COMMUN. MATH. SCI.
Vol. 20, No. 6, pp. 1659–1683

ERROR ANALYSIS OF GALERKIN SPECTRAL METHODS FOR NONLINEAR OPTIMAL CONTROL PROBLEMS WITH INTEGRAL CONTROL CONSTRAINT*

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Abstract. The error analysis of Galerkin spectral methods for integral control constrained nonlinear optimal control problems is investigated in this paper. At first, the optimality conditions of the optimal control problem are presented. More precisely, on the basis of the property of projection operator, a priori error analysis of Galerkin spectral discretization is derived. Moreover, a posteriori error analysis of state, control, adjoint state is established rigorously. Furthermore, for this nonlinear problem, detailed a posteriori error analysis of *hp* spectral element discretization for the optimal control problem is also proved. In the end, ample numerical experiments are presented to verify the theoretical analysis of Galerkin spectral discretization by using the efficient gradient projection algorithm.

Keywords. Nonlinear optimal control problem; control constraint; Galerkin spectral methods; a priori error analysis; a posteriori error analysis.

AMS subject classifications. 49J20; 65K10; 65N35.

1. Introduction

In recent decades, the numerical analysis of optimal control problems has already become a very attractive field of research. There exist a great number of works for the purpose of studying optimal control problems [1, 7, 9, 16, 22–24, 29]. More detailed research can be found in Hinze et al. [17], Lions [20], and Tröltzsch [32]. With the development of engineering applications, the problem of nonlinear optimal control has gradually stepped onto the stage of academic research, such as in [5, 6, 14, 18, 21, 25, 28] and so on. Tröltzsch [33] considered a parabolic optimal control problem with a nonlinear boundary condition and constraints on the control and the state. Arada, Casas and Tröltzsch [2] studied error analysis of distributed nonlinear optimal control problems governed by semilinear elliptic partial differential equations with pointwise constraint on the control. Liu and Yan [26] investigated a posteriori error analysis of nonlinear control problem. In [10], Chen and Lu considered a priori error analysis of semilinear parabolic control problem which is approximated by mixed finite element methods.

It is well known that spectral method [3,4,30,34] is very efficient numerical approach for solving partial differential equations, because it can obtain fast convergence and high order accuracy when the solutions have higher regularity. We must mention the works of Shen, Tang, and Wang [31]. Recently, spectral discretization for linear control problem has already been discussed in [15] successfully. In [11,12], a priori and a posteriori error estimates of Galerkin spectral approximation for optimal control problem with elliptic

*Received: January 08, 2021; Accepted (in revised form): January 17, 2022. Communicated by Jie Shen.
This work is supported by the State Key Program of National Natural Science Foundation of China (11931003) and National Natural Science Foundation of China (41974133, 11971410).
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PDE are proved rigorously. In [8], Chen, Huang, Yi, and Liu investigated the spectral discretization of integral control constrained Stokes optimal control problem, and proved a priori and a posteriori error analysis. In [35, 36], the authors investigated Galerkin spectral method for state constrained optimal control problem, and established a priori error estimates. A posteriori error estimates of hp spectral element methods for elliptic optimal control problem with state constraint are proved in [19], and then a priori and a posteriori error estimates of hp spectral element discretization for first bi-harmonic optimal control problem with integral state constraint are established in [13].

Motivated by above literature, the study of Galerkin spectral discretization for nonlinear optimal control problems is significant, because it can obtain high order accuracy. It is more difficult to study than linear optimal control problems because of the nonlinear function. The novelty of this article is analysing and discussing Galerkin spectral methods for nonlinear optimal control problem. Based on some properties of operators, a priori error analysis of spectral discretization is established in details. Moreover, a posteriori error analysis for this control problem is also deduced in L^2-H^1-norm and L^2-L^2-norm rigorously. Additionally, a posteriori error analysis of hp spectral element discretization is obtained in L^2-H^1-norm and L^2-L^2-norm, respectively. Finally, an efficient projection algorithm is presented, and the theoretical analysis is confirmed by numerical experiments.

Throughout this article, the following control constrained nonlinear elliptic optimal control problem is analyzed:

$$\min_{u\in K} J(y,u)=g(y)+h(u)$$

subject to

$$\begin{aligned} -\Delta y+\psi(y)&=f+u, \quad \text{in } \Omega,\\ y&=0, \quad \text{on } \partial\Omega, \end{aligned}$$

where the constraint set K is

$$K=\{v\in L^2(\Omega):\int_\Omega v\geq 0\}.$$

The details of the problem are presented in Section 2.

The rest of this article is organized as follows. In Section 2, we are going to construct the spectral discretization of the nonlinear optimal control problem. Based on the property of operators, a priori error analysis is derived rigorously in Section 3. Moreover, a posteriori error analysis is proved in L^2-H^1-norm and L^2-L^2-norm for the control problem in detail in Section 4. Furthermore, a posteriori error analysis for the nonlinear control problem discretized by hp spectral element methods in L^2-H^1-norm and L^2-L^2-norm is also investigated in Section 5. In Section 6, the gradient projection algorithm is presented and then some numerical experiments are carried out to verify that the theoretical analysis of Galerkin spectral approximation can obtain spectral accuracy. Finally, a brief conclusion and some future works are summarized in the last section.

There are some basic notations that will be used in the sequel. Let Ω be a bounded open set in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$. Introduce the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p=\sum\limits_{|\alpha|\leq m}\|D^\alpha v\|_{L^p(\Omega)}^p$ and a semi-norm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p=\sum\limits_{|\alpha|=m}\|D^\alpha v\|_{L^p(\Omega)}^p$. Set $W_0^{m,p}(\Omega)=\{w\in W^{m,p}(\Omega):$

$w|_{\partial\Omega}=0\}$. For $p=2$, we denote $H^m(\Omega)=W^{m,2}(\Omega)$, $H_0^m(\Omega)=W_0^{m,2}(\Omega)$, and $\|\cdot\|_m=\|\cdot\|_{m,2}$, $\|\cdot\|=\|\cdot\|_{0,2}$. Additionally, c and C denote different positive constants independent of discrete parameter.

2. Optimal control problems and its spectral approximation

In this section, we study Galerkin spectral approximation for control constrained optimal control problem governed by nonlinear elliptic equation. To begin with, we present a weak formulation and the optimality conditions of optimal control problem.

2.1. Optimal control problems. Let $Y=H_0^1(\Omega)$ be the state space, and $U=L^2(\Omega)$ be the control space, we investigate nonlinear optimal control problem as follows

$$\min_{u\in K} J(u,y)=\frac{1}{2}\int_\Omega (y-y_0)^2+\frac{\alpha}{2}\int_\Omega u^2, \tag{2.1}$$

subject to

$$\begin{aligned} -\Delta y+\psi(y)=f+u, \quad \text{in } \Omega,\\ y=0, \quad \text{on } \partial\Omega, \end{aligned} \tag{2.2}$$

and the constraint set is

$$K=\{v\in L^2(\Omega):\int_\Omega v\geq 0\},$$

where y_0 is the desired state, and α is the regularization parameter. Suppose that the function $\psi(\cdot)\in W^{1,\infty}(-R,R)$, for any $R>0$. $\psi'(y)\in L^2(\Omega)$ for any $y\in H^1(\Omega)$, and $\psi'\geq 0$.

Take

$$A(y,v)=\int_\Omega \nabla y\cdot\nabla v, \quad \forall\ y,v\in Y,$$

$$(p,q)=\int_\Omega pq, \quad \forall\ p,q\in U,$$

and there are two positive constants c and C such that $\forall\ y,v\in Y$

$$A(y,y)\geq c\|y\|_Y^2, \quad |A(y,v)|\leq C\|y\|_Y\|v\|_Y.$$

Thus the weak formulation of (2.2) is: for given functions f and u, seek $y(u)\in Y$ satisfying

$$A(y(u),v)+(\psi(y),v)=(f+u,v), \quad \forall\ v\in Y.$$

Hence, the nonlinear problem (2.1)-(2.2) can be shown as:

$$\min_{u\in K} J(y,u)=\frac{1}{2}\int_\Omega (y-y_0)^2+\frac{\alpha}{2}\int_\Omega u^2, \tag{2.3}$$

$$A(y(u),v)+(\psi(y),v)=(f+u,v), \quad \forall\ v\in Y. \tag{2.4}$$

It is well known [20] that the control problem has a solution (y,u). It follows from Lions [20] that the optimality conditions are presented as follows.

LEMMA 2.1. *If a pair (y,u) is the solution of the optimal control problem (2.3)-(2.4), then there exists an adjoint state $z \in Y$ such that the triplet (y,u,z) satisfies the following optimality conditions: $(OPT-CON)$*

$$\begin{cases} (a)\ A(y,v)+(\psi(y),v)=(f+u,v), \quad \forall\ v\in Y, \\ (b)\ A(q,z)+(\psi'(y)z,q)=(y-y_0,q), \quad \forall\ q\in Y, \\ (c)\ (\alpha u+z,\eta-u)_U\geq 0, \quad \forall\ \eta\in K, \end{cases} \tag{2.5}$$

and

$$\alpha u=\max\{0,\bar{z}\}-z=\theta-z, \tag{2.6}$$

where $\bar{z}=\frac{\int_\Omega z}{\int_\Omega 1}$ means the integral average on Ω.

2.2. Galerkin spectral discretization. We will establish Galerkin spectral discretization of the nonlinear control problem. Let $L_r(x_i)(i=1,2)$ be the r-th degree Legendre polynomial on the variable x_i. Set

$$X_M^i=span\{L_0(x_i),L_1(x_i),\cdots,L_M(x_i)\},\ M\geq 0,$$

and the product space is defined as

$$X_M=\prod_{i=1}^{2}X_M^i.$$

Take the approximation spaces $Y_M=X_M\cap Y$, $U_M=X_M\cap U$, and $K_M=U_M\cap K$. Here K_M and V_M are the spaces of control and state approximations, respectively. Then spectral discretization of the problem (2.3)-(2.4) is given as follows

$$\min_{u_M\in K_M} J_M(y_M,u_M)=\frac{1}{2}\int_\Omega (y_M-y_0)^2+\frac{\alpha}{2}\int_\Omega u_M^2, \tag{2.7}$$

$$A(y_M,v_M)+(\psi(y_M),v_M)=(f+u_M,v_M), \quad \forall\ v_M\in Y_M. \tag{2.8}$$

Analogously, the control problem has a solution (y_M,u_M), and the discrete optimality conditions are also gained as follows.

LEMMA 2.2. *If a pair (y_M,u_M) is the solution of the discrete optimal control problem (2.7)-(2.8), then there exists an adjoint state $z_M\in Y_M$ such that (y_M,u_M,z_M) satisfies the following optimality conditions: $(OPT-CON)_M$*

$$\begin{cases} (a)\ A(y_M,v_M)+(\psi(y_M),v_M)=(f+u_M,v_M), \quad \forall\ v_M\in Y_M, \\ (b)\ A(q_M,z_M)+(\psi'(y_M)z_M,q_M)=(y_M-y_0,q_M), \quad \forall\ q_M\in Y_M, \\ (c)\ (\alpha u_M+z_M,\eta_M-u_M)_U\geq 0, \quad \forall\ \eta_M\in K_M, \end{cases} \tag{2.9}$$

and

$$\alpha u_M=\max\{0,\overline{z_M}\}-z_M=\theta_M-z_M, \tag{2.10}$$

where $\overline{z_M}=\frac{\int_\Omega z_M}{\int_\Omega 1}$ means the integral average on Ω of z_M.

From [27], for Taylor expansion, assume that $\psi(\cdot)\in W^{2,\infty}(-R,R)$, and the property of ψ can be written as follows: for $\varphi\in Y_M$, we get

$$\psi(\varphi)-\psi(\rho)=-\tilde{\psi}'(\varphi)(\rho-\varphi)=-\psi'(\rho)(\rho-\varphi)+\tilde{\psi}''(\varphi)(\rho-\varphi)^2,$$

where

$$\tilde{\psi}'(\varphi)=\int_0^1\psi'(\varphi+s(\rho-\varphi))ds,\quad \tilde{\psi}''(\varphi)=\int_0^1(1-s)\psi''(\rho+s(\varphi-\rho))ds$$

are bounded functions in $\overline{\Omega}$.

3. A priori error analysis

A priori error analysis of Galerkin spectral discretization for the nonlinear optimal control problem will be derived rigorously in this section. Some important results of operators will be introduced to help us to establish the error analysis. They can be found in the reference books [4,31].

LEMMA 3.1. *For any $v\in L^2(\Omega)$, we define the orthogonal projection operator $P_M: L^2(\Omega)\to U_M$, which satisfies*

$$(v-P_Mv,w_M)=0,\quad \forall\ w_M\in U_M.$$

For all $v\in H^m(\Omega)$ $(m\geq 0)$, we have

$$\|v-P_Mv\|_{H^l(\Omega)}\leq CM^{l-m}\|v\|_{H^m(\Omega)},\quad 0\leq l\leq m.$$

LEMMA 3.2. *For any $v\in H_0^1(\Omega)$, we define the projection operator $P_{1,M}^0: H_0^1(\Omega)\to Y_M$ satisfying*

$$\int_\Omega \nabla(v-P_{1,M}^0v)\cdot\nabla w_M=0,\quad \forall\ w_M\in Y_M.$$

If $v\in H_0^1(\Omega)\cap H^m(\Omega), m\geq 1$, then we have that

$$\|v-P_{1,M}^0v\|_{H^\mu(\Omega)}\leq CM^{\mu-m}\|v\|_{H^m(\Omega)},\quad 0\leq\mu\leq 1.$$

The regularity of the control problem is presented as follows [11].

REMARK 3.1. Suppose that (y,z,u) satisfy the $(OPT-CON)$, and assume that f, y_0 and Ω are infinitely smooth, we can get the control $u\in C^\infty(\bar{\Omega})$.

The following auxiliary equations are important to derive a priori error analysis, seek $(y_M(u),\ z_M(u))$ such that

$$A(y_M(u),v_M)+(\psi(y_M(u)),v_M)=(f+u,v_M),\quad \forall\ v_M\in Y_M, \tag{3.1}$$

and

$$A(q_M,z_M(u))+(\psi'(y_M(u))z_M(u),q_M)=(y_M(u)-y_0,q_M),\quad \forall\ q_M\in Y_M. \tag{3.2}$$

The important analytical results are proved in the following theorem.

THEOREM 3.1. *Let (y,z,u) and (y_M,z_M,u_M) be the solutions of optimality conditions $(OPT-CON)$ and $(OPT-CON)_M$, respectively. Suppose that (y,z,u) is sufficiently regular, and then for any $m\geq 1$, there holds*

$$\|u-u_M\|_{L^2(\Omega)}+\|y-y_M\|_{H^1(\Omega)}+\|z-z_M\|_{H^1(\Omega)}\leq CM^{1-m}, \tag{3.3}$$

where $C>0$ is a constant independent of M.

Proof. We obviously know that

$$\|y-y_M\|_{H^1(\Omega)} \le \|y-y(u_M)\|_{H^1(\Omega)} + \|y(u_M)-y_M\|_{H^1(\Omega)}, \tag{3.4}$$

and

$$\|z-z_M\|_{H^1(\Omega)} \le \|z-z(u_M)\|_{H^1(\Omega)} + \|z(u_M)-z_M\|_{H^1(\Omega)}. \tag{3.5}$$

From the substraction of the $(OPT-CON)$ and auxiliary Equations (3.1)-(3.2), we can get

$$A(y-y_M(u),v_M)+(\psi(y)-\psi(y_M(u)),v_M)=0, \quad \forall\ v_M\in Y_M, \tag{3.6}$$

and

$$A(q_M,z-z_M(u))+(\psi'(y)z-\psi'(y_M(u))z_M(u),q_M)=(y-y_M(u),q_M), \quad \forall\ q_M\in Y_M. \tag{3.7}$$

We divided the proof process into the following five steps:

The first step: to estimate $\|y-y_M(u)\|_{H^1(\Omega)}$. Letting $v_M=y-y_M(u)$ in (3.6), we get

$$\begin{aligned}
&c\|y-y_M(u)\|^2_{H^1(\Omega)}\\
\le\ & A(y-y_M(u),y-y_M(u))+(\psi(y)-\psi(y_M(u)),y-y_M(u))\\
=\ & A(y-y_M(u),y-y_M(u)+w_M)+(\psi(y)-\psi(y_M(u)),y-y_M(u)+w_M)\\
=\ & A(y-y_M(u),y-y_M(u)+w_M)+(\widetilde{\psi}'(y)(y-y_M(u)),y-y_M(u)+w_M)\\
\le\ & \|y-y_M(u)\|_{H^1(\Omega)} \inf_{w_M\in Y_M}\|y-w_M\|_{H^1(\Omega)},
\end{aligned}$$

where we employ the fact that $\psi'(\cdot)\in W^{1,\infty}(\Omega)$, and $\psi'(\cdot)\ge 0$, then we obtain

$$\|y-y_M(u)\|_{H^1(\Omega)} \le C\inf_{w_M\in Y_M}\|y-w_M\|_{H^1(\Omega)} \le CM^{1-m}. \tag{3.8}$$

The second step: to estimate $\|z-z_M(u)\|_{H^1(\Omega)}$. Similarly, letting $q_M=z-z_M(u)$ in (3.7), we have

$$\begin{aligned}
&A(z-z_M(u),z-z_M(u))+(\psi'(y)z-\psi'(y_M(u))z_M(u),z-z_M(u))\\
=&(y-y_M(u),z-z_M(u)),
\end{aligned}$$

namely

$$\begin{aligned}
&A(z-z_M(u),z-z_M(u))+(\psi'(y_M(u))z-\psi'(y_M(u))z_M(u),z-z_M(u))\\
=&(y-y_M(u),z-z_M(u))+(\psi'(y_M(u))z-\psi'(y)z,z-z_M(u)),
\end{aligned}$$

combining (2.9) with (3.7), we gain

$$\begin{aligned}
c\|z-z_M(u)\|^2_{H^1(\Omega)} \le\ & A(z-z_M(u),z-z_M(u))+(\psi'(y_M(u))(z-z_M(u)),z-z_M(u))\\
=\ & (y-y_M(u),z-z_M(u))+(\psi'(y_M(u))z-\psi'(y)z,z-z_M(u))\\
\le\ & C\|\psi'(y)-\psi'(y_M(u))\|_{L^2(\Omega)}\|z\|_{H^1(\Omega)}\|z-z_M(u)\|_{H^1(\Omega)}\\
&+C\|z-z_M(u)\|_{L^2(\Omega)}\|y-y_M(u)\|_{L^2(\Omega)}
\end{aligned}$$

where we used the fact that $\psi'(\cdot)\in W^{1,\infty}(\Omega)$, and $\psi'(\cdot)\geq 0$, $\|z\|_{H^1(\Omega)}\leq C$, then the following inequality is obtained

$$\|z-z_M(u)\|_{H^1(\Omega)}\leq C\|y-y_M(u)\|_{L^2(\Omega)}\leq CM^{1-m}. \tag{3.9}$$

It follows from $(OPT-CON)_M$ and (3.1)-(3.2) that

$$A(y_M(u)-y_M,v_M)+(\psi(y_M(u))-\psi(y_M),v_M)=(u-u_M,v_M),\ \forall\ v_M\in Y_M, \tag{3.10}$$

and

$$\begin{aligned}&A(q_M,z_M(u)-z_M)+(\psi'(y_M(u))z_M(u)-\psi'(y_M)z_M,q_M)\\=&(y_M(u)-y_M,q_M),\ \forall\ q_M\in Y_M.\end{aligned} \tag{3.11}$$

The third step: estimating $\|y_M(u)-y_M\|_{H^1(\Omega)}$. Letting $v_M=y_M(u)-y_M$ in (3.10), then

$$\begin{aligned}&A(y_M(u)-y_M,y_M(u)-y_M)+(\psi(y_M(u))-\psi(y_M),y_M(u)-y_M)\\=&(u-u_M,y_M(u)-y_M),\end{aligned}$$

where we used $\psi'(\cdot)\geq 0$, and we have

$$\|y_M(u)-y_M\|_{H^1(\Omega)}\leq C\|u-u_M\|_{L^2(\Omega)}. \tag{3.12}$$

The fourth step: estimating $\|z_M(u)-z_M\|_{H^1(\Omega)}$. Letting $q_M=z_M(u)-z_M$ in (3.11), then

$$\begin{aligned}&A(z_M(u)-z_M,z_M(u)-z_M)+(\psi'(y_M(u))z_M(u)-\psi'(y_M)z_M,z_M(u)-z_M)\\=&(y_M(u)-y_M,z_M(u)-z_M),\end{aligned}$$

the above equation can be restated as

$$\begin{aligned}&A(z_M(u)-z_M,z_M(u)-z_M)+(\psi'(y_M(u))(z_M(u)-z_M),z_M(u)-z_M)\\=&(\psi'(y_M(u))z_M-\psi'(y_M)z_M,z_M(u)-z_M)+(y_M(u)-y_M,z_M(u)-z_M),\end{aligned}$$

then we have

$$\|z_M(u)-z_M\|_{H^1(\Omega)}\leq C\|y_M(u)-y_M\|_{L^2(\Omega)}\leq C\|u-u_M\|_{L^2(\Omega)}, \tag{3.13}$$

where we used

$$\begin{aligned}&(\psi'(y_M(u))z_M-\psi'(y_M)z_M,z_M(u)-z_M)\\\leq&C\|\psi'(y_M(u))z_M-\psi'(y_M)\|_{L^2(\Omega)}\|z_M\|_{H^1(\Omega)}\|z_M(u)-z_M\|_{H^1(\Omega)}\\\leq&C\|y_M(u)-y_M\|_{L^2(\Omega)}\|z_M(u)-z_M\|_{H^1(\Omega)}.\end{aligned}$$

Letting $v_M=z_M(u)-z_M$ in (3.10) and $q_M=y_M(u)-y_M$ in (3.11), then

$$\begin{aligned}&(u-u_M,z_M(u)-z_M)-(\psi(y_M(u))-\psi(y_M),z_M(u)-z_M)\\=&(y_M(u)-y_M,y_M(u)-y_M-(\psi'(y_M(u))z_M(u)-\psi'(y_M)z_M,y_M(u)-y_M),\end{aligned}$$

then the above equation can be written as

$$(u-u_M,z_M(u)-z_M)=(y_M(u)-y_M,y_M(u)-y_M)+(\psi(y_M(u))-\psi(y_M),z_M(u)-z_M)$$

$$-(\psi'(y_M(u))z_M(u)-\psi'(y_M)z_M, y_M(u)-y_M). \tag{3.14}$$

The fifth step: estimating $\|u-u_M\|_{L^2(\Omega)}$. It follows from (2.5)(c), (2.9)(c)that

$$\begin{aligned}
\alpha\|u-u_M\|^2_{L^2(\Omega)} &= \alpha(u-u_M, u-u_M) = (u-u_M, \alpha u - \alpha u_M)\\
&= (u-u_M, \alpha u + z - z + z_M - z_M - \alpha u_M)\\
&= (u-u_M, \alpha u + z) + (u-u_M, z_M - z) - (u-u_M, z_M + \alpha u_M)\\
&= (u-u_M, \alpha u + z) + (u-u_M, z_M - z) - (u-v_M + v_M - u_M, z_M + \alpha u_M)\\
&= -(u_M - u, \alpha u + z) + (u-u_M, z_M - z) - (u-v_M, z_M + \alpha u_M)\\
&\quad -(v_M - u_M, z_M + \alpha u_M)\\
&\leq (u-u_M, z_M - z_M(u)) + (u-u_M, z_M(u) - z).
\end{aligned}$$

Letting $v_M = P_M u \in U_M$, where P_M is defined in Lemma 3.1. Letting $w_M = 1$ in Lemma 3.1, then we have

$$\int (u - P_M u) = 0, \quad \int P_M u = \int u \geq 0,$$

thus, we have $P_M u \in K_M$. Finally, we get

$$\|u-u_M\|_{L^2(\Omega)} \leq C\|z - z_M(u)\|_{L^2(\Omega)}. \tag{3.15}$$

Combining with (3.4)-(3.15) to obtain the result (3.3), the proof is completed. □

4. A posteriori error analysis

We will prove a posteriori error analysis of optimal control problem rigorously in this section. Firstly, the L^2-H^1 posteriori error estimates are established, based on control error approximation using L^2-norm, and the both state error approximation using H^1-norm.

It is necessary to introduce the auxiliary equations as follows

$$A(y(u_M), v) + (\psi(y(u_M)), v) = (f + u_M, v), \quad \forall\ v \in Y, \tag{4.1}$$

and

$$A(q, z(u_M)) + (\psi'(y(u_M))z(u_M), q) = (y(u_M) - y_0, q), \quad \forall\ q \in Y. \tag{4.2}$$

Next, to derive the intermediate equations, from the $(OPT-CON)$ and auxiliary Equations (4.1)-(4.2), we have

$$A(y(u_M) - y, v) + (\psi(y(u_M)) - \psi(y), v) = (u_M - u, v), \quad \forall\ v \in Y, \tag{4.3}$$

and

$$A(q, z(u_M) - z) + (\psi'(y(u_M))z(u_M) - \psi'(y)z, q) = (y(u_M) - y, q), \quad \forall\ q \in Y. \tag{4.4}$$

Letting $v = y(u_M) - y$ in (4.3) to get

$$A(y(u_M) - y, y(u_M) - y) + (\psi(y(u_M)) - \psi(y), y(u_M) - y) = (u_M - u, y(u_M) - y),$$

since $(\psi(y(u_M)) - \psi(y), y(u_M) - y) = (\widetilde{\psi}'(y(u_M))(y(u_M) - y), y(u_M) - y)$ and $\psi'(y) \geq 0$, then we get

$$\|y(u_M) - y\|_{H^1(\Omega)} \leq c_1\|u_M - u\|_{L^2(\Omega)}. \tag{4.5}$$

Setting $q = z(u_M) - z$ in (4.4), we have

$$\begin{aligned} &A(z(u_M)-z, z(u_M)-z) + (\psi'(y(u_M))z(u_M) - \psi'(y(u_M))z, z(u_M)-z) \\ =&(y(u_M)-y, z(u_M)-z) - (\psi'(y(u_M))z - \psi'(y)z, z(u_M)-z). \end{aligned} \tag{4.6}$$

In fact

$$\begin{aligned} &(\psi'(y(u_M))z - \psi'(y)z, z(u_M)-z) \\ =&((\psi'(y(u_M)) - \psi'(y))z, z(u_M)-z) \\ \leq& C\|z\|_{L^\infty(\Omega)} \cdot \|\psi'(y(u_M)) - \psi'(y)\|_{L^2(\Omega)} \cdot \|z(u_M)-z\|_{L^2(\Omega)} \\ \leq& C\|y(u_M)-y\|_{L^2(\Omega)} \cdot \|z(u_M)-z\|_{L^2(\Omega)}. \end{aligned}$$

We can derive

$$\|z(u_M)-z\|_{H^1(\Omega)} \leq c_1 \|y(u_M)-y\|_{L^2(\Omega)} \leq c_1 \|u_M - u\|_{L^2(\Omega)}. \tag{4.7}$$

These intermediate error estimates are important for analysing a posteriori error analysis.

LEMMA 4.1. *Let (y_M, z_M, u_M) and $(y(u_M), z(u_M))$ be the solutions of optimality conditions $(OPT-CON)_M$ and auxiliary Equations (4.1)-(4.2), respectively. Then there holds*

$$\|y(u_M)-y_M\|_{H^1(\Omega)} + \|z(u_M)-z_M\|_{H^1(\Omega)} \leq C(\eta_1 + \eta_2), \tag{4.8}$$

where η_1, η_2 are defined as

$$\eta_1 = M^{-1}\|f + u_M + \Delta y_M - \psi(y_M)\|_{L^2(\Omega)},$$

and

$$\eta_2 = M^{-1}\|y_M - y_0 + \Delta z_M - \psi'(y_M)z_M\|_{L^2(\Omega)}.$$

Proof. In the light of the $(OPT-CON)_M$ and auxiliary Equations (4.1)-(4.2), we have

$$A(y(u_M)-y_M, v_M) + (\psi(y(u_M)) - \psi(y_M), v_M) = 0, \quad \forall\ v_M \in Y_M, \tag{4.9}$$

and

$$\begin{aligned} &A(q_M, z(u_M)-z_M) + (\psi'(y(u_M))z(u_M) - \psi'(y_M)z_M, q_M) \\ =&(y(u_M)-y_M, q_M), \forall\ q_M \in Y_M. \end{aligned} \tag{4.10}$$

Assume that $E_z = z(u_M) - z_M$ and $E_z^M = P_{1,M}^0 E_z$, where $P_{1,M}^0$ is defined in Lemma 3.2. According to (2.9)(b), (4.2), (4.10), we derive that

$$\begin{aligned} c\|z(u_M)-z_M\|_{H^1(\Omega)}^2 \leq& (\nabla E_z, \nabla E_z) + (\psi'(y(u_M))(z(u_M)-z_M), E_z) \\ =&(\nabla(E_z - E_z^M), \nabla E_z) + (\nabla E_z^M, \nabla E_z) \\ &+ (\psi'(y(u_M))z(u_M) - \psi'(y_M)z_M, E_z - E_z^M) \\ &+ (\psi'(y(u_M))z(u_M) - \psi'(y_M)z_M, E_z^M) \\ &+ ((\psi'(y_M) - \psi'(y(u_M)))z_M, E_z) \\ =&(y(u_M) - y_0 + \Delta z_M - \psi'(y_M)z_M, E_z - E_z^M) + (y(u_M) - y_M, E_z^M) \\ &+ ((\psi'(y_M) - \psi'(y(u_M)))z_M, E_z) \\ =&(y_M - y_0 + \Delta z_M - \psi'(y_M)z_M, E_z - E_z^M) + (y(u_M) - y_M, E_z) \\ &+ ((\psi'(y_M) - \psi'(y(u_M)))z_M, E_z) \end{aligned}$$

then applying Lemma 3.2, we get

$$\begin{aligned}c\|z(u_M)-z_M\|^2_{H^1(\Omega)} \le & C\|y_M-y_0+\Delta z_M-\psi'(y_M)z_M\|_{L^2(\Omega)}\|E_z-E_z^M\|_{L^2(\Omega)}\\ &+C\|z_M\|_{L^4(\Omega)}\cdot\|\psi'(y_M)-\psi'(y(u_M))\|_{L^2(\Omega)}\cdot\|E_z\|_{L^4(\Omega)}\\ &+C\|y_M-y(u_M)\|_{L^2(\Omega)}\cdot\|E_z\|_{L^2(\Omega)}\\ \le & C(\delta)M^{-2}\|y_M-y_0+\Delta z_M-\psi'(y_M)z_M\|^2_{L^2(\Omega)}\\ &+C(\delta)\|z_M\|^2_{H^1(\Omega)}\cdot\|\psi'(y_M)-\psi'(y(u_M))\|^2_{L^2(\Omega)}\\ &+C(\delta)\|y_M-y(u_M)\|^2_{L^2(\Omega)}+C\delta\|E_z\|^2_{H^1(\Omega)}\\ \le & C(\delta)M^{-2}\|y_M-y_0+\Delta z_M-\psi'(y_M)z_M\|^2_{L^2(\Omega)}\\ &+C(\delta)\|y_M-y(u_M)\|^2_{L^2(\Omega)}+C\delta\|z(u_M)-z_M\|^2_{H^1(\Omega)},\end{aligned}$$

where we have used the embedding theorem $\|v\|_{L^4(\Omega)}\le C\|v\|_{H^1(\Omega)}$ and $\psi(\cdot)\in W^{1,\infty}(\Omega)$ and the property $\|z_M\|_{H^1(\Omega)}\le C$. Hence we have

$$\begin{aligned}\|z(u_M)-z_M\|_{H^1(\Omega)}\le & CM^{-1}\|y_M-y_0+\Delta z_M-\psi'(y_M)z_M\|_{L^2(\Omega)}\\ &+C\|y_M-y(u_M)\|_{H^1(\Omega)}.\end{aligned} \tag{4.11}$$

Similarly, suppose that $E_y=y(u_M)-y_M$, and $E_y^M=P^0_{1,M}E_y\in Y_M$, where $P^0_{1,M}$ is defined in Lemma 3.2. According to (2.9)(a), (4.1), (4.9), we obtain

$$\begin{aligned}\|y(u_M)-y_M\|^2_{H^1(\Omega)}\le & (\nabla E_y,\nabla E_y)+(\psi(y(u_M))-\psi(y_M),E_y)\\ = & (\nabla(E_y-E_y^M),\nabla E_y)+(\psi(y(u_M))-\psi(y_M),E_y-E_y^M)\\ &+(\nabla E_y^M,\nabla E_y)+(\psi(y(u_M))-\psi(y_M),E_y^M)\\ = & (f+u_M+\Delta y_M-\psi(y_M),E_y-E_y^M)\\ \le & C(\delta)M^{-2}\|f+u_M+\Delta y_M-\psi(y_M)\|^2_{L^2(\Omega)}+\delta\|E_y\|^2_{H^1(\Omega)}\\ \le & C(\delta)M^{-2}\|f+u_M+\Delta y_M-\psi(y_M)\|^2_{L^2(\Omega)}+\delta\|y(u_M)-y_M\|^2_{H^1(\Omega)}.\end{aligned}$$

Hence we get

$$\|y(u_M)-y_M\|_{H^1(\Omega)}\le CM^{-1}\|f+u_M+\Delta y_M-\psi(y_M)\|_{L^2(\Omega)}. \tag{4.12}$$

Combing with (4.11)-(4.12), we arrive at

$$\|y(u_M)-y_M\|_{H^1(\Omega)}+\|z(u_M)-z_M\|_{H^1(\Omega)}\le C(\eta_1+\eta_2),$$

which completes the proof. □

THEOREM 4.1. *Let (y,z,u) and (y_M,z_M,u_M) be the solutions of $(OPT-CON)$ and $(OPT-CON)_M$, respectively. Then we derive*

$$\|u-u_M\|_{L^2(\Omega)}+\|y-y_M\|_{H^1(\Omega)}+\|z-z_M\|_{H^1(\Omega)}\le C(\eta_1+\eta_2), \tag{4.13}$$

where η_1,η_2 are presented in Lemma 4.1.

Proof. From [26], assume that the objective functional $J(\cdot)$ is locally convex in the neighborhood of the solution, then there exists a constant $c>0$ satisfying

$$(J'(v)-J'(w),v-w)\ge c\|v-w\|^2_{L^2(\Omega)},$$

then

$$\begin{aligned} c\|u-u_M\|^2_{L^2(\Omega)} \le & (J'(u),u-u_M)-(J'(u_M),u-u_M) \\ \le & -(J'(u_M),u-u_M) \\ = & (J'_M(u_M),u_M-u)+(J'_M(u_M)-J'(u_M),u-u_M) \\ \le & (\alpha u_M+z_M,v_M-u)+(J'_M(u_M)-J'(u_M),u-u_M) \\ = & (J'_M(u_M)-J'(u_M),u-u_M) \\ = & (z_M-z(u_M),u-u_M) \\ \le & C\|z(u_M)-z_M\|_{L^2(\Omega)}\|u-u_M\|_{L^2(\Omega)}. \end{aligned}$$

Here, using the following results:

$$(J'(u_M),u-w)=(\alpha u_M+z(u_M),u-w),$$

and

$$(J'_M(u_M),u_M-w_M)=(\alpha u_M+z_M,u_M-w_M),$$

letting $v_M=P_M u\in U_M$, where P_M is defined in Lemma 3.1, then we have $v_M=P_M u\in K_M$. Hence we arrive at

$$\|u-u_M\|_{L^2(\Omega)}\le C\|z_M-z(u_M)\|_{L^2(\Omega)}. \tag{4.14}$$

Employing the triangle inequality, we obtain

$$\|y-y_M\|_{H^1(\Omega)}\le\|y-y(u_M)\|_{H^1(\Omega)}+\|y(u_M)-y_M\|_{H^1(\Omega)}, \tag{4.15}$$

and

$$\|z-z_M\|_{H^1(\Omega)}\le\|z-z(u_M)\|_{H^1(\Omega)}+\|z(u_M)-z_M\|_{H^1(\Omega)}. \tag{4.16}$$

It follows from (4.15)-(4.16), and Lemma 4.1 that the error result (4.13) is proved. □

In real applications, we are mostly interested in computing the error estimates using L^2-norm to derive the estimators. We need to introduce the auxiliary problems to derive error analysis:

$$-\Delta\xi+\Psi\xi=f_1,\ \text{in}\ \Omega,\quad \xi|_{\partial\Omega}=0, \tag{4.17}$$

and

$$-\Delta\zeta+\psi'(y(u_M))\zeta=f_2,\ \text{in}\ \Omega,\quad \zeta|_{\partial\Omega}=0, \tag{4.18}$$

where

$$\Psi=\begin{cases}\dfrac{\psi(y(u_M))-\psi(y_M)}{y(u_M)-y_M},\ \text{if}\ y(u_M)\ne y_M,\\ \psi'(y_M),\ \text{if}\ y(u_M)=y_M.\end{cases}$$

From [23], the regularity results are given as follows:

LEMMA 4.2. *Let ξ, ζ be the solutions of (4.17)-(4.18), respectively. Suppose that $\psi\in W^{1,\infty}[0,\infty),\psi'\ge\gamma_0>0$ and Ω is a convex domain. Then we get*

$$\|\xi\|_{H^2(\Omega)}\le C\|f_1\|_{L^2(\Omega)},$$

and

$$\|\zeta\|_{H^2(\Omega)} \leq C\|f_2\|_{L^2(\Omega)}.$$

The following intermediate error estimates will be obtained, which are key to prove a posteriori error analysis.

LEMMA 4.3. *Let* (y_M, z_M, u_M) *and* $(y(u_M), z(u_M))$ *be the solutions of* $(OPT-CON)_M$ *and auxiliary system* (4.1)-(4.2)*, respectively. Then there holds*

$$\|y(u_M) - y_M\|_{L^2(\Omega)} + \|z(u_M) - z_M\|_{L^2(\Omega)} \leq C\eta_3, \tag{4.19}$$

where η_3 *is defined as*

$$\eta_3 = M^{-2}\|y_M - y_0 + \Delta z_M - \psi'(y_M)z_M\|_{L^2(\Omega)} + M^{-2}\|f + u_M + \Delta y_M - \psi(y_M)\|_{L^2(\Omega)}.$$

Proof. Suppose that ζ is the solution of (4.18) with $f_2 = z(u_M) - z_M$ and let $P^0_{1,M}$ be the projection operator defined in Lemma 3.2. According to (2.9)(b) and (4.2), (4.10), there holds

$$\begin{aligned}
\|z(u_M) - z_M\|^2_{L^2(\Omega)} =& (\nabla\zeta, \nabla(z(u_M) - z_M)) + (\psi'(y(u_M))(z(u_M) - z_M), \zeta) \\
=& (\nabla\zeta, \nabla(z(u_M) - z_M)) + (\psi'(y(u_M))z(u_M) - \psi'(y_M)z_M, \zeta) \\
&+ ((\psi'(y_M) - \psi'(y(u_M)))z_M, \zeta) \\
=& (\nabla(\zeta - P^0_{1,M}\zeta), \nabla(z(u_M) - z_M)) + (\nabla(P^0_{1,M}\zeta), \nabla(z(u_M) - z_M)) \\
&+ (\psi'(y(u_M)z(u_M) - \psi'(y_M)z_M, \zeta - P^0_{1,M}\zeta) \\
&+ (\psi'(y(u_M)z(u_M) - \psi'(y_M)z_M, P^0_{1,M}\zeta) \\
&+ ((\psi'(y_M) - \psi'(y(u_M)))z_M, \zeta) \\
=& (y(u_M) - y_0 + \Delta z_M - \psi'(y_M)z_M, \zeta - P^0_{1,M}\zeta) \\
&+ (y(u_M) - y_M, P^0_{1,M}\zeta) + ((\psi'(y_M) - \psi'(y(u_M)))z_M, \zeta) \\
=& (y_M - y_0 + \Delta z_M - \psi'(y_M)z_M, \zeta - P^0_{1,M}\zeta) + (y(u_M) - y_M, \zeta) \\
&+ ((\psi'(y_M) - \psi'(y(u_M)))z_M, \zeta)
\end{aligned}$$

then employing the Lemma 4.2, we obtain

$$\begin{aligned}
c\|z(u_M) - z_M\|^2_{L^2(\Omega)} \leq& C\|y_M - y_0 + \Delta z_M - \psi'(y_M)z_M\|_{L^2(\Omega)}\|\zeta - P^0_{1,M}\zeta\|_{L^2(\Omega)} \\
&+ C\|z_M\|_{L^4(\Omega)} \cdot \|\psi'(y_M) - \psi'(y(u_M))\|_{L^2(\Omega)} \cdot \|\zeta\|_{L^4(\Omega)} \\
&+ C\|y_M - y(u_M)\|_{L^2(\Omega)} \cdot \|\zeta\|_{L^2(\Omega)} \\
\leq& CM^{-2}\|y_M - y_0 + \Delta z_M - \psi'(y_M)z_M\|_{L^2(\Omega)} \cdot \|\zeta\|_{H^2(\Omega)} \\
&+ C\|z_M\|_{H^1(\Omega)} \cdot \|\psi'(y_M) - \psi'(y(u_M))\|_{L^2(\Omega)} \cdot \|\zeta\|_{\infty,\Omega} \\
&+ C\|y_M - y(u_M)\|_{L^2(\Omega)} \cdot \|\zeta\|_{L^2(\Omega)} \\
\leq& C(\delta)M^{-4}\|y_M - y_0 + \Delta z_M - \psi'(y_M)z_M\|^2_{L^2(\Omega)} \\
&+ C(\delta)\|y_M - y(u_M)\|^2_{L^2(\Omega)} + C\delta\|f_2\|^2_{L^2(\Omega)},
\end{aligned}$$

where we have applied the embedding theorem $\|v\|_{L^4(\Omega)} \leq C\|v\|_{1,\Omega}$ and $\psi(\cdot) \in W^{1,\infty}(\Omega)$ and $\|z_M\|_{H^1(\Omega)} \leq C$. Hence we have

$$\|z(u_M) - z_M\|_{L^2(\Omega)} \leq CM^{-2}\|y_M - y_0 + \Delta z_M - \psi'(y_M)z_M\|_{L^2(\Omega)} + C\|y_M - y(u_M)\|_{L^2(\Omega)}. \tag{4.20}$$

Analogously, let ξ be the solution of (4.17) with $f_1 = y(u_M) - y_M$, let $P^0_{1,M}$ be the projection operator defined in Lemma 3.2. It follows from (2.9)(a) and (4.1), (4.9) that

$$\begin{aligned}
\|y(u_M)-y_M\|^2_{L^2(\Omega)} =&(\nabla(y(u_M)-y_M),\nabla\xi)+(\psi(y(u_M))-\psi(y_M),\xi)\\
=&(\nabla(y(u_M)-y_M),\nabla(\xi-P^0_{1,M}\xi))+(\psi(y(u_M))-\psi(y_M),\xi-P^0_{1,M}\xi)\\
&+(\nabla(y(u_M)-y_M),\nabla(P^0_{1,M}\xi))+(\psi(y(u_M))-\psi(y_M),P^0_{1,M}\xi)\\
=&(f+u_M+\Delta y_M-\psi(y_M),\xi-P^0_{1,M}\xi)\\
\le& C\|f+u_M+\Delta y_M-\psi(y_M)\|_{L^2(\Omega)}\|\xi-P^0_{1,M}\xi\|_{L^2(\Omega)}\\
\le& C(\delta)M^{-4}\|f+u_M+\Delta y_M-\psi(y_M)\|^2_{L^2(\Omega)}+c\delta\|\xi\|^2_{H^2(\Omega)}\\
\le& C(\delta)M^{-4}\|f+u_M+\Delta y_M-\psi(y_M)\|^2_{L^2(\Omega)}+c\delta\|y(u_M)-y_M\|^2_{L^2(\Omega)}.
\end{aligned}$$

Hence we have

$$\|y(u_M)-y_M\|_{L^2(\Omega)} \le CM^{-2}\|f+u_M+\Delta y_M-\psi(y_M)\|_{L^2(\Omega)}. \tag{4.21}$$

This combined with (4.20)-(4.21) gives the result. This completes the proof. ∎

THEOREM 4.2. *Let (y,z,u) and (y_M,z_M,u_M) be the solutions of $(OPT-CON)$ and $(OPT-CON)_M$, respectively. Then we derive*

$$\|u-u_M\|_{L^2(\Omega)}+\|y-y_M\|_{L^2(\Omega)}+\|z-z_M\|_{L^2(\Omega)} \le C\eta_3, \tag{4.22}$$

where η_3 is defined in Lemma 4.3.

Proof. It is easy to derive the triangle inequality, we can obtain

$$\begin{aligned}
&\|y-y_M\|_{L^2(\Omega)} \le \|y-y(u_M)\|_{L^2(\Omega)}+\|y(u_M)-y_M\|_{L^2(\Omega)},\\
&\|z-z_M\|_{L^2(\Omega)} \le \|z-z(u_M)\|_{L^2(\Omega)}+\|z(u_M)-z_M\|_{L^2(\Omega)}.
\end{aligned} \tag{4.23}$$

Employing (4.23), and Lemma 4.3, which completes the proof of the analysis result. ∎

5. hp spectral element approximation

In this part, *hp* spectral element method is used to discretize the nonlinear control problem. More details can be referred to [11, 13, 19]. The *hp* spectral element approximation of optimal control problems (OCP) will be developed in this section.

The domain is divided into N_τ nonoverlapping subdomains (elements) $\tau_i, 1 \le i \le N_\tau$:

$$\overline{\Omega} = \bigcup_{i=0}^{N_\tau} \bar{\tau}_i,\ \tau_i \bigcap \tau_j = \emptyset,\ i \ne j,\ 1 \le i,j \le N_\tau.$$

Let $\mathcal{T} = \{\tau\}$ be a local quasi-uniform partitioning of Ω into nonoverlapping regular elements τ. Denote the reference element $\widehat{\tau} = (-1,1)^2$. Let $\mathcal{E}(\mathcal{T})$ be the set of all edges, and let $\mathcal{E}_0(\mathcal{T})$ be the set of all edges which do not lie on the boundary $\partial\Omega$. Each element τ can be the image of the reference element $\widehat{\tau}$ under an affine map $F_\tau : \hat{\tau} \to \tau$. Set $h_\tau :=$ diam τ and assume that the triangulation is γ-shape regular:

$$h_\tau^{-1}\|F'_\tau\| + h_\tau\|(F'_\tau)^{-1}\| \le \gamma. \tag{5.1}$$

For γ-shape regular meshes $\mathcal{T}$ of the domain Ω, we associate a polynomial degree $p_\tau \in \mathbb{N}_0$ with each element $\tau \in \mathcal{T}$; these polynomial degrees $\{p_\tau\}$ are collected into the polynomial degree vector $\boldsymbol{p} = \{p_\tau\}$. Then the spaces of *hp* spectral element approximation $U^{\boldsymbol{p}}(\mathcal{T},\Omega)$, $Y^{\boldsymbol{p}}(\mathcal{T},\Omega)$, $Y^{\boldsymbol{p}}_0(\mathcal{T},\Omega)$ are defined as follows:

$$U^{\boldsymbol{p}}(\mathcal{T},\Omega):=\{u\in L^2(\Omega):u|_\tau\circ F_\tau\in\mathcal{P}_{p_\tau}(\widehat{\tau})\},$$
$$Y^{\boldsymbol{p}}(\mathcal{T},\Omega):=\{u\in H^1(\Omega):u|_\tau\circ F_\tau\in\mathcal{P}_{p_\tau}(\widehat{\tau})\},$$
$$Y_0^{\boldsymbol{p}}(\mathcal{T},\Omega):=Y^{\boldsymbol{p}}(\mathcal{T},\Omega)\cap H_0^1(\Omega),$$

where $\mathcal{P}_{p_\tau}(\widehat{\tau})$ denotes the spaces of polynomials in $\widehat{\tau}$ of degree $\le p_\tau$ in each variable, respectively. For the polynomial degree distribution $\boldsymbol{p}$, similar to (5.1), we assume that the polynomial degrees of neighboring elements are comparable; i.e., there exists a constant $\chi>0$ such that

$$\chi^{-1}(p_\tau+1)\le p'_\tau+1\le\chi(p_\tau+1)\qquad\forall\,\tau,\tau'\in\mathcal{T},\quad\overline{\tau}\cap\overline{\tau}'\neq\emptyset.\tag{5.2}$$

The order of convergence can be obtained either by increasing the degree of the polynomials or by increasing the number of these nonoverlapping regular elements.

Let $K^{\boldsymbol{p}}:=K\bigcap U^{\boldsymbol{p}}$ and $Y^{\boldsymbol{p}}$ be the spaces of the control and state approximation, respectively. The *hp* spectral element discretization of (2.3)-(2.4) reads:

$$\min_{u_{hp}\in K^{\boldsymbol{p}}} J(u_{hp},y_{hp})=\frac{1}{2}\int_\Omega(y_{hp}-y_0)^2+\frac{\alpha}{2}\int_\Omega u_{hp}^2,\tag{5.3}$$

and

$$A(y_{hp},v_{hp})+(\psi(y_{hp}),v_{hp})=(f+u_{hp},v_{hp}),\forall\ v_{hp}\in Y^{\boldsymbol{p}}.\tag{5.4}$$

If a pair (y_{hp},u_{hp}) is the solution of optimal control problem (5.3)-(5.4), then there is an adjoint state $z_{hp}\in Y^{\boldsymbol{p}}$ such that (y_{hp},z_{hp},u_{hp}) satisfies the following optimality conditions $(OPT-CON)_{hp}$:

$$\begin{cases}(a)\ A(y_{hp},v_{hp})+(\psi(y_{hp}),v_{hp})=(f+u_{hp},v_{hp}),\quad\forall\ v_{hp}\in Y^{\boldsymbol{p}},\\(b)\ A(q_{hp},z_{hp})+(\psi'(y_{hp})z_{hp},q_{hp})=(y_{hp}-y_0,q_{hp}),\quad\forall\ q_{hp}\in Y^{\boldsymbol{p}},\\(c)\ (\alpha u_{hp}+z_{hp},\eta_{hp}-u_{hp})_U\ge0,\quad\forall\ \eta_{hp}\in K^{\boldsymbol{p}}.\end{cases}\tag{5.5}$$

In order to derive error analysis, we introduce auxiliary equations to find $(y(u_{hp}),z(u_{hp}))$:

$$\begin{aligned}&(a)\ A(y(u_{hp}),v)+(\psi(y(u_{hp})),v)=(f+u_{hp},v),\quad\forall\ v\in Y,\\&(b)\ A(q,z(u_{hp}))+(\psi'(y(u_{hp}))z(u_{hp}),q)=(y(u_{hp})-y_0,q),\quad\forall\ q\in Y.\end{aligned}\tag{5.6}$$

Next, we will introduce three lemmas [11, 19] which are helpful for analysing a posteriori error analysis.

LEMMA 5.1 (see [11]. Clément type quasi-interpolation). *Let $\mathcal{T}$ be a γ-shape regular triangulation of a domain $\Omega\subset\mathbb{R}^2$ and $\boldsymbol{p}$ be a polynomial degree distribution which is comparable. Then there exists a bounded linear operator $\Pi:L^1(\Omega)\to Y^{\boldsymbol{p}}(\mathcal{T},\Omega)$, and there exists a constant $C>0$, which depends only on γ, such that for every $u\in H^1(\Omega)$ and all elements $\tau\in\mathcal{T}$ and all edges $e\in\mathcal{E}(\tau)$,*

$$\|w-\Pi w\|_{L^2(\tau)}+\frac{h_\tau}{p_\tau}\|\nabla(w-\Pi w)\|_{L^2(\tau)}\le C\frac{h_\tau}{p_\tau}\|\nabla w\|_{L^2(\omega_\tau)},$$
$$\|w-\Pi w\|_{L^2(e)}\le C\sqrt{\frac{h_e}{p_e}}\|\nabla w\|_{L^2(\omega_e)},$$

where h_e is the length of the edge e and $p_e=\max(p_\tau,p_{\tau'})$, where τ,τ' are elements sharing the edge e, ω_τ, ω_e are patches covering τ and e with a few layers, respectively.

LEMMA 5.2 (see [11]. Scott-Zhang type quasi-interpolation). *Let $\mathcal{T}$ be a γ-shape regular triangulation of a domain $\Omega\subset\mathbb{R}^2$ and $\boldsymbol{p}$ be a polynomial degree distribution which is comparable. Then there exists a linear operator $\widetilde{\Pi}:H_0^1(\Omega)\to Y_0^{\boldsymbol{p}}(\mathcal{T},\Omega)$, and there exists a constant $C>0$, which depends only on γ, such that for every $u\in H_0^1(\Omega)$ and all elements $\tau\in\mathcal{T}$ and all edges $e\in\mathcal{E}(\tau)$,*

$$\|w-\widetilde{\Pi}w\|_{L^2(\tau)}+\frac{h_\tau}{p_\tau}\|\nabla(w-\widetilde{\Pi}w)\|_{L^2(\tau)}\le C\frac{h_\tau}{p_\tau}\|\nabla w\|_{L^2(\omega_\tau)},$$

$$\|w-\widetilde{\Pi}w\|_{L^2(e)}\le C\sqrt{\frac{h_e}{p_e}}\|\nabla w\|_{L^2(\omega_e)},$$

where h_e is the length of the edge e and $p_e=\max(p_\tau,p_{\tau'})$, where τ,τ' are elements sharing the edge e, ω_τ, ω_e are patches covering τ and e with a few layers, respectively.

LEMMA 5.3 (see [19]. New Scott-Zhang type quasi interpolation). *Let $\mathcal{T}$ be a $\gamma-shape$ regular triangulation of a domain $\Omega\subset\mathbb{R}^2$, and let $\boldsymbol{p}$ be a polynomial degree distribution which is comparable. Then there exists a bounded linear operator $\Lambda:H_0^1(\Omega)\cap H^2(\Omega)\to Y_0^{\boldsymbol{p}}(\mathcal{T},\Omega)\cap H_0^1(\Omega)$, and there exists a constant $C>0$, which depends only on χ, such that for every $u\in H_0^1(\Omega)\cap H^2(\Omega)$, all elements $\tau\in\mathcal{T}$, and all edges $e\in\mathcal{E}(\tau)$,*

$$\|w-\Lambda w\|_{L^2(\tau)}+\frac{h_\tau}{p_\tau}\|\nabla(w-\Lambda w)\|_{L^2(\tau)}\le C(\frac{h_\tau}{p_\tau})^2|w|_{H^2(\omega_\tau)},$$

$$\|w-\Lambda w\|_{L^2(e)}\le C(\frac{h_e}{p_e})^{\frac{3}{2}}|w|_{H^2(\omega_e)},$$

where h_e is the length of the edge e and $p_e=max(p_\tau,p_{\tau'})$, where τ,τ' are elements sharing the edge e, and w_τ,w_e are patches covering τ and e with a few layers, respectively.

It follows from $(OPT-CON)$ and auxiliary equations that

$$\begin{aligned}&(a)\,A(y-y(u_{hp}),v)+(\psi(y)-\psi(y(u_{hp})),v)=(u-u_{hp},v),\quad\forall\ v\in Y,\\&(b)\,A(q,z-z(u_{hp}))+(\psi'(y)z-\psi'(y(u_{hp}))z(u_{hp}),q)=(y-y(u_{hp}),q),\quad\forall\ q\in Y.\end{aligned}\tag{5.7}$$

Letting $v=y-y(u_{hp})$ in (5.7)(a), which implies

$$\|y-y(u_{hp})\|_{H^1(\Omega)}\le C\|u-u_{hp}\|_{L^2(\Omega)}.\tag{5.8}$$

Letting $q=z-z(u_{hp})$ in (5.7)(b), we get

$$\begin{aligned}&A(z-z(u_{hp}),z-z(u_{hp}))+(\psi'(y)z-\psi'(y(u_{hp}))z(u_{hp}),z-z(u_{hp}))\\&=(y-y(u_{hp}),z-z(u_{hp})),\end{aligned}$$

in fact

$$\begin{aligned}&A(z-z(u_{hp}),z-z(u_{hp}))+(\psi'(y(u_{hp}))(z-z(u_{hp})),z-z(u_{hp}))\\&=(y-y(u_{hp}),z-z(u_{hp}))-((\psi'(y)-\psi'(y(u_{hp})))z,z-z(u_{hp})),\end{aligned}$$

thus

$$\|z-z(u_{hp})\|_{H^1(\Omega)}\le C_1\|y-y(u_{hp})\|_{L^2(\Omega)}\le C_1\|u-u_{hp}\|_{L^2(\Omega)}.\tag{5.9}$$

Letting $v=z-z(u_{hp})$ in (5.7)(a) and $q=y-y(u_{hp})$ in (5.7)(b). Then we get

$$\begin{aligned}(u-u_{hp},z-z(u_{hp}))=&(y-y(u_{hp}),y-y(u_{hp}))-(\psi'(y)z-\psi'(y(u_{hp}))z(u_{hp}),y-y(u_{hp}))\\&-(\psi(y(u_{hp}))-\psi(y),z-z(u_{hp})).\end{aligned}$$

Applying the above notations, we now prove the important result.

LEMMA 5.4. *Assume that (y,z,u) and (y_{hp},z_{hp},u_{hp}) are the solutions of the optimality conditions $(OPT-CON)$ and $(OPT-CON)_{hp}$, respectively. Let $(y(u_{hp}),z(u_{hp}))$ be the solution of auxiliary equation, then there holds*

$$\|u-u_{hp}\|^2_{L^2(\Omega)} \leq C(\eta_4^2+\|z_{hp}-z(u_{hp})\|^2_{L^2(\Omega)}), \tag{5.10}$$

where

$$\eta_4^2 := \sum_{\tau\in\mathcal{T}} \frac{h_\tau^2}{p_\tau^2}\|\nabla(\alpha u_{hp}+z_{hp})\|^2_{L^2(\tau)}.$$

Proof. From [26], assume that the objective functional $J(\cdot)$ is locally convex in the neighborhood of the solution. Let $E^u=u-u_{hp}$, applying (2.5)(c) and (5.5)(c), we get

$$\begin{aligned}
c\|E^u\|^2_{L^2(\Omega)} \leq& (J'(u),E^u)-(J'(u_{hp}),E^u)\\
\leq& -(J'(u_{hp}),E^u)+(\alpha u_{hp}+z_{hp},\eta_{hp}-u_{hp})\\
=&(J'_{hp}(u_{hp}),-E^u)+(J'_{hp}(u_{hp})-J'(u_{hp}),E^u)+(\alpha u_{hp}+z_{hp},\eta_{hp}-u_{hp})\\
=&(\alpha u_{hp}+z_{hp},-E^u)+(z_{hp}-z(u_{hp}),E^u)+(\alpha u_{hp}+z_{hp},\eta_{hp}-u_{hp})\\
=&(\alpha u_{hp}+z_{hp},\eta_{hp}-u)+(z_{hp}-z(u_{hp}),E^u)\\
\leq&(\alpha u_{hp}+z_{hp},\eta_{hp}-u)+C(\delta)\|z_{hp}-z(u_{hp})\|^2_{L^2(\Omega)}+C\delta\|E^u\|^2_{L^2(\Omega)}.
\end{aligned}$$

Suppose that $\eta_{hp}=P_{hp}u$, where P_{hp} is the L^2-projection onto $U^{\boldsymbol{p}}$, and assume that $P_{hp}u\in K^{\boldsymbol{p}}$. Let Π be as defined in Lemma 5.1, we can gain

$$\begin{aligned}
c\|E^u\|^2_{L^2(\Omega)} \leq& \sum_{\tau\in\mathcal{T}}(\alpha u_{hp}+z_{hp},\eta_{hp}-u)_\tau+C(\delta)\|z_{hp}-z(u_{hp})\|^2_{H^1(\Omega)}+C\delta\|E^u\|^2_{L^2(\Omega)}\\
=&\sum_{\tau\in\mathcal{T}}(\alpha u_{hp}+z_{hp}-\Pi(u_{hp}+z_{hp}),P_{hp}(E^u)-(E^u))_\tau\\
&+C(\delta)\|z_{hp}-z(u_{hp})\|^2_{H^1(\Omega)}+\delta\|E^u\|^2_{L^2(\Omega)}\\
\leq& C(\delta)\sum_{\tau\in\mathcal{T}}\frac{h_\tau^2}{p_\tau^2}\|\nabla(\alpha u_{hp}+z_{hp})\|^2_{L^2(\tau)}+C(\delta)\|z_{hp}-z(u_{hp})\|^2_{H^1(\Omega)}\\
&+c\delta\|E^u\|^2_{L^2(\Omega)}.
\end{aligned}$$

Choosing $\delta=\frac{1}{2c}$, we get

$$\|u-u_{hp}\|^2_{L^2(\Omega)} \leq C(\eta_4^2+\|z_{hp}-z(u_{hp})\|^2_{L^2(\Omega)}),$$

which completes the proof. ∎

In the following lemma, the error estimates of the intermediate variables are derived for a posteriori error analysis rigorously.

LEMMA 5.5. *Assume that (y_{hp},z_{hp},u_{hp}) is the solution of the $(OPT-CON)_{hp}$, let $(y(u_{hp}),z(u_{hp}))$ be the solution of the auxiliary equation (5.6). Then there holds*

$$\|y(u_{hp})-y_{hp}\|^2_{H^1(\Omega)}+\|z(u_{hp})-z_{hp}\|^2_{H^1(\Omega)} \leq C(\eta_5^2+\eta_6^2), \tag{5.11}$$

where

$$\eta_5^2 := \sum_{\tau\in\mathcal{T}} \frac{h_\tau^2}{p_\tau^2} \|f + u_{hp} + \Delta y_{hp} - \psi(y_{hp})\|_{L^2(\tau)}^2 + \sum_{e\in\mathcal{E}_0(\mathcal{T})} \frac{h_e}{p_e} \|[\frac{\partial y_{hp}}{\partial n_e}]\|_{L^2(e)}^2,$$

and

$$\eta_6^2 := \sum_{\tau\in\mathcal{T}} \frac{h_\tau^2}{p_\tau^2} \|y_{hp} - y_0 + \Delta z_{hp} - \psi'(y_{hp}) z_{hp}\|_{L^2(\tau)}^2 + \sum_{e\in\mathcal{E}_0(\mathcal{T})} \frac{h_e}{p_e} \|[\frac{\partial z_{hp}}{\partial n_e}]\|_{L^2(e)}^2.$$

Proof. Set $E^z = z(u_{hp}) - z_{hp}$, and $E_I^z = \widetilde{\Pi} E^z$, where $\widetilde{\Pi}$ is as defined in Lemma 5.2, we obtain

$$\begin{aligned} c\|E^z\|_{H^1(\Omega)}^2 \le & A(E^z, E^z) + (\psi'(y(u_{hp}))(z(u_{hp}) - z_{hp}), E^z) \\ = & A(E^z, E^z - E_I^z) + A(E^z, E_I^z) + (\psi'(y(u_{hp})) z(u_{hp}) - \psi'(y_{hp}) z_{hp}, E^z - E_I^z) \\ & + (\psi'(y(u_{hp})) z(u_{hp}) - \psi'(y_{hp}) z_{hp}, E_I^z) + (\psi'(y_{hp}) z_{hp} - \psi'(y(u_{hp})) z_{hp}, E^z) \end{aligned}$$

which then yields

$$\begin{aligned} & c\|z(u_{hp}) - z_{hp}\|_{H^1(\Omega)}^2 \\ \le & \sum_{\tau\in\mathcal{T}} \int_\tau (y_{hp} - y_0 + \Delta z_{hp} - \psi'(y_{hp}) z_{hp})(E^z - E_I^z) + \sum_{e\in\mathcal{E}_0(\mathcal{T})} \int_e [\frac{\partial z_{hp}}{\partial n_e}](E^z - E_I^z) \\ & + (y(u_{hp}) - y_{hp}, E^z) + ((\psi'(y_{hp}) - \psi'(y(u_{hp}))) z_{hp}, E^z) \\ \le & C(\varepsilon)((\sum_{\tau\in\mathcal{T}} \frac{h_\tau^2}{p_\tau^2} \|y_{hp} - y_0 + \Delta z_{hp} - \psi'(y_{hp}) z_{hp}\|_{L(\tau)}^2 + \sum_{e\in\mathcal{E}_0(\mathcal{T})} \frac{h_e}{p_e} \|[\frac{\partial z_{hp}}{\partial n_e}]\|_{L(e)}^2) \\ & + \|y(u_{hp}) - y_{hp}\|_{L^2(\Omega)}^2 + \|y(u_{hp}) - y_{hp}\|_{L^2(\Omega)}^2 \|z_{hp}\|_{H^1(\Omega)}^2) + C\varepsilon\|E^z\|_{H^1(\Omega)}^2 \\ \le & C(\varepsilon)\eta_6^2 + C(\varepsilon)\|y(u_{hp}) - y_{hp}\|_{L^2(\Omega)}^2 + C\varepsilon\|z(u_{hp}) - z_{hp}\|_{H^1(\Omega)}^2, \end{aligned}$$

Hence, we get

$$\|z(u_{hp}) - z_{hp}\|_{H^1(\Omega)}^2 \le C\eta_6^2 + C\|y(u_{hp}) - y_{hp}\|_{L^2(\Omega)}^2. \tag{5.12}$$

Analogously, set $E^y = y(u_{hp}) - y_{hp}$, and $E_I^y = \widetilde{\Pi} E^y$, where $\widetilde{\Pi}$ is as defined in Lemma 5.2. Then, we obtain

$$\begin{aligned} C_1\|E^y\|_{H^1(\Omega)}^2 \le & A(E^y, E^y) + (\psi(y(u_{hp})) - \psi(y_{hp}), E^y) \\ = & (\nabla(E^y - E_I^y), \nabla E^y) + (\psi(y(u_{hp})) - \psi(y_{hp}), E^y - E_I^y) \\ & + (\nabla E_I^y, \nabla E^y) + (\psi(y(u_{hp})) - \psi(y_{hp}), E_I^y)) \\ = & (\nabla(E^y - E_I^y), \nabla E^y) + (\psi(y(u_{hp})) - \psi(y_{hp}), E^y - E_I^y) \\ = & A(E^y - E_I^y, E^y) + (\psi(y(u_{hp})) - \psi(y_{hp}), E^y - E_I^y) \end{aligned}$$

so we get

$$\begin{aligned}
C_1\|y(u_{hp})-y_{hp}\|^2_{H^1(\Omega)} \le & \sum_{\tau\in\mathcal{T}}\int_\tau (f+u_{hp}+\Delta y_{hp}-\psi(y_{hp}))(E^y-E^y_I) \\
& - \sum_{e\in\mathcal{E}_0(\mathcal{T})}\int_e [\frac{\partial y_{hp}}{\partial n_e}](E^y-E^y_I) \\
\le & C\sum_{\tau\in\mathcal{T}}\frac{h_\tau}{p_\tau}\|f+u_{hp}+\Delta y_{hp}-\psi(y_{hp})\|_{L^2(\tau)}\|\nabla E^y\|_{L^2(\omega_\tau)} \\
& + C\sum_{e\in\mathcal{E}_0(\mathcal{T})}\sqrt{\frac{h_e}{p_e}}\|[\frac{\partial y_{hp}}{\partial n_e}]\|_{L^2(e)}\|\nabla E^y\|_{L^2(\omega_e)} \\
\le & C(\vartheta)(\sum_{\tau\in\mathcal{T}}\frac{h_\tau^2}{p_\tau^2}\|\Delta y_{hp}-\psi(y_{hp})+f+u_{hp}\|^2_{L^2(\tau)} \\
& + \sum_{e\in\mathcal{E}_0(\mathcal{T})}\frac{h_e}{p_e}\|[\frac{\partial y_{hp}}{\partial n_e}]\|^2_{L^2(e)}) + c_1\vartheta\|E^y\|^2_{H^1(\Omega)}.
\end{aligned}$$

Choosing that $\vartheta = \frac{1}{2c_1}$, we have

$$\|y(u_{hp})-y_{hp}\|^2_{H^1(\Omega)} \le C\eta_5^2. \tag{5.13}$$

Then, we combine (5.12) and (5.13) to derive the estimates. □

THEOREM 5.1. *Assume that (y,z,u) is the solution of optimality conditions $(OPT-CON)$, and (y_{hp},z_{hp},u_{hp}) be the solution of optimality conditions (5.5). Then there holds*

$$\|y-y_{hp}\|^2_{H^1(\Omega)}+\|u-u_{hp}\|^2_{L^2(\Omega)}+\|z-z_{hp}\|^2_{H^1(\Omega)} \le C(\eta_4^2+\eta_5^2+\eta_6^2),$$

where $\eta_4^2,\eta_5^2,\eta_6^2$ are defined by Lemma 5.4 and Lemma 5.5.

Proof. It follows from triangle inequality that

$$\begin{aligned}
\|y-y_{hp}\|_{H^1(\Omega)} &\le \|y-y(u_{hp})\|_{H^1(\Omega)}+\|y(u_{hp})-y_{hp}\|_{H^1(\Omega)}, \\
\|z-z_{hp}\|_{H^1(\Omega)} &\le \|z-z(u_{hp})\|_{H^1(\Omega)}+\|z(u_{hp})-z_{hp}\|_{H^1(\Omega)}.
\end{aligned}$$

Using Lemma 5.4 and Lemma 5.5 proves the result. □

A posteriori error analysis in L^2-L^2-norms are very important for many applications. In the following, we introduce the auxiliary problems to help us to prove the error analysis.

$$-\Delta\xi+\Psi\xi = f_1, \text{ in } \Omega, \quad \xi|_{\partial\Omega}=0, \tag{5.14}$$

and

$$-\Delta\zeta+\psi'(y(u_{hp}))\zeta = f_2, \text{ in } \Omega, \quad \zeta|_{\partial\Omega}=0, \tag{5.15}$$

where

$$\Psi = \begin{cases} \dfrac{\psi(y(u_{hp}))-\psi(y_{hp})}{y(u_{hp})-y_{hp}}, \text{ if } y(u_{hp})\ne y_{hp}, \\ \psi'(y_{hp}), \text{ if } y(u_{hp})=y_{hp}. \end{cases}$$

Similar to Lemma 4.2, we can get the regularity results.

The following intermediate error estimates will be obtained, which are key to prove a posteriori error analysis.

LEMMA 5.6. *Let (y_{hp},z_{hp},u_{hp}) and $(y(u_{hp}),z(u_{hp}))$ be the solutions of $(OPT-CON)_{hp}$ and auxiliary system (5.6), respectively. Then there holds*

$$\|y(u_{hp})-y_{hp}\|^2_{L^2(\Omega)}+\|z(u_{hp})-z_{hp}\|^2_{L^2(\Omega)}\le C(\eta_7^2+\eta_8^2),$$

where the error indicators η_7^2,η_8^2 are defined as

$$\eta_7^2:=\sum_{\tau\in\mathcal{T}}\frac{h_\tau^4}{p_\tau^4}\|f+u_{hp}+\Delta y_{hp}-\psi(y_{hp})\|^2_{L^2(\tau)}+\sum_{e\in\mathcal{E}_0(\mathcal{T})}\frac{h_e^3}{p_e^3}\|[\frac{\partial y_{hp}}{\partial n_e}]\|^2_{L^2(e)},$$

and

$$\eta_8^2:=\sum_{\tau\in\mathcal{T}}\frac{h_\tau^4}{p_\tau^4}\|y_{hp}-y_0+\Delta z_{hp}-\psi'(y_{hp})z_{hp}\|^2_{L^2(\tau)}+\sum_{e\in\mathcal{E}_0(\mathcal{T})}\frac{h_e^3}{p_e^3}\|[\frac{\partial z_{hp}}{\partial n_e}]\|^2_{L^2(e)}.$$

Proof. Assume that ζ is the solution of (5.15) with $f_2=z(u_{hp})-z_{hp}$ and let Λ be the projection operator defined in Lemma 5.3. We derive that

$$\begin{aligned}\|z(u_{hp})-z_{hp}\|^2_{L^2(\Omega)}=&(\nabla\zeta,\nabla(z(u_{hp})-z_{hp}))+(\psi'(y(u_{hp}))(z(u_{hp})-z_{hp}),\zeta)\\=&(\nabla\zeta,\nabla(z(u_{hp})-z_{hp}))+(\psi'(y(u_{hp}))z(u_{hp})-\psi'(y_{hp})z_{hp},\zeta)\\&+((\psi'(y_{hp})-\psi'(y(u_{hp})))z_{hp},\zeta)\\=&(\nabla(\zeta-\Lambda\zeta),\nabla(z(u_{hp})-z_{hp}))+(\nabla(\Lambda\zeta),\nabla(z(u_{hp})-z_{hp}))\\&+(\psi'(y(u_{hp})z(u_{hp})-\psi'(y_{hp})z_{hp},\zeta-\Lambda\zeta)\\&+(\psi'(y(u_{hp})z(u_{hp})-\psi'(y_{hp})z_{hp},\Lambda\zeta)+((\psi'(y_{hp})-\psi'(y(u_{hp})))z_{hp},\zeta)\end{aligned}$$

then employing Lemma 5.3, we get

$$\begin{aligned}&c\|z(u_{hp})-z_{hp}\|^2_{L^2(\Omega)}\\&\le\sum_{\tau\in\mathcal{T}}\int_\tau(y_{hp}-y_0+\Delta z_{hp}-\psi'(y_{hp})z_{hp})(\zeta-\Lambda\zeta)+\sum_{e\in\mathcal{E}_0(\mathcal{T})}\int_e[\frac{\partial z_{hp}}{\partial n_e}](\zeta-\Lambda\zeta)\\&\quad+(y(u_{hp})-y_{hp},\zeta)+((\psi'(y_{hp})-\psi'(y(u_{hp})))z_{hp},\zeta)\\&\le C(\varepsilon)(\sum_{\tau\in\mathcal{T}}\frac{h_\tau^4}{p_\tau^4}\|y_{hp}-y_0+\Delta z_{hp}-\psi'(y_{hp})z_{hp}\|^2_{L(\tau)}+\sum_{e\in\mathcal{E}_0(\mathcal{T})}\frac{h_e^3}{p_e^3}\|[\frac{\partial z_{hp}}{\partial n_e}]\|^2_{L(e)})\\&\quad+C(\varepsilon)\|y(u_{hp})-y_{hp}\|^2_{L^2(\Omega)}+C(\varepsilon)\|y(u_{hp})-y_{hp}\|^2_{L^2(\Omega)}\|z_{hp}\|^2_{H^1(\Omega)}+C\varepsilon\|\zeta\|^2_{H^2(\Omega)}\\&\le C(\varepsilon)(\sum_{\tau\in\mathcal{T}}\frac{h_\tau^4}{p_\tau^4}\|y_{hp}-y_0+\Delta z_{hp}-\psi'(y_{hp})z_{hp}\|^2_{L(\tau)}+\sum_{e\in\mathcal{E}_0(\mathcal{T})}\frac{h_e^3}{p_e^3}\|[\frac{\partial z_{hp}}{\partial n_e}]\|^2_{L(e)})\\&\quad+C(\varepsilon)\|y(u_{hp})-y_{hp}\|^2_{L^2(\Omega)}+C\varepsilon\|z(u_{hp})-z_{hp}\|^2_{L^2(\Omega)},\end{aligned}$$

where we applied the embedding theorem $\|v\|_{L^4(\Omega)}\le C\|v\|_{1,\Omega}$ and $\psi(\cdot)\in W^{1,\infty}(\Omega)$ and $\|z_{hp}\|_{H^1(\Omega)}\le C$. Hence we have

$$\|z(u_{hp})-z_{hp}\|_{L^2(\Omega)}\le C(\sum_{\tau\in\mathcal{T}}\frac{h_\tau^4}{p_\tau^4}\|y_{hp}-y_0+\Delta z_{hp}-\psi'(y_{hp})z_{hp}\|^2_{L(\tau)}$$

$$+\sum_{e\in\mathcal{E}_0(\mathcal{T})}\frac{h_e^3}{p_e^3}\|[\frac{\partial z_{hp}}{\partial n_e}]\|_{L(e)}^2)+C\|y_{hp}-y(u_{hp})\|_{L^2(\Omega)}$$
$$\leq C\eta_8^2+C\|y_{hp}-y(u_{hp})\|_{L^2(\Omega)}. \tag{5.16}$$

Analogously, let ξ be the solution of (5.14) with $f_1=y(u_{hp})-y_{hp}$, let Λ be the projection operator defined in Lemma 5.3. It follows from (5.3)(a), (5.4) that

$$\begin{aligned}\|y(u_{hp})-y_{hp}\|_{L^2(\Omega)}^2 &=(\nabla(y(u_{hp})-y_{hp}),\nabla\xi)+(\psi(y(u_{hp}))-\psi(y_{hp}),\xi)\\
&=(\nabla(y(u_{hp})-y_{hp}),\nabla(\xi-\Lambda\xi))+(\psi(y(u_{hp}))-\psi(y_{hp}),\xi-\Lambda\xi)\\
&\quad+(\nabla(y(u_{hp})-y_{hp}),\nabla(\Lambda\xi))+(\psi(y(u_{hp}))-\psi(y_{hp}),\Lambda\xi)\\
&=\sum_{\tau\in\mathcal{T}}\int_\tau(f+u_{hp}+\Delta y_{hp}-\psi(y_{hp}))(\xi-\Lambda\xi)\\
&\quad+\sum_{e\in\mathcal{E}_0(\mathcal{T})}\int_e[\frac{\partial y_{hp}}{\partial n_e}](\xi-\Lambda\xi)\\
&\leq C(\delta)\{\sum_{\tau\in\mathcal{T}}\frac{h_\tau^4}{p_\tau^4}\|f+u_{hp}+\Delta y_{hp}-\psi(y_{hp})\|_{L^2(\tau)}^2\\
&\quad+\sum_{e\in\mathcal{E}_0(\mathcal{T})}\frac{h_e^3}{p_e^3}\|[\frac{\partial y_{hp}}{\partial n_e}]\|_{L^2(e)}^2\}+c\delta\|\xi\|_{H^2(\Omega)}^2\\
&\leq C(\delta)\{\sum_{\tau\in\mathcal{T}}\frac{h_\tau^4}{p_\tau^4}\|f+u_{hp}+\Delta y_{hp}-\psi(y_{hp})\|_{L^2(\tau)}^2\\
&\quad+\sum_{e\in\mathcal{E}_0(\mathcal{T})}\frac{h_e^3}{p_e^3}\|[\frac{\partial y_{hp}}{\partial n_e}]\|_{L^2(e)}^2\}+c\delta\|y(u_{hp})-y_{hp}\|_{L^2(\Omega)}^2.\end{aligned}$$

Hence we have

$$\begin{aligned}\|y(u_{hp})-y_{hp}\|_{L^2(\Omega)}^2 &\leq C(\sum_{\tau\in\mathcal{T}}\frac{h_\tau^4}{p_\tau^4}\|f+u_{hp}+\Delta y_{hp}-\psi(y_{hp})\|_{L^2(\tau)}^2\\
&\quad+\sum_{e\in\mathcal{E}_0(\mathcal{T})}\frac{h_e^3}{p_e^3}\|[\frac{\partial y_{hp}}{\partial n_e}]\|_{L^2(e)}^2)\\
&\leq C\eta_7^2.\end{aligned} \tag{5.17}$$

This combined with (5.16), (5.17) gives the theoretical result. ∎

THEOREM 5.2. *Let (y,p,u) and (y_{hp},p_{hp},u_{hp}) be the solutions of $(OPT-CON)$ and $(OPT-CON)_{hp}$, respectively. Then we derive*

$$\|u-u_{hp}\|_{L^2(\Omega)}^2+\|y-y_{hp}\|_{L^2(\Omega)}^2+\|z-z_{hp}\|_{L^2(\Omega)}^2\leq C(\eta_4^2+\eta_7^2+\eta_8^2),$$

where η_4 is defined in Lemma 5.4, η_7,η_8 are defined in Lemma 5.6.

Proof. According to the important triangle inequality, we obtain

$$\|y-y_{hp}\|_{L^2(\Omega)}\leq\|y-y(u_{hp})\|_{L^2(\Omega)}+\|y(u_{hp})-y_{hp}\|_{L^2(\Omega)},$$

$$\|z-z_{hp}\|_{L^2(\Omega)}\leq\|z-z(u_{hp})\|_{L^2(\Omega)}+\|z(u_{hp})-z_{hp}\|_{L^2(\Omega)}.$$

Employing Lemma 5.4 and Lemma 5.6 completes the analysis result. ∎

6. Numerical examples

In this section, we carry out two numerical experiments to confirm the theoretical analysis. The gradient projection algorithm is constructed and designed. The iterative scheme is $(n=0,1,2,\cdots)$

$$\begin{cases} (u^{n+\frac{1}{2}},v)=(u^n,v)-\rho_n(J'(u^n),v), \quad \forall\ v\in K, \\ u^{n+1}=P_K(u^{n+\frac{1}{2}}), \end{cases} \tag{6.1}$$

where ρ_n will be specified in the sequel.

It follows from [11,26] that the convergence result of the scheme is given as follows.

LEMMA 6.1. *Assume that the objective function J is locally uniformly convex near the solution, and J' is locally Lipschitz and monotone near the solution, there exist different positive constants C, c satisfying*

$$|J'(p)-J'(q)|\le C\|p-q\|_U, \quad \forall\ p,q\in U,$$
$$(J'(p)-J'(q),p-q)\ge c\|p-q\|_U^2, \quad \forall\ p,q\in U.$$

Then there exist $0<\delta<1$, $\epsilon>0$ such that

$$\|u-u^n\|\le\delta^n\|u-u^0\|, \quad n=0,1,2,\cdots,$$

provided $\rho_n\le\epsilon$.

REMARK 6.1. To ensure the convergence of the algorithm (6.1), we select suitable ρ_n to satisfy $0\le 1+\rho_n(C\rho_n-c)\le\delta$.

Suppose that $P_{K_M}:U_M\to K_M$ is the discrete projection operator, then it can infer that:

$$(P_{K_M}\omega-\omega,P_{K_M}\omega-\omega)=\min_{u\in K_M}(u-\omega,u-\omega),$$

which is equivalent to

$$(P_{K_M}\omega-\omega,v-P_{K_M}\omega)\ge 0, \quad \forall\ v\in K_M,$$

for given $\omega\in U_M$. It follows from (2.10) that

$$P_{K_M}u_M=-\min\{0,\overline{u_M}\}+u_M,$$

for any $u_M\in U_M$. Then we have by (6.1) that

$$u^{n+1}=P_{K_M}u^{n+\frac{1}{2}}=-\min\{0,\overline{u^{n+\frac{1}{2}}}\}+u^{n+\frac{1}{2}}.$$

In the following, we will investigate the nonlinear optimal control problem with $\alpha=1$:

$$\min_{u\in K}J(y,u)=\frac{1}{2}\int_\Omega(y-y_0)^2+\frac{1}{2}\int_\Omega(u-u_0)^2, \tag{6.2}$$

subject to

$$\begin{aligned} -\Delta y+y^3=f+u, &\quad \text{in } \Omega, \\ y=0, &\quad \text{on } \partial\Omega. \end{aligned} \tag{6.3}$$

Algorithm 1 Algorithm for optimal control problem.

1: Fix $\rho > 0$ and select an initial approximation $u_M^0 \in U_M$, seek $y_M^0 \in Y_M$ such that

$$(\nabla y_M^0, \nabla v_M) + (\psi(y_M^0), v_M) = (f + u_M^0, v_M), \quad \forall\, v_M \in Y_M.$$

2: Find $z_M^n \in Y_M$ such that

$$(\nabla q_M, \nabla z_M^n) + (\psi'(y_M^n) z_M^n, q_M) = (y_M^n - y_0, q_M), \quad \forall\, q_M \in Y_M.$$

3: Seek $u_M^{n+\frac{1}{2}}$ such that

$$(u_M^{n+\frac{1}{2}}, p_M) = (u_M^n, p_M) - \rho(z_M^n + \alpha u_M^n, p_M), \quad \forall\, p_M \in U_M,$$

then

$$u_M^{n+1} = P_{K_M}(u_M^{n+\frac{1}{2}}) = -\min\left(0, \overline{u_M^{n+\frac{1}{2}}}\right) + u_M^{n+\frac{1}{2}}.$$

4: Find $y_M^{n+1} \in Y_M$ such that

$$(\nabla y_M^{n+1}, \nabla v_M) + (\psi(y_M^{n+1}), v_M) = (f + u_M^{n+1}, v_M), \quad \forall\, v_M \in Y_M.$$

5: Stop if stopping criterion $\|u_M^{n+1} - u_M^n\|_{0,\Omega} \le Tol$ is satisfied. Otherwise take $n = n+1$ and then go to Step 2.

6.1. Example one. Based on the gradient projection algorithm, we now consider the problem on domain $\Omega = (-1,1)$ associated with the exact solutions

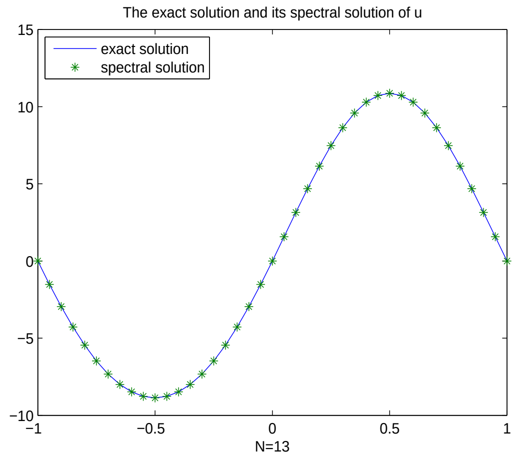
$$\begin{aligned}
&y = \sin \pi x, \quad z = 0,\\
&u = u_0 = -\Delta y + y^3 = \pi^2 \sin \pi x + \sin^3 \pi x,\\
&y_0 = \sin \pi x, \quad f = 0.
\end{aligned}$$

The numerical errors are presented in Table 6.1 to show that the errors decrease rapidly. We also plot the exact and discrete solutions when $M = 13$ in Figure 6.1. Based on Table 6.1 and Figure 6.1, we can get the fact that Galerkin spectral methods obtain high order accuracy for the nonlinear elliptic control problems when the solutions are sufficiently regular.

M	5	7	9	11	13
$\|u - u_M\|_{L^2(\Omega)}$	2.891e-001	6.920e-002	9.300e-003	8.005e-004	5.805e-005
$\|y - y_M\|_{H^1(\Omega)}$	9.200e-003	3.288e-004	7.618e-006	1.210e-007	2.107e-008

TABLE 6.1. *The values of discretization errors.*

6.2. Example two. We now consider the integral control-constrained nonlinear optimal control problem on domain $\Omega = (-1,1) \times (-1,1)$, associated with the exact



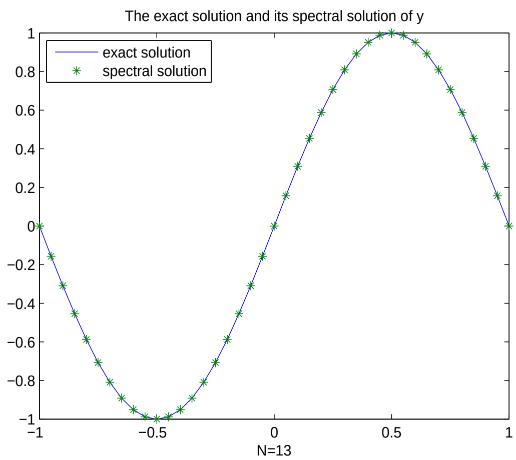


FIG. 6.1. *The exact solutions and its spectral solutions for the one-dimensional case*

solution

$$
\begin{aligned}
&y = \sin \pi x_1 \sin \pi x_2, \quad z = 0, \\
&u = u_0 = -\Delta y + y^3 = 2\pi^2 \sin \pi x_1 \sin \pi x_2 + (\sin \pi x_1 \sin \pi x_2)^3, \\
&y_0 = \sin \pi x_1 \sin \pi x_2, \quad f = 0.
\end{aligned}
$$

M	5	7	9	11	13
$\|u - u_M\|_{L^2(\Omega)}$	3.774e-001	8.480e-002	1.140e-002	9.804e-004	5.885e-005
$\|y - y_M\|_{H^1(\Omega)}$	1.240e-002	4.537e-004	1.061e-005	1.705e-007	2.107e-008

TABLE 6.2. *The values of discretization errors for the two-dimensional case.*

Employing the projection algorithm to solve the example, then the approximation errors are presented in Table 6.2. Finally, some numerical results confirm the theoretical analysis of spectral approximation.

7. Conclusion and future work

The gist of this article is analysing the Galerkin spectral discretization of the integral control constrained nonlinear optimal control problem. More precisely, a priori error estimates for the Galerkin spectral discretization are derived. Next, detailed a posteriori error estimates of the nonlinear control problem are established in L^2-H^1-norm and in L^2-L^2-norm. Additionally, a posteriori error estimates of hp spectral element discretization are also derived in L^2-H^1-norm and in L^2-L^2-norm rigorously. The theoretical analysis of Galerkin spectral approximation is confirmed by the numerical examples by the efficient gradient projection algorithm. The findings of this paper seem to be new, especially, error estimates of nonlinear optimal control problems. The work of this article is important and helpful for the future works.

In our future work, we will investigate more complex nonlinear optimal control problems approximated by Galerkin spectral methods and hp spectral element methods. Furthermore, we shall establish a priori and a posteriori error estimates of control problems.

Acknowledgement. We are very grateful to the anonymous referees for their valuable suggestions which have helped to improve the paper. This work is supported by the State Key Program of National Natural Science Foundation of China (Grant No. 11931003) and National Natural Science Foundation of China (Grant Nos. 41974133 and 11971410).

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