I}_1\|\left(\|\partial_t\mathbf{u}\|_2+\|\partial_{ttt}\mathbf{u}\|\right)\Big]\mathrm{d}\tau \\
\lesssim &\int_0^t\left(\epsilon^2+\frac{1}{\epsilon^2}\right)\sum_{i=0}^3\left\|\partial_t^i\mathbf{I}_1(\tau)\right\|_{3-i}^2\mathrm{d}\tau+C(M)t \\
\le &\, C_0(M_0)+C(M)t.
\end{aligned}$$

Plugging estimates of K_{13}-K_{16} into (4.8), we can derive (4.7). □

Now it suffices to deduce estimates on $\sum_{i=0}^3\left\|\partial_t^i s(t)\right\|_{3-i}^2$.

LEMMA 4.5. *Assume that the conditions in Theorem 1.2 hold, then we have for all $0\le t\le T^\epsilon$ that*

$$\sum_{i=0}^3\left\|\partial_t^i s(t)\right\|_{3-i}^2 \le C_0(M_0)+C(M)t. \tag{4.9}$$

Proof. First, we apply the operator ∂_x^α $(0\le|\alpha|\le 3)$ to (1.17), and multiply the resulting equations by $\partial_x^\alpha s$ in $L^2(\Omega)$ to achieve

$$\frac{1}{2}\int_\Omega |\partial_x^\alpha s|^2\mathrm{d}x=\frac{1}{2}\int_\Omega|\partial_x^\alpha s|^2(0)\mathrm{d}x+K_{17}, \tag{4.10}$$

where

$$K_{17}=\int_0^t\int_\Omega \partial_x^\alpha s\left[\partial_x^\alpha\left(P^{-1}I_0-P^{-1}\theta^4-\mathbf{u}\cdot\nabla s\right)-\left(\epsilon+\frac{1}{\epsilon}\right)\partial_x^\alpha\left(P^{-1}\mathbf{I}_1\cdot\mathbf{u}\right)\right]\mathrm{d}x\mathrm{d}\tau.$$

By using (2.1)-(2.5), (3.1), and integration by parts, we can get

$$\begin{aligned}
K_{17}\lesssim &\int_0^t\|s\|_3\Big[\left\|P^{-1}I_0\right\|_3+\left\|P^{-1}\theta^4\right\|_3+\|\mathbf{u}\|_3\|s\|_3+\|\partial_x^\alpha(\mathbf{u}\cdot\nabla s)-\mathbf{u}\cdot\partial_x^\alpha\nabla s\| \qquad (4.11)\\
&+\left(\epsilon+\frac{1}{\epsilon}\right)\left\|P^{-1}\mathbf{I}_1\cdot\mathbf{u}\right\|_3\Big]\mathrm{d}\tau \\
\lesssim &\int_0^t\|s\|_3\Big[\left(\|P\|_3+\|P\|_3^3\right)\left(\|I_0\|_3+\|\theta\|_3^4\right)+\|\mathbf{u}\|_3\|s\|_3 \qquad (4.12)\\
&+\left(\epsilon+\frac{1}{\epsilon}\right)\left(\|P\|_3+\|P\|_3^3\right)\|\mathbf{u}\|_3\|\mathbf{I}_1\|_3\Big]\mathrm{d}\tau \\
\lesssim &\int_0^t\left(\epsilon^2+\frac{1}{\epsilon^2}\right)\|\mathbf{I}_1\|_3^2\mathrm{d}\tau+C(M)t\le C_0(M_0)+C(M)t. \qquad (4.13)
\end{aligned}$$

The combination of (4.10) and (4.11) gives

$$\|s(t)\|_3^2 \le C_0(M_0) + C(M)t. \tag{4.14}$$

Second, applying the operator $\partial_x^\beta \partial_t$ $(0 \le |\beta| \le 2)$ to (1.17), multiplying by $\partial_x^\beta \partial_t s$, then integrating the result identity over $[0,t] \times \Omega$, we arrive at

$$\frac{1}{2}\int_\Omega |\partial_x^\beta \partial_t s|^2 \mathrm{d}x = \frac{1}{2}\int_\Omega |\partial_x^\beta \partial_t s|^2(0)\mathrm{d}x + \sum_{j=18}^{20} K_j, \tag{4.15}$$

where

$$K_{18} = -\int_0^t \int_\Omega \left\{ \frac{1}{2}\mathbf{u}\cdot\nabla\left(|\partial_x^\beta \partial_t s|^2\right) + \left[\partial_x^\beta(\mathbf{u}\cdot\nabla\partial_t s) - \mathbf{u}\cdot\nabla\partial_x^\beta\partial_t s\right]\partial_x^\beta \partial_t s \right\} \mathrm{d}x\mathrm{d}\tau,$$

$$K_{19} = -\int_0^t \int_\Omega \partial_x^\beta \partial_t s \partial_x^\beta \left(P^{-1}\partial_t I_0 - \partial_t \mathbf{u}\cdot\nabla s - P^{-2}\partial_t P I_0 + P^{-2}\partial_t P \theta^4 - 4P^{-1}\theta^3\partial_t\theta\right)\mathrm{d}x\mathrm{d}\tau,$$

$$K_{20} = \left(\epsilon + \frac{1}{\epsilon}\right)\int_0^t \int_\Omega \partial_x^\beta \partial_t s \partial_x^\beta \left(P^{-2}\partial_t P \mathbf{I}_1\cdot\mathbf{u} - P^{-1}\partial_t \mathbf{I}_1\cdot\mathbf{u} - P^{-1}\mathbf{I}_1\cdot\partial_t\mathbf{u}\right)\mathrm{d}x\mathrm{d}\tau.$$

To bound K_{18}-K_{20}, we employ (2.4), (2.5), (3.1), and integration by parts to obtain

$$\begin{aligned} K_{18} &\lesssim \int_0^t \int_\Omega |\mathrm{div}\mathbf{u}||\partial_x^\beta \partial_t s|^2 \mathrm{d}x\mathrm{d}\tau + \int_0^t \|\partial_x^\beta \partial_t s\| \left\|\partial_x^\beta(\mathbf{u}\cdot\nabla\partial_t s) - \mathbf{u}\cdot\nabla\partial_x^\beta\partial_t s\right\| \mathrm{d}\tau \\ &\lesssim \int_0^t \|\mathbf{u}\|_3 \|\partial_t s\|_2^2 \mathrm{d}\tau \le C(M)t, \end{aligned}$$

$$\begin{aligned} K_{19} \lesssim & \int_0^t \|\partial_t s\|_2 \Big[\left(\|P\|_2 + \|P\|_2^2\right)\left(\|\partial_t I_0\|_2 + \|\theta\|_2^3\|\partial_t\theta\|_2\right) \\ & + \left(\|P\|_2 + \|P\|_2^2\right)^2 \|\partial_t P\|_2 \left(\|I_0\|_2 + \|\theta\|_2^4\right) + \|\partial_t \mathbf{u}\|_2\|s\|_3 \Big]\mathrm{d}\tau \\ \le & C(M)t, \end{aligned}$$

and

$$\begin{aligned} K_{20} &\lesssim \left(\epsilon + \frac{1}{\epsilon}\right)\int_0^t \|\partial_t s\|_2 \Big[\left(\|P\|_2 + \|P\|_2^2\right)(\|\mathbf{u}\|_2\|\partial_t \mathbf{I}_1\|_2 + \|\partial_t\mathbf{u}\|_2\|\mathbf{I}_1\|_2) \\ &\quad + \left(\|P\|_2 + \|P\|_2^2\right)^2 \|\mathbf{u}\|_2\|\mathbf{I}_1\|_2\|\partial_t P\|_2\Big]\mathrm{d}\tau \\ &\lesssim \int_0^t \left(\epsilon^2 + \frac{1}{\epsilon^2}\right)\sum_{i=0}^3 \left\|\partial_t^i \mathbf{I}_1(\tau)\right\|_{3-i}^2 \mathrm{d}\tau + C(M)t \\ &\le C_0(M_0) + C(M)t. \end{aligned}$$

Combining the estimates on K_{18}-K_{20} and (4.15), we have

$$\|\partial_t s(t)\|_2^2 \le C_0(M_0) + C(M)t. \tag{4.16}$$

Moreover, we apply the operator $\partial_x^\eta \partial_{tt}$ $(0 \le |\eta| \le 1)$ to (1.17), and multiply it by $\partial_x^\eta \partial_{tt} s$ in $L^2(\Omega)$ to achieve

$$\frac{1}{2}\int_\Omega |\partial_x^\eta \partial_{tt} s|^2 \mathrm{d}x = \frac{1}{2}\int_\Omega |\partial_x^\eta \partial_{tt} s|^2(0)\mathrm{d}x + \sum_{j=21}^{23} K_j, \tag{4.17}$$

where

$$
\begin{aligned}
K_{21}= & -\int_0^t \int_{\Omega}\left\{\frac{1}{2} \mathbf{u} \cdot \nabla\left(|\partial_x^\eta \partial_{tt} s|^2\right)+[\partial_x^\eta(\mathbf{u} \cdot \nabla \partial_{tt} s)-\mathbf{u} \cdot \nabla \partial_x^\eta \partial_{tt} s] \partial_x^\eta \partial_{tt} s\right\} \mathrm{d} x \mathrm{d} \tau, \\
K_{22}= & \int_0^t \int_{\Omega} \partial_x^\eta \partial_{tt} s \partial_x^\eta\Big(2 P^{-3}|\partial_t P|^2 I_0-2 \partial_t \mathbf{u} \cdot \nabla \partial_t s-\partial_{tt} \mathbf{u} \cdot \nabla s-P^{-2} \partial_{tt} P I_0 \\
& -2 P^{-2} \partial_t P \partial_t I_0+P^{-1} \partial_{tt} I_0-2 P^{-3}|\partial_t P|^2 \theta^4+P^{-2} \partial_{tt} P \theta^4 \\
& +8 P^{-2} \theta^3 \partial_t P \partial_t \theta-12 P^{-1} \theta^2|\partial_t \theta|^2-4 P^{-1} \theta^3 \partial_{tt} \theta\Big) \mathrm{d} x \mathrm{d} \tau, \\
K_{23}= & \left(\epsilon+\frac{1}{\epsilon}\right) \int_0^t \int_{\Omega} \partial_x^\eta \partial_{tt} s \partial_x^\eta\Big(P^{-2} \partial_{tt} P \mathbf{I}_1 \cdot \mathbf{u}-2 P^{-3}|\partial_t P|^2 \mathbf{I}_1 \cdot \mathbf{u}+2 P^{-2} \partial_t P \partial_t \mathbf{I}_1 \cdot \mathbf{u} \\
& +2 P^{-2} \partial_t P \mathbf{I}_1 \cdot \partial_t \mathbf{u}-P^{-1} \partial_{tt} \mathbf{I}_1 \cdot \mathbf{u}-2 P^{-1} \partial_t \mathbf{I}_1 \cdot \partial_t \mathbf{u}-P^{-1} \mathbf{I}_1 \cdot \partial_{tt} \mathbf{u}\Big) \mathrm{d} x \mathrm{d} \tau .
\end{aligned}
$$

We utilize (2.3), (2.4), (2.5), (3.1), and integration by parts to infer that

$$
\begin{aligned}
K_{21} & \lesssim \int_0^t \int_{\Omega}|\operatorname{div} \mathbf{u}||\partial_x^\eta \partial_{tt} s|^2 \mathrm{d} x \mathrm{d} \tau+\int_0^t\|\partial_x^\eta \partial_{tt} s\|\|\partial_x^\eta(\mathbf{u} \cdot \nabla \partial_{tt} s)-\mathbf{u} \cdot \nabla \partial_x^\eta \partial_{tt} s\| \mathrm{d} \tau \\
& \lesssim \int_0^t\|\mathbf{u}\|_3\|\partial_{tt} s\|_1^2 \mathrm{d} \tau \leq C(M) t,
\end{aligned}
$$

$$
\begin{aligned}
K_{22} \lesssim & \int_0^t\|\partial_{tt} s\|_1\Big[(1+\|P\|_3)\left(\|\partial_{tt} I_0\|_1+\|\theta\|_3^3\|\partial_{tt} \theta\|_1\right)+\|\partial_t \mathbf{u}\|_2\|\partial_t s\|_2+\|\partial_{tt} \mathbf{u}\|_1\|s\|_3 \\
& +\left(\|P\|_2+\|P\|_2^2\right)\|\theta\|_2^2\|\partial_t \theta\|_2(\|\theta\|_2\|\partial_t P\|_2+\|\partial_t \theta\|_2) \\
& +\left(\|P\|_2+\|P\|_2^2\right)^2\|\partial_t P\|_2\|\partial_t I_0\|_2+\left(\|P\|_2+\|P\|_2^2\right)^3\|\partial_t P\|_2^2\left(\|I_0\|_2+\|\theta\|_2^4\right) \\
& +\|\partial_{tt} P\|_1\left(\|P\|_3\|I_0\|_2+\|I_0\|_3+\|P\|_3\|\theta\|_2^4+\|\theta\|_3^4\right)\Big] \mathrm{d} \tau \\
\leq & C(M) t,
\end{aligned}
$$

and

$$
\begin{aligned}
K_{23} \lesssim & \left(\epsilon+\frac{1}{\epsilon}\right) \int_0^t\|\partial_{tt} s\|_1\Big[(1+\|P\|_3)\|\mathbf{I}_1\|_3(\|\mathbf{u}\|_3\|\partial_{tt} P\|_1+\|\partial_{tt} \mathbf{u}\|_1) \\
& +\left(\|P\|_2+\|P\|_2^2\right)\|\partial_t \mathbf{I}_1\|_2\|\partial_t \mathbf{u}\|_2+\left(\|P\|_2+\|P\|_2^2\right)^2\|\partial_t P\|_2\Big(\|\partial_t \mathbf{I}_1\|_2\|\mathbf{u}\|_2 \\
& +\|\mathbf{I}_1\|_2\|\partial_t \mathbf{u}\|_2\Big)+\left(\|P\|_2+\|P\|_2^2\right)^3\|\partial_t P\|_2^2\|\mathbf{I}_1\|_2\|\mathbf{u}\|_2+(1+\|P\|_3)\|\partial_{tt} \mathbf{I}_1\|_1\|\mathbf{u}\|_3\Big] \mathrm{d} \tau \\
\lesssim & \int_0^t\left(\epsilon^2+\frac{1}{\epsilon^2}\right) \sum_{i=0}^3\left\|\partial_t^i \mathbf{I}_1(\tau)\right\|_{3-i}^2 \mathrm{d} \tau+C(M) t \\
\leq & C_0(M_0)+C(M) t .
\end{aligned}
$$

We plug the estimates on K_{21}-K_{23} into (4.17) to dedcue

$$
\|\partial_{tt} s(t)\|_1^2 \leq C_0(M_0)+C(M) t. \tag{4.18}
$$

Finally, we differentiate (1.17) with respect to t three times, multiply the resulting identity by $\partial_{ttt}s$ in $L^2(\Omega)$ to conclude that

$$\frac{1}{2}\int_\Omega |\partial_{ttt}s|^2 \mathrm{d}x = \frac{1}{2}\int_\Omega |\partial_{ttt}s|^2(0)\mathrm{d}x + \sum_{j=24}^{25} K_j, \tag{4.19}$$

where

$$\begin{aligned}
K_{24} = &\int_0^t\int_\Omega \partial_{ttt}s\Big(\mathbf{u}\cdot\nabla\partial_{ttt}s - 3\partial_t\mathbf{u}\cdot\nabla\partial_{tt}s - 3\partial_{tt}\mathbf{u}\cdot\nabla\partial_t s - \partial_{ttt}\mathbf{u}\cdot\nabla s\\
&-6P^{-4}\partial_t P|\partial_t P|^2 I_0 + 6P^{-3}\partial_t P\partial_{tt}PI_0 + 6P^{-3}|\partial_t P|^2\partial_t I_0 - P^{-2}\partial_{ttt}PI_0\\
&-3P^{-2}\partial_{tt}P\partial_t I_0 - 3P^{-2}\partial_t P\partial_{tt}I_0 + P^{-1}\partial_{ttt}I_0 + 6P^{-4}\partial_t P|\partial_t P|^2\theta^4\\
&-6P^{-3}\partial_t P\partial_{tt}P\theta^4 - 24P^{-3}|\partial_t P|^2\theta^3\partial_t\theta + P^{-2}\partial_{ttt}P\theta^4 + 12P^{-2}\partial_{tt}P\theta^3\partial_t\theta\\
&+36P^{-2}\theta^2\partial_t P|\partial_t\theta|^2 + 12P^{-2}\theta^3\partial_t P\partial_{tt}\theta - 24P^{-1}\theta\partial_t\theta|\partial_t\theta|^2\\
&-36P^{-1}\theta^2\partial_t\theta\partial_{tt}\theta - 4P^{-1}\theta^3\partial_{ttt}\theta\Big)\mathrm{d}x\mathrm{d}\tau,
\end{aligned}$$

$$\begin{aligned}
K_{25} = &\left(\epsilon+\frac{1}{\epsilon}\right)\int_0^t\int_\Omega \partial_{ttt}s\Big(P^{-2}\partial_{ttt}P\mathbf{I}_1\cdot\mathbf{u} - 6P^{-3}\partial_t P\partial_{tt}P\mathbf{I}_1\cdot\mathbf{u} + 3P^{-2}\partial_{tt}P\partial_t\mathbf{I}_1\cdot\mathbf{u}\\
&+3P^{-2}\partial_{tt}P\mathbf{I}_1\cdot\partial_t\mathbf{u} + 6P^{-4}\partial_t P|\partial_t P|^2\mathbf{I}_1\cdot\mathbf{u} - 6P^{-3}|\partial_t P|^2\partial_t\mathbf{I}_1\cdot\mathbf{u} - 6P^{-3}|\partial_t P|^2\mathbf{I}_1\cdot\partial_t\mathbf{u}\\
&+3P^{-2}\partial_t P\partial_{tt}\mathbf{I}_1\cdot\mathbf{u} + 6P^{-2}\partial_t P\partial_t\mathbf{I}_1\cdot\partial_t\mathbf{u} + 3P^{-2}\partial_t P\mathbf{I}_1\cdot\partial_{tt}\mathbf{u} - P^{-1}\partial_{ttt}\mathbf{I}_1\cdot\mathbf{u}\\
&-3P^{-1}\partial_{tt}\mathbf{I}_1\cdot\partial_t\mathbf{u} - 3P^{-1}\partial_t\mathbf{I}_1\cdot\partial_{tt}\mathbf{u} - P^{-1}\mathbf{I}_1\cdot\partial_{ttt}\mathbf{u}\Big)\mathrm{d}x\mathrm{d}\tau.
\end{aligned}$$

Similar as the terms K_{21}-K_{23}, K_{24} and K_{25} can be bounded as follows

$$\begin{aligned}
K_{24} \lesssim &\int_0^t \|\partial_{ttt}s\|\Big(\|\partial_{ttt}s\|\,\|\mathbf{u}\|_3 + \|\partial_t\mathbf{u}\|_2\,\|\partial_{tt}s\|_1 + \|\partial_{tt}\mathbf{u}\|_1\,\|\partial_t s\|_2 + \|\partial_{ttt}\mathbf{u}\|\,\|s\|_3\\
&+\|\partial_t P\|_2^3\,\|I_0\| + \|\partial_t P\|_2\,\|I_0\|_2\,\|\partial_{tt}P\| + \|\partial_t P\|_2^2\,\|\partial_t I_0\| + \|\partial_{ttt}P\|\,\|I_0\|_2\\
&+\|\partial_{tt}P\|\,\|\partial_t I_0\|_2 + \|\partial_t P\|_2\,\|\partial_{tt}I_0\| + \|\partial_{ttt}I_0\| + \|\partial_t P\|_2^3\,\|\theta\|_2^4 + \|\partial_t P\|_2\,\|\partial_{tt}P\|\,\|\theta\|_2^4\\
&+\|\partial_t P\|_2^2\,\|\theta\|_2^3\,\|\partial_t\theta\| + \|\partial_{ttt}P\|\,\|\theta\|_2^4 + \|\partial_{tt}P\|\,\|\theta\|_2^3\,\|\partial_t\theta\|_2 + \|\partial_t P\|\,\|\theta\|_2^2\,\|\partial_t\theta\|_2^2\\
&+\|\partial_t P\|_2\,\|\theta\|_2^3\,\|\partial_{tt}\theta\| + \|\theta\|_2\,\|\partial_t\theta\|_2^3 + \|\theta\|_2^2\,\|\partial_t\theta\|_2\,\|\partial_{tt}\theta\| + \|\theta\|_2^3\,\|\partial_{ttt}\theta\|\Big)\mathrm{d}\tau\\
\lesssim &\, C(M)t,
\end{aligned}$$

and

$$\begin{aligned}
K_{25} \lesssim &\left(\epsilon+\frac{1}{\epsilon}\right)\int_0^t \|\partial_{ttt}s\|\Big[\|\mathbf{I}_1\|_2\Big(\|\partial_t P\|_2\,\|\partial_{tt}P\|\,\|\mathbf{u}\|_2 + \|\partial_{ttt}P\|\,\|\mathbf{u}\|_2\\
&+\|\partial_{tt}P\|\,\|\partial_t\mathbf{u}\|_2 + \|\partial_t P\|_2^3\,\|\mathbf{u}\|_2 + \|\partial_t P\|_2^2\,\|\partial_t\mathbf{u}\| + \|\partial_t P\|_2\,\|\partial_{tt}\mathbf{u}\| + \|\partial_{ttt}\mathbf{u}\|\Big)\\
&+\|\partial_t\mathbf{I}_1\|_2\Big(\|\partial_{tt}P\|\,\|\mathbf{u}\|_2 + \|\partial_t P\|_2^2\,\|\mathbf{u}\|_2 + \|\partial_t P\|_2\,\|\partial_t\mathbf{u}\| + \|\partial_{tt}\mathbf{u}\|\Big)\\
&+\|\partial_{tt}\mathbf{I}_1\|\,(\|\partial_t P\|_2\,\|\mathbf{u}\|_2 + \|\partial_t\mathbf{u}\|_2) + \|\partial_{ttt}\mathbf{I}_1\|\,\|\mathbf{u}\|_2\Big]\mathrm{d}\tau
\end{aligned}$$

$$\begin{aligned}&\lesssim \int_0^t \left(\epsilon^2+\frac{1}{\epsilon^2}\right)\sum_{i=0}^{3}\left\|\partial_t^i \mathbf{I}_1(\tau)\right\|_{3-i}^2 \mathrm{d}\tau + C(M)t\\&\leq C_0(M_0)+C(M)t.\end{aligned}$$

We insert the above estimates into (4.19) to derive

$$\|\partial_{ttt}s\|^2 \leq C_0(M_0)+C(M)t. \tag{4.20}$$

Then (4.9) follows from (4.14), (4.16), (4.18), and (4.20). □

With Lemmas 3.1-4.5 in hand, we can deduce Theorem 1.3 immediately.

5. The proof of Theorem 1.2

The proof of Theorem 1.2 is actually based on [1, 10, 16, 19]. For the completeness of the paper, we sketch the proof here. Assume that Theorem 1.3 holds and $T^\epsilon < +\infty$ is the maximal life time of existence for the solution deduced in Theorem 1.1. Then we have for any $0 \leq t \leq \min\{1, T^\epsilon\}$ that

$$\begin{aligned}M^\epsilon(t) &\leq C_0(M^\epsilon(0)) + C(M^\epsilon(t))t\\&\leq C_0(M^\epsilon(0))\exp(C(M^\epsilon(t))t),\end{aligned} \tag{5.1}$$

where $M^\epsilon(0) \leq C_2$ for $0<\epsilon\leq 1$. In the sequence, we choose $C_3 > C_0(C_2)$ and next $T_1 \leq 1$ such that

$$C_0(C_2)\exp(T_1 C(C_3)) < C_3. \tag{5.2}$$

Let $t < \min\{T_1, T^\epsilon\}$, with (5.1) and (5.2) in hand, we can assert that $M^\epsilon(t) \neq C_3$. Besides, we can assume without restriction that $C_2 \leq C_3$, which implies that $M^\epsilon(0) \leq C_3$. Noticing the function $M^\epsilon(t)$ is continuous, one has

$$M^\epsilon(t) \leq C_3 \quad \text{for} \quad t < \min\{T_1, T^\epsilon\} \quad \text{and} \quad 0 < \epsilon \leq 1. \tag{5.3}$$

Thus we can claim that $T^\epsilon > T_1$ for $0<\epsilon\leq 1$. Otherwise, with (5.3) and Theorem 1.1 in hand, we can extend the time interval of existence to $[0, T_1]$, which contradicts to the maximality of T^ϵ. Thus, $M^\epsilon(t) \leq C_3$ for any $t \in [0, T_1]$, where T_1 is independent of $\epsilon \in (0, 1]$. Obviously, the conclusion is also true for $T^\epsilon = +\infty$ by applying the same argument. This completes the proof of Theorem 1.2.

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