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STABILITY OF THE PLANAR RAREFACTION WAVE TO THREE-DIMENSIONAL COMPRESSIBLE MODEL OF VISCOUS IONS MOTION*

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Abstract. The compressible Navier-Stokes-Poisson equations model the motion of viscous ions and play important roles in the study of self-gravitational viscous gaseous stars and in the simulations of charged particles in semiconductor devices and plasmas physics. This paper establishes the stability and precise large-time behavior of perturbations near the planar rarefaction wave to three-dimensional isentropic compressible Navier-Stokes-Poisson equations. The results presented in this paper are new. Previous studies focused on the one-dimensional compressible Navier-Stokes-Poisson equations and little has been done for the multi-dimensional case. In order to prove the desired asymptotic stability, we take into account both the effect of the self-consistent electrostatic potential and the decay rate of the planar rarefaction wave. Due to the complexity of the nonlinearity and the effect of the self-consistent electric field, the proof involves highly non-trivial a priori bounds.

Keywords. Navier-Stokes-Poisson equations; Planar rarefaction wave; Stability.

AMS subject classifications. 35B35; 35B40; 76N15.

1. Introduction

The three-dimensional (3D) compressible isentropic Navier-Stokes-Poisson system for viscous ions motion is given by

$$\begin{cases} \partial_t\rho+\mathrm{div}(\rho \mathrm{u})=0, \\ \partial_t(\rho \mathrm{u})+\mathrm{div}(\rho \mathrm{u}\otimes \mathrm{u})+\nabla p(\rho)=\rho\nabla\phi+\mu\Delta \mathrm{u}+(\mu+\nu)\nabla \mathrm{div}\mathrm{u}, \\ \Delta\phi=\rho-e^{-\phi}, \end{cases} \tag{1.1}$$

where $\rho(t,x_1,x_2,x_3)\geq 0$ denotes the density, $\mathrm{u}=\mathrm{u}(t,x_1,x_2,x_3)=(u_1,u_2,u_3)(t,x_1,x_2,x_3)$ denotes the velocity field and $\phi=\phi(t,x_1,x_2,x_3)$ is the electrostatic potential. The shear viscosity μ and the bulk viscosity ν both are constants satisfying the physical restrictions

$$\mu>0, \qquad \mu+\nu\geq 0.$$

The pressure $p=p(\rho)$ is given by the γ-law $p(\rho)=\frac{\rho^\gamma}{\gamma}$ with $\gamma\geq 1$ being the fluid constant. The spatial domain is taken to be $\mathbb{R}\times\mathbb{T}^2$, namely $x_1\in\mathbb{R}$ being the real line and $(x_2,x_3)\in\mathbb{T}^2:=(\mathbb{R}/\mathbb{Z})^2$ being a two-dimensional unit flat torus. (1.1) has many physical applications. For example, (1.1) models the transport of charged particles under the influence of the self-consistent electrostatic potential force as in the study of self-gravitational viscous gaseous stars. (1.1) is also useful in the simulations of the motion of charged particles in semiconductor devices and plasmas physics. Here we only consider a fluid description for ions, in which ions and electrons interact through the electrostatic potential. Electrons are assumed to be thermalized and follow a nondimensional Maxwell-Boltzmann distribution $\rho_e=e^{-\phi}$ connecting the scaled electron density ρ_e and the potential ϕ. Moreover, the electron potential is determined by the density and the

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background doping profile $b(x)$, namely, the third equation in (1.1) is $\Delta\phi=\rho-b(x)$. More background information on the compressible Navier-Stokes-Poisson equations can be found in, for example, [4, 8, 18, 29].

The compressible Navier-Stokes-Poisson system has recently attracted considerable interest and many important results have been established. In what follows, we only mention those closely related to our study. Assuming that the initial data are small perturbations near the non-vacuum constant states, Li, Matsumura and Zhang [20] were able to obtain the global existence and algebraic decay estimates for solutions of the 3D unipolar isentropic compressible Navier-Stokes-Poisson equations. Li and Zhang [22] showed the optimal decay rates of solutions in [20]. Li and Zhang [23] further obtained the decay rates for the derivatives of solutions when the initial perturbation is also in the negative Sobolev space $H^{-s}(\mathbb{R}^3)$ with $0\le s<3/2$. By analyzing the Green's function of the corresponding linearized equations, Wang and Wu [37] obtained the pointwise estimates of the solution to the unipolar isentropic compressible Navier-Stokes-Poisson system in $\mathbb{R}^n(n\ge 3)$. More recently Wang and Wang [38] established further new decay estimate of classical solutions to the unipolar isentropic compressible Navier-Stokes-Poisson equations in three and higher dimensions. More interesting results for unipolar non-isentropic, and bipolar isentropic and non-isentropic compressible Navier-Stokes-Poisson equations can be found in [21, 32, 33, 41, 42]. In addition, the global strong solution to the one-dimensional non-isentropic Navier-Stokes-Poisson system with large data for density-dependent viscosity was established in [34] and the nonexistence was discussed in [3]. The results mentioned above showed that the momentum of the Navier-Stokes-Poisson system decays at a slower rate than that of the compressible Navier-Stokes system in the absence of the electric field (see [20, 41]). This demonstrates that the electric field can affect the large-time behavior of the solution.

There are also substantial developments on the stability and large-time behavior around nonlinear wave patterns such as the stationary wave, discontinuity wave and the rarefaction wave. The stability of stationary states for the multi-dimensional isentropic compressible Navier-Stokes-Poisson system was studied by Tan, Wang and Wang in [31], and by Cai and Tan [2] in the case of non-flat doping profile and with an external force under the assumption that the gas states at far fields $\pm\infty$ are equal. Duan and Liu [10] were able to obtain the stability of rarefaction waves of the one-dimensional unipolar isentropic compressible Navier-Stokes-Poisson equations with different gas states at far fields. Another interesting and challenging problem is to study the stability of the isentropic compressible Navier-Stokes-Poisson equations on half space with different far field and different gas states at boundary. In general, the large-time behavior of solutions in the half space case is much more complicated due to boundary effect. For outflow problem on the unipolar isentropic compressible Navier-Stokes-Poisson system with doping profile, Jiang, Lai, Yin, and Zhu [17] and Wang, Zhang, and Zhang [35] studied the existence and stability of stationary solutions, respectively. Li and Zhu [25] investigated the asymptotic stability of the superposition of a stationary solution and a rarefaction wave for the out-flow problem of the unipolar compressible Navier-Stokes-Poisson equations in which the electron is assumed to be thermalized and follows the nondimensional Maxwell-Boltzmann distribution $\rho_e=e^{-\phi}$ relating the scaled electron density ρ_e and potential ϕ. We mention that there is a large literature on the stability of the two-fluids isentropic and non-isentropic compressible Navier-Stokes-Poisson system (see, e.g., [7, 11, 12, 15, 40]).

Most of the existing studies on the stability and large-time behavior near nonlinear wave patterns focus on the one-dimensional compressible Navier-Stokes-Poisson equations. Very little has been done for the corresponding multi-dimensional case. The goal of this work is to initiate the study on the three-dimensional compressible isentropic Navier-Stokes-Poisson equations. More precisely, we consider the time-asymptotic non-linear stability of the planar rarefaction wave to the initial value problem for the three-dimensional compressible isentropic Navier-Stokes-Poisson equations. The initial data to the model (1.1) are specified as follows:

$$(\rho,\mathrm{u})(x_1,x_2,x_3,0):=(\rho_0,u_{10},u_{20},u_{30})(x_1,x_2,x_3), \tag{1.2}$$

which satisfy

$$(\rho_0,u_{10},u_{20},u_{30})(x_1,x_2,x_3)\to(\rho_\pm,u_\pm,0,0) \quad \text{as } x_1\to\pm\infty. \tag{1.3}$$

In order to solve $(1.1)_3$, we also need

$$\lim_{x_1\to\pm\infty}\phi(t,x)=\phi_\pm. \tag{1.4}$$

Here $\rho_\pm$, $u_\pm$ and $\phi_\pm$ are constants satisfying

$$\rho_\pm>0, \quad \rho_\pm=e^{-\phi_\pm}. \tag{1.5}$$

The periodic boundary conditions are imposed on $(x_2,x_3)\in\mathbb{T}^2$ for the solution $(\rho,u_1,u_2,u_3,\phi)(t,x_1,x_2,x_3)$.

Duan and Liu in [10] have shown that the one-dimensional compressible Navier-Stokes-Poisson equations near the rarefaction wave converge to the Riemann problem on the corresponding one-dimensional hyperbolic conservation laws

$$\begin{cases}\rho_t+(\rho u)_{x_1}=0,\\ (\rho u)_t+(\rho u^2+p(\rho)+\rho)_{x_1}=0\end{cases} \tag{1.6}$$

with the initial data

$$(\rho_0^r,u_0^r)(x_1)=\begin{cases}(\rho_-,u_-), \text{ if } x_1<0,\\ (\rho_+,u_+), \text{ if } x_1>0,\end{cases} \tag{1.7}$$

and $\phi=-\ln\rho$. This hints that the large-time behavior of the solution to the compressible Navier-Stokes-Poisson Equations (1.1)-(1.5) is closely related to the Riemann problem on the corresponding three-dimensional compressible Euler equations

$$\begin{cases}\partial_t\rho+\operatorname{div}(\rho\mathrm{u})=0,\\ \partial_t(\rho\mathrm{u})+\operatorname{div}(\rho\mathrm{u}\otimes\mathrm{u})+\nabla(p(\rho)+\rho)=0,\end{cases} \tag{1.8}$$

with the Riemann initial data

$$(\rho_0,\mathrm{u}_0)(x_1)=\begin{cases}(\rho_-,u_-,0,0), \text{ if } x_1<0,\\ (\rho_+,u_+,0,0), \text{ if } x_1>0,\end{cases} \tag{1.9}$$

and $\phi=-\ln\rho$.

As we know from the results in [5, 6, 9, 14], there are essential differences between the one-dimensional Riemann problem (1.6)-(1.7) and the three-dimensional Riemann problem (1.8)-(1.9) even with (u_2,u_3)-component continuous on both sides of $x_1=0$ in (1.9). Partially motivated by [16, 24, 28, 39], we study the time-asymptotic nonlinear stability of the planar rarefaction wave to the three-dimensional compressible isentropic Navier-Stokes-Poisson Equations (1.1)-(1.5). We first give the description of the planar rarefaction wave to (1.8). It is well-known that the inviscid Euler system (1.6) is strictly hyperbolic for $\rho>0$ with two distinct eigenvalues

$$\lambda_1(\rho,u)=u-\sqrt{p'(\rho)+1},\quad \lambda_2(\rho,u)=u+\sqrt{p'(\rho)+1}$$

with the corresponding right eigenvectors denoted by $r_1(\rho,u)$ and $r_2(\rho,u)$, respectively, and both characteristic fields are genuinely nonlinear. The i-Riemann invariant $z_i(\rho,u)(i=1,2)$ is given by

$$z_i(\rho,u)=u+(-1)^{i+1}\int^{\rho}\sqrt{p'(\xi)+1}\,\xi^{-1}d\xi,$$

satisfying $\nabla_{(\rho,u)}z_i(\rho,u)\cdot r_i(\rho,u)\equiv 0,(i=1,2)$ for any ρ and u. Without loss of generality, we only consider the 2-rarefaction wave case. The cases of 1-rarefaction wave and the superposition of two rarefaction waves can be dealt with similarly. It is well-known that if the states $(\rho_\pm,u_\pm)$ satisfy

$$u_+-\int_{\rho_-}^{\rho_+}\sqrt{p'(\xi)+1}\,\xi^{-1}d\xi=u_-,\qquad \lambda_2(\rho_+,u_+)>\lambda_2(\rho_-,u_-),\tag{1.10}$$

i.e., 2-Riemann invariant $z_2(\rho,u)$ is constant and the second eigenvalue $\lambda_2(\rho,u)$ is expanding along the 2-rarefaction wave curve, then the Riemann problem (1.6)-(1.7) would admit a self-similar wave fan $(\rho^r,u^r)(x/t)$ which consists of only the constant states and the centered rarefaction waves (see, e.g., [19]). Then the planar rarefaction wave solution to the three-dimensional compressible Euler Equations (1.8)-(1.9) is defined by $(\rho^r,u^r,0,0)(x_1,t)$ with $(\rho^r,u^r)(x_1,t)$ being the one-dimensional rarefaction wave to (1.6)-(1.7). We also define $\phi^r=-\ln\rho^r$.

We shall use the following notation. Throughout this paper, C and c denote generic positive constants, which are independent of time t unless otherwise stated. Let $1\le p<\infty$, $L^p_{x_1}(\mathbb{R})$ and $L^p(\mathbb{R}\times\mathbb{T}^2)$ denote the space of Lebesgue measurable functions whose p-powers are integrable over $\mathbb{R}$ and $\mathbb{R}\times\mathbb{T}^2$, with the norm $\|\cdot\|_{L^p_{x_1}}=\left(\int_{\mathbb{R}}|\cdot|^p dx_1\right)^{\frac{1}{p}}$ and $\|\cdot\|_{L^p}=\left(\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\cdot|^p dx_1dx_2dx_3\right)^{\frac{1}{p}}$, respectively. For simplicity we denote dx_2dx_3 by dy, and $\|\cdot\|_{L^2}$ by $\|\cdot\|$. $L^\infty_{x_1}(\mathbb{R})$ and $L^\infty(\mathbb{R}\times\mathbb{T}^2)$ are the space of bounded measurable functions over $\mathbb{R}$ and $\mathbb{R}\times\mathbb{T}^2$, with the norm $\|\cdot\|_{L^\infty_{x_1}}=\text{ess sup}_{x_1\in\mathbb{R}}|\cdot|$ and $\|\cdot\|_{L^\infty}=\text{ess sup}_{(x_1,x_2,x_3)\in\mathbb{R}\times\mathbb{T}^2}|\cdot|$. For a non-negative integer k, $H^k(\mathbb{R}\times\mathbb{T}^2)$ denotes the standard Hilbert spaces of order k, and we write $\|\cdot\|_k$ for the usual norm of $H^k(\mathbb{R}\times\mathbb{T}^2)$. In addition, we denote by $C^0([0,T];H^k(\mathbb{R}\times\mathbb{T}^2))$ (resp. $L^2(0,T;H^k(\mathbb{R}\times\mathbb{T}^2))$) the space of continuous (resp. square integrable) functions on $[0,T]$ taking values in the space $H^k(\mathbb{R}\times\mathbb{T}^2)$.

We are now in a position to state the main results of this paper.

THEOREM 1.1. *Let the planar 2-rarefaction wave $(\rho^r,u^r,0,0)(x_1,t)$ connecting the constant states $(\rho_\pm,u_\pm,0,0)$ and satisfying (1.10) with $\rho_\pm>0$, and $\phi^r=-\ln\rho^r$. Suppose that the initial data satisfy $(\rho_0-\rho_0^r,u_{10}-u_0^r,u_{20},u_{30})\in L^2(\mathbb{R}\times\mathbb{T}^2)$, which is periodic in the directions $(x_2,x_3)\in\mathbb{T}^2$, and $(\nabla\rho_0,\nabla \mathrm{u}_0)\in H^1(\mathbb{R}\times\mathbb{T}^2)$,*

$$\|(\rho_0-\rho_0^r,u_{10}-u_0^r,u_{20},u_{30})\|+\|(\nabla\rho_0,\nabla \mathrm{u}_0)\|_1+\varepsilon\le\varepsilon_0,\tag{1.11}$$

where ε is defined in (2.5)$_2$, and the positive constant ε_0 is sufficiently small. Then the initial value problem (1.1)-(1.5) admits a unique global smooth solution $(\rho,\mathrm{u},\phi)=(\rho,u_1,u_2,u_3,\phi)$ satisfying

$$\begin{aligned}&(\rho-\rho^r,u_1-u_1^r,u_2,u_3,\phi-\phi^r)(t,x_1,x_2,x_3)\in C([0,+\infty);L^2(\mathbb{R}\times\mathbb{T}^2)),\\&(\nabla\rho,\nabla \mathrm{u},\nabla\phi)(t,x_1,x_2,x_3)\in C([0,+\infty);H^1(\mathbb{R}\times\mathbb{T}^2)),\\&(\nabla^3\mathrm{u},\nabla^3\phi)(t,x_1,x_2,x_3)\in L^2(0,+\infty;L^2(\mathbb{R}\times\mathbb{T}^2)).\end{aligned}$$

Moreover, it holds that

$$\lim_{t\to+\infty}\sup_{(x_1,x_2,x_3)\in\mathbb{R}\times\mathbb{T}^2}|(\rho,u_1,u_2,u_3,\phi)(t,x_1,x_2,x_3)-(\rho^r,u_1^r,0,0,\phi^r)(x)|=0. \quad (1.12)$$

Remark 1.1. Theorem 1.1 gives a first stability result of the planar rarefaction wave to the multi-dimensional system (1.1) with physical viscosities. Our stability analysis could also be applied to the time-asymptotic stability of the superposition of 1-rarefaction wave and 2-rarefaction wave to the three-dimensional compressible Navier-Stokes Equations (1.1) and the wave interaction estimates as in [26] will be considered additionally. Moreover, it is also interesting for researchers to study similar problems for the full compressible Navier-Stokes-Poisson system and the bipolar compressible Navier-Stokes-Poisson system. Finally, in this article we only consider the initial value problem to three-dimensional compressible Navier-Stokes-Poisson equations. However we should mention that the corresponding initial boundary value problem such as the out-flow problem and the inflow problem for the multi-dimensional compressible Navier-Stokes-Poisson equations is surely more difficult. These are expected to be done in the forthcoming papers.

The proof of Theorem 1.1 is outlined as follows. We use some of the ideas in [24] for the compressible Navier-Stokes system. However, due to the complexity of nonlinearity and the effect of the self-consistent electric field, it is highly non-trivial in establishing the suitable energy estimates for the compressible Navier-Stokes-Poisson system, as can be seen in the proof of Lemma 4.1. Our attention will be focused on the effect of the self-consistent electric field. Being different to the compressible Navier-Stokes equations, the main difficulty in this paper is to estimate the interacting terms of the potential function ϕ and the density under the case that the unknown ϕ has a slow time-decay rate (see [10]). To estimate those interacting terms, we use some of the ideas in [10] to make use of the good dissipation property of the Poisson equation by expanding the term $e^{-\phi}$, up to the third-order, around the rarefaction wave. We combine not only the equations of the density and the velocity to cancel the terms such as $(\mu+\lambda)\nabla\mathrm{div}\Psi\cdot\nabla\varphi+\mu\Delta\Psi\cdot\nabla\varphi$, but also the equations of the density and the electrostatic potential to cancel the terms such as $\nabla W\cdot\nabla\varphi$ in order to get the estimate of $\nabla\varphi$ and its derivatives.

The rest of the paper is organized as follows. In the next section, we first review a smooth approximate rarefaction wave which tends to the rarefaction wave fan uniformly as the time t tends to infinity. Then we reformulate the system for the perturbation around the approximate rarefaction wave in Section 3 and establish the a priori estimates for the perturbation in Section 4. Finally, in the last section, by applying these a priori estimates, we prove Theorem 1.1.

2. Smooth approximation rarefaction wave

In this section, we will construct a smooth approximation rarefaction wave in order to overcome the difficulty that the rarefaction wave is only Lipschitz continuous.

As in [26, 39], we construct a smooth approximation of the rarefaction waves through the Burgers equation. For this, we make use of the Riemann problem of the Burgers equation:

$$\begin{cases} w_t + w w_{x_1} = 0, \\ w(x_1,0) = w_0(x_1) = \begin{cases} w_-, \ x_1 < 0, \\ w_+, \ x_1 > 0. \end{cases} \end{cases} \tag{2.1}$$

If $w_- < w_+$, it is well known that problem (2.1) admits a continuous weak solution $w^r(\frac{x_1}{t})$ connecting w_- and w_+ (see, for instance, [30]), taking the form of

$$w^r\left(\frac{x_1}{t}\right) = \begin{cases} w_-, & x_1 < w_- t, \\ \frac{x_1}{t}, & w_- t \le x_1 \le w_+ t, \\ w_+, & x_1 > w_+ t. \end{cases} \tag{2.2}$$

Recall the definition of λ_2 and set $w_- = \lambda_2(\rho_-, u_-), w_+ = \lambda_2(\rho_+, u_+)$. It is easy to check that the 2-rarefaction wave $(\rho^r, u^r)(t, x_1) = (\rho^r, u^r)(x_1/t)$ to the Riemann problem (1.6)-(1.7) is given explicitly by $\lambda_2(\rho^r, u^r)(t, x_1) = w^r(t, x_1), z_2(\rho^r, u^r)(t, x) = z_2(\rho_\pm, u_\pm)$. That is, $(\rho^r, u^r)(t, x)$ satisfies the following Riemann problem of the Euler equations

$$\begin{cases} \rho^r_t + (\rho^r u^r)_{x_1} = 0, \\ \rho^r (u^r_t + u^r u^r_{x_1}) + (p'(\rho^r) + 1)\rho^r_{x_1} = 0, \end{cases} \tag{2.3}$$

and

$$(\rho, u)(0, x_1) = (\rho_0, u_0)(x_1) = \begin{cases} (\rho_-, u_-), & \text{if } x_1 < 0, \\ (\rho_+, u_+), & \text{if } x_1 > 0. \end{cases} \tag{2.4}$$

Further, let $\phi^r = -\ln \rho^r$.

We now turn to the approximate rarefaction wave for the Euler system (2.3)-(2.4). Here and in what follows, the constant states $(\rho_\pm, u_\pm)$ are fixed so that they are connected by the 2-rarefaction wave. Following [26, 27], we recall that $w^r(\frac{x_1}{t})$ may be approximated by a smooth function $w(x_1, t)$ which solves

$$\begin{cases} w_t + w w_{x_1} = 0, \\ w(x_1, 0) = w_0(x) = \frac{1}{2}(w_- + w_+) + \frac{1}{2}(w_+ - w_-)\tanh(\varepsilon x_1). \end{cases} \tag{2.5}$$

Here $0 < \varepsilon \le 1$ is a constant to be determined later on. Then, by the characteristic methods, the solution $w(t, x_1)$ of the problem (2.5) has the following properties and their proofs can be found in [26, 27].

Lemma 2.1. *Let $\tilde{w} = w_+ - w_- > 0$ be the wave strength of the 2-rarefaction wave. Then the problem (2.5) has a unique smooth solution $w(t, x)$ which satisfies the following properties*

(i) $w_- < w(t, x_1) < w_+$, $w_{x_1} > 0$ for $x_1 \in \mathbb{R}$ and $t \ge 0$.

(ii) For any $1 \le p \le +\infty$ there exists a constant C_p such that for $t > 0$,

$$\|w_{x_1}\|_{L^p_{x_1}} \le C_p \min\left\{\tilde{w}\varepsilon^{1-\frac{1}{p}}, \tilde{w}^{\frac{1}{p}} t^{-1+\frac{1}{p}}\right\},$$
$$\|\partial^i w\|_{L^p_{x_1}} \le C_p \min\left\{\tilde{w}\varepsilon^{i-\frac{1}{p}}, \varepsilon^{i-1-\frac{1}{p}} t^{-1}\right\}, \ \textit{where } i = 2, 3.$$

(iii) $\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}\left|w(t,x_1)-w^r\left(\frac{x_1}{t}\right)\right|=0.$

Now we shall approximate the rarefaction wave $(\rho^r,u^r,\phi^r)(\frac{x_1}{t})$ by the smooth function $(\bar\rho,\bar u,\bar\phi)(t,x_1)$, which can be constructed by

$$\lambda_2(\bar\rho,\bar u)(t,x_1)=w(1+t,x_1),\quad z_2(\bar\rho,\bar u)(t,x_1)=z_2(\rho_\pm,u_\pm),\quad \bar\phi=-\ln\bar\rho.$$

Here $w(t,x_1)$ is the smooth solution to the Burgers equation in (2.5). One can easily check that the above approximate rarefaction wave $(\bar\rho,\bar u,\bar\phi)(t,x_1)$ satisfies the system:

$$\begin{cases}\bar\rho_t+(\bar\rho\bar u)_{x_1}=0,\\ \bar\rho(\bar u_t+\bar u\bar u_{x_1})+p(\bar\rho)_{x_1}=-\bar\rho_{x_1},\end{cases}\tag{2.6}$$

with

$$(\bar\rho,\bar u)(0,x_1)=\begin{cases}(\rho_-,u_-),\text{ if } x_1<0,\\ (\rho_+,u_+),\text{ if } x_1>0,\end{cases}\tag{2.7}$$

and

$$\bar\phi=-\ln\bar\rho.\tag{2.8}$$

With Lemma 2.1 at our disposal, we have the following result concerning $(\bar\rho,\bar u,\bar\phi)(t,x_1)$.

LEMMA 2.2. *Set $\delta:=|\rho_+-\rho_-|+|u_+-u_-|$. The smoothed rarefaction wave $(\bar\rho,\bar u,\bar\phi)(t,x_1)$ which satisfies (2.6)-(2.8), possesses the following properties:*

(i) $\rho_{x_1}>0,\bar u_{x_1}>0$, and $\rho_-<\bar\rho(t,x_1)<\rho_+$, $u_-<\bar u(t,x_1)<u_+$ for $x_1\in\mathbb{R}$ and $t\ge 0$.

(ii) For any $1\le p\le+\infty$, there exists a constant C_p such that for $t>0$,

$$\|(\bar\rho_{x_1},\bar u_{x_1},\bar\phi_{x_1})\|_{L^p_{x_1}}\le C_p\min\left\{\delta\varepsilon^{1-\frac1p},\delta^{\frac1p}t^{-1+\frac1p}\right\},$$
$$\|(\partial^i_{x_1}\bar\rho,\partial^i_{x_1}\bar u,\partial^i_{x_1}\bar\phi)\|_{L^p_{x_1}}\le C_p\min\left\{\delta\varepsilon^{i-\frac1p},\varepsilon^{i-1-\frac1p}t^{-1}\right\},\ \text{here } i=2,3.$$

(iii) $\lim_{t\to+\infty}\sup_{x_1\in\mathbb{R}}\left|\left(\bar\rho,\bar u,\bar\phi\right)(t,x_1)-(\rho^r,u^r,\phi^r)\left(\frac{x_1}{t}\right)\right|=0.$

3. Reformulation of the problem

In this section, we reformulate the original problem (1.1)-(1.5) in terms of the perturbed variables. To begin with, we define the new unknowns (φ,Ψ,W) by

$$\varphi(t,x_1,x_2,x_3)=\rho(t,x_1,x_2,x_3)-\bar\rho(t,x_1),$$
$$\Psi(t,x_1,x_2,x_3)=(\psi_1,\psi_2,\psi_3)(t,x_1,x_2,x_3)=(u_1,u_2,u_3)(t,x_1,x_2,x_3)-(\bar u,0,0)(t,x_1),$$
$$W(t,x_1,x_2,x_3)=\phi(t,x_1,x_2,x_3)-\bar\phi(t,x_1).$$

Then from (1.1), (2.6) and (2.8), it is easy to check that the perturbed variable (φ,Ψ,W) satisfies

$$\begin{cases}\varphi_t+\rho\operatorname{div}\Psi+\rho_{x_2}\psi_2+\rho_{x_3}\psi_3+u_1\varphi_x=f,\\ \rho(\Psi_t+u_1\Psi_{x_1}+\psi_2\Psi_{x_2}+\psi_3\Psi_{x_3})+p'(\rho)\nabla\varphi=\mu\Delta\Psi+(\mu+\lambda)\nabla\operatorname{div}\Psi+\rho\nabla W+\rho g,\\ \Delta W=\varphi+\bar\rho\left(1-e^{-W}\right)-\bar\phi_{x_1x_1},\end{cases}\tag{3.1}$$

where f and g are given by

$$f=-\varphi\bar u_{x_1}-\psi_1\bar\rho_{x_1},$$

$$g=\Big(\frac{2\mu+\lambda}{\rho}\bar{u}_{x_1x_1},0,0\Big)^t-(\bar{u}_{x_1}\psi_1,0,0)^t-\Big(\Big(\frac{p'(\rho)}{\rho}-\frac{p'(\bar{\rho})}{\bar{\rho}}p'(\bar{\rho})\Big)\bar{\rho}_{x_1},0,0\Big)^t.$$

The initial conditions are

$$\begin{aligned}(\varphi,\Psi)(0,x_1,x_2,x_3)=:(\varphi_0,\Psi_0)(x_1,x_2,x_3)&=(\varphi_0,\psi_{10},\psi_{20},\psi_{30})(x_1,x_2,x_3)\\&=(\rho_0-\bar{\rho}_0,u_{10}-\bar{u}_0,u_{20},u_{30})(x_1,x_2,x_3),\end{aligned}\tag{3.2}$$

and the far-field condition becomes

$$\lim_{x_1\to\pm\infty}W(t,x_1,x_2,x_3)=0.\tag{3.3}$$

Our main result in terms of the perturbed variable $(\varphi,\Psi,W)(t,x_1,x_2,x_3)$ can then be restated as follows.

THEOREM 3.1. *Suppose that all the assumptions of Theorem 1.1 are met. Then there exists a unique global solution $(\varphi,\Psi,W)(t,x_1,x_2,x_3)$ to problem (3.1)-(3.3), satisfying*

$$\begin{aligned}&(\varphi,\Psi,W)(t,x_1,x_2,x_3)\in C([0,+\infty);H^2(\mathbb{R}\times\mathbb{T}^2)),\\&\nabla\varphi(t,x_1,x_2,x_3)\in L^2([0,+\infty);H^1(\mathbb{R}\times\mathbb{T}^2)),\\&(\nabla\Psi,\nabla W)(t,x_1,x_2,x_3)\in L^2(0,+\infty;H^2(\mathbb{R}\times\mathbb{T}^2)),\end{aligned}$$

and

$$\sup_{(x_1,x_2,x_3)\in\mathbb{R}\times\mathbb{T}^2}|(\varphi,\Psi,W)(t,x_1,x_2,x_3)|\to 0 \text{ as } t\to+\infty.\tag{3.4}$$

To prove this theorem, we shall employ the standard continuation argument based on a local existence theorem and on *a priori* estimates stated in the following proposition.

PROPOSITION 3.1. *Assume that $(\varphi,\Psi,W)(t,x_1,x_2,x_3)$ is the classical solution to problem* (3.1)-(3.3) *satisfying*

$$\begin{aligned}&(\varphi,\Psi,W)(t,x_1,x_2,x_3)\in C([0,T];H^2(\mathbb{R}\times\mathbb{T}^2)),\\&\nabla\varphi(t,x_1,x_2,x_3)\in L^2([0,T];H^1(\mathbb{R}\times\mathbb{T}^2)),\\&(\nabla\Psi,\nabla W)(t,x_1,x_2,x_3)\in L^2([0,T];H^2(\mathbb{R}\times\mathbb{T}^2))\end{aligned}$$

for any fixed $T>0$. Then there exists a suitably small constant $\varepsilon_1>0$ such that if

$$\sup_{0\le t\le T}\|(\varphi,\psi,W)(t)\|_2^2+\varepsilon\le\varepsilon_1,\tag{3.5}$$

then the following estimate holds:

$$\begin{aligned}&\sup_{0\le t\le T}\|(\varphi,\Psi,W)(t)\|_2^2+\int_0^T\Big[\|\bar{u}_x^{1/2}(\varphi,\psi_1)\|^2+\|(\nabla\varphi,\nabla\Psi,\nabla W)\|_1^2+\|(\nabla^3\Psi,\nabla^3W)\|^2\Big]dt\\&\le C[\|(\varphi_0,\Psi_0)\|_2^2+\varepsilon^{1/8}].\end{aligned}\tag{3.6}$$

4. The energy estimates

This section is devoted to the derivation of the *a priori* estimates for the unknown function $(\varphi,\Psi,W)(t,x_1,x_2,x_3)$ and their derivatives, that is, we are going to show Proposition 3.1. To derive those *a priori* estimates, we assume that there exists a solution $(\varphi,\Psi,W)(t,x_1,x_2,x_3)$ to the problem (3.1)-(3.3), such that

$$\begin{aligned}
&(\varphi,\Psi,W)(t,x_1,x_2,x_3)\in C([0,T];H^2(\mathbb{R}\times\mathbb{T}^2),\\
&\nabla\varphi(t,x_1,x_2,x_3)\in L^2([0,T];H^1(\mathbb{R}\times\mathbb{T}^2)),\\
&(\nabla\Psi,\nabla W)(t,x_1,x_2,x_3)\in L^2([0,T];H^2(\mathbb{R}\times\mathbb{T}^2)),\\
&\sup_{t\in[0,T]}\|(\varphi,\Psi,W)(t)\|_2\le\varepsilon_1
\end{aligned}$$

for any $T>0$. For the sake of simplicity, we set $E=\sup_{t\in[0,T]}\|(\varphi,\Psi,W)(t)\|_2$. Note that if ε_1 is suitably small, then the condition $\sup_{t\in[0,T]}\|(\varphi,\Psi,W)(t)\|_2\le\varepsilon_1$ and Sobolev embedding theorem:

$$\|f(x_1,x_2,x_3)\|_{L^\infty}\le C\|f(x_1,x_2,x_3)\|_2 \text{ for any} f(x_1,x_2,x_3)\in H^2(\mathbb{R}\times\mathbb{T}^2), \tag{4.1}$$

imply that

$$|\varphi|\le\frac{1}{2}\rho_-,$$

which, together with Lemma 2.2, yields

$$0<\frac{1}{2}\rho_-\le\rho\le\frac{1}{2}\rho_-+\rho_+. \tag{4.2}$$

We also have

$$|\mathrm{u}|=|(u_1,u_2,u_3)|\le C,\ |W|\le C. \tag{4.3}$$

For the sake of clarity, we will divide the proof of Proposition 3.1 into some lemmas. First, we establish the energy estimate for the unknown variable $(\varphi,\Psi,W)(t,x_1,x_2,x_3)$ to problem (3.1)-(3.3). For this, we introduce

$$\Phi(\rho,\bar\rho)=\int_{\bar\rho}^{\rho}\frac{p(\eta)-p(\bar n)}{\eta^2}d\eta.$$

Combining this and (4.2) yields

$$c\varphi^2\le\Phi(\rho,\bar\rho)\le C\varphi^2. \tag{4.4}$$

Next, multiplying (1.1) and $(3.1)_2$ by $\Phi(\rho,\bar\rho)$ and Ψ, respectively, and after tedious computations, we arrive at

$$\begin{aligned}
&(\frac{1}{2}\rho|\Psi|^2+\rho\Phi)_t+\mu|\nabla\Psi|^2+(\mu+\lambda)|\mathrm{div}\Psi|^2+\rho\bar u_x\psi_1^2+[p(\rho)-p(\bar\rho)-p'(\bar\rho)\varphi]\bar u_x\\
&=\mathrm{div}R_1+(2\mu+\lambda)\bar u_{xx}\psi_1+\rho\Psi\nabla W,
\end{aligned}\tag{4.5}$$

where

$$R_1=\mu(\psi_1\nabla\psi_1+\psi_2\nabla\psi_2+\psi_3\nabla\psi_3)+(\mu+\lambda)\Psi\mathrm{div}\Psi-\Big[\rho u\Phi+\frac{1}{2}\rho u|\Psi|^2+(p(\rho)-p(\bar\rho))\Psi\Big].$$

Then we have the following lemma.

LEMMA 4.1. *Assume that* $(\varphi,\Psi,W)(t,x_1,x_2,x_3)$ *is a solution to* (3.1)-(3.3), *satisfying the conditions in Proposition 3.1, then there exists a positive constant* C *such that the following estimate holds*

$$\sup_{0\le t\le T}[\|(\varphi,\Psi,W)(t)\|^2+\|\nabla W(t)\|^2]+\int_0^T\Big[\|\bar u_{x_1}^{\frac12}(\varphi,\psi_1)(t)\|^2+\|\nabla\Psi(t)\|^2\Big]dt$$
$$\le C(\|(\varphi_0,\Psi_0)\|^2+\|W_0\|_1^2+\varepsilon^{\frac18})+C(\eta+E+\varepsilon)\int_0^T\|(\nabla\varphi,\nabla W,\nabla^2W)(t)\|^2dt. \quad (4.6)$$

Here and in the subsequent, $\eta>0$ *is a suitably small constant.*

Proof. Integrating (4.5) with respect to (x_1,x_2,x_3) over $\mathbb{R}\times\mathbb{T}^2$ yields

$$\frac{d}{dt}\int_{\mathbb{T}^2}\int_{\mathbb{R}}(\frac12\rho\Psi^2+\rho\Phi)dx_1dy+\mu\|\nabla\Psi\|^2+(\mu+\lambda)\|\mathrm{div}\Psi\|^2+\int_{\mathbb{T}^2}\int_{\mathbb{R}}\bar u_{x_1}[\rho\psi_1^2+p(\rho)$$
$$-p(\bar\rho)-p'(\bar\rho)\varphi]dx_1dy=(2\mu+\lambda)\int_{\mathbb{T}^2}\int_{\mathbb{R}}\bar u_{x_1x_1}\psi_1dx_1dy+\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\Psi\nabla Wdx_1dy. \quad (4.7)$$

First, using (4.2) and (4.4), it is obvious that

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}}(\frac12\rho\Psi^2+\rho\Phi)dx_1dy\ge c(\|\varphi\|^2+\|\Psi\|^2) \quad (4.8)$$

and

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}}\bar u_x[\rho\bar u_x\psi_1^2+p(\rho)-p(\bar\rho)-p'(\bar\rho)\varphi]dx_1dy\ge c\int_{\mathbb{T}^2}\int_{\mathbb{R}}\bar u_{x_1}(\psi_1^2+\varphi^2)dx_1dy. \quad (4.9)$$

Second, we employ Lemma 2.2 and the Sobolev's inequality

$$\|f\|_{L^\infty_{x_1}}\le\sqrt2\|f\|_{L^2_{x_1}}^{\frac12}\|f_{x_1}\|_{L^2_{x_1}}^{\frac12} \quad (4.10)$$

for any $f(x_1)\in H^1(\mathbb{R})$, to obtain

$$\begin{aligned}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\bar u_{xx}\psi_1dx_1dy&\le C\int_{\mathbb{T}^2}\|\bar u_{xx}\|_{L^1_{x_1}}\|\psi_1\|_{L^\infty_{x_1}}dy\\&\le C\varepsilon^{\frac18}\int_{\mathbb{T}^2}(1+t)^{-\frac78}\|\psi_1\|_{L^2_{x_1}}^{\frac12}\|\psi_{1x_1}\|_{L^2_{x_1}}^{\frac12}dy\\&\le C\varepsilon^{\frac18}\int_{\mathbb{T}^2}[\|\psi_{1x_1}\|_{L^2_{x_1}}^2+C(1+t)^{-\frac76}\|\psi_1\|_{L^2_{x_1}}^{\frac23}]dy\\&\le C\varepsilon^{\frac18}[\|\psi_{x_1}\|^2+(1+t)^{-\frac98}+(1+t)^{-\frac54}\|\psi\|^2].\end{aligned} \quad (4.11)$$

Finally, we deal with the term $\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\Psi\nabla Wdx_1dy$. Noting $(3.1)_1$ and $(3.1)_3$, we have

$$\mathrm{div}(\rho\Psi)=-\varphi_t-(\bar u\varphi)_{x_1}, \quad (4.12)$$

and

$$\Delta W_t=\varphi_t+[\bar\rho(1-e^{-W})]_t-\bar\phi_{x_1x_1t}. \quad (4.13)$$

which, together with integration by parts, leads to

$$\begin{aligned}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\Psi\nabla W dx_1dy &= -\int_{\mathbb{T}^2}\int_{\mathbb{R}}\mathrm{div}(\rho\Psi)W dx_1dy\\ = -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\nabla W|^2dx_1dy &+ \int_{\mathbb{T}^2}\int_{\mathbb{R}}W\bar{\phi}_{x_1x_1t}dx_1dy\\ -\int_{\mathbb{T}^2}\int_{\mathbb{R}}W(\bar{\rho}(1-e^{-W}))_tdx_1dy &+ \int_{\mathbb{T}^2}\int_{\mathbb{R}}W(\bar{u}\varphi)_{x_1}dx_1dy. \end{aligned}\tag{4.14}$$

Invoking Lemma 2.2, by integration by parts, Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned}\int_{\mathbb{T}^2}\int_{\mathbb{R}}W\bar{\phi}_{x_1x_1t}dx_1dy &= -\int_{\mathbb{T}^2}\int_{\mathbb{R}}W_{x_1}\bar{\phi}_{x_1t}dx_1dy\\ &\le C\int_{\mathbb{T}^2}\int_{\mathbb{R}}|W_{x_1}(\bar{\rho}_{x_1}^2+\bar{\rho}_{x_1x_1}+\bar{\rho}_{x_1}\bar{u}_{x_1}+\bar{u}_{x_1x_1})|dx_1dy\\ &\le \eta\|W_{x_1}\|^2+C\varepsilon(1+t)^{-2}.\end{aligned}\tag{4.15}$$

For the third term on the right-hand side of (4.14), we note

$$1-e^{-W}=W-\frac{1}{2}W^2+R_2,\tag{4.16}$$

where R_2 is the Taylor remainder. This implies

$$\begin{aligned}-\int_{\mathbb{T}^2}\int_{\mathbb{R}}W(\bar{\rho}(1-e^{-W}))_tdx_1dy &= -\int_{\mathbb{T}^2}\int_{\mathbb{R}}W(\bar{\rho}(W-\frac{1}{2}W^2+R_2))_tdx_1dy\\ = -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\bar{\rho}W^2dx_1dy &+ \frac{1}{3}\frac{d}{dt}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\bar{\rho}W^3dx_1dy+H_1+H_2,\end{aligned}\tag{4.17}$$

where

$$\begin{aligned}H_1&=\int_{\mathbb{T}^2}\int_{\mathbb{R}}(\bar{\rho}\bar{u})_{x_1}(\frac{1}{2}W^2-\frac{1}{6}W^3)dx_1dy,\\ H_2&=\int_{\mathbb{T}^2}\int_{\mathbb{R}}W(\bar{\rho}\bar{u})_{x_1}R_2dx_1dy-\int_{\mathbb{T}^2}\int_{\mathbb{R}}W\bar{\rho}R_{2t}dx_1dy.\end{aligned}$$

We remark that H_1 cannot be directly controlled for the time being and it will be treated by cancelation with other terms later on. Here we first deal with H_2. Since, $\|W\|_{L^\infty}\le C$, it follows that

$$R_2\sim O(W^3),\ R_{2t}\sim O(W^2W_t).\tag{4.18}$$

Further, making use of Young's inequality, we have

$$H_2\le C(\|W^3\|^2+\|(\bar{\rho}\bar{u})_{x_1}W\|^2)+\eta\|W_t\|^2.\tag{4.19}$$

In addition, we get from $(3.1)_1$ and $(3.1)_3$ that

$$\begin{aligned}\|W_t\|^2+\|\nabla W_t\|^2 &\le C(\|\varphi_t\|^2+\|\bar{\rho}_tW\|^2+\|\bar{\phi}_{x_1x_1t}\|^2)\\ &\le C[\|\mathrm{div}(\varphi\Psi)\|^2+\|(\bar{u}\varphi)_{x_1}\|^2+\|\mathrm{div}(\bar{\rho}\Psi)\|^2+\|\bar{\rho}_tW\|^2+\|\bar{\phi}_{x_1x_1t}\|^2]\end{aligned}$$

$$\begin{aligned}&\leq C[\|(\nabla\varphi,\nabla\Psi)(t)\|^2+\|\bar{u}_{x_1}\varphi\|^2+\|\bar{\rho}_{x_1}\psi_1\|^2\\&\quad+\|(\bar{\rho}\bar{u})_{x_1}W\|^2+\|\bar{\phi}_{x_1x_1t}\|^2].\end{aligned}\tag{4.20}$$

Applying the interpolation inequality in the domain $\mathbb{D}=:\mathbb{R}\times\mathbb{T}^2$ in [1,36], that is,

$$\|g\|^2_{L^\infty(\mathbb{D})}\leq C(\|g\|_{L^2(\mathbb{D})}\|\nabla g\|_{L^2(\mathbb{D})}+\|\nabla g\|_{L^2(\mathbb{D})}\|\nabla^2 g\|_{L^2(\mathbb{D})})$$

for $g\in H^2(\mathbb{D})$, we have

$$\begin{aligned}\|W^3\|^2&\leq C\|W\|^4_{L^\infty}\|W\|^2\\&\leq C\|W\|^2(\|W\|^2\|\nabla W\|^2+\|\nabla W\|^2\|\nabla^2W\|^2)\\&\leq CE^2(\|\nabla W\|^2+\|\nabla^2 W\|^2).\end{aligned}\tag{4.21}$$

From Hölder's inequality and Lemma 2.2, one has

$$\begin{aligned}&\|\bar{u}_{x_1}\varphi\|^2+\|\bar{\rho}_{x_1}\psi_1\|^2+\|(\bar{\rho}\bar{u})_{x_1}W\|^2+\|\bar{\phi}_{x_1x_1t}\|^2\\&\leq C\|(\bar{\rho},\bar{u})_{x_1}\|^2_{L^\infty_{x_1}}\|(\varphi,\psi_1,W)\|^2+\|\bar{\phi}_{x_1x_1t}\|^2\\&\leq C\varepsilon(1+t)^{-2}\|(\varphi,\psi_1,W)\|^2+C\varepsilon(1+t)^{-2}.\end{aligned}\tag{4.22}$$

Inserting now inequalities (4.20)-(4.22) into (4.19), we then arrive at

$$\begin{aligned}H_2\leq\,& CE(\|\nabla W\|^2+\|\nabla^2W\|^2)+C\eta\|(\nabla\varphi,\nabla\Psi)(t)\|^2\\&+C\varepsilon(1+t)^{-2}\|(\varphi,\psi_1,W)(t)\|^2+C\varepsilon(1+t)^{-2}.\end{aligned}\tag{4.23}$$

We now turn to deal with the fourth term on the right-hand side of (4.14). Utilizing $(3.1)_3$, it follows that

$$\begin{aligned}&\int_{\mathbb{T}^2}\int_{\mathbb{R}}W(\bar{u}\varphi)_{x_1}dx_1dy\\&=\int_{\mathbb{T}^2}\int_{\mathbb{R}}(W\bar{u}\varphi_{x_1}+W\varphi\bar{u}_{x_1})dx_1dy\\&=\int_{\mathbb{T}^2}\int_{\mathbb{R}}W\bar{u}\left(\Delta W_{x_1}-(\bar{\rho}(1-e^{-W}))_{x_1}+\bar{\phi}_{x_1x_1x_1}\right)dx_1dy\\&\quad+\int_{\mathbb{T}^2}\int_{\mathbb{R}}W\bar{u}_{x_1}\left(\Delta W-\bar{\rho}(1-e^{-W})+\bar{\phi}_{x_1x_1}\right)dx_1dy=:H_3+H_4+H_5,\end{aligned}\tag{4.24}$$

where

$$\begin{aligned}H_3&=\frac{1}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\bar{u}_{x_1}(W^2_{x_1}-W^2_{x_2}-W^2_{x_3})dx_1dy-\int_{\mathbb{T}^2}\int_{\mathbb{R}}\bar{u}\bar{\phi}_{x_1x_1}W_{x_1}dx_1dy,\\H_4&=-\int_{\mathbb{T}^2}\int_{\mathbb{R}}W\bar{u}(\bar{\rho}(1-e^{-W}))_{x_1}dx_1dy,\\H_5&=-\int_{\mathbb{T}^2}\int_{\mathbb{R}}W\bar{u}_{x_1}\bar{\rho}(1-e^{-W})dx_1dy.\end{aligned}$$

We have used in (4.24) the following identities:

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}}W\bar{u}\Delta W_{x_1}dx_1dy+\int_{\mathbb{T}^2}\int_{\mathbb{R}}W\bar{u}_{x_1}\Delta Wdx_1dy=\frac{1}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\bar{u}_{x_1}(W^2_{x_1}-W^2_{x_2}-W^2_{x_3})dx_1dy,$$

and

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}} \bar{u}\bar{\phi}_{x_1x_1x_1} W dx_1 dy + \int_{\mathbb{T}^2}\int_{\mathbb{R}} \bar{u}_{x_1}\bar{\phi}_{xx} W dx_1 dy = -\int_{\mathbb{T}^2}\int_{\mathbb{R}} \bar{u}\bar{\phi}_{x_1x_1} W_{x_1} dx_1 dy.$$

By Lemmas 2.2, and Hölder's and Young's inequalities,

$$\begin{aligned} H_3 &\le C\int_{\mathbb{T}^2} \|\bar{u}_{x_1}\|_{L^\infty_{x_1}} \|\nabla W\|^2_{L^2_{x_1}} dy + C\int_{\mathbb{T}^2} \|W_{x_1}\|_{L^2_{x_1}} \|\bar{\phi}_{x_1x_1}\|_{L^2_{x_1}} dy \\ &\le \eta\|W_{x_1}\|^2 + C\varepsilon(1+t)^{-2} + C\varepsilon\|\nabla W\|^2. \end{aligned} \tag{4.25}$$

To estimate H_4 and H_5, we use the Taylor expansion (4.16) to get

$$\begin{aligned} H_4 &= -\int_{\mathbb{T}^2}\int_{\mathbb{R}} W\bar{u}[\bar{\rho}(W - \frac{1}{2}W^2 + R_2)]_{x_1} dx_1 dy \\ &= -\int_{\mathbb{T}^2}\int_{\mathbb{R}} \bar{\rho}_{x_1}\bar{u}W^2 dx_1 dy + \frac{1}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} \bar{\rho}_{x_1}\bar{u}W^3 dx_1 dy + \frac{1}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} (\bar{\rho}\bar{u})_{x_1} W^2 dx_1 dy \\ &\quad -\frac{1}{3}\int_{\mathbb{T}^2}\int_{\mathbb{R}} (\bar{\rho}\bar{u})_{x_1} W^3 dx_1 dy - \int_{\mathbb{T}^2}\int_{\mathbb{R}} W\bar{u}(\bar{\rho}R_2)_{x_1} dx_1 dy, \end{aligned}$$

and

$$\begin{aligned} H_5 &= -\int_{\mathbb{T}^2}\int_{\mathbb{R}} W\bar{u}_{x_1}\bar{\rho}(W - \frac{1}{2}W^2 + R_2) dx_1 dy \\ &= -\int_{\mathbb{T}^2}\int_{\mathbb{R}} \bar{\rho}_{x_1}\bar{u}W^2 dx_1 dy + \frac{1}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} \bar{\rho}_{x_1}\bar{u}W^3 dx_1 dy - \int_{\mathbb{T}^2}\int_{\mathbb{R}} \bar{\rho}\bar{u}_{x_1} W R_2 dx_1 dy. \end{aligned}$$

Then, with the help of integration by parts and (4.21), summation of H_1, H_4 and H_5 yields

$$\begin{aligned} H_1 + H_4 + H_5 &= -\int_{\mathbb{T}^2}\int_{\mathbb{R}} W\bar{u}(\bar{\rho}R_2)_{x_1} dx_1 dy - \int_{\mathbb{T}^2}\int_{\mathbb{R}} W\bar{u}_{x_1}\bar{\rho}R_2 dx_1 dy \\ &= \int_{\mathbb{T}^2}\int_{\mathbb{R}} W_{x_1}\bar{\rho}\bar{u}R_2 dx_1 dy \le \eta\|W_{x_1}\|^2 + C\|W^3\|^2 \\ &\le C(\eta + E^2)(\|\nabla W\|^2 + \|\nabla^2 W\|^2). \end{aligned} \tag{4.26}$$

Therefore, combining (4.15), (4.23), (4.25) and (4.26), and recalling (4.14), (4.17) and (4.24), we evaluate the term $\int_{\mathbb{T}^2}\int_{\mathbb{R}} \rho\Psi\nabla W dx_1 dy$ as

$$\begin{aligned} &\int_{\mathbb{T}^2}\int_{\mathbb{R}} \rho\Psi\nabla W dx_1 dy \\ &\le -\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\nabla W|^2 dx_1 dy + \frac{1}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} \bar{\rho}W^2 dx_1 dy - \frac{1}{3}\int_{\mathbb{T}^2}\int_{\mathbb{R}} \bar{\rho}W^3 dx_1 dy\right) \\ &\quad + C(\eta + E + \varepsilon)\|(\nabla\varphi, \nabla\Psi, \nabla W, \nabla^2 W)(t)\|^2 + C\varepsilon(1+t)^{-2} + C\varepsilon(1+t)^{-2}\|(\varphi, \psi_1, W)(t)\|^2, \end{aligned}$$

which is combined together with (4.7), (4.8), (4.9) and (4.11), then is integrated with respect to t, choosing $C\eta < \frac{1}{2}$ and finally implies (4.6) since $E + \varepsilon \le \varepsilon_1$ and ε_1 is assumed sufficiently small. This completes the proof of Lemma 4.1. ∎

LEMMA 4.2. *There exists a positive constant C such that for $0 \le t \le T$,*

$$\|(\varphi, \Psi, W, \nabla\varphi, \nabla W)(t)\|^2 + \int_0^t [\|\bar{u}_x^{1/2}(\varphi, \psi_1)(s)\|^2 + \|(\nabla\varphi, \nabla\Psi, \nabla W, \nabla^2 W)(s)\|^2] dt$$

$$\leq C[\|(\varphi_0,W_0)\|_1^2+\|\Psi_0\|^2+\varepsilon^{\frac{1}{8}}]+CE\int_0^t\|\nabla^3\Psi(s)\|^2dt. \tag{4.27}$$

Proof. Applying the operator ∇ to $(3.1)_1$ and then multiplying the resulting equation by $\nabla\varphi/\rho^2$ yield

$$\begin{aligned}
&(\frac{|\nabla\varphi|^2}{2\rho^2})_t+\mathrm{div}(\frac{\mathrm{u}|\nabla\varphi|^2}{2\rho^2})+\frac{(\mu+\lambda)\nabla\varphi\cdot\nabla\mathrm{div}\Psi+\mu\nabla\varphi\cdot\Delta\Psi}{(2\mu+\lambda)\rho}+\frac{\bar u_{x_1}\varphi_{x_1}^2}{\rho^2}\\
&\quad+\frac{\mu}{2\mu+\lambda}\Big[(\frac{\varphi_{x_1}\psi_{2x_1}}{\rho})_{x_2}+(\frac{\varphi_{x_1}\psi_{3x_1}}{\rho})_{x_3}+(\frac{\varphi_{x_2}\psi_{1x_2}}{\rho})_{x_1}+(\frac{\varphi_{x_2}\psi_{3x_2}}{\rho})_{x_3}\\
&\quad+(\frac{\varphi_{x_3}\psi_{1x_3}}{\rho})_{x_1}+(\frac{\varphi_{x_3}\psi_{2x_3}}{\rho})_{x_2}-(\frac{\varphi_{x_1}\psi_{1x_2}}{\rho})_{x_2}-(\frac{\varphi_{x_2}\psi_{2x_1}}{\rho})_{x_1}-(\frac{\varphi_{x_3}\psi_{3x_1}}{\rho})_{x_1}\\
&\quad-(\frac{\varphi_{x_3}\psi_{3x_2}}{\rho})_{x_2}-(\frac{\varphi_{x_1}\psi_{1x_3}}{\rho})_{x_3}-(\frac{\varphi_{x_2}\psi_{2x_3}}{\rho})_{x_3}\Big]\\
&=\frac{|\nabla\varphi|^2\mathrm{div}\Psi}{2\rho^2}+\frac{\bar u_{x_1}|\nabla\varphi|^2}{2\rho^2}-\frac{(\bar\rho_{x_1x_1}\psi_1+\bar u_{x_1x_1}\varphi)\varphi_{x_1}}{\rho^2}-\frac{\bar\rho_{x_1}\nabla\varphi\cdot\nabla\psi_1}{\rho^2}\\
&\quad-\frac{\bar\rho_{x_1}\varphi_{x_1}\mathrm{div}\Psi}{\rho^2}-\frac{(\varphi_{x_1}\nabla\psi_1+\varphi_{x_2}\nabla\psi_2+\varphi_{x_3}\nabla\psi_3)\cdot\nabla\varphi}{\rho^2}\\
&\quad-\frac{\mu\bar\rho_{x_1}[\varphi_{x_2}(\psi_{1x_2}-\psi_{2x_1})+\varphi_{x_3}(\psi_{1x_3}-\psi_{3x_1})]}{(2\mu+\lambda)\rho^2},
\end{aligned} \tag{4.28}$$

and multiplying $(3.1)_2$ by $\frac{\nabla\varphi}{\rho}$ gives

$$\begin{aligned}
&(\Psi\cdot\nabla\varphi)_t-\mathrm{div}(\Psi\varphi_t)+\frac{p'(\rho)}{\rho}|\nabla\varphi|^2-\frac{(\mu+\lambda)\nabla\mathrm{div}\Psi\cdot\nabla\varphi+\mu\Delta\Psi\cdot\nabla\varphi}{\rho}\\
&\quad+(u_1\varphi\psi_{2x_1})_{x_2}+(u_1\varphi\psi_{3x_1})_{x_3}-(u_1\varphi\psi_{2x_2})_{x_1}-(u_1\varphi\psi_{3x_3})_{x_1}\\
&=\rho(\mathrm{div}\Psi)^2+\nabla W\cdot\nabla\varphi+\frac{2\mu+\lambda}{\rho}\bar u_{x_1x_1}\varphi_{x_1}-\Big(\frac{p'(\rho)}{\rho}-\frac{p'(\bar\rho)}{\bar\rho}\Big)\bar\rho_{x_1}\varphi_{x_1}+\bar\rho_{x_1}\psi_1\mathrm{div}\Psi\\
&\quad-\bar u_{x_1}(\varphi\psi_{1x_1}-\psi_1\varphi_{x_1})+\psi_2(\varphi_{x_2}\psi_{1x_1}+\varphi_{x_2}\psi_{3x_3}-\varphi_{x_1}\psi_{1x_2}-\varphi_{x_3}\psi_{3x_2})\\
&\quad+\psi_3(\varphi_{x_3}\psi_{1x_1}+\varphi_{x_3}\psi_{2x_2}-\varphi_{x_1}\psi_{1x_3}-\varphi_{x_2}\psi_{2x_3})\\
&\quad+\varphi(\psi_{1x_2}\psi_{2x_1}+\psi_{1x_3}\psi_{3x_1}-\psi_{1x_1}\psi_{2x_2}-\psi_{1x_1}\psi_{3x_3}).
\end{aligned} \tag{4.29}$$

Then multiplying the equality (4.28) by $2\mu+\lambda$, and then adding the resulting equation and (4.29) together, and then integrating the final equation over $\mathbb{T}^2\times\mathbb{R}$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^2}\int_{\mathbb{R}}(\frac{2\mu+\lambda}{2\rho^2}|\nabla\varphi|^2+\Psi\cdot\nabla\varphi)dx_1dy+\int_{\mathbb{T}^2}\int_{\mathbb{R}}[\frac{2\mu+\lambda}{\rho^2}\bar u_{x_1}\varphi_{x_1}^2+\frac{p'(\rho)}{\rho}|\nabla\varphi|^2]dx_1dy\\
&=\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho(\mathrm{div}\Psi)^2dx_1dy+\int_{\mathbb{T}^2}\int_{\mathbb{R}}\nabla W\cdot\nabla\varphi dx_1dy+\int_{\mathbb{T}^2}\int_{\mathbb{R}}\Big\{\frac{|\nabla\varphi|^2\mathrm{div}\Psi}{2\rho^2}+\frac{\bar u_{x_1}|\nabla\varphi|^2}{2\rho^2}\\
&\quad-\frac{(\bar\rho_{x_1x_1}\psi_1+\bar u_{x_1x_1}\varphi)\varphi_{x_1}}{\rho^2}-\frac{\bar\rho_{x_1}\nabla\varphi\cdot\nabla\psi_1}{\rho^2}-\frac{\bar\rho_{x_1}\varphi_{x_1}\mathrm{div}\Psi}{\rho^2}+\frac{2\mu+\lambda}{\rho}\bar u_{x_1x_1}\varphi_{x_1}\\
&\quad-\frac{(\varphi_{x_1}\nabla\psi_1+\varphi_{x_2}\nabla\psi_2+\varphi_{x_3}\nabla\psi_3)\cdot\nabla\varphi}{\rho^2}-\frac{\mu\bar\rho_{x_1}[\varphi_{x_2}(\psi_{1x_2}-\psi_{2x_1})+\varphi_{x_3}(\psi_{1x_3}-\psi_{3x_1})]}{(2\mu+\lambda)\rho^2}\\
&\quad-(\tfrac{p'(\rho)}{\rho}-\tfrac{p'(\bar\rho)}{\bar\rho})\bar\rho_{x_1}\varphi_{x_1}+\bar\rho_{x_1}\psi_1\mathrm{div}\Psi-\bar u_{x_1}(\varphi\psi_{1x_1}-\psi_1\varphi_{x_1})+\psi_2(\varphi_{x_2}\psi_{1x_1}+\varphi_{x_2}\psi_{3x_3}
\end{aligned}$$

$$-\varphi_{x_1}\psi_{1x_2}-\varphi_{x_3}\psi_{3x_2})+\psi_3(\varphi_{x_3}\psi_{1x_1}+\varphi_{x_3}\psi_{2x_2}-\varphi_{x_1}\psi_{1x_3}-\varphi_{x_2}\psi_{2x_3})$$
$$+\varphi(\psi_{1x_2}\psi_{2x_1}+\psi_{1x_3}\psi_{3x_1}-\psi_{1x_1}\psi_{2x_2}-\psi_{1x_1}\psi_{3x_3})\Big\}dx_1dy. \tag{4.30}$$

Moreover, applying the operator ∇ to $(3.1)_3$, we obtain

$$\nabla\Delta W=\nabla\varphi+(\bar\rho_{x_1},0,0)^t(1-e^{-W})+\bar\rho\nabla(1-e^{-W})-(\bar\phi_{x_1x_1x_1},0,0)^t. \tag{4.31}$$

Then multiplying (4.31) by ∇W, and integrating the resulting equalities over $\mathbb{R}\times\mathbb{T}^2$ by parts, one has

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\nabla^2W|^2dx_1dy+\int_{\mathbb{T}^2}\int_{\mathbb{R}}\bar\rho e^{-W}|\nabla W|^2dx_1dy$$
$$=-\int_{\mathbb{T}^2}\int_{\mathbb{R}}\nabla W\cdot\nabla\varphi dx_1dy-\int_{\mathbb{T}^2}\int_{\mathbb{R}}\bar\rho_{x_1}(1-e^{-W})W_{x_1}dx_1dy+\int_{\mathbb{T}^2}\int_{\mathbb{R}}\bar\phi_{x_1x_1x_1}W_{x_1}dx_1dy,$$

which, together with (4.30) and (4.2), yields

$$\frac{d}{dt}\int_{\mathbb{T}^2}\int_{\mathbb{R}}(\frac{2\mu+\lambda}{2\rho^2}|\nabla\varphi|^2+\Psi\cdot\nabla\varphi)dx_1dy+[\|\bar u_x^{1/2}\varphi_{x_1}(t)\|^2+\|(\nabla\varphi,\nabla W,\nabla^2W)(t)\|^2]$$
$$\le C(\|\mathrm{div}\Psi\|^2+I_1+I_2+I_3+I_4+I_5). \tag{4.32}$$

Here

$$I_1=\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\bar\phi_{x_1x_1x_1}W_{x_1}|dx_1dy+\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\bar u_{x_1x_1}\varphi_{x_1}|dx_1dy,$$
$$I_2=\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\mathrm{div}\Psi||\nabla\varphi|^2dx_1dy+\int_{\mathbb{T}^2}\int_{\mathbb{R}}|(\varphi_{x_1}\nabla\psi_1+\varphi_{x_2}\nabla\psi_2+\varphi_{x_3}\nabla\psi_3)\cdot\nabla\varphi|dx_1dy,$$
$$I_3=\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\psi_2(\varphi_{x_2}\psi_{1x_1}+\varphi_{x_2}\psi_{3x_3}-\varphi_{x_1}\psi_{1x_2}-\varphi_{x_3}\psi_{3x_2})|dx_1dy$$
$$+\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\psi_3(\varphi_{x_3}\psi_{1x_1}+\varphi_{x_3}\psi_{2x_2}-\varphi_{x_1}\psi_{1x_3}-\varphi_{x_2}\psi_{2x_3})|dx_1dy$$
$$+\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\varphi(\psi_{1x_2}\psi_{2x_1}+\psi_{1x_3}\psi_{3x_1}-\psi_{1x_1}\psi_{2x_2}-\psi_{1x_1}\psi_{3x_3})|dx_1dy,$$
$$I_4=\int_{\mathbb{T}^2}\int_{\mathbb{R}}|(\bar\rho_{x_1x_1}\psi_1+\bar u_{x_1x_1}\varphi)\varphi_{x_1}|dx_1dy,$$

and

$$I_5=\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\bar\rho_{x_1}\varphi_{x_1}\mathrm{div}\Psi|dx_1dy+\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\bar\rho_{x_1}\psi_1\mathrm{div}\Psi|dx_1dy+\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\bar\rho_{x_1}\nabla\varphi\nabla\psi_1|dx_1dy$$
$$+\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\bar\rho_{x_1}\varphi_{x_1}\varphi|dx_1dy+\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\bar\rho_{x_1}W_{x_1}W|dx_1dy+\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\bar\rho_{x_1}\varphi_{x_2}(\psi_{1x_2}-\psi_{2x_1})|dx_1dy$$
$$+\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\bar\rho_{x_1}\varphi_{x_3}(\psi_{1x_3}-\psi_{3x_1})|dx_1dy+\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\bar u_{x_1}(\varphi\psi_{1x_1}-\psi_1\varphi_{x_1})|dx_1dy.$$

In the following we focus on the estimate of I_1-I_5 in (4.32). First, using the Hölder's, Young's inequalities and Lemma 2.2, we have

$$I_1\le C\int_{\mathbb{T}^2}\|W_{x_1}\|_{L^2_{x_1}}\|\bar\phi_{x_1x_1x_1}\|_{L^2_{x_1}}dy+C\int_{\mathbb{T}^2}\|\varphi_{x_1}\|_{L^2_{x_1}}\|\bar u_{x_1x_1}\|_{L^2_{x_1}}dy$$

$$\leq \frac{1}{8}\|W_{x_1}(t)\|^2 + \frac{1}{16}\|\varphi_{x_1}(t)\|^2 + C(\|\bar{u}_{x_1x_1}(t)\|^2 + \|\bar{\phi}_{x_1x_1x_1}(t)\|^2)$$
$$\leq \frac{1}{8}\|W_{x_1}(t)\|^2 + \frac{1}{16}\|\varphi_{x_1}(t)\|^2 + C\varepsilon(1+t)^{-2}. \tag{4.33}$$

By Sobolev's inequality (4.1) and Young's inequality,

$$\begin{aligned}\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\text{div}\Psi||\nabla\varphi|^2 dx_1 dy &\leq C\|\text{div}\Psi\|_{L^\infty}\|\nabla\varphi\|^2 \leq C\|\nabla\Psi\|_2\|\nabla\varphi\|^2\\ &\leq C(\|\nabla\Psi\| + \|\nabla^2\Psi\|)\|\nabla\varphi\|^2 + C\|\nabla^3\Psi\|\|\nabla\varphi\|^2\\ &\leq CE(\|\nabla^3\Psi(t)\|^2 + \|\nabla\varphi(t)\|^2).\end{aligned} \tag{4.34}$$

Similarly, we can deal with the other terms of I_2. Then, we have

$$I_2 \leq CE(\|\nabla^3\Psi\|^2 + \|\nabla\varphi\|^2). \tag{4.35}$$

Next, using Hölder's, Young's inequalities and Sobolev's inequality (4.1), we have

$$\begin{aligned}\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\psi_2\varphi_{x_2}\psi_{1x_1}| dx_1 dy &\leq \|\psi_2\|_{L^\infty}\|\psi_{1x_1}\|\|\varphi_{x_2}\|\\ &\leq \|\psi_2\|_2\|\psi_{1x_1}\|\|\varphi_{x_2}\|\\ &\leq CE(\|\nabla\varphi\|^2 + \|\nabla\psi_1\|^2).\end{aligned} \tag{4.36}$$

Similarly, we can deal with each term of I_3. Then, we have

$$I_3 \leq CE(\|\nabla\varphi\|^2 + \|\nabla\psi_1\|^2). \tag{4.37}$$

By Young's inequality, Hölder's inequality and Lemma 2.2,

$$\begin{aligned}I_4 &\leq C\int_{\mathbb{T}^2} \|\bar{\rho}_{x_1x_1}\|_{L^\infty_{x_1}}\|\psi_1\|_{L^2_{x_1}}\|\varphi_{x_1}\|_{L^2_{x_1}} dy + C\int_{\mathbb{T}^2} \|\bar{\rho}_{x_1x_1}\|_{L^\infty_{x_1}}\|\psi_1\|_{L^2_{x_1}}\|\varphi_{x_1}\|_{L^2_{x_1}} dy\\ &\leq \frac{1}{16}\|\nabla\varphi\|^2 + C\varepsilon(1+t)^{-2}\|(\varphi,\psi_1)(t)\|^2.\end{aligned} \tag{4.38}$$

Finally, we focus on I_5. First, it follows from Lemma 2.2 that

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\bar{u}_{x_1}||\nabla\varphi|^2 dx_1 dy \leq C\varepsilon\|\nabla\varphi\|^2.$$

Moreover, making use of Young's inequality, Hölder's inequality and Lemma 2.2, we have

$$\begin{aligned}\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\bar{\rho}_{x_1}\varphi_{x_1}\text{div}\Psi| dx_1 dy &\leq \int_{\mathbb{T}^2} \|\bar{\rho}_{x_1}\|_{L^\infty_{x_1}}\|\varphi_{x_1}\|_{L^2_{x_1}}\|\text{div}\Psi\|_{L^2_{x_1}} dy\\ &\leq C\|\text{div}\Psi_1\|^2 + C\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}\|\varphi_{x_1}\|^2,\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\bar{\rho}_{x_1}\varphi_{x_1}\varphi| dx_1 dy &\leq \int_{\mathbb{T}^2} \|\bar{\rho}_{x_1}\|_{L^\infty_{x_1}}\|\varphi_{x_1}\|_{L^2_{x_1}}\|\varphi\|_{L^2_{x_1}} dy\\ &\leq \frac{1}{40}\|\varphi_{x_1}\|^2 + C\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}\|\varphi\|^2.\end{aligned}$$

The other terms in I_5 can be analyzed similarly. Thus,

$$I_5 \leq C\|\nabla\Psi\|^2+\frac{1}{8}\|(\nabla\varphi,\nabla W)(t)\|^2+C\varepsilon\|\nabla\varphi(t)\|^2+C\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}\|\varphi,\psi_1,W)(t)\|^2. \tag{4.39}$$

Since $E+\varepsilon\leq\varepsilon_1$ and ε_1 is assumed sufficiently small, inserting the estimates of $I_i (i= 1,2,3,4,5)$ in (4.32), we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^2}\int_{\mathbb{R}}(\frac{2\mu+\lambda}{2\rho^2}|\nabla\varphi|^2+\Psi\cdot\nabla\varphi)dx_1dy+[\|\bar{u}_x^{1/2}\varphi_{x_1}\|^2+\|(\nabla\varphi,\nabla W,\nabla^2 W)(t)\|^2]\\
&\leq C\|\nabla\Psi\|^2+CE\|\nabla^3\Psi\|^2+C\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}\|(\varphi,\psi_1,W)(t)\|^2\\
&\quad +C\varepsilon(1+t)^{-2}\|(\varphi,\psi_1)(t)\|^2+C\varepsilon(1+t)^{-2}.
\end{aligned}$$

Integrating in t, and using (4.6), we have (4.27). This completes the proof of Lemma 4.2. ∎

LEMMA 4.3. *There exists a positive constant C such that for $0\leq t\leq T$,*

$$\begin{aligned}
&\|(\varphi,\Psi,W)(t)\|_1^2+\int_0^t[\|\bar{u}_x^{1/2}(\varphi,\Psi)(s)\|^2+\|\nabla\varphi(s)\|^2+\|(\nabla\Psi,\nabla W)(s)\|_1^2]ds\\
&\leq C[\|(\varphi_0,\Psi_0,W_0)\|_1^2+\varepsilon^{1/8}]+CE\int_0^t\|\nabla^3\Psi(s)\|^2ds.
\end{aligned} \tag{4.40}$$

Proof. Multiplying $(3.1)_2$ by $-\frac{\Delta\Psi}{\rho}$ gives

$$\begin{aligned}
&(\frac{|\nabla\Psi|^2}{2})_t-\mathrm{div}(\Psi_t\nabla\Psi+\frac{\mu+\lambda}{\rho}\mathrm{div}\Psi\nabla\mathrm{div}\Psi-\frac{\mu+\lambda}{\rho}\mathrm{div}\Psi\Delta\Psi)+\frac{1}{2}(u_1|\Psi_{x_2}|^2+u_1|\Psi_{x_3}|^2\\
&\quad -u_1|\Psi_{x_1}|^2)_{x_1}-(u_1\Psi_{x_1}\cdot\Psi_{x_2})_{x_2}-(u_1\Psi_{x_1}\cdot\Psi_{x_3})_{x_3}+\frac{1}{2}\bar{u}_{x_1}|\Psi_{x_1}|^2+\frac{\mu}{\rho}|\Delta\Psi|^2+\frac{\mu+\lambda}{\rho}|\nabla\mathrm{div}\Psi|^2\\
&=\frac{p'(\bar{\rho})}{\bar{\rho}}\nabla\varphi\cdot\Delta\Psi-\nabla W\Delta\Psi+\psi_2\Psi_{x_2}\cdot\Delta\Psi+\psi_3\Psi_{x_3}\cdot\Delta\Psi+\frac{\mu+\lambda}{\rho^2}\mathrm{div}\Psi\nabla\varphi\cdot\nabla\mathrm{div}\Psi\\
&\quad -\frac{\mu+\lambda}{\rho^2}\mathrm{div}\Psi\nabla\varphi\cdot\Delta\Psi+\frac{1}{2}\psi_{1x_1}(|\Psi_{x_2}|^2+|\Psi_{x_3}|^2-|\Psi_{x_1}|^2)-\psi_{1x_1}\psi_{2x_1}\psi_{2x_2}-\psi_{1x_3}\psi_{3x_1}\psi_{3x_3}\\
&\quad -\frac{2\mu+\lambda}{\rho}\bar{u}_{x_1x_1}\Delta\psi_1+\frac{1}{2}\bar{u}_{x_1}(|\Psi_{x_2}|^2+|\Psi_{x_3}|^2)+\bar{u}_{x_1}\psi_1\Delta\psi_1+(\frac{p'(\rho)}{\rho}-\frac{p'(\bar{\rho})}{\bar{\rho}})\bar{\rho}_{x_1}\Delta\psi_1\\
&\quad +\frac{\mu+\lambda}{\rho^2}\bar{\rho}_{x_1}\mathrm{div}\Psi\mathrm{div}\Psi_{x_1}-\frac{\mu+\lambda}{\rho^2}\bar{\rho}_{x_1}\mathrm{div}\Psi\Delta\psi_1.
\end{aligned}$$

Integrating over $\mathbb{R}\times\mathbb{T}^2$ and making use of (4.2) yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\Psi\|^2+\|\bar{u}_x^{1/2}\Psi_{x_1}\|^2+\|\Delta\Psi\|^2+\|\nabla\mathrm{div}\Psi\|^2\\
&\leq C\int_{\mathbb{T}^2}\int_{\mathbb{R}}(|\nabla\varphi\cdot\Delta\Psi|+|\nabla W\cdot\Delta\Psi|)dx_1dy+C\int_{\mathbb{T}^2}\int_{\mathbb{R}}(|\psi_2\Psi_{x_2}\cdot\Delta\Psi|\\
&\quad +|\psi_3\Psi_{x_3}\cdot\Delta\Psi|)dx_1dy+C\int_{\mathbb{T}^2}\int_{\mathbb{R}}(|\mathrm{div}\Psi\nabla\varphi\cdot\nabla\mathrm{div}\Psi|+|\mathrm{div}\Psi\nabla\varphi\cdot\Delta\Psi|)dx_1dy\\
&\quad +C\int_{\mathbb{T}^2}\int_{\mathbb{R}}(|\psi_{1x_1}|(|\Psi_{x_2}|^2+|\Psi_{x_3}|^2+|\Psi_{x_1}|^2)+|\psi_{1x_1}\psi_{2x_1}\psi_{2x_2}|+|\psi_{1x_3}\psi_{3x_1}\psi_{3x_3}|)dx_1dy\\
&\quad +C\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\bar{u}_{x_1x_1}\Delta\psi_1|dx_1dy+C\int_{\mathbb{T}^2}\int_{\mathbb{R}}[|\bar{u}_{x_1}|(|\Psi_{x_2}|^2+|\Psi_{x_3}|^2)+|\bar{u}_{x_1}\psi_1\Delta\psi_1|\\
&\quad +|\bar{\rho}_{x_1}\mathrm{div}\Psi\mathrm{div}\Psi_{x_1}|+|\bar{\rho}_{x_1}\mathrm{div}\Psi\Delta\psi_1|+|\bar{\rho}_{x_1}\varphi\Delta\psi_1|]dx_1dy:=\sum_{i=1}^{6}J_i.
\end{aligned} \tag{4.41}$$

We estimate each term on the right-hand side of (4.41) as follows. By Cauchy's and Young's inequalities,

$$J_1 \le \frac{1}{8}\|\Delta\Psi\|^2 + C\|(\nabla\varphi, \nabla W)(t)\|^2. \tag{4.42}$$

As in (4.34) and (4.36), we have

$$\begin{aligned} J_2 &\le C\|\psi_2\|_{L^\infty}\|\Psi_{x_2}\|\|\Delta\Psi\| + C\|\psi_3\|_{L^\infty}\|\Psi_{x_3}\|\|\Delta\Psi\| \\ &\le C\|\psi_2\|_2\|\Psi_{x_2}\|\|\Delta\Psi\| + C\|\psi_3\|_2\|\Psi_{x_3}\|\|\Delta\Psi\| \le CE(\|\nabla\Psi\|^2 + \|\Delta\Psi\|^2), \end{aligned} \tag{4.43}$$

and

$$\begin{aligned} J_3 &\le C\|\mathrm{div}\Psi\|_{L^\infty}\|\nabla\varphi\|\|\nabla\mathrm{div}\Psi\| + C\|\mathrm{div}\Psi\|_{L^\infty}\|\nabla\varphi\|\|\Delta\Psi\| \\ &\le C(\|\nabla\Psi\| + \|\nabla^2\Psi\|)\|\nabla\varphi\|\|\nabla^2\Psi\| + C\|\nabla^3\Psi\|\|\nabla\varphi\|\|\nabla^2\Psi\| \\ &\le CE(\|\nabla^3\Psi\|^2 + \|\nabla^2\Psi\|^2 + \|\nabla\varphi\|^2). \end{aligned} \tag{4.44}$$

Similarly, we can deal with J_4 as follows:

$$\begin{aligned} J_4 &\le C\|\psi_{1x_1}\|_{L^\infty}\|\nabla\Psi\|^2 + C\|\psi_{1x_1}\|_{L^\infty}\|\psi_{2x_1}\|\|\psi_{2x_2}\| + C\|\psi_{1x_1}\|_{L^\infty}\|\psi_{3x_1}\|\|\psi_{3x_3}\| \\ &\le C(\|\nabla\Psi\| + \|\nabla^2\Psi\|)\|\nabla\Psi\|^2 + C\|\nabla^3\Psi\|\|\nabla\Psi\|^2 \\ &\le CE(\|\nabla^3\Psi\|^2 + \|\nabla^2\Psi\|^2 + \|\nabla\Psi\|^2). \end{aligned} \tag{4.45}$$

By Cauchy's inequality, Young's inequality and Lemma 2.2,

$$J_5 \le \frac{1}{8}\|\Delta\psi_1\|^2 dt + C\varepsilon(1+t)^{-2}. \tag{4.46}$$

Finally, as in (4.39), we also have

$$J_6 \le C\varepsilon\|\nabla\Psi\|^2 + \frac{1}{8}\|\nabla^2\Psi(t)\|^2 + C\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}\|(\varphi,\psi_1,\nabla\Psi)(t)\|^2. \tag{4.47}$$

Inserting the estimates for $J_1 - J_6$ in (4.41) and using the elliptic estimate $\|\Delta\Psi\| \sim \|\nabla^2\Psi\|$, and noting $E+\varepsilon \le \varepsilon_1$ and ε_1 is assumed sufficiently small, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla\Psi(t)\|^2 + \|\bar{u}_x^{1/2}\Psi_{x_1}(t)\|^2 + \|\nabla^2\Psi(t)\|^2 \\ &\le C\|(\nabla\varphi,\nabla W)(t)\|^2 + C(E+\varepsilon)(\|\nabla^3\Psi(t)\|^2 + \|\nabla\Psi(t)\|^2) + C\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}\|(\varphi,\psi_1)(t)\|^2. \end{aligned}$$

Integrating in t, and using (4.27), we have (4.40). This completes the proof of Lemma 4.3. ∎

LEMMA 4.4. *There exists a positive constant C such that for $0 \le t \le T$,*

$$\begin{aligned} &\|(\varphi,\Psi,W)(t)\|_1^2 + \|\nabla^2\varphi(t)\|^2 + \int_0^t [\|\bar{u}_x^{1/2}(\varphi,\psi_1)(s)\|^2 + \|(\nabla\varphi,\nabla\Psi)(s)\|_1^2 + \|\nabla W(s)\|_2^2]dt \\ &\le C[\|(\Psi_0,W_0)\|_1^2 + \|\varphi_0\|_2^2 + \varepsilon^{1/8}] + CE\int_0^t \|\nabla^3\Psi(s)\|^2 ds. \end{aligned} \tag{4.48}$$

Proof. Applying the operator ∇^2 on $(3.1)_1$ and then multiplying the resulting equation by $\nabla^2\varphi/\rho^2$, we have

$$\left(\frac{|\nabla^2\varphi|^2}{2\rho^2}\right)_t + \frac{2\bar{u}_{x_1}|\nabla\varphi_{x_1}|^2}{\rho^2} + \frac{(\mu+\lambda)\nabla^2\varphi\cdot\nabla^2\mathrm{div}\Psi + \mu\nabla^2\varphi\cdot\nabla\Delta\Psi}{(2\mu+\lambda)\rho} + \mathrm{div}\left(\frac{\mathrm{u}|\nabla^2\varphi|^2}{2\rho^2}\right)$$

$$
\begin{aligned}
&+\frac{\mu}{2\mu+\lambda}\Big[(\frac{\nabla\varphi_{x_2}\cdot\nabla\psi_{1x_2}}{\rho})_{x_1}+(\frac{\nabla\varphi_{x_3}\cdot\nabla\psi_{1x_3}}{\rho})_{x_1}+(\frac{\nabla\varphi_{x_1}\cdot\nabla\psi_{2x_1}}{\rho})_{x_2}\\
&+(\frac{\nabla\varphi_{x_3}\cdot\nabla\psi_{2x_3}}{\rho})_{x_2}+(\frac{\nabla\varphi_{x_1}\cdot\nabla\psi_{3x_1}}{\rho})_{x_3}+(\frac{\nabla\varphi_{x_2}\cdot\nabla\psi_{3x_2}}{\rho})_{x_2}-(\frac{\nabla\varphi_{x_2}\cdot\nabla\psi_{2x_1}}{\rho})_{x_1}\\
&-(\frac{\nabla\varphi_{x_1}\cdot\nabla\psi_{1x_2}}{\rho})_{x_2}-(\frac{\nabla\varphi_{x_3}\cdot\nabla\psi_{3x_1}}{\rho})_{x_1}-(\frac{\nabla\varphi_{x_3}\cdot\nabla\psi_{3x_2}}{\rho})_{x_2}-(\frac{\nabla\varphi_{x_2}\cdot\nabla\psi_{2x_3}}{\rho})_{x_3}\\
&-(\frac{\nabla\varphi_{x_1}\cdot\nabla\psi_{1x_3}}{\rho})_{x_3}\Big]\\
=&K_1(t,x,y)+K_2(t,x,y), \qquad (4.49)
\end{aligned}
$$

where

$$
\begin{aligned}
K_1(t,x,y)=&\frac{|\nabla^2\varphi|^2\mathrm{div}\Psi}{2\rho^2}-\frac{\varphi_{x_1}\nabla\varphi_{x_1}\cdot\nabla\mathrm{div}\Psi}{\rho^2}-\frac{\varphi_{x_2}\nabla\varphi_{x_2}\cdot\nabla\mathrm{div}\Psi}{\rho^2}-\frac{\varphi_{x_3}\nabla\varphi_{x_3}\cdot\nabla\mathrm{div}\Psi}{\rho^2}\\
&+\frac{\mu\varphi_{x_1}}{(2\mu+\lambda)\rho^2}(\nabla\varphi_{x_2}\cdot\nabla\psi_{1x_2}+\nabla\varphi_{x_3}\cdot\nabla\psi_{1x_3}-\nabla\varphi_{x_2}\cdot\nabla\psi_{2x_1}-\nabla\varphi_{x_3}\cdot\nabla\psi_{3x_1})\\
&+\frac{\mu\varphi_{x_2}}{(2\mu+\lambda)\rho^2}(\nabla\varphi_{x_1}\cdot\nabla\psi_{2x_1}+\nabla\varphi_{x_2}\cdot\nabla\psi_{2x_3}-\nabla\varphi_{x_1}\cdot\nabla\psi_{1x_2}-\nabla\varphi_{x_3}\cdot\nabla\psi_{3x_2})\\
&+\frac{\mu\varphi_{x_3}}{(2\mu+\lambda)\rho^2}(\nabla\varphi_{x_1}\cdot\nabla\psi_{3x_1}+\nabla\varphi_{x_2}\cdot\nabla\psi_{3x_2}-\nabla\varphi_{x_2}\cdot\nabla\psi_{2x_3}-\nabla\varphi_{x_1}\cdot\nabla\psi_{1x_3})\\
&-\frac{\nabla\varphi\cdot(\mathrm{div}\Psi_{x_1}\nabla\varphi_{x_1}+\mathrm{div}\Psi_{x_2}\nabla\varphi_{x_2}+\mathrm{div}\Psi_{x_3}\nabla\varphi_{x_3})}{\rho^2}-\frac{\varphi_{x_1x_2}\nabla\psi_2\cdot\nabla\varphi_{x_1}}{\rho^2}\\
&-\frac{\varphi_{x_1x_3}\nabla\psi_3\cdot\nabla\varphi_{x_3}}{\rho^2}-\frac{\varphi_{x_2x_2}\nabla\psi_2\cdot\nabla\varphi_{x_2}+\varphi_{x_3x_3}\nabla\psi_3\cdot\nabla\varphi_{x_3}}{\rho^2}\\
&-\frac{(\psi_{2x_1}\nabla\varphi_{x_1}+\psi_{2x_2}\nabla\varphi_{x_2})\cdot\nabla\varphi_{x_2}}{\rho^2}-\frac{(\varphi_{x_1}\nabla^2\psi_1+\varphi_{x_2}\nabla^2\psi_2+\varphi_{x_3}\nabla^2\psi_3)\cdot\nabla^2\varphi}{\rho^2}\\
&-\frac{(\psi_{3x_1}\nabla\varphi_{x_1}+\psi_{3x_3}\nabla\varphi_{x_3})\cdot\nabla\varphi_{x_3}}{\rho^2}-\frac{(\psi_{1x_1}\nabla\varphi_{x_1}+\psi_{1x_2}\nabla\varphi_{x_2}+\psi_{1x_3}\nabla\varphi_{x_3})\cdot\nabla\varphi_{x_3}}{\rho^2}\\
&-\frac{\nabla\psi_1(\varphi_{x_1x_1}\nabla\varphi_{x_1}+\varphi_{x_1x_2}\nabla\varphi_{x_2}+\varphi_{x_1x_3}\nabla\varphi_{x_3})}{\rho^2},
\end{aligned}
$$

and

$$
\begin{aligned}
K_2(t,x,y)=&\frac{\bar u_{x_1}|\nabla^2\varphi|^2}{2\rho^2}-\frac{\bar\rho_{x_1x_1}\mathrm{div}\Psi\varphi_{x_1x_1}}{\rho^2}-\frac{2\bar\rho_{x_1}\nabla\varphi_{x_1}\cdot\nabla\mathrm{div}\Psi}{\rho^2}-\frac{\bar\rho_{x_1}\nabla^2\varphi\cdot\nabla^2\psi_1}{\rho^2}\\
&+\frac{\mu\bar\rho_x}{(2\mu+\lambda)\rho^2}(\nabla\varphi_{x_2}\cdot\nabla\psi_{1x_2}+\nabla\varphi_{x_3}\cdot\nabla\psi_{1x_3}-\nabla\varphi_{x_2}\cdot\nabla\psi_{2x_1}-\nabla\varphi_{x_3}\cdot\nabla\psi_{3x_1})\\
&-\frac{2\bar\rho_{x_1x_1}\nabla\psi_1\cdot\nabla\varphi_{x_1}}{\rho^2}-\frac{\bar u_{x_1x_1}\varphi_{x_1}\varphi_{x_1x_1}}{\rho^2}-\frac{2\bar u_{x_1x_1}\nabla\varphi\cdot\nabla\varphi_{x_1}}{\rho^2}\\
&-\frac{\bar\rho_{x_1x_1x_1}\psi_1\varphi_{xx}}{\rho^2}-\frac{\bar u_{x_1x_1x_1}\varphi\varphi_{x_1x_1}}{\rho^2}.
\end{aligned}
$$

On the other hand, dividing $(3.1)_2$ by ρ, applying the operator ∇ on the resulting equation and then multiplying the final equation by $\nabla^2\varphi$, we have

$$
\begin{aligned}
&(\nabla\Psi\cdot\nabla^2\varphi)_t+\frac{p'(\rho)}{\rho}|\nabla^2\varphi|^2-\frac{\mu\nabla^2\varphi\cdot\nabla\Delta\Psi+(\mu+\lambda)\nabla^2\varphi\cdot\nabla^2\mathrm{div}\Psi}{\rho}-\mathrm{div}(\varphi_{x_1t}\nabla\psi_1\\
&+\varphi_{x_2t}\nabla\psi_2+\varphi_{x_3t}\nabla\psi_3)+(u_1\nabla\varphi\cdot\Psi_{x_1x_2})_{x_2}+(u_1\nabla\varphi\cdot\Psi_{x_1x_3})_{x_3}-(u_1\nabla\varphi\cdot\Psi_{x_2x_2})_{x_1}\\
&-(u_1\nabla\varphi\cdot\Psi_{x_3x_3})_{x_1}\\
=&\rho\nabla\mathrm{div}\Psi\cdot\Delta\Psi+\nabla^2W\cdot\nabla^2\varphi+K_3(t,x,y)+K_4(t,x,y), \qquad (4.50)
\end{aligned}
$$

where

$$
\begin{aligned}
K_3(t,x,y)=&\,\mathrm{div}\Psi\nabla\varphi\cdot\Delta\Psi+\psi_2\nabla\varphi_{x_2}\cdot\Delta\Psi+\psi_3\nabla\varphi_{x_3}\cdot\Delta\Psi-\psi_{1x_1}\nabla\psi_1\cdot\nabla\varphi_{x_1}\\
&-\frac{p'''(\rho)}{\gamma-1}\varphi_{x_1}\nabla\varphi\cdot\nabla\varphi_{x_1}-\frac{\mu}{\rho^2}\Delta\psi_1\nabla\varphi\cdot\nabla\varphi_{x_1}-\frac{\mu+\lambda}{\rho}\mathrm{div}\Psi_{x_1}\nabla\varphi\cdot\nabla\varphi_{x_1}\\
&+\varphi_{x_2}\nabla\psi_2\cdot\Delta\Psi+\varphi_{x_3}\nabla\psi_3\cdot\Delta\Psi+\varphi_{x_1}\nabla\psi_1\cdot\Delta\Psi+\psi_{1x_2}\nabla\varphi\cdot\Psi_{x_1x_2}\\
&+\psi_{1x_3}\nabla\varphi\cdot\Psi_{x_1x_3}-\psi_{1x_1}\nabla\varphi\cdot\Psi_{x_2x_2}-\psi_{1x_1}\nabla\varphi\cdot\Psi_{x_3x_3}-\psi_{2x_1}\nabla\psi_1\cdot\nabla\varphi_{x_2}\\
&-\psi_{3x_1}\nabla\psi_1\cdot\nabla\varphi_{x_3}-\psi_2\nabla\Psi_{x_2}\cdot\nabla^2\varphi-\psi_3\nabla\Psi_{x_3}\cdot\nabla^2\varphi-\psi_{1x_2}\nabla\psi_2\cdot\nabla\varphi_{x_1}\\
&-\psi_{1x_3}\nabla\psi_3\cdot\nabla\varphi_{x_1}-\psi_{2x_2}\nabla\psi_2\cdot\nabla\varphi_{x_2}-\psi_{3x_3}\nabla\psi_3\cdot\nabla\varphi_{x_3}-\frac{p'''(\rho)}{\gamma-1}\varphi_{x_2}\nabla\varphi\cdot\nabla\varphi_{x_2}\\
&-\frac{p'''(\rho)}{\gamma-1}\varphi_{x_3}\nabla\varphi\cdot\nabla\varphi_{x_3}-\frac{\mu}{\rho^2}\Delta\psi_2\nabla\varphi\cdot\nabla\varphi_{x_2}-\frac{\mu}{\rho^2}\Delta\psi_3\nabla\varphi\cdot\nabla\varphi_{x_3}\\
&-\frac{\mu+\lambda}{\rho^2}\mathrm{div}\Psi_{x_2}\nabla\varphi\cdot\nabla\varphi_{x_2}-\frac{\mu+\lambda}{\rho^2}\mathrm{div}\Psi_{x_3}\nabla\varphi\cdot\nabla\varphi_{x_3},
\end{aligned}
$$

$$
\begin{aligned}
K_4(t,x,y)=&\,\bar\rho_{x_1}\mathrm{div}\Psi\Delta\psi_1+\bar u_{x_1}\nabla\varphi\cdot\Psi_{x_1x_1}+\bar\rho_{x_1x_1}\psi_1\Delta\psi_1+\bar u_{x_1x_1}\varphi\Delta\psi_1\\
&-(\frac{p'''(\rho)}{\gamma-1}-\frac{p'''(\bar\rho)}{\gamma-1})\bar\rho_{x_1}\varphi_{x_1x_1}-(\frac{p'(\rho)}{\rho}-\frac{p'(\bar\rho)}{\bar\rho})\bar\rho_{x_1x_1}\varphi_{x_1x_1}-\frac{\mu}{\rho^2}\bar\rho_{x_1}\nabla\varphi_{x_1}\cdot\Delta\Psi\\
&-\frac{2\mu+\lambda}{\rho^2}\bar u_{x_1x_1}\nabla\varphi\cdot\nabla\varphi_{x_1}+\bar u_{x_1}\varphi_{x_1}\Delta\psi_1+\bar\rho_{x_1}\nabla\psi_1\cdot\Delta\Psi-\bar u_{x_1}\Psi_{x_1}\cdot\nabla\varphi_{x_1}\\
&-\bar u_{x_1x_1}\psi_1\varphi_{x_1x_1}-\bar u_x\nabla\psi_1\cdot\nabla\varphi_x-\frac{2p'''(\rho)}{\gamma-1}\bar\rho_x\nabla\varphi\cdot\nabla\varphi_x-\frac{\mu+\lambda}{\rho^2}\bar\rho_x\nabla\varphi_x\cdot\nabla\mathrm{div}\Psi,
\end{aligned}
$$

and

$$
K_5(t,x,y)=-\frac{2\mu+\lambda}{\rho^2}\bar\rho_{x_1}\bar u_{x_1x_1}\varphi_{x_1x_1}+\frac{2\mu+\lambda}{\rho}\bar u_{x_1x_1x_1}\varphi_{x_1x_1}.
$$

Multiplying (4.49) by $2\mu+\lambda$, adding to (4.50), and then integrating the final equation over $\mathbb{R}\times\mathbb{T}^2$, one has

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathrm{T}^2}\int_{\mathrm{R}}(\frac{2\mu+\lambda}{2\rho^2}|\nabla^2\varphi|^2+\nabla\Psi\cdot\nabla^2\varphi)]dx_1dy\\
&\quad+\int_{\mathrm{T}^2}\int_{\mathrm{R}}[\frac{(2\mu+\lambda)}{2\rho^2}\bar u_x|\nabla\varphi_{x_1}|^2+\frac{p'(\rho)}{\rho}|\nabla^2\varphi|^2]dx_1dy\\
=&\int_{\mathrm{T}^2}\int_{\mathrm{R}}(\rho\nabla\mathrm{div}\Psi\cdot\Delta\Psi+\nabla^2W\cdot\nabla^2\varphi)dx_1dy+(2\mu+\lambda)\int_{\mathrm{T}^2}\int_{\mathrm{R}}(K_1(t,x,y)\\
&\quad+K_2(t,x,y))dx_1dy+\int_{\mathrm{T}^2}\int_{\mathrm{R}}(K_3(t,x,y)+K_4(t,x,y))dx_1dy. \qquad (4.51)
\end{aligned}
$$

Moreover, applying the operator ∇^2 to $(3.1)_3$, multiplying it by $\nabla^2 W$, and integrating the resulting equalities over $\mathbb{R}\times\mathbb{T}^2$ by parts, one has

$$
\begin{aligned}
&\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\nabla^3W|^2dx_1dy+\int_{\mathbb{T}^2}\int_{\mathbb{R}}\bar\rho e^{-W}|\nabla^2W|^2dx_1dy\\
=&-\int_{\mathbb{T}^2}\int_{\mathbb{R}}\nabla^2W\cdot\nabla^2\varphi dx_1dy+\int_{\mathbb{T}^2}\int_{\mathbb{R}}(K_6(t,x,y)+K_7(t,x,y)+K_8(t,x,y))dx_1dy,
\end{aligned}
$$

here

$$
K_6(t,x,y)=\bar\rho e^{-W}\nabla W\nabla W\nabla^2W,
$$

$$K_7(t,x,y)=-2\bar{\rho}_{x_1}e^{-W}\nabla W\cdot\sum_{i=1}^{3}\nabla W_{x_i}-\bar{\rho}_{x_1x_1}(1-e^{-W})\sum_{i=1}^{3}W_{x_1x_i},$$
$$K_8(t,x,y)=\bar{\phi}_{x_1x_1x_1x_1}(W_{x_1x_1}+W_{x_1x_2}+W_{x_1x_3})dx_1dy.$$

Then combining (4.51) and (4.52), and using (4.2) yield

$$\begin{aligned}&\frac{d}{dt}\int_{\mathrm{T}^2}\int_{\mathrm{R}}(\frac{2\mu+\lambda}{2\rho^2}|\nabla^2\varphi|^2+\nabla\Psi\cdot\nabla^2\varphi)]dx_1dy+[\|\bar{u}_x^{1/2}\nabla\varphi_{x_1}\|^2+\|(\nabla^2\varphi,\nabla^2W,\nabla^3W)\|^2]\\&\le C\|\nabla\mathrm{div}\Psi\|\|\Delta\Psi\|+C\int_{\mathrm{T}^2}\int_{\mathrm{R}}(|K_1(t,x,y)+K_3(t,x,y)+K_4(t,x,y)\\&\quad+K_6(t,x,y)+K_7(t,x,y)+K_8(t,x,y)|)dx_1dy.\end{aligned}\tag{4.52}$$

Now we estimate the terms on the right-hand side of (4.52). First, using Hölder's and Young's inequalities, (4.2) and (4.1), we have

$$\begin{aligned}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\frac{|\nabla^2\varphi|^2|\mathrm{div}\Psi|}{2\rho^2}dx_1dy&\le C\|\mathrm{div}\Psi\|_{L^\infty}\|\nabla^2\varphi\|^2\le C\|\mathrm{div}\Psi\|_2\|\nabla^2\varphi\|^2\\&\le C(\|\nabla\Psi\|+\|\nabla^2\Psi\|)\|\nabla^2\varphi\|^2+C\|\nabla^3\Psi\|\|\nabla^2\varphi\|^2\\&\le CE(\|\nabla^3\Psi(t)\|^2+\|\nabla^2\varphi(t)\|^2),\end{aligned}\tag{4.53}$$

and

$$\begin{aligned}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\frac{|\varphi_{x_1}\nabla\varphi_{x_1}\cdot\nabla\mathrm{div}\Psi|}{\rho^2}dx_1dy&\le C\|\varphi_{x_1}\|_{L^3}\|\nabla\mathrm{div}\Psi\|_{L^6}\|\nabla\varphi_{x_1}\|\\&\le C(\|\varphi_{x_1}\|+\|\nabla\varphi_{x_1}\|)\|\nabla^2\mathrm{div}\Psi\|\|\nabla\varphi_{x_1}\|\\&\le CE(\|\nabla^3\Psi(t)\|^2+\|\nabla^2\varphi(t)\|^2).\end{aligned}\tag{4.54}$$

Similarly, we can estimate the other terms of $\int_{\mathbb{T}^2}\int_{\mathbb{R}}|K_1(t,x,y)|dx_1dy$. Therefore, we have

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}}|K_1(t,x,y)|dx_1dy\le C\varepsilon_1(\|\nabla^3\Psi(t)\|^2+\|\nabla^2\varphi(t)\|^2).\tag{4.55}$$

Next, we bound $\int_{\mathbb{T}^2}\int_{\mathbb{R}}|K_2(t,x,y)|dx_1dy$. First, it follows from Lemma 2.2 that

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\bar{u}_{x_1}||\nabla^2\varphi|^2dx_1dy\le C\|\bar{u}_{x_1}\|_{L^\infty_{x_1}}\|\nabla^2\varphi\|^2\le C\varepsilon\|\nabla^2\varphi\|^2.$$

Moreover, making use of Young's inequality, Hölder's inequality and Lemma 2.2, we have

$$\begin{aligned}\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\bar{\rho}_{x_1x_1}\varphi_{x_1x_1}\mathrm{div}\Psi|dx_1dy&\le\int_{\mathbb{T}^2}\|\bar{\rho}_{x_1x_1}\|_{L^\infty_{x_1}}\|\varphi_{x_1x_1}\|_{L^2_{x_1}}\|\mathrm{div}\Psi_1\|_{L^2_{x_1}}dy\\&\le\frac{1}{16}\|\nabla^2\varphi\|^2+C\varepsilon(1+t)^{-2}\|\nabla\Psi\|^2,\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\bar{\rho}_{x_1}\nabla\varphi_{x_1}\nabla\mathrm{div}\Psi|dx_1dy&\le\int_{\mathbb{T}^2}\|\bar{\rho}_{x_1}\|_{L^\infty_{x_1}}\|\nabla\varphi_{x_1}\|_{L^2_{x_1}}\|\nabla\mathrm{div}\Psi\|_{L^2_{x_1}}dy\\&\le\frac{1}{16}\|\nabla\varphi_{x_1}\|^2+C\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}\|\nabla^2\Psi\|^2.\end{aligned}$$

The other terms in $\int_{\mathbb{T}^2}\int_{\mathbb{R}}|K_2(t,x,y)|dx_1dy$ can be analyzed similarly. Therefore,

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}}|K_2(t,x,y)|dx_1dy \le \frac{1}{8}\|\nabla^2\varphi\|^2 + C\varepsilon\|\nabla^2\varphi(t)\|^2 + C\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}\|\nabla\Psi(t)\|^2 + C\varepsilon(1+t)^{-2}\|(\varphi,\psi_1,\nabla\varphi,\nabla\Psi)(t)\|^2. \tag{4.56}$$

Furthermore, as in (4.55), we have

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}}|K_3(t,x,y)|dx_1dy \le CE(\|\nabla^3\Psi(t)\|^2 + \|\nabla^2\Psi(t)\|^2 + \|\nabla^2\varphi(t)\|^2), \tag{4.57}$$

and

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}}|K_6(t,x,y)|dx_1dy \le CE(\|\nabla^3 W(t)\|^2 + \|\nabla^2 W(t)\|^2 + \|\nabla W(t)\|^2). \tag{4.58}$$

As in (4.56), we have

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}}|K_4(t,x,y)|dx_1dy \le \frac{1}{8}\|\nabla^2\varphi(t)\|^2 + C\|\nabla^2\Psi(t)\|^2 + C\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}\|(\varphi,\nabla\varphi,\nabla\Psi)(t)\|^2 + C\varepsilon(1+t)^{-2}\|(\varphi,\psi_1,\nabla\varphi)(t)\|^2, \tag{4.59}$$

and

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}}|K_7(t,x,y)|dx_1dy \le \frac{1}{8}\|\nabla^2 W\|^2 + C\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}\|\nabla W(t)\|^2 + C\varepsilon(1+t)^{-2}\|W(t)\|^2. \tag{4.60}$$

Finally, as in (4.33), we can obtain

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}}|K_5(t,x,y)|dx_1dy \le \frac{1}{8}\|\nabla^2\varphi\|^2 + C\varepsilon(1+t)^{-2}, \tag{4.61}$$

and

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}}|K_8(t,x,y)|dx_1dy \le \frac{1}{8}\|\nabla^3 W\|^2 + C\varepsilon(1+t)^{-2}. \tag{4.62}$$

Inserting the estimates of $K_i(t,x,y)(i=1,2,\cdots,8)$ in (4.52), using the elliptic estimates $\|\Delta\Psi\| \sim \|\nabla^2\Psi\|$ and $\|\nabla\Delta\Psi\| \sim \|\nabla^3\Psi\|$, and noting $E+\varepsilon \le \varepsilon_1$ and ε_1 is assumed sufficiently small, we have

$$\frac{d}{dt}\int_{\mathbb{T}^2}\int_{\mathrm{R}}\left(\frac{2\mu+\lambda}{2\rho^2}|\nabla^2\varphi|^2 + \nabla\Psi\cdot\nabla^2\varphi)\right]dx_1dy + [\|\bar{u}_x^{1/2}\nabla\varphi_{x_1}\|^2 + \|(\nabla^2\varphi,\nabla^2 W,\nabla^3 W)\|^2] \le C\|\nabla^2\Psi\|^2 + C\varepsilon(1+t)^{-2}.$$

Integrating in t, and using (4.40), we have (4.48). This completes the proof of Lemma 4.4. ∎

LEMMA 4.5. *There exists a positive constant C such that, for $0 \le t \le T$,*

$$\|(\varphi,\Psi)(t)\|_2^2 + \|W(t)\|_1^2 + \int_0^t [\|\bar{u}_x^{1/2}(\varphi,\psi_1)(s)\|^2 + \|(\nabla\varphi,\nabla\Psi,\nabla W)(s)\|_1^2$$

$$+\|(\nabla^3\Psi,\nabla^3 W)(s)\|^2]ds\le C[\|(\varphi_0,\Psi_0)\|_2^2+\|W_0\|_1^2+\varepsilon^{1/8}]. \tag{4.63}$$

Proof. We first divide $(3.1)_2$ by ρ, apply the operator ∇ on the resulting equation and then multiply by $-\nabla\Delta\Psi$ to obtain

$$\begin{aligned}
&(\frac{|\nabla^2\Psi|^2}{2})_t+\frac{1}{2}\bar u_{x_1}|\Psi_{x_1x_1}|^2+\frac{\mu}{\rho}|\nabla\Delta\Psi|^2+\frac{\mu+\lambda}{\rho}|\nabla^2\mathrm{div}\Psi|^2-\mathrm{div}\Big[\psi_{1tx_1}\nabla\psi_{1x_1}\\
&\quad+\psi_{1tx_2}\nabla\psi_{1x_2}+\psi_{1tx_3}\nabla\psi_{1x_3}+\psi_{2tx_1}\nabla\psi_{2x_1}+\psi_{2tx_2}\nabla\psi_{2x_2}+\psi_{2tx_3}\nabla\psi_{2x_3}\\
&\quad+\psi_{3tx_1}\nabla\psi_{3x_1}+\psi_{3tx_2}\nabla\psi_{3x_2}+\psi_{3tx_3}\nabla\psi_{3x_3}+\frac{\mu+\lambda}{\rho}(\mathrm{div}\Psi_{x_1}\nabla\mathrm{div}\Psi_{x_1}+\mathrm{div}\Psi_{x_2}\nabla\mathrm{div}\Psi_{x_2}\\
&\quad+\mathrm{div}\Psi_{x_3}\nabla\mathrm{div}\Psi_{x_3})-\frac{\mu+\lambda}{\rho}(\mathrm{div}\Psi_{x_1}\Delta\mathrm{div}\Psi_{x_1}+\mathrm{div}\Psi_{x_2}\Delta\mathrm{div}\Psi_{x_2}+\mathrm{div}\Psi_{x_3}\Delta\mathrm{div}\Psi_{x_3})\Big]\\
&\quad+(\frac{u_1}{2}|\Psi_{x_2x_2}|^2+\frac{u_1}{2}|\Psi_{x_3x_3}|^2-\frac{u_1}{2}|\Psi_{x_1x_1}|^2)_{x_1}-(u_1\Psi_{x_1x_1}\cdot\Psi_{x_1x_2}+u_1\Psi_{x_1x_2}\cdot\Psi_{x_2x_2})_{x_2}\\
&\quad-(u_1\Psi_{x_1x_1}\cdot\Psi_{x_1x_3}+u_1\Psi_{x_1x_3}\cdot\Psi_{x_3x_3})_{x_3}\\
&=L_1(t,x,y)+L_2(t,x,y)+L_3(t,x,y)+L_4(t,x,y),
\end{aligned}\tag{4.64}$$

where

$$\begin{aligned}
L_1(t,x,y)&=\frac{p'(\rho)}{\rho}\nabla^2\varphi\cdot\nabla\Delta\Psi-\nabla^2W\cdot\nabla\Delta\Psi,\\
L_2(t,x,y)&=\frac{2\mu+\lambda}{\rho^2}\bar\rho_{x_1}\bar u_{x_1x_1}\Delta\psi_{1x_1}-\frac{2\mu+\lambda}{\rho}\bar u_{x_1x_1x_1}\Delta\psi_{1x_1},\\
L_3(t,x,y)&=\psi_{1x_1}\nabla\psi_1\cdot\nabla\Delta\psi_1+\psi_2\nabla\Psi_{x_2}\cdot\nabla\Delta\Psi-\psi_{1x_2}\Psi_{x_1x_2}\cdot\Delta\Psi-\psi_{1x_3}\Psi_{x_1x_3}\cdot\Delta\Psi\\
&\quad+\psi_3\nabla\Psi_{x_3}\cdot\nabla\Delta\Psi+\psi_{2x_1}\nabla\psi_1\cdot\nabla\Delta\psi_2+\psi_{3x_1}\nabla\psi_1\cdot\nabla\Delta\psi_3+\frac{1}{2}\psi_{1x_1}(|\Psi_{x_2x_2}|^2\\
&\quad+|\Psi_{x_3x_3}|^2-|\Psi_{x_1x_1}|^2)+\nabla\psi_2\cdot(\psi_{1x_2}\nabla\Delta\psi_1+\psi_{2x_2}\nabla\Delta\psi_2)+\nabla\psi_3\cdot(\psi_{1x_3}\nabla\Delta\psi_1\\
&\quad+\psi_{3x_3}\nabla\Delta\psi_3)+\frac{p'''(\rho)}{\gamma-1}\varphi_{x_1}\nabla\varphi\cdot\nabla\Delta\psi_1+\frac{\mu}{\rho^2}\Delta\psi_1\nabla\varphi\cdot\nabla\Delta\psi_1+\frac{p'''(\rho)}{\gamma-1}\varphi_{x_2}\nabla\varphi\cdot\nabla\Delta\psi_2\\
&\quad+\frac{\mu+\lambda}{\rho^2}\nabla\varphi\cdot(\mathrm{div}\Psi_{x_1}\nabla\psi_1+\mathrm{div}\Psi_{x_2}\nabla\Delta\psi_2+\mathrm{div}\Psi_{x_3}\nabla\Delta\psi_3)+\frac{\mu+\lambda}{\rho^2}\mathrm{div}\Psi_{x_1}\nabla\varphi\cdot\Delta\Psi_{x_1}\\
&\quad+\frac{p'''(\rho)}{\gamma-1}\varphi_{x_3}\nabla\varphi\cdot\nabla\Delta\psi_3+\frac{\mu}{\rho^2}\Delta\psi_3\nabla\varphi\cdot\nabla\Delta\psi_3+\frac{\mu+\lambda}{\rho^2}\nabla\varphi\cdot(\mathrm{div}\Psi_{x_2}\Delta\Psi_{x_2}\\
&\quad+\mathrm{div}\Psi_{x_3}\Delta\Psi_{x_3}-\mathrm{div}\Psi_{x_1}\nabla\mathrm{div}\Psi_{x_1}-\mathrm{div}\Psi_{x_2}\mathrm{div}\Psi_{x_2}-\mathrm{div}\Psi_{x_3}\mathrm{div}\Psi_{x_3}),
\end{aligned}$$

and

$$\begin{aligned}
L_4(t,x,y)&=\frac{1}{2}\bar u_{x_1}|\Psi_{x_2x_2}|^2+\frac{1}{2}\bar u_{x_1}|\Psi_{x_3x_3}|^2+\bar u_{x_1}\Psi_{x_1}\cdot\Delta\Psi_{x_1}\\
&\qquad+\bar u_{x_1x_1}\psi_1\Delta\psi_{1x_1}+\bar u_{x_1}\nabla\psi_1\cdot\nabla\Delta\psi_1\\
&\quad+(\frac{p'''(\rho)}{\gamma-1}-\frac{p'''(\bar\rho)}{\gamma-1})\bar\rho_{x_1}^2\Delta\psi_{1x_1}+(\frac{p'(\rho)}{\rho}-\frac{p'(\bar\rho)}{\bar\rho})\bar\rho_{x_1x_1}\Delta\psi_{1x_1}+\frac{\mu}{\rho^2}\bar\rho_{x_1}\Delta\Psi\cdot\Delta\Psi_{x_1}\\
&\quad+\frac{2\mu+\lambda}{\rho^2}\bar u_{x_1x_1}\nabla\varphi\cdot\nabla\Delta\psi_1+\frac{p'''(\rho)}{\gamma-1}\bar\rho_{x_1}\nabla\varphi\cdot(\Delta\Psi_{x_1}+\nabla\Delta\psi_1)+\frac{\mu}{\rho^2}\Delta\psi_2\nabla\varphi\cdot\nabla\Delta\psi_2\\
&\quad+\frac{\mu+\lambda}{\rho^2}\bar\rho_{x_1}\nabla\mathrm{div}\Psi\cdot(\nabla\Delta\psi_1-\nabla\mathrm{div}\Psi_{x_1})+\frac{\mu+\lambda}{\rho^2}\bar\rho_{x_1}\nabla\mathrm{div}\Psi\cdot\Delta\Psi_{x_1}.
\end{aligned}$$

Integrating the equation (4.64) over $\mathbb{R}\times\mathbb{T}^2$ yields that

$$\frac{1}{2}\frac{d}{dt}\|\nabla^2\Psi\|^2+[\|\bar u_{x_1}^{1/2}\Psi_{x_1x_1}(t)\|^2+\|\nabla\Delta\Psi(t)\|^2+\|\nabla^2\mathrm{div}\Psi(t)\|^2]$$

$$\leq C\int_{\mathbb{T}^2}\int_{\mathbb{R}}[|L_1(t,x,y)|+|L_2(t,x,y)|+|L_3(t,x,y)|+|L_4(t,x,y)|]dx_1dy. \tag{4.65}$$

Now we estimate the terms on the right-hand side of (4.65). As in (4.42) and (4.46),

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}}|L_1(t,x,y)|dx_1dy\leq\frac{1}{8}\|\nabla\Delta\Psi\|^2+C\|(\nabla^2\varphi,\nabla^2W)(t)\|^2, \tag{4.66}$$

and

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}}|L_2(t,x,y)|dx_1dy\leq\frac{1}{8}\|\nabla^3\psi_1\|^2dt+C\varepsilon(1+t)^{-2}. \tag{4.67}$$

As in (4.43), (4.44) and (4.45), we have

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}}|L_3(t,x,y)|dx_1dy\leq CE(\|\nabla^3\Psi\|^2+\|\nabla^2\Psi\|^2+\|\nabla^2\varphi\|^2). \tag{4.68}$$

Finally, similar as (4.39), we also have

$$\begin{aligned}\int_{\mathbb{T}^2}\int_{\mathbb{R}}|L_4(t,x,y)|dx_1dy\leq{}&\frac{1}{8}\|\nabla^3\Psi\|^2+C\varepsilon\|\nabla^2\Psi\|^2+C\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}\|(\nabla\varphi,\nabla\Psi,\nabla^2\Psi)(t)\|^2\\&+C\varepsilon(1+t)^{-2}\|(\varphi,\psi_1,\nabla\varphi)(t)\|^2+C\varepsilon(1+t)^{-3}\|\varphi(t)\|^2. \end{aligned}\tag{4.69}$$

Then putting (4.66)-(4.69) into (4.65), using the elliptic estimates $\|\Delta\Psi\|\sim\|\nabla^2\Psi\|$ and $\|\nabla\Delta\Psi\|\sim\|\nabla^3\Psi\|$, and and noting $E+\varepsilon\leq\varepsilon_1$ and ε_1 is assumed sufficiently small, it holds that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla^2\Psi(t)\|^2+[\|\bar{u}_{x_1}^{1/2}\Psi_{x_1x_1}(t)\|^2+\|\nabla^3\Psi(t)\|^2]\\&\leq C\|(\nabla^2\varphi,\nabla^2W)(t)\|^2+CE(\|\nabla^2\Psi\|^2+\|\nabla^2\varphi\|^2)+C\varepsilon\|\nabla^2\Psi\|^2+C\varepsilon(1+t)^{-3}\|\varphi(t)\|^2\\&\quad+C\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}\|(\nabla\varphi,\nabla\Psi,\nabla^2\Psi)(t)\|^2+C\varepsilon(1+t)^{-2}\|(\varphi,\psi_1,\nabla\varphi)(t)\|^2+C\varepsilon(1+t)^{-2}.\end{aligned}$$

Integrating the above inequality with respect to t, and using (4.48), we have (4.63). This completes the proof of Lemma 4.5. □

Proof of Proposition 3.1. First, from $(3.1)_3$ and Lemmas 2.2, we have

$$\|\nabla^2W\|_2^2\leq C\|\varphi\|^2+C\|W\|^2+C\varepsilon,$$

which, together with (4.63), yields

$$\begin{aligned}&\|(\varphi,\Psi,W)(t)\|_2^2+\int_0^t[\|\bar{u}_x^{1/2}(\varphi,\psi_1)(s)\|^2+\|(\nabla\varphi,\nabla\Psi,\nabla W)(s)\|_1^2+\|(\nabla^3\Psi,\nabla^3W)(s)\|^2]ds\\&\leq C[\|(\varphi_0,\Psi_0,W_0)\|_2^2+\varepsilon^{1/8}]\leq C[\|(\varphi_0,\Psi_0)\|_2^2+\varepsilon^{1/8}].\end{aligned}\tag{4.70}$$

Here we have used that

$$\|W_0\|_2^2\leq C\|\varphi_0\|^2+C\varepsilon,$$

which is derived by using $(3.1)_3$ and Lemma 2.2, we thus arrive finally at (3.6). The proof of Proposition 3.1 is complete.

5. The proof of Theorem 1.1

This section proves our main theorem. First, we focus on the proof of Theorem 3.1. To prove Theorem 3.1 we employ the standard continuation argument based on a local existence theorem and the *a priori* estimates. It should be noted that the uniform lower and upper bounds of the density function $\rho(t,x_1,x_2,x_3)$ in (4.2) guarantee the strict parabolicity of the momentum equation, which is crucial for the local and global-in-time existence of the classical solution to the system (3.1). Similar to [13], we can prove the local existence theorem, so we omit the details. On the other hand, the *a priori* estimates have been given in Proposition 3.1. Therefore, to complete the proof of Theorem 3.1, we need only to investigate the large-time behavior of the solution $(\varphi,\Psi,W)(t,x_1,x_2,x_3)$ to problem (3.1)- (3.3) as time tends to infinity.

Proof. **(The completion of the proof of Theorem 3.1.)** First we have, by (3.6),

$$\int_0^{+\infty}\|(\nabla\varphi,\nabla\Psi,\nabla W)(t)\|^2dt<+\infty. \tag{5.1}$$

To prove (3.4), we only need to show

$$\int_0^{+\infty}|\frac{d}{dt}\|(\nabla\varphi,\nabla\Psi,\nabla W)(t)\|^2|dt<+\infty. \tag{5.2}$$

In fact, by Cauchy's inequality, Lemma 2.2, (3.6) and the standard elliptic estimates, one has

$$\begin{aligned}
&\int_0^{+\infty}|\frac{d}{dt}\|\nabla\varphi\|^2|dt\\
&=\int_0^{+\infty}|\int_{\mathbb{T}^2}\int_{\mathbb{R}}\nabla\varphi\cdot\nabla\varphi_t dx_1dy|dt\\
&=2\int_0^{+\infty}|\int_{\mathbb{T}^2}\int_{\mathbb{R}}\mathrm{div}(\varphi_t\nabla\varphi)-\varphi_t\Delta\varphi dx_1dy|dt\\
&=2\int_0^{+\infty}|\int_{\mathbb{T}^2}\int_{\mathbb{R}}\varphi_t\Delta\varphi dx_1dy|dt\\
&=2\int_0^{+\infty}|\int_{\mathbb{T}^2}\int_{\mathbb{R}}(\rho\mathrm{div}\Psi+\psi_2\varphi_{x_2}+\psi_3\varphi_{x_3}+u_1\varphi_{x_1}+\bar\rho_{x_1}\psi_1+\bar u_{x_1}\varphi)\Delta\varphi dx_1dy|dt\\
&\le C\int_0^{+\infty}\int_{\mathbb{T}^2}\int_{\mathbb{R}}(|\nabla\Psi|^2+|\nabla\varphi|^2+\bar u_{x_1}\psi_1^2+\bar u_{x_1}\varphi^2+(\Delta\varphi)^2)dx_1dydt\\
&\le C[\|(\varphi_0,\Psi_0)\|_2^2+\varepsilon^{1/8}]<+\infty,
\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
&\int_0^{+\infty}|\frac{d}{dt}\|\nabla\Psi\|^2|dt\\
&=2\int_0^{+\infty}|\int_{\mathbb{T}^2}\int_{\mathbb{R}}\Psi_t\cdot\Delta\Psi dx_1dy|dt\\
&=C\int_0^{+\infty}\int_{\mathbb{T}^2}\int_{\mathbb{R}}(|\nabla\Psi|^2+\bar u_{x_1}\psi_1^2+|\nabla\varphi|^2+|\nabla W|^2+\bar\rho_{x_1}\varphi^2+|\nabla^2\Psi|^2+\bar u_{x_1x_1}^2)dx_1dydt\\
&\le C[\|(\varphi_0,\Psi_0)\|_2^2+\varepsilon^{1/8}]<+\infty.
\end{aligned} \tag{5.4}$$

Moreover, we also note

$$\begin{aligned}\int_0^{\infty}|\frac{d}{dt}\|\nabla W\|^2|dt &= 2\int_0^{+\infty}|\int_{\mathbb{T}^2}\int_{\mathbb{R}}\nabla W_t\cdot\nabla W dx_1 dy|dt\\ &\le C\int_0^{\infty}\|\nabla W(t)\|^2 dt + C\int_0^{\infty}\|\nabla W_t(t)\|^2 dt,\end{aligned}$$

which, together with (4.20) and (3.6), yields

$$\int_0^{\infty}|\frac{d}{dt}\|\nabla W\|^2|dt < +\infty. \tag{5.5}$$

Thus (5.3), (5.4) and (5.5) give (5.2). (5.1) and (5.2) imply that

$$\lim_{t\to+\infty}\|(\nabla\varphi,\nabla\Psi,\nabla W)(t,\cdot)\| = 0, \tag{5.6}$$

which, together with Sobolev's inequality and Proposition 3.1, yields

$$\lim_{t\to+\infty}\sup_{(x_1,x_2,x_3)\in\mathbb{R}\times\mathbb{T}^2}|(\varphi,\Psi,W)(t,x_1,x_2,x_3)| = 0. \tag{5.7}$$

This completes the proof of Theorem 3.1. ◻

Finally, we give the proof of Theorem 1.1.

Proof. **(Proof of Theorem 1.1.)** Due to Theorem 3.1, it remains to show (1.12). In fact, by (3.4) and (iii) in Lemma 2.2, we have

$$\begin{aligned}&\lim_{t\to+\infty}\sup_{(x_1,x_2,x_3)\in\mathbb{R}\times\mathbb{T}^2}|\rho(x_1,x_2,x_3,t)-\rho^r(x_1,t)|\\ =&\lim_{t\to+\infty}\sup_{(x_1,x_2,x_3)\in\mathbb{R}\times\mathbb{T}^2}|\varphi(x_1,x_2,x_3,t)+\bar{\rho}(x_1,t)-\rho^r(x_1,t)| = 0,\end{aligned}$$

$$\begin{aligned}&\lim_{t\to+\infty}\sup_{(x_1,x_2,x_3)\in\mathbb{R}\times\mathbb{T}^2}|\mathrm{u}(x_1,x_2,x_3,t)-(u^r,0,0)^t(x_1,t)|\\ =&\lim_{t\to+\infty}\sup_{(x_1,x_2,x_3)\in\mathbb{R}\times\mathbb{T}^2}|\Psi(x_1,x_2,x_3,t)+(\bar{u},0,0)^t(x_1,t)-(u^r,0,0)^t(x_1,t)| = 0,\end{aligned}$$

$$\begin{aligned}&\lim_{t\to+\infty}\sup_{(x_1,x_2,x_3)\in\mathbb{R}\times\mathbb{T}^2}|\phi(x_1,x_2,x_3,t)-\phi^r(x_1,t)|\\ =&\lim_{t\to+\infty}\sup_{(x_1,x_2,x_3)\in\mathbb{R}\times\mathbb{T}^2}|W(x_1,x_2,x_3,t)+\bar{\phi}(x_1,t)-\phi^r(x_1,t)| = 0.\end{aligned}$$

The proof of Theorem 1.1 is now completed. ◻

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