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STABILITY OF NON-DEGENERATE STATIONARY SOLUTION IN INFLOW PROBLEM FOR A 1-D RADIATION HYDRODYNAMICS MODEL*

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Abstract. This paper is concerned with the large-time behavior of the solutions to the inflow problem for a 1-D compressible viscous heat-conducting gas with radiation in the half line $(0,\infty)$. We first give the existence of non-degenerate (supersonic and subsonic) stationary solutions with the aid of center manifold theory. In addition, using an energy method, we show the time-asymptotic stability of the non-degenerate stationary solutions under smallness assumptions on the initial perturbation and the boundary data in the Sobolev space.

Keywords. Compressible radiation hydrodynamics; inflow problem, stationary solution, stability.

AMS subject classifications. 35Q30; 76N10; 35B35; 78A40; 76D33.

1. Introduction and main results

The equations describing the one-dimensional motion of a compressible viscous heat-conducting gas with radiation in Eulerian coordinates, can be written in the following form (see [1])

$$\begin{cases} \tilde\rho_t + (\tilde\rho\tilde u)_{\tilde x} = 0, \\ (\tilde\rho\tilde u)_t + (\tilde\rho\tilde u^2 + \tilde p)_{\tilde x} = \mu\tilde u_{\tilde x\tilde x}, \\ [\tilde\rho(\tilde e + \frac{\tilde u^2}{2})]_t + [\tilde\rho\tilde u(\tilde e + \frac{\tilde u^2}{2}) + \tilde p\tilde u]_{\tilde x} + \tilde q_{\tilde x} = \kappa\tilde\theta_{\tilde x\tilde x} + \mu(\tilde u\tilde u_{\tilde x})_{\tilde x}, \\ -\tilde q_{\tilde x\tilde x} + \tilde q + (\tilde\theta^4)_{\tilde x} = 0, \end{cases} \tag{1.1}$$

where the unknown functions are the density $\tilde\rho(\tilde x,t) > 0$, the velocity $\tilde u(\tilde x,t)$, the temperature $\tilde\theta(\tilde x,t) > 0$, and the radiative heat flux $\tilde q(\tilde x,t)$. Moreover, $\tilde e = e(\tilde\rho,\tilde\theta)$ and $\tilde p = p(\tilde\rho,\tilde\theta)$ is the internal energy and the pressure, while $\kappa > 0$ and $\mu > 0$ denote the heat-conductivity and the viscosity respectively.

We consider system (1.1) on $[0,\infty)$ supplemented with the far-field condition and the initial data

$$\begin{cases} (\tilde\rho,\tilde u,\tilde\theta)|_{t=0} = (\tilde\rho_0,\tilde u_0,\tilde\theta_0)(\tilde x), \quad \tilde x \in [0,\infty), \\ \lim_{\tilde x\to+\infty}(\tilde\rho,\tilde u,\tilde\theta,\tilde q)(\tilde x,t) = (\rho_+,u_+,\theta_+,0), \end{cases} \tag{1.2}$$

and the inflow boundary condition

$$\tilde\rho|_{\tilde x=0} = \rho_-, \quad \tilde u|_{\tilde x=0} = u_- > 0, \quad \tilde\theta|_{\tilde x=0} = \theta_-, \quad \tilde q|_{\tilde x=0} = 0, \tag{1.3}$$

where $\rho_\pm > 0$, $u_\pm$, $\theta_\pm > 0$ are prescribed constants.

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When the radiation effect is involved, the mathematical study for radiating gas models starts from Hamer′s work [2]. The model considered in [2] can be understood as Burgers equation coupled with an elliptic equation:

$$\begin{cases} w_t + f(w)_x + q_x = 0, \\ -q_{xx} + q + w_x = 0, \end{cases} \tag{1.4}$$

where w is a scalar unknown function. It is the simplest possible model and the third-order approximation of the compressible Euler system with radiation (see Appendix A in [3]):

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \\ [\rho(e + \frac{u^2}{2})]_t + [\rho u(e + \frac{u^2}{2}) + pu]_x + q_x = 0, \\ -q_{xx} + q + (\theta^4)_x = 0. \end{cases} \tag{1.5}$$

For Hamer′s model (1.4), Kawashima-Nishibata [4] proved asymptotic stability of shock profiles. Kawashima-Tanaka [5] showed the stability of rarefaction waves. Then this result has been extended to multi-D cases by Gao-Ruan-Zhu in [3, 6, 7]. Recently, Ohnawa [8] has extended the result in [4] to continuous shock cases.

On the other hand, there are also some results on the nonlinear stability of elementary waves for Euler system with radiation (1.5). In [9], the authors proved the global existence of shock profiles for the Euler-Poison system, and Lattanzio-Mascia-Serre [10] extended the proof to a general hyperbolic-elliptic system. Lin-Coulombel-Goudon studied the stability of shock profiles under the zero mass perturbation assumption in [11]. Then Nguyen-Plaza-Zumbrun removed the zero mass perturbation assumption by using a Green function method in [12]. The stability of a single "viscous contact wave" is studied in [13, 14] and the stability of a rarefaction wave is considered in [15]. Xie [16] proved the stability for the combination of viscous contact wave with rarefaction waves.

Moreover, for system (1.1) of compressible viscous heat-conducting gas with radiation, there are a few mathematical results for the stability of elementary waves. For Cauchy problem of system (1.1), Wang-Xie [1] proved the stability of a single viscous contact wave and Hong [17] showed the stability of the combination of contact discontinuity with rarefaction waves. Recently, for the outflow problem of system (1.1) in the half line, Choe-Hong-Kim [18] proved the existence, stability and convergence rate toward the non-degenerate stationary solution.

However, to the best of our knowledge, there is little work about the stability of nonlinear wave patterns for the inflow problem of system (1.1), so we consider the inflow problem, that is, system (1.1)-(1.3).

Throughout this paper, we assume that the pressure p and the inertial energy e are smooth functions of density ρ and temperature θ, and

$$p_\rho(\rho,\theta) > 0, \quad e_\theta(\rho,\theta) > 0. \tag{1.6}$$

Because the system (1.1)-(1.3) that we consider is in one dimension of the space variable $\tilde{x}$, it is convenient to use the following Lagrangian coordinate transformation (see [19]):

$$(t,x) \Rightarrow \left(t, \int_{(0,0)}^{(t,\tilde{x})} \tilde{\rho}(\tau,y)\,dy - \tilde{\rho}\tilde{u}(\tau,y)\,d\tau \right).$$

Thus, system (1.1)-(1.3) can be transformed into the following moving boundary problem in the Lagrangian coordinates:

$$\begin{cases} v_t - u_x = 0, & x > \sigma_- t, \quad t > 0, \\ u_t + p_x = \mu(\frac{u_x}{v})_x, \\ (e + \frac{u^2}{2})_t + (pu)_x + q_x = (\kappa \frac{\theta_x}{v} + \mu \frac{uu_x}{v})_x, \\ -\left(\frac{q_x}{v}\right)_x + vq + \left(\theta^4\right)_x = 0, \\ (v,u,\theta)\,|_{t=0} = (v_0,u_0,\theta_0)(x), \\ \lim_{x\to+\infty}(v,u,\theta,q)(x,t) = (v_+,u_+,\theta_+,0), \\ (v,u,\theta,q)\,|_{x=\sigma_- t} = (v_-,u_-,\theta_-,0), \quad u_- > 0, \end{cases} \tag{1.7}$$

where $v(x,t) = \tilde{\rho}^{-1}(\tilde{x},t)$, $u(x,t) = \tilde{u}(\tilde{x},t)$, $\theta(x,t) = \tilde{\theta}(\tilde{x},t)$, $q(x,t) = \tilde{q}(\tilde{x},t)$, $v_\pm = \rho_\pm^{-1}$, $\sigma_- = -\rho_- u_- < 0$.

To fix the moving boundary $x = \sigma_- t$, if we introduce a new variable $\xi = x - \sigma_- t$, then (1.7) yields the following inflow problem in half-line

$$\begin{cases} v_t - \sigma_- v_\xi - u_\xi = 0, & \xi > 0, \quad t > 0, \\ u_t - \sigma_- u_\xi + p_\xi = \mu(\frac{u_\xi}{v})_\xi, \\ (e + \frac{u^2}{2})_t - \sigma_-(e + \frac{u^2}{2})_\xi + (pu)_\xi + q_\xi = (\kappa \frac{\theta_\xi}{v} + \mu \frac{uu_\xi}{v})_\xi, \\ -\left(\frac{q_\xi}{v}\right)_\xi + vq + \left(\theta^4\right)_\xi = 0, \\ (v,u,\theta)\,|_{t=0} = (v_0,u_0,\theta_0)(\xi), \\ \lim_{\xi\to+\infty}(v,u,\theta,q)(\xi,t) = (v_+,u_+,\theta_+,0), \\ (v,u,\theta,q)\,|_{\xi=0} = (v_-,u_-,\theta_-,0), \quad u_- > 0. \end{cases} \tag{1.8}$$

Notation: Throughout this paper, $O(1), c$ or C denote a generic positive constant, and $c_i(\cdot,\cdot)$ or $C_i(\cdot,\cdot)(i \in Z_+)$ denote some generic constants depending only on the quantities listed in the parentheses. As long as no confusion arises, we denote the usual Sobolev space with norm $\|\cdot\|_k$ by $H^k := H^k(0,\infty)$ and $\|\cdot\|_0 = \|\cdot\|$ denotes the usual L^2-norm.

Our main results are as follows. The solution $(\hat{v},\hat{u},\hat{\theta},\hat{q})(\xi)(\xi = x - \sigma_- t)$ of system (1.8) must satisfy the system

$$\begin{aligned} & -\sigma_- \hat{v}_\xi - \hat{u}_\xi = 0, \qquad \xi \in (0,+\infty), \\ & -\sigma_- \hat{u}_\xi + \hat{p}_\xi = \mu(\frac{\hat{u}_\xi}{\hat{v}})_\xi, \\ & -\sigma_- \left(\hat{e} + \frac{\hat{u}^2}{2}\right)_\xi + (\hat{p}\hat{u})_\xi + \hat{q}_\xi = \left(\kappa \frac{\hat{\theta}_\xi}{\hat{v}} + \mu \frac{\hat{u}\hat{u}_\xi}{\hat{v}}\right)_\xi, \\ & -\left(\frac{\hat{q}_\xi}{\hat{v}}\right)_\xi + \hat{v}\hat{q} + \left(\hat{\theta}^4\right)_\xi = 0, \\ & (\hat{v},\hat{u},\hat{\theta},\hat{q})(0) = (v_-,u_-,\theta_-,0), \quad (\hat{v},\hat{u},\hat{\theta},\hat{q})(+\infty) = (v_+,u_+,\theta_+,0), \end{aligned} \tag{1.9}$$

where $\hat{p} = p(\hat{v},\hat{\theta})$, $\hat{e} = e(\hat{v},\hat{\theta})$.

We denote the sound speed and define the Mach number, respectively, by

$$c(v,\theta) = \sqrt{\frac{\partial p(\rho,s)}{\partial \rho}} = \sqrt{-v^2 \widetilde{p}_v(v,s)}, \quad M(v,u,\theta) = \frac{|u|}{c(v,\theta)}, \tag{1.10}$$

where $s = s(v,\theta)$ is the entropy.

Then, we first state the existence result of solutions $(\hat{v},\hat{u},\hat{\theta},\hat{q})(\xi)$ to the system (1.9):

THEOREM 1.1 (Existence of non-degenerate stationary solution). *Let $v_\pm > 0$, $u_- > 0$, $\theta_\pm > 0$. The necessary condition of the existence for the stationary solution to system (1.9) is*

$$\sigma_- = -\frac{u_-}{v_-} = -\frac{\hat{u}(\xi)}{\hat{v}(\xi)} = -\frac{u_+}{v_+}, \forall \xi > 0. \tag{1.11}$$

If $u_+ \leq 0$, there is no stationary solution to system (1.9).

For the case $M_+ \equiv M(v_+,u_+,\theta_+) \neq 1$, if $u_+ > 0$ and (1.6) hold, then there exists a positive constant δ_0 and a local manifold $\mathcal{M} \subset \mathcal{M}_{\delta_0} := \{(v,\theta) \in R_+^2 \,|\, 0 < |(v - v_+, \theta - \theta_+)| \leq \delta_0\}$ such that if $(v_-,\theta_-) \in \mathcal{M}$, then system (1.9) has a unique solution $(\hat{v},\hat{u},\hat{\theta},\hat{q})(\xi)$ satisfying

$$|\partial_\xi^k(\hat{v} - v_+, \hat{u} - u_+, \hat{\theta} - \theta_+, \hat{q})| \leq C\delta \exp(-\hat{c}\xi), \quad k = 0,1,2,\cdots, \tag{1.12}$$

where $\delta = |(v_- - v_+, \theta_- - \theta_+)|$ and $C, \hat{c}$ are positive constants.

REMARK 1.1. Inequality (1.12) shows that for $M_+ \neq 1$, the solution of system (1.9) converges to the spatial asymptotic state with an exponential rate, which is called non-degenerate stationary solution. For the case $M_+ = 1$, the solution of system (1.9) may converge with an algebraic rate, which is called the degenerate stationary solution. The case will be pursued by the authors in the future.

REMARK 1.2. Compared with the results of [20] and [21] for inflow problem of full compressible Navier-Stokes equations, inflow problem (1.8) is different in that there exists a stationary solution even for the case $M^+ > 1$.

Next, we state the result for the stability toward the non-degenerate stationary solution for inflow problem (1.8). We set the perturbation $(\phi,\psi,\zeta,\omega)(\xi,t)$ by

$$(\phi,\psi,\zeta,\omega)(\xi,t) = (v,u,\theta,q)(\xi,t) - (\hat{v},\hat{u},\hat{\theta},\hat{q})(\xi)$$

and the solution space $X(0,T)$ as

$$\begin{aligned} X(0,T) = \{(\phi,\psi,\zeta,\omega) \,|\, &(\phi,\psi,\zeta,\omega,\omega_\xi) \in C([0,T];H^1), \\ &\phi_\xi \in L_2(0,T;L_2),\ (\psi_\xi,\zeta_\xi,\omega,\omega_\xi) \in L_2(0,T;H^1)\} \end{aligned}$$

for any $0 < T \leq \infty$.

THEOREM 1.2 (Stability of the solution). *Let $v_\pm > 0, u_\pm > 0, \theta_\pm > 0$. Suppose that there exists a solution $(\hat{v},\hat{u},\hat{\theta},\hat{q})(\xi)$ to the system (1.9) satisfying (1.12). In addition, suppose that (1.6) holds and*

$$(\phi,\psi,\zeta)(\cdot,0) \in H^1,\ u_0(0) = u_-,\ \theta_0(0) = \theta_-.$$

Then, there exists a constant $\varepsilon_0 > 0$ such that if

$$\|(\phi,\psi,\zeta)(\cdot,0)\|_1 + \delta \leq \varepsilon_0,$$

where $\delta=|(v_- - v_+,\theta_- - \theta_+)|$, *then inflow problem* (1.8) *has a unique global stationary solution* $(v,u,\theta,q)(\xi,t)$ *satisfying* $(\phi,\psi,\zeta,\omega)\in X(0,\infty)$ *and*

$$\|(\phi,\psi,\zeta,\omega,\omega_\xi)(t)\|_1^2+\int_0^t \left(\|\phi_\xi(\tau)\|^2+\|(\psi_\xi,\zeta_\xi,\omega,\omega_\xi)(\tau)\|_1^2\right)d\tau \le C\|(\phi,\psi,\zeta)(\cdot,0)\|_1^2 \tag{1.13}$$

for any $t\in[0,\infty)$, *where* C *is a positive constant independent of* t *and* ε_0. *Moreover, the solution* $(v,u,\theta,q)(\xi,t)$ *tends time-asymptotically to the stationary solution* $(\hat v,\hat u,\hat\theta,\hat q)(\xi)$ *in the sense that*

$$\lim_{t\to\infty}\sup_{\xi\in(0,\infty)}|(v,u,\theta,q,q_\xi)(\xi,t)-(\hat v,\hat u,\hat\theta,\hat q,\hat q_\xi)(\xi)|=0. \tag{1.14}$$

Remark 1.3. Note that the stability analysis in Theorem 1.2 is the first result for inflow problem (1.8) of a compressible viscous heat-conducting gas with radiation in the half line. Moreover, we would like to emphasize that a similar stability result can be applied to the case with the combination of the following four basic waves: the 1, 3-rarefaction waves, stationary solution and viscous contact wave by similar arguments as in [22] or [23]. It will be left in the future work.

Here, we briefly review some main difficulties of our problem, compared to the inflow problem to compressible Navier-Stokes equations. As we know, when omitting the radiation effect, system (1.1) reduces to the classical compressible Navier-Stokes equations. For the inflow problem of compressible Navier-Stokes equations, there have been many mathematical studies about the existence and stability of the stationary solutions, please refer to [19, 24, 25]) for the isentropic case and to [20, 22, 23, 26–28] for the non-isentropic case. Compared to Navier-Stokes system, our problem is more general and more complex for the radiation effect is taken into account. For instance, in order to obtain the existence of stationary solutions, they, in [20], consider a 2×2 system of autonomous ordinary differential equations, but we have to introduce the new variable $\hat E$ (see (2.5)) to deduce the stationary equations to a 4×4 system of autonomous ordinary differential equations, and examine the dynamics around an equilibrium by applying the manifold theory (Section 2). Next, to show the stability of the stationary solutions by the elementary energy method, it is sufficient (see Proposition 3.1) to deduce certain uniform (with respect to the time t) a priori estimates on the perturbations (ϕ,ψ,ζ,ω) around stationary solutions $(\hat v,\hat u,\hat\theta,\hat q)$. In the first step of a priori estimates, comparing with the Navier-Stokes equations, the main difficulty is to control the energy form (3.8) so that we get the uniform estimate for L_2-norm of the perturbations, which is not trivial due to control of the new leading term $-\frac{\zeta\omega_\xi}{\theta}$ (see (3.16) in Section 3). Moreover, to prove the uniform estimate for $\|\zeta_\xi(t)\|$, we have to control the more difficult leading term $-\zeta_{\xi\xi}\left(\frac{q_\xi}{e_\theta(v,\theta)}-\frac{\hat q_\xi}{e_\theta(\hat v,\hat\theta)}\right)$ (see (3.32) in Section 3), which is also not trivial.

2. The existence of non-degenerate stationary solutions

It is known that the following relationship between thermodynamic variables (p,v,e, θ and s) is established (see [21]).

$$s_v(v,\theta)=p_\theta(v,\theta),\quad s_\theta(v,\theta)=\frac{e_\theta(v,\theta)}{\theta},\quad e_v(v,\theta)=\theta p_\theta(v,\theta)-p(v,\theta), \tag{2.1}$$

or

$$\begin{cases}\widetilde e_v(v,s)=-\widetilde p(v,s), & \widetilde e_s(v,s)=\theta,\\ \widetilde p_v(v,s)=p_v(v,\theta)-\frac{\theta(p_\theta(v,\theta))^2}{e_\theta(v,\theta)}, & \widetilde p_s(v,s)=\frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)},\\ \widetilde\theta_v(v,s)=-\frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)}, & \widetilde\theta_s(v,s)=\frac{\theta}{e_\theta(v,\theta)}.\end{cases} \tag{2.2}$$

From (2.2) and (1.6) we have

$$\widetilde{p}_v(v,s)=p_v(v,\theta)-\frac{\theta(p_\theta(v,\theta))^2}{e_\theta(v,\theta)}<0,$$

$$\begin{aligned}&\widetilde{e}_{ss}(v,s)=\frac{\theta}{e_\theta(v,\theta)}>0,\ \widetilde{e}_{vs}(v,s)=-\frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)},\\&\widetilde{e}_{vv}(v,s)=-p_v(v,\theta)+\frac{\theta(p_\theta(v,\theta))^2}{e_\theta(v,\theta)}>0,\end{aligned}\tag{2.3}$$

which means that $\widetilde{e}(v,s)$ is a convex function with respect to (v,s).

2.1. Reformulation of stationary problem. Integrating $(1.9)_{1-4}$ over $[\xi,\infty)$ yields

$$\begin{aligned}&-\sigma_-(\hat{v}-v_+)-(\hat{u}-u_+)=0,\ \xi\geq 0,\\&-\sigma_-(\hat{u}-u_+)+(\hat{p}-p_+)=\mu\frac{\hat{u}_\xi}{\hat{v}},\\&-\sigma_-\left((\hat{e}-e_+)+\frac{1}{2}(\hat{u}^2-u_+^2)\right)+(\hat{p}\hat{u}-p_+u_+)+\hat{q}=\kappa\frac{\hat{\theta}_\xi}{\hat{v}}+\mu\frac{\hat{u}\hat{u}_\xi}{\hat{v}},\\&\frac{\hat{q}_\xi}{\hat{v}}+\int_\xi^\infty(\hat{v}\hat{q})(y)dy-\left(\hat{\theta}^4-\theta_+^4\right)=0,\end{aligned}\tag{2.4}$$

where $e_+=e(v_+,\theta_+)$, $p_+=p(v_+,\theta_+)$.

Let $\xi=0$ in (2.4); then we get $\sigma_-v_++u_+=\sigma_-v_-+u_-=0$ and $(2.4)_1$ yields (1.11). If $u_+\leq 0$, there is no stationary solution of system (1.9).

Suppose that $u_+>0$. Setting $\hat{E}(\xi)=-\int_\xi^\infty(\hat{v}\hat{q})(y)dy$, and using (2.4) and (1.11) yields

$$\begin{aligned}&\hat{v}_\xi=-\frac{\hat{v}}{\mu\sigma_-}\left(\sigma_-^2(\hat{v}-v_+)+(\hat{p}-p_+)\right),\\&\hat{\theta}_\xi=\frac{\hat{v}}{\kappa}\left(-\sigma_-(\hat{e}-e_+)-\sigma_-p_+(\hat{v}-v_+)+\frac{\sigma_-^3}{2}(\hat{v}-v_+)^2+\hat{q}\right),\\&\hat{q}_\xi=\hat{v}\left(\hat{\theta}^4-\theta_+^4+\hat{E}\right),\quad \hat{E}_\xi=\hat{v}\hat{q},\end{aligned}\tag{2.5}$$

where we used

$$\mu\frac{\hat{u}\hat{u}_\xi}{\hat{v}}=\mu\sigma_-^2\hat{v}_\xi=-\sigma_-\hat{v}\left(\sigma_-^2(\hat{v}-v_+)+(\hat{p}-p_+)\right).$$

Moreover, we have

$$(\hat{v},\hat{\theta},\hat{q})(0)=(v_-,\theta_-,0),\quad (\hat{v},\hat{\theta},\hat{q},\hat{E})(\infty)=(v_+,\theta_+,0,0).\tag{2.6}$$

To discuss the solvability of system (2.5), (2.6) near the infinity asymptotic state $(v_+,\theta_+,0,0)$, we need to introduce the stationary perturbation variables given by

$$(\widetilde{v},\widetilde{\theta},\widetilde{q},\widetilde{E}):=(\hat{v},\hat{\theta},\hat{q},\hat{E})-(v_+,\theta_+,0,0).$$

Then, system (2.5), (2.6) is transformed into the vector equations for $(\widetilde{v},\widetilde{\theta},\widetilde{q},\widetilde{E})$

$$\frac{d}{d\xi}\begin{pmatrix}\widetilde{v}\\ \widetilde{\theta}\\ \widetilde{q}\\ \widetilde{E}\end{pmatrix}=J_+\begin{pmatrix}\widetilde{v}\\ \widetilde{\theta}\\ \widetilde{q}\\ \widetilde{E}\end{pmatrix}+\begin{pmatrix}g_1(\widetilde{v},\widetilde{\theta})\\ g_2(\widetilde{v},\widetilde{\theta},\widetilde{q})\\ g_3(\widetilde{v},\widetilde{\theta},\widetilde{E})\\ g_4(\widetilde{v},\widetilde{q})\end{pmatrix},\quad \xi>0 \tag{2.7}$$
$$(\widetilde{v},\widetilde{\theta},\widetilde{q})(0)=(v_--v_+,\theta_--\theta_+,0),\quad (\widetilde{v},\widetilde{\theta},\widetilde{q},\widetilde{E})(\infty)=(0,0,0,0),$$

where J_+ is Jacobian matrix at an equilibrium point $(0,0,0,0)$ defined by

$$J_+=\begin{pmatrix}-\frac{v_+}{\mu\sigma_-}\left(\sigma_-^2+p_v^+\right) & -\frac{v_+}{\mu\sigma_-}p_\theta^+ & 0 & 0\\ -\frac{\sigma_-v_+}{\kappa}\left(e_v^++p_+\right) & -\frac{\sigma_-v_+}{\kappa}e_\theta^+ & \frac{v_+}{\kappa} & 0\\ 0 & 4v_+\theta_+^3 & 0 & v_+\\ 0 & 0 & v_+ & 0\end{pmatrix}\equiv\begin{pmatrix}a_{11} & a_{12} & 0 & 0\\ a_{21} & a_{22} & \frac{v_+}{\kappa} & 0\\ 0 & 4v_+\theta_+^3 & 0 & v_+\\ 0 & 0 & v_+ & 0\end{pmatrix} \tag{2.8}$$

and $g_i(i=1,2)$ are nonlinear terms such that

$$\begin{aligned}
g_1(\widetilde{v},\widetilde{\theta})=&-\frac{\widetilde{v}}{\mu\sigma_-}\left(\sigma_-^2\widetilde{v}+(\hat{p}-p_+)\right)\\
&-\frac{v_+}{\mu\sigma_-}(\hat{p}-p_+-p_v^+\widetilde{v}-p_\theta^+\widetilde{\theta})=O(\widetilde{v}^2+\widetilde{\theta}^2),\\
g_2(\widetilde{v},\widetilde{\theta},\widetilde{q})=&-\frac{\sigma_-\widetilde{v}}{\kappa}(\hat{e}-e_+)-\frac{\sigma_-v_+}{\kappa}(\hat{e}-e_+-e_v^+\widetilde{v}-e_\theta^+\widetilde{\theta})\\
&-\frac{\sigma_-p_+}{\kappa}\widetilde{v}^2+\frac{\sigma_-^3}{2\kappa}\widetilde{v}^3-\frac{\widetilde{v}\widetilde{q}}{\kappa}=O(\widetilde{v}^2+\widetilde{\theta}^2+\widetilde{q}^2),\\
g_3(\widetilde{v},\widetilde{\theta},\widetilde{E})=&\,\widetilde{v}\left((\widetilde{\theta}+\theta_+)^4-\theta_+^4+\widetilde{E}\right)+v_+\left((\widetilde{\theta}+\theta_+)^4-\theta_+^4-4\theta_+^3\widetilde{\theta}\right)\\
=&\,O(\widetilde{v}^2+\widetilde{\theta}^2+\widetilde{E}^2),\\
g_4(\widetilde{v},\widetilde{q})=&\,\widetilde{v}\widetilde{q}=O(\widetilde{v}^2+\widetilde{q}^2),
\end{aligned}$$

where $p_v^+=p_v(v_+,\theta_+)$, $e_v^+=e_v(v_+,\theta_+)$ and so on.

2.2. Proof of Theorem 1.1. By (2.8), we have

$$J_+-\lambda I=\begin{pmatrix}a_{11}-\lambda & a_{12} & 0 & 0\\ a_{21} & a_{22}-\lambda & \frac{v_+}{\kappa} & 0\\ 0 & 4v_+\theta_+^3 & -\lambda & v_+\\ 0 & 0 & v_+ & -\lambda\end{pmatrix}$$

and the characteristic determinant of J_+ is

$$\begin{aligned}
|J_+-\lambda I|=&(-\lambda)\begin{vmatrix}a_{11}-\lambda & a_{12} & 0\\ a_{21} & a_{22}-\lambda & \frac{v_+}{\kappa}\\ 0 & 4v_+\theta_+^3 & -\lambda\end{vmatrix}-v_+\begin{vmatrix}a_{11}-\lambda & a_{12} & 0\\ a_{21} & a_{22}-\lambda & 0\\ 0 & 4v_+\theta_+^3 & v_+\end{vmatrix}\\
=&\left(\lambda^2-v_+^2\right)\begin{vmatrix}a_{11}-\lambda & a_{12}\\ a_{21} & a_{22}-\lambda\end{vmatrix}+\frac{4v_+^2\theta_+^3}{\kappa}\lambda(a_{11}-\lambda).
\end{aligned}$$

Assume that $u_+>0$ and (1.6). Then, the eigenvalues $\lambda_i(i=1,\cdots,4)$ of J_+ must satisfy

$$\lambda^4+b_1\lambda^3+b_2\lambda^2+b_3\lambda+b_4=0, \tag{2.9}$$

where

$$\begin{aligned} &b_1=-(a_{11}+a_{22}), \quad b_2=(a_{11}a_{22}-a_{12}a_{21})-v_+^2-\frac{4v_+^2\theta_+^3}{\kappa}, \\ &b_3=-v_+^2b_1+\frac{4v_+^2\theta_+^3}{\kappa}a_{11}, \quad b_4=-v_+^2(a_{11}a_{22}-a_{12}a_{21}). \end{aligned} \tag{2.10}$$

Noticing (see (2.2), (2.1)) that

$$\widetilde{p}_v(v_+,s_+)=p_v^+-\frac{\theta_+(p_\theta^+)^2}{e_\theta^+} \quad \text{and} \quad e_v^+=\theta_+p_\theta^+-p^+,$$

and using (2.8), we get

$$\begin{aligned} a_{11}+a_{22}&=-\frac{v_+}{\mu\sigma_-}\left(\sigma_-^2+\widetilde{p}_v(v_+,s_+)+\frac{\theta_+(p_\theta^+)^2}{e_\theta^+}\right)-\frac{\sigma_-v_+}{\kappa}e_\theta^+, \\ a_{11}a_{22}-a_{12}a_{21}&\equiv\frac{v_+^2}{\mu\kappa}\left(\frac{u_+^2}{v_+^2}+p_v^+\right)e_\theta^+-\frac{v_+^2}{\mu\kappa}\left(e_v^++p_+\right)p_\theta^+ \\ &=\frac{v_+^2}{\mu\kappa}\left(\frac{u_+^2}{v_+^2}+\widetilde{p}_v(v_+,s_+)\right)e_\theta^+, \\ a_{11}&=-\frac{v_+}{\mu\sigma_-}\left(\sigma_-^2+\widetilde{p}_v(v_+,s_+)+\frac{\theta_+(p_\theta^+)^2}{e_\theta^+}\right). \end{aligned} \tag{2.11}$$

Using (1.10) and (1.11), we have

$$M_+>1(<1)\leftrightarrows\left(\sigma_-^2+\widetilde{p}_v(v_+,s_+)\right)>0(<0). \tag{2.12}$$

From Vieta's formula, the roots of system (2.9) have the following properties:

$$\begin{aligned} &\lambda_1+\lambda_2+\lambda_3+\lambda_4=-b_1, \\ &\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4=b_2, \\ &\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4=-b_3, \\ &\lambda_1\lambda_2\lambda_3\lambda_4=b_4. \end{aligned} \tag{2.13}$$

For the case $M_+>1$: Using (1.6), (2.11), (2.12) and $\sigma_-<0$, we obtain from (2.10)

$$b_1<0, \quad b_3>0, \quad b_4<0,$$

which implies, together with (2.13)

$$\begin{aligned} &\lambda_1\lambda_2\lambda_3\lambda_4<0, \\ &\lambda_1\lambda_2(\lambda_3+\lambda_4)+(\lambda_1+\lambda_2)\lambda_3\lambda_4<0, \\ &\lambda_1+\lambda_2+\lambda_3+\lambda_4>0. \end{aligned} \tag{2.14}$$

The first inequality of (2.14) implies (2.9) doesn't have any zero real root and we can assume $\lambda_1\lambda_2<0$, $\lambda_3\lambda_4>0$ without loss of generality. Moreover, using the second and third inequalities of (2.14), we have $\lambda_3+\lambda_4>0$. Therefore, without loss of generality, we can assume

$$\lambda_1<0,\ \lambda_2>0,\ Re\lambda_3>0 \text{ and } Re\lambda_4>0.$$

To make the manifold theory directly applicable, we need to reduce system (2.7) to the block diagonal form. From elementary linear algebra, there is a real nonsingular matrix $Q=(q_{ij})_{4\times 4}$ such that

$$Q^{-1}J_+Q=\mathrm{diag}(\lambda_1,\mathrm{A}), \tag{2.15}$$

where A is a 3×3 matrix corresponding to the eigenvalues $\lambda_i(i=2,3,4)$. Therefore, the linear transformation

$$\begin{pmatrix}\widetilde{V}\\ \widetilde{\Theta}\\ \widetilde{Q}\\ \widetilde{\Xi}\end{pmatrix}=Q^{-1}\begin{pmatrix}\widetilde{v}\\ \widetilde{\theta}\\ \widetilde{q}\\ \widetilde{E}\end{pmatrix}$$

applied to system (2.7) yields the equivalent boundary value problem

$$\frac{d}{d\xi}\begin{pmatrix}\widetilde{V}\\ \widetilde{\Theta}\\ \widetilde{Q}\\ \widetilde{\Xi}\end{pmatrix}=\mathrm{diag}(\lambda_1,\mathrm{A})\begin{pmatrix}\widetilde{V}\\ \widetilde{\Theta}\\ \widetilde{Q}\\ \widetilde{\Xi}\end{pmatrix}+\begin{pmatrix}H_1(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})\\ H_2(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})\\ H_3(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})\\ H_4(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})\end{pmatrix},\quad \xi>0, \tag{2.16}$$

$$\begin{aligned}&q_{11}\widetilde{V}(0)+q_{12}\widetilde{\Theta}(0)+q_{13}\widetilde{Q}(0)+q_{14}\widetilde{\Xi}(0)=v_--v_+,\\ &q_{21}\widetilde{V}(0)+q_{22}\widetilde{\Theta}(0)+q_{23}\widetilde{Q}(0)+q_{24}\widetilde{\Xi}(0)=\theta_--\theta_+,\\ &q_{31}\widetilde{Q}(0)+q_{32}\widetilde{\Theta}(0)+q_{33}\widetilde{Q}(0)+q_{34}\widetilde{\Xi}(0)=0,\end{aligned} \tag{2.17}$$

$$(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})(\infty)=(0,0,0,0), \tag{2.18}$$

where $H_i(i=1,\cdots,4)$ are defined by

$$\begin{pmatrix}H_1(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})\\ H_2(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})\\ H_3(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})\\ H_4(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})\end{pmatrix}=Q^{-1}\begin{pmatrix}g_1(\widetilde{v},\widetilde{\theta})\\ g_2(\widetilde{v},\widetilde{\theta},\widetilde{q})\\ g_3(\widetilde{v},\widetilde{\theta},\widetilde{E})\\ g_4(\widetilde{v},\widetilde{q})\end{pmatrix}.$$

For the sake of technique only, it is convenient to introduce an undetermined parameter $\widetilde{E}_0:=\widetilde{E}(0)$, simultaneously add the auxiliary boundary condition $q_{41}\widetilde{V}(0)+q_{42}\widetilde{\Theta}(0)+q_{43}\widetilde{Q}(0)+q_{44}\widetilde{\Xi}(0)=\widetilde{E}_0$ which is combined with (2.17) and described very succinctly as

$$\begin{pmatrix}\widetilde{V}(0)\\ \widetilde{\Theta}(0)\\ \widetilde{Q}(0)\\ \widetilde{\Xi}(0)\end{pmatrix}=Q^{-1}\begin{pmatrix}v_--v_+\\ \theta_--\theta_+\\ 0\\ \widetilde{E}_0\end{pmatrix}. \tag{2.19}$$

Since the previous argument proceeds inductively to yield the fact that J_+ has one negative eigenvalue λ_1 as well as three eigenvalues with positive real part. By virtue of the manifold theory in [29], there exist a C^∞ local stable manifold $W^s_{loc}(0,0,0,0)$ corresponding to λ_1 and a C^∞ local unstable manifold $W^u_{loc}(0,0,0,0)$ corresponding to $\lambda_i (i=2,3,4)$. More specifically, $W^s_{loc}(0,0,0,0)$ can locally be represented by a graph over the $\widetilde{V}$ variable, i.e.,

$$\begin{aligned}W^s_{loc}(0,0,0,0) = \{&(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi}) \in R^4 \,|\, \exists C^\infty \text{ functions } h^s_{\widetilde{\Theta}},\ h^s_{\widetilde{Q}} \text{ and } h^s_{\widetilde{\Xi}} \text{ such that}\\ &\widetilde{\Theta} = h^s_{\widetilde{\Theta}}(\widetilde{V}), \widetilde{Q} = h^s_{\widetilde{Q}}(\widetilde{V}) \text{and} \widetilde{\Xi} = h^s_{\widetilde{\Xi}}(\widetilde{V}) \text{ with } h^s_{\widetilde{\Theta}}(0) = Dh^s_{\widetilde{\Theta}}(0) = 0,\\ &h^s_{\widetilde{Q}}(0) = Dh^s_{\widetilde{Q}}(0) = 0 \text{ and } h^s_{\widetilde{\Xi}}(0) = Dh^s_{\widetilde{\Xi}}(0) = 0, \text{ for } |\widetilde{V}| \text{ sufficiently small}\}.\end{aligned}$$

Furthermore, if $(\widetilde{V}(0),\widetilde{\Theta}(0),\widetilde{Q}(0),\widetilde{\Xi}(0))$ is located on the stable manifold $W^s_{loc}(0,0,0,0)$, then problem (2.16), (2.18) and (2.19) has a unique smooth solution $(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})$ which approaches the origin $(0,0,0,0)$ at an exponential rate asymptotically as $\xi \to \infty$, i.e.,

$$|\partial^k_x(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})(\xi)| \le C|\widetilde{V}(0)|e^{-c\xi}, \text{ for } k = 0,1,2,\cdots. \tag{2.20}$$

Next, we assert that if

$$(\widetilde{V}(0),\widetilde{\Theta}(0),\widetilde{Q}(0),\widetilde{\Xi}(0)) \in \{(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi}) \in R^4 \,|\, \widetilde{\Theta} = h^s_{\widetilde{\Theta}}(\widetilde{V}),\ \widetilde{Q} = h^s_{\widetilde{Q}}(\widetilde{V}),\ \widetilde{\Xi} = h^s_{\widetilde{\Xi}}(\widetilde{V})\}, \tag{2.21}$$

original stationary problem (1.9) with $|v_- - v_+| + |\theta_- - \theta_+| \ll 1$ is equivalent to boundary value problem (2.16), (2.18) and (2.19) with $|\widetilde{V}(0)| \ll 1$. It suffices to show that $\widetilde{V}(0)$ depends locally on the original data $(v_- - v_+, \theta_- - \theta_+)$ in a continuous differentiable way.

Using (2.21), the first and second equations in (2.17) can be rewritten as

$$\begin{aligned}q_{11}\widetilde{V}(0) + q_{12}h^s_{\widetilde{\Theta}}(\widetilde{V}(0)) + q_{13}h^s_{\widetilde{Q}}(\widetilde{V}(0)) + q_{14}h^s_{\widetilde{\Xi}}(\widetilde{V}(0)) = v_- - v_+,\\ q_{21}\widetilde{V}(0) + q_{22}h^s_{\widetilde{\Theta}}(\widetilde{V}(0)) + q_{23}h^s_{\widetilde{Q}}(\widetilde{V}(0)) + q_{24}h^s_{\widetilde{\Xi}}(\widetilde{V}(0)) = \theta_- - \theta_+.\end{aligned} \tag{2.22}$$

If we do not maintain

$$q_{11} = q_{21} = 0, \tag{2.23}$$

then by using implicit function theorem, one easily solves equation (2.22) for $\widetilde{V}(0)$ to obtain a unique C^1 function of $(v_- - v_+, \theta_- - \theta_+)$ in a neighborhood of the origin $(0,0)$. Thus, by using differential mean value theorem, we have

$$|\widetilde{V}(0)| \le C(|v_- - v_+| + |\theta_- - \theta_+|) \tag{2.24}$$

if $|v_- - v_+| + |\theta_- - \theta_+| \ll 1$. This implies the assertion mentioned at the beginning of this paragraph holds. In addition, from (2.24), it follows that condition (2.20) is also equivalent to (1.12). If (2.23) holds, then we prove

$$q_{11} = q_{21} = q_{31} = q_{41} = 0,$$

which is impossible since the matrix Q is nonsingular. By premultiplying both sides of equality (2.15) by Q and using (2.8), we immediately deduce that $J_+Q = Q\text{diag}(\lambda_1, \mathrm{A})$

including the following algebraic equations:

$$\begin{cases} a_{11}q_{11}+a_{12}q_{21}=\lambda_1 q_{11}, \\ a_{21}q_{11}+a_{22}q_{21}+\frac{v_+}{\kappa}q_{31}=\lambda_1 q_{21}, \\ 4v_+\theta_+^3 q_{21}+v_+q_{41}=\lambda_1 q_{31}, \quad v_+q_{31}=\lambda_1 q_{41} \end{cases} \tag{2.25}$$

according to the definition of matrix multiplication. By (2.25) and (2.23), we get (2.23). For the case $M_+>1$, the proof of Theorem 1.1 is completed.

For the case $M_+<1$: Using (1.6), (2.11), (2.12) and $\sigma_-<0$, we obtain from (2.10)

$$b_2<0, \quad b_4>0,$$

which implies, together with (2.13)

$$\begin{aligned} &\lambda_1\lambda_2\lambda_3\lambda_4>0, \\ &\lambda_1\lambda_2+\lambda_3\lambda_4+(\lambda_1+\lambda_2)(\lambda_3+\lambda_4)<0. \end{aligned} \tag{2.26}$$

Using (2.26), we deduce that (2.9) doesn't have any zero real root and the following possible cases:

$$\begin{aligned} &(1) \quad \lambda_1\lambda_2>0,\ \lambda_3\lambda_4>0,\ (\lambda_1+\lambda_2)(\lambda_3+\lambda_4)<0, \\ &(2) \quad \lambda_1\lambda_2>0,\ \lambda_3\lambda_4<0. \end{aligned} \tag{2.27}$$

Therefore, we can assume from (2.27), without the loss of generality,

$$Re\lambda_1<0,\ Re\lambda_2<0,\ \lambda_3>0 \text{ and } \lambda_4>0. \tag{2.28}$$

Using (2.28), by similar arguments as above, we can prove Theorem 1.1 in the case of $M_+<1$ (see [18] for more detail).

3. Stability of non-degenerate stationary solution

Rewriting $(1.8)_3$ as

$$e_t-\sigma_-e_\xi+pu_\xi+q_\xi=\kappa(\frac{\theta_\xi}{v})_\xi+\mu\frac{u_\xi^2}{v}, \tag{3.1}$$

it is convenient to work with the equation for the entropy s and the temperature θ

$$\begin{aligned} &s_t-\sigma_-s_\xi+\frac{q_\xi}{\theta}=\kappa\left(\frac{\theta_\xi}{v\theta}\right)_\xi+\kappa\frac{\theta_\xi^2}{v\theta^2}+\mu\frac{u_\xi^2}{v\theta}, \\ &\theta_t-\sigma_-\theta_\xi+\frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)}u_\xi+\frac{1}{e_\theta(v,\theta)}q_\xi=\frac{\kappa}{e_\theta(v,\theta)}\left(\frac{\theta_\xi}{v}\right)_\xi+\frac{\mu}{e_\theta(v,\theta)}\frac{u_\xi^2}{v}, \end{aligned} \tag{3.2}$$

where we used $\widetilde{e}_v(v,s)=-\widetilde{p}(v,s)$ and $\widetilde{e}_s(v,s)=\theta$ due to (2.2). Moreover, rewriting $(1.9)_3$ as

$$-\sigma_-\hat{e}_\xi+\hat{p}\hat{u}_\xi+\hat{q}_\xi=\kappa\left(\frac{\hat{\theta}_\xi}{\hat{v}}\right)_\xi+\mu\frac{\hat{u}_\xi^2}{\hat{v}},$$

it is convenient to work with the equation for $\hat{s}=s(\hat{v},\hat{\theta})$ and $\hat{\theta}$

$$\begin{aligned}
&-\sigma_- s(\hat{v},\hat{\theta})_\xi+\frac{\hat{q}_\xi}{\hat{\theta}}=\kappa\left(\frac{\hat{\theta}_\xi}{\hat{v}\hat{\theta}}\right)_\xi+\kappa\frac{\hat{\theta}_\xi^2}{\hat{v}\tilde{\theta}^2}+\mu\frac{\hat{u}_\xi^2}{\hat{v}\hat{\theta}},\\
&-\sigma_-\hat{\theta}_\xi+\frac{\hat{\theta}p_\theta(\hat{v},\hat{\theta})}{e_\theta(\hat{v},\hat{\theta})}\hat{u}_\xi+\frac{1}{e_\theta(\hat{v},\hat{\theta})}\hat{q}_\xi=\frac{\kappa}{e_\theta(\hat{v},\hat{\theta})}\left(\frac{\hat{\theta}_\xi}{\hat{v}}\right)_\xi+\frac{\mu}{e_\theta(\hat{v},\hat{\theta})}\frac{\hat{u}_\xi^2}{\hat{v}}.
\end{aligned}\tag{3.3}$$

By (1.8), (1.9), (3.2) and (3.3), we have

$$\begin{cases}
\phi_t-\sigma_-\phi_\xi-\psi_\xi=0,\\
\psi_t-\sigma_-\psi_\xi+(p-\hat{p})_\xi=\mu\left(\dfrac{u_\xi}{v}-\dfrac{\hat{u}_\xi}{\hat{v}}\right)_\xi,\\
\zeta_t-\sigma_-\zeta_\xi+\left(\dfrac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)}u_\xi-\dfrac{\hat{\theta}p_\theta(\hat{v},\hat{\theta})}{e_\theta(\hat{v},\hat{\theta})}\hat{u}_\xi\right)+\left(\dfrac{1}{e_\theta(v,\theta)}q_\xi-\dfrac{1}{e_\theta(\hat{v},\hat{\theta})}\hat{q}_\xi\right)\\
\quad=\kappa\left(\dfrac{1}{e_\theta(v,\theta)}\left(\dfrac{\theta_\xi}{v}\right)_\xi-\dfrac{1}{e_\theta(\hat{v},\hat{\theta})}\left(\dfrac{\hat{\theta}_\xi}{\hat{v}}\right)_\xi\right)+\mu\left(\dfrac{1}{e_\theta(v,\theta)}\dfrac{u_\xi^2}{v}-\dfrac{1}{e_\theta(\hat{v},\hat{\theta})}\dfrac{\hat{u}_\xi^2}{\hat{v}}\right),\\
-\left(\dfrac{q_\xi}{v}-\dfrac{\hat{q}_\xi}{\hat{v}}\right)_\xi+(vq-\hat{v}\hat{q})+\left(\theta^4-\hat{\theta}^4\right)_\xi=0,\\
(\phi,\psi,\zeta)\,|_{t=0}=(\phi_0,\psi_0,\zeta_0)(\xi)\equiv(v_0-\hat{v},u_0-\hat{u},\theta_0-\hat{\theta})(\xi),\\
\lim\limits_{\xi\to+\infty}(\phi,\psi,\zeta,\omega)(\xi,t)=(0,0,0,0),\\
(\phi,\psi,\zeta,\omega)\,|_{\xi=0}=(0,0,0,0).
\end{cases}\tag{3.4}$$

3.1. A priori estimate. To prove Theorem 1.2, we show the following a priori estimate:

Proposition 3.1 (A priori estimate). *Besides the assumptions of Theorem 1.2, suppose that $(\phi,\psi,\zeta,\omega)\in X(0,T)$ is a stationary solution to system (3.4) for some positive constant $T>0$. Then, there exists a constant $\varepsilon_1>0$ such that if*

$$\sup_{0\le t\le T}\|(\varphi,\psi,\zeta,\omega,\omega_\xi)(t)\|_1\le\varepsilon_1\quad\textit{and}\quad\delta=|(v_--v_+,\theta_--\theta_+)|\le\varepsilon_1,\tag{3.5}$$

then for any $t\in[0,T]$, holds that

$$\|(\phi,\psi,\zeta,\omega,\omega_\xi)(t)\|_1^2+\int_0^t\|\phi_\xi(\tau)\|^2d\tau+\int_0^t\|(\psi_\xi,\zeta_\xi,\omega,\omega_\xi)(\tau)\|_1^2d\tau\le C\|(\phi,\psi,\zeta)(0)\|_1^2.\tag{3.6}$$

In this subsection, we will prove Proposition 3.1. For notational simplicity, we introduce $A\lesssim B$ if $A\le C_0B$ holds uniformly on the constant C_0 independently of t,ξ,T,ε_1.

We first give the following energy estimate.

Lemma 3.1. *Under the assumptions of Proposition 3.1, holds that*

$$\|(\phi,\psi,\zeta)(t)\|^2+\int_0^t\|(\psi_\xi,\zeta_\xi,\omega,\omega_\xi)(\tau)\|^2d\tau\lesssim\|(\phi,\psi,\zeta)(0)\|^2+\varepsilon_1\int_0^t\|\phi_\xi(\tau)\|^2d\tau.\tag{3.7}$$

Proof. Let

$$\mathcal{E}=\widetilde{e}(v,s)-\widetilde{e}(\hat{v},\hat{s})+\frac{\psi^2}{2}+\widetilde{p}(\hat{v},\hat{s})(v-\hat{v})-\hat{\theta}(s-\hat{s}), \tag{3.8}$$

where $\hat{s}=s(\hat{v},\hat{\theta})$, then using (3.1)-(3.4) yields

$$\begin{aligned}\mathcal{E}_t-\sigma_-\mathcal{E}_\xi=&-pu_\xi+\hat{p}\hat{u}_\xi+\kappa\left(\frac{\theta_\xi}{v}-\frac{\hat{\theta}_\xi}{\hat{v}}\right)_\xi+\mu\left(\frac{u_\xi^2}{v}-\frac{\hat{u}_\xi^2}{\hat{v}}\right)-\omega_\xi\\&-\psi(p-\hat{p})_\xi+\mu\psi\left(\frac{u_\xi}{v}-\frac{\hat{u}_\xi}{\hat{v}}\right)_\xi-\sigma_-\widetilde{p}(\hat{v},\hat{s})_\xi(v-\hat{v})+\hat{p}\psi_\xi-\sigma_-\hat{\theta}_\xi(s-\hat{s})\\&-\kappa\hat{\theta}\left(\frac{\theta_\xi}{v\theta}-\frac{\hat{\theta}_\xi}{\hat{v}\hat{\theta}}\right)_\xi-\kappa\hat{\theta}\left(\frac{\theta_\xi^2}{v\theta^2}-\frac{\hat{\theta}_\xi^2}{\hat{v}\hat{\theta}^2}\right)-\mu\hat{\theta}\left(\frac{u_\xi^2}{v\theta}-\frac{\hat{u}_\xi^2}{\hat{v}\hat{\theta}}\right)+\hat{\theta}\left(\frac{q_\xi}{\theta}-\frac{\hat{q}_\xi}{\hat{\theta}}\right).\end{aligned} \tag{3.9}$$

Noticing that

$$-pu_\xi+\hat{p}\hat{u}_\xi-\psi(p-\hat{p})_\xi+\hat{p}\psi_\xi=-[\psi(p-\hat{p})]_\xi-(p-\hat{p})\,\hat{u}_\xi,$$

$$\begin{aligned}&\left(\frac{\theta_\xi}{v}-\frac{\hat{\theta}_\xi}{\hat{v}}\right)_\xi-\hat{\theta}\left(\frac{\theta_\xi}{v\theta}-\frac{\hat{\theta}_\xi}{\hat{v}\hat{\theta}}\right)_\xi-\hat{\theta}\left(\frac{\theta_\xi^2}{v\theta^2}-\frac{\hat{\theta}_\xi^2}{\hat{v}\hat{\theta}^2}\right)=\left(1-\frac{\hat{\theta}}{\theta}\right)\left(\frac{\theta_\xi}{v}\right)_\xi\\&=\left(\frac{\zeta\zeta_\xi}{\theta v}\right)_\xi+\frac{\zeta}{\theta}\left(\frac{\hat{\theta}_\xi}{v}\right)_\xi+\frac{\hat{\theta}_\xi\zeta\zeta_\xi}{v\theta^2}-\frac{\hat{\theta}\zeta_\xi^2}{v\theta^2},\end{aligned}$$

$$\begin{aligned}&\psi\left(\frac{u_\xi}{v}-\frac{\hat{u}_\xi}{\hat{v}}\right)_\xi+\left(\frac{u_\xi^2}{v}-\frac{\hat{u}_\xi^2}{\hat{v}}\right)-\hat{\theta}\left(\frac{u_\xi^2}{v\theta}-\frac{\hat{u}_\xi^2}{\hat{v}\hat{\theta}}\right)=\psi\left(\frac{u_\xi}{v}-\frac{\hat{u}_\xi}{\hat{v}}\right)_\xi+\left(1-\frac{\hat{\theta}}{\theta}\right)\frac{u_\xi^2}{v}\\&=\left(\psi\left(\frac{u_\xi}{v}-\frac{\hat{u}_\xi}{\hat{v}}\right)\right)_\xi-2\frac{\hat{u}_\xi\zeta\psi_\xi}{v\theta}+\frac{\hat{u}_\xi^2\zeta}{v\theta}-\frac{\hat{\theta}\psi_\xi^2}{v\theta}+\frac{\phi\psi_\xi\hat{u}_\xi}{v\hat{v}},\end{aligned}$$

$$\begin{aligned}\widetilde{p}(\hat{v},\hat{s})_\xi(v-\hat{v})-\hat{\theta}_\xi(s-\hat{s})=&\left(\widetilde{p}_v(\hat{v},\hat{s})(v-\hat{v})+\widetilde{p}_s(\hat{v},\hat{s})(s-\hat{s})\right)\hat{v}_\xi\\&-\left(\widetilde{\theta}_v(\hat{v},\hat{s})(v-\hat{v})+\widetilde{\theta}_s(\hat{v},\hat{s})(s-\hat{s})\right)\hat{s}_\xi,\end{aligned}$$

and using (3.3) and (1.9), by the same lines as in [17, (3.12)], we drive from (3.9) that

$$\mathcal{E}_t-\sigma_-\mathcal{E}_\xi+\mu\frac{\hat{\theta}}{v\theta}\psi_\xi^2+\kappa\frac{\hat{\theta}}{v\theta^2}\zeta_\xi^2=-\frac{\zeta}{\theta}\omega_\xi+\Pi_{1\xi}+\Pi_2-\hat{u}_\xi\Delta_1+\sigma_-\hat{s}_\xi\Delta_2, \tag{3.10}$$

where

$$\begin{aligned}\Pi_1=&\,\mu\psi\left(\frac{u_\xi}{v}-\frac{\hat{u}_\xi}{\hat{v}}\right)+\kappa\frac{\zeta\zeta_\xi}{\theta v}-\psi(p(v,\theta)-p(\hat{v},\hat{\theta})),\\\Pi_2=&-2\mu\frac{\hat{u}_\xi\zeta\psi_\xi}{v\theta}+\mu\hat{u}_\xi^2\zeta\left(\frac{1}{v\theta}-\frac{1}{\hat{v}\hat{\theta}}\right)+\mu\frac{\phi\psi_\xi\hat{u}_\xi}{v\hat{v}}\\&+\kappa\zeta\left(\frac{1}{\theta}\left(\frac{\hat{\theta}_\xi}{v}\right)_\xi-\frac{1}{\hat{\theta}}\left(\frac{\hat{\theta}_\xi}{\hat{v}}\right)_\xi\right)+\kappa\frac{\hat{\theta}_\xi\zeta\zeta_\xi}{v\theta^2},\\\Delta_1=&\,\widetilde{p}(v,s)-\widetilde{p}(\hat{v},\hat{s})-\widetilde{p}_v(\hat{v},\hat{s})(v-\hat{v})-\widetilde{p}_s(\hat{v},\hat{s})(s-\hat{s}),\\\Delta_2=&\,\theta-\hat{\theta}-\widetilde{\theta}_v(\hat{v},\hat{s})(v-\hat{v})-\widetilde{\theta}_s(\hat{v},\hat{s})(s-\hat{s}).\end{aligned}$$

Using (2.3), $\widetilde{e}_v(v,s)=-\widetilde{p}(v,s)$ and $\widetilde{e}_s(v,s)=\theta$, it is easy to check that

$$(\phi^2+\psi^2+\zeta^2)\lesssim\mathcal{E}\lesssim(\phi^2+\psi^2+\zeta^2). \tag{3.11}$$

Using $(\phi,\psi,\zeta)|_{\xi=0}=(0,0,0)$ and (3.11), we get $\mathcal{E}|_{\xi=0}=\Pi_1|_{\xi=0}=0$. So, integrating (3.10) over $(\xi,t)\in(0,\infty)\times(0,t)$ yields

$$\begin{aligned}&\|(\phi,\psi,\zeta)(t)\|^2+\int_0^t\int_0^\infty\left(\mu\frac{\hat{\theta}}{v\theta}\psi_\xi^2+\kappa\frac{\hat{\theta}}{v\theta^2}\zeta_\xi^2\right)d\xi d\tau-\int_0^t\int_0^\infty\omega\left(\frac{\zeta}{\theta}\right)_\xi d\xi d\tau\\ \lesssim&\|(\phi,\psi,\zeta)(0)\|^2+\int_0^t\int_0^\infty(|\Pi_2|+|\hat{u}_\xi\Delta_1|+|\hat{s}_\xi\Delta_2|)d\xi d\tau.\end{aligned} \tag{3.12}$$

Noticing that by (3.5), there exist positive constants C and c such that

$$c\le v(\xi,t),\theta(\xi,t)\le C \tag{3.13}$$

if we choose ε_1 to be small, we have

$$\begin{aligned}|\Pi_2|\lesssim&|\zeta||\hat{u}_\xi||\psi_\xi|+|\zeta||\hat{u}_\xi|^2(|\phi|+|\zeta|)+|\psi_\xi||\phi||\hat{u}_\xi|\\&+|\zeta|\Big(|\hat{\theta}_{\xi\xi}|(|\phi|+|\zeta|)+|\hat{\theta}_\xi|(|\phi_\xi|+|\hat{v}_\xi|(|\phi|+|\zeta|))\Big)+|\hat{\theta}_\xi||\phi||\zeta_\xi|,\\ |\Delta_1|\lesssim&(\phi^2+\zeta^2),\quad |\Delta_2|\lesssim(\phi^2+\zeta^2).\end{aligned} \tag{3.14}$$

By using (3.14), (1.12) and the inequality $|f(\xi)|\le|f(0)|+\sqrt{\xi}\|f_\xi\|$, we have

$$\begin{aligned}&\int_0^\infty|\Pi_2|d\xi\lesssim\delta\int_0^\infty e^{-\hat{c}\xi}(\phi^2+\zeta^2)d\xi+\delta\|(\phi_\xi,\psi_\xi,\zeta_\xi)\|^2\lesssim\delta\|(\phi_\xi,\psi_\xi,\zeta_\xi)\|^2,\\&\int_0^\infty|\hat{u}_\xi\Delta_1|d\xi\lesssim\delta\int_0^\infty e^{-\hat{c}\xi}(\phi^2+\zeta^2)d\xi\lesssim\delta\|(\phi_\xi,\zeta_\xi)\|^2,\\&\int_0^\infty|\hat{s}_\xi\Delta_2|d\xi\lesssim\int_0^\infty(|\hat{v}_\xi|+|\hat{\theta}_\xi|)(\phi^2+\zeta^2)\lesssim\delta\|(\phi_\xi,\zeta_\xi)\|^2.\end{aligned} \tag{3.15}$$

Noticing that

$$-\left(\frac{\zeta}{\theta}\right)_\xi=-\frac{\zeta\hat{\theta}_\xi}{\theta\hat{\theta}}-\frac{\hat{\theta}}{4\theta^5}\left(\frac{q_\xi}{v}-\frac{\hat{q}_\xi}{\hat{v}}\right)_\xi-\hat{\theta}\left(\frac{vq}{4\theta^5}-\frac{\hat{v}\hat{q}}{4\hat{\theta}^5}\right)$$

due to $(3.4)_4$ and using $\omega|_{\xi=0}=0$, we get

$$\begin{aligned}&-\int_0^\infty\omega\left(\frac{\zeta}{\theta}\right)_\xi d\xi\\ =&\int_0^\infty\frac{\hat{\theta}}{4v\theta^5}\omega_\xi^2d\xi+\int_0^\infty\frac{v\hat{\theta}}{4\theta^5}\omega^2d\xi\underbrace{-\int_0^\infty\omega\frac{\zeta\hat{\theta}_\xi}{\theta\hat{\theta}}d\xi+\int_0^\infty\frac{\hat{\theta}}{4v\theta^5}\omega_\xi\left(\frac{1}{v}-\frac{1}{\hat{v}}\right)\hat{q}_\xi d\xi}_{I_1}\\&\underbrace{+\int_0^\infty\omega\left(\frac{q_\xi}{v}-\frac{\hat{q}_\xi}{\hat{v}}\right)\left(\frac{\hat{\theta}}{4\theta^5}\right)_\xi d\xi-\int_0^\infty\omega\hat{\theta}\hat{q}\left(\frac{v}{4\theta^5}-\frac{\hat{v}}{4\hat{\theta}^5}\right)d\xi}_{I_2}.\end{aligned} \tag{3.16}$$

By using (3.13), (1.12), (3.5) and the inequality $|f(\xi)|\le|f(0)|+\sqrt{\xi}\|f_\xi\|$, we have

$$\begin{aligned}|I_1|&\lesssim\int_0^\infty\Big(|\omega||\zeta||\hat\theta_\xi|+|\omega_\xi||\phi||\hat q_\xi|\Big)d\xi\\&\lesssim\delta\int_0^\infty e^{-\hat c\xi}(\omega^2+\zeta^2+\omega_\xi^2+\phi^2)d\xi\lesssim\delta\|(\phi_\xi,\zeta_\xi,\omega_\xi)\|^2,\\|I_2|&\lesssim\int_0^\infty\Big[|\omega|(|\omega_\xi|+|\phi||\hat q_\xi|)(|\zeta_\xi|+|\hat\theta_\xi|)+|\omega||\hat q|(|\phi|+|\zeta|)\Big]d\xi\\&\lesssim(\varepsilon_1+\delta)\|(\zeta_\xi,\omega,\omega_\xi)\|^2+\delta\int_0^\infty e^{-\hat c\xi}(\omega^2+\phi^2+\zeta^2)d\xi\lesssim\varepsilon_1\|(\phi_\xi,\zeta_\xi,\omega,\omega_\xi)\|^2.\end{aligned}\tag{3.17}$$

By using (3.15), (3.16) and (3.17), we obtain (3.7) from (3.12). □

Next, we estimate $\|(\phi_\xi,\psi_\xi)(t)\|$.

Lemma 3.2. *Under the assumptions of Proposition 3.1, it holds that*

$$\|(\phi_\xi,\psi_\xi)(t)\|^2+\int_0^t\|(\phi_\xi,\psi_{\xi\xi})(\tau)\|^2d\tau\lesssim\|(\phi,\psi,\zeta,\phi_\xi,\psi_\xi)(0)\|^2.\tag{3.18}$$

Proof. Applying ∂_ξ to $(3.4)_1$ and multiplying the resulting equality by $\mu\frac{\phi_\xi}{v^2}$ yields

$$\mu\left(\frac{\phi_\xi^2}{2v^2}\right)_t-\sigma_-\mu\left(\frac{\phi_\xi^2}{2v^2}\right)_\xi+\mu\frac{u_\xi\phi_\xi^2}{v^3}-\mu\frac{\phi_\xi\psi_{\xi\xi}}{v^2}=0.\tag{3.19}$$

Multiplying $(3.4)_2$ by $\frac{\phi_\xi}{v}$ yields

$$\left(\frac{\phi_\xi\psi}{v}\right)_t-\left(\frac{\phi_t\psi}{v}\right)_\xi+\frac{(p-\hat p)_\xi\phi_\xi}{v}+\frac{\psi_\xi^2}{v}=-\frac{\hat u_\xi\phi_\xi\psi}{v^2}+\frac{\hat v_\xi\psi\psi_\xi}{v^2}+\mu\left(\frac{u_\xi}{v}-\frac{\hat u_\xi}{\hat v}\right)_\xi\frac{\phi_\xi}{v}.\tag{3.20}$$

Subtracting (3.19) and (3.20), we have

$$\begin{aligned}&\left(\frac{\mu\phi_\xi^2}{2v^2}-\frac{\phi_\xi\psi}{v}\right)_t-\left(\frac{\sigma_-\mu\phi_\xi^2}{2v^2}-\frac{\phi_t\psi}{v}\right)_\xi-\frac{p_v(v,\theta)}{v}\phi_\xi^2\\&=\underbrace{\frac{\psi_\xi^2}{v}+\frac{\hat u_\xi\phi_\xi\psi}{v^2}-\frac{\hat v_\xi\psi\psi_\xi}{v^2}+\mu\frac{\hat v_\xi\phi_\xi\psi_\xi}{v^3}}_{I_3}\\&\quad\underbrace{-\mu\frac{\hat u_\xi\phi_\xi^2}{v^3}+\mu\left(\frac{\hat u_\xi\phi}{v\hat v}\right)_\xi\frac{\phi_\xi}{v}+\frac{p_\theta(v,\theta)}{v}\phi_\xi\zeta_\xi}_{I_4}\\&\quad+\underbrace{\Big[(p_v(v,\theta)-p_v(\hat v,\hat\theta))\hat v_\xi+(p_\theta(v,\theta)-p_\theta(\hat v,\hat\theta))\hat\theta_\xi\Big]\frac{\phi_\xi}{v}}_{I_5}.\end{aligned}\tag{3.21}$$

Integrating (3.21) over $(\xi,t)\in(0,\infty)\times(0,t)$, and using $\psi|_{\xi=0}=0$, $p_v(v,\theta)<0$ and (3.13), we get

$$\begin{aligned}&\|\phi_\xi(t)\|^2+\int_0^t\|\phi_\xi(\tau)\|^2d\tau\\&\lesssim\|(\phi_\xi,\psi)(0)\|^2+\|\psi(t)\|^2+\frac{|\sigma_-|}{v_-^2}\int_0^t\phi_\xi^2(0,\tau)d\tau+\sum_{i=3}^5\int_0^t\int_0^\infty|I_i|d\xi d\tau.\end{aligned}\tag{3.22}$$

By using $-\sigma_-\phi_\xi(0,t)=\psi_\xi(0,t)$ due to $(3.4)_1$ and $\phi(0,t)=0$ and the inequality $|f(\xi)|\le\sqrt{2}\|f\|^{\frac12}\|f_\xi\|^{\frac12}$, we have

$$\int_0^t \phi_\xi^2(0,\tau)d\tau=\frac{1}{\sigma_-^2}\int_0^t \psi_\xi^2(0,\tau)d\tau\lesssim\eta^{-1}\int_0^t\|\psi_\xi\|^2d\tau+\eta\int_0^t\|\psi_{\xi\xi}\|^2d\tau \tag{3.23}$$

for any $\eta>0$.

By using (3.13), (1.12), (3.5) and the inequality $|f(\xi)|\le|f(0)|+\sqrt{\xi}\|f_\xi\|$, we have

$$\begin{aligned}\int_0^\infty|I_3|d\xi&\lesssim\int_0^\infty\left(\psi_\xi^2+|\hat u_\xi||\phi_\xi||\psi|+|\hat v_\xi||\psi||\psi_\xi|+|\hat v_\xi||\phi_\xi||\psi_\xi|\right)d\xi\\&\lesssim\|\psi_\xi\|^2+\delta\int_0^\infty e^{-\hat c\xi}(\phi_\xi^2+\psi^2+\psi_\xi^2)d\xi\lesssim\|\psi_\xi\|^2+\delta\|(\phi_\xi,\psi_\xi)\|^2,\\ \int_0^\infty|I_4|d\xi&\lesssim\int_0^\infty\left[|\hat u_\xi||\phi_\xi|^2+(|\hat u_{\xi\xi}|+|\hat v_\xi|)|\phi||\phi_\xi|+|\phi_\xi||\zeta_\xi|\right]d\xi\\&\lesssim\delta\int_0^\infty e^{-\hat c\xi}(\phi^2+\phi_\xi^2)d\xi+\|\phi_\xi\|\|\zeta_\xi\|\lesssim\delta\|\phi_\xi\|^2+\eta\|\phi_\xi\|^2+\eta^{-1}\|\zeta_\xi\|^2,\\ \int_0^\infty|I_5|d\xi&\lesssim\int_0^\infty(|\hat v_\xi|+|\hat\theta_\xi|)(|\phi|+|\zeta|)|\phi_\xi|d\xi\\&\lesssim\delta\int_0^\infty e^{-\hat c\xi}(\phi^2+\zeta^2+\phi_\xi^2)d\xi\lesssim\delta\|(\phi_\xi,\zeta_\xi)\|^2,\end{aligned} \tag{3.24}$$

for any $\eta>0$.

Combining (3.22)-(3.24) and using (3.7) yields

$$\|\phi_\xi(t)\|^2+\int_0^t\|\phi_\xi(\tau)\|^2d\tau\lesssim(1+\eta^{-1})\|(\phi,\psi,\zeta,\phi_\xi)(0)\|^2+\eta\int_0^t\|\psi_{\xi\xi}(\tau)\|^2d\tau \tag{3.25}$$

for any $\eta>0$.

Multiplying $(3.4)_2$ by $-\psi_{\xi\xi}$, we have

$$\begin{aligned}&\frac12\left(\psi_\xi^2\right)_t+\left(\frac{\sigma_-}{2}\psi_\xi^2-\psi_t\psi_\xi\right)_\xi+\frac{\mu}{v}\psi_{\xi\xi}^2\\=&\underbrace{(p-\hat p)_\xi\psi_{\xi\xi}-\mu\psi_{\xi\xi}\hat u_{\xi\xi}\left(\frac1v-\frac1{\hat v}\right)+\mu\psi_{\xi\xi}\left(\frac{u_\xi v_\xi}{v^2}-\frac{\hat u_\xi\hat v_\xi}{\hat v^2}\right)}_{I_6}.\end{aligned} \tag{3.26}$$

Integrating (3.26) over $(0,\infty)\times(0,t)$, and using $\sigma_-<0$ and $\psi_t(0,t)=0$, we get

$$\|\psi_\xi(t)\|^2+\int_0^t\|\psi_{\xi\xi}(\tau)\|^2d\tau\lesssim\|\psi_\xi(0)\|^2+\int_0^t\int_0^\infty|I_6|d\xi d\tau. \tag{3.27}$$

Noticing that

$$(p-\hat p)_\xi=p_v(v,\theta)\phi_\xi+(p_\theta(v,\theta)\zeta_\xi+(p_v(v,\theta)-p_v(\hat v,\hat\theta))\hat v_\xi+(p_\theta(v,\theta)-p_\theta(\hat v,\hat\theta))\hat\theta_\xi,$$

$$\begin{aligned}|I_6|\lesssim&\left[|\phi_\xi|+|\zeta_\xi|+(|\phi|+|\zeta|)(|\hat v_\xi|+|\hat\theta_\xi|)\right]|\psi_{\xi\xi}|+|\hat u_{\xi\xi}||\phi||\psi_{\xi\xi}|\\&+|\psi_{\xi\xi}|(|\psi_\xi v_\xi|+|\hat u_\xi\phi_\xi|+|\phi||\hat u_\xi||\hat v_\xi|),\end{aligned}$$

and using (3.13), (1.12) and (3.5), we have

$$\begin{aligned}\int_0^\infty |I_6|d\xi \lesssim & \|\psi_{\xi\xi}\|(\|\phi_\xi\|+\|\zeta_\xi\|)+\|\psi_{\xi\xi}\|^{\frac{3}{2}}\|\psi_\xi\|^{\frac{1}{2}}\|\phi_\xi\| \\ & +\delta\int_0^\infty e^{-\hat{c}\xi}(\phi^2+\zeta^2+\psi_{\xi\xi}^2+\phi_\xi^2+\psi_\xi^2)d\xi \\ \lesssim & \eta\|\psi_{\xi\xi}\|^2+\eta^{-1}\|(\phi_\xi,\zeta_\xi)\|^2+\varepsilon_1\|(\phi_\xi,\psi_\xi,\zeta_\xi,\psi_{\xi\xi})\|^2 \end{aligned} \tag{3.28}$$

for any $\eta>0$.

By using (3.28), (3.7) and (3.25), we obtain from (3.27)

$$\begin{aligned}&\|\psi_\xi(t)\|^2+\int_0^t\|\psi_{\xi\xi}(\tau)\|^2d\tau\lesssim\|\psi_\xi(0)\|^2+\int_0^t\|(\phi_\xi,\psi_\xi,\zeta_\xi)(\tau)\|^2d\tau \\ &\lesssim\|(\phi,\psi,\zeta,\psi_\xi)(0)\|^2+\int_0^t\|\phi_\xi(\tau)\|^2d\tau\lesssim\|(\phi,\psi,\zeta,\phi_\xi,\psi_\xi)(0)\|^2.\end{aligned} \tag{3.29}$$

By (3.29) and (3.25), we get (3.18). □

Last, we estimate $\|\zeta_\xi(t)\|$.

LEMMA 3.3. *Under the assumptions of Proposition 3.1, it holds that*

$$\|\zeta_\xi(t)\|^2+\int_0^t\|(\zeta_{\xi\xi},\omega_{\xi\xi},\omega_\xi)(\tau)\|^2d\tau\lesssim\|(\phi,\psi,\zeta)(0)\|_1^2. \tag{3.30}$$

Proof. Multiplying $(3.4)_3$ by $-\zeta_{\xi\xi}$, we have

$$\begin{aligned}&\frac{1}{2}(\zeta_\xi^2)_t+\left(\frac{\sigma_-}{2}\zeta_\xi^2-\zeta_t\zeta_\xi\right)_\xi+\frac{\kappa}{ve_\theta(v,\theta)}\zeta_{\xi\xi}^2-\zeta_{\xi\xi}\left(\frac{q_\xi}{e_\theta(v,\theta)}-\frac{\hat{q}_\xi}{e_\theta(\hat{v},\hat{\theta})}\right) \\ &=\underbrace{-\kappa\zeta_{\xi\xi}\tilde{\theta}_{\xi\xi}\left(\frac{1}{e_\theta(v,\theta)v}-\frac{1}{e_\theta(\hat{v},\hat{\theta})\hat{v}}\right)+\kappa\zeta_{\xi\xi}\left(\frac{\theta_\xi v_\xi}{e_\theta(v,\theta)v^2}-\frac{\hat{\theta}_\xi\hat{v}_\xi}{e_\theta(\hat{v},\hat{\theta})\hat{v}^2}\right)}_{I_7} \\ &\quad\underbrace{+\left(\frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)}u_\xi-\frac{\hat{\theta}p_\theta(\hat{v},\hat{\theta})}{e_\theta(\hat{v},\hat{\theta})}\hat{u}_\xi\right)\zeta_{\xi\xi}-\mu\zeta_{\xi\xi}\left(\frac{u_\xi^2}{e_\theta(v,\theta)v}-\frac{\hat{u}_\xi^2}{e_\theta(\hat{v},\hat{\theta})\hat{v}}\right)}_{I_8}.\end{aligned} \tag{3.31}$$

Noticing that

$$\begin{aligned}\zeta_\xi&=\frac{1}{4\theta^3}\left(\frac{q_\xi}{v}-\frac{\hat{q}_\xi}{\hat{v}}\right)_\xi-\frac{vq-\hat{v}\hat{q}}{4\theta^3}-\frac{(\theta^3-\hat{\theta}^3)}{\theta^3}\hat{\theta}_\xi \\ &=\frac{\omega_{\xi\xi}}{4\theta^3v}-\frac{v\omega}{4\theta^3}-\frac{\omega_\xi v_\xi}{4\theta^3v^2}-\frac{1}{4\theta^3}\left(\frac{\phi\hat{q}_\xi}{v\hat{v}}\right)_\xi-\frac{\phi\hat{q}}{4\theta^3}-\frac{(\theta^3-\hat{\theta}^3)}{\theta^3}\hat{\theta}_\xi\end{aligned}$$

due to $(3.4)_4$, we have

$$\begin{aligned}-\int_0^\infty\zeta_{\xi\xi}\left(\frac{q_\xi}{e_\theta(v,\theta)}-\frac{\hat{q}_\xi}{e_\theta(\hat{v},\hat{\theta})}\right)d\xi&=\int_0^\infty\frac{\zeta_\xi\omega_{\xi\xi}}{e_\theta(v,\theta)}d\xi+\int_0^\infty\zeta_\xi\omega_\xi\left(\frac{1}{e_\theta(v,\theta)}\right)_\xi d\xi \\ &+\int_0^\infty\zeta_\xi\left[\left(\frac{1}{e_\theta(v,\theta)}-\frac{1}{e_\theta(\hat{v},\hat{\theta})}\right)\hat{q}_\xi\right]_\xi d\xi+\zeta_\xi(0,t)\left(\frac{q_\xi}{e_\theta(v,\theta)}-\frac{\hat{q}_\xi}{e_\theta(\hat{v},\hat{\theta})}\right)(0,t)\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \frac{\omega_{\xi\xi}^2}{4\theta^3 v e_\theta(v,\theta)} d\xi + \int_0^\infty \frac{v}{4\theta^3 e_\theta(v,\theta)} \omega_\xi^2 d\xi + \underbrace{\int_0^\infty \left(\frac{v}{4\theta^3 e_\theta(v,\theta)}\right)_\xi \omega\omega_\xi d\xi}_{I_9} \\
&\quad \underbrace{- \int_0^\infty \left[\frac{\omega_\xi v_\xi}{4\theta^3 v^2} + \frac{1}{4\theta^3}\left(\frac{\phi\hat{q}_\xi}{v\hat{v}}\right)_\xi + \frac{\phi\hat{q}}{4\theta^3} + \frac{(\theta^3-\hat{\theta}^3)}{\theta^3}\hat{\theta}_\xi\right] \frac{\omega_{\xi\xi}}{e_\theta(v,\theta)} d\xi}_{I_{10}} \\
&\quad \underbrace{+ \int_0^\infty \zeta_\xi \omega_\xi \left(\frac{1}{e_\theta(v,\theta)}\right)_\xi d\xi + \int_0^\infty \zeta_\xi \left[\left(\frac{1}{e_\theta(v,\theta)} - \frac{1}{e_\theta(\hat{v},\hat{\theta})}\right)\hat{q}_\xi\right]_\xi d\xi}_{I_{11}} \\
&\quad + \underbrace{\zeta_\xi(0,t)\omega_\xi(0,t)}_{I_{12}},
\end{aligned} \tag{3.32}
$$

where we used

$$
-\int_0^\infty \frac{v\omega}{4\theta^3} \frac{\omega_{\xi\xi}}{e_\theta(v,\theta)} d\xi = \int_0^\infty \frac{v}{4\theta^3 e_\theta(v,\theta)} \omega_\xi^2 d\xi + \int_0^\infty \left(\frac{v}{4\theta^3 e_\theta(v,\theta)}\right)_\xi \omega\omega_\xi d\xi.
$$

Integrating (3.31) over $(0,\infty)\times(0,t)$, and using (3.32), $\sigma_- < 0$, $\zeta_t(0,t)=0$, $e_\theta(v,\theta) > 0$ and (3.13), we get

$$
\|\zeta_\xi(t)\|^2 + \int_0^t \|(\zeta_{\xi\xi},\omega_{\xi\xi},\omega_\xi)(\tau)\|^2 d\tau \lesssim \|\zeta_\xi(0)\|^2 + \sum_{i=7}^{8}\int_0^t\int_0^\infty |I_i| d\xi d\tau + \sum_{i=9}^{12}\int_0^t |I_i| d\tau. \tag{3.33}
$$

Noticing that

$$
\begin{aligned}
|I_7| &\lesssim |\zeta_{\xi\xi}||\hat{\theta}_{\xi\xi}|(|\phi|+|\zeta|) + |\zeta_{\xi\xi}||\theta_\xi v_\xi - \hat{\theta}_\xi \hat{v}_\xi| + |\zeta_{\xi\xi}||\hat{\theta}_\xi||\hat{v}_\xi|(|\phi|+|\zeta|), \\
|I_8| &\lesssim |\zeta_{\xi\xi}|[|\psi_\xi| + (|\phi|+|\zeta|)|\hat{u}_\xi|] + |\zeta_{\xi\xi}|\left[|u_\xi^2 - \hat{u}_\xi^2| + (|\phi|+|\zeta|)|\hat{u}_\xi|^2\right],
\end{aligned}
$$

and using (3.13), (1.12) and (3.5), we have

$$
\begin{aligned}
\int_0^\infty |I_7| d\xi &\lesssim \delta \int_0^\infty e^{-\hat{c}\xi}(\zeta_{\xi\xi}^2 + \phi^2 + \zeta^2 + \phi_\xi^2 + \zeta_\xi^2) d\xi \lesssim \varepsilon_1 \|(\zeta_{\xi\xi},\phi_\xi,\zeta_\xi)\|^2, \\
\int_0^\infty |I_8| d\xi &\lesssim \|\zeta_{\xi\xi}\|\|\psi_\xi\| + \|\zeta_{\xi\xi}\|\|\psi_\xi\|^{\frac{3}{2}}\|\psi_{\xi\xi}\|^{\frac{1}{2}} + \delta\int_0^\infty e^{-\hat{c}\xi}(\zeta_{\xi\xi}^2 + \phi^2 + \zeta^2 + \psi_\xi^2) d\xi \\
&\lesssim \eta\|\psi_{\xi\xi}\|^2 + \eta^{-1}\|\psi_\xi\|^2 + \varepsilon_1\|(\phi_\xi,\psi_\xi,\zeta_\xi,\psi_{\xi\xi},\zeta_{\xi\xi})\|^2
\end{aligned} \tag{3.34}
$$

for any $\eta > 0$.

Moreover, using (3.13), (1.12) and (3.5) yields

$$
\begin{aligned}
|I_9| &\lesssim \int_0^\infty (|v_\xi| + |\theta_\xi|)|\omega||\omega_\xi| d\xi \lesssim \varepsilon_1 \|(\phi_\xi,\zeta_\xi,\omega,\omega_\xi)\|^2, \\
|I_{10}| &\lesssim \int_0^\infty \left[|\omega_\xi|(|\phi_\xi|+|\hat{v}_\xi|) + |\phi_\xi||\hat{q}_\xi| + |\phi|(|\hat{q}_{\xi\xi}|+|\hat{q}_\xi|+|\phi_\xi|+|\hat{v}_\xi|) + |\zeta||\hat{\theta}_\xi|\right]|\omega_{\xi\xi}| d\xi \\
&\lesssim \|\omega_\xi\|^{\frac{1}{2}}\|\omega_{\xi\xi}\|^{\frac{3}{2}}\|\phi_\xi\| + \|\phi\|^{\frac{1}{2}}\|\phi_\xi\|^{\frac{3}{2}}\|\omega_{\xi\xi}\| + \delta\|(\omega_\xi,\omega_{\xi\xi},\phi_\xi,\zeta_\xi)\|^2 \lesssim \varepsilon_1\|(\omega_\xi,\omega_{\xi\xi},\phi_\xi,\zeta_\xi)\|^2,
\end{aligned} \tag{3.35}
$$

$$
\begin{aligned}
|I_{11}| &\lesssim \int_0^\infty |\zeta_\xi||\omega_\xi|\left(|\phi_\xi|+|\zeta_\xi|+|\hat{v}_\xi|+|\hat{\theta}_\xi|\right) d\xi \\
&\quad + \int_0^\infty |\zeta_\xi|[(|\phi|+|\zeta|)|\hat{q}_{\xi\xi}| + (|\phi_\xi|+|\zeta_\xi|+|\phi|+|\zeta|)|\hat{q}_\xi|] d\xi \\
&\lesssim \|\omega_\xi\|^{\frac{1}{2}}\|\omega_{\xi\xi}\|^{\frac{1}{2}}\|\zeta_\xi\|\|\phi_\xi\| + \delta\|(\zeta_\xi,\omega_\xi,\phi_\xi)\|^2 \lesssim \varepsilon_1\|(\zeta_\xi,\omega_\xi,\phi_\xi)\|^2, \\
|I_{12}| &\lesssim \|\zeta_\xi\|^{\frac{1}{2}}\|\zeta_{\xi\xi}\|^{\frac{1}{2}}\|\omega_\xi\|^{\frac{1}{2}}\|\omega_{\xi\xi}\|^{\frac{1}{2}} \lesssim \eta\|(\zeta_{\xi\xi},\omega_{\xi\xi})\|^2 + \eta^{-1}\|(\zeta_\xi,\omega_\xi)\|^2
\end{aligned} \tag{3.36}
$$

for any $\eta>0$.

Substituting (3.34)-(3.36) into (3.33), we get

$$\|\zeta_\xi(t)\|^2+\int_0^t\|(\zeta_{\xi\xi},\omega_{\xi\xi},\omega_\xi)(\tau)\|^2d\tau\lesssim\|\zeta_\xi(0)\|^2+\int_0^t\|(\phi_\xi,\psi_\xi,\zeta_\xi,\psi_{\xi\xi},\omega_\xi)\|^2d\tau. \tag{3.37}$$

By (3.37), (3.18) and (3.7), we get (3.30). □

Proof. **(Proof of Proposition 3.1.)** By Lemmas 3.1- 3.3, we obtain

$$\|(\phi,\psi,\zeta)(t)\|_1^2+\int_0^t\|\phi_\xi(\tau)\|^2d\tau+\int_0^t\|(\psi_\xi,\zeta_\xi,\omega,\omega_\xi)(\tau)\|_1^2d\tau\lesssim\|(\phi,\psi,\zeta)(0)\|_1^2. \tag{3.38}$$

We rewrite $(3.4)_4$ as

$$-\left(\frac{\omega_\xi}{v}\right)_\xi+v\omega=\left((\frac{1}{v}-\frac{1}{\hat{v}})\hat{q}_\xi\right)_\xi-\phi\hat{q}-\left(\theta^4-\hat{\theta}^4\right)_\xi. \tag{3.39}$$

Multiplying (3.39) by ω and integrating the resulting equality over $\xi\in(0,\infty)$, we have

$$\int_0^\infty\left(\frac{\omega_\xi^2}{v}+v\omega^2\right)d\xi=\underbrace{-\int_0^\infty(\frac{1}{v}-\frac{1}{\hat{v}})\hat{q}_\xi\omega_\xi d\xi}_{I_{13}}\underbrace{-\int_0^\infty\phi\hat{q}\omega d\xi}_{I_{14}}+\underbrace{\int_0^\infty\left(\theta^4-\hat{\theta}^4\right)\omega_\xi d\xi}_{I_{15}}. \tag{3.40}$$

Using (3.13), (1.12) and (3.5) yields

$$\begin{aligned}&|I_{13}|+|I_{14}|\lesssim\int_0^\infty|\phi||\hat{q}_\xi||\omega_\xi|d\xi+\int_0^\infty|\phi||\hat{q}||\omega|d\xi\lesssim\varepsilon_1\|(\phi_\xi,\omega_\xi)\|^2,\\&|I_{15}|\lesssim\int_0^\infty|\zeta||\omega_\xi|d\xi\lesssim\eta\|\omega_\xi\|^2+\eta^{-1}\|\zeta\|^2\end{aligned} \tag{3.41}$$

for any $\eta>0$. By (3.40), (3.13) and (3.41), we get

$$\|\omega_\xi\|^2+\|\omega\|^2\lesssim\|(\phi_\xi,\zeta)\|^2. \tag{3.42}$$

Multiplying (3.39) by ω and integrating the resulting equality over $\xi\in(0,\infty)$, we have

$$\begin{aligned}\int_0^\infty\left(\frac{\omega_{\xi\xi}^2}{v}+v\omega_\xi^2\right)d\xi=&\underbrace{\int_0^\infty\frac{v_\xi}{v^2}\omega_\xi\omega_{\xi\xi}d\xi}_{I_{16}}\underbrace{-\int_0^\infty v\omega\omega_{\xi\xi}d\xi}_{I_{17}}\\&\underbrace{-\int_0^\infty\left((\frac{1}{v}-\frac{1}{\hat{v}})\hat{q}_\xi\right)_\xi\omega_{\xi\xi}d\xi}_{I_{18}}+\underbrace{\int_0^\infty\phi\hat{q}\omega_{\xi\xi}d\xi}_{I_{19}}+\underbrace{\int_0^\infty\left(\theta^4-\hat{\theta}^4\right)_\xi\omega_{\xi\xi}d\xi}_{I_{20}}.\end{aligned} \tag{3.43}$$

By using (3.13), (1.12), (3.5) and (3.42), we have

$$\begin{aligned}&|I_{16}|\lesssim\int_0^\infty|v_\xi||\omega_\xi||\omega_{\xi\xi}|d\xi\lesssim\|\phi_\xi\|\|\omega_\xi\|^{\frac{1}{2}}\|\omega_{\xi\xi}\|^{\frac{3}{2}}+\delta\|\omega_\xi\|\|\omega_{\xi\xi}\|\lesssim\varepsilon_1\|(\omega_\xi,\omega_{\xi\xi})\|^2,\\&|I_{17}|\lesssim\int_0^\infty|\omega||\omega_{\xi\xi}|d\xi\lesssim\|\omega\|\|\omega_{\xi\xi}\|\lesssim\eta\|\omega_{\xi\xi}\|^2+\eta^{-1}\|\omega\|^2,\end{aligned}$$

$$\begin{aligned}|I_{18}| &\lesssim \int_0^\infty [|\phi||\hat{q}_{\xi\xi}|+(|\phi_\xi|+|\phi|)|\hat{q}_\xi|]\,|\omega_{\xi\xi}|d\xi \lesssim \varepsilon_1\|(\phi_\xi,\omega_{\xi\xi})\|^2,\\ |I_{19}| &\lesssim \int_0^\infty |\phi||\hat{q}||\omega_{\xi\xi}|d\xi \lesssim \delta\|(\phi_\xi,\omega_{\xi\xi})\|^2,\\ |I_{20}| &\lesssim \int_0^\infty (|\zeta_\xi|+|\zeta|)|\omega_{\xi\xi}|d\xi \lesssim \eta\|\omega_{\xi\xi}\|^2+\eta^{-1}\|\zeta\|_1^2\end{aligned} \tag{3.44}$$

for any $\eta>0$. By (3.43), (3.13) and (3.44), we get

$$\|\omega_{\xi\xi}\|^2+\|\omega_\xi\|^2 \lesssim \|(\omega,\phi_\xi,\zeta,\zeta_\xi)\|^2. \tag{3.45}$$

By (3.38), (3.42) and (3.45), we get (3.6). The proof of Proposition 3.1 is completed. □

3.2. Proof of Theorem 1.2. In this subsection, we prove Theorem 1.2. Since it is easy to check the following local (in time) existence, we just state it and omit its proof for brevity.

Proposition 3.2 (Local existence). *Suppose that the initial data* (ϕ_0,ψ_0,ζ_0) *satisfy*

$$2\underline{m}\le v_0(\xi),\theta_0(\xi)\le \frac{1}{2}\overline{m} \quad \text{and} \quad (\phi,\psi,\zeta)(\xi,0)\in H^1.$$

Then, the system (3.4) admits a unique stationary solution $(\phi,\psi,\zeta,\omega)\in X(0,t_1)$ *for sufficiently small* $t_1>0$ *satisfying*

$$\begin{aligned}&\underline{m}\le v(\xi,t),\theta(\xi,t)\le \overline{m},\\ &\|(\phi,\psi,\zeta,\omega,\omega_\xi)(t)\|_1^2\le C\|(\phi,\psi,\zeta)(0)\|_1^2\end{aligned}$$

for all $0\le t\le t_1$, *where* C *is a positive constant depending on* $\underline{m},\overline{m}$ *and independent of* $t,\|(\phi,\psi,\zeta)(0)\|_1$. *Here* t_1 *depends only on* $\underline{m},\overline{m},\|(\phi,\psi,\zeta)(0)\|_1$.

Using the local existence of solution (see Proposition 3.2) and a priori estimate of solution (see Proposition 3.1), we can show the existence of stationary solution $(v,u,\theta,q)(\xi,t)$ to inflow problem (1.8) on the $[0,\infty)$ satisfying $(\phi,\psi,\zeta,\omega)\in X(0,\infty)$ and (1.13) by standard continuum argument. It is easy to identify (1.14), using (1.13) and the inequality $\|f\|_{L_\infty}\le\sqrt{2}\|f\|^{\frac{1}{2}}\|f_\xi\|^{\frac{1}{2}}$. Moreover, the proof of uniqueness for solution is standard.

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