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GLOBAL WEAK SOLUTIONS TO A THREE-DIMENSIONAL COMPRESSIBLE NON-NEWTONIAN FLUID*

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Abstract. The paper concerns on the existence of global weak solutions to a compressible non-Newtonian fluid with the power-law type. The main contribution of this paper is to handle the power-law structure with the exponent $r > \frac{12\gamma}{5\gamma-3}$ when the pressure is related to ρ^γ with $\gamma > 1$. The exponent r is forced by the convective term and the convergent argument of approximate solution. Inspired by the weak formulation of the momentum equation, the existence of global weak solutions is proved relying on the Faedo-Galerkin method, weak compactness techniques and the monotonicity method.

Keywords. Compressible non-Newtonian fluids; global weak solution; the monotonicity method.

AMS subject classifications. 76A05; 35Q35.

1. Introduction

The motion of a three-dimensional compressible non-Newtonian fluid is governed by the following generalized Navier-Stokes system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{div}(|\mathbb{D}u|^{r-2}\mathbb{D}u + \eta(|\operatorname{div} u|)\operatorname{div} u \mathbb{I}), \end{cases} \tag{1.1}$$

where $\nabla = \nabla_x$, $\operatorname{div} = \operatorname{div}_x$, $(x,t) \in \Omega \times \mathbb{R}_+$, ρ, u and p denote the density, velocity and pressure, respectively. $\mathbb{I}$ is the 3×3 identity matrix. The set $\Omega \subset \mathbb{R}^3$ is a bounded domain with a regular boundary $\partial\Omega$ (of class, say $C^{2+\nu}$, $\nu > 0$, taken for convenience). The tensor $\mathbb{D}u = \frac{1}{2}(\nabla u + \nabla^T u)$ is a symmetric part of the velocity gradient. The bulk viscosity coefficient η is a continuous function of $|\operatorname{div} u|$. The positive constant r satisfies $r > 1$.

The main reason to investigate the non-Newtonian fluid with the power-law type is the phenomena of rapidly increasing fluid viscosity under various stimuli such as shear rate, electric or magnetic field. The power-law models are quite popular among rheologists, in chemical engineering and colloidal mechanics.

It is an important issue to study the global existence of weak solutions to the system (1.1), and significant progress has been made recently on this topic. Ladyzhenskaya initiated the incompressible non-Newtonian fluids of power-law type in [18,19], where the global existence of weak solutions for the exponent $r \geqslant 1 + \frac{2d}{d+2}$ (d stands for space dimension) was proved for Dirichlet boundary conditions. The modern state of the art in the mathematical theory of non-Newtonian fluids is described in [20], where homogeneous incompressible fluids are studied in sufficient details, while compressible non-Newtonian fluids are hardly considered (only very weak measure-valued solutions are obtained). By Lipschitz truncation methods, Frehse-Málek-Steinhauer [11] established the weak solutions for the constant exponent $r > \frac{2d}{d+2}$ with $d \geqslant 2$. Wolf [26] proved existence of weak solutions to an incompressible homogenous fluid with shear rate dependent viscosity for $r > \frac{2(d+1)}{d+2}$ without assumptions on shape and size of Ω. The existence of global weak

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solutions with Dirichlet boundary conditions for $r > \frac{2(d+1)}{d+2}$ was achieved in [2] by Lipschitz truncation and local pressure methods. For the nonhomogenous incompressible non-Newtonian fluids, the first result goes back to [10], where existence of Dirichlet weak solutions was obtained for $r \geqslant \frac{12}{5}$ if $d=3$, existence of space-periodic weak solutions for $r \geqslant 2$ with some regularity properties of weak solutions was achieved in [15] for $r \geqslant \frac{20}{9}$ with $d=3$. For the details on the existence, uniqueness, regularity and long-time behavior of solutions to the initial-boundary value problem of incompressible non-Newtonian fluids, one can refer the papers [1, 12, 13] and the references therein.

For the compressible non-Newtonian fluids, Mamontov [21] established the global existence of sufficiently regular solutions to two-dimensional and three-dimensional equations provided that the initial density is without vacuum. Later, Mamontov [22, 23] (see also the references therein) considered a model with linear pressure equation and an exponential dependence of the viscosity on the velocity gradient in two-dimensional domains. Zhikov-Pastukhova [28] obtained the existence of weak solutions to the initial-boundary-value problem for multidimensional fluids under some restrictions. Feireisl-Liao-Málek [9] studied the large-data existence result of weak solutions to a compressible non-Newtonian fluid with nonlinear constitutive equation guaranteeing the divergence of the velocity field remains bounded, provided that the initial density is strictly positive. Fang-Kong-Liu [4] investigated the existence of weak solutions to a one-dimensional full compressible non-Newtonian fluid. Recently, Fang-Guo [5], Shi-Wang-Zhang [25] discussed the stability of rarefaction waves for the isentropic and nonisentropic compressible non-Newtonian fluids, respectively. Guo-Dong-Liu [14] studied the existence and large-time behaviors of boundary layer solution of the inflow problem on the half space for an isentropic compressible non-Newtonian fluid. For the details on the existence of strong solutions to the one-dimensional compressible non-Newtonian fluid, one can refer the papers [6, 27] and the references therein.

The analysis of the degenerations and nonlinearity in system (1.1) requires some special attentions. The major concerns are stated as follows.

(1) The degeneration of the initial density containing vacuum. It is rather difficult and challenging to investigate the global existence of weak solutions to the system (1.1). And the possible appearance of vacuum in the fluid density (that is, the fluid density is zero) is one of the essential difficulties in the analysis of the well-posedness and related problems. For the compressible Navier-Stokes equation, the artificial viscosity terms and artificial pressure terms were introduced by Lions in [17] and improved by Feireisl in [8].

(2) The strong degeneration and nonlinearity of the elliptic operator $\operatorname{div}(|\mathbb{D}u|^{r-2}\mathbb{D}u)$ in momentum Equations $(1.1)_2$. Mathematically, this degeneration and nonlinearity leads to a difficulty which makes it hard to find a reasonable way to get the convergence of velocity under the uniform bound.

(3) The strong nonlinearity for the pressure term $p(\rho)$. It should be pointed out here that unlike the case of $p(\rho)=\rho^\gamma$, despite the weak regularizing effect on solutions, the pressure term $p(\rho)$ also causes some troubles in the strong convergence of the density. Therefore, much attention needs to be paid in order to control the strong nonlinearity of the pressure term.

Recently, there is some interesting work that overcomes these difficulties mentioned above. Zhikov-Pastukhova [28] handled the strong degeneration and nonlinearity of the term $\operatorname{div}(|\mathbb{D}u|^{r-2}\mathbb{D}u)$ by regularization and convex analysis under the assumption that $r>3$. Feireisl-Liao-Málek [9] handled the term $\operatorname{div}((1+|\mathbb{D}^d u|^2)^{\frac{r-2}{2}}\mathbb{D}^d u)$ based on the assumption that the divergence of the velocity field is under all circumstances bounded and that the initial density is without vacuum. However, those results require $r>3$ or allow $\mathrm{div}((1+|\mathbb{D}^d u|^2)^{\frac{r-2}{2}}\mathbb{D}^d u)$ in the case the initial density being with lower bound and upper bound.

In this paper, we consider the global weak solution to a three-dimensional compressible non-Newtonian fluid with large data. More precisely, we study the initial-boundary value problem of system (1.1) in a bounded domain $\Omega\subset\mathbb{R}^3$ subjected with the initial data

$$(\rho,\rho u)|_{t=0}=(\rho_0,m_0)(x)\ (x\in\Omega), \tag{1.2}$$

and the no-slip boundary condition

$$u|_{\partial\Omega}=0. \tag{1.3}$$

The aim of this paper is to establish the global existence theory of weak solutions to the problem (1.1)-(1.3). We are interested in the case that $\frac{12\gamma}{5\gamma-3}<r<3$ and the case that $r\geqslant 3$, when the pressure $p(\rho)$ is a positive function of the density satisfying

$$\begin{cases} p'(s)\geqslant a_1 s^{\gamma-1} \ \text{ for all } s>0, & p(s)\leqslant a_2 s^{\gamma} \text{ for all } s\geqslant 0,\\ p\in C[0,\infty)\cap C^1(0,\infty), & p(0)=0 \end{cases} \tag{1.4}$$

for some positive constants $a_1,\ a_2,\gamma$.

The global existence of weak solutions is proved with the help of the Faedo-Galerkin method, the vanishing viscosity method and the monotonicity method. To prove the existence of weak solution, we use an approximation scheme similar to that in [7], which consists of Faedo-Galerkin approximation, artificial viscosity and artificial pressure. Then, we get an improvement on the integrability of density, which can ensure the effectiveness and convergence of our approximation scheme. More specifically, we show that the uniform bound of ρ^γ in $L^\theta(\Omega\times(0,T))$, with some $\theta>1$ (stated in Lemma 5.1) depends on the value of r and q given in Theorem 2.1 and Theorem 2.2. To overcome the difficulty arising from possible large oscillations of the density ρ, we adopt the method in Lions [17] and Feireisl [7], which is based on the celebrated weak continuity of the effective viscous flux $p(\rho)\mathbb{I}-[|\mathbb{D}u|^{r-2}\mathbb{D}u+\eta(|\mathrm{div}u|)\mathrm{div}u\mathbb{I}]$ and the properties of convex function. The nonlinear term $|\mathbb{D}u|^{r-2}\mathbb{D}u+\eta(|\mathrm{div}u|)\mathrm{div}u\mathbb{I}$ is going to be dealt with by a weak formulation of the momentum equation arising in [9].

The rest of this paper is organized as follows. In Section 2, we introduce the definition of weak solutions to the compressible non-Newtonian fluids, and also state the main existence results. In Section 3, a series of a priori estimates on the solution is derived. In Section 4, we construct a three-level approximation scheme inspired by [8,9] for the problem (1.1)-(1.3). In Section 5, we prove Theorem 2.1 through a vanishing viscosity and vanishing artificial pressure limit passage using the weak convergence method. Finally, the proof of Theorem 2.2 is given in Section 6.

2. Main results

The aim of this section is to give the definition of the weak solution to the problem (1.1)-(1.3) and state the main results.

Before we state our main results, we need to specify the definition of weak solutions. It is necessary to require that the weak solutions should satisfy the natural energy estimates. From the viewpoint of physics, the conservation laws on mass, momentum and energy also should be satisfied at least in the sense of distributions. Based on those considerations, the definition of reasonable global-in-time weak solutions goes as follows.

DEFINITION 2.1. *Let $r>1$. A pair of functions (ρ,u) is said to be a weak solution to the problem (1.1)-(1.3) on the time interval $(0,T)$ for any fixed $T>0$ if the following conditions hold:*

- $$\begin{aligned}&\rho\geqslant 0,\ \rho\in C([0,T];L^1(\Omega))\cap L^\infty(0,T;L^\gamma(\Omega)),\\&u\in L^r(0,T;W_0^{1,r}(\Omega)),\ \eta(|divu|)|divu|^2\in L^1(\Omega\times(0,T));\end{aligned}$$
- *the density $\rho\geqslant 0$ and the velocity u satisfy (1.1) in $\mathcal{D}'(\Omega\times(0,T))$ and*
$$\int_0^\tau\int_\Omega(\rho\partial_t\varphi+\rho u\cdot\nabla\varphi)\,dxdt=\left(\int_\Omega\rho\varphi dx\right)\Big|_0^\tau$$
for any $\tau\in[0,T]$ and any test function $\varphi\in C^\infty(\Omega\times[0,T])$ with $\varphi(x,0)=\varphi(x,T)=0$ for $x\in\Omega$;
- *the functions ρ and ρu satisfy the initial conditions in the following weak sense*
$$ess\lim_{t\to0^+}\int_\Omega(\rho,\rho u)(x,t)\omega(x)dx=\int_\Omega(\rho_0,m_0)\omega dx$$
holds for any $\omega\in C_0^\infty(\Omega)$.

REMARK 2.1. Suppose that $r>\frac{12\gamma}{5\gamma-3}$. Then one can get the following three properties.

(1) The energy inequality
$$\int_\Omega\left(\frac12\rho|u|^2+\rho P(\rho)\right)dx\Big|_0^\tau+\int_0^\tau\int_\Omega(|\mathbb{D}u|^r+\eta(|\mathrm{div}u|)|\mathrm{div}u|^2)dxdt\leqslant 0$$
holds for a.e. $\tau\in[0,T]$, where $P(\rho)=\int_1^\rho\frac{p(z)}{z^2}dz$ is the elastic potential.

(2) Inspired by Feireisl-Liao-Málek [9], any weak solution in Definition 2.1 satisfies the following weak formulation of the momentum equation
$$\begin{aligned}&[\frac12\int_\Omega\rho|u|^2dx]\Big|_0^\tau-[\int_\Omega\rho u\cdot\varphi dx]\Big|_0^\tau+\int_0^\tau\int_\Omega[\rho u\cdot\partial_t\varphi+\rho u\otimes u:\nabla\varphi]dxdt\\&\quad+\int_0^\tau\int_\Omega|\mathbb{D}u|^{r-2}\mathbb{D}u:\mathbb{D}(u-\varphi)dxdt+\int_0^\tau\int_\Omega p(\rho)\mathrm{div}(\varphi-u)dxdt\\&\leqslant\int_0^\tau\int_\Omega[\Lambda(\mathrm{div}\varphi)-\Lambda(\mathrm{div}u)]dxdt\end{aligned}$$
for any $\tau\in[0,T]$ and any test function $\varphi\in C_c^\infty(\Omega\times[0,T];\mathbb{R}^3)$, where $\Lambda'(z)=\eta(z)z$ and $\Lambda''(z)\geqslant 0$ for any $z>0$.

(3) It follows immediately from (1.1) that any weak solution in Definition 2.1 belongs to the class
$$\rho\in C([0,T];L^\gamma_{weak}(\Omega)),\quad\rho u\in C([0,T];L^{\frac{2\gamma}{\gamma+1}}_{weak}(\Omega)).$$
Consequently, the initial conditions (1.2) make sense. Accordingly, the initial data ρ_0 and m_0 are supposed to comply with compatibility conditions of the form
$$\rho_0\in L^\gamma(\Omega),\ \rho_0\geqslant0,\ m_0(x)=0\text{ whenever }\rho_0(x)=0,\ \frac{|m_0|^2}{\rho}\in L^1(\Omega).$$

Now, we are in a position to address our main results on the global existence of weak solution to the problem (1.1)-(1.3) in this paper.

THEOREM 2.1 (Main Theorem). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\nu}$ for some $\nu > 0$ and $\eta(z) = |z|^{q-1}$. Suppose that the following conditions hold:*

(i) the pressure $p(\rho)$ is given by $p(\rho) = \rho^\gamma$ with the adiabatic exponent $\gamma > 1$;

(ii) the initial data satisfy

$$\begin{cases} \rho_0 \in L^\gamma(\Omega), \rho_0 \geqslant 0 \text{ on } \Omega, \\ \frac{|m_0|^2}{\rho_0} \in L^1(\Omega); \end{cases}$$

(iii) the positive constants r and q satisfy the case that

$$\frac{12\gamma}{5\gamma-3} < r < 3 \text{ and } q > \frac{r+2\gamma-1}{(r-1)(2\gamma-1)}$$

or the case that

$$r \geqslant 3 \text{ and } q > 1.$$

Then, the problem (1.1)-(1.3) has a weak solution (ρ, u) on $\Omega \times (0,T)$ for any given $T > 0$.

REMARK 2.2. The solution constructed in Theorem 2.1 admits that

$$\operatorname{div} u \in L^{q+1}(\Omega \times (0,T)) \text{ and } \nabla u \in L^r(\Omega \times (0,T)).$$

Moreover,

$$\rho^\gamma \in L^{\frac{q+1}{q}}(\Omega \times (0,T))$$

holds for the case that $\frac{12\gamma}{5\gamma-3} < r < 3$ and $\max\{r-1, \gamma, \frac{(6+r)\gamma}{(5r-6)\gamma-3r}\} < q \leqslant \frac{11\gamma}{\gamma-3}$, and

$$\rho^\gamma \in L^{\frac{r}{r-1}}(\Omega \times (0,T))$$

holds for the case that $\frac{12\gamma}{5\gamma-3} < r < 3$ and $q \geqslant \frac{r+2\gamma-1}{(r-1)(2\gamma-1)}$ and the case that $r \geqslant 3$ and $q > 1$. The details can be found in Lemma 5.1.

THEOREM 2.2 (Main Theorem). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\nu}$ for some $\nu > 0$ and $\eta(z) = |z|^{q-1}$. Suppose that the following conditions hold:*

(i) the pressure $p(\rho)$ is given by

$$\begin{cases} p'(s) \geqslant a_1 s^{\gamma-1} \text{ for all } s > 0, \quad p(s) \leqslant a_2 s^\gamma \text{ for all } s \geqslant 0, \\ p \in C[0,\infty) \cap C^1(0,\infty), \qquad p(0) = 0 \end{cases}$$

with the adiabatic exponent $\gamma > 1$;

(ii) the initial data satisfy

$$\begin{cases} \rho_0 \in L^\gamma(\Omega), \ \rho_0 \geqslant \underline{\rho} > 0 \text{ on } \Omega, \\ u_0 \in L^r(\Omega); \end{cases}$$

(iii) the positive constants r and q satisfy the case that

$$\frac{12\gamma}{5\gamma-3} < r < 3 \text{ and } q > \frac{r+2\gamma-1}{(r-1)(2\gamma-1)}$$

or the case that

$$r \geqslant 3 \text{ and } q > 1.$$

Then, the problem (1.1)-(1.3) has a weak solution (ρ, u) on $\Omega \times (0,T)$ for any given $T > 0$.

REMARK 2.3. The solution constructed in Theorem 2.1 and Theorem 2.2 satisfies the continuity equation in the sense of re-normalized solutions, more specifically,

$$\partial_t[b(\rho)] + \mathrm{div}[b(\rho)u] + (b'(\rho)\rho - b(\rho))\mathrm{div} u = 0$$

holds in the sense of distributions for any $b \in C^1(\mathbb{R})$ such that $|b'(z)z| \leqslant c|z|^{\frac{\gamma}{2}}$ for z larger than some positive constant z_0.

REMARK 2.4. Our results also hold for the bulk viscosity coefficient $\eta(z) \sim z^{q-1}$ for any $z > 0$, where the symbol $\sim$ refers that $\eta(z) + z\eta'(z) > 0$ for any $z > 0$ and there exist positive constants C_1 and C_2 such that

$$C_1 z^{q-1} \leqslant \eta(z) \leqslant C_2 z^{q-1}$$

holds for any $z > 0$.

REMARK 2.5. When $r > \frac{12\gamma}{5\gamma - 3}$, the direct calculation gives that

$$\mathrm{div}(\rho u \otimes u) \in L^{\frac{r}{r-1}}(0,T; W^{-1,\frac{r}{r-1}}(\Omega))$$

since $u \in L^r(0,T; W^{1,r}(\Omega))$.

REMARK 2.6. Our method also works for the case with nonzero external force f in the momentum equation. It is obvious that in our analysis the presence of the external force does not add any additional difficulty, and usually can be dealt with by using classical Young's inequality under the assumptions that $f \in L^{\frac{r}{r-1}}(\Omega \times (0,T))$.

Next, we give the inverse Hölder inequality, which can be found in [24]. The inverse Hölder inequality is a critical key to deal with convergence of pressure by using the convex analysis. Here the proof of Lemma 2.1 is given for completeness.

LEMMA 2.1 ([24]). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a regular boundary $\partial\Omega$, $p, q \in \mathbb{R}$ with $0 < p < 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}} \left(\int_\Omega |g|^q dx\right)^{\frac{1}{q}} \leqslant \int_\Omega |fg| dx$$

holds for any measurable functions f, g with $\int_\Omega |f|^p dx$, $\int_\Omega |fg| dx$ being finite and $\int_\Omega |g|^q dx > 0$ being finite.

Proof. Since $0 < p < 1$, it is deduced that $\frac{1}{p} > 1$ and

$$\begin{aligned}
\int_\Omega |f|^p dx &= \int_\Omega \frac{|fg|^p}{|g|^p} dx \quad (\text{ set } \frac{0}{0} = 0) \\
&\leqslant \left(\int_\Omega (|fg|^p)^{\frac{1}{p}} dx\right)^p \left(\int_\Omega (|g|^{-p})^{\frac{1}{1-p}} dx\right)^{1-p} \\
&= \left(\int_\Omega (|fg|^p)^{\frac{1}{p}} dx\right)^p \left(\int_\Omega |g|^{-\frac{p}{1-p}} dx\right)^{1-p}
\end{aligned}$$

by the Hölder inequality. So,

$$\left(\int_\Omega |f|^p dx\right)\left(\int_\Omega |g|^{-\frac{p}{1-p}} dx\right)^{p-1} \leqslant \left(\int_\Omega (|fg|^p)^{\frac{1}{p}} dx\right)^p,$$

which gives that

$$\left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}}\left(\int_\Omega |g|^{-\frac{p}{1-p}} dx\right)^{-\frac{1-p}{p}} \leqslant \int_\Omega |fg| dx. \tag{2.1}$$

The assumption $\frac{1}{p}+\frac{1}{q}=1$ implies that $q=-\frac{p}{1-p}$, and the desired result follows from (2.1) directly. ∎

Finally, we give a key tool to deal with convergence of the nonlinear term by using the monotonicity method. Let $A_\varepsilon(x,\xi)\colon \Omega\times\mathbb{R}^d\to\mathbb{R}^3$ be Carathéodory vector functions. These vector functions are assumed to satisfy the minimal monotonicity and convergence conditions

$$\begin{aligned}&(A_\varepsilon(x,\xi)-A_\varepsilon(x,\eta))\cdot(\xi-\eta)\geqslant 0, A_\varepsilon(x,0)\equiv 0,\\ &|A_\varepsilon(x,\xi)|\leqslant c_0(|\xi|)<\infty, \lim_{\varepsilon\to 0} A_\varepsilon(x,\xi)=A(x,\xi)\end{aligned}$$

for a.e. $x\in\Omega$ and any $\xi,\eta\in\mathbb{R}^3$. In fact, the minimal monotonicity of the Carathéodory vector function is the foundation for the monotonicity method.

LEMMA 2.2 ([29]). *Suppose that $v_\varepsilon \rightharpoonup v$, $A_\varepsilon(x,v_\varepsilon)\rightharpoonup z$ in $L^1(\Omega)$. Let $K\subset\Omega$ be a measurable set such that $z\cdot v\in L^1(K)$. Then*

$$\liminf_{\varepsilon\to 0}\int_K A_\varepsilon(x,v_\varepsilon)\cdot v_\varepsilon dx \geqslant \int_K z\cdot v dx.$$

In particular,

$$z|_K = A|_K, \quad A=A(x,v)$$

when $\liminf_{\varepsilon\to 0}\int_K A_\varepsilon(x,v_\varepsilon)\cdot v_\varepsilon dx=\int_K z\cdot v dx$.

3. A priori estimates

The energy associated with the problem (1.1)-(1.3) takes the form

$$E(t)=\int_\Omega\left(\frac{1}{2}\rho|u|^2+\rho P(\rho)\right)dx$$

where $P(\rho)=\int_1^\rho \frac{p(z)}{z^2}dz$ is the elastic potential. If the fluid is smooth, the energy balance

$$\frac{dE}{dt}+\int_\Omega \left(|\mathbb{D}u|^r+\eta(|\mathrm{div}u|)|\mathrm{div}u|^2\right)dx=0 \tag{3.1}$$

holds. The energy conservation (3.1) implies that

$$\int_\Omega\left(\frac{1}{2}\rho|u|^2+\rho P(\rho)\right)dx\leqslant\int_\Omega\left(\rho_0P(\rho_0)+\frac{1}{2}\rho_0|u_0|^2\right)dx. \tag{3.2}$$

The assumption (1.4) implies that there is a positive constant c such that

$$\rho P(\rho)\geqslant c\rho^{\gamma}-\frac{1}{c} \text{ for any } \rho\geqslant 0.$$

Thus, (3.2) ensures that ρ^{γ} and $\rho|u|^2$ are bounded in $L^{\infty}(0,T;L^1(\Omega))$. Hence,

$$\rho\in L^{\infty}(0,T;L^{\gamma}(\Omega)),\quad \rho u\in L^{\infty}(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega)).$$

Next, we turn to the estimates on the velocity. Indeed, integrating (3.1) over $\Omega\times(0,T)$, one can get that

$$\begin{aligned}&\int_{\Omega}\left(\frac{1}{2}\rho|u|^2+\rho P(\rho)\right)dx+\int_0^T\int_{\Omega}\left(|\mathbb{D}u|^r+\eta(|\mathrm{div}u|)|\mathrm{div}u|^2\right)dxdt\\ \leqslant&\int_{\Omega}\left(\rho_0P(\rho_0)+\frac{1}{2}\rho_0|u_0|^2\right)dx\end{aligned}$$

and so

$$u\in L^r(0,T;W_0^{1,r}(\Omega)),\ \eta(|\mathrm{div}u|)|\mathrm{div}u|^2\in L^1(\Omega\times(0,T)).$$

Moreover, one can use Hölder inequality to get that

$$\|\rho u\|_{L^r(0,T;L^{c_1}(\Omega))}\leqslant\|\rho\|_{L^{\infty}(0,T;L^{\gamma}(\Omega))}\|u\|_{L^r(0,T;L^{c_2}(\Omega))}$$

where c_1 and c_2 depend on the value of r.

In summary, if $\rho_0|u_0|^2\in L^1(\Omega)$ and $\rho_0P(\rho_0)\in L^1(\Omega)$, the weak solution to the problem (1.1)-(1.3) satisfies the following estimates

$$\begin{cases}\rho P(\rho)\text{ is bounded in }L^{\infty}(0,T;L^1(\Omega)),\\ \rho\text{ is bounded in }L^{\infty}(0,T;L^{\gamma}(\Omega)),\\ \rho u\text{ is bounded in }L^{\infty}(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega))\cap L^r(0,T;L^{c_1}(\Omega)),\\ u\text{ is bounded in }L^r(0,T;W_0^{1,r}(\Omega)),\\ \eta(|\mathrm{div}u|)|\mathrm{div}u|^2\text{ is bounded in }L^1(\Omega\times(0,T)),\end{cases}$$

where c_1 depends on the value of r.

4. The approximation scheme and approximation solutions

Inspired by [16] and Chapter 7 in [8], we introduce the following approximate system

$$\begin{cases}\partial_t\rho+\mathrm{div}(\rho u)=\epsilon\Delta\rho,\\ \partial_t(\rho u)+\mathrm{div}(\rho u\otimes u)+\nabla(p+\delta\rho^{\beta})+\epsilon\nabla u\cdot\nabla\rho=\mathrm{div}((\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}\mathbb{D}u+\eta(|\mathrm{div}u|)\mathrm{div}u\mathbb{I})\end{cases}\tag{4.1}$$

with the initial-boundary conditions

$$\begin{cases}\nabla\rho\cdot n|_{\partial\Omega}=0,\ \rho|_{t=0}=\rho_{0,\delta},\\ u|_{\partial\Omega}=0,\qquad \rho u|_{t=0}=m_{0,\delta},\end{cases}\tag{4.2}$$

where the operator $\Delta=\Delta_x$, ϵ and δ are two positive parameters, $\beta>0$ is a fixed constant large enough, and n is the unit outer normal of $\partial\Omega$. The initial data are chosen in such a way that

$$\begin{cases} \rho_{0,\delta} \in C^3(\overline{\Omega}), & 0 < \delta \leqslant \rho_{0,\delta} \leqslant \delta^{-\frac{1}{2\beta}}; \\ \rho_{0,\delta} \to \rho_0 \text{ in } L^\gamma(\Omega), & |\{x \in \Omega : \rho_{0,\delta}(x) < \rho_0(x)\}| \to 0 \text{ as } \delta \to 0; \\ \delta \int_\Omega \rho_{0,\delta}^\beta dx \to 0 \text{ as } \delta \to 0; & \\ m_{0,\delta} = \begin{cases} m_0, & \text{if } \rho_{0,\delta} \geqslant \rho_0, \\ 0, & \text{if } \rho_{0,\delta} < \rho_0. \end{cases} & \end{cases} \tag{4.3}$$

Modified by Section 4.1 in [9] and Proposition 7.2 in [8], we know that the approximate problem (4.1)-(4.3) with fixed positive parameters ϵ and δ can be solved by means of a modified Faedo-Galerkin method. The detailed explanation is given below.

Since $C_0^\infty(\Omega)$ is compactly and densely embedded in the Hilbert space $L^2(\Omega)$, we can choose a countable set $\{w_n\}_{n=1}^\infty \subset C_0^\infty(\Omega)$ as an orthogonal basis in the inner product $<\cdot,\cdot>_{L^2(\Omega)}$. Let X_n be the linear span of $\{w_1, w_2, \cdots, w_n\}$. For a given $u \in C([0,T]; X_n)$, let $\rho = \rho[u]$ be the unique solution to

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = \epsilon \Delta \rho, \\ \rho|_{t=0} = \rho_{0,\delta}, \\ \nabla \rho \cdot n|_{\partial\Omega} = 0. \end{cases} \tag{4.4}$$

Thanks to the standard results for the parabolic equation (see e.g. [3]), the mapping assigning the velocity field u to the solution $\rho = \rho[u]$ is continuous, from $L^1([0,T]; X_n)$ to $C([0,T]; X_n)$. Using the same arguments as in Chapter 7 of [8], we may find an approximate solution $u_n \in C([0,T]; X_n)$ required to satisfy the following integral equation

$$\begin{aligned} &\left(\int_\Omega \rho u_n \cdot \varphi dx\right)\Big|_0^\tau + \int_0^\tau \int_\Omega (-\rho u_n \otimes u_n : \nabla \varphi)\, dx dt - \int_0^\tau \int_\Omega (p(\rho) + \delta \rho^\beta) \operatorname{div} \varphi dx dt \\ &- \epsilon \int_0^\tau \int_\Omega \nabla \rho \cdot \nabla u_n \cdot \varphi dx dt + \int_0^\tau \int_\Omega (\delta + |\mathbb{D} u_n|^2)^{\frac{r-2}{2}} \mathbb{D} u_n : \mathbb{D} \varphi dx dt \\ &+ \int_0^\tau \int_\Omega \eta(|\operatorname{div} u_n|) \operatorname{div} u_n \operatorname{div} \varphi dx dt = 0 \ (\forall \tau \in [0,T]) \end{aligned} \tag{4.5}$$

for all $\varphi \in X_n$, where $\rho = \rho_n = \rho[u_n]$. Differentiating Equation (4.5) with respect to τ, integrating by parts and then choosing the test function $\varphi = u_n$, we obtain the modified energy equality for (ρ_n, u_n):

$$\begin{aligned} &\left(\int_\Omega \left(\frac{1}{2}\rho_n |u_n|^2 + \rho_n P(\rho_n) + \frac{\delta}{\beta - 1}\rho_n^\beta\right) dx\right)\Big|_0^\tau + \epsilon \int_0^\tau \int_\Omega \left(\frac{p'(\rho_n)}{\rho_n} + \delta\beta(\rho_n)^{\beta-2}\right) |\nabla \rho_n|^2 dx \\ &+ \int_0^\tau \int_\Omega \left(|\mathbb{D} u_n|^r + \eta(|\operatorname{div} u_n|)|\operatorname{div} u_n|^2\right) dx dt = 0 \ (\forall \tau \in [0,T]). \end{aligned} \tag{4.6}$$

Now, our goal is to identify a limit for $n \to \infty$ of the approximate solutions (ρ_n, u_n) as a solution of problem (4.1)-(4.3). In order to achieve this, some additional estimates are needed. It is easy to see that the energy Equation (4.6) yields that

$$\sqrt{\delta}\rho_n^{\frac{\beta}{2}} \quad \text{is bounded in } L^2(0,T; W^{1,2}(\Omega)).$$

Evoking the imbedding inequality, one can get that

$$\rho_n^\beta \quad \text{is bounded in } L^1(0,T; L^3(\Omega)).$$

Moreover,

$$\rho_n^\beta \text{ is bounded in } L^\infty(0,T;L^1(\Omega))$$

and so $\|\rho_n^\beta\|_{L^2(\Omega)} \leqslant C(\Omega)\|\rho_n^\beta\|_{L^1(\Omega)}^{\frac{1}{4}}\|\rho_n^\beta\|_{L^3(\Omega)}^{\frac{3}{4}}$. Consequently,

$$\|\rho_n\|_{L^{\beta+1}(\Omega\times(0,T))} \leqslant C(\epsilon,\delta,T,m_0,\rho_0),$$

provided that $\beta\geqslant 3$. Besides, (4.4) multiplied on ρ_n yields

$$\begin{aligned}\epsilon\int_0^T \|\nabla\rho_n(t)\|_{L^2(\Omega)}^2 dt &\leqslant \frac{1}{2}\|\rho_0\|_{L^2(\Omega)}^2 + C\int_0^T\int_\Omega \rho_n^2|\mathrm{div}u_n|dxdt \\ &\leqslant C(\rho_0) + C(\|\rho_n\|_{L^{\frac{2r}{r-1}}(\Omega)}^2 + \|\nabla u\|_{L^r(\Omega\times(0,T))}^r) \quad (\beta\geqslant\frac{2r}{r-1}).\end{aligned}$$

Thus, we deduce the estimate

$$\sqrt{\epsilon}\|\nabla\rho_n\|_{L^2(\Omega\times(0,T))} \leqslant C(\delta,T,m_0,\rho_0),$$

provided that $\beta\geqslant\max\{3,\gamma,\frac{2r}{r-1}\}$.

In conclusion, we put together all estimates obtained above.

Proposition 4.1. *Let $\beta\geqslant\max\{3,\gamma,\frac{2r}{r-1}\}$. Suppose the assumption in Theorem 2.1 is satisfied with pressure $p(\rho)$ satisfying (1.4). Then the solutions (ρ_n,u_n) to the approximate problem (4.1)-(4.3) satisfy the following estimates:*

$$\|\rho_n\|_{L^\infty(0,T;L^\gamma(\Omega))} \leqslant C(\delta,T,m_0,\rho_0), \tag{4.7}$$
$$\epsilon\|\rho_n\|_{L^\infty(0,T;L^\beta(\Omega))} \leqslant C(\delta,T,m_0,\rho_0), \tag{4.8}$$
$$\|\rho_n\|_{L^{\beta+1}(\Omega\times(0,T))} \leqslant C(\delta,T,m_0,\rho_0), \tag{4.9}$$
$$\|\sqrt{\rho_n}u_n\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C(\delta,T,m_0,\rho_0), \tag{4.10}$$
$$\|u_n\|_{L^r(0,T;W_0^{1,r}(\Omega))} \leqslant C(\delta,T,m_0,\rho_0), \tag{4.11}$$
$$\|\eta(|divu_n|)|divu_n|^2\|_{L^1(\Omega\times(0,T))} \leqslant C(\delta,T,m_0,\rho_0), \tag{4.12}$$
$$\sqrt{\epsilon}\|\nabla\rho_n\|_{L^2(\Omega\times(0,T))} \leqslant C(\delta,T,m_0,\rho_0), \tag{4.13}$$

where all the constants are independent of n.

It follows from Equation $(4.4)_1$ and the estimates obtained in Proposition 4.1 that $\partial_t\rho$ is bounded in $L^2(0,T;W^{-1,2}(\Omega))$ provided $\beta\geqslant\max\{3,\gamma,\frac{2r}{r-1}\}$. According to the Aubin-Lions Lemma, the sequence $\{\rho_n\}_{n=1}^\infty$ contains a subsequence (not relabeled) such that

$$\rho_n\to\rho \text{ strongly in } L^\beta(\Omega\times(0,T)), \tag{4.14}$$

where ρ is a non-negative function. Analogously, we can find a subsequence (not relabeled) such that

$$u_n\to u \text{ weakly in } L^r(0,T;W_0^{1,r}(\Omega)), \tag{4.15}$$

where the limit velocity u satisfies the boundary condition $u|_{\partial\Omega}=0$ in the sense of traces. It deduces from (4.14) and (4.15) that

$$\rho_n u_n\to\rho u \text{ weakly}^* \text{ in } L^\infty(0,T;L^{\frac{2\gamma}{2\gamma+1}}(\Omega)),$$

$$\eta(\mathrm{div}u_n)\mathrm{div}u_n \to \overline{\eta(\mathrm{div}u)\mathrm{div}u} \text{ weakly in } L^{\frac{q+1}{q}}(\Omega\times(0,T))$$

according to the estimates (4.10), (4.7), (4.11) and (4.12). Furthermore,

$$p(\rho_n)\to p(\rho) \text{ strongly in } L^s(\Omega\times(0,T)) \quad (s\in(1,\frac{\beta+1}{\gamma}))$$

and

$$(\rho_n)^\beta \to \rho^\beta \text{ strongly in } L^s(\Omega\times(0,T)) \quad (s\in(1,\frac{\beta+1}{\beta}))$$

provided $\beta\geqslant\max\{3,\frac{2r}{r-1},\gamma\}$.

In particular, we have proved that the limit function (ρ,u) solves problem (4.4) in $\mathcal{D}'(\Omega\times(0,T))$. In order to continue, we have to show that (4.4) holds in the strong sense and the corresponding result can be found in [8].

LEMMA 4.1 ([8]). *There exist $\alpha>1$ and $s>2$ such that*

$$\begin{aligned}&\partial_t\rho_n,\ \Delta\rho_n \text{ are bounded in } L^\alpha(\Omega\times(0,T)),\\ &\nabla\rho_n \text{ is bounded in } L^s(0,T;L^2(\Omega))\end{aligned}$$

independently of n. Accordingly, the limit function ρ belongs to the same class and satisfies the Equation $(4.4)_1$ for a.a $(x,t)\in\Omega\times(0,T)$ together with the homogeneous Neumann boundary conditions in the sense of traces. In particular,

$$\nabla\rho_n\to\nabla\rho \text{ strongly in } L^2(\Omega\times(0,T))$$

and

$$\nabla\rho_n\cdot\nabla u_n\to\nabla\rho\cdot\nabla u \text{ weakly in } L^{\frac{2r}{r+2}}(\Omega\times(0,T)). \tag{4.16}$$

Next, we turn to the limit of the approximate velocity. One can deduce from (4.5) that the following inequality

$$\begin{aligned}&\left(\frac{1}{2}\int_\Omega\rho|u|^2dx\right)\Big|_0^\tau-\left(\int_\Omega\rho u\cdot\varphi dx\right)\Big|_0^\tau+\int_0^\tau\int_\Omega(\rho u\cdot\partial_t\varphi+\rho u\otimes u:\nabla\varphi)dxdt\\ &\quad+\int_0^\tau\int_\Omega(p(\rho)+\delta\rho^\beta)(\mathrm{div}\varphi-\mathrm{div}u)dxdt-\int_0^\tau\int_\Omega\epsilon\nabla\rho\cdot\nabla u\cdot\varphi dxdt\\ &\quad+\int_0^\tau\int_\Omega\left(\overline{(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}|\mathbb{D}u|^2}-\overline{(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}\mathbb{D}u}:\mathbb{D}\varphi\right)dxdt\\ &\leqslant\int_0^\tau\int_\Omega[\Lambda(\mathrm{div}\varphi)-\Lambda(\mathrm{div}u)]dxdt \text{ for a.e. } \tau\in[0,T]\end{aligned}\tag{4.17}$$

for all $\varphi\in C_c^\infty(\Omega\times[0,T])$, with weak limit $\overline{(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}|\mathbb{D}u|^2}$ of sequence $(\delta+|\mathbb{D}u_n|^2)^{\frac{r-2}{2}}\mathbb{D}u_n:\mathbb{D}u_n$ being a measure on $\Omega\times[0,T]$. Using the idea of Zhikov and Pastukhova in [28], we intend to show $\overline{(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}|\mathbb{D}u|^2}=(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}|\mathbb{D}u|^2$ by considering a family of regularized kernels

$$\eta_h(t):=\frac{1}{h}\mathbb{I}_{[-h,0]}(t) \text{ and } \eta_{-h}(t):=\frac{1}{h}\mathbb{I}_{[0,h]}(t)\ (h>0),$$

together with the cut-off functions

$$\xi_\sigma \in C_c^\infty(0,\tau),\quad 0\leqslant \xi \leqslant 1,\quad \xi_\sigma(t)=1 \text{ whenever } t\in[\sigma,\tau-\sigma] \text{ and } \sigma>0.$$

Noticing that $\eta_h * u = \frac{1}{h}\int_t^{t+h} u ds \in W^{1,r}(0,T;W_0^{1,r}(\Omega))$, we can take the quantities

$$\varphi_{h,\sigma} = \xi_\sigma \eta_{-h} * \eta_h * (\xi_\sigma u)\ (\sigma,h>0)$$

as test functions in (4.17). Obviously, one can obtain that

$$\lim_{\sigma\to 0}\lim_{h\to 0}\int_0^\tau \int_\Omega (\Lambda(\mathrm{div}\varphi_{h,\sigma})-\Lambda(\mathrm{div}u))dxdt=0,$$
$$\left(\int_\Omega \rho u\cdot\varphi_{h,\sigma}dx\right)\Big|_0^\tau=0 \text{ (for all } \sigma,h>0),$$
$$\lim_{\sigma\to 0}\lim_{h\to 0}\int_0^\tau \int_\Omega \left(\overline{(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}|\mathbb{D}u|^2}-\overline{(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}\mathbb{D}u}:\mathbb{D}\varphi_{h,\sigma}\right)dxdt\geqslant 0.$$

Moreover, one can observe that

$$\begin{aligned}\int_0^\tau\int_\Omega \rho u\cdot\partial_t\varphi_{h,\sigma}dxdt=&\int_0^\tau\int_\Omega \rho u\cdot\partial_t\xi_\sigma\eta_{-h}*\eta_h*(\xi_\sigma u)dxdt\\&+\int_{\mathbb{R}^1}\int_\Omega(\eta_h*(\rho\xi_\sigma u))\cdot\partial_t[\eta_h*(\xi_\sigma u)]dxdt,\end{aligned}\tag{4.18}$$

where the members on the right-hand side of (4.18) are estimated as

$$\begin{aligned}&\lim_{\sigma\to 0}\lim_{h\to 0}\int_0^\tau\int_\Omega \rho u\cdot\partial_t\xi_\sigma\eta_{-h}*\eta_h*(\xi_\sigma u)dxdt\\=&\lim_{\sigma\to 0}\int_0^\tau(\frac{1}{2}\int_\Omega\rho|u|^2dx)\partial_t|\xi_\sigma|^2dt\\=&-\left(\frac{1}{2}\int_\Omega\rho|u|^2dx\right)\Big|_0^\tau\end{aligned}\tag{4.19}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^1}\int_\Omega(\eta_h*(\rho\xi_\sigma u))\cdot\partial_t[\eta_h*(\xi_\sigma u)]dxdt\\=&-\int_{\mathbb{R}^1}\int_\Omega\partial_t[\eta_h*(\rho\xi_\sigma u)]\cdot[\eta_h*(\xi_\sigma u)]dxdt\\=&-\int_{\mathbb{R}^1}\int_\Omega\frac{(\rho\xi_\sigma u)(t+h)-(\rho\xi_\sigma u)(t)}{h}\cdot[\eta_h*(\xi_\sigma u)]dxdt\\=&-\int_{\mathbb{R}^1}\int_\Omega\frac{(\rho\xi_\sigma u)(t+h)-(\rho\xi_\sigma u)(t)}{h}\cdot[\eta_h*(\xi_\sigma u)]dxdt\\&+\int_{\mathbb{R}^1}\int_\Omega\rho\frac{(\xi_\sigma u)(t+h)-(\xi_\sigma u)(t)}{h}\cdot[\eta_h*(\xi_\sigma u)]dxdt-\int_{\mathbb{R}^1}\int_\Omega\frac{1}{2}\rho\partial_t|\eta_h*(\xi_\sigma u)|^2dxdt\\=&-\int_{\mathbb{R}^1}\int_\Omega\frac{\rho(t+h)-\rho(t)}{h}(\xi_\sigma u)(t+h)\cdot[\eta_h*(\xi_\sigma u)]dxdt-\int_{\mathbb{R}^1}\int_\Omega\frac{1}{2}\rho\partial_t|\eta_h*(\xi_\sigma u)|^2dxdt.\end{aligned}\tag{4.20}$$

Note that

$$\partial_t(\eta_h*\rho)+\mathrm{div}[\eta_h*(\rho u)]=\epsilon\Delta(\eta_h*\rho).$$

Setting

$$(\rho,u)(t)=(\rho_{0,\delta},0) \text{ for } t<0 \text{ and } (\rho,u)(t)=(\rho_{0,\delta}(T),0) \text{ for } t>T,$$

one can obtain that

$$\begin{aligned}
&-\int_{\mathbb{R}^1}\int_\Omega \frac{\rho(t+h)-\rho(t)}{h}(\xi_\sigma u)(t+h)\cdot[\eta_h*(\xi_\sigma u)]dxdt\\
=&-\int_{\mathbb{R}^1}\int_\Omega \frac{\partial}{\partial t}(\eta_h*\rho)(\xi_\sigma u)(t+h)\cdot[\eta_h*(\xi_\sigma u)]dxdt\\
=&\int_{\mathbb{R}^1}\int_\Omega \operatorname{div}(\eta_h*(\rho u))(\xi_\sigma u)(t+h)\cdot[\eta_h*(\xi_\sigma u)]dxdt\\
&-\epsilon\int_{\mathbb{R}^1}\int_\Omega \Delta(\eta_h*\rho)(\xi_\sigma u)(t+h)\cdot[\eta_h*(\xi_\sigma u)]dxdt\\
=&-\int_{\mathbb{R}^1}\int_\Omega \eta_h*(\rho u)\cdot\nabla[(\xi_\sigma u)(t+h)\cdot[\eta_h*(\xi_\sigma u)]]dxdt\\
&-\epsilon\int_{\mathbb{R}^1}\int_\Omega \Delta(\eta_h*\rho)(\xi_\sigma u)(t+h)[\eta_h*(\xi_\sigma u)]dxdt
\end{aligned}$$

and

$$\begin{aligned}
&-\int_{\mathbb{R}^1}\int_\Omega \frac{1}{2}\rho\partial_t|\eta_h*(\xi_\sigma u)|^2dxdt\\
=&-\int_{\mathbb{R}^1}\int_\Omega \frac{1}{2}\partial_t[\rho|\eta_h*(\xi_\sigma u)|^2]dxdt+\int_{\mathbb{R}^1}\int_\Omega \frac{1}{2}\partial_t\rho|\eta_h*(\xi_\sigma u)|^2dxdt\\
=&-\int_{\mathbb{R}^1}\int_\Omega \frac{1}{2}\operatorname{div}(\rho u)|\eta_h*(\xi_\sigma u)|^2dxdt+\int_{\mathbb{R}^1}\int_\Omega \frac{1}{2}\epsilon\Delta\rho|\eta_h*(\xi_\sigma u)|^2dxdt\\
=&\int_{\mathbb{R}^1}\int_\Omega \frac{1}{2}\rho u\cdot\nabla|\eta_h*(\xi_\sigma u)|^2dxdt+\epsilon\int_{\mathbb{R}^1}\int_\Omega \frac{1}{2}\Delta\rho|\eta_h*(\xi_\sigma u)|^2dxdt.
\end{aligned}$$

Thus, it is not difficult to get that

$$\begin{aligned}
&\lim_{\sigma\to0}\lim_{h\to0}\Big(\int_0^\tau\int_\Omega \rho u\otimes u:\nabla\varphi_{\sigma,h}dxdt+\int_{\mathbb{R}^1}\int_\Omega \frac{1}{2}\rho u\cdot\nabla|\eta_h*(\xi_\sigma u)|^2dxdt\\
&+\epsilon\int_{\mathbb{R}^1}\int_\Omega \frac{1}{2}\Delta\rho|\eta_h*(\xi_\sigma u)|^2dxdt-\int_0^\tau\int_\Omega \epsilon\nabla\rho\cdot\nabla u\cdot\varphi_{\sigma,h}dxdt\\
&-\epsilon\int_{\mathbb{R}^1}\int_\Omega \Delta(\eta_h*\rho)(\xi_\sigma u)(t+h)\cdot[\eta_h*(\xi_\sigma u)]dxdt\\
&-\int_{\mathbb{R}^1}\int_\Omega \eta_h*(\rho u)\cdot\nabla[(\xi_\sigma u)(t+h)\cdot[\eta_h*(\xi_\sigma u)]]dxdt\Big)=0.
\end{aligned}\tag{4.21}$$

Based on the estimates (4.16)-(4.21), one can take the aforementioned calculation to arrive at

$$\int_0^\tau\int_\Omega\left(\overline{(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}|\mathbb{D}u|^2}-\overline{(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}\mathbb{D}u}:\mathbb{D}u\right)dxdt\leqslant0.$$

Then, it is deduced from Lemma 2.2 that

$$\overline{(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}\mathbb{D}u}=(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}\mathbb{D}u$$

and so

$$\overline{(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}|\mathbb{D}u|^2}=(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}|\mathbb{D}u|^2.$$

Next, one can find that

$$\overline{\eta(\mathrm{div}u)\mathrm{div}u}=\eta(\mathrm{div}u)\mathrm{div}u,$$

since $\eta(z)=z^{q-1}$, $\Lambda'(z)=\eta(z)z$ and $\Lambda''(z)\geqslant 0$ for any $z>0$.

According to the fact that the function Λ is continuous and convex, one can get the existence result for the problem (4.1)-(4.3) achieved in this section, which is stated in Proposition 4.2.

PROPOSITION 4.2. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with $C^{2+\nu}$ boundary with $\nu>0$. Let $\epsilon>0$, $\delta>0$ and $\beta>\max\{3,\frac{2r}{r-1},\gamma\}$ be fixed. Suppose the assumption in Theorem 2.1 is satisfied with pressure $p(\rho)$ satisfying (1.4). Then there exists a weak solution (ρ,u) to the approximate problem (4.1)-(4.3) such that*

$$\begin{aligned}
&\|\rho\|_{L^\infty(0,T;L^\gamma(\Omega))}\leqslant C(\delta,m_0,\rho_0),\\
&\delta\|\rho\|^\beta_{L^\infty(0,T;L^\beta(\Omega))}\leqslant C(\delta,m_0,\rho_0),\\
&\|\rho\|_{L^{\beta+1}(\Omega\times(0,T))}\leqslant C(\delta,T,m_0,\rho_0),\\
&\|\sqrt{\rho}u\|_{L^\infty(0,T;L^2(\Omega))}\leqslant C(\delta,m_0,\rho_0),\\
&\|u\|_{L^r(0,T;W_0^{1,r}(\Omega))}\leqslant C(\delta,m_0,\rho_0),\\
&\|\eta(|divu|)|divu|^2\|_{L^1(\Omega\times(0,T))}\leqslant C(\delta,m_0,\rho_0),\\
&\sqrt{\epsilon}\|\nabla\rho\|_{L^2(\Omega\times(0,T))}\leqslant C(\delta,T,m_0,\rho_0),
\end{aligned}$$

and the following weak formulation of the momentum equation

$$\begin{aligned}
&\left(\frac{1}{2}\int_\Omega\rho|u|^2dx\right)\Big|_0^\tau-\left(\int_\Omega\rho u\cdot\varphi dx\right)\Big|_0^\tau+\int_0^\tau\int_\Omega(\rho u\cdot\partial_t\varphi+\rho u\otimes u:\nabla\varphi)dxdt\\
&\quad+\int_0^\tau\int_\Omega(p(\rho)+\delta\rho^\beta)(div\varphi-divu)dxdt-\int_0^\tau\int_\Omega\epsilon\nabla\rho\cdot\nabla u\cdot\varphi dxdt\\
&\quad+\int_0^\tau\int_\Omega\Big((\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}|\mathbb{D}u|^2-(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}\mathbb{D}u:\mathbb{D}\varphi\Big)dxdt\\
&\leqslant\int_0^\tau\int_\Omega(\Lambda(div\varphi)-\Lambda(divu))dxdt \ \textit{for a.a.}\ \tau\in[0,T]
\end{aligned}$$

holds for any test function $\varphi\in C_c^\infty(\Omega\times[0,T];\mathbb{R}^3)$.

In the next two steps, in order to obtain the weak solution of the problem (1.1)-(1.3), we need to take the vanishing limits of the artificial viscosity $\epsilon\to 0$ and artificial pressure coefficient $\delta\to 0$ in the solutions to the approximate problem (4.1)-(4.3). In some sense, the step of taking $\epsilon\to 0$ is much easier than the step of taking $\delta\to 0$ due to the higher integrability of ρ, based on the technique used in the step of taking $n\to\infty$. Hence, we are going to omit the step of taking $\epsilon\to 0$ and focus on the step of taking $\delta\to 0$, since the techniques used in those two procedures are rather similar. Here, we state the result without proof as follows after taking $\epsilon\to 0$.

PROPOSITION 4.3. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with $C^{2+\nu}$ boundary with $\nu>0$. Let $\delta>0$ and $\beta>\max\{3,\frac{2r}{r-1},\gamma\}$ be fixed. Suppose the assumption in Theorem 2.1 is satisfied* *with pressure $p(\rho)$ satisfying (1.4). Then the problem (1.1)-(1.3) admits an approximate solution (ρ,u) with parameter δ (as the limit of the solutions to the approximate problem (4.1)-(4.3) when $\epsilon\to 0$) in the following sense:*

(1) *the density ρ is a non-negative function, and*

$$\rho\in C([0,T];L^{\beta}_{weak}(\Omega))$$

satisfying the initial condition in (4.2);

(2) *the functions (ρ,u) solve the equations*

$$\begin{cases}\partial_t\rho+div(\rho u)=0,\\ \partial_t(\rho u)+div(\rho u\otimes u)+\nabla(p+\delta\rho^{\beta})=div((\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}\mathbb{D}u+\eta(|divu|)divu\mathbb{I})\end{cases} \tag{4.22}$$

in $\mathcal{D}'(\Omega\times(0,T))$; moreover, $\rho\in L^{\beta+1}(\Omega\times(0,T))$ and the Equation $(4.22)_1$ holds in the sense of renormalized solutions in $\mathcal{D}'(\mathbb{R}^3\times(0,T))$ provided (ρ,u) were prolonged to be zero on $\mathbb{R}^3\backslash\Omega$;

(3) *the functions (ρ,u) satisfy the estimates*

$$\begin{aligned}&\|\rho\|_{L^\infty(0,T;L^\gamma(\Omega))}\leqslant C(T,m_0,\rho_0),\\ &\delta\|\rho\|^{\beta}_{L^\infty(0,T;L^\beta(\Omega))}\leqslant C(T,m_0,\rho_0),\\ &\delta\|\rho\|^{\beta+1}_{L^{\beta+1}(\Omega\times(0,T))}\leqslant C(\delta,T,m_0,\rho_0),\\ &\|\sqrt{\rho}u\|_{L^\infty(0,T;L^2(\Omega))}\leqslant C(T,m_0,\rho_0),\\ &\|u\|_{L^r(0,T;W^{1,r}_0(\Omega))}\leqslant C(T,m_0,\rho_0),\\ &\|\eta(|divu|)|divu|^2\|_{L^1(\Omega\times(0,T))}\leqslant C(T,m_0,\rho_0).\end{aligned}$$

In fact, the following weak formulation of the momentum equation

$$\begin{aligned}&\left(\frac{1}{2}\int_\Omega\rho|u|^2dx\right)\Big|_0^\tau-\left(\int_\Omega\rho u\cdot\varphi dx\right)\Big|_0^\tau+\int_0^\tau\int_\Omega(\rho u\cdot\partial_t\varphi+\rho u\otimes u:\nabla\varphi)dxdt\\ &\quad+\int_0^\tau\int_\Omega(p(\rho)+\delta\rho^\beta)(div\varphi-divu)dxdt\\ &\quad+\int_0^\tau\int_\Omega\Big((\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}|\mathbb{D}u|^2-(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}\mathbb{D}u:\mathbb{D}\varphi\Big)dxdt\\ &\leqslant\int_0^\tau\int_\Omega(\Lambda(div\varphi)-\Lambda(divu))dxdt\ for\ a.a.\ \tau\in[0,T]\end{aligned}\tag{4.23}$$

holds for any test function $\varphi\in C_c^\infty(\Omega\times[0,T];\mathbb{R}^3)$.

5. The limit of vanishing artificial pressure

In this section, we take the limit as $\delta\to 0$ to eliminate the $\delta-$dependent terms appearing in (4.22). Denote by $\{\rho_\delta,u_\delta\}_{\delta>0}$ the sequence of approximate solutions obtained in Proposition 4.3. Except the possible oscillation effects on density, the strong non-linear term on velocity is also a major issue of this section. To deal with these difficulties, we employ a variant of well-known Feireils-Lions method [8,17] and [28] in our *new* context.

5.1. The density estimates. Our aim in this subsection is to derive uniform estimates on ρ_δ. To begin with this, we introduce a linear operator $\mathcal{B}=[\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3]$ defined as

$$\mathcal{B}:\{f\in L^s(\Omega):\ |\Omega|^{-1}\int_\Omega f dx=0\}\mapsto W_0^{1,s}(\Omega)\ (1<s<\infty)$$

satisfying

$$\begin{aligned}&\mathrm{div}\mathcal{B}(g)=g,\quad \mathcal{B}(f)|_{\partial\Omega}=0,\\&\|\mathcal{B}(g)\|_{W^{1,s}(\Omega)}\leqslant C(p)\|g\|_{L^s(\Omega)}\quad (1<s<\infty),\\&\|\mathcal{B}(\mathrm{div}g)\|_{L^s(\Omega)}\leqslant C(p)\|g\|_{L^s(\Omega)}\quad (1<s<\infty),\end{aligned}$$

where $g\in L^s(\Omega)$ and $g\cdot n|_{\partial\Omega}=0$. Since the continuity Equation $(4.22)_1$ is satisfied in the sense of renormalized solutions in $\mathcal{D}'(\mathbb{R}^3\times(0,T))$, we can regularize the Equation $(4.22)_1$ to get that

$$\partial_t<b(\rho_\delta)>_\sigma+\mathrm{div}[<b(\rho_\delta)>_\sigma u_\delta]+<(b'(\rho_\delta)\rho_\delta-b(\rho_\delta))\mathrm{div}u_\delta>_\sigma=r_\sigma \text{ a.e. on } \mathbb{R}^3 \tag{5.1}$$

with

$$r_\sigma\to 0 \text{ strongly in } L^\lambda(\mathbb{R}^3\times(0,T)) \text{ as } \sigma\to 0\ (\frac{1}{\lambda}=\frac{1}{r}+\frac{1}{\gamma}). \tag{5.2}$$

We are going to use the operator $\mathcal{B}$ to improve the estimates of the density component by constructing multiplications of the form

$$\varphi_i(x,t)=\psi(t)\mathcal{B}_i[<b(\rho_\delta)>_\sigma-\frac{1}{|\Omega|}\int_\Omega<b(\rho_\delta)>_\sigma dx]\ (i=1,2,3) \tag{5.3}$$

with $\psi\in\mathcal{D}(0,T)$. Obviously, the functions φ_i are smooth with respect to the $x-$variable while $\partial_t\varphi_i$ are bounded in $L^2(0,T;W^{1,\lambda}(\Omega))$ in view of (5.1) and (5.2). Consequently, the quantities φ_i $(i=1,2,3)$ may be used as test functions for the Equation $(4.22)_2$. Taking (5.1) into account, one can arrive at the following formula

$$\begin{aligned}&\int_0^T\psi\int_\Omega(p(\rho_\delta)+\delta\rho_\delta^\beta)<b(\rho_\delta)>_\sigma dxdt=\int_0^T\psi\int_\Omega(p(\rho_\delta)+\delta\rho_\delta^\beta)dx\fint_\Omega<b(\rho_\delta)>_\sigma dxdt\\&\quad+\int_0^T\psi\int_\Omega(\delta+|\mathbb{D}u_\delta|^2)^{\frac{r-2}{2}}\mathbb{D}u_\delta:\mathbb{D}\mathcal{B}[<b(\rho_\delta)>_\sigma-\fint_\Omega<b(\rho_\delta)>_\sigma dx]dxdt\\&\quad+\int_0^T\psi\int_\Omega\eta(\mathrm{div}u_\delta)\mathrm{div}u_\delta\mathbb{I}:\mathbb{D}\mathcal{B}[<b(\rho_\delta)>_\sigma-\fint_\Omega<b(\rho_\delta)>_\sigma dx]dxdt\\&\quad+\int_0^T\psi\int_\Omega\rho_\delta u_\delta\cdot\mathcal{B}[<(b'(\rho_\delta)\rho_\delta-b(\rho_\delta))\mathrm{div}u_\delta>_\sigma-\fint_\Omega<(b'(\rho_\delta)\rho_\delta-b(\rho_\delta))\mathrm{div}u_\delta>_\sigma dx]dxdt\\&\quad+\int_0^T\psi\int_\Omega\rho_\delta u_\delta\cdot\mathcal{B}[\mathrm{div}(<b(\rho_\delta)>_\sigma u_\delta)]dxdt-\int_0^T\psi\int_\Omega\rho_\delta u_\delta\cdot\mathcal{B}[r_\sigma+\fint_\Omega r_\sigma dx]dxdt\\&\quad-\int_0^T\psi_t\int_\Omega\rho_\delta u_\delta\cdot\mathcal{B}[<b(\rho_\delta)>_\sigma-\fint_\Omega<b(\rho_\delta)>_\sigma dx]dxdt\\&\quad-\int_0^T\psi\int_\Omega\rho_\delta u_\delta\otimes u_\delta:\nabla\mathcal{B}[<b(\rho_\delta)>_\sigma-\fint_\Omega<b(\rho_\delta)>_\sigma dx]dxdt.\end{aligned}$$

Due to the fact that (5.2), $u_\delta\in L^r(0,T;W_0^{1,r}(\Omega))$ and $\rho_\delta\in L^\infty(0,T;L^\beta(\Omega))$, we can pass to the limit for $\sigma\to 0$ in (5.3) in two different cases according to the value of r and q.

Case 1. $\frac{12\gamma}{5\gamma-3}<r<3$ *and* $\max\{r-1,\gamma,\frac{(6+r)\gamma}{(5r-6)\gamma-3r}\}<q\leqslant\frac{11\gamma}{\gamma-3}$. Note that $\frac{12\gamma}{5\gamma-3}<3$ if and only if $\gamma>3$. Moreover, $\frac{12\gamma}{5\gamma-3}>\frac{12}{5}$ for any $\gamma>1$. We approximate the function $b(z)=z^{\frac{\gamma}{q}}$ $(q>\gamma)$ to deduce that

$$\begin{aligned}
&\int_0^T\psi\int_\Omega(p(\rho_\delta)+\delta\rho_\delta^\beta)(\rho_\delta)^{\frac{\gamma}{q}}dxdt=\int_0^T\psi\int_\Omega(p(\rho_\delta)+\delta\rho_\delta^\beta)dx\fint_\Omega(\rho_\delta)^{\frac{\gamma}{q}}dxdt\\
&+\int_0^T\psi\int_\Omega(\delta+|\mathbb{D}u_\delta|^2)^{\frac{r-2}{2}}\mathbb{D}u_\delta:\mathbb{D}\mathcal{B}[(\rho_\delta)^{\frac{\gamma}{q}}-\fint_\Omega(\rho_\delta)^{\frac{\gamma}{q}}dx]dxdt\\
&+\int_0^T\psi\int_\Omega\eta(\mathrm{div}u_\delta)\mathrm{div}u_\delta\mathbb{I}:\mathbb{D}\mathcal{B}[(\rho_\delta)^{\frac{\gamma}{q}}-\fint_\Omega(\rho_\delta)^{\frac{\gamma}{q}}dx]dxdt\\
&+(1-\frac{\gamma}{q})\int_0^T\psi\int_\Omega\rho_\delta u_\delta\cdot\mathcal{B}[\rho_\delta^{\frac{\gamma}{q}}\mathrm{div}u_\delta-\fint_\Omega\rho_\delta^{\frac{\gamma}{q}}\mathrm{div}u_\delta dx]dxdt\\
&+\int_0^T\psi\int_\Omega\rho_\delta u_\delta\cdot\mathcal{B}[\mathrm{div}(\rho_\delta^{\frac{\gamma}{q}}u_\delta)]dxdt-\int_0^T\psi_t\int_\Omega\rho_\delta u_\delta\cdot\mathcal{B}[(\rho_\delta)^{\frac{\gamma}{q}}-\fint_\Omega(\rho_\delta)^{\frac{\gamma}{q}}dx]dxdt\\
&-\int_0^T\psi\int_\Omega\rho_\delta u_\delta\otimes u_\delta:\nabla\mathcal{B}[(\rho_\delta)^{\frac{\gamma}{q}}-\fint_\Omega(\rho_\delta)^{\frac{\gamma}{q}}dx]dxdt:=\sum_{i=1}^7 I_i. \qquad (5.4)
\end{aligned}$$

Now, each term on the right hand of (5.4) can be estimated one by one as follows:

$$\begin{aligned}
|I_1|=&|\int_0^T\psi\int_\Omega(p(\rho_\delta)+\delta\rho_\delta^\beta)dx\fint_\Omega\rho_\delta^{\frac{\gamma}{q}}dxdt|\\
\leqslant& C\int_0^T(\int_\Omega(p(\rho_\delta)+\delta\rho_\delta^\beta)dx)(\int_\Omega\rho_\delta^\gamma dx)^{\frac{1}{q}}dt\\
\leqslant& C(\rho_0,m_0,T),\\
|I_2|=&|\int_0^T\psi\int_\Omega(\delta+|\mathbb{D}u_\delta|^2)^{\frac{r-2}{2}}\mathbb{D}u_\delta:\mathbb{D}\mathcal{B}[\rho_\delta^{\frac{\gamma}{q}}-\fint_\Omega\rho_\delta^{\frac{\gamma}{q}}dx]dxdt|\\
\leqslant& C\textstyle\int_0^T[\|\mathbb{D}u_\delta\|_{L^r(\Omega)}^{r-1}\|\mathcal{B}[\rho_\delta^{\frac{\gamma}{q}}-\fint_\Omega\rho_\delta^{\frac{\gamma}{q}}dx]\|_{W^{1,r}(\Omega)}+\|\mathbb{D}u_\delta\|_{L^r(\Omega)}\|\mathcal{B}[\rho_\delta^{\frac{\gamma}{q}}-\fint_\Omega\rho_\delta^{\frac{\gamma}{q}}dx]\|_{W^{1,r}(\Omega)}]dt\\
\leqslant& C\int_0^T[\|\mathbb{D}u_\delta\|_{L^r(\Omega)}^{r-1}\|\rho_\delta^{\frac{\gamma}{q}}\|_{L^r(\Omega)}+\|\mathbb{D}u_\delta\|_{L^r(\Omega)}\|\rho_\delta^{\frac{\gamma}{q}}\|_{L^r(\Omega)}]dt\\
\leqslant& C(\epsilon)\int_0^T\|\mathbb{D}u_\delta\|_{L^r(\Omega)}^r dt+\epsilon\int_0^T[\|\rho_\delta^{\frac{\gamma}{q}}\|_{L^r(\Omega)}^r+\|\rho_\delta^{\frac{\gamma}{q}}\|_{L^r(\Omega)}^{\frac{r}{r-1}}]dtdt\\
\leqslant&\varepsilon\int_0^T\int_\Omega\rho_\delta^{\frac{q+1}{q}\gamma}dxdt+C(\rho_0,m_0,T)
\end{aligned}$$

provided that $q\geqslant r-1$,

$$\begin{aligned}
|I_3|=&|\int_0^T\psi\int_\Omega\eta(\mathrm{div}u_\delta)\mathrm{div}u_\delta\mathbb{I}:\mathbb{D}\mathcal{B}[\rho_\delta^{\frac{\gamma}{q}}-\fint_\Omega\rho_\delta^{\frac{\gamma}{q}}dx]dxdt|\\
\leqslant& C\int_0^T\|\mathrm{div}u_\delta\|_{L^{q+1}(\Omega)}^q\|\rho_\delta^{\frac{\gamma}{q}}\|_{L^{q+1}(\Omega)}dt\\
\leqslant&\varepsilon\int_0^T\int_\Omega\rho_\delta^{\frac{q+1}{q}\gamma}dxdt+C(\rho_0,m_0,T),\\
|I_4|=&|1-\frac{\gamma}{q}||\int_0^T\psi\int_\Omega\rho_\delta u_\delta\cdot\mathcal{B}[(\rho_\delta^{\frac{\gamma}{q}}\mathrm{div}u_\delta)-\fint_\Omega\rho_\delta^{\frac{\gamma}{q}}\mathrm{div}u_\delta dx]dxdt|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\int_0^T \|\rho_\delta\|^{\frac12}_{L^{\frac{q+1}{q}\gamma}(\Omega)} \|\mathcal{B}[(\rho_\delta^{\frac{\gamma}{q}}\mathrm{div}u_\delta)-\fint_\Omega \rho_\delta^{\frac{\gamma}{q}}\mathrm{div}u_\delta dx]\|_{L^{\frac{2(q+1)\gamma}{(q+1)\gamma-q}}(\Omega)}dt\\
&\leqslant \varepsilon\int_0^T\int_\Omega \rho_\delta^{\frac{q+1}{q}\gamma}dxdt+C(\rho_0,m_0,T)
\end{aligned}
$$

provided that $q\leqslant\frac{11\gamma}{\gamma-3}$,

$$
\begin{aligned}
|I_5|=&|\int_0^T\psi\int_\Omega \rho_\delta u_\delta\cdot\mathcal{B}[\mathrm{div}(\rho_\delta^{\frac{\gamma}{q}}u_\delta)]dxdt|\\
\leqslant& C\int_0^T \|\rho_\delta\|^{\frac12}_{L^{\frac{q+1}{q}\gamma}(\Omega)}\|\sqrt{\rho_\delta}u_\delta\|_{L^2(\Omega)}\|\mathcal{B}[\mathrm{div}(\rho_\delta^{\frac{\gamma}{q}}u_\delta)]\|_{L^{\frac{2(q+1)\gamma}{(q+1)\gamma-q}}(\Omega)}dt\\
\leqslant& C\int_0^T \|\rho_\delta\|^{\frac12}_{L^{\frac{q+1}{q}\gamma}(\Omega)}\|\rho_\delta^{\frac{\gamma}{q}}u_\delta\|_{L^{\frac{2(q+1)\gamma}{(q+1)\gamma-q}}(\Omega)}dt\\
\leqslant& \varepsilon\int_0^T\int_\Omega \rho_\delta^{\frac{q+1}{q}\gamma}dxdt+C(\rho_0,m_0,T)
\end{aligned}
$$

provided that $q\geqslant\frac{(6+r)\gamma}{(5r-6)\gamma-3r}$ for the case $\frac{12\gamma}{5\gamma-3}<r<3$ by using the fact that $1\leqslant\frac{3r}{5r-6}\leqslant\frac32$ on the interval $[2,3]$,

$$
\begin{aligned}
|I_6|=&|\int_0^T\psi_t\int_\Omega \rho_\delta u_\delta\cdot\mathcal{B}[\rho_\delta^{\frac{\gamma}{q}}-\fint_\Omega\rho_\delta^{\frac{\gamma}{q}}dx]dxdt|\\
\leqslant& C\int_0^T|\psi_t|\|\rho_\delta\|^{\frac12}_{L^{\frac{q+1}{q}\gamma}(\Omega)}\|\sqrt{\rho_\delta}u_\delta\|_{L^2(\Omega)}\|\mathcal{B}[\rho_\delta^{\frac{\gamma}{q}}-\fint_\Omega\rho_\delta^{\frac{\gamma}{q}}dx]\|_{L^{\frac{2(q+1)\gamma}{(q+1)\gamma-q}}(\Omega)}dt\\
\leqslant& C\int_0^T|\psi_t|\|\rho_\delta\|^{\frac12}_{L^{\frac{q+1}{q}\gamma}(\Omega)}\|\mathcal{B}[\rho_\delta^{\frac{\gamma}{q}}-\fint_\Omega\rho_\delta^{\frac{\gamma}{q}}dx]\|_{W^{1,q+1}(\Omega)}dt\\
\leqslant& C\int_0^T|\psi_t|\|\rho_\delta\|^{\frac12}_{L^{\frac{q+1}{q}\gamma}(\Omega)}\|\rho_\delta\|^{\frac{\gamma}{q}}_{L^{\frac{q+1}{q}\gamma}(\Omega)}dt\\
\leqslant& \varepsilon\int_0^T\int_\Omega \rho_\delta^{\frac{q+1}{q}\gamma}dxdt+C\int_0^T|\psi_t|^{\frac{2(q+1)\gamma}{2(\gamma-1)q+\gamma}}dt
\end{aligned}
$$

for any fixed $q>1$, and

$$
\begin{aligned}
|I_7|=&|\int_0^T\psi\int_\Omega \rho_\delta u_\delta\otimes u_\delta:\nabla\mathcal{B}[\rho_\delta^{\frac{\gamma}{q}}-\fint_\Omega\rho_\delta^{\frac{\gamma}{q}}dx]dxdt|\\
\leqslant& C\int_0^T\|\rho_\delta\|^{\frac12}_{L^{\frac{q+1}{q}\gamma}(\Omega)}\|\nabla u_\delta\|_{L^r(\Omega)}\|\rho_\delta^{\frac{\gamma}{q}}\|_{L^{q+1}(\Omega)}dt\\
\leqslant& \varepsilon\int_0^T\int_\Omega \rho_\delta^{\frac{q+1}{q}\gamma}dxdt+C(\rho_0,m_0,T)
\end{aligned}
$$

provided that $q\geqslant\frac{(6+r)\gamma}{(5r-6)\gamma-3r}$ for the case $\frac{12\gamma}{5\gamma-3}<r<3$. Then,

$$
\int_0^T\int_\Omega(\rho_\delta^{\frac{q+1}{q}\gamma}+\delta\rho_\delta^{\beta+\frac{\gamma}{q}})dxdt\leqslant C
$$

holds for $\frac{12\gamma}{5\gamma-3}<r<3$ and $\max\{r-1,\gamma,\frac{(6+r)\gamma}{(5r-6)\gamma-3r}\}<q\leqslant\frac{11\gamma}{\gamma-3}$, by taking suitable small $\varepsilon>0$.

Case 2. $\frac{12\gamma}{5\gamma-3}<r<3$ and $q\geqslant\frac{r+2\gamma-1}{(r-1)(2\gamma-1)}$. We approximate the function $b(z)=z^\theta$ for some suitable θ to deduce that

$$\begin{aligned}
&\int_0^T\psi\int_\Omega(p(\rho_\delta)+\delta\rho_\delta^\beta)(\rho_\delta)^\theta dxdt=\int_0^T\psi\int_\Omega(p(\rho_\delta)+\delta\rho_\delta^\beta)dx\fint_\Omega(\rho_\delta)^\theta dxdt\\
&\quad+\int_0^T\psi\int_\Omega(\delta+|\mathbb{D}u_\delta|^2)^{\frac{r-2}{2}}\mathbb{D}u_\delta:\mathbb{D}\mathcal{B}[(\rho_\delta)^\theta-\fint_\Omega(\rho_\delta)^\theta dx]dxdt\\
&\quad+\int_0^T\psi\int_\Omega\eta(\mathrm{div}u_\delta)\mathrm{div}u_\delta\mathbb{I}:\mathbb{D}\mathcal{B}[(\rho_\delta)^\theta-\fint_\Omega(\rho_\delta)^\theta dx]dxdt\\
&\quad+(1-\frac{\gamma}{q})\int_0^T\psi\int_\Omega\rho_\delta u_\delta\cdot\mathcal{B}[\rho_\delta^\theta\mathrm{div}u_\delta-\fint_\Omega\rho_\delta^\theta\mathrm{div}u_\delta dx]dxdt\\
&\quad+\int_0^T\psi\int_\Omega\rho_\delta u_\delta\cdot\mathcal{B}[\mathrm{div}(\rho_\delta^\theta u_\delta)]dxdt-\int_0^T\psi_t\int_\Omega\rho_\delta u_\delta\cdot\mathcal{B}[(\rho_\delta)^\theta-\fint_\Omega(\rho_\delta)^\theta dx]dxdt\\
&\quad-\int_0^T\psi\int_\Omega\rho_\delta u_\delta\otimes u_\delta:\nabla\mathcal{B}[(\rho_\delta)^\theta-\fint_\Omega(\rho_\delta)^\theta dx]dxdt:=\sum_{i=1}^7 I_i'.
\end{aligned}$$

The estimate of I_i' $(i=1,2,\cdots,7)$ can be obtained by the similar way in the case $q>\gamma$ and also see in [7]. So, there exists C, independent of $\delta>0$, such that

$$\int_0^T\int_\Omega(\rho_\delta^{\gamma+\theta}+\delta\rho_\delta^{\beta+\theta})dxdt\leqslant C$$

with $0<\theta\leqslant\min\{1,\frac{\gamma}{r},\frac{\gamma}{q+1},\frac{5r-3}{3r}\gamma-1\}$. Since $r>\frac{12\gamma}{5\gamma-3}$, one can get

$$\mathrm{div}(\rho_\delta u_\delta\otimes u_\delta)\in L^{\frac{r}{r-1}}(0,T;W^{-1,\frac{r}{r-1}}(\Omega)).$$

So, one can increase the integrability of ρ as

$$\int_0^T\int_\Omega(\rho_\delta^{\frac{r}{r-1}\gamma}+\delta\rho_\delta^{\beta+\frac{\gamma}{r-1}})dxdt\leqslant C$$

by iterating the above step.

Case 3. $r\geqslant3$ *and* $q>1$. We can take the same argument to increase the integrability of ρ as

$$\int_0^T\int_\Omega(\rho_\delta^{\frac{r}{r-1}\gamma}+\delta\rho_\delta^{\beta+\frac{\gamma}{r-1}})dxdt\leqslant C.$$

It should be mentioned here that

$$|\int_0^T\psi\int_\Omega\rho_\delta u_\delta\cdot\mathcal{B}[(\rho_\delta^\theta\mathrm{div}u_\delta)-\fint_\Omega\rho_\delta^\theta\mathrm{div}u_\delta dx]dxdt|$$

is estimated in the iterating step, provided that $q\geqslant\frac{r+2\gamma-1}{(r-1)(2\gamma-1)}$ if $\frac{12\gamma}{5\gamma-3}<r<3$ and $q>1$ if $r\geqslant3$.

Consequently, we have the following result.

Lemma 5.1. *Let the assumption in Theorem 2.1 be satisfied with pressure $p(\rho)$ satisfying (1.4). Then there exists a positive constant C, independent of δ, such that*

$$\int_0^T\int_\Omega(\rho_\delta^{\frac{q+1}{q}\gamma}+\delta\rho_\delta^{\beta+\frac{\gamma}{q}})dxdt\leqslant C$$

holds for the case that $\frac{12\gamma}{5\gamma-3}<r<3$ *and* $\max\{r-1,\gamma,\frac{(6+r)\gamma}{(5r-6)\gamma-3r}\}<q\leqslant\frac{11\gamma}{\gamma-3}$, *and*

$$\int_0^T\int_\Omega(\rho_\delta^{\frac{r}{r-1}\gamma}+\delta\rho_\delta^{\beta+\frac{\gamma}{r-1}})dxdt\leqslant C$$

holds for the case that $\frac{12\gamma}{5\gamma-3}<r<3$ *and* $q\geqslant\frac{r+2\gamma-1}{(r-1)(2\gamma-1)}$ *and the case that* $r\geqslant 3$ *and* $q>1$.

5.2. The limit passage. Passing to subsequences as the case may be, we use the uniform energy estimates in Proposition 4.3 to get that

$$\begin{aligned}
&\rho_\delta\to\rho \text{ strongly in } C([0,T];L^\gamma_{weak}(\Omega)),\\
&u_\delta\to u \text{ weakly in } L^r(0,T;W_0^{1,r}(\Omega)),\\
&\rho_\delta u_\delta\to\rho u \text{ in } C([0,T];L^{\frac{2\gamma}{\gamma+1}}_{weak}(\Omega)),\\
&\rho_\delta^\gamma\to\overline{\rho^\gamma} \text{ weakly in } L^{\frac{r}{r-1}}(\Omega\times(0,T))
\end{aligned}$$

the case that $\frac{12\gamma}{5\gamma-3}<r<3$ and $q\geqslant\frac{r+2\gamma-1}{(r-1)(2\gamma-1)}$ and the case that $r\geqslant 3$ and $q>1$, or

$$\rho_\delta^\gamma\to\overline{\rho^\gamma} \text{ weakly in } L^{\frac{q+1}{q}}(\Omega\times(0,T))$$

the case that $\frac{12\gamma}{5\gamma-3}<r<3$ and $\max\{r-1,\gamma,\frac{(6+r)\gamma}{(5r-6)\gamma-3r}\}<q\leqslant\frac{11\gamma}{\gamma-3}$.

In fact, the limits ρ and ρu satisfy the initial condition (1.2)-(1.3). The convergence

$$\rho_\delta u_\delta^i u_\delta^j\to\rho u_i u_j \text{ in } \mathcal{D}'(\Omega\times(0,T))\ (i,j=1,2,3),$$

requires that $r>\frac{12\gamma}{5\gamma-3}$ which is our assumption. Moreover, Lemma 5.1 implies that

$$\delta\rho_\delta^\beta\to 0 \text{ in } L^1(\Omega\times(0,T)).$$

Consequently, ρ and u satisfy

$$\partial_t\rho+\mathrm{div}(\rho u)=0$$

and

$$\partial_t(\rho u)+\mathrm{div}(\rho u\otimes u)+\nabla\overline{p(\rho)}=\mathrm{div}(\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u+\eta(\mathrm{div}u)\mathrm{div}u\mathbb{I}})\tag{5.5}$$

in $\mathcal{D}'(\Omega\times(0,T))$, where the term $\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u+\eta(\mathrm{div}u)\mathrm{div}u\mathbb{I}}$ refers to the weak limit of the sequence $(\delta+|\mathbb{D}u_\delta|^2)^{\frac{r-2}{2}}\mathbb{D}u_\delta+\eta(|\mathrm{div}u_\delta|)\mathrm{div}u_\delta\mathbb{I}$.

Now, the remaining question is to prove the following two equalities

$$\overline{p(\rho)}=p(\rho) \text{ and } \overline{|\mathbb{D}u|^{r-2}\mathbb{D}u+\eta(\mathrm{div}u)\mathrm{div}u\mathbb{I}}=|\mathbb{D}u|^{r-2}\mathbb{D}u+\eta(\mathrm{div}u)\mathrm{div}u\mathbb{I}.$$

For this purpose, we need all the possibilities available.

5.3. The amplitude of oscillations. For the cut-off operators introduced by Jiang and Zhang in [16] and [8], we consider a family of functions

$$T_k(z)=kT(\frac{z}{k}) \text{ for } z\in\mathbb{R}\quad(k=1,2,\cdots),\tag{5.6}$$

where $T\in C^\infty(\mathbb{R})$ is chosen so that

$$T(z)=z \text{ for } z\leqslant 1,\ T(z)=2 \text{ for } z\geqslant 3,\ T \text{ concave}.$$

LEMMA 5.2. *Let T_k be the cut-off functions introduced in (5.6) and the assumption in Theorem 2.1 be satisfied with pressure $p(\rho)$ satisfying (1.4). Then there exists a positive constant C, independent of k, such that*

$$\limsup_{\delta\to 0}\|T_k(\rho_\delta)-T_k(\rho)\|_{L^{\frac{q+1}{q}\gamma}(\Omega\times(0,T))}\leqslant C$$

holds for the case that $\frac{12\gamma}{5\gamma-3}<r<3$ and $\max\{r-1,\gamma,\frac{(6+r)\gamma}{(5r-6)\gamma-3r}\}<q\leqslant\frac{11\gamma}{\gamma-3}$, and

$$\limsup_{\delta\to 0}\|T_k(\rho_\delta)-T_k(\rho)\|_{L^{\gamma+1}(\Omega\times(0,T))}\leqslant C$$

holds for the case that $\frac{12\gamma}{5\gamma-3}<r<3$ and $q\geqslant\frac{r+2\gamma-1}{(r-1)(2\gamma-1)}$ and the case that $r\geqslant 3$ and $q>1$.

Proof. Similarly to Section 5.3 in [8], we consider the operators

$$\mathcal{A}_i[v]=\Delta^{-1}[\partial_{x_i}v]\quad(i=1,2,3),$$

where Δ^{-1} stands for the inverse of the Laplace operator on $\mathbb{R}^3$. To be more specific, the Fourier symbol of $\mathcal{A}_i$ is

$$\hat{\mathcal{A}}_j[\xi]=\frac{-i\xi_j}{|\xi|^2}\quad(j=1,2,3).$$

Notice that $\mathrm{div}\mathcal{A}[v]=v$ and $\Delta\mathcal{A}_i=\partial_i$. The classical Mikhlin multiplier theorem yields that

$$\begin{cases}\|\mathcal{A}_i[v]\|_{W^{1,s}(\Omega)}\leqslant C(s,\Omega)\|v\|_{L^s(\mathbb{R}^3)}, & 1<s<\infty,\\ \|\mathcal{A}_i[v]\|_{L^\alpha(\Omega)}\leqslant C\|\mathcal{A}_i[v]\|_{W^{1,s}(\Omega)}\leqslant C(s,\alpha,\Omega)\|v\|_{L^s(\mathbb{R}^3)}, & \frac{1}{\alpha}\geqslant\frac{1}{s}-\frac{1}{3},\\ \|\mathcal{A}_i[v]\|_{L^\infty(\Omega)}\leqslant C(s,\Omega)\|v\|_{L^s(\mathbb{R}^3)}, & s>3.\end{cases}$$

Here, we give the proof for the case that $\frac{12\gamma}{5\gamma-3}<r<3$ and $\max\{r-1,\gamma,\frac{(6+r)\gamma}{(5r-6)\gamma-3r}\}<q\leqslant\frac{11\gamma}{\gamma-3}$. The quantities

$$\varphi_i(x,t)=\psi(t)h(x)\mathcal{A}_i[T_k^{\frac{\gamma}{q}}(\rho_\delta)]\ (i=1,2,3)$$

with $\psi\in\mathcal{D}(0,T)$ and $h\in\mathcal{D}(\Omega)$, can be taken as test function for the equation

$$\begin{aligned}&\partial_t(\rho_\delta u_\delta)+\mathrm{div}(\rho_\delta u_\delta\otimes u_\delta)+\nabla p(\rho_\delta)+\delta\nabla\rho_\delta^\beta\\ &=\mathrm{div}((\delta+|\mathbb{D}u_\delta|^2)^{\frac{r-2}{2}}\mathbb{D}u_\delta+|\mathrm{div}u_\delta|^{q-1}\mathrm{div}u_\delta\mathbb{I})\end{aligned}$$

using the fact that ρ_δ and u_δ are the renormalized solutions of the continuity Equation $(4.22)_1$ in $\mathcal{D}'(\mathbb{R}^3\times(0,T))$. A lengthy but straightforward computation shows that

$$\begin{aligned}&\int_0^T\int_\Omega\psi(t)h(x)(p(\rho_\delta)+\delta\rho_\delta^\beta)T_k^{\frac{\gamma}{q}}(\rho_\delta)dxdt\\ &\quad-\int_0^T\int_\Omega\psi(t)h(x)((\delta+|\mathbb{D}u_\delta|^2)^{\frac{r-2}{2}}\mathbb{D}u_\delta+|\mathrm{div}u_\delta|^{q-1}\mathrm{div}u_\delta\mathbb{I}):\nabla\mathcal{A}[T_k^{\frac{\gamma}{q}}(\rho_\delta)]dxdt\\ &=\int_0^T\int_\Omega\psi(t)((\delta+|\mathbb{D}u_\delta|^2)^{\frac{r-2}{2}}\mathbb{D}u_\delta+|\mathrm{div}u_\delta|^{q-1}\mathrm{div}u_\delta\mathbb{I})\nabla h\cdot\mathcal{A}[T_k^{\frac{\gamma}{q}}(\rho_\delta)]dxdt\\ &\quad-\int_0^T\int_\Omega\psi(t)(p(\rho_\delta)+\delta\rho_\delta^\beta)\nabla h\cdot\mathcal{A}[T_k^{\frac{\gamma}{q}}(\rho_\delta)]dxdt\end{aligned}$$

$$
\begin{aligned}
&-\int_0^T\int_\Omega h\rho_\delta u_\delta\cdot\{\psi_t\mathcal{A}[T_k^{\frac{\gamma}{q}}(\rho_\delta)]+\psi\mathcal{A}[(T_k^{\frac{\gamma}{q}}(\rho_\delta)-\frac{\gamma}{q}T_k^{\frac{\gamma}{q}-1}(\rho_\delta)T_k'(\rho_\delta)\rho_\delta)\mathrm{div}u_\delta]\}dxdt\\
&-\int_0^T\int_\Omega\psi(t)\rho_\delta u_\delta\otimes u_\delta\nabla h\cdot\mathcal{A}[T_k^{\frac{\gamma}{q}}(\rho_\delta)]dxdt\\
&+\int_0^T\int_\Omega\psi(t)u_\delta^i\{T_k^{\frac{\gamma}{q}}(\rho_\delta)R_{i,j}[h\rho_\delta u_\delta^j]-h\rho_\delta u_\delta^jR_{i,j}[T_k^{\frac{\gamma}{q}}(\rho_\delta)]\}dxdt
\end{aligned}
\tag{5.7}
$$

where the operators $R_{i,j}[v]=\partial_{x_j}\mathcal{A}_i[v]$ and the summation convention is used to simplify notations.

On the other hand, we consider the equation

$$
\partial_t(\rho u)+\mathrm{div}(\rho u\otimes u)+\nabla\overline{p(\rho)}=\mathrm{div}(\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u+\eta(\mathrm{div}u)\mathrm{div}u\mathbb{I}})
\tag{5.8}
$$

with the test function $\varphi_i(x,t)=\psi(t)h(x)\mathcal{A}_i[\overline{T_k^{\frac{\gamma}{q}}(\rho)}]$ $(i=1,2,3)$. Following the argument of Section 4.3 in [7], one can arrive at

$$
\begin{aligned}
&\int_0^T\int_\Omega\psi(t)h(x)[\overline{p(\rho)}\overline{T_k^{\frac{\gamma}{q}}(\rho)}-\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u+\eta(\mathrm{div}u)\mathrm{div}u\mathbb{I}}:\nabla\mathcal{A}[\overline{T_k^{\frac{\gamma}{q}}(\rho)}]dxdt\\
=&\int_0^T\int_\Omega\psi(t)\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u+\eta(\mathrm{div}u)\mathrm{div}\mathbb{I}}\nabla h\cdot\mathcal{A}[\overline{T_k^{\frac{\gamma}{q}}(\rho)}]dxdt\\
&-\int_0^T\int_\Omega\psi(t)\overline{p(\rho)}\nabla h\cdot\mathcal{A}[\overline{T_k^{\frac{\gamma}{q}}(\rho)}]dxdt\\
&-\int_0^T\int_\Omega h\rho u\cdot\{\psi_t\mathcal{A}[\overline{T_k^{\frac{\gamma}{q}}(\rho)}]+\psi\mathcal{A}[\overline{(T_k^{\frac{\gamma}{q}}(\rho)-\frac{\gamma}{q}T_k^{\frac{\gamma}{q}-1}(\rho)T_k'(\rho)\rho)\mathrm{div}u}]\}dxdt\\
&-\int_0^T\int_\Omega\psi(t)\rho u\otimes u\nabla h\cdot\mathcal{A}[\overline{T_k^{\frac{\gamma}{q}}(\rho)}]dxdt\\
&+\int_0^T\int_\Omega\psi(t)u^i\{\overline{T_k^{\frac{\gamma}{q}}(\rho)}R_{i,j}[h\rho u^j]-h\rho u^jR_{i,j}[\overline{T_k^{\frac{\gamma}{q}}(\rho)}]\}dxdt.
\end{aligned}
$$

The Div-Curl Lemma can be used in order to show that the right-hand side of (5.7) converges to that of (5.8), that is,

$$
\begin{aligned}
&\lim_{\delta\to0}\int_0^T\int_\Omega\psi(t)h(x)p(\rho_\delta)T_k^{\frac{\gamma}{q}}(\rho_\delta)dxdt\\
&\quad-\lim_{\delta\to0}\int_0^T\int_\Omega\psi(t)h(x)((\delta+|\mathbb{D}u_\delta|^2)^{\frac{r-2}{2}}\mathbb{D}u_\delta+\eta(\mathrm{div}u_\delta)\mathrm{div}u_\delta\mathbb{I}):\nabla\mathcal{A}[T_k^{\frac{\gamma}{q}}(\rho_\delta)dxdt\\
=&\int_0^T\int_\Omega\psi(t)h(x)[\overline{p(\rho)}\overline{T_k^{\frac{\gamma}{q}}(\rho)}-\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u+\eta(\mathrm{div}u)\mathrm{div}u\mathbb{I}}:\nabla\mathcal{A}[\overline{T_k^{\frac{\gamma}{q}}(\rho)}]]dxdt.
\end{aligned}
$$

In other words,

$$
\begin{aligned}
&\lim_{\delta\to0}\int_0^T\int_\Omega\psi(t)h(x)p(\rho_\delta)T_k^{\frac{\gamma}{q}}(\rho_\delta)dxdt-\int_0^T\int_\Omega\psi(t)h(x)\overline{p(\rho)}\overline{T_k^{\frac{\gamma}{q}}(\rho)}dxdt\\
=&\lim_{\delta\to0}\int_0^T\int_\Omega\psi(t)h(x)((\delta+|\mathbb{D}u_\delta|^2)^{\frac{r-2}{2}}\mathbb{D}u_\delta+\eta(\mathrm{div}u_\delta)\mathrm{div}u_\delta\mathbb{I}):\nabla\mathcal{A}[T_k^{\frac{\gamma}{q}}(\rho_\delta)]dxdt
\end{aligned}
$$

$$-\int_0^T\int_\Omega \psi(t)h(x)\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u+\eta(|\mathrm{div}u|)\mathrm{div}u\mathbb{I}}:\nabla\mathcal{A}[\overline{T_k^{\frac{\gamma}{q}}(\rho)}]dxdt.$$

The convexity of functions $z\to p(z)$ and $z\to -T_k^{\frac{\gamma}{q}}(z)$, implies that

$$\begin{aligned}
&\limsup_{\delta\to 0+}\int_0^T\int_\Omega\left(p(\rho_\delta)T_k^{\frac{\gamma}{q}}(\rho_\delta)-\overline{p(\rho)}\overline{T_k^{\frac{\gamma}{q}}(\rho)}\right)dxdt\\
&=\limsup_{\delta\to 0+}\int_0^T\int_\Omega(p(\rho_\delta)-p(\rho))(T_k^{\frac{\gamma}{q}}(\rho_\delta)-T_k^{\frac{\gamma}{q}}(\rho))dxdt\\
&\quad+\int_0^T\int_\Omega(\overline{p(\rho)}-p(\rho))(T_k^{\frac{\gamma}{q}}(\rho)-\overline{T_k^{\frac{\gamma}{q}}(\rho)})dxdt\\
&\geqslant\limsup_{\delta\to 0+}\int_0^T\int_\Omega(p(\rho_\delta)-p(\rho))(T_k^{\frac{\gamma}{q}}(\rho_\delta)-T_k^{\frac{\gamma}{q}}(\rho))dxdt.
\end{aligned}$$

According to the fact

$$p(y)-p(z)=\int_z^y p'(s)ds\geqslant a_1\int_z^y s^{\gamma-1}ds\geqslant a_1\int_z^y(s-z)^{\gamma-1}ds=\frac{a_1}{\gamma}(y-z)^\gamma$$

and

$$y^\alpha-z^\alpha=\alpha\int_z^y s^{\alpha-1}ds\geqslant\alpha\int_z^y s^{\alpha-1}ds\geqslant\alpha(y-z)^\alpha\ \ (\alpha>1)$$

holds for all $y\geqslant z\geqslant 0$, one can get that

$$[p(y)-p(z)][T_k^{\frac{\gamma}{q}}(y)-T_k^{\frac{\gamma}{q}}(z)]\geqslant C|y-z|^\gamma|T_k^{\frac{\gamma}{q}}(y)-T_k^{\frac{\gamma}{q}}(z)|\geqslant C|T_k^{\frac{\gamma}{q}}(y)-T_k^{\frac{\gamma}{q}}(z)|^{q+1}$$

holds for all $y,z\geqslant 0$. So,

$$\begin{aligned}
&\lim_{\delta\to 0}\int_0^T\int_\Omega[p(\rho_\delta)T_k^{\frac{\gamma}{q}}(\rho_\delta)-\overline{p(\rho)}\overline{T_k^{\frac{\gamma}{q}}(\rho)}]dxdt\\
&=\lim_{\delta\to 0}\int_0^T\int_\Omega((\delta+|\mathbb{D}u_\delta|^2)^{\frac{r-2}{2}}\mathbb{D}u_\delta+\eta(\mathrm{div}u_\delta)\mathrm{div}u_\delta\mathbb{I}):\nabla\mathcal{A}[T_k^{\frac{\gamma}{q}}(\rho_\delta)]dxdt\\
&\quad-\int_0^T\int_\Omega\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u+\eta(|\mathrm{div}u|)\mathrm{div}u\mathbb{I}}:\nabla\mathcal{A}[\overline{T_k^{\frac{\gamma}{q}}(\rho)}]dxdt\\
&=\lim_{\delta\to 0}\int_0^T\int_\Omega((\delta+|\mathbb{D}u_\delta|^2)^{\frac{r-2}{2}}\mathbb{D}u_\delta+\eta(\mathrm{div}u_\delta)\mathrm{div}u_\delta\mathbb{I}):\nabla\mathcal{A}[T_k^{\frac{\gamma}{q}}(\rho_\delta)-\overline{T_k^{\frac{\gamma}{q}}(\rho)}]dxdt\\
&\quad+\lim_{\delta\to 0}\int_0^T\int_\Omega((\delta+|\mathbb{D}u_\delta|^2)^{\frac{r-2}{2}}\mathbb{D}u_\delta+\eta(\mathrm{div}u_\delta)\mathrm{div}u_\delta\mathbb{I}):\nabla\mathcal{A}[\overline{T_k^{\frac{\gamma}{q}}(\rho)}]dxdt\\
&\quad-\int_0^T\int_\Omega\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u+\eta(\mathrm{div}u)\mathrm{div}u\mathbb{I}}:\nabla\mathcal{A}[\overline{T_k^{\frac{\gamma}{q}}(\rho)}]dxdt\\
&=\lim_{\delta\to 0}\int_0^T\int_\Omega((\delta+|\mathbb{D}u_\delta|^2)^{\frac{r-2}{2}}\mathbb{D}u_\delta+\eta(\mathrm{div}u_\delta)\mathrm{div}u_\delta\mathbb{I}):\nabla\mathcal{A}[T_k^{\frac{\gamma}{q}}(\rho_\delta)-\overline{T_k^{\frac{\gamma}{q}}(\rho)}]dxdt\\
&\leqslant\limsup_{\delta\to 0}\{\|\mathbb{D}u_\delta\|^{r-1}_{L^r(\Omega\times(0,T))}\|\nabla\mathcal{A}[T_k^{\frac{\gamma}{q}}(\rho_\delta)-\overline{T_k^{\frac{\gamma}{q}}(\rho)}]\|_{L^r(\Omega\times(0,T))}+\\
&\quad+\|\mathrm{div}u_\delta\|^q_{L^{q+1}(\Omega\times(0,T))}\|\nabla\mathcal{A}[T_k^{\frac{\gamma}{q}}(\rho_\delta)-\overline{T_k^{\frac{\gamma}{q}}(\rho)}]\|_{L^{q+1}(\Omega\times(0,T))}\}
\end{aligned}$$

$$\leqslant C\overline{\lim_{\delta\to 0}}[\|T_k^{\frac{\gamma}{q}}(\rho_\delta)-T_k^{\frac{\gamma}{q}}(\rho)\|_{L^r(\Omega\times(0,T))}+\|T_k^{\frac{\gamma}{q}}(\rho_\delta)-\overline{T_k^{\frac{\gamma}{q}}(\rho)}\|_{L^r(\Omega\times(0,T))}$$
$$+\|T_k^{\frac{\gamma}{q}}(\rho_\delta)-T_k^{\frac{\gamma}{q}}(\rho)\|_{L^{q+1}(\Omega\times(0,T))}+\|T_k^{\frac{\gamma}{q}}(\rho)-\overline{T_k^{\frac{\gamma}{q}}(\rho)}\|_{L^{q+1}(\Omega\times(0,T))}].$$

Hence, there exists a constant C independent of k such that

$$\overline{\lim_{\delta\to 0}}\|T_k^{\frac{\gamma}{q}}(\rho_\delta)-T_k^{\frac{\gamma}{q}}(\rho)\|_{L^{q+1}(\Omega\times(0,T))}\leqslant C,$$

which, in particular, implies that

$$\overline{\lim_{\delta\to 0}}\int_0^T\int_\Omega |T_k(\rho_\delta)-T_k(\rho)|^{\frac{q+1}{q}\gamma}dxdt$$
$$\leqslant \overline{\lim_{\delta\to 0}}\frac{q}{\gamma}\int_0^T\int_\Omega |T_k^{\frac{\gamma}{q}}(\rho)+T_k^{\frac{\gamma}{q}}(\rho_\delta)|^{(\frac{q}{\gamma}-1)\frac{q+1}{q}\gamma}|T_k^{\frac{\gamma}{q}}(\rho_\delta)-T_k^{\frac{\gamma}{q}}(\rho)|^{\frac{q+1}{q}\gamma}dxdt$$
$$\leqslant C$$

under consideration the function $f(z)=z^{\frac{q}{\gamma}}$. Thus, there exists a constant C independent of k such that

$$\limsup_{\delta\to 0}\|T_k(\rho_\delta)-T_k(\rho)\|_{L^{\frac{q+1}{q}\gamma}(\Omega\times(0,T))}\leqslant C.$$

For the case that $\frac{12\gamma}{5\gamma-3}<r<3$ and $q\geqslant\frac{r+2\gamma-1}{(r-1)(2\gamma-1)}$ and the case that $r\geqslant 3$ and $q>1$, the quantities $\varphi_i(t,x)=\psi(t)h(x)\mathcal{A}_i[T_k(\rho_\delta)]$ $(i=1,2,3)$ with $\psi\in\mathcal{D}(0,T)$ and $h\in\mathcal{D}(\Omega)$, can be taken as test function for the equation

$$\partial_t(\rho_\delta u_\delta)+\mathrm{div}(\rho_\delta u_\delta\otimes u_\delta)+\nabla p(\rho_\delta)+\delta\nabla\rho_\delta^\beta$$
$$=\mathrm{div}((\delta+|\mathbb{D}u_\delta|^2)^{\frac{r-2}{2}}\mathbb{D}u_\delta+|\mathrm{div}u_\delta|^{q-1}\mathrm{div}u_\delta\mathbb{I}).$$

Using the similar argument in the case that $\frac{12\gamma}{5\gamma-3}<r<3$ and $\max\{r-1,\gamma,\frac{(6+r)\gamma}{(5r-6)\gamma-3r}\}<q\leqslant\frac{11\gamma}{\gamma-3}$, one can get that there exists a constant C, independent of k, such that

$$\overline{\lim_{\delta\to 0}}\|T_k(\rho_\delta)-T_k(\rho)\|_{L^{\gamma+1}(\Omega\times(0,T))}\leqslant C.$$

Therefore, the proof of Lemma 5.2 is completed. □

With the help of Lemma 5.2, one can derive that that the limit functions ρ and u satisfy the continuity Equation $(1.1)_1$ in the sense of renormalized solutions by taking the similar argument to prove Proposition 6.3 in [8].

We are going to complete the proof of Theorem 2.1. To this end, we start with the momentum Equations $(1.1)_2$.

5.4. The momentum equation. The first key point is to prove $\overline{\rho^\gamma}=\rho^\gamma$. The approximation solutions ρ_δ and u_δ satisfy

$$\partial_t\left(\frac{\rho_\delta|u_\delta|^2}{2}+\frac{\rho_\delta^\gamma}{\gamma-1}+\frac{\delta}{\beta-1}\rho_\delta^\beta\right)+\mathrm{div}(\frac{1}{2}\rho_\delta|u_\delta|^2u_\delta+\frac{\gamma}{\gamma-1}\rho_\delta^\gamma u_\delta+\frac{\beta}{\beta-1}\rho_\delta^\beta u_\delta)$$
$$-\mathrm{div}((\delta+|\mathbb{D}u_\delta|^2)^{\frac{r-2}{2}}\mathbb{D}u_\delta+|\mathrm{div}u_\delta|^{q-1}\mathrm{div}u_\delta\mathbb{I})u_\delta)+(\delta+|\mathbb{D}u_\delta|^2)^{\frac{r-2}{2}}|\mathbb{D}u_\delta|^2+|\mathrm{div}u_\delta|^{q+1}=0.$$

in the sense of distributions. Deducing from Lemma 2.2 and standard compactness lemmas of the Aubin-Lions type, we can get that

$$\partial_t\left(\frac{1}{2}\rho|u|^2+\frac{1}{\gamma-1}\overline{\rho^\gamma}\right)+\mathrm{div}(\frac{1}{2}\rho|u|^2u+\frac{\gamma}{\gamma-1}\overline{\rho^\gamma}u)$$
$$-\mathrm{div}(\overline{(|\mathbb{D}u|^{r-2}\mathbb{D}u+|\mathrm{div}u|^{q-1}\mathrm{div}u\mathbb{I})\ u})+\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u+|\mathrm{div}u|^{q-1}\mathrm{div}u\mathbb{I}:\mathbb{D}u}\leqslant 0 \quad (5.9)$$

holds in the sense of distributions. Now, we use φu_h as the test function of (5.5) with $\varphi\in C^\infty(\overline{\Omega\times(0,T)})$ with

$$u_h=\frac{1}{h}\int_t^{t+h}u(\cdot,s)ds$$

being the Steklov average, and pass to the limit as $h\to 0$. Note that

$$\mathrm{div}u\in L^{q+1}(\Omega\times(0,T))\text{ and }\nabla u\in L^r(\Omega\times(0,T)).$$

Moreover,

$$\rho^\gamma\in L^{\frac{r}{r-1}}(\Omega\times(0,T))$$

holds for the case that $r>\frac{12\gamma}{5\gamma-3}$ and $q\geqslant\frac{r+2\gamma-1}{(r-1)(2\gamma-1)}$ and the case that $r\geqslant 3$ and $q>1$, or

$$\rho^\gamma\in L^{\frac{q+1}{q}}(\Omega\times(0,T))$$

holds for the case that $\frac{12\gamma}{5\gamma-3}<r<3$ and $\max\{r-1,\gamma,\frac{(6+r)\gamma}{(5r-6)\gamma-3r}\}<q\leqslant\frac{11\gamma}{\gamma-3}$. The above passage to the limit gives that

$$\partial_t\left(\frac{1}{2}\rho|u|^2\right)+\mathrm{div}(\frac{1}{2}\rho|u|^2u)+\nabla\overline{\rho^\gamma}\cdot u-\mathrm{div}(\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u+|\mathrm{div}u|^{q-1}\mathrm{div}u\mathbb{I}}\ u)$$
$$+\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u+|\mathrm{div}u|^{q-1}\mathrm{div}u\mathbb{I}}:\mathbb{D}u=0 \quad (5.10)$$

also holds in the sense of distributions. Comparing (5.9) and (5.10), one can derive the inequality

$$\partial_t\overline{\rho^\gamma}+\gamma\mathrm{div}(\overline{\rho^\gamma}u)-\mathrm{div}\left(\overline{(|\mathbb{D}u|^{r-2}\mathbb{D}u+|\mathrm{div}u|^{q-1}\mathrm{div}u\mathbb{I})\ u}-\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u+|\mathrm{div}u|^{q-1}\mathrm{div}u\mathbb{I}}\ u\right)$$
$$\leqslant(\gamma-1)\nabla\overline{\rho^\gamma}\cdot u \quad (5.11)$$

holds in the sense of distributions. Moreover, the functions ρ and u satisfy the continuity Equation $(1.1)_1$ in the sense of renormalized solutions. That is,

$$\partial_t\rho^\gamma+\gamma\mathrm{div}(\rho^\gamma u)=(\gamma-1)\nabla\rho^\gamma\cdot u \quad (5.12)$$

holds in the sense of distributions. Since $\gamma>1$, we know that $\overline{\rho^\gamma}-\rho^\gamma\geqslant 0$ by convexity. So, it is deduced from (5.11) and (5.12) that

$$\partial_t(\overline{\rho^\gamma}-\rho^\gamma)+\gamma\mathrm{div}((\overline{\rho^\gamma}-\rho^\gamma)u)$$
$$-\mathrm{div}\left(\overline{(|\mathbb{D}u|^{r-2}\mathbb{D}u+|\mathrm{div}u|^{q-1}\mathrm{div}u\mathbb{I})\ u}-\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u+|\mathrm{div}u|^{q-1}\mathrm{div}u\mathbb{I}}\ u\right)$$
$$\leqslant(\gamma-1)\nabla(\overline{\rho^\gamma}-\rho^\gamma)\cdot u$$

holds in the sense of of distributions. Thus, it is deduced from Hölder inequality that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega(\overline{\rho^\gamma}-\rho^\gamma)\,dx\leqslant-(\gamma-1)\int_\Omega(\overline{\rho^\gamma}-\rho^\gamma)\,\mathrm{div}u dx\\ \leqslant{}& C\int_\Omega(\overline{\rho^\gamma}-\rho^\gamma)\,|\mathrm{div}u|dx.\end{aligned}$$

So, the condition $(\overline{\rho^\gamma}-\rho^\gamma)|_{t=0}=0$ ensures that

$$\int_\Omega(\overline{\rho^\gamma}-\rho^\gamma)\,dx\leqslant C\left(\int_\Omega(\overline{\rho^\gamma}-\rho^\gamma)^\theta\,dx\right)^{\frac{1}{\theta}} \tag{5.13}$$

holds for $t\in[0,T]$, where $\theta=\max\{\frac{q+1}{q},\frac{r}{r-1}\}$. For any given number $\alpha>0$,

$$\begin{aligned}&\left(\int_0^t\int_\Omega(\overline{\rho^\gamma}-\rho^\gamma)^\theta\,dxds\right)^{\frac{1}{\theta}}\\ ={}&\left[\left(\int_0^t\int_\Omega\left((\overline{\rho^\gamma}-\rho^\gamma)^{-\alpha\theta}\right)^{-\frac{1}{\alpha}}dxds\right)^{-\alpha}\times\left(\int_0^t\int_\Omega\left((\overline{\rho^\gamma}-\rho^\gamma)^{\alpha\theta}\right)^{\frac{1}{1+\alpha}}dxds\right)^{1+\alpha}\right.\\ &\left.\times\left(\int_0^t\int_\Omega\left((\overline{\rho^\gamma}-\rho^\gamma)^{\alpha\theta}\right)^{\frac{1}{1+\alpha}}dxds\right)^{-(1+\alpha)}\right]^{-\frac{1}{\alpha\theta}}\\ \leqslant{}&\left(\int_0^t\int_\Omega 1dxds\right)^{-\frac{1}{\alpha}}\left(\int_0^t\int_\Omega\left((\overline{\rho^\gamma}-\rho^\gamma)^{\alpha\theta}\right)^{\frac{1}{1+\alpha}}dxds\right)^{\frac{1+\alpha}{\alpha}}\\ \leqslant{}& C\left(\int_0^t\int_\Omega(\overline{\rho^\gamma}-\rho^\gamma)^{\frac{\alpha\theta}{1+\alpha}}dxds\right)^{\frac{1+\alpha}{\alpha\theta}}\end{aligned} \tag{5.14}$$

holds for $t\in[0,T]$, where $q=-\frac{1}{\alpha}$ and $p=\frac{1}{1+\alpha}$ in the inverse Hölder inequality stated in Lemma 2.1. Taking $\alpha=\frac{1}{\theta-1}$ such that $\frac{\alpha\theta}{1+\alpha}=1$, it is follows from (5.13) and (5.14) that

$$\int_\Omega(\overline{\rho^\gamma}-\rho^\gamma)\,dx\leqslant C\int_0^t\int_\Omega(\overline{\rho^\gamma}-\rho^\gamma)\,dxds. \tag{5.15}$$

Thus, Gronwall's inequality and the condition $(\overline{\rho^\gamma}-\rho^\gamma)|_{t=0}=0$ ensure that

$$\int_\Omega(\overline{\rho^\gamma}-\rho^\gamma)\,dx=0 \text{ a.e on } [0,T],$$

and so $\overline{\rho^\gamma}=\rho^\gamma$ a.e on Ω.

The second key equality $\overline{|\mathbb{D}u|^r}=|\mathbb{D}u|^r$ is obtained by applying the technique in Section 3. Taking the limit in (4.23) as $\delta\to0$, we can obtain the following inequality

$$\begin{aligned}&\left(\frac{1}{2}\int_\Omega\rho|u|^2dx\right)\Big|_0^\tau-\left(\int_\Omega\rho u\cdot\varphi dx\right)\Big|_0^\tau+\int_0^\tau\int_\Omega(\rho u\cdot\partial_t\varphi+\rho u\otimes u:\nabla\varphi)dxdt\\ &+\int_0^\tau\int_\Omega\rho^\gamma(\mathrm{div}\varphi-\mathrm{div}u)dxdt+\int_0^\tau\int_\Omega\left(\overline{|\mathbb{D}u|^r}-\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u}:\mathbb{D}\varphi\right)dxdt\\ \leqslant{}&\int_0^\tau\int_\Omega(\Lambda(\mathrm{div}\varphi)-\Lambda(\mathrm{div}u))dxdt \text{ for } a.e.\tau\in[0,T]\end{aligned} \tag{5.16}$$

holds for all $\varphi\in C_c^\infty(\Omega\times[0,T])$ with the weak limit $\overline{|\mathbb{D}u|^r}$ of the sequence $(\delta+|\mathbb{D}u_\delta|^2)^{\frac{r-2}{2}}\mathbb{D}u_\delta:\mathbb{D}u_\delta$ being a measure on $\Omega\times[0,T]$. Using the idea of regularized kernels in [28] again, a family of regularizing kernels

$$\eta_h(t):=\frac{1}{h}\mathbb{I}_{[-h,0]}(t) \text{ and } \eta_{-h}(t):=\frac{1}{h}\mathbb{I}_{[0,h]}(t)\ (h>0),$$

is considered together with the cut-off functions

$$\xi_\sigma\in C_c^\infty(0,\tau),\ \ 0\leqslant\xi\leqslant 1,\ \ \xi_\sigma(t)=1 \ \text{ whenever } t\in[\sigma,\tau-\sigma] \text{ and } \sigma>0.$$

Noticing that $\eta_h*u=\frac{1}{h}\int_t^{t+h}uds\in W^{1,r}(0,T;W_0^{1,r}(\Omega))$, we can take the quantities

$$\varphi_{h,\sigma}=\xi_\sigma\eta_{-h}*\eta_h*(\xi_\sigma u)\ (\sigma,h>0)$$

as test functions in (5.5). Obviously,

$$\begin{aligned}
&\lim_{\sigma\to0}\lim_{h\to0}\int_0^\tau\int_\Omega(\Lambda(\mathrm{div}\varphi_{h,\sigma})-\Lambda(\mathrm{div}u))dxdt=0,\\
&\left(\int_\Omega\rho u\cdot\varphi_{h,\sigma}dx\right)\Big|_0^\tau=0\ (\text{for all } \sigma,h>0),\\
&\lim_{\sigma\to0}\lim_{h\to0}\int_0^\tau\int_\Omega\left(\overline{|\mathbb{D}u|^r}-\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u}:\mathbb{D}\varphi_{h,\sigma}\right)dxdt\geqslant0.
\end{aligned}$$

On one hand

$$\begin{aligned}
\int_0^\tau\int_\Omega\rho u\cdot\partial_t\varphi_{h,\sigma}dxdt=&\int_0^\tau\int_\Omega\rho u\cdot\partial_t\xi_\sigma\eta_{-h}*\eta_h*(\xi_\sigma u)dxdt\\
&+\int_{\mathbb{R}^1}\int_\Omega(\eta_h*(\rho\xi_\sigma u))\cdot\partial_t[\eta_h*(\xi_\sigma u)]dxdt.
\end{aligned}\tag{5.17}$$

On the other hand, the terms on the left-hand side of (5.17) can be estimated as follows:

$$\begin{aligned}
&\lim_{\sigma\to0}\lim_{h\to0}\int_0^\tau\int_\Omega\rho u\cdot\partial_t\xi_\sigma\eta_{-h}*\eta_h*(\xi_\sigma u)dxdt\\
=&\lim_{\sigma\to0}\int_0^\tau(\frac{1}{2}\int_\Omega\rho|u|^2dx)\partial_t|\xi_\sigma|^2dt=-[\frac{1}{2}\int_\Omega\rho|u|^2dx]\Big|_0^\tau
\end{aligned}\tag{5.18}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^1}\int_\Omega(\eta_h*(\rho\xi_\sigma u))\cdot\partial_t[\eta_h*(\xi_\sigma u)]dxdt\\
=&-\int_{\mathbb{R}^1}\int_\Omega\partial_t[\eta_h*(\rho\xi_\sigma u)]\cdot[\eta_h*(\xi_\sigma u)]dxdt\\
=&-\int_{\mathbb{R}^1}\int_\Omega\frac{(\rho\xi_\sigma u)(t+h)-(\rho\xi_\sigma u)(t)}{h}\cdot[\eta_h*(\xi_\sigma u)]dxdt\\
=&-\int_{\mathbb{R}^1}\int_\Omega\frac{(\rho\xi_\sigma u)(t+h)-(\rho\xi_\sigma u)(t)}{h}\cdot[\eta_h*(\xi_\sigma u)]dxdt\\
&+\int_{\mathbb{R}^1}\int_\Omega\rho\frac{(\xi_\sigma u)(t+h)-(\xi_\sigma u)(t)}{h}\cdot[\eta_h*(\xi_\sigma u)]dxdt-\int_{\mathbb{R}^1}\int_\Omega\frac{1}{2}\rho\partial_t|\eta_h*(\xi_\sigma u)|^2dxdt
\end{aligned}\tag{5.19}$$

$$=-\int_{\mathbb{R}^1}\int_\Omega \frac{\rho(t+h)-\rho(t)}{h}(\xi_\sigma u)(t+h)\cdot[\eta_h*(\xi_\sigma u)]dxdt-\int_{\mathbb{R}^1}\int_\Omega \frac{1}{2}\rho\partial_t|\eta_h*(\xi_\sigma u)|^2dxdt.$$

Note that

$$\partial_t(\eta_h*\rho)+\mathrm{div}[\eta_h*(\rho u)]=0.$$

We also set

$$(\rho,u)(t)=(\rho_{0,\delta},0)\text{ for }t<0\text{ and }(\rho,u)(t)=(\rho_{0,\delta}(T),0)\text{ for }t>T.$$

So, we can obtain that

$$\begin{aligned}
&-\int_{\mathbb{R}^1}\int_\Omega \frac{\rho(t+h)-\rho(t)}{h}(\xi_\sigma u)(t+h)\cdot[\eta_h*(\xi_\sigma u)]dxdt\\
=&-\int_{\mathbb{R}^1}\int_\Omega \frac{\partial}{\partial t}(\eta_h*\rho)(\xi_\sigma u)(t+h)\cdot[\eta_h*(\xi_\sigma u)]dxdt\\
=&\int_{\mathbb{R}^1}\int_\Omega \mathrm{div}(\eta_h*(\rho u))(\xi_\sigma u)(t+h)\cdot[\eta_h*(\xi_\sigma u)]dxdt\\
=&-\int_{\mathbb{R}^1}\int_\Omega \eta_h*(\rho u)\cdot\nabla[(\xi_\sigma u)(t+h)\cdot[\eta_h*(\xi_\sigma u)]]dxdt
\end{aligned}$$

and

$$\begin{aligned}
&-\int_{\mathbb{R}^1}\int_\Omega \frac{1}{2}\rho\partial_t|\eta_h*(\xi_\sigma u)|^2dxdt\\
=&-\int_{\mathbb{R}^1}\int_\Omega \frac{1}{2}\partial_t[\rho(\eta_h*(\xi_\sigma u))^2]dxdt+\int_{\mathbb{R}^1}\int_\Omega \frac{1}{2}\partial_t\rho|\eta_h*(\xi_\sigma u)|^2dxdt\\
=&-\frac{1}{2}\int_{\mathbb{R}^1}\int_\Omega \mathrm{div}(\rho u)|\eta_h*(\xi_\sigma u)|^2dxdt\\
=&\int_{\mathbb{R}^1}\int_\Omega \frac{1}{2}\rho u\cdot\nabla|\eta_h*(\xi_\sigma u)|^2dxdt.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\lim_{\sigma\to0}\lim_{h\to0}\Big(\int_0^\tau\int_\Omega \rho u\otimes u:\nabla\varphi_{\sigma,h}dxdt+\int_{\mathbb{R}^1}\int_\Omega \frac{1}{2}\rho u\cdot\nabla[\eta_h*(\xi_\sigma u)]^2dxdt\\
&-\int_{\mathbb{R}^1}\int_\Omega \eta_h*(\rho u)\cdot\nabla[(\xi_\sigma u)(t+h)\cdot[\eta_h*(\xi_\sigma u)]]dxdt=0.
\end{aligned}\tag{5.20}$$

According to the estimate (5.16)-(5.20), one can arrive at

$$\int_0^\tau\int_\Omega\left(\overline{|\mathbb{D}u|^r}-\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u}:\mathbb{D}u\right)dxdt\leqslant0.$$

It is also deduced from Lemma 2.2 that

$$\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u}=|\mathbb{D}u|^{r-2}\mathbb{D}u\quad\text{and}\quad\overline{|\mathbb{D}u|^r}=|\mathbb{D}u|^r.$$

Recalling that $\eta(z)=z^{q-1}$, $\Lambda'(z)=\eta(z)z$ and $\Lambda''(z)\geqslant0$ for any $z>0$, one can deduce from the convexity analysis that

$$\overline{\eta(\mathrm{div}u)\mathrm{div}u}=\eta(\mathrm{div}u)\mathrm{div}u.$$

Finally, the fact that Λ is a convex and low continuous function, helps us to complete the proof of Theorem 2.1.

6. Proof of Theorem 2.2

Taking the same idea as in Section 4, we also introduce an approximate problem which consists of a system of regularized equations

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = \epsilon \Delta \rho, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla(p + \delta \rho^\beta) + \epsilon \nabla u \cdot \nabla \rho = \mathrm{div}((\delta + |\mathbb{D}u|^2)^{\frac{r-2}{2}} \mathbb{D}u + \eta(|\mathrm{div}u|)\mathrm{div}u\mathbb{I}) \end{cases} \tag{6.1}$$

with the initial-boundary conditions

$$\begin{cases} \nabla \rho \cdot n|_{\partial\Omega} = 0, \ \rho|_{t=0} = \rho_{0,\delta}, \\ u|_{\partial\Omega} = 0, \qquad \rho u|_{t=0} = m_{0,\delta}, \end{cases} \tag{6.2}$$

where the operator $\Delta = \Delta_x$, ϵ and δ are two positive parameters, $\beta > 0$ is a fixed constant large enough, and n is the unit outer normal of $\partial\Omega$. The initial data are chosen in such a way that

$$\begin{cases} \rho_{0,\delta} \in C^3(\overline{\Omega}), & 0 < \underline{\rho} \leqslant \rho_{0,\delta} \leqslant \delta^{-\frac{1}{2\beta}}; \\ \rho_{0,\delta} \to \rho_0 \text{ in } L^\gamma(\Omega), & |\{x \in \Omega : \rho_{0,\delta}(x) < \rho_0(x)\}| \to 0 \text{ as } \delta \to 0; \\ \delta \int_\Omega \rho_{0,\delta}^\beta dx \to 0 \text{ as } \delta \to 0; & \\ m_{0,\delta} = \begin{cases} m_0, & \text{if } \rho_{0,\delta} \geqslant \rho_0, \\ 0, & \text{if } \rho_{0,\delta} < \rho_0. \end{cases} & \end{cases} \tag{6.3}$$

According to the proof of Theorem 2.1, the approximate problem (6.1)-(6.3) with fixed positive parameters ϵ and δ can be solved by means of a modified Faedo-Galerkin method, and the vanishing limits of the artificial viscosity $\epsilon \to 0$ and artificial pressure coefficient $\delta \to 0$ in the solutions to the approximate problem (6.1)-(6.3) can be handled by a similar way. Moreover, the limit (ρ, u) of the approximate solutions $\{\rho_\delta, u_\delta\}_{\delta > 0}$ (maybe subsequence) is proved to satisfy the continuity Equation $(1.1)_1$ in the sense of renormalized solutions. And we can obtain the following inequality as $\delta \to 0$

$$\begin{aligned} &\left(\frac{1}{2}\int_\Omega \rho|u|^2 dx\right)\Big|_0^\tau - \left(\int_\Omega \rho u \cdot \varphi dx\right)\Big|_0^\tau + \int_0^\tau \int_\Omega (\rho u \cdot \partial_t \varphi + \rho u \otimes u : \nabla \varphi) dxdt \\ &\quad + \int_0^\tau \int_\Omega \left(\overline{p(\rho)}\mathrm{div}\varphi - \overline{p(\rho)\mathrm{div}u}\right) dxdt + \int_0^\tau \int_\Omega \left(\overline{|\mathbb{D}u|^r} - \overline{|\mathbb{D}u|^{r-2}\mathbb{D}u} : \mathbb{D}\varphi\right) dxdt \\ &\leqslant \int_0^\tau \int_\Omega (\Lambda(\mathrm{div}\varphi) - \Lambda(\mathrm{div}u)) dxdt \text{ for } a.e. \tau \in [0,T] \end{aligned} \tag{6.4}$$

holds for all $\varphi \in C_c^\infty(\Omega \times [0,T])$ with the weak limit $\overline{|\mathbb{D}u|^r}$ of the sequence $(\delta + |\mathbb{D}u_\delta|^2)^{\frac{r-2}{2}} \mathbb{D}u_\delta : \mathbb{D}u_\delta$ being a measure on $\Omega \times [0,T]$. Taking the regularized kernels in [28] again, we can arrive at

$$\int_0^\tau \int_\Omega (\overline{p(\rho)}\mathrm{div}u - \overline{p(\rho)\mathrm{div}u}) dxdt \leqslant 0 \text{ for a.a } \tau \in [0,T]. \tag{6.5}$$

In order to finish the proof, we have to establish point-wise convergence of the densities ρ_δ. Since the limit (ρ, u) satisfy the continuity Equation $(1.1)_1$ in the sense of renormalized solutions, we know that

$$\partial_t[\overline{\rho P(\rho)}] + \mathrm{div}[\overline{\rho P(\rho)}u] + \overline{p(\rho)\mathrm{div}u} = 0 \tag{6.6}$$

and

$$\partial_t[\rho P(\rho)] + \mathrm{div}[\rho P(\rho) u] + p(\rho)\mathrm{div}u = 0, \tag{6.7}$$

where $P(\rho)=\int_1^\rho \frac{p(z)}{z^2}dz$. Taking (6.5) into account, we conclude that

$$\left(\int_\Omega \left(\overline{\rho P(\rho)}-\rho P(\rho)\right)dx\right)\Big|_0^\tau = -\int_0^\tau\int_\Omega (\overline{p(\rho)\mathrm{div}u}-p(\rho)\mathrm{div}u)dxdt$$
$$\leqslant -\int_0^\tau\int_\Omega (\overline{p(\rho)}-p(\rho))\mathrm{div}u dxdt, \tag{6.8}$$

where

$$[\overline{\rho P(\rho)}-\rho P(\rho)](0,\cdot)=0.$$

The convexity of the function $zP(z)$ implies that there exists a certain $\alpha>0$ such that

$$\int_\Omega \left(\overline{\rho P(\rho)}-\rho P(\rho)\right)dx \geqslant \alpha \limsup_{\delta\to 0}\int_\Omega |\rho_\delta-\rho|^2 dx,$$

while

$$-\int_0^\tau\int_\Omega (\overline{p(\rho)}-p(\rho))\mathrm{div}u dxdt = -\lim_{\delta\to 0}\int_0^\tau\int_\Omega (p(\rho_\delta)-p(\rho))\mathrm{div}u dxdt$$
$$\leqslant \lim_{\delta\to 0}\int_0^\tau\int_\Omega p'(\rho)(\rho_\delta-\rho)\mathrm{div}u dxdt + C\limsup_{\delta\to 0}\int_0^\tau\int_\Omega |\rho_\delta-\rho|^2 dxdt.$$

Thus, one can use (6.8), together with the standard Gronwall argument, to conclude that

$$\overline{\rho P(\rho)}=\rho P(\rho).$$

In particular,

$$\rho_\delta \to \rho \text{ in } L^2(\Omega\times(0,T)), \quad \overline{p(\rho)}=p(\rho),$$
$$\int_0^\tau\int_\Omega (\overline{p(\rho)}-p(\rho))\mathrm{div}u dxdt=0.$$

Then, one can arrive at

$$\int_0^\tau\int_\Omega \left(\overline{|\mathbb{D}u|^r}-\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u}:\mathbb{D}u\right)dxdt \leqslant 0.$$

Using the fact that

$$\int_0^\tau\int_\Omega \left(\overline{|\mathbb{D}u|^r}-\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u}:\mathbb{D}u\right)dxdt \geqslant 0,$$

one can get that

$$\overline{|\mathbb{D}u|^{r-2}\mathbb{D}u}=|\mathbb{D}u|^{r-2}\mathbb{D}u \text{ and } \overline{|\mathbb{D}u|^r}=|\mathbb{D}u|^r.$$

Finally,

$$\overline{\eta(\mathrm{div}u)\mathrm{div}u}=\eta(\mathrm{div}u)\mathrm{div}u$$

is deduced from the fact that $\eta(z)=|z|^{q-1}$, $\Lambda'(z)=\eta(z)z$ and $\Lambda''(z)\geqslant 0$ for any $z>0$.

Therefore, the proof of Theorem 2.2 is completed.

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