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BLOW-UP TIME OF STRONG SOLUTIONS TO A BIOLOGICAL NETWORK FORMATION MODEL IN HIGH SPACE DIMENSIONS*

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Abstract. We investigate the possible blow-up of strong solutions to a biological network formation model originally introduced by [D. Cai and D. Hu, Phys. Rev. Lett., 111:138701, 2013]. The model is represented by an initial and boundary value problem for an elliptic-parabolic system with cubic nonlinearity. We obtain an algebraic equation for the possible blow-up time of strong solutions. The equation yields information on how various given data may contribute to the blow-up of solutions. As a by-product of our development, we establish a $W^{1,q}$ estimate for solutions to an elliptic equation which shows the explicit dependence of the upper bound on the elliptic coefficients.

Keywords. Biological network formation; blow-up time; existence.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with $C^{1,1}$ boundary $\partial\Omega$ and T a positive number. Set $\Omega_T = \Omega\times(0,T)$. We study the possible blow-up of strong solutions to the system

$$-\operatorname{div}[(I+\mathbf{m}\otimes\mathbf{m})\nabla p] = S(x) \quad \text{in } \Omega_T, \tag{1.1}$$

$$\partial_t\mathbf{m} - D^2\Delta\mathbf{m} + |\mathbf{m}|^{2(\gamma-1)}\mathbf{m} = E^2(\mathbf{m}\cdot\nabla p)\nabla p \quad \text{in } \Omega_T, \tag{1.2}$$

coupled with the initial and boundary conditions

$$p(x,t)=0,\ \ \mathbf{m}(x,t)=0, \quad (x,t)\in\Sigma_T\equiv\partial\Omega\times(0,T), \tag{1.3}$$

$$\mathbf{m}(x,0)=\mathbf{m}_0(x), \qquad x\in\Omega \tag{1.4}$$

for given functions $S(x),\mathbf{m}_0(x)$ and physical parameters D,E,γ with properties:

(H1) $\mathbf{m}_0(x)\in\left(W_0^{1,\infty}(\Omega)\right)^N$, $N\geq 3$, $S(x)\in L^{\frac{4qN}{N+4q}}(\Omega)$ for some $q>1+\frac{N}{2}$; and

(H2) $D,E\in(0,\infty),\gamma\in(\frac{1}{2},\infty)$.

This system was originally derived in ([13, 14]) as the formal gradient flow of the continuous version of a cost functional describing formation of biological transportation networks on discrete graphs. In this context, the scalar function $p=p(x,t)$ is the pressure due to Darcy's law, while the vector-valued function $\mathbf{m}=\mathbf{m}(x,t)$ is the conductance vector. The function $S(x)$ is the time-independent source term. More detailed information on the biological relevance of the problem can be found in [1, 2, 9, 11].

We are interested in the mathematical analysis of the problem.

A pair $(\mathbf{m},p)$ is said to be a weak solution to (1.1)-(1.4) if the following conditions (D1)-(D3) hold.

(D1) We have

$$\mathbf{m}\in L^\infty\left(0,T;\left(W_0^{1,2}(\Omega)\cap L^{2\gamma}(\Omega)\right)^N\right),\ \ \partial_t\mathbf{m}\in\left(L^2(\Omega_T)\right)^N,$$
$$p\in L^\infty(0,T;W_0^{1,2}(\Omega)),\ \ \text{and } (\mathbf{m}\cdot\nabla p)\in L^\infty(0,T;L^2(\Omega));$$

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(D2) $\mathbf{m}(x,0)=\mathbf{m}_0$ in $C\left([0,T];\left(L^2(\Omega)\right)^N\right)$;

(D3) Equations (1.1) and (1.2) are satisfied in the sense of distributions.

The existence of a weak solution was first established in [10]. It was based upon the following a priori estimates

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\mathbf{m}(x,\tau)|^2dx + D^2\int_{\Omega_\tau}|\nabla\mathbf{m}|^2dxdt + E^2\int_{\Omega_\tau}(\mathbf{m}\cdot\nabla p)^2dxdt \\
&\qquad + \int_{\Omega_\tau}|\mathbf{m}|^{2\gamma}dxdt + 2E^2\int_{\Omega_\tau}|\nabla p|^2dxd\tau \\
&= \frac{1}{2}\int_\Omega |\mathbf{m}_0|^2dx + 2E^2\int_{\Omega_\tau}S(x)pdxdt,
\end{aligned} \tag{1.5}$$

$$\begin{aligned}
&\int_{\Omega_\tau}|\partial_t\mathbf{m}|^2dxdt + \frac{D^2}{2}\int_\Omega|\nabla\mathbf{m}(x,\tau)|^2dx + \frac{E^2}{2}\int_\Omega(\mathbf{m}\cdot\nabla p)^2dx \\
&\qquad + \frac{E^2}{2}\int_\Omega|\nabla p|^2dx + \frac{1}{2\gamma}\int_\Omega|\mathbf{m}|^{2\gamma}dx \\
&= \frac{D^2}{2}\int_\Omega|\nabla\mathbf{m}_0|^2dx + \frac{E^2}{2}\int_\Omega(\mathbf{m}_0\cdot\nabla p_0)^2dx + \frac{1}{2\gamma}\int_\Omega|\mathbf{m}_0|^{2\gamma}dx \\
&\qquad + \frac{E^2}{2}\int_\Omega|\nabla p_0|^2dx,
\end{aligned} \tag{1.6}$$

where $\tau\in(0,T], \Omega_\tau=\Omega\times(0,\tau)$, and p_0 is the solution of the boundary value problem

$$-\mathrm{div}[(I+\mathbf{m}_0\otimes\mathbf{m}_0)\nabla p_0]=S(x), \quad \text{in } \Omega, \tag{1.7}$$

$$p_0=0 \quad \text{on } \partial\Omega. \tag{1.8}$$

The preceding equations are due to the fact that our system can be viewed as a (formal) $L^2(\Omega)$ constrained gradient flow and the energy associated with it is nonincreasing [10, 11].

Partial regularity of weak solutions was addressed in [17, 21] for $N\leq 3$. If the space dimension is 2, a classical solution was obtained in [23] for the stationary case and in [20] for the time-dependent case. Finite time extinction or break-down of solutions in the spatially one-dimensional setting for certain ranges of the relaxation exponent γ was carefully studied in [11]. Further modeling analysis and numerical results can be found in [1, 9]. We also mention that the question of existence in the case where $\gamma=\frac{1}{2}$ is addressed in [11]. In this case, the term $|\mathbf{m}|^{2(\gamma-1)}\mathbf{m}$ is not continuous at $\mathbf{m}=0$. Nevertheless, the general regularity theory remains fundamentally incomplete. In particular, it is not known whether or not weak solutions develop singularities in high space dimensions $N\geq 3$ even though a blow-up criterion was obtained in [16] when $\Omega=\mathbb{R}^3$.

A strong solution is a weak solution with the additional property

(D4) $\mathbf{m}$ is Hölder continuous in $\overline{\Omega_T}$.

Obviously, Equation (1.1) becomes uniformly elliptic under (D4). More importantly, under (D4) and the assumption

(H3) $\partial\Omega$ is $C^{1,1}$,

the result in ([19], p.82) becomes applicable. That is, for each $s>1$ there is a positive number c determined by N, s, Ω, and the Hölder continuity of $\mathbf{m}$ such that

$$\|\nabla p\|_{s,\Omega}\leq c\|S\|_{\frac{sN}{s+N},\Omega}. \tag{1.9}$$

This, in turn, will further improve the regularity of $(\mathbf{m},p)$. In fact, one can easily infer from the results in [4, 15] that the system (1.1)-(1.2) is satisfied a.e on Ω_T. See [21] for details.

Our main result is:

THEOREM 1.1. *Let (H1)-(H3) be satisfied. Then there is a positive number $T_{\max}$ such that problem (1.1)-(1.4) has a strong solution in Ω_T for each $T < T_{\max}$. The number $T_{\max}$ is the unique solution of Equation (4.10) below.*

By carefully examining its proof, we can reformulate (4.10) in the form

$$\left(\sum_{i=1}^{7} T_{\max}^{a_i} \|S\|_{\frac{4qN}{N+4q},\Omega}^{b_i} \|\nabla \mathbf{m}_0\|_{\infty,\Omega}^{c_i}\right)^s \sum_{i=8}^{19} T_{\max}^{a_i} \|S\|_{\frac{4qN}{N+4q},\Omega}^{b_i} \|\nabla \mathbf{m}_0\|_{\infty,\Omega}^{c_i} = d_0, \tag{1.10}$$

where all the exponents $s>0, a_i>0, b_i\geq 0, c_i \geq 0,\ i=1,\cdots,19$, are determined by N,q,γ only, while $d_0>0$ also depends on Ω and the physical parameters D,E in the problem in addition to N,q,γ. The left-hand side of (1.10) as a function of $T_{\max}$ strictly increases from 0 to ∞ as $T_{\max}$ goes from 0 to ∞. Thus, (1.10) has a unique solution.

The local existence of a strong solution was already obtained in Theorem 1.7 of [21]. The novelty here is the explicit dependence of $T_{\max}$ on the given data. Obviously, the smaller $\|S\|_{\frac{4qN}{N+4q},\Omega}$ and $\|\nabla \mathbf{m}_0\|_{\infty,\Omega}$ are, the longer the life span of our strong solution is. We must point out that the theorem does not say if the blow-up does occur at $T_{\max}$. Thus, $T_{\max}$ only serves as a lower bound for the blow-up time if it exists.

To describe the mathematical difficulty involved in our problem, first notice the term $(\mathbf{m}\cdot\nabla p)\nabla p$ in (1.2) represents a cubic nonlinearity. This type of nonlinearity is still not well-studied in the literature. Second, the elliptic coefficients in (1.1) satisfy

$$|\xi|^2 \leq (I+\mathbf{m}\otimes\mathbf{m})\xi\cdot\xi \leq (1+|\mathbf{m}|^2)|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N. \tag{1.11}$$

Hence, Equation (1.1) is singular unless $\mathbf{m}$ is bounded, which is not known a priori. However, existing regularity results for degenerate and/or singular elliptic equations [12] often require that the largest eigenvalue λ_l and the smallest eigenvalue λ_s of the coefficient matrix satisfy

$$\lambda_l \leq c\lambda_s \text{ and } \int_{B_r(y)} \lambda_s dx \int_{B_r(y)} dx \frac{1}{\lambda_s} dx \leq c r^{2N} \quad \text{for each ball } B_r(y)\subset\Omega.$$

Here and in what follows the letter c denotes a generic positive number. Thus our problem does not fit into the classical framework.

To gain some insights into our problem, we take a look at the one-dimensional case

$$-[(1+m^2)p_x]_x = S(x) \quad \text{in } (0,1)\times(0,\infty), \tag{1.12}$$

$$m_t - D^2 m_{xx} + |m|^{2(\gamma-1)} m = E^2 m p_x^2 \quad \text{in } (0,1)\times(0,\infty), \tag{1.13}$$

$$p(0,t) = p(1,t) = 0 \quad \text{on } (0,\infty), \tag{1.14}$$

$$m(0,t) = m(1,t) = 0 \quad \text{on } (0,\infty), \tag{1.15}$$

$$m(x,0) = m_0(x) \quad \text{on } (0,1). \tag{1.16}$$

According to Rolle's theorem, for each $t\in(0,\infty)$ there is an $x^*(t)\in(0,1)$ such that

$$p_x(x^*(t),t) = 0.$$

For $x\in(0,1)$ we integrate (1.12) over $(x^*(t),x)$ to derive

$$p_x=-\frac{1}{1+m^2}\int_{x^*(t)}^{x}S(y)dy. \tag{1.17}$$

Subsequently,

$$\|p_x\|_\infty\le\|S\|_1.$$

Substitute (1.17) into (1.13) to obtain

$$m_t-D^2m_{xx}+|m|^{2(\gamma-1)}m=\frac{E^2m}{(1+m^2)^2}\left(\int_{x^*(t)}^{x}S(y)dy\right)^2.$$

We multiply through the above equation by m_t and integrate to obtain

$$\begin{aligned}&\int_0^1 m_t^2dx+\frac{D^2}{2}\frac{d}{dt}\int_0^1 m_x^2dx+\frac{1}{2\gamma}\frac{d}{dt}\int_0^1 m^{2\gamma}dx\\ &=E^2\int_0^1\frac{mm_t}{(1+m^2)^2}\left(\int_{x^*(t)}^{x}S(y)dy\right)^2dx\\ &\le E^2\|S\|_1^2\int_0^1\frac{m^2}{(1+m^2)^2}dx.\end{aligned}$$

This implies that blow-up in m does not occur. The key here is (1.17), which is not available in high space dimensions. To seek a substitute, we have developed the following theorem which seems to be of interest in its own right.

THEOREM 1.2. *Assume that (H3) holds,*

$$\mathbf{w}\in\left(W^{1,\ell}(\Omega)\right)^N \text{ for some } \ell>N, \text{ and } S(x)\in L^{\frac{Nq}{N+q}}(\Omega) \text{ for some } q>\tfrac{N}{N-1}\ . \tag{1.18}$$

Let p be the solution of the boundary value problem

$$-\mathrm{div}\,[(I+\mathbf{w}(x)\otimes\mathbf{w}(x))\nabla p]=S(x)\quad in\ \Omega, \tag{1.19}$$

$$p=0\quad on\ \partial\Omega. \tag{1.20}$$

Then there is a positive number $c=(N,\Omega,q,\ell)$ such that

$$\|\nabla p\|_{q,\Omega}\le c\left(1+\|\mathbf{w}\|_{(W^{1,\ell}(\Omega))^N}\right)^{s_1}\left(\|\nabla p\|_{1,\Omega}+\|S\|_{\frac{Nq}{N+q},\Omega}\right), \tag{1.21}$$

where $s_1=\frac{5N(2\ell-N+N\ell)(Nq-N)}{q(\ell-N)}$.

The advantage of this theorem over the classical result (1.9) is that it gives the explicit dependence of the upper bound on the coefficient matrix. This is very crucial to our applications.

Condition on q in (1.18) is to ensure that $\frac{Nq}{N+q}>1$. When $N=2$, a version of (1.21) was obtained in [20] by deriving an equation for the term $(I+\mathbf{w}\otimes\mathbf{w})\nabla p\cdot\nabla p$. Unfortunately, this technique only works for $N=2$. Our approach here is largely inspired by the papers [3, 4, 6].

The rest of the paper is organized as follows. A refinement of a classical uniform bound for solutions to a linear parabolic equation is given in Section 2. The proof of Theorem 1.2 is presented in Section 3. Our main theorem is established in Section 4.

Finally, we remark that unless stated otherwise, our generic constant c depends only on N,q,Ω and the three physical parameters D,E,γ in the problem. In particular, it does not depend on $T,\mathbf{m}_0(x)$, or $S(x)$. The following two inequalities are frequently used without acknowledgment:

$$(a+b)^\alpha \le a^\alpha + b^\alpha \quad \text{for } a\ge 0, b\ge 0, \alpha\in(0,1),$$
$$(a+b)^\alpha \le 2^{\alpha-1}(a^\alpha + b^\alpha) \quad \text{for } a\ge 0, b\ge 0, \alpha\ge 1.$$

2. Preliminary results

In this section, we collect a few relevant known results. The first lemma contains some elementary inequalities whose proof can be found in ([18], p. 146-148).

LEMMA 2.1. *Let x,y be any two vectors in $\mathbb{R}^N$. Then:*

(i) *For $\gamma\ge 1$,*

$$\left(\left(|x|^{2\gamma-2}x-|y|^{2\gamma-2}y\right)\cdot(x-y)\right)\ge \frac{1}{2^{2\gamma-1}}|x-y|^{2\gamma};$$

(ii) *For $\frac{1}{2}<\gamma\le 1$,*

$$(|x|+|y|)^{2-2\gamma}\left(\left(|x|^{2\gamma-2}x-|y|^{2\gamma-2}y\right)\cdot(x-y)\right)\ge(2\gamma-1)|x-y|^2.$$

The next lemma plays a central role in our main result.

LEMMA 2.2. *Let $h(\tau)$ be a continuous non-negative function defined on $[0,T_0]$ for some $T_0>0$. Suppose that there exist three positive numbers ε,δ,b such that*

$$h(\tau)\le \varepsilon h^{1+\delta}(\tau)+b \quad \text{for each } \tau\in[0,T_0].$$

Then

$$h(\tau)\le \frac{1}{[\varepsilon(1+\delta)]^{\frac{1}{\delta}}}\equiv h_0 \quad \text{for each } \tau\in[0,T_0]$$

whenever

$$\varepsilon\le \frac{\delta^\delta}{(b+\delta)^\delta(1+\delta)^{1+\delta}} \quad \text{and} \quad h(0)\le h_0.$$

We will use the proof of this lemma given in [22].

The following lemma can be found in ([5], p.12).

LEMMA 2.3. *Let $\{y_n\},n=0,1,2,\cdots,$ be a sequence of positive numbers satisfying the recursive inequalities*

$$y_{n+1}\le cb^n y_n^{1+\alpha} \quad \text{for some } b>1, c,\alpha\in(0,\infty).$$

If

$$y_0\le c^{-\frac{1}{\alpha}}b^{-\frac{1}{\alpha^2}},$$

then $\lim_{n\to\infty}y_n=0$.

LEMMA 2.4. *Let $u_0 \in L^\infty(\Omega)$ and $g, f, |\mathbf{g}|^2 \in L^q(\Omega_T)$ for some*

$$q > 1 + \frac{N}{2}. \tag{2.1}$$

Assume that u is a sub-solution of the problem

$$\partial_t u - D^2 \Delta u = gu + f + \operatorname{div}\mathbf{g} \quad \text{in } \Omega_T, \tag{2.2}$$

$$u = 0 \quad \text{on } \Sigma_T, \tag{2.3}$$

$$u = u_0 \quad \text{on } \Omega. \tag{2.4}$$

Then there exists a positive number $c = c(D, N, \Omega, q)$ such that

$$\sup_{\Omega_T} u \le 2\sup_{\Omega} u_0 + c\left(\|g\|_{q,\Omega_T}^{\frac{(q-1)s_0}{2q}} + 1 \right) \|u^+\|_{\frac{2q}{q-1},\Omega_T} + \|f\|_{q,\Omega_T} T^{\frac{1}{2s_0}} + \|\mathbf{g}\|_{2q,\Omega_T}, \tag{2.5}$$

where

$$s_0 = \frac{q(N+2)}{2q - N - 2}. \tag{2.6}$$

This lemma is essentially known. The interest here lies in the fact that it gives the precise dependence of the uniform upper bound on T and the given functions. This is very important to our late development.

Proof. The proof is based upon the De Giorgi iteration scheme. Let

$$k \ge 2\sup_{\Omega} u_0 \tag{2.7}$$

be selected as below. Define

$$k_n = k - \frac{k}{2^{n+1}}, \quad n = 0, 1, 2, \cdots.$$

Use $(u - k_{n+1})^+$ as a test function in (2.2) and use the fact that $(u - k_{n+1})^+|_{t=0} = 0$ to get

$$\frac{1}{2}\sup_{0\le t\le T} \int_\Omega \left[(u - k_{n+1})^+\right]^2 dx + D^2 \int_{\Omega_T} \left|\nabla(u - k_{n+1})^+\right|^2 dxdt$$
$$\le 2\int_{\Omega_T} gu(u - k_{n+1})^+ dxdt + 2\int_{\Omega_T} f(u - k_{n+1})^+ dxdt - 2\int_{\Omega_T} \mathbf{g}\cdot\nabla(u - k_{n+1})^+ dxdt. \tag{2.8}$$

Set

$$Q_n = \{(x,t) \in \Omega_T : u(x,t) \ge k_n\}.$$

Then we easily see from Young's inequality ([8], p.145) that

$$\left|\int_{\Omega_T} \mathbf{g}\cdot\nabla(u - k_{n+1})^+ dxdt\right| \le \varepsilon \int_{\Omega_T} \left|\nabla(u - k_{n+1})^+\right|^2 dxdt + \frac{c}{\varepsilon}\|\mathbf{g}\|_{2,Q_{n+1}}^2, \quad \varepsilon > 0. \tag{2.9}$$

By Hölder's inequality,

$$\left|\int_{\Omega_T} f(u - k_{n+1})^+ dxdt\right| \le \|f\|_{\frac{2(N+2)}{N+4},Q_{n+1}} \|(u - k_{n+1})^+\|_{\frac{2(N+2)}{N},\Omega_T}. \tag{2.10}$$

Note that

$$\begin{aligned}\left[(u-k_n)^+\right]^2 &\geq (u-k_n)^+(u-k_{n+1})^+\\ &\geq u\left(1-\frac{k_n}{k_{n+1}}\right)(u-k_{n+1})^+\\ &\geq \frac{1}{2^{n+2}}u(u-k_{n+1})^+.\end{aligned}$$

Consequently,

$$\begin{aligned}\left|\int_{\Omega_T} gu(u-k_{n+1})^+dxdt\right| &\leq 2^{n+2}\int_{\Omega_T}|g|\left[(u-k_n)^+\right]^2dxdt\\ &\leq 2^{n+2}\|g\|_{q,\Omega_T}y_n^{\frac{q-1}{q}},\end{aligned}\tag{2.11}$$

where

$$y_n=\int_{\Omega_T}\left[(u-k_n)^+\right]^{\frac{2q}{q-1}}dxdt.$$

Substitute (2.9), (2.10), and (2.11) into (2.8) and choose ε suitably small in the resulting inequality to derive

$$\begin{aligned}&\sup_{0\leq t\leq T}\int_\Omega\left[(u-k_{n+1})^+\right]^2dx+\int_{\Omega_T}\left|\nabla(u-k_{n+1})^+\right|^2dxdt\\ &\leq c2^n\|g\|_{q,\Omega_T}y_n^{\frac{q-1}{q}}+c\|f\|_{\frac{2(N+2)}{N+4},Q_{n+1}}\|(u-k_{n+1})^+\|_{\frac{2(N+2)}{N},\Omega_T}+c\|\mathbf{g}\|^2_{2,Q_{n+1}}.\end{aligned}$$

It follows from Poincaré's inequality that

$$\begin{aligned}&\int_{\Omega_T}\left[(u-k_{n+1})^+\right]^{2+\frac{4}{N}}dxdt\\ &\leq\int_0^T\left(\int_\Omega\left[(u-k_{n+1})^+\right]^2dx\right)^{\frac{2}{N}}\left(\int_\Omega\left[(u-k_{n+1})^+\right]^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}}dt\\ &\leq c\left(\sup_{0\leq t\leq T}\int_\Omega\left[(u-k_{n+1})^+\right]^2dx\right)^{\frac{2}{N}}\int_{\Omega_T}\left|\nabla(u-k_{n+1})^+\right|^2dxdt\\ &\leq c\left(2^n\|g\|_{q,\Omega_T}y_n^{\frac{q-1}{q}}+\|f\|_{\frac{2(N+2)}{N+4},Q_{n+1}}\|(u-k_{n+1})^+\|_{\frac{2(N+2)}{N},\Omega_T}+\|\mathbf{g}\|^2_{2,Q_{n+1}}\right)^{\frac{N+2}{N}}\\ &\leq c2^{\frac{n(N+2)}{N}}\|g\|^{\frac{N+2}{N}}_{q,\Omega_T}y_n^{\frac{(q-1)(N+2)}{qN}}+c\|f\|^{\frac{N+2}{N}}_{\frac{2(N+2)}{N+4},Q_{n+1}}\|(u-k_{n+1})^+\|^{\frac{N+2}{N}}_{\frac{2(N+2)}{N},\Omega_T}+c\|\mathbf{g}\|^{\frac{2(N+2)}{N}}_{2,Q_{n+1}}\\ &\leq\varepsilon\int_{\Omega_T}\left[(u-k_{n+1})^+\right]^{2+\frac{4}{N}}dxdt+\frac{c}{\varepsilon}\|f\|^{\frac{2(N+2)}{N}}_{\frac{2(N+2)}{N+4},Q_{n+1}}+c\|\mathbf{g}\|^{\frac{2(N+2)}{N}}_{2,Q_{n+1}}\\ &\quad+c2^{\frac{n(N+2)}{N}}\|g\|^{\frac{N+2}{N}}_{q,\Omega_T}y_n^{\frac{(q-1)(N+2)}{qN}},\quad\varepsilon>0.\end{aligned}$$

Consequently,

$$\begin{aligned}\int_{\Omega_T}\left[(u-k_{n+1})^+\right]^{2+\frac{4}{N}}dxdt &\leq c2^{\frac{n(N+2)}{N}}\|g\|^{\frac{N+2}{N}}_{q,\Omega_T}y_n^{\frac{(q-1)(N+2)}{qN}}\\ &\quad+c\|f\|^{\frac{2(N+2)}{N}}_{\frac{2(N+2)}{N+4},Q_{n+1}}+c\|\mathbf{g}\|^{\frac{2(N+2)}{N}}_{2,Q_{n+1}}.\end{aligned}\tag{2.12}$$

The last two terms in the above inequality can be estimated as follows:

$$\begin{aligned}\|f\|_{\frac{2(N+2)}{N+4},Q_{n+1}}^{\frac{2(N+2)}{N}} &= \left(\int_{Q_{n+1}} |f|^{\frac{2(N+2)}{N+4}}\,dxdt\right)^{\frac{N+4}{N}}\\ &\le \|f\|_{q,\Omega_T}^{\frac{2(N+2)}{N}} |Q_{n+1}|^{\frac{N+4}{N}-\frac{2(N+2)}{Nq}},\\ \|\mathbf{g}\|_{2,Q_{n+1}}^{\frac{2(N+2)}{N}} &\le \|\mathbf{g}\|_{2q,\Omega_T}^{\frac{2(N+2)}{N}} |Q_{n+1}|^{\frac{N+2}{N}-\frac{N+2}{Nq}}.\end{aligned}$$

Use this in (2.12) to obtain

$$\begin{aligned}\int_{\Omega_T}\left[(u-k_{n+1})^+\right]^{2+\frac{4}{N}}dxdt \le c2^{\frac{n(N+2)}{N}}\|g\|_{q,\Omega_T}^{\frac{N+2}{N}} y_n^{\frac{(q-1)(N+2)}{qN}} + c\|f\|_{q,\Omega_T}^{\frac{2(N+2)}{N}} |Q_{n+1}|^{\frac{N+4}{N}-\frac{2(N+2)}{Nq}}\\ +c\|\mathbf{g}\|_{2q,\Omega_T}^{\frac{2(N+2)}{N}} |Q_{n+1}|^{\frac{N+2}{N}-\frac{N+2}{Nq}}.\end{aligned}\tag{2.13}$$

By (2.1),

$$\frac{q}{q-1} < \frac{N+2}{N}.$$

This, together with (2.13), implies

$$\begin{aligned}y_{n+1} &= \int_{\Omega_T}\left[(u-k_{n+1})^+\right]^{\frac{2q}{q-1}}dxdt\\ &\le \left(\int_{\Omega_T}\left[(u-k_{n+1})^+\right]^{2+\frac{4}{N}}dxdt\right)^{\frac{qN}{(q-1)(N+2)}} |Q_{n+1}|^{1-\frac{qN}{(q-1)(N+2)}}\\ &\le c2^{\frac{qn}{q-1}}\|g\|_{q,\Omega_T}^{\frac{q}{q-1}} y_n |Q_{n+1}|^{1-\frac{qN}{(q-1)(N+2)}}\\ &\quad +c\|f\|_{q,\Omega_T}^{\frac{2q}{q-1}} |Q_{n+1}|^{\frac{(N+4)q-2(N+2)}{(q-1)(N+2)}+1-\frac{qN}{(q-1)(N+2)}}\\ &\quad +c\|\mathbf{g}\|_{2q,\Omega_T}^{\frac{2q}{q-1}} |Q_{n+1}|^{\frac{(N+2)q-N-2}{(q-1)(N+2)}+1-\frac{qN}{(q-1)(N+2)}}\\ &= c2^{\frac{qn}{q-1}}\|g\|_{q,\Omega_T}^{\frac{q}{q-1}} y_n |Q_{n+1}|^{\alpha} + c\|f\|_{q,\Omega_T}^{\frac{2q}{q-1}} |Q_{n+1}|^{1+2\alpha}\\ &\quad +c\|\mathbf{g}\|_{2q,\Omega_T}^{\frac{2q}{q-1}} |Q_{n+1}|^{1+\alpha},\end{aligned}\tag{2.14}$$

where

$$\alpha = \frac{2q-N-2}{(q-1)(N+2)} = \frac{q}{(q-1)s_0} > 0.\tag{2.15}$$

We easily see that

$$y_n \ge \int_{Q_{n+1}} (k_{n+1}-k_n)^{\frac{2q}{q-1}}dxdt = \frac{k^{\frac{2q}{q-1}}}{2^{\frac{2q(n+2)}{q-1}}}|Q_{n+1}|.$$

Therefore,

$$\begin{aligned}|Q_{n+1}|^{\alpha} &\le \frac{2^{\frac{2q(n+2)\alpha}{q-1}}}{k^{\frac{2q\alpha}{q-1}}} y_n^{\alpha},\\ |Q_{n+1}|^{1+\alpha} &\le \frac{2^{\frac{2q(n+2)(1+\alpha)}{q-1}}}{k^{\frac{2q(1+\alpha)}{q-1}}} y_n^{1+\alpha},\\ |Q_{n+1}|^{1+2\alpha} \le |Q_{n+1}|^{1+\alpha}|\Omega_T|^{\alpha} &\le \frac{2^{\frac{2q(n+2)}{q-1}(1+\alpha)}}{k^{\frac{2q}{q-1}(1+\alpha)}}|\Omega_T|^{\alpha} y_n^{1+\alpha}.\end{aligned}$$

Collecting the preceding three estimates in (2.14) to get

$$\begin{aligned} y_{n+1} \leq & \frac{c2^{\frac{(2\alpha+1)qn}{q-1}}\|g\|_{q,\Omega_T}^{\frac{q}{q-1}}}{k^{\frac{2q\alpha}{q-1}}} y_n^{1+\alpha} \\ & + \frac{c2^{\frac{2(\alpha+1)qn}{q-1}}\left(\|f\|_{q,\Omega_T}^{\frac{2q}{q-1}} T^{\alpha} + \|\mathbf{g}\|_{2q,\Omega_T}^{\frac{2q}{q-1}}\right)}{k^{\frac{2q}{q-1}(1+\alpha)}} y_n^{1+\alpha}. \end{aligned} \tag{2.16}$$

We choose k so large that

$$\frac{\|f\|_{q,\Omega_T}^{\frac{2q}{q-1}} T^{\alpha} + \|\mathbf{g}\|_{2q,\Omega_T}^{\frac{2q}{q-1}}}{k^{\frac{2q}{q-1}}} \leq 1. \tag{2.17}$$

Use this in (2.16) to get

$$y_{n+1} \leq \frac{c2^{\frac{2(\alpha+1)qn}{q-1}}\left(\|g\|_{q,\Omega_T}^{\frac{q}{q-1}} + 1\right)}{k^{\frac{2q\alpha}{q-1}}} y_n^{1+\alpha}.$$

According to Lemma 2.3, if we further require k to satisfy

$$y_0 \leq c\left(\frac{k^{\frac{2q\alpha}{q-1}}}{c\left(\|g\|_{q,\Omega_T}^{\frac{q}{q-1}} + 1\right)}\right)^{\frac{1}{\alpha}},$$

then

$$\sup_{\Omega_T} u \leq k. \tag{2.18}$$

In view of (2.17) and (2.7), it is enough for us to take

$$k = 2\sup_{\Omega} u_0 + c\left(\|g\|_{q,\Omega_T}^{\frac{1}{2\alpha}} + 1\right) y_0^{\frac{q-1}{2q}} + \|f\|_{q,\Omega_T} T^{\frac{1}{2s_0}} + \|\mathbf{g}\|_{2q,\Omega_T}.$$

Note that

$$y_0^{\frac{q-1}{2q}} \leq \left(\int_{\Omega_T} (u^+)^{\frac{2q}{q-1}}\right)^{\frac{q-1}{2q}} = \|u^+\|_{\frac{2q}{q-1},\Omega_T}.$$

This, combined with (2.18) and (2.15), gives (2.5). The proof is complete. □

If

$$\sup_{\Omega_T} u = \|u\|_{\infty,\Omega_T},$$

we can apply the interpolation inequality ([8],p.146) to obtain

$$\|u^+\|_{\frac{2q}{q-1},\Omega_T} \leq \|u\|_{\frac{2q}{q-1},\Omega_T} \leq \varepsilon\|u\|_{\infty,\Omega_T} + \frac{1}{\varepsilon^{\frac{q+1}{q-1}}}\|u\|_{1,\Omega_T}, \quad \varepsilon > 0.$$

Plug this into (2.5) to get

$$\|u\|_{\infty,\Omega_T} \le c\left(\|g\|_{q,\Omega_T}^{\frac{1}{2\alpha}}+1\right)\left(\varepsilon\|u\|_{\infty,\Omega_T}+\frac{1}{\varepsilon^{\frac{q+1}{q-1}}}\|u\|_{1,\Omega_T}\right)$$
$$+2\sup_\Omega u_0+\|f\|_{q,\Omega_T}T^{\frac{1}{2s_0}}+\|\mathbf{g}\|_{2q,\Omega_T}.$$

Take ε so that the coefficient of the term $\|u\|_{\infty,\Omega_T}$ on the right-hand side of the above inequality

$$c\left(\|g\|_{q,\Omega_T}^{\frac{1}{2\alpha}}+1\right)\varepsilon=\frac{1}{2}$$

to drive

$$\|u\|_{\infty,\Omega_T} \le 4\sup_\Omega u_0+c\left(\|g\|_{q,\Omega_T}^{s_0}+1\right)\|u\|_{1,\Omega_T}+2\|f\|_{q,\Omega_T}T^{\frac{1}{2s_0}}+2\|\mathbf{g}\|_{2q,\Omega_T}. \tag{2.19}$$

Here we have used (2.15).

Lemma 2.5. *Assume that (H3) holds. Let u be the solution of the problem*

$$\begin{aligned} \partial_t u-D^2\Delta u &= f \quad \text{in } \Omega_T,\\ u&=0 \quad \text{on } \Sigma_T,\\ u&=u_0 \quad \text{on } \Omega, \end{aligned}$$

where

$$u_0\in W_0^{1,\infty}(\Omega), \quad f\in L^{2q}(\Omega_T) \quad \text{for some } q>1+\tfrac{N}{2}.$$

Then there is a positive number $c=c(\Omega,N,q)$ such that

$$\|\nabla u\|_{\infty,\Omega_T}\le c\|\nabla u_0\|_{\infty,\Omega}+c\|f\|_{2q,\Omega_T}. \tag{2.20}$$

This lemma is known. In fact, it is not difficult for us to see (2.20). Indeed, u_{x_i} satisfies (2.2) with f being replaced by f_{x_i} and g, $\mathbf{g}$ being 0. A full proof can be inferred from Proposition 2.3 in [22].

3. $W^{1,q}$ estimates for elliptic equations

Before we prove Theorem 1.2, we recall some results from [3].

Definition 3.1. *A function $k(x)$ on $\mathbb{R}^N\setminus\{0\}$ is called a Calderón-Zygmund kernel (in short, C-Z kernel) if:*

(i) $k\in C^\infty(\mathbb{R}^N\setminus\{0\})$;

(ii) $k(x)$ *is homogeneous of degree $-N$, i.e.,* $k(tx)=t^{-N}k(x)$;

(iii) $\int_{\partial B_1(0)}k(x)d\mathcal{H}^{N-1}=0$.

The most fundamental result concerning C-Z kernels [3] is the following:

Lemma 3.1. *Given a C-Z kernel $k(x)$, we define*

$$K_\varepsilon f(x)=\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}k(x-y)f(y)dy \quad \text{for } \varepsilon>0 \text{ and } f\in L^q(\mathbb{R}^N) \text{ with } q\in(1,\infty).$$

Then:

(CZ1) *For each* $f \in L^q(\mathbb{R}^N)$ *there exists a function* $Kf \in L^q(\mathbb{R}^N)$ *such that*

$$\lim_{\varepsilon\to 0}\|K_\varepsilon f - Kf\|_{q,\mathbb{R}^N} = 0.$$

In this case we use the notation

$$Kf(x) = \text{P.V.} k * f(x) = \text{P.V.}\int_{\mathbb{R}^N} k(x-y)f(y)dy.$$

(CZ2) K *is a bounded operator on* $L^q(\mathbb{R}^N)$*. More precisely, we have*

$$\|Kf\|_{q,\mathbb{R}^N} \le c\left(\int_{\partial B_1(0)} k^2(x)d\mathcal{H}^{N-1}\right)^{\frac{1}{2}}\|f\|_{q,\mathbb{R}^N},$$

where the positive number c *depends only on* N, q.

We are ready to prove Theorem 1.2.

Proof. **(Proof of Theorem 1.2.)** As in [3,6], the proof comprises a local interior estimate and a boundary estimate. To establish the former, we fix $x_0 \in \Omega$. Let $0 < \delta < r$ with $B_r(x_0) \subset \Omega$. Pick a smooth cutoff function $\xi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\begin{aligned} \xi &= 1 \quad \text{on } B_\delta(x_0), \\ \xi &= 0 \quad \text{outside } B_r(x_0), \\ 0 \le \xi &\le 1 \quad \text{on } B_r(x_0), \\ |\nabla\xi| &\le \frac{c}{r-\delta} \quad \text{on } B_r(x_0). \end{aligned}$$

Set

$$u = p\xi.$$

We can easily verify that u satisfies the equation

$$-\text{div}\left[(I + \mathbf{w}(x)\otimes\mathbf{w}(x))\nabla u\right] = -\text{div}\left[p(I + \mathbf{w}(x)\otimes\mathbf{w}(x))\nabla\xi\right] + F \quad \text{in } \mathbb{R}^N,$$

where

$$F = \xi S(x) - \nabla\xi(I + \mathbf{w}(x)\otimes\mathbf{w}(x))\nabla p.$$

Set

$$A(x_0) = I + \fint_{B_r(x_0)} \mathbf{w}(x)dx \otimes \fint_{B_r(x_0)} \mathbf{w}(x)dx.$$

Then we can write the above equation in the form

$$\begin{aligned} &-\text{div}[A(x_0)\nabla u(x)] \\ &= \text{div}\left[((\mathbf{w}(x) - \fint_{B_r(x_0)} \mathbf{w}(x)dx)\cdot\nabla u(x))\mathbf{w}(x)\right] \\ &\quad + \text{div}\left[(\fint_{B_r(x_0)} \mathbf{w}(x)dx\cdot\nabla u(x))(\mathbf{w}(x) - \fint_{B_r(x_0)} \mathbf{w}(x)dx)\right] \\ &\quad - \text{div}\left[p(x)(I + \mathbf{w}(x)\otimes\mathbf{w}(x))\nabla\xi(x)\right] + F(x). \end{aligned}$$

Recall that the fundamental solution of the equation $-\mathrm{div}[A(x_0)\nabla v(x)]=0$ is given by

$$\Gamma(x_0,x)=\frac{1}{(N-2)\omega_N\sqrt{\det A(x_0)}}\left(\sum_{i,j=1}^{N}A_{ij}(x_0)x_ix_j\right)^{\frac{2-N}{2}},$$

where $A_{ij}(x_0)$ is the co-factor of the entry that lies in the ith row and the jth column in the matrix $A(x_0)$ and ω_N is the surface area of the unit sphere. Then we have the frequently used representation formula

$$u(y)=-\int_{B_r(x_0)}\Gamma(x_0,x-y)\mathrm{div}[A(x_0)\nabla u(x)]dx \tag{3.1}$$

whenever u is compactly supported in $B_r(x_0)$. We can easily verify that

$$\frac{\partial^2}{\partial x_i\partial x_j}\Gamma(x_0,x),\ i,j=1,\cdots,N,$$

are C-Z kernels [3,6]. Our key observation is the following:

LEMMA 3.2. *There is a positive number $c=c(N)$ such that*

$$\left|\frac{\partial^2\Gamma(x_0,x)}{\partial x_i\partial x_j}\right|\le c\left(1+\left|\fint_{B_r(x_0)}\mathbf{w}(x)dx\right|^2\right)^{\frac{5N-2}{2}}\le c(1+\|\mathbf{w}\|_{\infty,\Omega})^{5N-2}\quad\text{for } x\in\partial B_1(0).$$

Proof. Obviously,

$$\begin{aligned}|y|^2&\le\left(I+\fint_{B_r(x_0)}\mathbf{w}(x)dx\otimes\fint_{B_r(x_0)}\mathbf{w}(x)dx\right)y\cdot y\\&\le\left(1+\left|\fint_{B_r(x_0)}\mathbf{w}(x)dx\right|^2\right)|y|^2\quad\text{for all } y\in\mathbb{R}^N.\end{aligned}$$

This implies

$$\text{the largest eigenvalue of } A(x_0)\le 1+\left|\fint_{B_r(x_0)}\mathbf{w}(x)dx\right|^2, \tag{3.2}$$

$$\text{the smallest eigenvalue of } A(x_0)\ge 1. \tag{3.3}$$

Hence,

$$\begin{aligned}\det A(x_0)&=\text{the product of the eigenvalues, counted with multiplicity}\\&\in\left(1,\left(1+\left|\fint_{B_r(x_0)}\mathbf{w}(x)dx\right|^2\right)^N\right).\end{aligned}\tag{3.4}$$

We calculate

$$\partial_{x_k}\Gamma(x_0,x)=-\frac{\sum_{i=1}^N(A_{ik}(x_0)+A_{ki}(x_0))x_i}{2\omega_N\sqrt{\det A(x_0)}\left(\sum_{i,j=1}^N A_{ij}(x_0)x_ix_j\right)^{\frac{N}{2}}}, \tag{3.5}$$

$$\begin{aligned}\partial^2_{x_kx_\ell}\Gamma(x_0,x)=&-\frac{A_{\ell k}(x_0)+A_{k\ell}(x_0)}{2\omega_N\sqrt{\det A(x_0)}\left(\sum_{i,j=1}^N A_{ij}(x_0)x_ix_j\right)^{\frac{N}{2}}}\\&+\frac{N\sum_{i=1}^N(A_{ik}(x_0)+A_{ki}(x_0))x_i\sum_{j=1}^N(A_{j\ell}(x_0)+A_{\ell j}(x_0))x_\ell}{4\omega_N\sqrt{\det A(x_0)}\left(\sum_{i,j=1}^N A_{ij}(x_0)x_ix_j\right)^{\frac{N+2}{2}}}.\end{aligned}$$

Note that

$$A^{-1}(x_0)=\frac{1}{\det A(x_0)}\begin{pmatrix}A_{11}&\cdots&A_{N1}\\ \vdots&&\vdots\\ A_{1N}&\cdots&A_{NN}\end{pmatrix}.$$

It is easy to see that

$$\sum_{i,j=1}^N A_{ij}(x_0)x_ix_j=\det A(x_0)A^{-1}(x_0)x\cdot x.$$

Recall that λ is an eigenvalue of an invertible matrix A if and only $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. With this in mind, we derive from (3.2), (3.3), and (3.4) that

$$\begin{aligned}&\sum_{i,j=1}^N A_{ij}(x_0)x_ix_j\leq\det A(x_0)|x|^2\leq\left(1+\left|\fint_{B_r(x_0)}\mathbf{w}(x)dx\right|^2\right)^N|x|^2,\\&\sum_{i,j=1}^N A_{ij}(x_0)x_ix_j\geq\frac{\det A(x_0)}{1+\left|\fint_{B_r(x_0)}\mathbf{w}(x)dx\right|^2}|x|^2\geq\frac{1}{1+\left|\fint_{B_r(x_0)}\mathbf{w}(x)dx\right|^2}|x|^2.\end{aligned} \tag{3.6}$$

Remember that the determinant of an $n\times n$ matrix A is the signed sum over all possible products of n entries of A with exactly one entry being selected from each row and from each column of A. Thus,

$$|A_{ij}(x_0)|\leq(N-1)!\left(1+\left|\fint_{B_r(x_0)}\mathbf{w}(x)dx\right|^2\right)^{N-1}. \tag{3.7}$$

Here we have used the fact that each entry in $A(x_0)$ is bounded by $1+\left|\fint_{B_r(x_0)}\mathbf{w}(x)dx\right|^2$. We are ready to estimate for $x\in\partial B_1(0)$ that

$$\begin{aligned}\left|\partial^2_{x_kx_\ell}\Gamma(x_0,x)\right|\leq&\, c\left(1+\left|\fint_{B_r(x_0)}\mathbf{w}(x)dx\right|^2\right)^{N-1+\frac{N}{2}}\\&+c\left(1+\left|\fint_{B_r(x_0)}\mathbf{w}(x)dx\right|^2\right)^{2(N-1)+\frac{N+2}{2}}\\\leq&\, c\left(1+\left|\fint_{B_r(x_0)}\mathbf{w}(x)dx\right|^2\right)^{\frac{5N-2}{2}}.\end{aligned} \tag{3.8}$$

The proof is complete. ∎

Return to the proof of Lemma 1.2. Since u is compactly supported in $B_r(x_0)$, we have from (3.1) that

$$
\begin{aligned}
u(y)=&\int_{B_r(x_0)}((\mathbf{w}(x)-\fint_{B_r(x_0)}\mathbf{w}(x)dx)\cdot\nabla u(x))\mathbf{w}(x)\cdot\nabla_x\Gamma(x_0,y-x)dx\\
&+\int_{B_r(x_0)}(\fint_{B_r(x_0)}\mathbf{w}(x)dx\cdot\nabla u(x))(\mathbf{w}(x)-\fint_{B_r(x_0)}\mathbf{w}(x)dx)\cdot\nabla_x\Gamma(x_0,y-x)dx\\
&+\int_{B_r(x_0)}p(x)(I+\mathbf{w}(x)\otimes\mathbf{w}(x))\nabla\xi(x)\cdot\nabla_x\Gamma(x_0,y-x)dx\\
&+\int_{B_r(x_0)}F(x)\Gamma(x_0,y-x)dx.
\end{aligned}
$$

Differentiate the above equation with respect to y_i to obtain

$$
\begin{aligned}
u_{y_i}(y)=&-\int_{B_r(x_0)}((m_h(x)-m_h(x_0))u_{x_h}(x))m_j(x)\frac{\partial^2}{\partial x_i\partial x_j}\Gamma(x_0,y-x)dx\\
&-\int_{B_r(x_0)}(m_h(x_0)u_{x_h}(x))(m_j(x)-m_j(x_0))\frac{\partial^2}{\partial x_i\partial x_j}\Gamma(x_0,y-x)dx\\
&-\int_{B_r(x_0)}p(x)(I+\mathbf{w}(x)\otimes\mathbf{w}(x))\nabla\xi(x)\cdot\nabla_x\Gamma_{x_i}(x_0,y-x)dx\\
&-\int_{B_r(x_0)}F(x)\Gamma_{x_i}(x_0,y-x)dx.
\end{aligned}\tag{3.9}
$$

According to Morrey's inequality ([7], p.143), for $\ell>N$ there is a positive number $c=c(N,\ell)$ such that

$$
\left|\mathbf{w}(x)-\fint_{B_r(x_0)}\mathbf{w}(x)dx\right|\leq cr^{1-\frac{N}{\ell}}\|\nabla\mathbf{w}\|_{\ell,B_r(x_0)}\ \text{ on } B_r(x_0).
$$

With this in mind, we apply (CZ2) to (3.9) to deduce

$$
\begin{aligned}
&\|\nabla u\|_{q,B_r(x_0)}\\
\leq\ & cr^{1-\frac{N}{\ell}}\sum_{i,j=1}^{N}\left\|\frac{\partial^2}{\partial x_i\partial x_j}\Gamma(x_0,x)\right\|_{2,\partial B_1(0)}\|\mathbf{w}\|_{\infty,B_r(x_0)}\|\nabla\mathbf{w}\|_{\infty,B_r(x_0)}\|\nabla u\|_{q,B_r(x_0)}\\
&+\frac{c}{r-\delta}\sum_{i,j=1}^{N}\left\|\frac{\partial^2}{\partial x_i\partial x_j}\Gamma(x_0,x)\right\|_{2,\partial B_1(0)}(1+\|\mathbf{w}\|^2_{\infty,B_r(x_0)})\|p\|_{q,B_r(x_0)}\\
&+c\left\|\int_{B_r(x_0)}F(x)\Gamma_{x_i}(y,y-x)dx\right\|_{q,B_r(x_0)}\\
\leq\ & cr^{1-\frac{N}{\ell}}(1+\|\mathbf{w}\|_{\infty,B_r(x_0)})^{5N-1}\|\nabla\mathbf{w}\|_{\infty,B_r(x_0)}\|\nabla u\|_{q,B_r(x_0)}\\
&+\frac{c}{r-\delta}\|p\|_{q,B_r(x_0)}\left(1+\|\mathbf{w}\|_{\infty,B_r(x_0)}\right)^{5N}+\left\|\int_{B_r(x_0)}F(x)\Gamma_{x_i}(x_0,y-x)dx\right\|_{q,B_r(x_0)}.
\end{aligned}\tag{3.10}
$$

The last step is due to (3.8). To estimate the last term in the above inequality, we derive from (3.5), (3.6), and (3.7) that

$$|\Gamma_{x_i}(x_0, y-x)| \le \frac{c\left(1+\left|\fint_{B_r(x_0)} \mathbf{w}(x)dx\right|^2\right)^{\frac{3N-2}{2}}}{|x-y|^{N-1}}.$$

Consequently,

$$\begin{aligned}
&\left|\int_{B_r(x_0)} F(x)\Gamma_{x_i}(x_0, y-x)dx\right| \\
&\le c\left(1+\|\mathbf{w}\|_{\infty,B_r(x_0)}\right)^{3N-2} \int_{B_r(x_0)} \frac{|\xi S(x)| + |\nabla \xi|\left(1+\|\mathbf{w}\|^2_{\infty,B_r(x_0)}\right)|\nabla p|}{|x-y|^{N-1}} dx.
\end{aligned}$$

By the remark following Lemma 7.12 in [8], we obtain

$$\begin{aligned}
\left\|\int_{B_r(x_0)} F(x)\Gamma_{x_i}(x_0, y-x)dx\right\|_{q,B_r(x_0)} &\le c\left(1+\|\mathbf{w}\|_{\infty,B_r(x_0)}\right)^{3N-2}\|S\|_{\frac{Nq}{N+q},B_r(x_0)} \\
&\quad + \frac{c\left(1+\|\mathbf{w}\|_{\infty,B_r(x_0)}\right)^{3N}}{r-\delta}\|\nabla p\|_{\frac{Nq}{N+q},B_r(x_0)}.
\end{aligned}$$

Plug this into (3.10) and then use the definition of ξ to obtain

$$\begin{aligned}
\|\nabla p\|_{q,B_\delta(x_0)} \le\ & cr^{1-\frac{N}{\ell}}(1+\|\mathbf{w}\|_{\infty,B_r(x_0)})^{5N-1}\|\nabla \mathbf{m}\|_{q,B_r(x_0)}\|\nabla p\|_{q,B_r(x_0)} \\
&+ \frac{cr^{1-\frac{N}{\ell}}}{r-\delta}(1+\|\mathbf{w}\|_{\infty,B_r(x_0)})^{5N-1}\|\nabla \mathbf{m}\|_{q,B_r(x_0)}\|p\|_{q,B_r(x_0)} \\
&+ \frac{c}{r-\delta}\left(1+\|\mathbf{w}\|_{\infty,B_r(x_0)}\right)^{5N}\|p\|_{q,B_r(x_0)} \\
&+ c\left(1+\|\mathbf{w}\|_{\infty,B_r(x_0)}\right)^{3N-2}\|S\|_{\frac{Nq}{N+q},B_r(x_0)} \\
&+ \frac{c}{r-\delta}\left(1+\|\mathbf{w}\|_{\infty,B_r(x_0)}\right)^{3N}\|\nabla p\|_{\frac{Nq}{N+q},B_r(x_0)}.
\end{aligned} \tag{3.11}$$

Set

$$\begin{aligned}
K_1 =\ & r^{1-\frac{N}{\ell}}(1+\|\mathbf{w}\|_{\infty,B_r(x_0)})^{5N-1}\|\nabla \mathbf{m}\|_{q,B_r(x_0)}, \qquad (3.12)\\
K_2 =\ & r^{1-\frac{N}{\ell}}(1+\|\mathbf{w}\|_{\infty,B_r(x_0)})^{5N-1}\|\nabla \mathbf{m}\|_{q,B_r(x_0)}\|p\|_{q,B_r(x_0)} \\
&+ \left(1+\|\mathbf{w}\|_{\infty,B_r(x_0)}\right)^{5N}\|p\|_{q,B_r(x_0)} + \left(1+\|\mathbf{w}\|_{\infty,B_r(x_0)}\right)^{3N}\|\nabla p\|_{\frac{Nq}{N+q},B_r(x_0)}, \\
K_3 =\ & \left(1+\|\mathbf{w}\|_{\infty,B_r(x_0)}\right)^{3N-2}\|S\|_{\frac{Nq}{N+q},B_r(x_0)}.
\end{aligned}$$

We can write (3.11) as

$$\|\nabla p\|_{q,B_\delta(x_0)} \le cK_1\|\nabla p\|_{q,B_r(x_0)} + \frac{c}{r-\delta}K_2 + cK_3. \tag{3.13}$$

Define

$$r_n = r - \frac{r}{2^{n+1}}, \quad n = 0, 1, 2, \cdots.$$

Take $(\delta,r)=(r_n,r_{n+1})$ in (3.13) and keep in mind the fact that K_1,K_2, and K_3 are all increasing with r to get

$$\begin{aligned}\|\nabla p\|_{q,B_{r_n}(x_0)}&\leq cK_1\|\nabla p\|_{q,B_{r_{n+1}}(x_0)}+\frac{c}{r_{n+1}-r_n}K_2+cK_3\\&\leq cK_1\|\nabla p\|_{q,B_{r_{n+1}}(x_0)}+\frac{c2^{n+2}}{r}K_2+cK_3.\end{aligned}$$

By iteration,

$$\|\nabla p\|_{q,B_{\frac{r}{2}}(x_0)}\leq (cK_1)^n\|\nabla p\|_{q,B_{r_n}(x_0)}+\frac{c}{r}K_2\sum_{i=0}^{n-1}(2cK_1)^i+cK_3\sum_{i=0}^{n-1}(cK_1)^i. \quad (3.14)$$

In view of (3.12), we can take r so that

$$c2K_1\leq\frac{1}{2}. \quad (3.15)$$

Then let $n\to\infty$ in (3.14) to get

$$\begin{aligned}\|\nabla p\|_{q,B_{\frac{r}{2}}(x_0)}&\leq\frac{c}{r}K_2+cK_3\\&=cr^{-\frac{N}{\ell}}(1+\|\mathbf{w}\|_{\infty,B_r(x_0)})^{5N-1}\|\nabla\mathbf{m}\|_{q,B_r(x_0)}\|p\|_{q,B_r(x_0)}\\&\quad+\frac{c}{r}\left(1+\|\mathbf{w}\|_{\infty,B_r(x_0)}\right)^{5N}\|p\|_{q,B_r(x_0)}\\&\quad+\frac{c}{r}\left(1+\|\mathbf{w}\|_{\infty,B_r(x_0)}\right)^{3N}\|\nabla p\|_{\frac{Nq}{N+q},B_r(x_0)}\\&\quad+c\left(1+\|\mathbf{w}\|_{\infty,B_r(x_0)}\right)^{3N-2}\|S\|_{\frac{Nq}{N+q},B_r(x_0)}.\end{aligned} \quad (3.16)$$

By virtue of (3.12), for (3.15) to hold, it is enough for us to take

$$cr^{1-\frac{N}{\ell}}\left(1+\|\mathbf{w}\|_{W^{1,\ell}(\Omega)}\right)^{5N}=\frac{1}{4}. \quad (3.17)$$

This, combined with (3.16), yields

$$\begin{aligned}\|\nabla p\|_{q,B_{\frac{r}{2}}(x_0)}&\leq c\left(1+\|\mathbf{w}\|_{W^{1,\ell}(\Omega)}\right)^{\frac{5N(2\ell-N)}{\ell-N}}\|p\|_{q,\Omega}\\&\quad+c\left(1+\|\mathbf{w}\|_{W^{1,\ell}(\Omega)}\right)^{\frac{N(8\ell-3N)}{\ell-N}}\|\nabla p\|_{\frac{Nq}{N+q},\Omega}\\&\quad+c(1+\|\mathbf{w}\|_{\infty,\Omega})^{3N-2}\|S\|_{\frac{Nq}{N+q},\Omega}\\&\leq c\left(1+\|\mathbf{w}\|_{W^{1,\ell}(\Omega)}\right)^{\frac{5N(2\ell-N)}{\ell-N}}\left(\|\nabla p\|_{\frac{Nq}{N+q},\Omega}+\|S\|_{\frac{Nq}{N+q},\Omega}\right).\end{aligned}$$

Here we have applied Poincaré's inequality to p and the Sobolev embedding theorem to $\mathbf{w}$.

If $x_0\in\partial\Omega$, the same estimate still holds with $B_r(x_0)$ (resp. $B_{\frac{r}{2}}(x_0)$) being replaced by $B_r(x_0)\cap\Omega$ (resp. $B_{\frac{r}{2}}(x_0)\cap\Omega$). This can be achieved by the classical technique of flattening $B_r(x_0)\cap\partial\Omega$ and then turning x_0 into an interior point ([8], p.300). Also see [22] for a rather detailed implementation of the technique. We shall omit it here.

Finally, let r be determined by (3.17). There is an integer j with the property

$$j-1\leq\frac{\operatorname{diam}(\Omega)}{r}\leq j. \quad (3.18)$$

We can find at most $(2j)^N$ balls $\{B_r(x_0^{(i)})\}$ with

$$\Omega\subset\cup_{i=1}^{(2j)^N}B_{\frac{r}{2}}(x_0^{(i)}).$$

Consequently,

$$\begin{aligned}\|\nabla p\|_{q,\Omega}&\leq\sum_{i=1}^{(2j)^N}\|\nabla p\|_{q,B_{\frac{r}{2}}(x_0^{(i)})}\\&\leq cj^N\left(1+\|\mathbf{w}\|_{W^{1,\ell}(\Omega)}\right)^{\frac{5N(2\ell-N)}{\ell-N}}\left(\|\nabla p\|_{\frac{Nq}{N+q},\Omega}+\|S\|_{\frac{Nq}{N+q},\Omega}\right).\end{aligned}\tag{3.19}$$

Observe from (3.18) and (3.17) that

$$\begin{aligned}j^N&\leq\frac{c}{r^N}+c\\&\leq c\left(1+\|\nabla\mathbf{w}\|_{W^{1,\ell}(\Omega)}\right)^{\frac{5N^2\ell}{\ell-N}}+c.\end{aligned}$$

Substitute this into (3.19) to obtain

$$\|\nabla p\|_{q,\Omega}\leq c(1+\|\nabla\mathbf{w}\|_{W^{1,\ell}(\Omega)})^{\frac{5N(2\ell-N+N\ell)}{\ell-N}}\left(\|\nabla p\|_{\frac{Nq}{N+q},\Omega}+\|S\|_{\frac{Nq}{N+q},\Omega}\right).\tag{3.20}$$

On account of the condition on q in (1.18) and the interpolation inequality ([8], p.146), we have

$$\|\nabla p\|_{\frac{Nq}{N+q},\Omega}\leq\varepsilon\|\nabla p\|_{q,\Omega}+\frac{1}{\varepsilon^{\frac{(N-1)q-N}{q}}}\|\nabla p\|_{1,\Omega},\quad\varepsilon>0.$$

Plug this into (3.20) and choose ε appropriately in the resulting inequality to get

$$\|\nabla p\|_{q,\Omega}\leq c\left(1+\|\nabla\mathbf{w}\|_{W^{1,\ell}(\Omega)}\right)^{\frac{5N(2\ell-N+N\ell)(Nq-N)}{q(\ell-N)}}\left(\|\nabla p\|_{1,\Omega}+\|S\|_{\frac{Nq}{N+q},\Omega}\right).$$

The proof is complete. ∎

Note that if we wish to further weaken $\mathbf{w}$ to a VMO function as in [3,4,6] we will run into a technical problem, which is that we do not know how the constant c in inequality (2.3) of [3] depends on $\|k\|_{2,\partial B_1(0)}$.

4. Blow-up time

In this section we offer the proof of the main theorem.

Assume that $(\mathbf{m},p)$ is a strong solution to (1.1)-(1.4). The existence of such an "approximate" solution will be made clear later.

By virtue of the boundary condition (1.3), we have

$$\|\mathbf{m}\|_{W^{1,\ell}(\Omega)}\leq c\|\nabla\mathbf{m}\|_{\infty,\Omega}.$$

Use p as a test function in (1.19) to get

$$\int_\Omega|\nabla p|^2dx\leq\int_\Omega S(x)pdx\leq\|S\|_{2,\Omega}\|p\|_{2,\Omega}\leq c\|S\|_{2,\Omega}\|\nabla p\|_{2,\Omega},$$

from whence follows

$$\|\nabla p\|_{2,\Omega}\leq c\|S\|_{2,\Omega}.$$

Thus we can write (1.21) as

$$\|\nabla p\|_{4q,\Omega} \le c(1+\|\nabla \mathbf{m}\|_{\infty,\Omega})^{s_1}\|S\|_{\frac{4Nq}{N+4q},\Omega}.$$

Here we have replaced q by $4q$. Take the $(4q)$-th power, integrate over $(0,T)$, and then take the $(4q)$-th root to derive

$$\|\nabla p\|_{4q,\Omega_T} \le cT^{\frac{1}{4q}}(1+\|\nabla \mathbf{m}\|_{\infty,\Omega})^{s_1}\|S\|_{\frac{4Nq}{N+4q},\Omega}. \tag{4.1}$$

The rest of the proof of Theorem 1.1 is divided into several lemmas.

LEMMA 4.1. *We have*

$$\sup_{\Omega_T}|\mathbf{m}| \le cT^{\frac{s_0}{4q}}\left(T^{\frac{1}{2}}\|\mathbf{m}_0\|_{2,\Omega}+T\|S\|_{2,\Omega}\right)\|\nabla p\|_{4q,\Omega_T}^{s_0} + c\left(\|\mathbf{m}_0\|_{\infty,\Omega}+T^{\frac{1}{2}}\|\mathbf{m}_0\|_{2,\Omega}+T\|S\|_{2,\Omega}\right), \tag{4.2}$$

where s_0 is given as (2.6).

Proof. Take the dot product of (1.2) with $\mathbf{m}$ to derive

$$\frac{1}{2}\partial_t|\mathbf{m}|^2-\frac{D^2}{2}\Delta|\mathbf{m}|^2+D^2|\nabla\mathbf{m}|^2+|\mathbf{m}|^{2\gamma}=E^2(\mathbf{m}\cdot\nabla p)^2\le E^2|\nabla p|^2|\mathbf{m}|^2 \quad \text{in } \Omega_T.$$

Drop the two non-negative terms on the left-hand side and then apply (2.19) to derive

$$\sup_{\Omega_T}|\mathbf{m}|^2 \le c\sup_{\Omega}|\mathbf{m}_0|^2+c\left(\|\nabla p\|_{2q,\Omega_T}^{2s_0}+1\right)\|\mathbf{m}\|_{2,\Omega_T}^2. \tag{4.3}$$

We can deduce from (1.5) that

$$\sup_{0\le t\le T}\int_\Omega |\mathbf{m}(x,t)|^2dx \le c\int_\Omega|\mathbf{m}_0(x)|^2dx+cT\|S(x)\|_{2,\Omega}^2.$$

Use this in (4.3) to get (4.2). ◻

LEMMA 4.2. *We have*

$$\|\nabla\mathbf{m}\|_{\infty,\Omega_T}\le cG(T)+cF(T)\|\nabla p\|_{4q,\Omega_T}^{s_4}, \tag{4.4}$$

where s_4 is a positive number determined by N,q,γ,

$$\begin{aligned}F(T)=&T^{\frac{s_0}{4q}}\left(T^{\frac{1}{2}}\|\mathbf{m}_0\|_{2,\Omega}+T\|S(x)\|_{2,\Omega}\right)\\&+\left(\|\mathbf{m}_0\|_{\infty,\Omega}+T^{\frac{1}{2}}\|\mathbf{m}_0\|_{2,\Omega}+T\|S(x)\|_{2,\Omega}\right)\\&+T^{\frac{s_0(2\gamma-1)+2}{4q}}\left(T^{\frac{2\gamma-1}{2}}\|\mathbf{m}_0\|_{2,\Omega}^{2\gamma-1}+T^{2\gamma-1}\|S(x)\|_{2,\Omega}^{2\gamma-1}\right), \quad \textit{and}\end{aligned}$$

$$G(T)=\|\nabla\mathbf{m}_0\|_{\infty,\Omega}+F(T)+T^{\frac{1}{2q}}\left(\|\mathbf{m}_0\|_{\infty,\Omega}^{2\gamma-1}+T^{\frac{2\gamma-1}{2}}\|\mathbf{m}_0\|_{2,\Omega}^{2\gamma-1}+T^{2\gamma-1}\|S(x)\|_{2,\Omega}^{2\gamma-1}\right).$$

Proof. We can write (1.2) in the form

$$\partial_t m_i - D^2\Delta m_i = E^2(\mathbf{m}\cdot\nabla p)p_{x_i}-|\mathbf{m}|^{2(\gamma-1)}m_i \quad \text{in } \Omega_T, \quad i=1,\ldots,N.$$

This puts us in a position to apply Lemma 2.5. Upon doing so, we arrive at

$$\|\nabla m_i\|_{\infty,\Omega_T} \le c\|\nabla \mathbf{m}_0\|_{\infty,\Omega} + c\left\|E^2(\mathbf{m}\cdot\nabla p)p_{x_i} - |\mathbf{m}|^{2(\gamma-1)}m_i\right\|_{2q,\Omega_T}. \tag{4.5}$$

We estimate from (4.2) that

$$\begin{aligned}
&\left\|E^2(\mathbf{m}\cdot\nabla p)p_{x_i} - |\mathbf{m}|^{2(\gamma-1)}m_i\right\|_{2q,\Omega_T}\\
&\le c\|\mathbf{m}\|_{\infty,\Omega_T}\|\nabla p\|^2_{4q,\Omega_T} + cT^{\frac{1}{2q}}\|\mathbf{m}\|^{2\gamma-1}_{\infty,\Omega_T}\\
&\le cT^{\frac{s_0}{4q}}\left(T^{\frac12}\|\mathbf{m}_0\|_{2,\Omega} + T\|S(x)\|_{2,\Omega}\right)\|\nabla p\|^{s_0+2}_{4q,\Omega_T}\\
&\quad + c\left(\|\mathbf{m}_0\|_{\infty,\Omega} + T^{\frac12}\|\mathbf{m}_0\|_{2,\Omega} + T\|S(x)\|_{2,\Omega}\right)\|\nabla p\|^2_{4q,\Omega_T}\\
&\quad + cT^{\frac{s_0(2\gamma-1)+2}{4q}}\left(T^{\frac{2\gamma-1}{2}}\|\mathbf{m}_0\|^{2\gamma-1}_{2,\Omega} + T^{2\gamma-1}\|S(x)\|^{2\gamma-1}_{2,\Omega}\right)\|\nabla p\|^{s_0(2\gamma-1)}_{4q,\Omega_T}\\
&\quad + cT^{\frac{1}{2q}}\left(\|\mathbf{m}_0\|^{2\gamma-1}_{\infty,\Omega} + T^{\frac{2\gamma-1}{2}}\|\mathbf{m}_0\|^{2\gamma-1}_{2,\Omega} + T^{2\gamma-1}\|S(x)\|^{2\gamma-1}_{2,\Omega}\right).
\end{aligned} \tag{4.6}$$

Set

$$s_4 = \max\{s_0+2, s_0(2\gamma-1)\}.$$

Then Young's inequality asserts

$$\begin{aligned}
\|\nabla p\|^{s_0+2}_{4q,\Omega_T} &\le c\|\nabla p\|^{s_4}_{4q,\Omega_T} + c,\\
\|\nabla p\|^{2}_{4q,\Omega_T} &\le c\|\nabla p\|^{s_4}_{4q,\Omega_T} + c,\\
\|\nabla p\|^{s_0(2\gamma-1)}_{4q,\Omega_T} &\le c\|\nabla p\|^{s_4}_{4q,\Omega_T} + c.
\end{aligned}$$

Combining this with (4.6) and (4.5) yields (4.4). □

Use (4.4) in (4.1) to obtain

$$\|\nabla p\|_{4q,\Omega_T} \le cG_1(T) + cF_1(T)\|\nabla p\|^{s_5}_{4q,\Omega_T}, \tag{4.7}$$

where

$$G_1(T) = T^{\frac{1}{4q}}\left(G^{s_1}(T)+1\right)\|S\|_{\frac{4Nq}{N+4q},\Omega}, \quad F_1(T) = T^{\frac{1}{4q}}F^{s_1}(T)\|S\|_{\frac{4Nq}{N+4q},\Omega}, \quad s_5 = s_1 s_4.$$

Note that we can represent $G_1(T)$ as the sum of 12 terms, each of which is of the form

$$T^a\|S\|^b_{\frac{4Nq}{N+4q},\Omega}\|\nabla \mathbf{m}_0\|^c_{\infty,\Omega}$$

with $a>0, b\ge 0, c\ge 0$ being determined by N, q, γ only. The same can be done for $F_1(T)$ except that there are only seven terms in $F_1(T)$. It follows from (4.7) that

$$\|\nabla p\|_{4q,\Omega_\tau} \le cF_1(T)\|\nabla p\|^{s_5}_{4q,\Omega_\tau} + cG_1(T) \quad \text{for each } \tau\in[0,T]. \tag{4.8}$$

By letting $h(\tau) = \|\nabla p\|_{4q,\Omega_\tau}$, we can put ourselves in the situation of Lemma 2.2. As in the proof of the lemma in [22], we consider the function $g(h) = cF_1(T)h^{s_5} - h + cG_1(T)$ on $[0,\infty)$. It follows from (4.8) that

$$g(h(\tau)) \ge 0 \quad \text{for each } \tau\in[0,T]. \tag{4.9}$$

We compute

$$g'(h)=cs_5F_1(T)h^{s_5-1}-1.$$

Thus, $g(h)$ is decreasing on $\left(0,\frac{1}{(cs_5F_1(T))^{\frac{1}{s_5-1}}}\right)$ and increasing on $\left(\frac{1}{(cs_5F_1(T))^{\frac{1}{s_5-1}}},\infty\right)$. The minimum value of g is given by

$$m_g\equiv g\left(\frac{1}{(cs_5F_1(T))^{\frac{1}{s_5-1}}}\right)=-\frac{s_5-1}{s_5(cs_5F_1(T))^{\frac{1}{s_5-1}}}+cG_1(T).$$

It is easy to see that there is a unique solution $T_{\max}$ to the equation

$$\frac{s_5-1}{s_5(cs_5F_1(T_{\max}))^{\frac{1}{s_5-1}}cG_1(T_{\max})}=1. \tag{4.10}$$

We have

$$m_g<0 \quad \text{for each } T<T_{\max}.$$

Fix $T<T_{\max}$. Observe that $\|\nabla p\|_{4q,\Omega_\tau}$ is a continuous function of τ and

$$\lim_{\tau\to 0}\|\nabla p\|_{4q,\Omega_\tau}=0.$$

We can infer from (4.9) that

$$\|\nabla p\|_{4q,\Omega_\tau}<\frac{1}{(cs_5F_1(T))^{\frac{1}{s_5-1}}} \quad \text{for each } \tau\le T.$$

In particular,

$$\|\nabla p\|_{4q,\Omega_T}<\frac{1}{(cs_5F_1(T))^{\frac{1}{s_5-1}}}. \tag{4.11}$$

Then (D4) follows from the classical regularity result for linear parabolic equations in ([15], p.204). We can easily transform (4.10) into (1.10).

The existence of a solution in the preceding calculations can be established via the Leray-Schauder fixed point theorem ([8], p.280). To this end, let q be given as in (H1) and set

$$\mathcal{B}=L^{4q}\left(0,T;W_0^{1,4q}(\Omega)\right).$$

Then define an operator $\mathcal{T}$ from $\mathcal{B}$ into itself as follows: Let $p\in\mathcal{B}$. We say $w=\mathbb{T}(p)$ if w is the unique solution of the problem

$$\begin{aligned}-\operatorname{div}((I+\mathbf{n}\otimes\mathbf{n})\nabla w)&=S(x) \quad \text{in } \Omega_T,\\ w&=0 \quad \text{on } \Sigma_T,\end{aligned}$$

where $\mathbf{n}$ solves the problem

$$\partial_t\mathbf{n}-D^2\Delta\mathbf{n}+|\mathbf{n}|^{2(\gamma-1)}\mathbf{n}=E^2(\mathbf{n}\cdot\nabla p)\nabla p \quad \text{in } \Omega_T, \tag{4.12}$$

$$\mathbf{n}=0 \quad \text{on } \Sigma_T, \tag{4.13}$$

$$\mathbf{n}(x,0)=\mathbf{m}_0(x) \quad \text{on } \Omega. \tag{4.14}$$

We have:

LEMMA 4.3. *Let (H1)-(H3) hold. Then there is a unique weak solution* $\mathbf{n}$ *to* (4.12)-(4.14) *in the space* $C\left([0,T];\left(L^2(\Omega)\right)^N\right)\cap L^2\left((0,T);\left(W_0^{1,2}(\Omega)\right)^N\right)$. *Furthermore,*

$$\mathbf{n} \text{ is Hölder continuous in } \overline{\Omega_T} \text{ with } \mathbf{n}\in L^\infty(0,T;W_0^{1,\infty}(\Omega)). \tag{4.15}$$

We postpone the proof of this lemma to the end of the section. Equipped with this lemma, we can claim that $\mathcal{T}$ is well-defined. Under (4.15) and (H3), we can appeal to (1.9), thereby yielding

$$\|\nabla w\|_{4q,\Omega}\le c\|S(x)\|_{\frac{4Nq}{N+4q},\Omega},$$

from whence follows

$$\|\nabla w\|_{4q,\Omega_T}\le cT^{\frac{1}{4q}}\|S(x)\|_{\frac{4Nq}{N+4q},\Omega}.$$

According to the Leray-Schauder fixed point theorem, for $\mathcal{T}$ to have a fixed point, we must verify

(C1) $\mathcal{T}$ is continuous;

(C2) $\mathcal{T}$ maps bounded sets into precompact ones;

(C3) There is a constant c such that

$$\|p\|_{\mathcal{B}}\le c$$

for all $p\in\mathcal{B}$ and $\sigma\in[0,1]$ satisfying

$$p=\sigma\mathcal{T}(p). \tag{4.16}$$

To see (C2), suppose

$$p_n\to p \text{ weakly in } L^{4q}\left(0,T;W_0^{1,4q}(\Omega)\right).$$

Denote by $\mathbf{n}_n$ the solution to (4.12)-(4.14) with p being replaced by p_n. In view of (4.15), we can extract a sub-sequence of $\{\mathbf{n}_n\}$, not relabeled, such that

$$\mathbf{n}_n\to\mathbf{n} \text{ uniformly in } \left(C(\overline{\Omega_T})\right)^N.$$

Note that we have

$$-\operatorname{div}[(I+\mathbf{n}_n\otimes\mathbf{n}_n)\nabla p_n]=S(x) \text{ in } \Omega_T. \tag{4.17}$$

Thus, we can pass to the limit in the above equation to get

$$-\operatorname{div}[(I+\mathbf{n}\otimes\mathbf{n})\nabla p]=S(x) \text{ in } \Omega_T.$$

Subtract this equation from (4.17) to derive

$$-\operatorname{div}[(I+\mathbf{n}_n\otimes\mathbf{n}_n)\nabla(p_n-p)]=\operatorname{div}[(\mathbf{n}_n\otimes\mathbf{n}_n-\mathbf{n}\otimes\mathbf{n})\nabla p] \text{ in } \Omega_T.$$

Once again, we can use (1.9) to get

$$\|\nabla(p_n-p)\|_{4q,\Omega_T}\le c\|(\mathbf{n}_n\otimes\mathbf{n}_n-\mathbf{n}\otimes\mathbf{n})\nabla p\|_{4q,\Omega_T}\le c\|\mathbf{n}_n\otimes\mathbf{n}_n-\mathbf{n}\otimes\mathbf{n}\|_{\infty,\Omega_T}\to 0.$$

That is, (C2) holds. Each problem in the definition of $\mathcal{T}$ has a unique solution. This together with (C2) implies (C1).

We easily see that (4.16) is equivalent to the problem

$$\begin{aligned}
-\mathrm{div}((I+\mathbf{m}\otimes\mathbf{m})\nabla p)&=\sigma S(x) \ \text{ in } \Omega_T,\\
\partial_t\mathbf{m}-D^2\Delta\mathbf{m}+|\mathbf{m}|^{2(\gamma-1)}\mathbf{m}&=E^2(\mathbf{m}\cdot\nabla p)\nabla p \ \text{ in } \Omega_T,\\
p&=0 \ \text{ on } \Sigma_T,\\
\mathbf{m}&=0 \ \text{ on } \Sigma_T,\\
\mathbf{m}(x,0)&=\mathbf{m}_0(x) \ \text{ on } \Omega.
\end{aligned}$$

Invoking the proof of (4.11), we obtain (C3) for each $T<T_{\max}$.

We are ready to prove Lemma 4.3.

Proof. **(Proof of Lemma 4.3.)** We first establish the uniqueness assertion. Suppose that (4.12)-(4.14) has two solutions, say, $\mathbf{n}_1,\mathbf{n}_2$. Then $\mathbf{n}\equiv\mathbf{n}_1-\mathbf{n}_2$ satisfies

$$\partial_t\mathbf{n}-D^2\Delta\mathbf{n}+|\mathbf{n}_1|^{2(\gamma-1)}\mathbf{n}_1-|\mathbf{n}_2|^{2(\gamma-1)}\mathbf{n}_2=E^2(\mathbf{n}\cdot\nabla p)\nabla p \ \text{ in } \Omega_T, \tag{4.18}$$

$$\mathbf{n}=0 \ \text{ on } \Sigma_T, \tag{4.19}$$

$$\mathbf{n}(x,0)=0 \ \text{ on } \Omega. \tag{4.20}$$

Recall from Lemma 2.1 that

$$\left(|\mathbf{n}_1|^{2(\gamma-1)}\mathbf{n}_1-|\mathbf{n}_2|^{2(\gamma-1)}\mathbf{n}_2\right)\cdot\mathbf{n}\geq 0.$$

With this in mind, we take the dot product of (4.18) with $\mathbf{n}$ to derive

$$\frac{1}{2}\sup_{0\leq t\leq T}\int_\Omega|\mathbf{n}|^2dx+D^2\int_{\Omega_T}|\nabla\mathbf{n}|^2dxdt\leq 2E^2\int_{\Omega_T}(\mathbf{n}\cdot\nabla p)^2dxdt.$$

By Poincaré's inequality,

$$\begin{aligned}
\int_{\Omega_T}|\mathbf{n}|^{2+\frac{4}{N}}dxdt&\leq\int_0^T\left(\int_\Omega|\mathbf{n}|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}}\left(\int_\Omega|\mathbf{n}|^2dx\right)^{\frac{2}{N}}dt\\
&\leq c\sup_{0\leq t\leq T}\left(\int_\Omega|\mathbf{n}|^2dx\right)^{\frac{2}{N}}\int_{\Omega_T}|\nabla\mathbf{n}|^2dxdt\\
&\leq c\left(\int_{\Omega_T}(\mathbf{n}\cdot\nabla p)^2dxdt\right)^{\frac{N+2}{N}}\\
&\leq c\int_{\Omega_T}|\mathbf{n}|^{2+\frac{4}{N}}dxdt\left(\int_{\Omega_T}|\nabla p|^{N+2}dxdt\right)^{\frac{2}{N}}.
\end{aligned}\tag{4.21}$$

Here c depends only on N,D,E. Obviously, we can pick a positive number $\tau\leq T$ such that

$$c\left(\int_{\Omega_\tau}|\nabla p|^{N+2}dxdt\right)^{\frac{2}{N}}<1.$$

Then (4.21) implies

$$\mathbf{n}=0 \ \text{ in } \Omega\times[0,\tau].$$

If $\tau < T$, then we apply the preceding proof to the problem on $\Omega\times(\tau,T)$. In a finite number of steps, we can achieve

$$\mathbf{n}=0 \quad \text{in } \Omega_T.$$

To obtain (4.15), it is enough for us to show that

$$\mathbf{n}\in\left(L^\infty(\Omega_T)\right)^N. \tag{4.22}$$

To see this, we write (4.12) as

$$\partial_t\mathbf{n}-D^2\Delta\mathbf{n}=E^2(\mathbf{n}\cdot\nabla p)\nabla p-|\mathbf{n}|^{2(\gamma-1)}\mathbf{n}\in L^q(\Omega_T).$$

Since $q>1+\frac{N}{2}$, we can invoke the classical result in ([15], p.204) and Lemma 2.5 to conclude (4.15). As for (4.22), we take the dot product of (4.12) with $\mathbf{n}$ to deduce

$$\partial_t|\mathbf{n}|^2-D^2\Delta|\mathbf{n}|^2\le 2E^2|\nabla p|^2|\mathbf{n}|^2 \quad \text{in } \Omega_T.$$

We can infer (4.22) from Lemma 2.4.

The existence of a weak solution to (4.12)-(4.14) can also be established via the Leray-Schauder fixed point theorem. In this case, we define an operator B from $\left(L^\infty(\Omega_T)\right)^N$ into itself as follows: For each $\mathbf{m}\in\left(L^\infty(\Omega_T)\right)^N$ we let $\mathbf{w}=B(\mathbf{m})$ be the unique solution of the problem

$$\begin{aligned}\partial_t\mathbf{w}-D^2\Delta\mathbf{w}&=E^2(\mathbf{m}\cdot\nabla p)\nabla p-|\mathbf{m}|^{2(\gamma-1)}\mathbf{m} \quad \text{in } \Omega_T,\\ \mathbf{w}&=0 \quad \text{on } \Sigma_T,\\ \mathbf{w}&=\mathbf{m}_0(x) \quad \text{on } \Omega.\end{aligned} \tag{4.23}$$

The term on the right-hand side of (4.23) lies in $L^{2q}(\Omega_T)$. Thus $\mathbf{w}$ is Hölder continuous on $\overline{\Omega_T}$. With this in mind, we can easily verify that (C1)-(C3) are all satisfied by B. This completes the proof. □

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