COMMUN. MATH. SCI.
Vol. 20, No. 6, pp. 1685–1701

EXPONENTIAL DECAY FOR A CLASS OF NON-LOCAL NON-LINEAR SCHRÖDINGER EQUATIONS WITH LOCALISED DAMPING*

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Abstract. In this paper we study the exponential decay of both the charge and the free energy for solutions of a family of non-linear, non-local Schrödinger equations with localised damping on the whole line. We first establish an observability inequality for the linear flow, from which we obtain the result in the linear case. Then we consider the non-linear case and by perturbative arguments we obtain the exponential decay for solutions with small initial data. Finally we discuss qualitative aspects of the dynamics and show that the stabilisation rate becomes smaller as the free damping region is chosen around the origin.

Keywords. Stabilisation; localised damping; nonlinear Schrödinger; Hartree potential.

AMS subject classifications. 35Q55; 93D15; 93B05.

1. Introduction

We are mainly concerned with the exponential decay of solutions for the following class of 1-D Schrödinger equation posed in the Sobolev space $\{\phi\in H^1(\mathbb{R}):\int\mu(x)|\phi|^2<\infty\}$, where μ is a positive regular function satisfying $\mu(x)\equiv|x|$, for $|x|>1$:

$$iu_t=-u_{xx}+\mu(x)u-ia(x)u+m(u)u,\, x\in\mathbb{R},\, t>0. \tag{1.1}$$

Here, the term $-ia(x)u$ models the mechanism of dissipation of the system. Since we are interested in damping terms of localised nature, throughout this article we will assume that $a\in W^{1,\infty}(\mathbb{R})$ satisfies $a(x)\geq 0$ for $x\in\mathbb{R}$, $a(x)\equiv 0$ for $x\in[K_1,K_2]$ and $a(x)\geq\alpha$ for $x\in\mathbb{R}\setminus[\widetilde{K_1},\widetilde{K_2}]$, with $[K_1,K_2]\subseteq(\widetilde{K_1},\widetilde{K_2})$. On the other hand, the non-linear term is of non-local nature:

$$m(\phi)(x)=\int\varrho(x,y)|\phi(y)|^2dy,$$

where the kernel satisfies the estimate $|\varrho(x,y)|\leq\mu(y)$.

The model Equation (1.1) was first used in [6] to handle the controllability of the self-consistent 1-D Schrödinger-Poisson equation. The motivation of this choice is as follows: When coupling the Schrödinger equation with the Poisson equation in the whole line, we get a Hartree type potential $V(u)=\frac{1}{2}\left(|x|*\left(\mathcal{D}-|u|^2\right)\right)$, where $\mathcal{D}(x)$ denotes the fixed positively charged background or *impurities*, see [8] and references therein for semiconductor models. After a suitable splitting, $V(u)$ reads as

$$V(u)=q\mu(x)+\int\frac{1}{2}\left(|x-y|-\mu(x)\right)\left(\mathcal{D}(y)-|u(y)|^2\right)dy,$$

where $q=\|\mathcal{D}\|_{L^1(\mathbb{R})}-\|\phi\|^2_{L^2(\mathbb{R})}$ is a constant that depends on the difference between the *impurities* and the size of the initial datum. Besides, due to the regularity requirements

*Received: September 02, 2021; Accepted (in revised form): January 17, 2022. Communicated by Peter Markowich.

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of the unique continuation technique used in Lemma 2.3, the regular function $\mu(x)$ appears as a regularised approximation of a locally constant electric field, which is modelled with $|x|$. It is also worth mentioning that, since the impurities give rise to a bounded potential

$$V_d(x)=\frac{1}{2}\int(|x-y|-\mu(x))\mathcal{D}(y)dy,$$

and hence enters the model equation as a bounded multiplication operator, and since our results remain valid for bounded perturbations, there is no loss of generality by restricting to the case $\mathcal{D}\equiv 0$. Let us finally mention that results on controllability/stabilisation with local nonlinearities as $|u|^{2\sigma}u$ are extensively developed, see [7,9,11], and will therefore not be taken into account.

Regarding the problem of controllability/stabilisation for Schrödinger–like equations, we shall mention the work [1] in which the authors present stabilisation results for the free Schrödinger equation in the complement of a bounded interval. Concerning bounded domains, in [11] the authors show exact controllability results in H^s for the Schrödinger equation

$$iu_t=-\triangle u+\gamma|u|^2u, \qquad x\in\Omega\subseteq\mathbb{R}^N,$$

where $s>N/2$, or $0\leq s<N/2$ with $1\leq N<2s+2$, or $s=0,1$ with $N=2$.

In [12], the author considers the equation in the whole space

$$iu_t=-\triangle u+\lambda|u|^{2/N}u, \qquad x\in\mathbb{R}^N,$$

$N=1,2,3$, with a dissipation term given by a complex constant $\lambda\in\mathbb{C}$, and shows that for $\mathrm{Im}(\lambda)<0$ the solutions with small initial data are globally defined and decay in L^∞ as $(t\log(t))^{-N/2}$.

Concerning the results that include a *localised* damping, we shall mention [4] in which the authors consider a *defocusing* NLS equation

$$iu_t=-\triangle u+|u|^2u-i\eta(x)u, \qquad x\in\mathbb{R}^2,$$

and show the exponential decay for the L^2-norm of solutions (here $\eta\in W^{1,\infty}(\mathbb{R}^2)$ with $\eta(x)\geq\eta_0>0$ for $|x|>1$). In [9], the author establishes the exponential decay of the L^2 norm for the solutions of the cubic-like Schrödinger equation

$$iu_t=-u_{xx}+\lambda|u|^{\alpha-1}u-i\eta(x)u, \qquad x\in\mathbb{R},$$

in both *focusing* and *defocusing* cases ($\lambda<0$ and $\lambda>0$ respectively). Moreover, in [3], the authors treat also the case in which the damping is given by $ib(x)|u|^2u$ and show that the L^2–norm of the solutions decay with a polynomial rate.

It should be noted that the key ingredient in achieving exponential stabilisation is to first establish the existence of an observability inequality, which at the level of L^2, as in the papers cited above, means:

$$\int_0^T\int_{|x|\leq 1}|u(x,t)|^2dxdt\leq C(T)\int_0^T\int\eta(x)|u(x,t)|^2dxdt.$$

Concerning nonlocal interactions with unbounded kernels, such as happens when coupling with the Poisson equation in one spatial dimension, in [6] we treat the problem of exact internal controllability for the nonlocal, nonlinear Schrödinger equation

$$iu_t=-u_{xx}+\mu(x)u+m(u)u, \qquad x\in\mathbb{R},$$

posed in the Sobolev space $\{\phi \in H^1(\mathbb{R}): \int \mu(x)|\phi|^2\}$, in which μ is a positive regular function that coincides with $|x|$ away from the origin. In this work, we show an observability inequality at the $\mathcal{H}$–level which, as we have pointed out above, together with the unique continuation property, are the starting points for the present work. Finally, we mention that in [2] we give numerical evidence of the exponential decay for the solutions of the model equation considered.

The paper is organised as follows. In Section 1 we state the problem. In Section 2, we treat the existence of dynamics and establish the exponential decay of the total charge and the free energy (given by the L^2 and $\mathcal{H}$ norms, respectively) for the linear evolution. Section 3 is devoted to the nonlinear problem: We first establish the local existence of solutions for arbitrary initial data and then show the existence of an invariant set in which the evolution shows exponential decay rates. In Section 4 we deal with the problem of obtaining bounds for the stabilisation rate: We first discuss qualitative aspects of the dynamics and then show that the rate becomes smaller as the free–damping region is chosen closer to the origin.

Notation 1.1. *Throughout this article we will use the notation:*

$$\{f,g\} := \mathrm{Re}\langle f,g\rangle = \mathrm{Re}\int f g^* dx.$$

2. Stabilisation: linear problem

2.1. Existence of dynamics. In order to get the exponential decay we shall show the global existence of the solutions for the linear problem given by the equation:

$$iv_t = -v_{xx} + \mu(x)v(x) - ia(x)v, \qquad x\in\mathbb{R}, t>t_0, \tag{2.1}$$

together with the initial datum $v(t_0)=\phi\in\mathcal{H}$, in which the localised damping satisfies $a\in W^{1,\infty}(\mathbb{R})$, $a(x)\geq 0$ for $x\in\mathbb{R}$, $a(x)\equiv 0$ for $x\in[K_1,K_2]$ and $a(x)\geq\alpha>0$ for $x\in\mathbb{R}\setminus[\widetilde{K_1},\widetilde{K_2}]$, with $[K_1,K_2]\subseteq(\widetilde{K_1},\widetilde{K_2})$:

This problem is posed in the energy space, $\mathcal{H} := \{\phi\in H^1(\mathbb{R}): \|\phi\|^2_{L^2_\mu} := \int|\phi|^2\mu<\infty\}$ with $\|\phi\|^2_{\mathcal{H}} := \|\phi_x\|^2_{L^2} + \|\phi\|^2_{L^2_\mu}$, here $\mu(x)\in C^\infty(\mathbb{R})$, $\mu\geq\max\{|x|,1\}$ and $\mu(x)=|x|$ for $|x|\geq 2$.

We introduce the notation for the linear operator $L: D(L)\to L^2$ given by $L(\phi) := -\phi_{xx}+\mu\phi$. Following [6] we know that L is a self–adjoint operator with compact resolvent; in addition, a straightforward computation shows the identity $\|\phi\|^2_{\mathcal{H}} = \langle\phi, L(\phi)\rangle$ and therefore we have $\mathcal{H}=D(L^{1/2})$, $\mathcal{H}'=D(L^{-1/2})$ and $L:\mathcal{H}\to\mathcal{H}'$.

Since $a\in \mathrm{W}^{1,\infty}$, we have that the damping term a acts as a bounded operator in $\mathcal{H}$. According to a standard perturbative argument, see [10], it follows that $-iL-a$ generates a strongly continuous semigroup.

Thus, we have proved the existence of dynamics:

Theorem 2.1. *For any $\phi\in\mathcal{H}$ there exists a unique solution v of Equation (2.1) satisfying $v(0)=\phi$ and $v\in \mathrm{C}([0,\infty),\mathcal{H})\cap C^1([0,\infty),\mathcal{H}')$. Besides, the solution satisfies: $\|v(t)\|_{\mathcal{H}}\leq C(t)\|\phi\|_{\mathcal{H}}$.*

In order to complete the well–posedness result we shall consider the continuity with respect to the damping term; since a sharper result concerning some continuity in the rates is valid we will give the details after we have proved the exponential decay of the solutions, see Lemma 4.1.

Once the well–posedness result is established, we take into account the exponential decay of the energy, given by the $\mathcal{H}$–norm. To get this result, the main challenge is to manage the current-like terms, such as $\langle a_x v, v_x\rangle$. This will be accomplished by first showing the exponential decay of the total charge, from where a uniform-in-time bound for the kinetic energy $\|v_x\|_{L^2}^2$ is easily deduced, see Corollary 2.1.

2.2. Exponential decay for the total charge. To deal with the evolution of the L^2–norm of solutions to the linear problem (2.1) we develop a useful *observability inequality.* To start with, we present some lemmas:

LEMMA 2.1. *Let $\phi\in\mathcal{H}$ and let $v\in C([0,+\infty),\mathcal{H})$ be the solution of the problem (2.1). Then the total charge satisfies:*

$$\frac{d}{dt}\|v\|_{L^2}^2 = -2\int a(x)|v(x,t)|^2dx \tag{2.2}$$

Proof. It is a straightforward computation and will be omitted. □

LEMMA 2.2 (Observability inequality: L^2-level). *Let $T>0$ be fixed. Then there exists a positive constant $C=C(T)$ such that for any $t_0\in\mathbb{R}$ and $\phi\in\mathcal{H}$ the solution v of (2.1) with initial datum $v(t_0)=\phi$ satisfies the estimate:*

$$\int_{t_0}^{t_0+T}\int_{\mathbb{R}}|v(x,t)|^2dxdt \le C(T)\int_{t_0}^{t_0+T}\int_{\mathbb{R}}a(x)|v(x,t)|^2dxdt. \tag{2.3}$$

Proof. We argue by contradiction. Let us suppose that (2.3) is not true. Let $J=[t_0,t_0+T]$, with $t_0\ge 0$, and let $\phi_k\in\mathcal{H}$ be a sequence of initial data with $\|\phi_k\|_{\mathcal{H}}\le 1$ such that the corresponding solutions of (2.1) $v_k\in C(J,\mathcal{H})$ satisfies the estimate:

$$\int_J\int_{\mathbb{R}}|v_k(x,t)|^2dxdt \ge k\int_J\int_{\mathbb{R}}a(x)|v_k(x,t)|^2dxdt. \tag{2.4}$$

Since previous estimate is equivalent to

$$\int_J\int_{\mathbb{R}}a(x)|v_k(x,t)|^2dxdt \le \frac{1}{k}\int_J\int_{\mathbb{R}}|v_k(x,t)|^2dxdt$$

and since the total charge is a nonincreasing function, we get the uniform bound

$$\int_J\int_{\mathbb{R}}a(x)|v_k(x,t)|^2dxdt \le \frac{1}{k}\int_J\int_{\mathbb{R}}|v_k(x,t)|^2dxdt \le \frac{1}{k}T\|\phi_k\|_{L^2}^2 \le \frac{1}{k}T$$

from where we deduce the convergence, strong in $L^2(J,L^2)$, $a^{1/2}v_k\to 0$ valid for $a\in L^\infty$, and conclude:

$$av_k\to 0. \tag{2.5}$$

Since the sequence of initial data is uniformly bounded and $\mathcal{H}$ being a Hilbert space, there exist $\phi\in\mathcal{H}$ and a subsequence, still denoted $\{\phi_k\}_{k\in\mathbb{N}}$, such that $\phi_k\rightharpoonup\phi$ weak in $\mathcal{H}$; since the embedding $\mathcal{H}\subseteq L^2$ is compact, we have $\phi_k\to\phi$ strong in L^2. Let $\{v_k\}_{k\in\mathbb{N}}$ be the sequence of the solutions satisfying $v_k(t_0)=\phi_k$ $(t_0=\min(J))$, using the well-posedness result of Theorem 2.1 we know that for a fixed $T>0$ the solution satisfies $\|v_k(t)\|_{\mathcal{H}}\le M(T)$, valid for all $t\in J$; in addition, $L^2(J,\mathcal{H})$ being a Hilbert space, there exist $w\in L^2(J,\mathcal{H})$ and a subsequence, still denoted $\{v_k\}_{k\in\mathbb{N}}$, such that $v_k\rightharpoonup w$ weak in $L^2(J,\mathcal{H})$. From the compact embedding $\mathcal{H}\subseteq L^2$ we have $v_k\to w$ strong in $L^2(J,L^2)$ and also, from uniqueness of weak limits in L^2, we have $w(0)=\phi$.

Since $a \in L^\infty$ we get $av_k \to aw$ strong in $L^2(J,L^2)$, taking into account the convergence (2.5) and the uniqueness of the strong limits we deduce that $aw \equiv 0$ in $L^2(J,L^2)$; in addition, using that $w(t) \in \mathcal{H} \subseteq H^1$, for $t \in J$, we conclude that $w(t): \mathbb{R} \to \mathbb{C}$ is a continuous function and therefore $a(x)w(x,t) = 0$ is valid in $\mathbb{R} \times J$. It is worth remarking that $w(x,t) = 0$ is valid for $x \in \mathbb{R} \setminus [\widetilde{K_1}, \widetilde{K_2}]$.

Up to now we have shown that w satisfies both the equation $iw_t = -w_{xx} + \mu w$ and the condition $w(x,t) = 0$ in $[\widetilde{K_1}, \widetilde{K_2}] \times J$, in order to derive a contradiction we are willing to use the unique continuation property; with this in mind we consider the auxiliary problem, where $\psi \in C_0^\infty(\mathbb{R})$ is given by $\psi(x) = 1$ for $x \in [\widetilde{K_1}, \widetilde{K_2}]$ and $\psi(x) = 0$ for $x \in [-1+\widetilde{K_1}, 1+\widetilde{K_2}]$:

$$\begin{aligned} &iW_t = -W_{xx} + \mu(x)\psi(x)W \\ &W(x,t_0) = w(x,t_0). \end{aligned}$$

Since $\psi w = w$ we have that w is a solution of the auxiliary problem, in which the initial datum is of compact support. We then apply the results of Proposition 2.3 of [11] with $A(x,t) = \mu(x)\psi(x) \in C_0^\infty(\mathbb{R})$ and $B(x,t) = 0$ to conclude that $w \in C^\infty(\mathbb{R} \times (t_0, t_0+T))$. By the unique continuation property we deduce $w \equiv 0$ on $\mathbb{R} \times (t_0, t_0+T)$. In view of inequality (2.4) we conclude that this is a contradiction. Then, we have shown the observability inequality. $\square$

Using previous estimates, we can adapt the proof given in [3] about the exponential decay of the L^2-norm of the solutions for the linear problem.

Theorem 2.2. *Let $a \in W^{1,\infty}(\mathbb{R})$ be a localised damping with $a(x) = 0$ for $x \in [K_1, K_2]$ and $a(x) \geq \alpha > 0$ for $x \in \mathbb{R} \setminus [\widetilde{K_1}, \widetilde{K_2}]$, with $[K_1, K_2] \subseteq (\widetilde{K_1}, \widetilde{K_2})$. Then, there are constants C and β such that, for any $\phi \in \mathcal{H}$, the solution $v \in \mathrm{C}([0,+\infty), \mathcal{H})$ of the problem (2.1) with initial datum ϕ, satisfies:*

$$\|v\|_{L^2}(t) \leq Ce^{-\beta t}\|\phi\|_{L^2}.$$

As it was stated before, the proof is similar to the proof of Th. 3.1 in [3] and will be omitted. However, the stronger result for the $\mathcal{H}-$norm is based on a slight generalisation of this result and will be given in detail, see Theorem 2.3.

Below are some useful estimates that follow directly from previous theorem:

Corollary 2.1. *Let $\phi \in \mathcal{H}$ and let $v \in C([0,+\infty), \mathcal{H})$ be the solution of the linear problem (2.1) with initial datum ϕ. Then the following estimates are valid:*

- $\|v_x\|_{L^2}(t) \leq \|\phi\|_{H^1} + C \cdot (\|a_x\|_{L^\infty} + \|\mu_x\|_{L^\infty}) \cdot \|e^{-\beta t}\|_{L^1(0,+\infty)} \cdot \|\phi\|_{L^2}$.
- $\|v\|_{L^\infty}(t) \leq Ce^{-\beta/2t}\|\phi\|_{\mathcal{H}}$.

Proof. The first assertion is obtained as follows. We compute the time derivative of the kinetic energy $\frac{1}{2}\frac{d}{dt}\|v_x\|^2_{L^2} = \{v_{xt}, v_x\}$, see Notation (1.1), and then apply an integration by parts to obtain the identity $\{v_{xt}, v_x\} = -\{v_t, v_{xx}\}$. Thus, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v_x\|^2_{L^2} &= -\{iv_{xx} - i\mu v - av, v_{xx}\} \\ &= -\{i\mu_x v, v_x\} - \{a_x v, v_x\} - \{av_x, v_x\} \end{aligned}$$

from where we deduce

$$\|v_x\|_{L^2} \cdot \frac{d}{dt}\|v_x\|_{L^2} \leq |\{(i\mu_x + a_x)v, v_x\}| \leq (\|\mu_x\|_{L^\infty} + \|a_x\|_{L^\infty}) \cdot \|v\|_{L^2} \cdot \|v_x\|_{L^2},$$

$$\frac{d}{dt}\|v_x\|_{L^2} \le C\cdot(\|\mu_x\|_{L^\infty}+\|a_x\|_{L^\infty})\cdot\|\phi\|_{L^2}\cdot e^{-\beta t}.$$

Integrating in the time interval $[0,t]$, we obtain the first estimate.

The second assertion is a direct consequence of the well known estimate $\|\psi\|^2_{L^\infty} \le C\|\psi_x\|_{L^2}\cdot\|\psi\|_{L^2}$. ∎

2.3. Main result. We turn to the main result of this section, that is the exponential decay at the $\mathcal{H}-$level whose proof relies on the following *observability inequality* at the $\mathcal{H}-$level, see [6], inequality (3.14) of Lemma 3.4 for details.

LEMMA 2.3 (Observability inequality: $\mathcal{H}-$level). *Let $\psi \in \mathrm{W}^{1,\infty}(\mathbb{R})$ be a positive function satisfying $\psi(x)\equiv 0$ for $x\in[K_1,K_2]$ and $\psi(x)\equiv 1$ for $x\in\mathbb{R}\setminus[\widetilde{K_1},\widetilde{K_2}]$, with $[K_1,K_2]\subseteq(\widetilde{K_1},\widetilde{K_2})$. Then for any fixed $T>0$ there exists a constant $C_o=C_o(T)$ such that for any time interval J with $|J|=T$ and any $\phi\in\mathcal{H}$ the solution $v\in\mathrm{C}(J,\mathcal{H})$ of (2.1) with initial datum ϕ satisfies the estimate:*

$$\int_J \|v\|^2_{\mathcal{H}}dt \le C_o\int_J \|\psi v\|^2_{\mathcal{H}}dt. \tag{2.6}$$

The stabilisation result is obtained as follows.

THEOREM 2.3. *Let $a\in\mathrm{W}^{1,\infty}(\mathbb{R})$ be a localised (positive) damping satisfying $\alpha(x)=0$ for $x\in[K_1,K_2]$ and $a(x)\ge\alpha>0$ for $x\in\mathbb{R}\setminus[\widetilde{K_1},\widetilde{K_2}]$, with $[K_1,K_2]\subseteq(\widetilde{K_1},\widetilde{K_2})$. Then there are constants C and η such that for any initial datum ϕ the solution v of (2.1) satisfies the estimate, valid for $t\ge 0$:*

$$\|v\|_{\mathcal{H}}(t)\le Ce^{-\eta t}\|\phi\|_{\mathcal{H}}.$$

Proof. Since we are in the linear case we take $\|\phi\|_{\mathcal{H}}\le 1$ and consider $v\in\mathrm{C}(0,\infty,\mathcal{H})$ the solution of (2.1), with initial datum ϕ. The starting point is the identity

$$\int_0^T \|v\|^2_{\mathcal{H}}dt = \int_0^T \|v_x\|^2_{L^2}dt + \int_0^T \|v\|^2_{L^2_\mu}dt$$

from where we deduce the estimate, in which $R=\mathbb{R}\setminus[\widetilde{K_1},\widetilde{K_2}]$:

$$\begin{aligned}\int_0^T \|v\|^2_{\mathcal{H}}dt &= \int_0^T\int_{[\widetilde{K_1},\widetilde{K_2}]}\left(\mu|v|^2+|v_x|^2\right)dxdt + \int_0^T\int_{x\in R}\left(\mu|v|^2+|v_x|^2\right)dxdt\\ &\le \int_0^T\int_{[\widetilde{K_1},\widetilde{K_2}]}\left(\mu|v|^2+|v_x|^2\right)dxdt + \frac{1}{\alpha}\int_0^T\int_{x\in R} a\left(\mu|v|^2+|v_x|^2\right)dxdt\\ &\le \int_0^T\int_{[\widetilde{K_1},\widetilde{K_2}]}\left(\mu|v|^2+|v_x|^2\right)dxdt + \frac{1}{\alpha}\int_0^T\int_{\mathbb{R}} a\left(\mu|v|^2+|v_x|^2\right)dxdt.\end{aligned}$$

On the other hand, since

$$\frac{d}{dt}\frac{1}{2}\|v\|^2_{\mathcal{H}} = -\{a_x v, v_x\} - \int_{\mathbb{R}} a\left(\mu|v|^2+|v_x|^2\right)dx$$

we have

$$\int_0^T\int_{\mathbb{R}} a\left(\mu|v|^2+|v_x|^2\right)dxdt = -\int_0^T\{a_x v,v_x\}dt + \frac{1}{2}\|v\|^2_{\mathcal{H}}(0) - \frac{1}{2}\|v\|^2_{\mathcal{H}}(T). \tag{2.7}$$

Thus, we combine these results in order to produce the estimate

$$\int_0^T \|v\|_{\mathcal{H}}^2 \leq \int_0^T \int_{[\widetilde{K_1},\widetilde{K_2}]} \left(\mu|v|^2+|v_x|^2\right) - \frac{1}{\alpha}\int_0^T \{a_x v, v_x\} + \frac{1}{2\alpha}\|v\|_{\mathcal{H}}^2(0) - \frac{1}{2\alpha}\|v\|_{\mathcal{H}}^2(T). \tag{2.8}$$

To introduce the observability inequality (2.6), we take $\psi \in C^\infty(\mathbb{R})$ such that $\|\psi^2 - a\|_{W^{1,2}([\widetilde{K_1},\widetilde{K_2}])} < \varepsilon$ and $\psi^2(x) = \alpha$ for $x \notin [\widetilde{K_1},\widetilde{K_2}]$. Since

$$\|\psi v\|_{\mathcal{H}}^2 = \|\psi v\|_{L^2_\mu}^2 + \|\psi v_x\|_{L^2}^2 + \|\psi_x v\|_{L^2}^2 + 2\{\psi\psi_x v, v_x\}$$

we deduce an estimate for the first term in the right-hand side of (2.8):

$$\begin{aligned}\int_0^T \int_{[\widetilde{K_1},\widetilde{K_2}]} \left(\mu|v|^2+|v_x|^2\right) dxdt \leq C_o \int_0^T \int a(x)\left(\mu|v(x,t)|^2+|v_x(x,t)|^2\right) dxdt \\ + C_o\int_0^T \{a_x v, v_x\}(t)dt + C_o\int_0^T \int (\psi_x(x))^2 |v(x,t)|^2 dxdt.\end{aligned}$$

We recall that, in view of Theorem 2.2, the last two terms are uniformly bounded in time; in addition, from Corollary 2.1, we have the following estimate, in which $C_1 = C_1(\|a\|_{\mathrm{W}^{1,\infty}}, \|\mu_x\|_{L^\infty})$ is a constant:

$$\begin{aligned}&\int_0^T \int (\psi_x(x))^2 |v(x,t)|^2 dxdt \leq C_1 \|e^{-2\beta t}\|_{L^1[0,+\infty)}\\ &\int_0^T |\{a_x v, v_x\}(t)|\, dt \leq C_1 \|e^{-\beta t}\|_{L^1[0,+\infty)}.\end{aligned} \tag{2.9}$$

This leads to the estimate

$$\begin{aligned}\int_0^T \int_{x\notin R} \left(\mu|v|^2+|v_x|^2\right) dxdt \leq C_o \int_0^T \int a(x)\left(\mu|v(x,t)|^2+|v_x(x,t)|^2\right) dxdt \\ + C_o C_1 \|e^{-\beta t}\|_{L^1[0,+\infty)}.\end{aligned} \tag{2.10}$$

Since the total energy is not a decreasing function of time, it is necessary to reformulate the arguments of [3] to bound the term $T\|v\|_{\mathcal{H}}^2(T)$. With this in mind we write:

$$\|v\|_{\mathcal{H}}^2(T) = \|v\|_{\mathcal{H}}^2(s) - 2\int_s^T \{a_x v, v_x\}dt - 2\int_s^T \int a\left(\mu|v|^2+|v_x|^2\right) dxdt,$$

valid for $s \in [0,T]$. Integrating in the interval $[0,T]$ and neglecting the last term, we get the estimate

$$T\|v\|_{\mathcal{H}}^2(T) \leq C_2 \|te^{-\beta t}\|_{L^1[0,+\infty)} + \int_0^T \|v\|_{\mathcal{H}}^2(s)ds \tag{2.11}$$

in which the current–like term $\{a_x v, v_x\}$ was treated as in (2.9) and $C_2 = C_2(\|\mu_x\|_{L^\infty}, \|a_x\|_{L^\infty})$ is a constant.

Replacing the last term in (2.11) by the estimate (2.8), neglecting the negative terms and using the estimates (2.9)–(2.10), we obtain

$$T\|v\|_{\mathcal{H}}^2(T) \leq \frac{1}{2\alpha}\|v\|_{\mathcal{H}}^2(0) + C_1\left(C_o + \frac{1}{\alpha}\right)\|e^{-\beta t}\|_{L^1[0,+\infty)} + C_2\|te^{-\beta t}\|_{L^1[0,+\infty)}$$

$$+C_o\int_0^T\int a\left(\mu|v|^2+|v_x|^2\right).$$

Thus, we replace the last term using identity (2.7) to get the following inequality, in which we have used estimate (2.9),

$$\begin{aligned}T\|v\|^2_{\mathcal{H}}(T)\leq &\frac{1}{2\alpha}\|v\|^2_{\mathcal{H}}(0)+\frac{C_o}{2}\|v\|^2_{\mathcal{H}}(0)-\frac{C_o}{2}\|v\|^2_{\mathcal{H}}(T)\\ &+2C_1\,(C_o+\frac{1}{2\alpha})\,\|e^{-\beta t}\|_{L^1[0,+\infty)}+C_2\|te^{-\beta t}\|_{L^1[0,+\infty)}.\end{aligned}\tag{2.12}$$

Next, we introduce the notation $C(T)=2C_1\,(C_o+\frac{1}{2\alpha})\,\|e^{-\beta t}\|_{L^1[0,+\infty)}$ $+C_2\|te^{-\beta t}\|_{L^1[0,+\infty)}$ and write previous expression as follows:

$$\|v\|^2_{\mathcal{H}}(T)\leq\frac{1}{T}\left(\frac{1}{2\alpha}+\frac{C_o}{2}\right)\|v\|^2_{\mathcal{H}}(0)-\frac{C_o}{2T}\|v\|^2_{\mathcal{H}}(T)+\frac{C(T)}{T},$$

from where we conclude

$$\left(1+\frac{C_o}{2T}\right)\|v\|^2_{\mathcal{H}}(T)\leq\frac{1}{T}\left(\frac{1}{2\alpha}+\frac{C_o}{2}\right)\|v\|^2_{\mathcal{H}}(0)+\frac{C(T)}{T}.\tag{2.13}$$

We now recall that we are interested in showing the exponential decay of $\|v\|^2_{\mathcal{H}}(t)$. For this purpose we fix $T>(2\alpha)^{-1}$ and introduce the constants

$$\begin{aligned}\gamma_1&:=\frac{\frac{1}{2\alpha T}+\frac{C_o}{2T}}{1+\frac{C_o}{2T}}=\frac{1+\alpha C_o}{2\alpha T+\alpha C_o}<1,\\ d_1&:=\frac{C(T)}{T\,(1+\frac{C_o}{2T})}=\frac{2C(T)}{2T+C_o},\end{aligned}$$

which lead us to write the bound (2.13) in the form

$$\|v\|^2_{\mathcal{H}}(T)\leq\gamma_1\|v\|^2_{\mathcal{H}}(0)+d_1.\tag{2.14}$$

We obtain the exponential decay by means of an inductive argument, typical from semigroup theory, which consists in getting bounds for time intervals of the form $J_k:=[kT,(k+1)T]$. With this in mind we consider the time interval $[T,2T]$ and adapt previous computations to obtain the following estimate, similar to the one previously given in (2.12) for the interval $[0,T]$,

$$\begin{aligned}2T\|v\|^2_{\mathcal{H}}(2T)\leq&\left(\frac{1}{2\alpha}+\frac{C_o}{2}\right)\|v\|^2_{\mathcal{H}}(T)-\frac{C_o}{2}\|v\|^2_{\mathcal{H}}(2T)\\ &+2(C_o+\frac{1}{2\alpha})\,\|e^{-\beta t}\|_{L^1[T,+\infty)}+C_2\|(t-T)e^{-\beta t}\|_{L^1[T,+\infty)}.\end{aligned}$$

It is worth remarking that the constant $C_o=C_o(T)$ given by Lemma 2.3 depends only on the length of the interval and therefore is the same for all of $J_k=[kT,(k+1)T]$.

Recalling the exponential decay of both terms $\|e^{-\beta t}\|_{L^1[T,+\infty)}=e^{-\beta T}\|e^{-\beta t}\|_{L^1[0,+\infty)}$ and $\|(t-T)e^{-\beta t}\|_{L^1[T,+\infty)}=e^{-\beta T}\|te^{-\beta t}\|_{L^1[0,+\infty)}$, we are led to obtain the bound

$$\left(1+\frac{C_o}{4T}\right)\|v\|^2_{\mathcal{H}}(2T)\leq\frac{1}{2T}\left(\frac{1}{2\alpha}+\frac{C_o}{2}\right)\|v\|^2_{\mathcal{H}}(T)+e^{-\beta T}\frac{C(T)}{2T}.$$

As we did for $[0,T]$, we introduce the constants

$$\gamma_2 := \frac{\frac{1}{4\alpha T} + \frac{C_o}{4T}}{1+\frac{C_o}{4T}} = \frac{1+\alpha C_o}{4\alpha T + \alpha C_o},$$
$$d_2 := \frac{C(T)}{2T\left(1+\frac{C_o}{4T}\right)} = \frac{2C(T)}{4T+C_o},$$

and write

$$\|v\|^2_{\mathcal{H}}(2T) \le \gamma_2 \|v\|^2_{\mathcal{H}}(T) + d_2 e^{-\beta T}. \tag{2.15}$$

Applying inductively previous estimates to the interval $[(k-1)T,kT]$, we get sequences $\{\gamma_k\}_k$ and $\{d_k\}_k$ given by

$$\gamma_k := \frac{\frac{1}{2k\alpha T} + \frac{C_o}{2kT}}{1+\frac{C_o}{2kT}} = \frac{1+\alpha C_o}{2k\alpha T + \alpha C_o},$$
$$d_k := \frac{2C(T)}{2kT\left(1+\frac{C_o}{2kT}\right)} = \frac{2C(T)}{2kT+C_o},$$

such that the following estimate holds

$$\|v\|^2_{\mathcal{H}}((k+1)T) \le \gamma_{k+1} \|v\|^2_{\mathcal{H}}(kT) + d_{k+1} e^{-k\beta T}. \tag{2.16}$$

Since both sequences satisfy $1 > \gamma_1 > \gamma_2 > \cdots > \gamma_k$ and $d_1 > d_2 > \cdots > d_k$, by means of an inductive argument we get the upper bound, in which $\zeta = \max\{\gamma_1, e^{-\beta T}\} < 1$,

$$\|v\|^2_{\mathcal{H}}((k+1)T) \le \zeta^k \left(\gamma_1 \|v\|^2_{\mathcal{H}}(0) + (k+1)d_1\right)$$

from where we deduce $\left(\|v\|^2_{\mathcal{H}}((k+1)T)\right)^{1/(k+1)} \to \zeta < 1$ and therefore we conclude that there exists a constant $\widetilde{C}$ such that $\|v\|^2_{\mathcal{H}}(kT) \le \widetilde{C}\|v\|^2_{\mathcal{H}}(0)\zeta^k$. We finally take $t>0$, write $t = kT + r$ with $r \in [0,T)$, and apply previous estimates to obtain

$$\|v\|^2_{\mathcal{H}}(kT+r) \le \widetilde{C}\|v\|^2_{\mathcal{H}}(r)\zeta^k \le \widetilde{C}\|v\|^2_{\mathcal{H}}(r)\zeta^{-1}\zeta^{k+\frac{r}{T}}.$$

In addition, from (2.7)–(2.9) we have the uniform estimate, valid for $r \in [0,T]$:

$$\|v\|^2_{\mathcal{H}}(r) \le \|v\|^2_{\mathcal{H}}(0)(1+T)C_1.$$

Collecting previous estimates and setting $C = \widetilde{C}C_1\zeta^{-1}(1+T), \eta = -\frac{\ln(\zeta)}{T}$, we obtain the estimate

$$\|v\|^2_{\mathcal{H}}(t) \le Ce^{-\eta t}.$$

This finishes the proof. ∎

3. Non-linear problem

3.1. Local existence. The local existence for the non-linear problem is derived from standard arguments, see the Appendix for details.

3.2. Global existence and exponential decay. Once the local existence is established we consider the exponential decay for solutions with small initial data. The key idea here is to show the existence of a suitable bounded invariant set; this will allow us to get both the global existence and the exponential decay.

THEOREM 3.1. *There exist $\varepsilon>0$, $C>0$ and $\gamma>0$ such that for $\phi\in\mathcal{H}$ with $\|\phi\|_{\mathcal{H}}<\varepsilon$ the local solution u for the nonlinear problem (A.1), given by Theorem A.1, satisfies both $u\in \mathrm{C}([0,\infty),\mathcal{H})\cap C^1([0,\infty),\mathcal{H}')$ and the exponential decay:*

$$\|u\|_{\mathcal{H}}(t)<Ce^{-\gamma t}\|\phi\|_{\mathcal{H}}.$$

Proof. The starting point is the semigroup estimate $\|U_a(t)\phi\|_{\mathcal{H}}\le Ce^{-\eta t}\|\phi\|_{\mathcal{H}}$ given by Theorem 2.3. We thus introduce the norm in $\mathcal{H}$: $\|\phi\|_\eta:=\sup\limits_{t\ge 0}\|e^{\eta t}U_a(t)\phi\|_{\mathcal{H}}$. A straightforward computation shows the following well–known identities:

$$\|\phi\|_{\mathcal{H}}\le\|\phi\|_\eta\le C\|\phi\|_{\mathcal{H}} \tag{3.1a}$$

$$\|U_a(\tau)\phi\|_\eta\le e^{-\eta\tau}\|\phi\|_\eta \tag{3.1b}$$

We then apply a fixed-point argument to bounded sets of $\mathcal{H}_\eta:=(\mathcal{H},\|\cdot\|_\eta)$, with $\eta>0$. Let $r,\delta>0$ be positive constants and set the Banach space $X_{r,\delta}=\mathrm{C}([0,\delta],B_r)$ equipped with the norm $\|v\|_{L^\infty(0,\delta,\mathcal{H}_\eta)}$. We also consider the fixed-point operator:

$$\Gamma(v)(t):=U_a(t)v(0)-i\int_0^t U_a(t-s)N(v(s))ds.$$

Applying $\|\cdot\|_\eta$ and using the estimates of (3.1) and Lemma A.1, we get the following estimate, here C represents different constants, none of them depending upon r,δ:

$$\begin{aligned}\|\Gamma(v)(t)\|_\eta&\le\|U_a(t)v(0)\|_\eta+\int_0^t\|U_a(t-s)N(v(s))\|_\eta ds\\&\le e^{-\eta t}\|v(0)\|_\eta+C\int_0^t\|N(v(s))\|_{\mathcal{H}}ds\\&\le e^{-\eta t}\|v(0)\|_\eta+C\int_0^t\|v(s)\|_{\mathcal{H}}^3ds\\&\le e^{-\eta t}\|v(0)\|_\eta+C\int_0^t\|v(s)\|_\eta^3ds.\end{aligned}$$

Since $v:[0,\delta]\to B_r$ we have the estimate, valid for $r>0$ and for $t\in[0,\delta]$, $\|\Gamma(v)(t)\|_\eta\le r\left(e^{-\eta t}+Cr^2t\right)$. We set $g(t):=e^{-\eta t}+Cr^2t$. Since $g(0)=1$ and $e^{-\eta t}\le 1-\eta t+\frac{1}{2}\eta^2t^2$ we have $g(t)<1-(\eta-Cr^2)t+\frac12\eta^2t^2$. We thus take

$$r^2<\frac{\eta}{C}\quad\text{and}\quad\delta<2\eta^{-2}(\eta-Cr^2) \tag{3.2}$$

from where we conclude that $g(t)\le 1$ is valid for $t\in[0,\delta]$. This shows that $\Gamma(X_{r,\delta})\subseteq X_{r,\delta}$.

Next, we consider the Lipschitz property of Γ. To this end we take $v,w\in X_{r,\delta}$ such that $v(0)=w(0)$ and compute: $\Gamma(v)(t)-\Gamma(w)(t)=\int_0^t U_a(t-s)\left(N(v(s))-N(w(s))\right)ds$. Proceeding as above we get:

$$\|\Gamma(v)(t)-\Gamma(w)(t)\|_\eta\le\int_0^t\|U_a(t-s)\left(N(v(s))-N(w(s))\right)\|_\eta ds$$

$$\leq Cr^2 \int_0^t \|v(s)-w(s)\|_\eta ds$$
$$\leq Cr^2\delta\|v-w\|_{L^\infty(0,\delta,B_r)}$$

from where we obtain $\|\Gamma(v)-\Gamma(w)\|_{L^\infty(0,\delta,B_r)} \leq Cr^2\delta\|v-w\|_{L^\infty(0,\delta,B_r)}$. Taking (r,δ) such that $Cr^2\delta<1$, together with the conditions (3.2), we have that, for $\phi\in B_r$, $\Gamma_\phi : X_{r,\delta}\to X_{r,\delta}$ is a well defined Lipschitz continuous operator with $\|\Gamma_\phi\|_{\mathrm{Lip}}<1$ and therefore exists $w\in \mathrm{C}([0,\delta],B_r)$, the unique fixed point of Γ_Φ. This means that, for $\phi\in B_r$, the local solution given by Theorem A.1 remains in B_r as $t\in[0,\delta]$. We shall show that w can be extended to $[0,+\infty)$. From Duhamel's formula we have the identity, valid in $[0,\delta]$: $w(t)=U_a(t)\phi - i\int_0^t U_a(t-s)N(w(s))ds$, in which $U_a(t)$ is the semigroup generated by the linear term. Taking norm in $\mathcal{H}$, we get the estimate, valid for $t\in[0,\delta]$:

$$\|w\|_\eta(t)\leq \|U_a(t)\phi\|_\eta + \int_0^t \|U_a(t-s)N(w(s))\|_\eta ds.$$

Using the estimate of Lemma A.1 and the exponential decay of the linear evolution, we obtain:

$$\|w\|_\eta(t)\leq e^{-\eta t}\|\phi\|_\eta + Cr^2\int_0^t e^{-\eta(t-s)}\|w(s)\|_\eta ds,$$

valid for $t\in[0,\delta]$, which is equivalent to the estimate:

$$e^{\eta t}\|w\|_\eta(t)\leq \|\phi\|_\eta + Cr^2\int_0^t e^{\eta s}\|w(s)\|_\eta ds.$$

Using the result of Gromwall lemma with $z(t)=e^{\eta t}\|w\|_\eta(t)$, we deduce $e^{\eta t}\|w\|_{\mathcal{H}}(t)\leq \|\phi\|_\eta e^{Cr^2 t}$, which means:

$$\|w\|_\eta(t)\leq \|\phi\|_\eta e^{-(\eta - Cr^2)t}.$$

Let $\gamma := \eta - Cr^2$, since (r,δ) satisfy estimate (3.2), we have $\gamma>0$ and therefore we also obtain the exponential decay valid for $t\in[0,\delta]$: $\|w\|_\eta(t)\leq\|\phi\|_\eta e^{-\gamma t}$. This shows that the $\mathcal{H}_\eta$–ball with radius r is an invariant set for the non-linear flow; from where we conclude that for any $\phi\in\mathcal{H}$ with $\|\phi\|_\eta<r$ the solution u is globally defined and satisfies the estimate:

$$\|u\|_\eta(t)\leq C\|\phi\|_\eta e^{-\gamma t},$$

for all $t>0$.

Since $\|u\|_\eta$ and $\|u\|_{\mathcal{H}}$ are equivalent norms, the proof is complete. □

4. Stabilisation rates: qualitative aspects of the dynamics

Since we have so far only used the *size* of the damping term, as $\|a\|_{\mathrm{W}^{1,\infty}}$, and not its precise location to derive the estimates, it is natural to expect that, for terms of similar size, the values of the rates are strongly dependent on the configuration. In this section we will focus on how rates vary as a function of the location of the free damping region. More precisely, we will take damping terms satisfying:

$$a\in \mathrm{W}^{1;\infty},\, a\equiv 0 \text{ in } [K_1;K_2],\, a\equiv\alpha \text{ in } [\widetilde{K_1};\widetilde{K_2}]^c,\, \widetilde{K_1}<K_1<K_2<\widetilde{K_2}, \tag{4.1}$$

and show that when the free–damping region is chosen as $[-K_2;K_2]$ and for large times, the rates for charge and energy could be made smaller than those given in Theorems 2.2–2.3; we also provide estimates for the length of these time intervals. Finally, let us mention that the following analysis is based on the spectral decomposition of the operator $L_0(\phi)=-\phi_{xx}+|x|\phi$ on the space $\mathcal{H}_0:=(\mathcal{H},\|\cdot\|_0)$ where $\|\phi\|_0^2:=\langle\phi;L_0\phi\rangle$, we refer [6] for details on the spectral decomposition; however; it is worth remarking that $L-L_0$ is the multiplication operator with compact support given by $\mu(x)-|x|$ and that $\|\cdot\|_0$ and $\|\cdot\|_{\mathcal{H}}$ are equivalent norms.

We start with the following lemma relating the rates of solutions with different damping terms for the linear Equation (2.1); this will provide both the continuity of the rates with respect to the size of the damping term and also the basic estimates involved in the proof of the main result, see Theorem 4.1. We consider the rates for both the charge and the energy.

LEMMA 4.1. *Let $a_0,a_1\in \mathrm{W}^{1,\infty}(\mathbb{R})$ both satisfying (4.1), let $\phi\in\mathcal{H}$ and let u_0, u_1 be the related solution of (2.1) with initial datum ϕ. Let also β_0 be the decaying rate given by Theorem 2.2 for $\|u_0\|_{L^2}$, and η_0 be the decaying rate given by Theorem 2.3 for $\|u_0\|_{\mathcal{H}}$. Then, the following estimates hold, in which C and D are constants:*

$$\|u_0(t)-u_1(t)\|_{L^2}\le\|\phi\|_{L^2}\left(e^{tC\|a_0-a_1\|_{L^\infty}}-1\right)e^{-\beta_0 t}.$$
$$\|u_0(t)-u_1(t)\|_{\mathcal{H}}\le\|\phi\|_{\mathcal{H}}\left(e^{tD\|a_0-a_1\|_{W^{1,\infty}}}-1\right)e^{-\eta_0 t}.$$

Proof. It is based upon a classical argument stemming from Duhamel's identity. We outline some details for further reference.

Let $U_j(t)$, $j=0,1$, denote the C_0–semigroups of contractions with generators $iL-A_j$, in which $L(\phi)=-\phi_{xx}+\mu\phi$ and A_j is the multiplication operator given by a_j; let also δA be the multiplication operator given by $\delta a=a_1-a_0$. The starting point is the identity:

$$U_1(t)\phi-U_0(t)\phi=\int U_0(t-s)\,\delta A\,U_1(s)\phi ds. \tag{4.2}$$

Setting $E^{(0)}(t)=U_0(t)\phi$ and $E^{(m+1)}(t)=\int_0^t U_0(t-s)\,\delta A\,E^{(m)}(s)ds$, $m\ge 0$, we shall show the convergence of the formal series: $E^{(\infty)}=\sum_{m=1} E^{(m)}$.

Let $M\ge 1$ and $\gamma>0$ be such that $\|U_0(t)\phi\|_X\le Me^{-\gamma t}\|\phi\|_X$, here $X=L^2$ or $X=\mathcal{H}$. Taking norm in X and performing an inductive argument, we get the estimate:

$$\|E^{(m)}(t)\|_X\le e^{-\gamma t}\,\|\phi\|_{L^2}\,\frac{M^m t^m}{m!}\,\|\delta A\|^m_{\mathcal{L}(X)}. \tag{4.3}$$

On the other hand, a straightforward computation gives the estimates $\|\delta A\|_{\mathcal{L}(L^2)}\le\|\delta a\|_{L^\infty}$ and $\|\delta A\|_{\mathcal{L}(\mathcal{H})}\le 2\|\delta a\|_{\mathrm{W}^{1,\infty}}$. Using the estimates $\|U_0(t)\phi\|_{L^2}\le C_0e^{-\beta t}\|\phi\|_{L^2}$ and $\|U_0(t)\phi\|_{\mathcal{H}}\le D_0e^{-\eta t}\|\phi\|_X$ of Theorems 2.2–2.3 respectively, where C_0 and D_0 are the bounding constants, we obtain the desired estimates. ∎

REMARK 4.1. In the nonlinear case, identity (4.2) and the related error term become:

$$u(t)\phi-U_0(t)\phi=\int U_0(t-s)\left(\delta A u(s)+N(u(s))\right)ds.$$
$$E^{(m+1)}(t)=\int_0^t U_0(t-s)\left(\delta A\,E^{(m)}(s)+N(E^{(m)}(s))\right)ds.$$

Following Lemma A.1 and taking $CR^2 < \varepsilon$, we get the estimate for the nonlinear term: $\|N(\phi)\|_{\mathcal{H}} \le \varepsilon \|\phi\|_{\mathcal{H}}$ valid for $\|\phi\|_{\mathcal{H}} < R$.

Thus, for small initial data, the perturbation $\delta A + N$ satisfies the estimate:

$$\|(\delta A + N)(\phi)\|_{\mathcal{H}} \le \left(\|\delta A\|_{\mathcal{L}(\mathcal{H})} + \varepsilon\right) \|\phi\|_{\mathcal{H}}.$$

From Theorem 3.1 we have the exponential decay for the solutions in the nonlinear equation, from where we deduce the estimate:

$$\|E^{(m)}(t)\|_X \le e^{-\gamma t} \|\phi\|_{L^2} \frac{M^m t^m}{m!} \left(\|\delta A\|_{\mathcal{L}(\mathcal{H})} + \varepsilon\right)^m, \tag{4.4}$$

which is similar to the one given in (4.3).

The main result relies upon the use of the method outlined in the proof of Lemma 4.1 for some suitable splitting of the given damping operator. Previous remark lead us to state the result for the nonlinear problem. This will be done with the aim of the behaviour of the Airy functions involved in the spectral decomposition of the linear term $L_0(\phi) = -\phi_{xx} + |x|\phi$. The following lemma gives the basic estimates.

Lemma 4.2. *Let the damping term a satisfy (4.1). Let $A(\phi) := a\phi$ be the related multiplication operator. Then, for any $\varepsilon > 0$, there exists $0 < K_2$ such that for $J = [-K_2, K_2]$ the following estimates hold, in which P_0 is the projection on the first eigenfunction of L_0 and $X = L^2$ or $X = \mathcal{H}$:*

$$\begin{aligned} \|AP_0\|_X &\le \varepsilon, \\ \|P_0 A P_0\|_X &\in \varepsilon^2 [C_1; C_2], \end{aligned}$$

here C_j are constants depending only on φ_0.

Proof. We first notice that the first eigenfunction of L_0 is given by:

$$\varphi_0(x) = c_0 \operatorname{Ai}(|x| - \lambda_0),$$

in which λ_0 is the first zero of $\operatorname{Ai}'(-x)$ and $c_0 = (2\lambda_0)^{-1/2} |\operatorname{Ai}(-\lambda_0)|^{-1}$ is the normalization constant; in addition, we have $P_0\phi = \langle \varphi_0; \phi\rangle \varphi_0$ and thus $AP_0\phi = \langle \varphi_0; \phi \rangle A\varphi_0$.

Since $a \equiv 0$ in $[-K_2; K_2]$, we have

$$\|A\varphi_0\|_{L^2}^2 = \int_{|x| \ge K_2} a^2(x) \varphi_0^2(x) dx \le \alpha^2 \|\varphi_0\|_{L^2(|x| \ge K_2)}^2.$$

We now consider $X = \mathcal{H}_0$ and we recall that $L_0(\phi) = -\phi_{xx} + |x|\phi$. Since $\|\phi\|_{\mathcal{H}_0}^2 = \langle \phi; L_0 \phi\rangle$, we get:

$$\begin{aligned} \|A\varphi_0\|_{\mathcal{H}_0}^2 \le{}& \|a\|_{L^\infty}^2 \|\varphi_0\|_{L^2_\mu(|x|\ge K_2)}^2 + \|a_x\|_{L^\infty}^2 \|\varphi_0\|_{L^2(|x| \ge K_2)}^2 \\ &+ \|a\|_{L^\infty}^2 \|(\varphi_0)_x\|_{L^2(|x|\ge K_2)}^2 + 2\|a_x\|_{L^\infty} \|a\|_{L^\infty} \|\varphi_0\|_{L^2(|x|\ge K_2)} \|(\varphi_0)_x\|_{L^2(|x| \ge K_2)}, \end{aligned}$$

from where we deduce:

$$\|A\varphi_0\|_{\mathcal{H}_0}^2 \le 3 \|a\|_{\mathrm{W}^{1,\infty}}^2 \left(\|\varphi_0\|_{L^2_\mu(|x| \ge K_2)}^2 + \|(\varphi_0)_x\|_{L^2(|x|\ge K_2)}^2 \right).$$

For the quadratic term, we have the identity:

$$\langle \varphi_0; A\varphi_0 \rangle = \int_{|x| \ge K_2} a(x) \varphi_0^2(x) dx = \int_{\widetilde{K}_2 \ge |x| \ge K_2} a(x) \varphi_0^2(x) dx + \alpha \int_{|x| \ge \widetilde{K}_2} \varphi_0^2(x) dx,$$

from where we deduce the estimates

$$\alpha\|\varphi_0\|^2_{L^2(|x|\geq \widetilde{K}_2)} \leq \langle \varphi_0; A\varphi_0\rangle \leq \alpha\|\varphi_0\|^2_{L^2(|x|\geq K_2)}.$$

In order to obtain the required estimates we take advantage of the following properties of the Airy function, which are valid for $x>1$ and $\xi=\frac{2}{3}x^{3/2}$, see [13]:

$$\mathrm{Ai}(x)=x^{-1/4}e^{-\xi}F_1(x), \quad \|F_1\|_{L^\infty}\leq \frac{1}{2\sqrt{\pi}}.$$
$$\mathrm{Ai}'(x)=-x^{1/4}e^{-\xi}F_2(x), \quad \|F_2\|_{L^\infty}\leq \frac{1}{4\sqrt{\pi}}.$$
$$\int_{z>M}\mathrm{Ai}(z)^2dz=\mathrm{Ai}'(M)^2-M\,\mathrm{Ai}(M)^2$$
$$\int_{z>M} z\,\mathrm{Ai}(z)^2dz=\frac{1}{3}M\,\mathrm{Ai}'(M)^2-\frac{1}{3}M^2\,\mathrm{Ai}(M)^2-\frac{1}{3}\mathrm{Ai}(M)\mathrm{Ai}'(M)$$
$$\int_{z>M}\mathrm{Ai}'(z)^2dz=-\frac{1}{3}M\,\mathrm{Ai}'(M)^2+\frac{1}{3}M^2\,\mathrm{Ai}(M)^2-\frac{2}{3}\mathrm{Ai}(M)\mathrm{Ai}'(M)$$

and obtain the inequalities:

$$\|\varphi_0\|^2_{L^2(|x|>K_2)}\leq (20\pi\lambda_0)^{-1}\,\mathrm{Ai}(-\lambda_0)^2(K_2-\lambda_0)^{1/2}e^{-4/3(K_2-\lambda_0)^{3/2}}.$$
$$\|\varphi_0\|^2_{L^2_\mu(|x|>K_2)}+\|(\varphi_0)_x\|^2_{L(|x|>K_2)}\leq (8\pi\lambda_0)^{-1}\,\mathrm{Ai}(-\lambda_0)^2e^{-4/3(K_2-\lambda_0)^{3/2}}.$$

Taking $K_2\gg 1$ such that

$$\alpha C(\lambda_0)(K_2-\lambda_0)^{1/2}e^{-4/3(K_2-\lambda_0)^{3/2}}<e^{-(K_2-\lambda_0)^{3/2}}<\varepsilon^2, \tag{4.5}$$

we conclude the required inequalities:

$$\|AP_0\|_X\leq \varepsilon,$$

$$\|P_0AP_0\|_X\in \varepsilon^2[C_1;C_2].$$

This finishes the proof. ∎

The main result of this section is based on the continuity of the rates with respect to the damping term given by Lemma 4.1 and expresses that, for finite times, the closer the origin is to the non-damping region, the smaller the rates are.

THEOREM 4.1. *Let the damping term $a\in W^{1,\infty}$ satisfy the assumptions* (4.1)*. Let $\phi\in\mathcal{H}$ and u be the solution of the nonlinear Equation* (A.1) *with small initial datum ϕ. Let $T>0$ and $\zeta>0$ be fixed, then there exists $K\gg 1$, $0<\nu\ll 1$ and $S(t)$ a C_0–semigroup such that: $e^{\nu t}\|S(t)\phi\|_X\geq \|P_0\phi\|_X$ and, for $t\in[0;T]$, the solution $u(t)$ satisfies the estimate:*

$$e^{\nu t}\|u(t)-S(t)\phi\|_X\leq \zeta.$$

Proof. To start with, we consider the split: $A=A_0+\delta A$, in which $A_0=P_0AP_0+P_1AP_1$ and $\delta A=P_1AP_0+P_0AP_1$. Since A_0 is a bounded perturbation of A (in both $X=L^2$ and $X=\mathcal{H}$), the operator $iL-A_0$ indeed generates a C_0–semigroup of contractions, which will be denoted by $S(t)$. Let $X_0=\text{span}\{\varphi_0\}$ and $X_1=X_0^{\perp}$, valid in both $X=L^2$ and $X=\mathcal{H}$, since X_0 and X_1 are A_0–invariant subspaces, we have that $P_j S(t)=S(t)P_j$, for $j=0,1$, and therefore:

$$P_0 S(t)\phi = e^{-\nu_X t+i\lambda_0 t}P_0\phi,$$
$$P_1 S(t)\phi = S(t)P_1\phi,$$

where $\nu_X=\|P_0 A P_0\|_X$.

Since the damping term satisfies assumptions (4.1), we apply the results of Lemma 4.2 and get the inequalities:

$$\|\delta A\|_X=\|P_1 A P_0+P_0 A P_1\|_X\le \|A P_0\|_X\le C\varepsilon.$$
$$\|P_0 A P_0\|_X\sim\varepsilon^2,$$

with ε to be especified and $C=C(\alpha,\varphi_0)$ a constant. Let θ_X expresses any of the rates $\{\beta,\eta,\gamma\}$ provided by Theorems 2.2–2.3–3.1, we have the following estimates for the semigroups:

$$P_0 S(t)\phi\sim e^{-\varepsilon^2 t}\phi,$$
$$\|P_1 S(t)\phi\|_X\le e^{-\theta_X t}M\|\phi\|_{L^2}.$$

Thus, setting $\varepsilon<\min\{\beta^{1/2};\eta^{1/2};\gamma^{1/2}\}$, we deduce the estimates for the semigroup: $\|S(t)\phi\|_X\le e^{-\varepsilon^2 t}M\|\phi\|_X$ and $\|S(t)P_0\phi\|_X\sim e^{-\varepsilon^2 t}\|\phi\|_X$. From previous estimates and recalling the inequality (4.4), we have $\|\delta A\|_X\le C\varepsilon$, from where we get:

$$\|E^{(m)}(t)\|_X\le e^{-\varepsilon^2 t}\|\phi\|_X\frac{(\widetilde{M}\varepsilon t)^m}{m!}.$$

Applying the results of Lemma 4.1, we get the bound:

$$\|u(t)-S(t)\phi\|_X\le e^{-\varepsilon^2 t}(e^{\widetilde{M}\varepsilon t}-1)\|\phi\|_X.$$

Finally, we take: $K_2>\lambda_0+\left(\ln(\frac{\widetilde{M}T}{\varsigma})\right)^{2/3}$, $\nu=e^{-(K_2-\lambda_0)^{3/2}}$ and $\varepsilon=\nu^{1/2}$, and the result follows from estimate (4.5). ∎

Appendix. Local existence for the nonlinear problem. For the sake of completeness, and since the local existence of solutions for the non-linear case are derived from standard arguments mainly from fixed-point techniques, we devote this appendix to present the related details.

To start with, we focus on the local existence of solutions for the problem given by the equation

$$iu_t=-u_{xx}+\mu(x)u-i\,a(x)u+N(u), \tag{A.1}$$

together with the initial condition $u(x,0)=u_0(x_0)\in\mathcal{H}$; here, the non-linearity N is of non-local nature and is given by $N(\phi)=\phi m(\phi)$ with

$$m(\phi)(x):=\int \varrho(x,y)|\phi(y)|^2dy, \tag{A.2}$$

In the sequel we made the following assumptions on the kernel,

$$|\varrho(x,y)|\le\mu(x),\quad |\varrho_x(x,y)|\le C, \tag{A.3}$$

this choice is motivated by the Schrödinger–Poisson equation whose kernel is given by $\varrho(x,y)=\frac{1}{2}(|x-y|-|x|)$, see [5].

We also recall that $\mu(x)\in C^\infty(\mathbb{R})$, $\mu\geq\max\{|x|,1\}$ with $\mu(x)=|x|$ for $|x|\geq 2$, and $a\in \mathrm{W}^{1,\infty}(\mathbb{R})$, $\alpha(x)=0$ for $x\in[K_1,K_2]$ and $a(x)\geq\alpha>0$ for $x\in\mathbb{R}\setminus[\widetilde{K_1},\widetilde{K_2}]$, with $[K_1,K_2]\subseteq(\widetilde{K_1},\widetilde{K_2})$.

As usual, the nonlinear result is obtained from the linear case by means of perturbative arguments. With these ideas in mind we present some useful estimates.

Lemma A.1. *Let $m(\phi)$ be given by (A.2), with the kernel satisfying the estimates (A.3), let also $N(\phi)=\phi m(\phi)$. Then for $\phi,\psi\in\mathcal{H}$ the following estimates hold, in which C are different constants depending only upon the kernel ϱ.*

- $\|N(\phi)\|_{\mathcal{H}}\leq C\|\phi\|_{\mathcal{H}}^3$.
- $\|N(\phi)-N(\psi)\|_{\mathcal{H}}\leq CR^2\|\phi-\psi\|_{\mathcal{H}}$, *valid for* $\|\phi\|_{\mathcal{H}},\|\psi\|_{\mathcal{H}}<R$.

Proof. It is a straightforward computation and will be omitted. ◻

We now move to the local existence result.

Theorem A.1. *For any $\phi\in\mathcal{H}$ there exists $\delta=\delta(\phi)>0$ such that the nonlinear Equation (A.1) with initial datum $u(0)=\phi$ has a unique solution $u\in \mathrm{C}([0,\delta],\mathcal{H})\cap C^1([0,\delta],\mathcal{H}')$.*

Proof. It is based on a fixed-point argument. Let $\phi\in\mathcal{H}$ and let $U_a(t)$ be the semigroup generated by $-iL-a$. For $\varepsilon,\tau>0$ we define $X_{\varepsilon,\tau}(\phi)=\{w\in \mathrm{C}([0,\tau],\mathcal{H})\cap C^1([0,\tau],\mathcal{H}'):\|v-U_a(t)\phi\|_{L^\infty([0,\tau],\mathcal{H})}\leq\varepsilon\}$ and equipped it with $\|w\|_{L^\infty(0,\tau,\mathcal{H})}$. We also define the fixed-point operator:

$$\Gamma(w)(t):=U_a(t)\phi-i\int_0^t U_a(t-s)N(w(s))ds.$$

Since $U_a(t)$ is a C_0–semigroup it is clear that $\Gamma(X_{\varepsilon,\tau}(\phi))\subseteq \mathrm{C}([0,\tau],\mathcal{H})\cap C^1([0,\tau],\mathcal{H}')$. In order to show the invariance of $X_{\varepsilon,\tau}(\phi)$ for a suitable choice of the parameters we consider the estimate, in which we have used Lemma A.1:

$$\begin{aligned}\|\Gamma(w)(t)-U_a(t)\phi\|_{\mathcal{H}}&\leq\int_0^t\|N(w(s))\|_{\mathcal{H}}\\&\leq C\int_0^t\|w(s)\|_{\mathcal{H}}^3\\&\leq C\tau\left(\|\phi\|_{\mathcal{H}}+\varepsilon\right)^3\end{aligned}$$

from where we conclude $\|\Gamma(w)-U_a(t)\phi\|_{L^\infty([0,\tau],\mathcal{H})}\leq C\tau\left(\|\phi\|_{\mathcal{H}}+\varepsilon\right)^3$.

On the other hand, in order to get the Lipschitz constant we take $w_1,w_2\in X_{\varepsilon,\tau}(\phi)$ and apply previous estimates. We thus have:

$$\begin{aligned}\|\Gamma(w_1)(t)-\Gamma(w_2)(t)\|_{\mathcal{H}}&\leq\int_0^t\|U_a(t-s)\left(N(w_1(s))-N(w_2(s))\right)\|_{\mathcal{H}}ds\\&\leq C\left(\|\phi\|_{\mathcal{H}}+\varepsilon\right)^2\int_0^t\|w_1-w_2\|_{\mathcal{H}}(s)ds\\&\leq C\left(\|\phi\|_{\mathcal{H}}+\varepsilon\right)^2\tau\|w_1-w_2\|_{L^\infty([0,\tau],\mathcal{H})}\end{aligned}$$

and therefore $\|\Gamma\|_{\mathrm{Lip}}\leq C\left(\|\phi\|_{\mathcal{H}}+\varepsilon\right)^2\tau$.

We finally take, for $\phi \in \mathcal{H}$, some $\varepsilon > 0$ and get $\delta > 0$ such that $C\left(\|\phi\|_{\mathcal{H}} + \varepsilon\right)^2 \delta < 1$ from where we conclude that $\Gamma : X_{\varepsilon,\delta}(\phi) \to X_{\varepsilon,\delta}(\phi)$ is a contraction and therefore there exists $u \in \mathrm{C}([0,\delta], \mathcal{H}) \cap C^1([0,\delta], \mathcal{H}')$ such that $u(t) = U_a(t)\phi - i\int_0^t U_a(t-s)N(u(s))ds$. This finishes the proof. ∎

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