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RAREFIED GAS DYNAMICS WITH EXTERNAL FIELDS UNDER SPECULAR REFLECTION BOUNDARY CONDITION*

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Abstract. We consider the Boltzmann equation with external fields in strictly convex domains with the specular reflection boundary condition. We construct classical C^1 solutions away from the grazing set under the assumption that the external field is C^2 and the normal derivative of the field is positive and bounded away from zero.

Keywords. Boltzmann; boundary; specular; regularity; field.

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1. Introduction

Kinetic theory studys the time evolution of a large number of particles modeled by a distribution function in the phase space: $F(t,x,v)$ for $(t,x,v)\in[0,\infty)\times\Omega\times\mathbb{R}^3$, where Ω is an open bounded subset of $\mathbb{R}^3$. Dynamics and collision processes of dilute charged particles with a field E can be modeled by the Vlasov-Boltzmann equation

$$\partial_t F + v\cdot\nabla_x F + E\cdot\nabla_v F = Q(F,F). \tag{1.1}$$

The collision operator measures "the change rate" in binary collisions and takes the form of

$$\begin{aligned} Q(F_1,F_2)(v) &:= Q_{\text{gain}}(F_1,F_2) - Q_{\text{loss}}(F_1,F_2) \\ &:= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u)\cdot\omega)[F_1(u')F_2(v') - F_1(u)F_2(v)]\mathrm{d}\omega\mathrm{d}u, \end{aligned} \tag{1.2}$$

where $u' = u - [(u-v)\cdot\omega]\omega$ and $v' = v + [(u-v)\cdot\omega]\omega$.

Here, $B(v-u,\omega) = |v-u|^\kappa q_0(\frac{v-u}{|v-u|}\cdot\omega)$, $0\le\kappa\le 1$ (hard potential), and $0\le q_0(\frac{v-u}{|v-u|}\cdot\omega)\le C|\frac{v-u}{|v-u|}\cdot\omega|$ (angular cutoff).

The collision operator enjoys collision invariance: for any measurable function G,

$$\int_{\mathbb{R}^3}\begin{bmatrix}1 & v & \frac{|v|^2-3}{2}\end{bmatrix} Q(G,G)\mathrm{d}v = \begin{bmatrix}0 & 0 & 0\end{bmatrix}. \tag{1.3}$$

It is well-known that a global Maxwellian μ satisfies $Q(\mu,\mu)=0$ where

$$\mu(v) := \frac{1}{(2\pi)^{3/2}}\exp\left(-\frac{|v|^2}{2}\right). \tag{1.4}$$

Throughout this paper we assume that Ω is a bounded open subset of $\mathbb{R}^3$ and there exists a C^3 function $\xi:\mathbb{R}^3\to\mathbb{R}$ such that $\Omega=\{x\in\mathbb{R}^3 : \xi(x)<0\}$, and $\partial\Omega=\{x\in\mathbb{R}^3 : \xi(x)=0\}$. Moreover we assume the domain is *strictly convex*:

$$\sum_{i,j}\partial_{ij}\xi(x)\zeta_i\zeta_j \ge C_\xi|\zeta|^2 \text{ for all } \zeta\in\mathbb{R}^3 \text{ and for all } x\in\bar{\Omega}=\Omega\cup\partial\Omega. \tag{1.5}$$

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We assume that

$$\nabla\xi(x)\neq 0 \text{ when } |\xi(x)|\ll 1, \tag{1.6}$$

and we define the outward normal as $n(x)=\frac{\nabla\xi(x)}{|\nabla\xi(x)|}$ at the boundary. The boundary of the phase space $\gamma:=\{(x,v)\in\partial\Omega\times\mathbb{R}^3\}$ can be decomposed as

$$\begin{aligned}
\gamma_- &= \{(x,v)\in\partial\Omega\times\mathbb{R}^3 : n(x)\cdot v<0\}, \quad \text{(the incoming set)},\\
\gamma_+ &= \{(x,v)\in\partial\Omega\times\mathbb{R}^3 : n(x)\cdot v>0\}, \quad \text{(the outcoming set)},\\
\gamma_0 &= \{(x,v)\in\partial\Omega\times\mathbb{R}^3 : n(x)\cdot v=0\}, \quad \text{(the grazing set)}.
\end{aligned} \tag{1.7}$$

In general the boundary condition is imposed only for the incoming set γ_- for general kinetic PDEs. In this paper we consider a so-called specular reflection boundary condition

$$F(t,x,v)=F(t,x,R_xv) \text{ on } (x,v)\in\gamma_-, \text{ where } R_xv:=v-2n(x)(n(x)\cdot v). \tag{1.8}$$

Physically this represents when a gas particle hits the boundary, it bounces back with the opposite normal velocity and the same tangential velocity, just like a billiard ball. Previous studies on the Boltzmann equation with specular reflection boundary conditions can be found in [8, 10, 15–17]. For other important physical boundary conditions, such as the diffuse boundary condition, we refer to [1–4, 8, 10] and the references therein.

Due to the importance of the Boltzmann equation in the mathematical theory and application, there have been explosive research activities in analytic study of the equation. Notably the nonlinear energy method has led to solutions of many open problems including global strong solution of Boltzmann equation coupled with either the Poisson equation or the Maxwell system for electromagnetism when the initial data are close to the Maxwellian μ in periodic box (no boundary). See [7] and the references therein. In many important physical applications, e.g. semiconductor and tokamak, the charged dilute gas is confined within a container, and its interaction with the boundary plays a crucial role both in physics and mathematics.

However, in general, higher regularity may not be expected for solutions of the Boltzmann equation in physical bounded domains. Such a drastic difference of solutions with boundaries had been demonstrated as the formation and propagation of discontinuity in non-convex domains [5, 18], and the non-existence of some second order derivatives at the boundary in convex domains [8]. Evidently the nonlinear energy method is not generally available to the boundary problems. In order to overcome such critical difficulty, Guo developed a L^2-L^∞ framework in [10] to study global solutions of the Boltzmann equation with various boundary conditions. The core of the method lies in a direct approach (without taking derivatives) to achieve a pointwise bound using trajectory of the transport operator, which leads to substantial development in various directions including [5, 6, 8, 9, 14]. In [8], with the aid of some distance function towards the grazing set, the weighted classical C^1 solutions of Boltzmann equation ($E\equiv 0$ in (1.1)) were constructed under various boundary conditions.

In this paper, we extend a result of [8] to the Boltzmann Equation (1.1) with a given external field ($E\neq 0$) satisfying a crucial sign condition on the boundary:

$$E(t,x)\cdot n(x)>C_E>0 \quad \text{for all } t \text{ and all } x\in\partial\Omega. \tag{1.9}$$

One of the major difficulties when dealing with a field $E\neq 0$ is that trajectories are curved and behave in a very complicated way when they hit the boundary.

Let's clarify some notations. For any function $z(x,v):\bar{\Omega}\times\mathbb{R}^3\to\mathbb{R}$, denote

$$\|z\|_\infty = \sup_{(x,v)\in\Omega\times\mathbb{R}^3}|z(x,v)|.$$

And for any function $g(t,x):[0,T]\times\bar{\Omega}\to\mathbb{R}$, denote

$$\|g\|_{L^\infty_{t,x}} = \sup_{(t,x)\in[0,T]\times\bar{\Omega}}|g(t,x)|, \text{ and } \|g\|_{C^n_{t,x}} = \sum_{0\le\alpha+\beta\le n}\sup_{(t,x)\in[0,T]\times\bar{\Omega}}|\partial_t^\alpha\partial_x^\beta g(t,x)|.$$

Our main result is a weighted C^1 estimate for the solution of (1.1) with specular boundary condition (1.8) in a short time. To state the main result, we introduce a distance function $\alpha(t,x,v)$ towards the grazing set γ_0:

$$\alpha(t,x,v)\sim\Big[|v\cdot\nabla\xi(x)|^2+\xi(x)^2-2(v\cdot\nabla^2\xi(x)\cdot v)\xi(x)-2(E(t,\overline{x})\cdot\nabla\xi(\overline{x}))\xi(x)\Big]^{1/2} \tag{1.10}$$

for $x\in\Omega$ close to boundary, where $\overline{x}:=\{\bar{x}\in\partial\Omega:d(x,\bar{x})=d(x,\partial\Omega)\}$ is uniquely defined. The precise definition of α can be found in (3.3). Note that $\alpha|_{\gamma_-}\sim|n(x)\cdot v|$. Similar distance functions towards γ_0 were used in [8, 11, 13]. With the weight α, we establish the main theorem:

THEOREM 1.1 (Weighted C^1 Estimate). *Suppose E satisfies the sign condition (1.9), and*

$$\|E\|_{C^2_{t,x}}<\infty. \tag{1.11}$$

Assume $F_0=\sqrt{\mu}f_0\ge0$, for $2<\beta<3$, $0<\theta<\frac{1}{4}$, and $b>1$,

$$\left\|\frac{\alpha^{\beta-1}}{\langle v\rangle^b}\partial_x f_0\right\|_\infty+\left\|\frac{\alpha^{\beta-2}}{\langle v\rangle^{b-2}}\partial_v f_0\right\|_\infty+\left\|e^{\theta|v|^2}f_0\right\|_\infty<\infty, \tag{1.12}$$

and the compatibility condition

$$f_0(x,v)=f_0(x,R_xv)\quad \text{on } (x,v)\in\gamma_-. \tag{1.13}$$

Then there exists a unique solution $F(t)=\sqrt{\mu}f(t)$ for $0\le t\le T$ with $T\ll1$ to the system (1.1), (1.8) that satisfies, for some $\varpi>0$ big enough, and for all $0\le t\le T$,

$$\begin{aligned}&\left\|e^{-\varpi\langle v\rangle t}\frac{\alpha^\beta}{\langle v\rangle^{b+1}}\partial_x f(t)\right\|_\infty+\left\|e^{-\varpi\langle v\rangle t}\frac{\alpha^{\beta-1}}{\langle v\rangle^{b-1}}\partial_v f(t)\right\|_\infty\\ &\lesssim\left\|\frac{\alpha^{\beta-1}}{\langle v\rangle^b}\partial_x f_0\right\|_\infty+\left\|\frac{\alpha^{\beta-2}}{\langle v\rangle^{b-2}}\partial_v f_0\right\|_\infty+P\left(\left\|e^{\theta|v|^2}f_0\right\|_\infty\right)\end{aligned} \tag{1.14}$$

for some polynomial P. Furthermore, if $f_0\in C^1$, then $f\in C^1$ away from the grazing set γ_0.

The proof of Theorem 1.1 devotes a nontrivial extension of the result in [8]. The idea is to use Duhamel's formula to expand f along the characteristics to the initial data and then take derivatives. To do this, we need to define the generalized characteristics as follows:

DEFINITION 1.1. *For any* $(t,x,v)\in[0,T]\times\Omega\times\mathbb{R}^3$, *let* $(X(s;t,x,v),V(s;t,x,v))$ *denote the characteristics*

$$\frac{d}{ds}\begin{bmatrix}X(s;t,x,v)\\V(s;t,x,v)\end{bmatrix}=\begin{bmatrix}V(s;t,x,v)\\E(s,X(s;t,x,v))\end{bmatrix}\quad \textit{for } 0\le s,t\le T, \tag{1.15}$$

with $(X(t;t,x,v),V(t;t,x,v))=(x,v)$.

We define the backward exit time $t_{\mathbf{b}}(t,x,v)$ *as*

$$t_{\mathbf{b}}(t,x,v):=\sup\{s\ge 0: X(\tau;t,x,v)\in\Omega \ \ \textit{for all } \tau\in(t-s,t)\}. \tag{1.16}$$

Furthermore, we define $x_{\mathbf{b}}(t,x,v):=X(t-t_{\mathbf{b}}(t,x,v);t,x,v)$, *and* $v_{\mathbf{b}}(t,x,v):=V(t-t_{\mathbf{b}}(t,x,v);t,x,v)$.

Now let $(t^0,x^0,v^0)=(t,x,v)$. *We define the specular cycles, for* $\ell\ge 0$,

$$\begin{aligned}&(t^{\ell+1},x^{\ell+1},v^{\ell+1})\\&=(t^\ell-t_{\mathbf{b}}(t^\ell,x^\ell,v^\ell),x_{\mathbf{b}}(t^\ell,x^\ell,v^\ell),v_{\mathbf{b}}(t^\ell,x^\ell,v^\ell)-2n(x^{\ell+1})(v_{\mathbf{b}}(t^\ell,x^\ell,v^\ell)\cdot n(x^{\ell+1}))).\end{aligned}$$

And we define the generalized characteristics as

$$\begin{aligned}X_{\mathbf{cl}}(s;t,x,v)=&\sum_\ell \mathbf{1}_{[t^{\ell+1},t^\ell)}(s)X(s;t^\ell,x^\ell,v^\ell),\quad V_{\mathbf{cl}}(s;t,x,v)\\=&\sum_\ell \mathbf{1}_{[t^{\ell+1},t^\ell)}(s)V(s;t^\ell,x^\ell,v^\ell).\end{aligned} \tag{1.17}$$

The key component of the proof is to estimate the derivatives of the backward trajectory

$$\frac{\partial(X_{\mathbf{cl}}(s;t,x,v),V_{\mathbf{cl}}(s;t,x,v))}{\partial(x,v)}.$$

This is done through the matrix method where we estimate the multiplication of $\ell^*(s;t,x,v)$ many Jacobian matrices

$$\prod_{\ell=0}^{\ell^*(s;t,x,v)}\frac{\partial(t^{\ell+1},x^{\ell+1},v^{\ell+1})}{\partial(t^\ell,x^\ell,v^\ell)}. \tag{1.18}$$

Here $\ell^*(s;t,x,v)$ is the number of bounces it takes for the backward trajectory to reach time s from time t, which can be shown to have order $\ell^*(s;t,x,v)\sim\frac{|t-s||v|}{\alpha(t,x,v)}$. And for each bounce, we can calculate the Jacobian matrix $\frac{\partial(t^{\ell+1},x^{\ell+1},v^{\ell+1})}{\partial(t^\ell,x^\ell,v^\ell)}$ explicitly.

One major difficulty here, comparing to the Boltzmann equation ($E=0$ in (1.1)), is the field E is time dependent, thus the characteristics ODE (1.15) is not autonomous. This results in the fact that the $\frac{\partial(x^{\ell+1},v^{\ell+1})}{\partial t^\ell}$ derivatives in the first column of the matrix $\frac{\partial(t^{\ell+1},x^{\ell+1},v^{\ell+1})}{\partial(t^\ell,x^\ell,v^\ell)}$ are not trivially equal to 0, and need careful analysis.

We estimate (1.18) by diagonalizing each matrix and multiplying them together. Here we point out the importance of the external field E satisfying the regularity assumption

$$\|E(t,x)\|_{C^2_{t,x}}<\infty. \tag{1.19}$$

Without such C^2 regularity of E, it seems that from our analysis the derivatives $\frac{\partial(n(x^{\ell+1})\cdot v^{\ell+1})}{\partial(t^\ell,x^\ell)}$ can only be bounded as $|\frac{\partial(n(x^{\ell+1})\cdot v^{\ell+1})}{\partial(t^\ell,x^\ell)}|\lesssim|t^\ell-t^{\ell+1}|$. And this bound will cause the multiplication of the ℓ^* many eigenvalues of the matrices to behave as

$$\prod_{\ell=0}^{\ell^*(s;t,x,v)} \mathrm{eig}\left|\frac{\partial(t^{\ell+1},x^{\ell+1},v^{\ell+1})}{\partial(t^\ell,x^\ell,v^\ell)}\right|\sim(1+\sqrt{\alpha})^{\ell^*}\sim(1+\sqrt{\alpha})^{\frac{1}{\alpha}}\to\infty,$$

as $\alpha\to0$, where $\alpha=\alpha(t,x,v)$ in (1.10). This blow up will result in the bound on (1.18) becoming too singular and makes it impossible for us to close the estimate. In order to avoid such blow up, we utilize a crucial cancellation property (5.60), and find that as long as the external field E satisfies the regularity assumption (1.19), we can improve the estimate and achieve the bound $|\frac{\partial(n(x^{\ell+1})\cdot v^{\ell+1})}{\partial(t^\ell,x^\ell)}|\lesssim|t^\ell-t^{\ell+1}|^2$. This extra smallness turns out to be just enough to control the accumulation in the many multiplications of eigenvalues:

$$\prod_{\ell=0}^{\ell^*(s;t,x,v)} \mathrm{eig}\left|\frac{\partial(t^{\ell+1},x^{\ell+1},v^{\ell+1})}{\partial(t^\ell,x^\ell,v^\ell)}\right|\sim(1+\alpha)^{\frac{1}{\alpha}}<C. \tag{1.20}$$

With this bound and additional cancellations between two adjacent matrices (5.78), we carefully analyze the multiplications of the matrices and eventually achieve the key estimate in Theorem 5.1.

Let's also address some other important differences when comparing the equation (1.1) with the Boltzmann equation ($E=0$). Because of the presence of the field E and its sign condition (1.9), we can achieve a better bound on the time gap

$$|t^\ell-t^{\ell+1}|\lesssim|n(x^\ell)\cdot v^{\ell+1}|$$

when v is small (5.2). This is because when the velocity is small, the field would always "push" the trajectory back to the boundary in a short time. This fact would eventually allow us to get the bound

$$|\partial_v X_{\mathbf{cl}}(s;t,x,v)|\lesssim\frac{1}{\langle v\rangle}$$

in Theorem 5.1, which does not blow up when $|v|\to0$, and this turns out to be necessary for us to close the estimate.

When taking derivatives to the Duhamel's formula of $f(t,x,v)$ in (6.3), if $E\neq0$, an extra term would come up as (6.4). In order to bound this term we have to additionally estimate the derivatives $\partial_x t^\ell$ and $\partial_v t^\ell$, for any $1\le\ell\le\ell^*$. Those estimates are consequences of the matrix method and are obtained in (5.97) and (5.98):

$$|\partial_x t^\ell|\lesssim\frac{1}{\alpha^2},\quad |\partial_v t^\ell|\lesssim\frac{1}{\alpha}.$$

It's also important to note that in (6.4), we have $|R_{x^\ell}v^\ell-v^\ell|=2|(n(x^\ell)\cdot v^\ell)|\sim\alpha$. Thus the extra regularity we get by multiplying α^β to $\partial_x f$ and $\alpha^{\beta-1}$ to $\partial_v f$ will bound the term as

$$\sum_{1\le\ell\le\ell^*}(\alpha^\beta|\partial_x t^\ell|+\alpha^{\beta-1}|\partial_v t^\ell|)\max_\ell|R_{x^\ell}v^\ell-v^\ell|\lesssim\frac{1}{\alpha}\left(\alpha^\beta\frac{1}{\alpha^2}+\alpha^{\beta-1}\frac{1}{\alpha}\right)\alpha\lesssim\alpha^{\beta-2}<C,$$

as long as $\beta>2$.

2. Local existence and in-flow problems with external fields

In this section we state some standard results which we will need to prove Theorem 1.1. Let $F(t,x,v)=\sqrt{\mu}f(t,x,v)$. Then the corresponding problem to (1.1), (1.8) becomes

$$\partial_t f+v\cdot\nabla_x f+E\cdot\nabla_v f-\frac{v}{2}\cdot Ef=\Gamma_{\text{gain}}\,(f,f)-\nu(\sqrt{\mu}f)f. \tag{2.1}$$

Here (cf. [12])

$$\begin{aligned}\nu(\sqrt{\mu}f)(v)&:=\frac{1}{\sqrt{\mu(v)}}Q_{\text{loss}}(\sqrt{\mu}f,\sqrt{\mu}f)(v)\\&=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-u|^\kappa q_0\big(\frac{v-u}{|v-u|}\cdot\omega\big)\sqrt{\mu(u)}f(u)\mathrm{d}\omega\mathrm{d}u,\end{aligned}\tag{2.2}$$

and the gain term of the nonlinear Boltzmann operator is given by

$$\begin{aligned}\Gamma_{\text{gain}}(f_1,f_2)(v)&:=\frac{1}{\sqrt{\mu}}Q_{\text{gain}}(\sqrt{\mu}f_1,\sqrt{\mu}f_2)(v)\\&=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-u|^\kappa q_0\big(\frac{v-u}{|v-u|}\cdot\omega\big)\sqrt{\mu(u)}f_1(u')f_2(v')\mathrm{d}\omega\mathrm{d}u.\end{aligned}\tag{2.3}$$

And the specular reflection boundary condition in terms of f is

$$f(t,x,v)=f(t,x,R_xv),\quad \text{on } (x,v)\in\gamma_-. \tag{2.4}$$

We first state a local existence result which is standard:

Lemma 2.1 (Local Existence). *Suppose $\|E\|_{L^\infty_{t,x}}<\infty$, and $\|e^{\theta|v|^2}f_0\|_\infty<\infty$, $0<\theta<\frac{1}{4}$. And f_0 satisfies the compatibility condition (1.13). Then there exists $0<T\ll 1$ small enough such that $f\in L^\infty([0,T)\times\Omega\times\mathbb{R}^3)$ solves the Equation (2.1) with specular boundary condition (2.4).*

Proof. Let $f^0=\sqrt{\mu}$. We start with the sequence for $m\geq 0$

$$(\partial_t+v\cdot\nabla_x+E\cdot\nabla_v-\frac{v}{2}\cdot E+\nu(\sqrt{\mu}f^m))f^{m+1}=\Gamma_{\text{gain}}(f^m,f^m), \tag{2.5}$$

with the initial data $f^m(0,x,v)=f_0(x,v)$, and boundary condition for all $(x,v)\in\gamma_-$ be

$$\begin{aligned}f^1(t,x,v)&=f_0(x,R_xv),\\f^{m+1}(t,x,v)&=f^m(t,x,R_xv),m\geq 1.\end{aligned}\tag{2.6}$$

Then (see Lemma 7 in [8], for example)

$$\sup_m\sup_{0\leq t\leq T}\|e^{\theta'|v|^2}f^m(t)\|_\infty\lesssim\|e^{\theta|v|^2}f_0\|_\infty<\infty, \tag{2.7}$$

where $\theta'=\theta-T$. From (2.7) we have, up to a subsequence, the weak-$*$ convergence:

$$e^{\theta'|v|^2}f^m(t,x,v)\overset{*}{\rightharpoonup}e^{\theta'|v|^2}f(t,x,v) \tag{2.8}$$

in $L^\infty([0,T)\times\Omega\times\mathbb{R}^3)\cap L^\infty([0,T)\times\gamma)$ for some f. And it's easy to show f is the solution of (2.1) with specular boundary condition (2.4). ∎

We need some bound on the derivatives of the nonlocal term:

LEMMA 2.2. *Let* $[Y,W]=[Y(x,v),W(x,v)]\in\Omega\times\mathbb{R}^3$. *For* $0<\theta<\frac{1}{4}$ *and* $\partial_{\mathbf{e}}\in\{\partial_t,\nabla_x,\nabla_v\}$,

$$\begin{aligned}&|\partial_{\mathbf{e}}\Gamma_{\text{gain}}(g,g)(Y,W)|\\ &\lesssim |\partial_{\mathbf{e}}Y|||e^{\theta|v|^2}g||_\infty\int_{\mathbb{R}^3}\frac{e^{-C_\theta|u-W|^2}}{|u-W|^{2-\kappa}}|\nabla_x g(Y,u)|\mathrm{d}u\\ &\quad+|\partial_{\mathbf{e}}W|||e^{\theta|v|^2}g||_\infty\int_{\mathbb{R}^3}\frac{e^{-C_\theta|u-W|^2}}{|u-W|^{2-\kappa}}|\nabla_v g(Y,u)|\mathrm{d}u+\langle v\rangle^\kappa e^{-\theta|v|^2}|\partial_{\mathbf{e}}W|||e^{\theta|v|^2}g||_\infty^2.\end{aligned}$$

Proof. See [8]. ∎

We need a result for the corresponding inflow problem to (2.1). Consider

$$\partial_t f+v\cdot\nabla_x f+E\cdot\nabla_v f+\nu f=H, \tag{2.9}$$

where $H=H(t,x,v)$ and $\nu=\nu(t,x,v)$ are given. Let $\tau_1(x)$ and $\tau_2(x)$ be the unit tangential vectors to $\partial\Omega$ satisfying

$$\tau_1(x)\cdot n(x)=0=\tau_2(x)\cdot n(x)\text{ and }\tau_1(x)\times\tau_2(x)=n(x). \tag{2.10}$$

And let $\partial_{\tau_i}g$ be the tangential derivative at direction τ_i for g defined on $\partial\Omega$. Define

$$\nabla_x g=\sum_{i=1}^2\tau_i\partial_{\tau_i}g-\frac{n}{n\cdot v_{\mathbf{b}}}\Big\{\partial_t g+\sum_{i=1}^2(v_{\mathbf{b}}\cdot\tau_i)\partial_{\tau_i}g+\nu g-H+E\cdot\nabla_v g\Big\}. \tag{2.11}$$

PROPOSITION 2.1. *Assume the compatibility condition*

$$f_0(x,v)=g(0,x,v)\quad\textit{for}\quad(x,v)\in\gamma_-.$$

Let $p\in[1,\infty)$ *and* $0<\theta<1/4$. $|\nu(t,x,v)|\lesssim\langle v\rangle$. $\|E\|_{L^\infty_{t,x}}+\|\nabla_x E\|_{L^\infty_{t,x}}<\infty$.

Assume

$$\begin{aligned}&\nabla_x f_0,\nabla_v f_0,\in L^p(\Omega\times\mathbb{R}^3),\\ &\nabla_v g,\partial_{\tau_i}g\in L^p([0,T]\times\gamma_-),\\ &\frac{n(x)}{n(x)\cdot v}\Big\{\partial_t g+\sum_{i=1}^2(v\cdot\tau_i)\partial_{\tau_i}g+\nu g-H+E\cdot\nabla_v g\Big\}\in L^p([0,T]\times\gamma_-),\\ &\frac{n(x)\cdot\iint\partial_x E}{n(x)\cdot v}\Big\{\partial_t g+\sum_{i=1}^2(v\cdot\tau_i)\partial_{\tau_i}g-\nu g+H\Big\}\in L^p([0,T]\times\gamma_-),\\ &\nabla_x H,\nabla_v H\in L^p([0,T]\times\Omega\times\mathbb{R}^3),\\ &e^{-\theta|v|^2}\nabla_x\nu,e^{-\theta|v|^2}\nabla_v\nu\in L^p([0,T]\times\Omega\times\mathbb{R}^3),\\ &e^{\theta|v|^2}f_0\in L^\infty(\Omega\times\mathbb{R}^3),e^{\theta|v|^2}g\in L^\infty([0,T]\times\gamma_-),\\ &e^{\theta|v|^2}H\in L^\infty([0,T]\times\Omega\times\mathbb{R}^3).\end{aligned}$$

Then for any $T>0$, *there exists a unique solution* f *to* (2.9), *such that* $f,\partial_t,\nabla_x f,\nabla_v f\in C^0([0,T];L^p(\Omega\times\mathbb{R}^3))$ *and their traces satisfy*

$$\begin{aligned}\nabla_v f|_{\gamma_-}&=\nabla_v g,\nabla_x f|_{\gamma_-}=\nabla_x g,\quad\textit{on}\quad\gamma_-,\\ \nabla_x f(0,x,v)&=\nabla_x f_0,\nabla_v f(0,x,v)=\nabla_v f_0,\quad\textit{in}\quad\Omega\times\mathbb{R}^3,\\ \partial_t f(0,x,v)&=\partial_t f_0,\quad\textit{in}\quad\Omega\times\mathbb{R}^3,\end{aligned} \tag{2.12}$$

where $\nabla_x g$ is given by (2.11).

Proof. *See [3].* □

3. Velocity lemma and the nonlocal to local estimate

Recall the definition of specular trajectories in (1.17). In this section we prove some properties of the specular trajectories which are crucial in order to establish the main result.

Let's give the precise definition for the weight function α. We first need a cutoff function: for any $\epsilon > 0$, let $\chi_\epsilon : [0,\infty) \to [0,\infty)$ be a smooth function satisfying:

$$\begin{aligned} &\chi_\epsilon(x) = x \text{ for } 0 \le x \le \frac{\epsilon}{4}, \\ &\chi_\epsilon(x) = C_\epsilon \text{ for } x \ge \frac{\epsilon}{2}, \\ &\chi_\epsilon(x) \text{ is increasing for } \frac{\epsilon}{4} < x < \frac{\epsilon}{2}, \\ &\chi'_\epsilon(x) \le 1. \end{aligned} \tag{3.1}$$

Let $d(x,\partial\Omega) := \inf_{y\in\partial\Omega} \|x-y\|$. And for any $\delta > 0$, let

$$\Omega^\delta := \{x \in \Omega : d(x,\partial\Omega) < \delta\}.$$

Since $\partial\Omega$ is C^2, we claim that if $\delta \ll 1$ is small enough we have:

$$\begin{aligned} &\text{for any } x \in \Omega^\delta \text{ there exists a unique } \bar{x} \in \partial\Omega \text{ such that } d(x,\bar{x}) = d(x,\partial\Omega), \\ &\text{moreover } \sup_{x\in\Omega^\delta} |\nabla_x \bar{x}| < \infty. \end{aligned} \tag{3.2}$$

To prove the claim, we have by (1.6) WLOG locally, we can assume η takes the form $\eta(x_\parallel) = (x_{\parallel,1}, x_{\parallel,2}, \bar{\eta}(x_{\parallel,1}, x_{\parallel,2}))$, and $\bar{x} = \eta(\bar{x}_\parallel) = (\bar{x}_{\parallel,1}, \bar{x}_{\parallel,2}, \bar{\eta}(\bar{x}_{\parallel,1}, \bar{x}_{\parallel,2}))$. Denote $\partial_i \bar{\eta} = \frac{\partial}{\partial x_{\parallel,i}} \bar{\eta}(x_{\parallel,1}, x_{\parallel,2})$, and $\partial_{i,j}\bar{\eta} = \frac{\partial^2}{\partial x_{\parallel,i}\partial x_{\parallel,j}} \bar{\eta}(x_{\parallel,1}, x_{\parallel,2})$. Now since $|\eta(\bar{x}_\parallel) - x|^2 = \inf_{y\in\partial\Omega} |y - x|^2$, $\bar{x}_\parallel$ satisfies

$$\omega(x_1,x_2,x_3,\bar{x}_{\parallel,1},\bar{x}_{\parallel,2}) = \begin{bmatrix} (\bar{x}_{\parallel,1} - x_1) + (\bar{\eta}(\bar{x}_{\parallel,1},\bar{x}_{\parallel,2}) - x_3)\partial_1\bar{\eta}(\bar{x}_{\parallel,1},\bar{x}_{\parallel,2}) \\ (\bar{x}_{\parallel,2} - x_2) + (\bar{\eta}(\bar{x}_{\parallel,1},\bar{x}_{\parallel,2}) - x_3)\partial_2\bar{\eta}(\bar{x}_{\parallel,1},\bar{x}_{\parallel,2}) \end{bmatrix} = 0.$$

Since

$$\begin{aligned} \det(\frac{\partial\omega}{\partial x_\parallel}) &= \det \begin{bmatrix} 1 + (\partial_1\bar{\eta})^2 + (\bar{\eta} - x_3)\partial_{1,1}\bar{\eta} & \partial_2\bar{\eta}\partial_1\bar{\eta} + (\bar{\eta} - x_3)\partial_{1,2}\bar{\eta} \\ \partial_1\bar{\eta}\partial_2\bar{\eta} + (\bar{\eta} - x_3)\partial_{1,2}\bar{\eta} & 1 + (\partial_2\bar{\eta})^2 + (\bar{\eta} - x_3)\partial_{1,2}\bar{\eta} \end{bmatrix} \\ &= (1 + (\partial_1\bar{\eta})^2)(1 + (\partial_2\bar{\eta})^2) - (\partial_1\bar{\eta}\partial_2\bar{\eta})^2 + O(|\bar{\eta} - x_3|) \\ &= 1 + (\partial_1\bar{\eta})^2 + (\partial_2\bar{\eta})^2 + O(|\bar{\eta} - x_3|) > 0, \end{aligned}$$

if $|\bar{\eta}(x_\parallel) - x_3|$ is small enough. By the implicit function theorem $(\bar{x}_{\parallel,1}, \bar{x}_{\parallel,2})$ are functions of x_1, x_2, x_3 if x is close enough to $\partial\Omega$.

Moreover,

$$\begin{aligned} \frac{\partial \bar{x}_\parallel}{\partial x_j} &= -(\frac{\partial\omega}{\partial \bar{x}_\parallel})^{-1} \cdot \frac{\partial\omega}{\partial x_j} \\ &= \frac{1}{\det(\frac{\partial\omega}{\partial\bar{x}_\parallel})} \begin{bmatrix} 1 + (\partial_2\bar{\eta})^2 + (\bar{\eta} - x_3)\partial_{1,2}\bar{\eta} & -\partial_2\bar{\eta}\partial_1\bar{\eta} - (\bar{\eta} - x_3)\partial_{1,2}\bar{\eta} \\ -\partial_1\bar{\eta}\partial_2\bar{\eta} - (\bar{\eta} - x_3)\partial_{1,2}\bar{\eta} & 1 + (\partial_1\bar{\eta})^2 + (\bar{\eta} - x_3)\partial_{1,1}\bar{\eta} \end{bmatrix} \cdot \frac{\partial\omega}{\partial x_j} \end{aligned}$$

is bounded as $\frac{\partial \omega}{\partial x_j}$ is bounded and $\det(\frac{\partial \omega}{\partial \bar{x}})$ is bounded from below if x is close enough to the boundary. Therefore $|\nabla_x \bar{x}|$ is bounded. This proves (3.2).

Now define

$$\beta(t,x,v)=\left[|v\cdot\nabla\xi(x)|^2+\xi(x)^2-2(v\cdot\nabla^2\xi(x)\cdot v)\xi(x)-2(E(t,\bar{x})\cdot\nabla\xi(\bar{x}))\xi(x)\right]^{1/2},$$

for all $(x,v)\in\Omega^\delta\times\mathbb{R}^3$. Let $\delta':=\min\{|\xi(x)|:x\in\Omega,d(x,\partial\Omega)=\delta\}$, and let $\chi_{\delta'}$ be a smooth cutoff function satisfying (3.1), then define

$$\alpha(t,x,v):=\begin{cases}(\chi_{\delta'}(\beta(t,x,v))) & x\in\Omega^\delta,\\ C_{\delta'} & x\in\Omega\setminus\Omega^\delta.\end{cases} \tag{3.3}$$

The following lemmas about α are important for our estimate:

LEMMA 3.1 (Velocity lemma near boundary). *Suppose $E(t,x)$ satisfies $\|E\|_{C^1}<\infty$ and the sign condition (1.9). Then α is continuous, and for $\delta\ll 1$ small enough, we have for any $(t,x,v)\in[0,T]\times\Omega\times\mathbb{R}^3$, and $0\le s<t$, α satisfies*

$$e^{-C\int_s^t(|V_{\mathbf{cl}}(\tau')|+1)d\tau'}\alpha(s,X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))\le\alpha(t,x,v)\le e^{C\int_s^t(|V_{\mathbf{cl}}(\tau')|+1)d\tau'}\alpha(s,X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s)), \tag{3.4}$$

for any $C\ge\frac{C_\xi(\|E\|_{L^\infty_{t,x}}+\|\nabla E\|_{L^\infty_{t,x}}+\|\partial_t E\|_{L^\infty_{t,x}}+1)}{C_E}$, where $C_\xi>0$ is a large constant depending only on ξ. Here $(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))=(X_{\mathbf{cl}}(s;t,x,v),V_{\mathbf{cl}}(s;t,x,v))$ as defined in (1.17).

Similar estimates have been used in [11] and then in [8,13].

Proof. See [3]. ◻

LEMMA 3.2. *Suppose E satisfies (1.9), then for any $y\in\bar{\Omega}$, $1<\beta<3$, $0<\kappa\le 1$, and $\theta>0$ we have*

$$\int_{\mathbb{R}^3}\frac{e^{-\theta|v-u|^2}}{|v-u|^{2-\kappa}[\alpha(s,y,u)]^\beta}du\le C\left(\frac{1}{(|v|^2\xi(y)+c(y))^{\frac{\beta-1}{2}}}+1\right), \tag{3.5}$$

where $c(y)=\xi(y)^2-C_E\xi(y)$.

Proof. See [3]. ◻

LEMMA 3.3.

(1) *Let $(t,x,v)\in[0,T]\times\Omega\times\mathbb{R}^3$, $1<\beta<3$, $0<\kappa\le 1$. Suppose E satisfies (1.9) and (1.11), then for $\varpi\gg 1$ large enough, we have for any $0<\delta\ll 1$,*

$$\begin{aligned}&\int_{\max\{0,t-t_{\mathbf{b}}\}}^t\int_{\mathbb{R}^3}e^{-\int_s^t\frac{\varpi}{2}\langle V(\tau;t,x,v)\rangle d\tau}\frac{e^{-\frac{C_\theta}{2}|V(s)-u|^2}}{|V(s)-u|^{2-\kappa}}\frac{1}{(\alpha(s,X(s),u))^\beta}duds\\ &\lesssim e^{2C_\xi\frac{\|\nabla E\|_\infty+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}}{C_E}}\frac{\delta^{\frac{3-\beta}{2}}}{\langle v\rangle^2(C_E+1)^{\frac{\beta-1}{2}}(\alpha(t,x,v))^{\beta-2}(\|E\|^2_{L^\infty_{t,x}}+1)^{\frac{3-\beta}{2}}}\\ &\quad+\frac{(\|E\|^2_{L^\infty_{t,x}}+1)^{\beta-1}}{C_E^{\beta-1}\delta^{\beta-1}(\alpha(t,x,v))^{\beta-1}}\frac{2}{\varpi},\end{aligned} \tag{3.6}$$

where $(X(s),V(s))=(X(s;t,x,v),V(s;t,x,v))$ as in (1.15).

(2) *Let* $[X_{\mathbf{cl}}(s;t,x,v),V_{\mathbf{cl}}(s;t,x,v)]$ *be the specular backward trajectory as in* (1.17). *Let* $Z(s,x,v)\geq 0$ *be any bounded non-negative function in the phase space.*

For any $\varepsilon>0$, *there exists* $l\gg 1$ *such that for any* $r>0$,

$$\int_0^t\int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)}\frac{e^{-\theta|V_{\mathbf{cl}}(s;t,x,v)-u|^2}}{|V_{\mathbf{cl}}(s;t,x,v)-u|^{2-\kappa}}\frac{\langle u\rangle^r}{\langle v\rangle^r}\frac{Z(s,x,v)}{\big[\alpha(s,X_{\mathbf{cl}}(s;t,x,v),u)\big]^{\beta}}\mathrm{d}u\mathrm{d}s$$
$$\lesssim \frac{O(\varepsilon)}{\langle v\rangle\big[\alpha(t,x,v)\big]^{\beta-1}}\sup_{0\leq s\leq t}\big\{e^{-\frac{l}{2}\langle v\rangle(t-s)}Z(s,x,v)\big\}. \tag{3.7}$$

Proof. **(Proof of (1) Lemma 3.3.)** The proof is similar to the proof of Lemma 11 in [3], but with some modifications made in order to achieve (3.7) later. We separate the proof into several cases.

In *Step 1, Step 2, Step 3* we prove (3.6) for the case when $x\in\partial\Omega$ and $t\leq t_{\mathbf{b}}$.

In *Step 4* we prove (3.6) for the case when $x\in\partial\Omega$ and $t>t_{\mathbf{b}}$.

In *Step 5* we prove (3.6) for the case when $x\in\Omega$ and $t\leq t_{\mathbf{b}}$.

In *Step 6* we prove (3.6) for the case when $x\in\Omega$ and $t>t_{\mathbf{b}}$.

Step 1. Let's first start with the case $t\geq t_{\mathbf{b}}$ and prove (3.6), Let's shift the time variable: $s\mapsto t-t_{\mathbf{b}}+s$, and let $\tilde{X}(s)=X(t-t_{\mathbf{b}}+s)$, $\tilde{V}(s)=V(t-t_{\mathbf{b}}+s)$. Then $s\in[0,t_{\mathbf{b}}]$ and from (3.5) we only need to bound the integral

$$\int_0^{t_{\mathbf{b}}} e^{-\int_{t-t_{\mathbf{b}}+s}^t\frac{\varpi}{2}\langle V(\tau;t,x,v)\rangle d\tau}\frac{1}{\Big[|\tilde{V}(s)|^2\xi(\tilde{X}(s))+\xi^2(\tilde{X}(s))-C_E\xi(\tilde{X}(s))\Big]^{\frac{\beta-1}{2}}}ds. \tag{3.8}$$

Let's assume $x\in\partial\Omega$ and $v\cdot\nabla\xi(x)>0$. Then by the velocity lemma (Lemma 3.1) we have $v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})<0$.

Claim: for any $0<\delta\ll 1$ small enough, if we let

$$\sigma_1=\delta\frac{v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})}{|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1},\ \text{and}\ \sigma_2=\delta\frac{v\cdot\nabla\xi(x)}{|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1}, \tag{3.9}$$

then $|\xi(\tilde{X}(s)|$ is monotonically increasing on $[0,\sigma_1]$, and monotonically decreasing on $[t_{\mathbf{b}}-\sigma_2,t_{\mathbf{b}}]$. Moreover, we have the following bounds:

$$\begin{aligned}|\xi(\tilde{X}(\sigma_1))|&\geq\frac{\delta(v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}}))^2}{2(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)},\\ |\xi(\tilde{X}(\sigma_2))|&\geq\frac{\delta(v\cdot\nabla\xi(x))^2}{2(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)},\end{aligned} \tag{3.10}$$

$$\begin{aligned}|\xi(\tilde{X}(s)|&\leq\frac{3\delta(v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}}))^2}{2(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)},s\in[0,\sigma_1],\\ |\xi(\tilde{X}(s)|&\leq\frac{3\delta(v\cdot\nabla\xi(x))^2}{2(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)},s\in[t_{\mathbf{b}}-\sigma_2,t_{\mathbf{b}}],\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}|\tilde{V}(s)\cdot\nabla\xi(\tilde{X}(s))|&\geq\frac{|v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})|}{2},s\in[0,\sigma_1],\\ |\tilde{V}(s)\cdot\nabla\xi(\tilde{X}(s))|&\geq\frac{|v\cdot\nabla\xi(x)|}{2},s\in[t_{\mathbf{b}}-\sigma_2,t_{\mathbf{b}}].\end{aligned} \tag{3.12}$$

To prove the claim we first note that $\frac{d}{ds}\xi(\tilde{X}(s))|_{s=0}=v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})<0$, and

$$\begin{aligned}\frac{d^2}{d^2s}\xi(\tilde{X}(s)))=&\frac{d}{ds}(\tilde{V}(s)\cdot\nabla\xi(\tilde{X}(s)))\\ =&\tilde{V}(s)\cdot\nabla^2\xi(\tilde{X}(s))\cdot\tilde{V}(s)+E(s,\tilde{X}(s))\cdot\nabla\xi(\tilde{X}(s))\\ \le& C(|\tilde{V}(s)|^2+\|E\|_{L^\infty_{t,x}})\\ \le& C(2|v|^2+2(t_{\mathbf{b}}\|E\|_{L^\infty_{t,x}})^2+\|E\|_{L^\infty_{t,x}})\\ \le& C_1(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1),\end{aligned}\tag{3.13}$$

for some $C_1>0$. Thus if δ small enough, we have $\frac{d}{ds}\xi(\tilde{X}(s))<0$ for all $s\in[0,\delta\frac{|v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})|}{|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1}]$. Therefore $\xi(\tilde{X}(s))$ is decreasing on $[0,\sigma_1]$.

Similarly $\frac{d}{ds}\xi(\tilde{X}(s))|_{s=t_{\mathbf{b}}}=v\cdot\nabla\xi(x)>0$, and since $|\frac{d^2}{d^2s}\xi(\tilde{X}(s)))|\lesssim(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)$ we have that $\frac{d}{ds}\xi(\tilde{X}(s))>0$ for all $s\in[t_{\mathbf{b}}-\delta\frac{|v\cdot\nabla\xi(v)|}{|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1},t_{\mathbf{b}}]$ if δ small enough. Therefore $\xi(\tilde{X}(s))$ is increasing on $[t_{\mathbf{b}}-\sigma_2,t_{\mathbf{b}}]$.

Next we establish the bounds (3.10), (3.11), and (3.12). By (3.13), we have

$$\begin{aligned}|\xi(\tilde{X}(\sigma_1))|=&\int_0^{\sigma_1}-\tilde{V}(s)\cdot\nabla\xi(\tilde{X}(s))ds\\ =&\int_0^{\sigma_1}\left(\int_0^s-\frac{d}{d\tau}(\tilde{V}(\tau)\cdot\nabla\xi(\tilde{X}(\tau)))d\tau-v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})\right)ds\\ \ge&\int_0^{\sigma_1}\left(|v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})|-C_1(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)s\right)ds\\ =&\sigma_1|v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})|-\frac{\sigma_1^2}{2}C_1(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)\\ =&\sigma_1\left(|v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})|-\frac{\delta C_1}{2}|v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})|\right)\\ \ge&\frac{\sigma_1}{2}|v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})|=\frac{\delta(v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}}))^2}{2(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)}.\end{aligned}$$

And by the same argument we have $|\xi(\tilde{X}(\sigma_2))|\ge\frac{\delta(v\cdot\nabla\xi(x))^2}{2(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)}$ for $\delta\ll1$. This proves (3.10).

To prove (3.11), we have from (3.13), for $s\in[0,\sigma_1]$,

$$\begin{aligned}|\xi(\tilde{X}(s)|\le& s\left(|v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})|+\frac{\delta C_1}{2}|v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})|\right)\\ \le&\frac{3s}{2}|v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})|\le\frac{3\delta(v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}}))^2}{2(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)},\end{aligned}$$

and $|\xi(\tilde{X}(s)|\le\frac{3\delta(v\cdot\nabla\xi(x))^2}{2(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)}$ for $s\in[t_{\mathbf{b}}-\sigma_2,t_{\mathbf{b}}]$. This proves (3.11).

Finally for (3.12), again from (3.13),

$$\begin{aligned}|\tilde{V}(s)\cdot\nabla\xi(\tilde{X}(s))|\ge&|v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})|-\int_0^{\sigma_1}C_1(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)ds\\ \ge&|v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})|-C_1\delta|v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})|\ge\frac{|v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})|}{2}.\end{aligned}$$

And similarly $|\tilde{V}(s)\cdot\nabla\xi(\tilde{X}(s))|\geq \frac{|v\cdot\nabla\xi(x)|}{2}$ for $s\in[t_{\mathbf{b}}-\delta_2,t_{\mathbf{b}}]$. This proves the claim.

Step 2. Recall the definition of σ_1,σ_2 in (3.9), and C_E in (1.9). In this step we establish the lower bound:

$$|\xi(\tilde{X}(s))|>\frac{C_E}{10}(\sigma_2)^2, \text{for all } s\in[\sigma_1,t_{\mathbf{b}}-\sigma_2]. \tag{3.14}$$

Suppose towards contradiction that $I:=\{s\in[\sigma_1,t_{\mathbf{b}}-\sigma_2]:|\xi(\tilde{X}(s))|\leq\frac{C_E}{10}(\sigma_2)^2\}\neq\emptyset$. Then from (3.4) and (3.10) we have

$$\begin{aligned}\frac{C_E}{10}(\sigma_2)^2\leq&\delta^2\frac{C_E}{10}\frac{(v\cdot\nabla\xi(x))^2}{|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1}\\ \leq&\delta^2\frac{C_E}{10}e^{C_\xi\frac{\|\nabla E\|_{L^\infty_{t,x}}+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}}{C_E}}\frac{(v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}}))^2}{|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1}\\ \leq&2\delta\frac{C_E}{10}e^{C_\xi\frac{\|\nabla E\|_{L^\infty_{t,x}}+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}}{C_E}}|\xi(\tilde{X}(\sigma_1))|\\ <&|\xi(\tilde{X}(\sigma_1))|,\end{aligned}$$

if $\delta\ll 1$. So $\sigma_1\notin I$. Let $s^*:=\min\{s\in I\}$ be the minimum of such s. Then clearly

$$\frac{d}{ds}\xi(\tilde{X}(s))|_{s=s^*}=\tilde{V}(s^*)\cdot\nabla\xi(\tilde{X}(s^*))\geq 0.$$

Now expanding around $\tilde{X}(s)$, we have

$$E(s,\tilde{X}(s))\cdot\nabla\xi(\tilde{X}(s))=E(s,\overline{\tilde{X}(s)})\cdot\nabla\xi(\overline{\tilde{X}(s)})+c(\tilde{X}(s))\cdot\xi(\tilde{X}(s)), \tag{3.15}$$

with $|c(\tilde{X}(s))|<\frac{C_\xi(\|E\|_{L^\infty_{t,x}}+\|\nabla E\|_{L^\infty_{t,x}})}{C_E}$. Thus

$$\begin{aligned}&\frac{d}{ds}(\tilde{V}(s)\cdot\nabla\xi(\tilde{X}(s)))\\ &=\tilde{V}(s)\cdot\nabla^2\xi(\tilde{X}(s))\cdot\tilde{V}(s)+E(s,\tilde{X}(s))\cdot\nabla\xi(\tilde{X}(s))\\ &=\tilde{V}(s)\cdot\nabla^2\xi(\tilde{X}(s))\cdot\tilde{V}(s)+E(s,\overline{\tilde{X}(s)})\cdot\nabla\xi(\overline{\tilde{X}(s)})+c(\tilde{X}(s))\cdot\xi(\tilde{X}(s))\\ &\geq C_E-\frac{C_\xi(\|E\|_{L^\infty_{t,x}}+\|\nabla E\|_{L^\infty_{t,x}})}{C_E}|\xi(\tilde{X}(s))|,\end{aligned} \tag{3.16}$$

so

$$\begin{aligned}&\frac{d}{ds}(\tilde{V}(s)\cdot\nabla\xi(\tilde{X}(s)))|_{s=s^*}\\ \geq&C_E-\delta^2\frac{C_\xi(\|E\|_{L^\infty_{t,x}}+\|\nabla E\|_{L^\infty_{t,x}})}{C_E}\frac{C_E}{10}\frac{(v\cdot\nabla\xi(x))^2}{|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1}\geq\frac{C_E}{2},\end{aligned}$$

for $\delta\ll 1$ small enough. Then we have $\frac{d}{ds}(\tilde{V}(s)\cdot\nabla\xi(\tilde{X}(s)))$ is increasing on the interval $[s^*,t_{\mathbf{b}}]$ as $|\xi(\tilde{X}(s))|$ is decreasing. So

$$\frac{d}{ds}(\tilde{V}(s)\cdot\nabla\xi(\tilde{X}(s)))\geq\frac{C_E}{2},\quad s\in[s^*,t_{\mathbf{b}}].$$

And therefore

$$
\begin{aligned}
|\xi(\tilde{X}(s^*))| &= \int_{s^*}^{t_{\mathbf{b}}} \tilde{V}(s)\cdot\nabla\xi(\tilde{X}(s))ds \\
&= \int_{s^*}^{t_{\mathbf{b}}} \left(\int_{s^*}^{s} \frac{d}{d\tau}(\tilde{V}(\tau)\cdot\nabla\xi(\tilde{X}(\tau)))d\tau + \tilde{V}(s^*)\cdot\nabla\xi(\tilde{X}(s^*)) \right) ds \\
&\geq \int_{s^*}^{t_{\mathbf{b}}} (s-s^*)\frac{C_E}{2}ds = \frac{C_E}{4}(t_{\mathbf{b}}-s^*)^2 \geq \frac{C_E}{4}(\sigma_2)^2,
\end{aligned}
$$

which is a contradiction. Therefore we conclude (3.14).

Step 3. Let's split the time integration (3.8) as

$$
\begin{aligned}
&\int_0^{t_{\mathbf{b}}} e^{-\int_{t-t_{\mathbf{b}}+s}^{t} \frac{\varpi}{2}\langle V(\tau;t,x,v)\rangle d\tau} \frac{1}{\left[|\tilde{V}(s)|^2\xi(\tilde{X}(s)) + \xi^2(\tilde{X}(s) - C_E\xi(\tilde{X}(s))\right]^{\frac{\beta-1}{2}}} ds \\
&= \int_0^{\sigma_1} + \int_{\sigma_1}^{t_{\mathbf{b}}-\sigma_2} + \int_{t_{\mathbf{b}}-\sigma_2}^{t_{\mathbf{b}}} = (\mathbf{I}) + (\mathbf{II}) + (\mathbf{III}). \qquad (3.17)
\end{aligned}
$$

Let's first estimate $(\mathbf{I})$, $(\mathbf{III})$:

From *Step 2* we have that $|\xi(\tilde{X}(s)|$ is monotonically increasing on $[0,\sigma_1]$ and $[t_{\mathbf{b}}-\sigma_2, t_{\mathbf{b}}]$, so we have the change of variables:

$$
ds = \frac{d|\xi|}{|\tilde{V}(s)\cdot\nabla\xi(\tilde{X}(s))|}.
$$

Using this change of variable and the bounds (3.11), (3.12), and $|\tilde{V}(s)|^2+1 \gtrsim |v|^2+1$, $(\mathbf{I})$ is bounded by

$$
\begin{aligned}
(\mathbf{I}) &\leq \int_0^{\sigma_1} \frac{1}{\left[|\tilde{V}(s)|^2\xi(\tilde{X}(s)) + \xi^2(\tilde{X}(s) - C_E\xi(\tilde{X}(s))\right]^{\frac{\beta-1}{2}}} ds \\
&\leq \int_0^{\frac{3\delta(v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}}))^2}{2(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)}} \frac{1}{|\tilde{V}(s)\cdot\nabla\xi(\tilde{X}(s))|((C_E+|\tilde{V}(s)|)|\xi|)^{\frac{\beta-1}{2}}} d|\xi| \\
&\lesssim \int_0^{\frac{3\delta(v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}}))^2}{2(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)}} \frac{2}{|v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})|((C_E+|v|^2)|\xi|)^{\frac{\beta-1}{2}}} d|\xi| \\
&= \frac{2}{|v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})|(C_E+|v|^2)^{\frac{\beta-1}{2}}} \left[|\xi|^{\frac{3-\beta}{2}}\right]_0^{\frac{3\delta(v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}}))^2}{2(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)}} \\
&= \frac{2^{\frac{\beta-1}{2}}\delta^{\frac{3-\beta}{2}}}{(C_E+|v|^2)^{\frac{\beta-1}{2}}|v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}})|^{\beta-2}(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)^{\frac{3-\beta}{2}}} \\
&\lesssim \frac{e^{2C_\xi\frac{\|\nabla E\|_{L^\infty_{t,x}}+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}}{C_E}}\delta^{\frac{3-\beta}{2}}}{(C_E+|v|^2)^{\frac{\beta-1}{2}}(\alpha(t,x,v))^{\beta-2}(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)^{\frac{3-\beta}{2}}} \\
&\lesssim e^{2C_\xi\frac{\|\nabla E\|_{L^\infty_{t,x}}+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}}{C_E}} \frac{\delta^{\frac{3-\beta}{2}}}{\langle v\rangle^2(C_E+1)^{\frac{\beta-1}{2}}(\alpha(t,x,v))^{\beta-2}(\|E\|^2_{L^\infty_{t,x}}+1)^{\frac{3-\beta}{2}}}.
\end{aligned}
\qquad (3.18)
$$

And by the same computation we get

$$(\mathbf{III}) \lesssim e^{2C_\xi \frac{\|\nabla E\|_{L^\infty_{t,x}} + \|E\|^2_{L^\infty_{t,x}} + \|E\|_{L^\infty_{t,x}}}{C_E}} \frac{\delta^{\frac{3-\beta}{2}}}{\langle v\rangle^2 (C_E+1)^{\frac{\beta-1}{2}} (\alpha(t,x,v))^{\beta-2} (\|E\|^2_{L^\infty_{t,x}}+1)^{\frac{3-\beta}{2}}}. \tag{3.19}$$

Finally for (**II**), using the lower bound for $|\xi(\tilde{X}(s))|$ in (3.14), we have

$$\begin{aligned}
(\mathbf{II}) &= \int_{\sigma_1}^{\sigma_2} e^{-\int_{t-t_{\mathbf{b}}+s}^{t} \frac{\varpi}{2}\langle V(\tau;t,x,v)\rangle d\tau} \frac{1}{\left[|\tilde{V}(s)|^2 \xi(\tilde{X}(s)) + \xi^2(\tilde{X}(s) - C_E \xi(\tilde{X}(s))\right]^{\frac{\beta-1}{2}}} ds \\
&\leq \int_0^{t_{\mathbf{b}}} e^{-\int_{t-t_{\mathbf{b}}+s}^{t} \frac{\varpi}{2}\langle V(\tau;t,x,v)\rangle d\tau} \frac{1}{|(|\tilde{V}(s)|^2 + C_E)\xi(\tilde{X}(s))|^{\frac{\beta-1}{2}}} ds \\
&\lesssim \frac{1}{C_E^{\beta-1} (\langle v\rangle \sigma_2)^{\beta-1}} \int_0^{t_{\mathbf{b}}} e^{\int_{t-t_{\mathbf{b}}+s}^{t} \frac{\varpi}{2} d\tau} ds \\
&\lesssim \frac{(|v| + \|E\|_{L^\infty_{t,x}} + \|E\|^2_{L^\infty_{t,x}} + 1)^{\beta-1}}{C_E^{\beta-1} \langle v\rangle^{\beta-1} \delta^{\beta-1} (\alpha(t,x,v))^{\beta-1}} \int_0^{t_{\mathbf{b}}} e^{(s-t_{\mathbf{b}})\frac{\varpi}{2}} ds \\
&\lesssim \frac{(|v| + \|E\|_{L^\infty_{t,x}} + \|E\|^2_{L^\infty_{t,x}} + 1)^{\beta-1}}{C_E^{\beta-1} \langle v\rangle^{\beta-1} \delta^{\beta-1} (\alpha(t,x,v))^{\beta-1}} \frac{2}{\varpi}.
\end{aligned} \tag{3.20}$$

This proves (3.6) for the case $x \in \partial\Omega$ and $t \leq t_{\mathbf{b}}$.

Step 4. Now suppose $x \in \partial\Omega$ and $t_{\mathbf{b}} > t$. It suffices to bound the integral:

$$\int_0^t e^{-\int_s^t \frac{\varpi}{2}\langle V(\tau;t,x,v)\rangle d\tau} \frac{1}{[|V(s)|^2 \xi(X(s)) + \xi^2(X(s) - C_E \xi(X(s))]^{\frac{\beta-1}{2}}} ds. \tag{3.21}$$

Denote

$$X(0;t,x,v) = x_0, V(0;t,x,v) = v_0.$$

Let

$$\sigma_2 = \delta \frac{v \cdot \nabla \xi(x)}{|v|^2 + \|E\|_{L^\infty_{t,x}} + \|E\|^2_{L^\infty_{t,x}} + 1}$$

as defined in (3.9). If

$$\sigma_2 \geq t,$$

then from *Step 2* $|\xi(X(s))|$ is decreasing on $[0,t]$, and by (3.11), (3.12), and the bound for (**III**) (3.19), we get the desired estimate. Now we assume

$$\sigma_2 < t.$$

So from (3.10) we have

$$|\xi(X(\sigma_2))| \geq \frac{\delta (v \cdot \nabla \xi(x))^2}{2(|v|^2 + \|E\|_{L^\infty_{t,x}} + \|E\|^2_{L^\infty_{t,x}} + 1)}. \tag{3.22}$$

Now if $|\xi(x_0)|\leq\delta\frac{\alpha^2(t,x,v)}{10(|v|^2+\|E\|_{L^\infty_{t,x}}+\|E\|^2_{L^\infty_{t,x}}+1)}$,

$$\begin{aligned}\alpha^2(t,x,v)&\lesssim e^{C_\xi\frac{\|\nabla E\|_{L^\infty_{t,x}}+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}}{C_E}}\alpha^2(0,x_0,v_0)\\&\lesssim e^{C_\xi\frac{\|\nabla E\|_{L^\infty_{t,x}}+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}}{C_E}}((\nabla\xi(x_0)\cdot v_0)^2+(|v_0|^2+|\xi(x_0)|+\|E\|_{L^\infty_{t,x}})|\xi(x_0)|)\\&\lesssim e^{C_\xi\frac{\|\nabla E\|_{L^\infty_{t,x}}+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}}{C_E}}(\nabla\xi(x_0)\cdot v_0)^2+\delta\alpha^2(t,x,v). \qquad (3.23)\end{aligned}$$

So

$$\frac{1}{2}\alpha(t,x,v)\lesssim e^{C_\xi\frac{\|\nabla E\|_{L^\infty_{t,x}}+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}}{C_E}}|\nabla\xi(x_0)\cdot v_0|, \qquad (3.24)$$

if $\delta\ll 1$ is small enough.

Claim:

$$\nabla\xi(x_0)\cdot v_0<0.$$

Since otherwise by (3.16) we have

$$\frac{d}{ds}|\xi(X(s))|<0,$$

for all $s\in[0,t]$, so $|\xi(X(s))|$ is always decreasing, which contradicts (3.22).

Therefore $\nabla\xi(x_0)\cdot v_0<0$, and we can run the same argument from *Step 1*, *Step 2*, *Step 3* with $\nabla\xi(x_{\mathbf{b}})\cdot v_{\mathbf{b}}$ replaced by $\nabla\xi(x_0)\cdot v_0$, and by (3.24) we get the same estimate.

If $|\xi(x_0)|>\delta\frac{\alpha^2(t,x,v)}{10(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)}$, then we have

$$\frac{C_E\sigma_2^2}{10}=\delta^2\frac{C_E}{10}\frac{(v\cdot\nabla\xi(x))^2}{|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1}<C_E\delta|\xi(x_0)|<|\xi(x_0)|, \qquad (3.25)$$

for $\delta\ll 1$ small enough. Therefore by (3.22) and the same argument in *Step 3* we get the same lower bound

$$|\xi(s)|>\frac{C_E}{10}(\sigma_2)^2, \text{for all } s\in[0,t-\sigma_2]. \qquad (3.26)$$

And therefore we get the desired estimate.

Step 5. We now consider the case when $x\in\Omega$ and $t\geq t_{\mathbf{b}}$. We need to bound the integral (3.8). Let

$$\sigma_1=\delta\frac{v_{\mathbf{b}}\cdot\nabla(x_{\mathbf{b}})}{|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1},$$

as defined in (3.10). If

$$\sigma_1\geq t,$$

then from *Step 2* $|\xi(\tilde{X}(s))|$ is increasing on $[0,t_{\mathbf{b}}]$, and by (3.11), (3.12), and the bound for $(\mathbf{I})$ in (3.18), we get the desired estimate.

Now we assume

$$\sigma_1 < t.$$

So from (3.10) we have

$$|\xi(\tilde{X}(\sigma_1))| \geq \frac{\delta(v_{\mathbf{b}}\cdot\nabla\xi(x_{\mathbf{b}}))^2}{2(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)}. \tag{3.27}$$

Now if

$$|\xi(x)| \leq \delta \frac{\alpha^2(t,x,v)}{10(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)}, \tag{3.28}$$

we have

$$\begin{aligned}\alpha^2(t,x,v) &\leq (\nabla\xi(x)\cdot v)^2 + C(|v|^2+\|E\|_{L^\infty_{t,x}}+1)|\xi(x)| \\ &\leq (\nabla\xi(x)\cdot v)^2 + \delta\alpha^2(t,x,v) \leq (\nabla\xi(x)\cdot v)^2 + \frac{1}{10}\alpha^2(t,x,v),\end{aligned} \tag{3.29}$$

if $\delta \ll 1$ is small enough. So

$$\frac{1}{2}\alpha(t,x,v) \leq |\nabla\xi(x)\cdot v|. \tag{3.30}$$

Claim:

$$\nabla\xi(x)\cdot v > 0.$$

Since otherwise by (3.16) we have

$$\frac{d}{ds}|\xi(\tilde{X}(s))| > 0,$$

for all $s\in[0,t_{\mathbf{b}}]$, so $|\xi(\tilde{X}(s))|$ is always increasing, thus

$$|\xi(\tilde{X}(s))| \leq \delta \frac{\alpha^2(t,x,v)}{10(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)},$$

for all $s\in[0,t_{\mathbf{b}}]$, which contradicts (3.27).

Therefore $\nabla\xi(x)\cdot v>0$, and we can run the same argument from *Step 2*, *Step 3*, *Step 4*, and by (3.30) we get the same estimate.

If

$$|\xi(x)| > \delta \frac{\alpha^2(t,x,v)}{10(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)}, \tag{3.31}$$

we claim:

$$|\xi(\tilde{X}(s))| \geq \delta^2 \frac{\alpha^2(t,x,v)}{|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1}, \tag{3.32}$$

for all $s\in[\sigma_1,t_{\mathbf{b}}]$. Since otherwise let

$$s^* := \min\{s\in[\sigma_1,t] : |\xi(\tilde{X}(s))| < \delta^2 \frac{\alpha^2(t,x,v)}{|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1}\}.$$

From (3.27) we have $s^* > \sigma_1$, and

$$\frac{d}{ds}|\xi(\tilde{X}(s^*))| < 0.$$

And from (3.16) we have

$$\frac{d^2}{ds^2}|\xi(\tilde{X}(s))| < 0,$$

for all $s \in [s^*, t]$. So $|\xi(\tilde{X}(s))|$ is always decreasing on $[s^*, t_{\mathbf{b}}]$. Therefore

$$|\xi(x)| = |\xi(\tilde{X}(t_{\mathbf{b}}))| < |\xi(\tilde{X}(s^*))| < \delta^2 \frac{\alpha^2(t,x,v)}{|v|^2 + \|E\|^2_{L^\infty_{t,x}} + \|E\|_{L^\infty_{t,x}} + 1},$$

which contradicts (3.31). Therefore the lower bound (3.32) and the estimates (3.20), (3.18) give the desired bound.

Step 6. Finally we consider the case $x \in \Omega$ and $t < t_{\mathbf{b}}$. First suppose

$$|\xi(x)| \leq \delta \frac{\alpha^2(t,x,v)}{10(|v|^2 + \|E\|^2_{L^\infty_{t,x}} + \|E\|_{L^\infty_{t,x}} + 1)}.$$

From (3.30) we have

$$\frac{\alpha(t,x,v)}{2} \leq |v \cdot \nabla \xi(x)|.$$

If $v \cdot \nabla \xi(x) > 0$, then by (3.16) we have $\xi(X(t+t')) = 0$ for some $t' \lesssim \frac{\delta}{C_E^2} < 1$. Therefore we can extend the trajectory until it hits the boundary and conclude the desired bound from *Step 3*.

If $v \cdot \nabla \xi(x) < 0$, again by (3.16) we have $|\xi(X(s))|$ is increasing on $[0,t]$ and $|V(s) \cdot \nabla \xi(X(s))|$ is decreasing on $[0,t]$. Therefore using the change of variable $s \mapsto |\xi|$:

$$\begin{aligned}
&\int_0^t e^{-\int_s^t \frac{\varpi}{2}\langle V(\tau;t,x,v)\rangle d\tau} \frac{1}{[|V(s)|^2 \xi(X(s)) + \xi^2(X(s) - C_E \xi(X(s))]^{\frac{\beta-1}{2}}} ds \\
&\lesssim \int_0^{\delta \frac{\alpha^2(t,x,v)}{10(|v|^2 + \|E\|^2_{L^\infty_{t,x}} + \|E\|_{L^\infty_{t,x}} + 1)}} \frac{1}{|V(s) \cdot \nabla \xi(X(s))| (C_E |\xi|)^{\frac{\beta-1}{2}}} d|\xi| \\
&\lesssim \int_0^{\delta \frac{\alpha^2(t,x,v)}{10(|v|^2 + \|E\|^2_{L^\infty_{t,x}} + \|E\|_{L^\infty_{t,x}} + 1)}} \frac{1}{|v \cdot \nabla \xi(x)| (C_E |\xi|)^{\frac{\beta-1}{2}}} d|\xi| \\
&\lesssim \int_0^{\delta \frac{\alpha^2(t,x,v)}{10(|v|^2 + \|E\|^2_{L^\infty_{t,x}} + \|E\|_{L^\infty_{t,x}} + 1)}} \frac{1}{|\alpha(t,x,v) (C_E |\xi|)^{\frac{\beta-1}{2}}} d|\xi| \\
&\lesssim \frac{\delta^{\frac{3-\beta}{2}}}{C_E^{\frac{\beta-1}{2}} (\alpha(t,x,v))^{\beta-2} (|v|^2 + \|E\|^2_{L^\infty_{t,x}} + \|E\|_{L^\infty_{t,x}} + 1)^{\frac{3-\beta}{2}}},
\end{aligned} \tag{3.33}$$

which is the desired estimate.

Now suppose

$$|\xi(x)| > \delta \frac{\alpha^2(t,x,v)}{10(|v|^2 + \|E\|^2_{L^\infty_{t,x}} + \|E\|_{L^\infty_{t,x}} + 1)}, \tag{3.34}$$

and

$$|\xi(x_0)| \le \delta \frac{\alpha^2(t,x,v)}{10(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)}.$$

Then by (3.24) we have

$$\frac{\alpha(t,x,v)}{2} \lesssim e^{C_\xi \frac{\|\nabla E\|_{L^\infty_{t,x}}+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}}{C_E}} |\nabla\xi(x_0)\cdot v_0|. \tag{3.35}$$

Now if $v_0\cdot\nabla\xi(x_0)>0$, then from (3.16) we have $|\xi(X(s))|$ is decreasing for all $s\in[0,t]$. And this contradicts with (3.34). So we must have

$$v_0\cdot\nabla\xi(x_0)<0.$$

Then we can define $\sigma_1=\delta\frac{|v_0\cdot\nabla\xi(x_0)|}{|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1}$ as before. Now if $\sigma_1\ge t$ then $|\xi(X(s))|$ is increasing on $[0,t]$, using the change of variable $x\mapsto|\xi|$ and the estimate (3.18) and (3.35) we get the desired bound.

If $\sigma_1<t$, then from (3.10) we have

$$|\xi(X(\sigma_1))| \ge \delta \frac{(v_0\cdot\nabla\xi(x_0))^2}{2(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)}.$$

And then from the argument for (3.32) we get

$$|\xi(X(s))| \ge \delta^2 \frac{\alpha^2(t,x,v)}{|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1},$$

for all $s\in[\sigma_1,t]$. This lower bound combined with the estimate (3.20), (3.18) gives the desired bound.

Finally we are left with the case

$$|\xi(x_0)| > \delta \frac{\alpha^2(t,x,v)}{10(|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1)}.$$

Then again, from the argument for (3.32) we get

$$|\xi(X(s))| \ge \delta^2 \frac{\alpha^2(t,x,v)}{|v|^2+\|E\|^2_{L^\infty_{t,x}}+\|E\|_{L^\infty_{t,x}}+1},$$

for all $s\in[0,t]$. This lower bound combined with the estimate (3.20) gives the desired bound. ∎

Proof. **(Proof of (2) Lemma 3.3.)** Since $\frac{\langle u\rangle^r}{\langle v\rangle^r}\lesssim\frac{\langle u\rangle^r}{\langle V_{\mathbf{cl}}(s)\rangle^r}\lesssim\{1+|V_{\mathbf{cl}}(s)-u|^2\}^{\frac{r}{2}}$ and $\langle V_{\mathbf{cl}}(s)-u\rangle^r e^{-\theta|V_{\mathbf{cl}}(s)-u|^2}\lesssim e^{-C_{\theta,r}|V_{\mathbf{cl}}(s)-u|^2}$, it suffices to consider the case $r=0$. It is important to control the *number of bounces*,

$$\ell_*(s)=\ell_*(s;t,x,v)\in\mathbb{N} \quad \text{such that} \quad t^{\ell_*+1}\le s<t^{\ell_*}.$$

An important consequence of Velocity lemma is that for the specular cycles

$$\alpha(s,X_{\mathbf{cl}}(s;t,x,v),V_{\mathbf{cl}}(s;t,x,v)) \gtrsim e^{-\mathcal{C}\langle v\rangle||t-s|}\alpha(t,x,v),$$

and therefore for the specular cycles

$$\begin{aligned}\ell_*(s;t,x,v) &\le \frac{|t-s|}{\min_{0\le\ell\le\ell_*(s;t,x,v)}|t^\ell-t^{\ell+1}|} \lesssim \frac{|t-s|}{\min_{0\le\ell\le\ell_*(s;t,x,v)}\frac{\alpha(t^\ell,x^\ell,v^\ell)}{|v^\ell|^2}}\\ &\lesssim \frac{|t-s|(|v|^2+1)}{\alpha(t,x,v)}e^{\mathcal{C}\langle v\rangle(t-s)}. \end{aligned}\tag{3.36}$$

For fixed (t,x,v) we use the following notation $\alpha(s):=\alpha(s;t,x,v):=\alpha(X_{\mathbf{cl}}(s;t,x,v),V_{\mathbf{cl}}(s;t,x,v))$.

Now we consider the estimate (3.7). From (3.18), (3.19), and (3.20) we have

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^3}e^{-l\langle v\rangle(t-s)}\frac{e^{-\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}\frac{Z(s,x,v)}{\left[\alpha(s,X_{\mathbf{cl}}(s;t,x,v),u)\right]^\beta}\mathrm{d}u\mathrm{d}s\\
&\lesssim\sum_{\ell=0}^{\ell_*(0;t,x,v)}\int_{t^{\ell+1}}^{t^\ell}\int_{\mathbb{R}^3}e^{-l\langle v\rangle(t-s)}\frac{e^{-\theta|v^\ell-u|^2}}{|v^\ell-u|^{2-\kappa}}\frac{Z(s,x,v)}{\left[\alpha(s,X_{\mathbf{cl}}(s;t,x,v),u)\right]^\beta}\mathrm{d}u\mathrm{d}s\\
&\lesssim\sup_{0\le s\le t}\{e^{-\frac{l}{2}\langle v\rangle(t-s)}Z(s,x,v)\}\times\sum_{\ell=0}^{\ell_*(0;t,x,v)}\Bigg(e^{-\frac{l}{2}\langle v\rangle(t-t^\ell)}\frac{\delta^{\frac{3-\beta}{2}}}{\langle v\rangle^2(\alpha(t^\ell,x^\ell,v^\ell))^{\beta-2}}\\
&\qquad+\frac{1}{\delta^{\beta-1}(\alpha(t^\ell,x^\ell,v^\ell))^{\beta-1}}\int_{t^{\ell+1}}^{t^\ell}e^{-\frac{l}{2}\langle v\rangle(t-s)}ds\Bigg)\\
&\lesssim\sup_{0\le s\le t}\{e^{-\frac{l}{2}\langle v\rangle(t-s)}Z(s,x,v)\}\times\sum_{\ell=0}^{\ell_*(0;t,x,v)}\Bigg(e^{-\frac{l}{4}\langle v\rangle(t-t^\ell)}\frac{\delta^{\frac{3-\beta}{2}}}{\langle v\rangle^2(\alpha(t,x,v))^{\beta-2}}\\
&\qquad+\frac{e^{C\langle v\rangle|t-t^\ell|}}{\delta^{\beta-1}(\alpha(t,x,v))^{\beta-1}}\int_{t^{\ell+1}}^{t^\ell}e^{-\frac{l}{2}\langle v\rangle(t-s)}ds\Bigg).\end{aligned}\tag{3.37}$$

Clearly

$$\begin{aligned}&\sum_{\ell=0}^{\ell_*(0;t,x,v)}\frac{e^{C\langle v\rangle|t-t^\ell|}}{\delta^{\beta-1}(\alpha(t,x,v))^{\beta-1}}\int_{t^{\ell+1}}^{t^\ell}e^{-\frac{l}{2}\langle v\rangle(t-s)}ds\\
&\lesssim\frac{1}{\delta^{\beta-1}(\alpha(t,x,v))^{\beta-1}}\int_0^t e^{-\frac{l}{4}\langle v\rangle(t-s)}ds\\
&\lesssim\frac{1}{l\langle v\rangle\delta^{\beta-1}(\alpha(t,x,v))^{\beta-1}}.\end{aligned}\tag{3.38}$$

And for $\sum_{\ell=0}^{\ell_*(0;t,x,v)}e^{-\frac{l}{4}\langle v\rangle(t-t^\ell)}\frac{\delta^{\frac{3-\beta}{2}}}{\langle v\rangle^2(\alpha(t,x,v))^{\beta-2}}$, we let $\tilde{\ell}$ be the bounce that $t^{\tilde{\ell}}\ge t-\frac{1}{\langle v\rangle}$ and $t^{\tilde{\ell}+1}<t-\frac{1}{\langle v\rangle}$, and decompose $\sum_{\ell=0}^{\ell_*(0;t,x,v)}=\sum_{\ell=0}^{\tilde{\ell}}+\sum_{\ell=\tilde{\ell}+1}^{\ell_*(0;t,x,v)}$. Then from (3.36)

$$\sum_{\ell=0}^{\tilde{\ell}}e^{-\frac{l}{4}\langle v\rangle(t-t^\ell)}\le|\tilde{\ell}|\lesssim\frac{1/\langle v\rangle}{\alpha(t,x,v)/|v|^2}\lesssim\frac{|v|}{\alpha(t,x,v)}.$$

For $\ell\ge\tilde{\ell}+1$, we have

$$|t-t^{\ell+1}|\le|t-t^\ell|+|t^\ell-t^{\ell+1}|\le|t-t^\ell|+C\frac{1}{\langle v\rangle}\le|t-t^\ell|+C|t-t^\ell|=(1+C)|t-t^\ell|.$$

Thus

$$\begin{aligned}\sum_{\ell=\tilde{\ell}+1}^{\ell_*(0;t,x,v)} e^{-\frac{l}{4}\langle v\rangle(t-t^\ell)} &\le \sum_{\ell=\tilde{\ell}+1}^{\ell_*(0;t,x,v)} e^{-\frac{l}{8}\langle v\rangle(t-t^\ell)} e^{-\frac{l}{8(1+C)}\langle v\rangle(t-t^{\ell+1})}\\
&\le \max_\ell \left\{ \frac{e^{-\frac{l}{8}\langle v\rangle(t-t^\ell)}}{|t^\ell - t^{\ell+1}|}\right\} \sum_{\ell=0}^{\ell_*} |t^\ell - t^{\ell+1}| e^{-\frac{l}{8(1+C)}\langle v\rangle(t-t^{\ell+1})}\\
&\lesssim \frac{\langle v\rangle^2 e^{-\frac{l}{8}\langle v\rangle(t-t^\ell)} e^{C\langle v\rangle(t-t^\ell)}}{\alpha(t,x,v)} \int_0^t e^{-\frac{l}{8(1+C)}\langle v\rangle(t-s)} ds\\
&\lesssim \frac{\langle v\rangle(1+C)}{l\alpha(t,x,v)}.\end{aligned}$$

Therefore

$$\sum_{\ell=0}^{\ell_*(0;t,x,v)} e^{-\frac{l}{4}\langle v\rangle(t-t^\ell)} \frac{\delta^{\frac{3-\beta}{2}}}{\langle v\rangle^2(\alpha(t,x,v))^{\beta-2}} \lesssim \frac{\delta^{\frac{3-\beta}{2}}}{\langle v\rangle(\alpha(t,x,v))^{\beta-1}}. \tag{3.39}$$

Combining (3.37), (3.38) and (3.39) we prove (3.7). ∎

4. Moving frame for specular cycles

We use the moving frame for the specular cycles introduced in [8]. We denote the standard spherical coordinate $\mathbf{x}_\parallel = \mathbf{x}_\parallel(\omega) = (\mathbf{x}_{\parallel,1}, \mathbf{x}_{\parallel,2})$ for $\omega \in \mathbb{S}^2$

$$\omega = (\cos \mathbf{x}_{\parallel,1}(\omega) \sin \mathbf{x}_{\parallel,2}(\omega), \sin \mathbf{x}_{\parallel,1}(\omega) \sin \mathbf{x}_{\parallel,2}(\omega), \cos \mathbf{x}_{\parallel,2}(\omega)),$$

where $\mathbf{x}_{\parallel,1}(\omega) \in [0,2\pi)$ is the azimuth and $\mathbf{x}_{\parallel,2}(\omega) \in [0,\pi)$ is the inclination.

We define an orthonormal basis of $\mathbb{R}^3$, $\{\hat{r}(\omega), \hat{\phi}(\omega), \hat{\theta}(\omega)\}$, with $\hat{r}(\omega) := \omega$ and

$$\begin{aligned}\hat{\phi}(\omega) &:= (\cos \mathbf{x}_{\parallel,1}(\omega) \cos \mathbf{x}_{\parallel,2}(\omega), \sin \mathbf{x}_{\parallel,1}(\omega) \cos \mathbf{x}_{\parallel,2}(\omega), -\sin \mathbf{x}_{\parallel,2}(\omega)),\\
\hat{\theta}(\omega) &:= (-\sin \mathbf{x}_{\parallel,1}(\omega), \cos \mathbf{x}_{\parallel,1}(\omega), 0).\end{aligned}$$

Moreover, $\hat{r} \times \hat{\phi} = \hat{\theta}$, $\hat{\phi} \times \hat{\theta} = \hat{r}$, $\hat{\theta} \times \hat{r} = \hat{\phi}$, and

$$\partial_{\mathbf{x}_{\parallel,1}} \hat{r} = \sin \mathbf{x}_{\parallel,2}\ \hat{\theta}, \quad \partial_{\mathbf{x}_{\parallel,2}} \hat{r} = \hat{\phi}, \tag{4.1}$$

where $\partial_{\mathbf{x}_{\parallel,1}} \hat{r}$ does not vanish (non-degenerate) away from $\mathbf{x}_{\parallel,2} = 0$ or π.

Without loss of generality we assume $\mathbf{0} = (0,0,0) \in \Omega$. For

$$\mathbf{p} = (z,w) \in \partial\Omega \times \mathbb{S}^2 \quad \text{with} \quad n(z)\cdot w = 0,$$

we define the north pole $\mathcal{N}_\mathbf{p} \in \partial\Omega$ and the south pole $\mathcal{S}_\mathbf{p} \in \partial\Omega$ as

$$\mathcal{N}_\mathbf{p} := |\mathcal{N}_\mathbf{p}|(n(z)\times w) \in \partial\Omega, \quad \mathcal{S}_\mathbf{p} := -|\mathcal{S}_\mathbf{p}|(n(z)\times w) \in \partial\Omega,$$

where $\partial_{\mathbf{x}_{\parallel,1}} \hat{r}$ is degenerate. We define the straight-line $\mathcal{L}_\mathbf{p}$ passing both poles

$$\mathcal{L}_\mathbf{p} := \{\tau \mathcal{N}_\mathbf{p} + (1-\tau)\mathcal{S}_\mathbf{p} : \tau \in \mathbb{R}\}.$$

LEMMA 4.1. *Assume Ω is convex (1.5). Fix $\mathbf{p} = (z,w) \in \partial\Omega \times \mathbb{S}^2$ with $n(z)\cdot w = 0$.*

(i) *There exists a smooth map (spherical-type coordinate)*

$$\begin{aligned} \eta_{\mathbf{p}} \ : \quad & [0,2\pi)\times(0,\pi) && \rightarrow \ \partial\Omega\backslash\{\mathcal{N}_{\mathbf{p}},\mathcal{S}_{\mathbf{p}}\}, \\ \mathbf{x}_{\|_{\mathbf{p}}} := & (\mathbf{x}_{\|_{\mathbf{p}},1},\mathbf{x}_{\|_{\mathbf{p}},2}) && \mapsto \quad \eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}}), \end{aligned} \tag{4.2}$$

which is one-to-one and onto. Here on $[0,2\pi)\times(0,\pi)$ *we have* $\partial_i\eta_{\mathbf{p}} := \frac{\partial \eta_{\mathbf{p}}}{\partial \mathbf{x}_{\|_{\mathbf{p}},i}} \neq 0$ *and*

$$\frac{\partial \eta_{\mathbf{p}}}{\partial \mathbf{x}_{\|_{\mathbf{p}},1}}(\mathbf{x}_{\|_{\mathbf{p}}}) \times \frac{\partial \eta_{\mathbf{p}}}{\partial \mathbf{x}_{\|_{\mathbf{p}},2}}(\mathbf{x}_{\|_{\mathbf{p}}}) \neq 0. \tag{4.3}$$

We define

$$\mathbf{n}_{\mathbf{p}} := n\circ\eta_{\mathbf{p}} : [0,2\pi)\times(0,\pi) \rightarrow \mathbb{S}^2.$$

(ii) *We define the* $\mathbf{p}-$*spherical coordinate in the tubular neighborhood of the boundary:*

For $\delta>0,\ \delta_1>0,\ C>0,$ *we have a smooth one-to-one and onto map*

$$\begin{aligned} \Phi_{\mathbf{p}} : [0,C\delta)\times[0,2\pi)\times(\delta_1,\pi-\delta_1)\times\mathbb{R}\times\mathbb{R}^2 \ &\rightarrow \ \{x\in\bar{\Omega} : |\xi(x)|<\delta\}\backslash B_{C\delta_1}(\mathcal{L}_{\mathbf{p}})\times\mathbb{R}^3, \\ (\mathbf{x}_{\perp_{\mathbf{p}}},\mathbf{x}_{\|_{\mathbf{p}},1},\mathbf{x}_{\|_{\mathbf{p}},2},\mathbf{v}_{\perp_{\mathbf{p}}},\mathbf{v}_{\|_{\mathbf{p}},1},\mathbf{v}_{\|_{\mathbf{p}},2}) \ &\mapsto \ \Phi_{\mathbf{p}}(\mathbf{x}_{\perp_{\mathbf{p}}},\mathbf{x}_{\|_{\mathbf{p}},1},\mathbf{x}_{\|_{\mathbf{p}},2},\mathbf{v}_{\perp_{\mathbf{p}}},\mathbf{v}_{\|_{\mathbf{p}},1},\mathbf{v}_{\|_{\mathbf{p}},2}), \end{aligned}$$

where $B_{C\delta_1}(\mathcal{L}_{\mathbf{p}}) := \{x\in\mathbb{R}^3 : |x-y|<C\delta_1 \text{ for some } y\in\mathcal{L}_{\mathbf{p}}\}$.
Explicitly,

$$\Phi_{\mathbf{p}}(\mathbf{x}_{\perp_{\mathbf{p}}},\mathbf{x}_{\|_{\mathbf{p}}},\mathbf{v}_{\perp_{\mathbf{p}}},\mathbf{v}_{\|_{\mathbf{p}}}) := \begin{bmatrix} \mathbf{x}_{\perp_{\mathbf{p}}}[-\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})] + \eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}}) \\ \mathbf{v}_{\perp_{\mathbf{p}}}[-\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})] + \mathbf{v}_{\|_{\mathbf{p}}}\cdot\nabla\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}}) + \mathbf{x}_{\perp_{\mathbf{p}}}\mathbf{v}_{\|_{\mathbf{p}}}\cdot\nabla[-\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})] \end{bmatrix}, \tag{4.4}$$

where $\nabla\eta_{\mathbf{p}} = (\partial_1\eta_{\mathbf{p}},\partial_2\eta_{\mathbf{p}}) = (\frac{\partial\eta_{\mathbf{p}}}{\partial\mathbf{x}_{\|_{\mathbf{p}},1}},\frac{\partial\eta_{\mathbf{p}}}{\partial\mathbf{x}_{\|_{\mathbf{p}},2}})$ *and* $\nabla\mathbf{n}_{\mathbf{p}} = (\partial_1\mathbf{n}_{\mathbf{p}},\partial_2\mathbf{n}_{\mathbf{p}}) = (\frac{\partial\mathbf{n}_{\mathbf{p}}}{\partial\mathbf{x}_{\|_{\mathbf{p}},1}},\frac{\partial\mathbf{n}_{\mathbf{p}}}{\partial\mathbf{x}_{\|_{\mathbf{p}},2}})$.

The Jacobian matrix is

$$\begin{aligned} &\frac{\partial\Phi(\mathbf{x}_\perp,\mathbf{x}_\|,\mathbf{v}_\perp,\mathbf{v}_\|)}{\partial(\mathbf{x}_\perp,\mathbf{x}_\|,\mathbf{v}_\perp,\mathbf{v}_\|)} \\ &= \left[\begin{array}{ccc|ccc} n & \overbrace{\begin{matrix}\frac{\partial\eta}{\partial\mathbf{x}_{\|,1}} \\ +\mathbf{x}_\perp\frac{\partial n}{\partial\mathbf{x}_{\|,1}}\end{matrix} \quad \begin{matrix}\frac{\partial\Phi_1(\mathbf{x}_\perp,\mathbf{x}_\|)}{\partial(\mathbf{x}_\perp,\mathbf{x}_\|)}\end{matrix}}^{} & \begin{matrix}\frac{\partial\eta}{\partial\mathbf{x}_{\|,2}} \\ +\mathbf{x}_\perp\frac{\partial n}{\partial\mathbf{x}_{\|,2}}\end{matrix} & & \mathbf{0}_{3,3} & \\ \hline -\mathbf{v}_\|\cdot\nabla_{\mathbf{x}_\|}\mathbf{n} & \begin{matrix}-\mathbf{v}_\perp\frac{\partial\mathbf{n}}{\partial\mathbf{x}_{\|,1}} \\ +\mathbf{v}_\|\cdot\nabla_{\mathbf{x}_\|}\frac{\partial\eta}{\partial\mathbf{x}_{\|,1}} \\ -\mathbf{x}_\perp\mathbf{v}_\|\cdot\nabla_{\mathbf{x}_\|}\frac{\partial\mathbf{n}}{\partial\mathbf{x}_{\|,1}}\end{matrix} & \begin{matrix}-\mathbf{v}_\perp\frac{\partial\mathbf{n}}{\partial\mathbf{x}_{\|,2}} \\ +\mathbf{v}_\|\cdot\nabla_{\mathbf{x}_\|}\frac{\partial\eta}{\partial\mathbf{x}_{\|,2}} \\ -\mathbf{x}_\perp\mathbf{v}_\|\cdot\nabla_{\mathbf{x}_\|}\frac{\partial\mathbf{n}}{\partial\mathbf{x}_{\|,2}}\end{matrix} & n & \begin{matrix}\frac{\partial\eta}{\partial\mathbf{x}_{\|,1}} \\ +\mathbf{x}_\perp\frac{\partial n}{\partial\mathbf{x}_{\|,1}}\end{matrix} \quad \frac{\partial\Phi_1(\mathbf{x}_\perp,\mathbf{x}_\|)}{\partial(\mathbf{x}_\perp,\mathbf{x}_\|)} & \begin{matrix}\frac{\partial\eta}{\partial\mathbf{x}_{\|,2}} \\ +\mathbf{x}_\perp\frac{\partial n}{\partial\mathbf{x}_{\|,2}}\end{matrix} \end{array}\right]. \end{aligned} \tag{4.5}$$

We fix an inverse map

$$\Phi_{\mathbf{p}}^{-1} : \{x\in\bar{\Omega} : |\xi(x)|<\delta\}\backslash B_{C\delta'}(\mathcal{L}_{\mathbf{p}})\times\mathbb{R}^3 \ \rightarrow \ [0,C\delta)\times[0,2\pi)\times(\delta_1,\pi-\delta_1)\times\mathbb{R}\times\mathbb{R}^2.$$

In general this choice is not unique but once we fix the range as above then an inverse map is uniquely determined.

We denote, for $(x,v)\in\{x\in\bar{\Omega} : |\xi(x)|<\delta\}\backslash B_{C\delta'}(\mathcal{L}_{\mathbf{p}})\times\mathbb{R}^3$

$$(\mathbf{x}_{\perp_{\mathbf{p}}},\mathbf{x}_{\|_{\mathbf{p}},1},\mathbf{x}_{\|_{\mathbf{p}},2},\mathbf{v}_{\perp_{\mathbf{p}}},\mathbf{v}_{\|_{\mathbf{p}},1},\mathbf{v}_{\|_{\mathbf{p}},2}) = \Phi_{\mathbf{p}}^{-1}(x,v).$$

(iii) *Let* $\mathbf{q}=(y,u)\in\partial\Omega\times\mathbb{S}^2$ *with* $n(y)\cdot u=0$ *and* $|\mathbf{p}-\mathbf{q}|\ll 1$ *and*

$$\Phi_{\mathbf{p}}(\mathbf{x}_{\perp_{\mathbf{p}}},\mathbf{x}_{\|_{\mathbf{p}}},\mathbf{v}_{\perp_{\mathbf{p}}},\mathbf{v}_{\|_{\mathbf{p}}}) = (x,v) = \Phi_{\mathbf{q}}(\mathbf{x}_{\perp_{\mathbf{q}}},\mathbf{x}_{\|_{\mathbf{q}}},\mathbf{v}_{\perp_{\mathbf{q}}},\mathbf{v}_{\|_{\mathbf{q}}}).$$

Then

$$\frac{\partial(\mathbf{x}_{\perp_{\mathbf{p}}},\mathbf{x}_{\|_{\mathbf{p}}},\mathbf{v}_{\perp_{\mathbf{p}}},\mathbf{v}_{\|_{\mathbf{p}}})}{\partial(\mathbf{x}_{\perp_{\mathbf{q}}},\mathbf{x}_{\|_{\mathbf{q}}},\mathbf{v}_{\perp_{\mathbf{q}}},\mathbf{v}_{\|_{\mathbf{q}}})} = \nabla\Phi_{\mathbf{q}}^{-1}\nabla\Phi_{\mathbf{p}} = \mathbf{Id}_{6,6} + O_\xi(|\mathbf{p}-\mathbf{q}|)\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & & & \\ 0 & 1 & 1 & & \mathbf{0}_{3,3} & \\ 0 & 1 & 1 & & & \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & |v| & |v| & 0 & 1 & 1 \\ 0 & |v| & |v| & 0 & 1 & 1 \end{array}\right]. \tag{4.6}$$

Proof. See [8]. ∎

LEMMA 4.2.

(i) *For* $|\xi(X_{\mathbf{cl}}(s;t,x,v))|<\delta$ *and* $|X_{\mathbf{cl}}(s;t,x,v)-\mathcal{L}_{\mathbf{p}}|>C\delta_1$ *we define*

$$\begin{aligned}(\mathbf{X}_{\mathbf{p}}(s;t,x,v),\mathbf{V}_{\mathbf{p}}(s;t,x,v)) &:= \Phi_{\mathbf{p}}^{-1}(X_{\mathbf{cl}}(s;t,x,v),V_{\mathbf{cl}}(s;t,x,v)) \\ &:= (\mathbf{x}_{\perp_{\mathbf{p}}}(s;t,x,v),\mathbf{x}_{\|_{\mathbf{p}}}(s;t,x,v),\mathbf{v}_{\perp_{\mathbf{p}}}(s;t,x,v),\mathbf{v}_{\|_{\mathbf{p}}}(s;t,x,v)).\end{aligned}$$

Then $|v|\simeq|\mathbf{V}_{\mathbf{p}}|$ *and*

$$\begin{bmatrix}\dot{\mathbf{x}}_{\perp_{\mathbf{p}}}\\ \dot{\mathbf{x}}_{\|_{\mathbf{p}}}\\ \dot{\mathbf{v}}_{\perp_{\mathbf{p}}}\\ \dot{\mathbf{v}}_{\|_{\mathbf{p}}}\end{bmatrix}(s;t,x,v) = \begin{bmatrix}\mathbf{v}_{\perp_{\mathbf{p}}}\\ \mathbf{v}_{\|_{\mathbf{p}}}\\ F_{\perp_{\mathbf{p}}}(\mathbf{x}_{\mathbf{p}},\mathbf{v}_{\mathbf{p}})\\ F_{\|_{\mathbf{p}}}(\mathbf{x}_{\mathbf{p}},\mathbf{v}_{\mathbf{p}})\end{bmatrix}(s;t,x,v). \tag{4.7}$$

Here

$$\begin{aligned}F_{\perp_{\mathbf{p}}} &= F_{\perp_{\mathbf{p}}}(\mathbf{x}_{\perp_{\mathbf{p}}},\mathbf{x}_{\|_{\mathbf{p}}},\mathbf{v}_{\|_{\mathbf{p}}}) \\ &= \sum_{j,k=1}^{2}\mathbf{v}_{\|_{\mathbf{p}},k}\mathbf{v}_{\|_{\mathbf{p}},j}\ \partial_j\partial_k\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})\cdot\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}}) - \mathbf{x}_{\perp_{\mathbf{p}}}\sum_{k=1}^{2}\mathbf{v}_{\|_{\mathbf{p}},k}(\mathbf{v}_{\|_{\mathbf{p}}}\cdot\nabla)\partial_k\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})\cdot\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}}) \\ &\quad - E(s,-\mathbf{x}_\perp\mathbf{n}(\mathbf{x}_\|)+\eta(\mathbf{x}_\|))\cdot\mathbf{n}(\mathbf{x}_\|),\end{aligned} \tag{4.8}$$

where

$$\sum_{j,k=1}^{2}\mathbf{v}_{\|_{\mathbf{p}},k}\mathbf{v}_{\|_{\mathbf{p}},j}\ \partial_j\partial_k\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})\cdot\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}}) \lesssim_\xi -|\mathbf{v}_\||^2,$$

and

$$\begin{aligned}F_{\|_{\mathbf{p}}} &= F_{\|_{\mathbf{p}}}(\mathbf{x}_{\perp_{\mathbf{p}}},\mathbf{x}_{\|_{\mathbf{p}}},\mathbf{v}_{\perp_{\mathbf{p}}},\mathbf{v}_{\|_{\mathbf{p}}}) \\ &= \sum_{i=1,2}G_{\mathbf{p},ij}(\mathbf{x}_{\perp_{\mathbf{p}}},\mathbf{x}_{\|_{\mathbf{p}}})\frac{(-1)^i}{\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})\cdot(\partial_1\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})\times\partial_2\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}}))} \\ &\quad\times\big\{2\mathbf{v}_{\perp_{\mathbf{p}}}\mathbf{v}_{\|_{\mathbf{p}}}\cdot\nabla\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}}) - \mathbf{v}_{\|_{\mathbf{p}}}\cdot\nabla^2\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})\cdot\mathbf{v}_{\|_{\mathbf{p}}} + \mathbf{x}_{\perp_{\mathbf{p}}}\mathbf{v}_{\|_{\mathbf{p}}}\cdot\nabla^2\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})\cdot\mathbf{v}_{\|_{\mathbf{p}}} \\ &\qquad - E(s,-\mathbf{x}_\perp\mathbf{n}(\mathbf{x}_\|)+\eta(\mathbf{x}_\|))\big\}\cdot\big\{\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})\times\partial_{i+1}\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})\big\},\end{aligned} \tag{4.9}$$

where a smooth bounded function $G_{\mathbf{p},ij}(\mathbf{x}_{\perp_{\mathbf{p}}},\mathbf{x}_{\|_{\mathbf{p}}})$ *is specified in* (4.16).

(ii) *For* $\tau\in(t^{\ell+1},t^\ell)$, *if the* $\mathbf{p}^\ell$*−spherical coordinate is well-defined in* $[\tau,t^\ell)$ *then*

$$[\mathbf{X}_\ell(\tau;t,x,v),\mathbf{V}_\ell(\tau;t,x,v)] \equiv [\mathbf{X}_\ell(\tau;t^\ell,0,\mathbf{x}^\ell_{\|_\ell},\mathbf{v}^\ell_{\perp_\ell},\mathbf{v}^\ell_{\|_\ell}),\mathbf{V}_\ell(\tau;t^\ell,0,\mathbf{x}^\ell_{\|_\ell},\mathbf{v}^\ell_{\perp_\ell},\mathbf{v}^\ell_{\|_\ell})]$$

and, for $\partial_{\mathbf{v}^\ell_\ell}=[\partial_{\mathbf{v}^\ell_{\perp_\ell}},\partial_{\mathbf{v}^\ell_{\|_\ell}}]$,

$$\begin{bmatrix} |\partial_{\mathbf{x}^\ell_{\|_\ell}}\mathbf{X}_\ell(\tau)| & |\partial_{\mathbf{v}^\ell_\ell}\mathbf{X}_\ell(\tau)| \\ |\partial_{\mathbf{x}^\ell_{\|_\ell}}\mathbf{V}_\ell(\tau)| & |\partial_{\mathbf{v}^\ell_\ell}\mathbf{V}_\ell(\tau)| \end{bmatrix} \lesssim \begin{bmatrix} 1 & |\tau-t^\ell| \\ (O_{\xi,\|\nabla E\|_{L^\infty_{t,x}}}(1)+|v|^2)|\tau-t^\ell| & 1 \end{bmatrix}. \tag{4.10}$$

For $t^{\ell+1}<\tau<s<t^\ell$ *then*

$$\begin{aligned} &[\mathbf{X}_\ell(\tau;t,x,v),\mathbf{V}_\ell(\tau;t,x,v)] \\ &\equiv [\mathbf{X}_\ell(\tau;s,\mathbf{X}_\ell(s;t,x,v),\mathbf{V}_\ell(s;t,x,v)),\mathbf{V}_\ell(\tau;s,\mathbf{X}_\ell(s;t,x,v),\mathbf{V}_\ell(s;t,x,v))], \end{aligned}$$

and

$$\begin{bmatrix} |\partial_{\mathbf{X}_\ell(s)}\mathbf{X}_\ell(\tau)| & |\partial_{\mathbf{V}_\ell(s)}\mathbf{X}_\ell(\tau)| \\ |\partial_{\mathbf{X}_\ell(s)}\mathbf{V}_\ell(\tau)| & |\partial_{\mathbf{V}_\ell(s)}\mathbf{V}_\ell(\tau)| \end{bmatrix} \lesssim \begin{bmatrix} 1 & |\tau-s| \\ (O_{\xi,\|\nabla E\|_{L^\infty_{t,x}}}(1)+|v|^2)|\tau-s| & 1 \end{bmatrix}. \tag{4.11}$$

Moreover, for either $[\partial_{\mathbf{X}},\partial_{\mathbf{V}}]=[\partial_{\mathbf{x}^\ell_{\|\ell}},\partial_{\mathbf{v}^\ell_{\perp\ell}},\partial_{\mathbf{v}^\ell_{\|\ell}}]$ *or* $[\partial_{\mathbf{X}},\partial_{\mathbf{V}}]=[\partial_{\mathbf{X}_\ell(s)},\partial_{\mathbf{V}_\ell(s)}]$

$$\begin{bmatrix} |\partial_{\mathbf{X}}F(\tau)| & |\partial_{\mathbf{V}}F(\tau)| \\ |\frac{d}{d\tau}\partial_{\mathbf{X}}F(\tau)| & |\frac{d}{d\tau}\partial_{\mathbf{V}}F(\tau)| \end{bmatrix} \lesssim \begin{bmatrix} O_{\xi,\|\nabla E\|_{L^\infty_{t,x}}}(1)+|v|^2 & O_{\xi,\|\nabla E\|_{L^\infty_{t,x}}}(1)+|v| \\ O_{\xi,\|\nabla E\|_{L^\infty_{t,x}}}(1)+|v|^3 & O_{\xi,\|\nabla E\|_{L^\infty_{t,x}}}(1)+|v|^2 \end{bmatrix}. \tag{4.12}$$

Proof. From $\dot{v}=0$ and the second equation of (4.4) we get

$$\begin{aligned} E(s,-\mathbf{x}_\perp\mathbf{n}(\mathbf{x}_\|)+\eta(\mathbf{x}_\|)) =& \dot{\mathbf{v}}_\perp(s)[-\mathbf{n}(\mathbf{x}_\|(s))]-2\mathbf{v}_\perp(s)\mathbf{v}_\|\cdot\nabla\mathbf{n}(\mathbf{x}_\|(s))+\dot{\mathbf{v}}_\|(s)\cdot\nabla\eta(\mathbf{x}_\|(s)) \\ &+\mathbf{v}_\|\cdot\nabla^2\eta(\mathbf{x}_\|)\cdot\mathbf{v}_\|-\mathbf{x}_\perp\dot{\mathbf{v}}_\|\cdot\nabla\mathbf{n}(\mathbf{x}_\|)-\mathbf{x}_\perp\mathbf{v}_\|\cdot\nabla^2\mathbf{n}(\mathbf{x}_\|)\cdot\mathbf{v}_\|. \end{aligned} \tag{4.13}$$

We take the inner product with $\mathbf{n}(\mathbf{x}_\|(s))$ to the above equation to have

$$\begin{aligned} \dot{\mathbf{v}}_\perp(s) =& -E(s,-\mathbf{x}_\perp\mathbf{n}(\mathbf{x}_\|)+\eta(\mathbf{x}_\|))\cdot\mathbf{n}(\mathbf{x}_\|)+[\mathbf{v}_\|\cdot\nabla^2\eta(\mathbf{x}_\|)\cdot\mathbf{v}_\|]\cdot\mathbf{n}(\mathbf{x}_\|) \\ &-\mathbf{x}_\perp[\mathbf{v}_\|\cdot\nabla^2\mathbf{n}(\mathbf{x}_\|)\cdot\mathbf{v}_\|]\cdot\mathbf{n}(\mathbf{x}_\|) \\ :=& F_\perp(\mathbf{v}_\perp,\mathbf{v}_\|,\mathbf{x}_\|), \end{aligned} \tag{4.14}$$

where we have used the fact $\nabla\mathbf{n}\perp\mathbf{n}$ and $\nabla\eta\perp\mathbf{n}$.

Since $0=\xi(\eta(\mathbf{x}_\|))$ we take $\mathbf{x}_{\|,i}$ and $\mathbf{x}_{\|,j}$ derivatives to have

$$0=\partial_{\mathbf{x}_{\|,j}}\Big[\sum_k\partial_k\xi\partial_{\mathbf{x}_{\|,i}}\eta_k\Big]=\sum_{k,m}\partial_k\partial_m\xi\partial_{\mathbf{x}_{\|,j}}\eta_m\partial_{\mathbf{x}_{\|,i}}\eta_k+\sum_k\partial_k\xi\partial_{\mathbf{x}_{\|,i}}\partial_{\mathbf{x}_{\|,j}}\eta_k,$$

and from the convexity (1.5) and $\mathbf{n}=\nabla\xi/|\nabla\xi|$,

$$\begin{aligned} \big[\mathbf{v}_\|\cdot\nabla^2\eta\cdot\mathbf{v}_\|\big]\cdot\mathbf{n} =& \sum_{i,j,k}\frac{\mathbf{v}_{\|,i}\partial_k\xi\partial_i\partial_j\eta_k\mathbf{v}_{\|,j}}{|\nabla\xi|} = -\sum_{i,j,k,m}\frac{\{\mathbf{v}_{\|,i}\partial_i\eta_m\}\partial_k\partial_m\xi\{\partial_j\eta_m\mathbf{v}_{\|,j}\}}{|\nabla\xi|} \\ &\lesssim_\xi -|\mathbf{v}_\||^2. \end{aligned}$$

Define $a_{ij}(\mathbf{x}_\|)$ via

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \partial_1\mathbf{n}\cdot\partial_1\mathbf{n} & \partial_1\mathbf{n}\cdot\partial_2\mathbf{n} \\ \partial_2\mathbf{n}\cdot\partial_1\mathbf{n} & \partial_2\mathbf{n}\cdot\partial_2\mathbf{n} \end{bmatrix} \begin{bmatrix} \partial_1\eta\cdot\partial_1\eta & \partial_1\eta\cdot\partial_2\eta \\ \partial_2\eta\cdot\partial_1\eta & \partial_2\eta\cdot\partial_2\eta \end{bmatrix}^{-1},$$

where $\det(\partial_i\eta\cdot\partial_j\eta)=|\partial_1\eta\times\partial_2\eta|^2\neq 0$ due to (4.3). Then $\nabla\mathbf{n}$ is generated by $\nabla\eta$:

$$-\partial_i\mathbf{n}(\mathbf{x}_\|)=\sum_k a_{ik}(\mathbf{x}_\|)\partial_k\eta(\mathbf{x}_\|).$$

We take the inner product (4.13) with $(-1)^{i+1}(\mathbf{n}(\mathbf{x}_\|)\times\partial_i\mathbf{n}(\mathbf{x}_\|))$ to have

$$\begin{aligned}&\sum_k(\delta_{ki}+\mathbf{x}_\perp a_{ki})\dot{\mathbf{v}}_{\|,k}\\&=\frac{(-1)^{i+1}}{-\mathbf{n}(\mathbf{x}_\|)\cdot(\partial_1\eta(\mathbf{x}_\|)\times\partial_2\eta(\mathbf{x}_\|))}\\&\quad\times\Big\{-2\mathbf{v}_\perp\mathbf{v}_\|\cdot\nabla\mathbf{n}(\mathbf{x}_\|)+\mathbf{v}_\|\cdot\nabla^2\eta(\mathbf{x}_\|)\cdot\mathbf{v}_\|-\mathbf{x}_\perp\mathbf{v}_\|\cdot\nabla^2\mathbf{n}(\mathbf{x}_\|)\cdot\mathbf{v}_\|\\&\quad-E(s,-\mathbf{x}_\perp\mathbf{n}(\mathbf{x}_\|)+\eta(\mathbf{x}_\|))\Big\}\cdot(-\mathbf{n}(\mathbf{x}_\|)\times\partial_{i+1}\eta(\mathbf{x}_\|)),\end{aligned}$$

where we used the notational convention for $\partial_{i+1}\eta$, the index $i+1$ mod 2 . For $|\xi(x)|\ll 1$(and therefore $|\mathbf{x}_\perp|\ll 1$) the matrix $\delta_{ki}+\mathbf{x}_\perp a_{ki}$ is invertible: there exists the inverse matrix G_{ij} such that $\sum_i(\delta_{ki}+\mathbf{x}_\perp a_{ki}(\mathbf{x}_\|))G_{ij}(\mathbf{x}_\perp,\mathbf{x}_\|)=\delta_{kj}$. Therefore we have

$$\begin{aligned}\dot{\mathbf{v}}_{\|,j}&=\sum_i G_{ij}(\mathbf{x}_\perp,\mathbf{x}_\|)\frac{(-1)^{i+1}}{-\mathbf{n}(\mathbf{x}_\|)\cdot(\partial_1\eta(\mathbf{x}_\|)\times\partial_2\eta(\mathbf{x}_\|))}\\&\quad\times\Big\{-2\mathbf{v}_\perp\mathbf{v}_\|\cdot\nabla\mathbf{n}(\mathbf{x}_\|)+\mathbf{v}_\|\cdot\nabla^2\eta(\mathbf{x}_\|)\cdot\mathbf{v}_\|-\mathbf{x}_\perp\mathbf{v}_\|\cdot\nabla^2\mathbf{n}(\mathbf{x}_\|)\cdot\mathbf{v}_\|\\&\quad-E(s,-\mathbf{x}_\perp\mathbf{n}(\mathbf{x}_\|)+\eta(\mathbf{x}_\|))\Big\}\cdot(-\mathbf{n}(\mathbf{x}_\|)\times\partial_{i+1}\eta(\mathbf{x}_\|))\\&:=F_{\|,j}(\mathbf{x}_\perp,\mathbf{x}_\|,\mathbf{v}_\perp,\mathbf{v}_\|).\end{aligned}\tag{4.15}$$

Here

$$\begin{aligned}&\begin{bmatrix}G_{11}&G_{12}\\G_{21}&G_{22}\end{bmatrix}\\&=\frac{1}{1+\mathbf{x}_\perp(a_{11}+a_{22})+(\mathbf{x}_\perp)^2(a_{11}a_{22}-a_{12}a_{21})}\begin{bmatrix}1+\mathbf{x}_\perp a_{22}&-\mathbf{x}_\perp a_{12}\\-\mathbf{x}_\perp a_{21}&1+\mathbf{x}_\perp a_{11}\end{bmatrix},\\&\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\\&=\frac{1}{|\partial_1\eta|^2|\partial_2\eta|^2-(\partial_1\eta\cdot\partial_2\eta)^2}\\&\quad\times\begin{bmatrix}|\partial_1\mathbf{n}|^2|\partial_2\eta|^2-(\partial_1\mathbf{n}\cdot\partial_2\mathbf{n})(\partial_1\eta\cdot\partial_2\eta)&-|\partial_1\mathbf{n}|^2(\partial_1\eta\cdot\partial_2\eta)+(\partial_1\mathbf{n}\cdot\partial_2\mathbf{n})|\partial_1\eta|^2\\(\partial_1\mathbf{n}\cdot\partial_2\mathbf{n})|\partial_2\eta|^2-|\partial_2\mathbf{n}|^2(\partial_1\eta\cdot\partial_2\eta)&-(\partial_1\mathbf{n}\cdot\partial_2\mathbf{n})(\partial_1\eta\cdot\partial_2\eta)+|\partial_2\mathbf{n}|^2|\partial_1\eta|^2\end{bmatrix}.\end{aligned}\tag{4.16}$$

To complete the proof of (4.7), from $\dot{x}=v$ and $\dot{v}=E$, we have

$$\begin{aligned}v&=-\mathbf{v}_\perp\mathbf{n}+\mathbf{v}_{||}\cdot\nabla\eta+\mathbf{x}_\perp[-\nabla n(\mathbf{x}_{||})]\dot{\mathbf{x}}_{||}\\&=\dot{\mathbf{x}}_\perp(-\mathbf{n}(\mathbf{x}_{||}))+\mathbf{x}_\perp[-\nabla\mathbf{n}(\mathbf{x}_{||})]\dot{\mathbf{x}}_{||}+\nabla\eta\dot{\mathbf{x}}_{||}\\E(s,-\mathbf{x}_\perp\mathbf{n}(\mathbf{x}_\|)+\eta(\mathbf{x}_\|))&=\dot{\mathbf{v}}_\perp(-\mathbf{n}(\mathbf{x}_{||}))-\mathbf{v}_\perp\nabla\mathbf{n}\dot{\mathbf{x}}_{||}+\dot{\mathbf{v}}_{||}\nabla\eta+\mathbf{v}_{||}\nabla^2\eta\dot{\mathbf{x}}_{||}\\&\quad+\dot{\mathbf{x}}_\perp\mathbf{v}_{||}[-\nabla\mathbf{n}(\mathbf{x}_{||})]+\mathbf{x}_\perp\dot{\mathbf{v}}_{||}[-\nabla\mathbf{n}(\mathbf{x}_{||})]+\mathbf{x}_\perp\mathbf{v}_{||}[-\nabla^2 n]\dot{\mathbf{x}}_{||}.\end{aligned}$$

We therefore conclude that $\dot{\mathbf{x}}_\perp=\mathbf{v}_\perp$, and $\dot{\mathbf{x}}_{||}=\mathbf{v}_{||}$ from $\Phi_{\mathbf{p}}^{-1}$. We then solve $\dot{\mathbf{v}}_\perp$ and $\dot{\mathbf{v}}_{||}$ to obtain (4.7).

Now we prove (4.10) and (4.11). From (4.8) and (4.9), $\dot{\mathbf{x}}_{\|_\ell} = \mathbf{v}_{\|_\ell}$, $\dot{\mathbf{x}}_{\perp_\ell} = \mathbf{v}_{\perp_\ell}$ and $\dot{\mathbf{v}}_{\perp_\ell} = F_{\perp_\ell}$ and $\dot{\mathbf{v}}_{\|_\ell} = F_{\|_\ell}$. Denote $\partial = [\frac{\partial}{\partial \mathbf{x}^\ell_{\|_\ell}}, \frac{\partial}{\partial \mathbf{v}^\ell_{\perp_\ell}}, \frac{\partial}{\partial \mathbf{v}^\ell_{\|_\ell}}]$. From (4.8) and (4.9),

$$\begin{bmatrix} |\partial F_\perp| \\ |\partial F_\|| \end{bmatrix} \lesssim \begin{bmatrix} (O_{\xi,\|\nabla E\|_{L^\infty_{t,x}}}(1) + |V(\tau)|^2)\{|\partial \mathbf{x}_\perp| + |\partial \mathbf{x}_\||\} + |V(\tau)||\partial \mathbf{v}_\|| \\ (O_{\xi,\|\nabla E\|_{L^\infty_{t,x}}}(1) + |V(\tau)|^2)\{|\partial \mathbf{x}_\perp| + |\partial \mathbf{x}_\||\} + |V(\tau)|\{|\partial \mathbf{v}_\perp| + |\partial \mathbf{v}_\||\} \end{bmatrix}. \tag{4.17}$$

Now we use a single (rough) bound of
$|\partial F_\perp| + |\partial F_\|| \lesssim (O_{\xi,\|\nabla E\|_{L^\infty_{t,x}}}(1) + |V(\tau)|^2)\{|\partial \mathbf{x}_\perp| + |\partial \mathbf{x}_\||\} + |V(\tau)|\{|\partial \mathbf{v}_\perp| + |\partial \mathbf{v}_\||\}$
to have

$$\begin{aligned} &\frac{d}{d\tau}\{|\partial \mathbf{v}_{\perp_\ell}(\tau)| + |\partial \mathbf{v}_{\|_\ell}(\tau)|\} \\ &\lesssim |\partial F_{\perp_\ell}(\tau)| + |\partial F_{\|_\ell}(\tau)| \\ &\lesssim (O_{\xi,\|\nabla E\|_{L^\infty_{t,x}}}(1) + |V(\tau)|^2)\big\{|\partial \mathbf{x}_{\perp_\ell}(\tau)| + |\partial \mathbf{x}_{\|_\ell}(\tau)|\big\} + |V(\tau)|\big\{|\partial \mathbf{v}_{\perp_\ell}(\tau)| + |\partial \mathbf{v}_{\|_\ell}(\tau)|\big\}. \end{aligned}$$

Combining with $\frac{d}{d\tau}[\mathbf{x}_{\perp_\ell}(\tau), \mathbf{x}_{\|_\ell}(\tau)] = [\mathbf{v}_{\perp_\ell}(\tau), \mathbf{v}_{\|_\ell}(\tau)]$ yields

$$\begin{aligned} &\frac{d}{d\tau}\begin{bmatrix} |\partial \mathbf{x}_{\perp_\ell}(\tau)| + |\partial \mathbf{x}_{\|_\ell}(\tau)| \\ |\partial \mathbf{v}_{\perp_\ell}(\tau)| + |\partial \mathbf{v}_{\|_\ell}(\tau)| \end{bmatrix} \\ &\lesssim_\xi \begin{bmatrix} 0 & 1 \\ (O_{\xi,\|\nabla E\|_{L^\infty_{t,x}}}(1) + |V(\tau)|^2) & |V(\tau)| \end{bmatrix} \begin{bmatrix} |\partial \mathbf{x}_{\perp_\ell}(\tau)| + |\partial \mathbf{x}_{\|_\ell}(\tau)| \\ |\partial \mathbf{v}_{\perp_\ell}(\tau)| + |\partial \mathbf{v}_{\|_\ell}(\tau)| \end{bmatrix}. \end{aligned} \tag{4.18}$$

Now for $M \gg 1$, let's first prove (4.10) for $|v| < M$. From (4.18) we have

$$\begin{aligned} &|\partial \mathbf{X}_\ell(\tau)| + |\partial \mathbf{V}_\ell(\tau)| \\ &\lesssim 1 + \int_\tau^{t^\ell} \Big(1 + O_{\xi,\|\nabla E\|_{L^\infty_{t,x}}}(1) + |V(\tau')| + |V(\tau')|^2\Big) |\partial \mathbf{X}_\ell(\tau')| + |\partial \mathbf{V}_\ell(\tau')| d\tau' \\ &\lesssim 1 + \int_\tau^{t^\ell} \Big(1 + O_{\xi,\|\nabla E\|_{L^\infty_{t,x}}}(1) + M^2\Big) |\partial \mathbf{X}_\ell(\tau')| + |\partial \mathbf{V}_\ell(\tau')| d\tau'. \end{aligned}$$

From Gronwall's inequality we have

$$|\partial \mathbf{X}_\ell(\tau)| + |\partial \mathbf{V}_\ell(\tau)| \lesssim_{\xi,\|\nabla E\|_{L^\infty_{t,x}},M} 1. \tag{4.19}$$

For $\partial_{\mathbf{v}} = [\frac{\partial}{\partial \mathbf{v}^\ell_{\perp_\ell}}, \frac{\partial}{\partial \mathbf{v}^\ell_{\|_\ell}}]$, from (4.19) we have

$$|\partial_{\mathbf{v}} \mathbf{X}_\ell(\tau)| \le \int_\tau^{t^\ell} |\partial_{\mathbf{v}} \mathbf{V}_\ell(\tau')| d\tau' \lesssim_{\xi,\|\nabla E\|_{L^\infty_{t,x}},M} |\tau - t^\ell|. \tag{4.20}$$

And for $\partial_{\mathbf{x}} = \frac{\partial}{\partial \mathbf{x}^\ell_{\|_\ell}}$, from (4.18), (4.19) we have

$$\begin{aligned} |\partial_{\mathbf{x}} \mathbf{V}_\ell(\tau)| &\le \int_\tau^{t^\ell} \Big(O_{\xi,\|\nabla E\|_{L^\infty_{t,x}}}(1) + |V(\tau')|^2\Big) |\partial_{\mathbf{x}} \mathbf{X}_\ell(\tau')| + |V(\tau')||\partial_{\mathbf{x}} \mathbf{V}_\ell(\tau')| d\tau' \\ &\lesssim \Big(O_{\xi,\|\nabla E\|_{L^\infty_{t,x}}}(1) + |v|^2\Big) |\tau - t^\ell| + M \int_\tau^{t^\ell} |\partial_{\mathbf{x}} \mathbf{V}_\ell(\tau')| d\tau'. \end{aligned}$$

From Gronwall's inequality we have

$$|\partial_{\mathbf{x}}\mathbf{V}_\ell(\tau)| \lesssim_{\xi,\|\nabla E\|_{L^\infty_{t,x}},M} \Big(O_{\xi,\|\nabla E\|_{L^\infty_{t,x}}}(1)+|v|^2\Big)|\tau-t^\ell|. \tag{4.21}$$

Combining (4.19), (4.20), and (4.21) we prove (4.10) for $|v|<M$.

For the case $|v|\geq M\gg 1$, we have $|V(\tau)|<2|v|$, so

$$\frac{d}{d\tau}\begin{bmatrix}|\partial\mathbf{x}_{\perp_\ell}(\tau)|+|\partial\mathbf{x}_{\parallel_\ell}(\tau)|\\ |\partial\mathbf{v}_{\perp_\ell}(\tau)|+|\partial\mathbf{v}_{\parallel_\ell}(\tau)|\end{bmatrix} \lesssim_\xi \begin{bmatrix}0 & 1\\ (O_{\xi,\|\nabla E\|_{L^\infty_{t,x}}}(1)+|v|^2) & |v|\end{bmatrix}\begin{bmatrix}|\partial\mathbf{x}_{\perp_\ell}(\tau)|+|\partial\mathbf{x}_{\parallel_\ell}(\tau)|\\ |\partial\mathbf{v}_{\perp_\ell}(\tau)|+|\partial\mathbf{v}_{\parallel_\ell}(\tau)|\end{bmatrix}.$$

By Lemma 5.2 we prove our claim (4.10) for the case $|v|\geq M$. The proof of (4.11) is exactly same but we use $\partial=[\partial_{\mathbf{X}_\ell}(s),\partial_{\mathbf{V}_\ell}(s)]$ to conclude the proof.

We prove the first row of (4.12) by (4.17). By taking the time derivative to (4.8), (4.9) and applying (4.7) we prove the second row of (4.12). ∎

5. Derivative estimate for the generalized characteristics

The main goal of this section is to prove the following key estimate for the derivatives of the generalized characteristics $(X_{\mathbf{cl}}(s;t,x,v),V_{\mathbf{cl}}(s;t,x,v))$ defined in (1.17).

THEOREM 5.1. *There exists $C=C(\Omega,E)>0$ such that for all $(t,x,v)\in[0,T]\times\bar{\Omega}\times\mathbb{R}^3$, $0\leq s\leq t$, with $s\neq t^\ell$ for $\ell=1,2,\cdots,\ell_*$*

$$\begin{aligned}|\partial_x X_{\mathbf{cl}}(s;t,x,v)| &\lesssim e^{C|v|(t-s)}\frac{|v|+1}{\alpha(t,x,v)},\\ |\partial_v X_{\mathbf{cl}}(s;t,x,v)| &\lesssim e^{C|v|(t-s)}\frac{1}{|v|+1},\\ |\partial_x V_{\mathbf{cl}}(s;t,x,v)| &\lesssim e^{C|v|(t-s)}\frac{|v|^3+1}{\alpha^2(t,x,v)},\\ |\partial_v V_{\mathbf{cl}}(s;t,x,v)| &\lesssim e^{C|v|(t-s)}\frac{|v|+1}{\alpha(t,x,v)}.\end{aligned} \tag{5.1}$$

In order to achieve this, we need a crucial bound on the backward exit time:

LEMMA 5.1. *Suppose $E(t,x)\cdot n(x)>c_E$ for all $x\in\partial\Omega$, then there exists $C=C(\Omega,E)\gg 1$ and $0<T\ll 1$ such that for any $(t,x,v)\in[0,T]\times\overline{\Omega}\times\mathbb{R}^3$, $t^1(t,x,v)>0$,*

$$\frac{|t-t^1|}{|\mathbf{v}^1_\perp|}+\frac{|t-t^1||v|}{|\mathbf{v}^1_\perp|}+\frac{|t-t^1||v|^2}{|\mathbf{v}^1_\perp|}<C. \tag{5.2}$$

And for $(t,x,v)\in[0,T]\times\gamma_+\times\mathbb{R}^3$, $t^1(t,x,v)<0$,

$$\frac{|t|}{|\mathbf{v}_\perp|}+\frac{|t||v|}{|\mathbf{v}_\perp|}+\frac{|t||v|^2}{|\mathbf{v}_\perp|}<C. \tag{5.3}$$

Proof. Let $N>10(\|E\|_{L^\infty_{t,x}}+1)$ be fixed. Let's first consider the case $(t,x,v)\in[0,T]\times\overline{\Omega}\times\mathbb{R}^3$, $t^1(t,x,v)>0$, and prove

$$\frac{|t-t^1|}{|\mathbf{v}^1_\perp|}\lesssim 1 \text{ for all } |v|<N. \tag{5.4}$$

From (4.8) we have

$$F_\perp(s)<-c_\xi|\mathbf{v}_\parallel|^2-c_E+C_\xi\mathbf{x}_\perp|\mathbf{v}_\parallel|^2.$$

By choosing $T<\frac{c_\xi}{4NC_\xi}$, we have $\mathbf{x}_\perp<2NT<\frac{c_\xi}{2C_\xi}$, thus

$$F_\perp(s)<-c_E-c_\xi|\mathbf{v}_\parallel|^2+\frac{c_\xi}{2}|\mathbf{v}_\parallel|^2<-c_E,\ \text{for all } t^1<s<t. \tag{5.5}$$

Therefore

$$\begin{aligned}0<\mathbf{x}_\perp(t)&=\int_{t^1}^t \mathbf{v}_\perp(s)ds\\&=\int_{t^1}^t\left(-\mathbf{v}_\perp^1+\int_{t^1}^s F_\perp(\tau)d\tau\right)ds\\&=(t-t^1)(-\mathbf{v}_\perp^1)+\int_{t^1}^t\int_{t^1}^t F_\perp(\tau)d\tau ds.\end{aligned} \tag{5.6}$$

So from (5.5) and (5.6),

$$\frac{c_E}{2}(t-t^1)^2<-\int_{t^1}^t\int_{t^1}^t F_\perp(\tau)d\tau ds<|t-t^1||\mathbf{v}_\perp^1|. \tag{5.7}$$

Therefore $\frac{c_E}{2}(t-t^1)<|\mathbf{v}_\perp^1|$, and this proves (5.4).

Next, for $|v|\geq N$, let $d=\max_{x,y\in\overline{\Omega}}|x-y|$, then $\xi(X(t+t'))=0$ for some $t'<\frac{2d}{N}$ by extending the field as $E(s,x)=E(T,x)$ for $s>T$ if necessary. So we can, without loss of generality, assume $x\in\partial\Omega$. We claim

$$\frac{|t-t^1||v|^2}{|\mathbf{v}_\perp^1|}\lesssim 1 \text{ for all } |v|\geq N. \tag{5.8}$$

Since $(x,v)\in\gamma_+$ we have

$$\begin{aligned}0=&\xi(x^1)\\=&\xi(x)-\int_{t^1}^t\nabla\xi(X(s))\cdot V(s)ds\\=&-(t-t^1)v\cdot\nabla\xi(x)+\int_{t^1}^t\int_s^t\big(V(\tau)\cdot\nabla^2\xi(X(\tau))\cdot V(\tau)+E(\tau,X(\tau))\cdot\nabla\xi(X(\tau))\big)d\tau ds.\end{aligned} \tag{5.9}$$

Note that for $T<\frac{N}{4\|E\|_{L^\infty_{t,x}}}$, $\frac{|v|}{2}<|V(\tau)|<2|v|$ for all $\tau\in[t^1,t]$. Thus from (5.9)

$$\begin{aligned}&|t-t^1|(v\cdot\nabla\xi(x))\\\geq&\frac{C}{8}|t-t^1|^2|v|^2+\int_{t^1}^t\int_s^t E(\tau,X(\tau))\cdot\nabla\xi(X(\tau))d\tau ds\\\geq&\frac{C}{8}|t-t^1|^2|v|^2+\frac{|t-t^1|^2}{2}E(t,x)\cdot\nabla\xi(x)\\&-\int_{t^1}^t\int_s^t\int_\tau^t\frac{d}{d\tau'}(E(\tau',X(\tau'))\cdot\nabla\xi(X(\tau')))d\tau'd\tau ds\\\geq&\frac{C}{8}|t-t^1|^2|v|^2-|t-t^1|^3C_{E,\xi}(1+|v|)\\\geq&|t-t^1|^2\left(\frac{C}{8}|v|^2-|t-t^1|C_{E,\xi}(1+|v|)\right).\end{aligned} \tag{5.10}$$

Since $|v|\geq N$, we have $\frac{C}{8}|v|^2-|t-t^1|C_{E,\xi}(1+|v|)>\frac{C}{20}|v|^2$. Therefore (5.10) gives

$$|v\cdot\nabla\xi(x)|>\frac{C}{20}|t-t^1||v|^2. \tag{5.11}$$

Then using the velocity lemma we have $|t-t^1||v|^2\lesssim|v\cdot\nabla\xi(x)|\lesssim|\mathbf{v}_\perp^1|$, and we conclude (5.8).

Now combining (5.4) and (5.8) we actually have for all $(x,v)\in\gamma_+$,

$$\frac{|t-t^1|}{|\mathbf{v}_\perp^1|}+\frac{|t-t^1||v|^2}{|\mathbf{v}_\perp^1|}\lesssim 1.$$

Therefore

$$\frac{|t-t^1||v|}{|\mathbf{v}_\perp^1|}\leq\max\{\frac{|t-t^1|}{|\mathbf{v}_\perp^1|},\frac{|t-t^1||v|^2}{|\mathbf{v}_\perp^1|}\}\lesssim 1,$$

and we conclude (5.2).

The proof of (5.3) is similar. If $|v|<N$, we have

$$0<\mathbf{x}_\perp(0)=-\int_0^t\mathbf{v}_\perp(s)ds=-\int_0^t\left(\mathbf{v}_\perp-\int_s^t F_\perp(\tau)d\tau\right)ds=-t\mathbf{v}_\perp+\int_0^t\int_s^t F_\perp(\tau)d\tau ds, \tag{5.12}$$

So same as (5.7) we have

$$\frac{c_E}{2}t^2<-\int_0^t\int_s^t F_\perp(\tau)d\tau ds<t|\mathbf{v}_\perp|.$$

Therefore $\frac{c_E}{2}t<|\mathbf{v}_\perp|$. And if $|v|>N$, similarly we get

$$\begin{aligned}0>&\xi(X(0))\\ =&\xi(x)-\int_0^t\nabla\xi(X(s))\cdot V(s)ds\\ =&-|t|(v\cdot\nabla\xi(x))+\int_0^t\int_s^t\big(V(\tau)\cdot\nabla^2\xi(X(\tau))\cdot V(\tau)+E(\tau,X(\tau))\cdot\nabla\xi(X(\tau))\big)d\tau ds.\end{aligned} \tag{5.13}$$

Then by the same argument as lines between (5.9) and (5.11) we get $|v\cdot\nabla\xi(x)|>\frac{C}{20}t|v|^2$, and this proves (5.3). ∎

We need a version of Gronwall's inequality for matrices:

Lemma 5.2. *Let $m>0$, $a(\tau),b(\tau),f(\tau),g(\tau)\geq 0$ for all $0\leq\tau\leq t$, and satisfy $|v|>M\gg 1$, and*

$$\begin{bmatrix}a(\tau)\\ b(\tau)\end{bmatrix}\lesssim\begin{bmatrix}0 & 1\\ m+|v|^2 & |v|\end{bmatrix}\begin{bmatrix}\int_\tau^t a(\tau')\mathrm{d}\tau'\\ \int_\tau^t b(\tau')\mathrm{d}\tau'\end{bmatrix}+\begin{bmatrix}g(t-\tau)\\ h(t-\tau)\end{bmatrix}$$

then

$$\begin{aligned}\begin{bmatrix}a(\tau)\\ b(\tau)\end{bmatrix}\lesssim&\ e^{C(\tau-t)}\begin{bmatrix}1 & |\tau-t|\\ |v|^2|\tau-t| & 1\end{bmatrix}\begin{bmatrix}g(0)\\ h(0)\end{bmatrix}\\ &+\int_t^\tau e^{C(\tau-\tau')}\begin{bmatrix}1 & |\tau-\tau'|\\ |v|^2|\tau-\tau'| & 1\end{bmatrix}\begin{bmatrix}|g'(t-\tau')|\\ |h'(t-\tau')|\end{bmatrix}\mathrm{d}\tau'.\end{aligned} \tag{5.14}$$

Proof. First we consider $A^\varepsilon, B^\varepsilon$ solving, for $\varepsilon > 0$,

$$\begin{bmatrix} A^\varepsilon(\tau) \\ B^\varepsilon(\tau) \end{bmatrix} = C \begin{bmatrix} 0 & 1 \\ m+|v|^2 & |v| \end{bmatrix} \begin{bmatrix} \int_\tau^t A^\varepsilon(\tau')\mathrm{d}\tau' \\ \int_\tau^t B^\varepsilon(\tau')\mathrm{d}\tau' \end{bmatrix} + \begin{bmatrix} g(t-\tau) \\ h(t-\tau) \end{bmatrix} + \begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix}. \tag{5.15}$$

We claim that

$$\begin{aligned} \begin{bmatrix} A^\varepsilon(\tau) \\ B^\varepsilon(\tau) \end{bmatrix} &\lesssim e^{C(\tau-t)} \begin{bmatrix} 1 & |\tau-t| \\ |v|^2|\tau-t| & 1 \end{bmatrix} \begin{bmatrix} g(0)+\varepsilon \\ h(0)+\varepsilon \end{bmatrix} \\ &\quad + \int_t^\tau e^{C(\tau-\tau')} \begin{bmatrix} 1 & |\tau-\tau'| \\ |v|^2|\tau-\tau'| & 1 \end{bmatrix} \begin{bmatrix} |g'(t-\tau')| \\ |h'(t-\tau')| \end{bmatrix} \mathrm{d}\tau'. \end{aligned} \tag{5.16}$$

We consider the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ m+|v|^2 & |v| \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ m+|v|^2 & |v| \end{bmatrix}$. Denote

$$r_1 := \frac{1+\sqrt{5+\frac{4m}{|v|^2}}}{2}, \; r_2 := \frac{1-\sqrt{5+\frac{4m}{|v|^2}}}{2}, \; r_3 := \frac{1}{\sqrt{5+\frac{4m}{|v|^2}}}.$$

Then we diagonalize this matrix as

$$\begin{bmatrix} 0 & 1 \\ m+|v|^2 & |v| \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ r_1|v| & r_2|v| \end{bmatrix} \begin{bmatrix} r_1|v| & 0 \\ 0 & r_2|v| \end{bmatrix} \begin{bmatrix} -r_2r_3 & r_3\frac{1}{|v|} \\ r_1r_3 & -r_3\frac{1}{|v|} \end{bmatrix}.$$

Denote $\begin{bmatrix} \mathcal{A}^\varepsilon(\tau) \\ \mathcal{B}^\varepsilon(\tau) \end{bmatrix} := \begin{bmatrix} -r_2r_3 & r_3\frac{1}{|v|} \\ r_1r_3 & -r_3\frac{1}{|v|} \end{bmatrix} \begin{bmatrix} A^\varepsilon(\tau) \\ B^\varepsilon(\tau) \end{bmatrix}$ and rewrite the equations as

$$\frac{d}{d\tau} \begin{bmatrix} \mathcal{A}^\varepsilon(\tau) \\ \mathcal{B}^\varepsilon(\tau) \end{bmatrix} = C \begin{bmatrix} r_1|v| & 0 \\ 0 & r_2|v| \end{bmatrix} \begin{bmatrix} \mathcal{A}^\varepsilon(\tau) \\ \mathcal{B}^\varepsilon(\tau) \end{bmatrix} + \begin{bmatrix} -r_2r_3 & r_3\frac{1}{|v|} \\ r_1r_3 & -r_3\frac{1}{|v|} \end{bmatrix} \begin{bmatrix} g'(t-\tau) \\ h'(t-\tau) \end{bmatrix}.$$

Directly we compute

$$\begin{aligned} \begin{bmatrix} \mathcal{A}^\varepsilon(\tau) \\ \mathcal{B}^\varepsilon(\tau) \end{bmatrix} &= \begin{bmatrix} e^{Cr_1|v|(\tau-t)}\mathcal{A}^\varepsilon(t) \\ e^{C_\xi r_2|v|(\tau-t)}\mathcal{B}^\varepsilon(t) \end{bmatrix} \\ &\quad + \int_t^\tau \begin{bmatrix} e^{Cr_2|v|(\tau-\tau')} & 0 \\ 0 & e^{Cr_2|v|(\tau-\tau')} \end{bmatrix} \begin{bmatrix} -r_2r_3 & r_3\frac{1}{|v|} \\ r_1r_3 & -r_3\frac{1}{|v|} \end{bmatrix} \begin{bmatrix} g'(t-\tau') \\ h'(t-\tau') \end{bmatrix} \mathrm{d}\tau'. \end{aligned}$$

Then

$$\begin{aligned} \begin{bmatrix} A^\varepsilon(\tau) \\ B^\varepsilon(\tau) \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ r_1|v| & r_2|v| \end{bmatrix} \begin{bmatrix} \mathcal{A}^\varepsilon(\tau) \\ \mathcal{B}^\varepsilon(\tau) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ r_1|v| & r_2|v| \end{bmatrix} \begin{bmatrix} e^{Cr_1|v|(\tau-t)} & 0 \\ 0 & e^{Cr_2|v|(\tau-t)} \end{bmatrix} \begin{bmatrix} -r_2r_3 & r_3\frac{1}{|v|} \\ r_1r_3 & -r_3\frac{1}{|v|} \end{bmatrix} \begin{bmatrix} A^\varepsilon(t) \\ B^\varepsilon(t) \end{bmatrix} \\ &\quad + \int_t^\tau \begin{bmatrix} 1 & 1 \\ r_1|v| & r_2|v| \end{bmatrix} \begin{bmatrix} e^{Cr_1|v|(\tau-\tau')} & 0 \\ 0 & e^{Cr_2|v|(\tau-\tau')} \end{bmatrix} \\ &\qquad \times \begin{bmatrix} -r_2r_3 & r_3\frac{1}{|v|} \\ r_1r_3 & -r_3\frac{1}{|v|} \end{bmatrix} \begin{bmatrix} g'(t-\tau') \\ h'(t-\tau') \end{bmatrix} \mathrm{d}\tau'. \end{aligned}$$

Directly, the RHS equals

$$\begin{bmatrix} r_3\big(r_1e^{Cr_2|v|(\tau-t)}-r_2e^{Cr_1|v|(\tau-t)}\big) & \frac{r_3}{|v|}\big(e^{Cr_1|v|(\tau-t)}-e^{Cr_2|v|(\tau-t)}\big) \\ -r_1r_2r_3|v|\big(e^{Cr_1|v|(\tau-t)}-e^{Cr_2|v|(\tau-t)}\big) & r_3\big(r_1e^{Cr_2|v|(\tau-t)}-r_2e^{Cr_1|v|(\tau-t)}\big)\end{bmatrix}\begin{bmatrix}A^\varepsilon(t)\\ B^\varepsilon(t)\end{bmatrix}$$
$$+\int_t^\tau \begin{bmatrix} r_3\big(r_1e^{Cr_2|v|(\tau-\tau')}-r_2e^{Cr_1|v|(\tau-\tau')}\big) & \frac{r_3}{|v|}\big(e^{Cr_1|v|(\tau-\tau')}-e^{Cr_2|v|(\tau-\tau')}\big) \\ -r_1r_2r_3|v|\big(e^{Cr_1|v|(\tau-\tau')}-e^{Cr_2|v|(\tau-\tau')}\big) & r_3\big(r_1e^{Cr_2|v|(\tau-\tau')}-r_2e^{Cr_1|v|(\tau-\tau')}\big)\end{bmatrix}$$
$$\times\begin{bmatrix}g'(t-\tau')\\ h'(t-\tau')\end{bmatrix}\mathrm{d}\tau'.$$

Since $|v|>M$, we have $|r_1-r_2|\lesssim 1$, so by expansion we have $|e^{Cr_1|v|(\tau-t)}-e^{Cr_2|v|(\tau-t)}|\lesssim_{C^{\xi,\delta}}|v||\tau-t|e^{C_{\xi,\delta}|v|(\tau-t)}$. Therefore we conclude (5.16).

Now we claim

$$a(\tau)\le A(\tau),\quad b(\tau)\le B(\tau),\quad \text{for all}\quad \tau\le t. \tag{5.17}$$

First we claim that $a(\tau)\le A^\varepsilon(\tau)$ and $b(\tau)\le B^\varepsilon(\tau)$ for all τ. Otherwise, we should have at least for some time τ_0 such that $a(\tau)\le A^\varepsilon(\tau)$ and $b(\tau)\le B^\varepsilon(\tau)$ for $\tau_0\le\tau\le t$ but either $a(\tau)>A^\varepsilon(\tau)$ or $b(\tau)>B^\varepsilon(\tau)$ for a small neighborhood of $\tau>\tau_0$. Especially either $a(\tau_0)=A^\varepsilon(\tau_0)$ or $b(\tau_0)=B^\varepsilon(\tau_0)$. But this is impossible. Since

$$\begin{bmatrix}A^\varepsilon(\tau)-a(\tau)\\ B^\varepsilon(\tau)-b(\tau)\end{bmatrix}\ge C\begin{bmatrix}0 & 1\\ m+|v|^2 & |v|\end{bmatrix}\begin{bmatrix}\int_\tau^t(A^\varepsilon(\tau')-a(\tau'))\mathrm{d}\tau'\\ \int_\tau^t(B^\varepsilon(\tau')-b(\tau'))\mathrm{d}\tau'\end{bmatrix}+\begin{bmatrix}\varepsilon\\ \varepsilon\end{bmatrix},$$

we have $\begin{bmatrix}A^\varepsilon(\tau)-a(\tau)\\ B^\varepsilon(\tau)-b(\tau)\end{bmatrix}\ge\begin{bmatrix}\varepsilon\\ \varepsilon\end{bmatrix}>0$ as $\tau\to\tau_0^+$. Then we prove the inequalities (5.17) by letting $\varepsilon\to 0$. Finally we prove the claim (5.14) from (5.16) and (5.17) and letting $\varepsilon\to 0$. ∎

Proof. **(Proof of Theorem 5.1.)** First we consider the case of $t<t_{\mathbf{b}}(t,x,v)$. Directly

$$\left|\frac{\partial(X_{\mathbf{cl}}(s;t,x,v),V_{\mathbf{cl}}(s,t,x,v))}{\partial(t,x,v)}\right|\lesssim\begin{bmatrix}|v|+(t-s) & 1 & (t-s)\\ \|E\|_{L^\infty_{t,x}}+(t-s) & |v|+(t-s) & 1\end{bmatrix}.$$

The computation will be the same as that we will get for (5.30).

Now we consider the case of $t\ge t_{\mathbf{b}}(t,x,v)$. We split our proof into 10 steps.

Step 1. Moving frames and grouping with respect to the scaling $t|v|=L_\xi$, with fixed $0<L_\xi\ll 1$.

Fix $(t,x,v)\in[0,\infty)\times\bar\Omega\times\mathbb{R}^3$. Also we fix a small constant δ such that $\delta\ll\|E\|_{L^\infty_{t,x}}$. We define, at the boundary,

$$\mathbf{r}^\ell:=\frac{|\mathbf{v}_\perp^\ell|}{|v^\ell|}. \tag{5.18}$$

Bounces ℓ (and (t^ℓ,x^ℓ,v^ℓ)) are categorized as *Type I*, *Type II*, or *Type III*:

$$\begin{aligned}&\text{all the bounces } \ell \text{ are } \textit{Type I} \text{ if and only if } |v|\le\delta,\\ &\text{a bounce } \ell \text{ is } \textit{Type II} \text{ if and only if } |v|>\delta, \mathbf{r}^\ell\le\sqrt{\delta},\\ &\text{a bounce } \ell \text{ is } \textit{Type III} \text{ if and only if } |v|>\delta, \mathbf{r}^\ell>\sqrt{\delta}.\end{aligned} \tag{5.19}$$

Now we choose $T < \frac{\sqrt{\delta}}{\|E\|^2_{L^\infty_{t,x}}+1}$. Then if $|v|\leq \delta$, we have

$$\max_{t^{\ell+1}\leq s\leq t^\ell} |\xi(X_{\mathbf{cl}}(s;t^\ell,x^\ell,v^\ell))| \leq |v|T + \|E\|_{L^\infty_{t,x}} T^2 \leq 2\delta.$$

And if $|v|>\delta, \mathbf{r}^\ell \leq \sqrt{\delta}$, we have from (5.2)

$$\max_{t^{\ell+1}\leq s\leq t^\ell} |\xi(X_{\mathbf{cl}}(s;t^\ell,x^\ell,v^\ell))| \lesssim |t^\ell - t^{\ell+1}|^2|v^\ell|^2 + (\|E\|^2_{L^\infty_{t,x}}+1)T^2 \lesssim \left(\frac{|\mathbf{v}^\ell_\perp|}{|v^\ell|}\right)^2 + \delta \lesssim \delta.$$

Therefore if a bounce ℓ is *Type I* or *Type II* then $\max_{t^{\ell+1}\leq\tau\leq t^\ell}|\xi(X_{\mathbf{cl}}(\tau;t,x,v))|\leq C\delta$.

Now we assign a coordinate chart for each bounce ℓ (moving frames). For *Type I* bounces ℓ in (5.19) we let $\mathbf{p}^\ell = (z^\ell, w^\ell)$ with $z^\ell = x^\ell$ and $w^\ell = \tau_1(x^\ell)$. We choose $\mathbf{p}^\ell-$spherical coordinate in Lemma 4.1 and (4.4) with this $\mathbf{p}^\ell$.

For *Type II* bounce ℓ, we choose $\mathbf{p}^\ell := (z^\ell, w^\ell)$ on $\partial\Omega\times\mathbb{S}^2$ with $n(z^\ell)\cdot w^\ell = 0$

$$z^\ell = x^\ell, \quad w^\ell = \frac{v^\ell - (v^\ell\cdot n(z^\ell))n(z^\ell)}{|v^\ell - (v^\ell\cdot n(z^\ell))n(z^\ell)|}. \tag{5.20}$$

Note that, by the definition of *Type I* bounce, $|v^\ell - (v^\ell\cdot n(z^\ell)n(z^\ell))|^2 = |v|^2 - |\mathbf{v}^\ell_\perp|^2 \gtrsim |v|^2(1-\delta)\gtrsim_\delta |v|^2$ and hence w^ℓ is well-defined.

Moreover for *Type I* and *Type II* bounces

$$|X_{\mathbf{cl}}(s;t,x,v) - \mathcal{L}_{\mathbf{p}^\ell}| \gtrsim C_\delta > 0, \tag{5.21}$$

for $|v||t^\ell - s| \leq \frac{1}{100}\min_{x\in\partial\Omega}|x|$. This is due to the fact that the projection of $V_{\mathbf{cl}}(s)$ on the plane passing z^ℓ and perpendicular to $n(z^\ell)\times w^\ell$ is at most $|v|$ magnitude but the distance from z^ℓ to the origin(the projection of poles $\mathcal{N}_{\mathbf{p}^\ell}$ and $\mathcal{S}_{\mathbf{p}^\ell}$) has lower bound $\frac{1}{10}\min_{x\in\partial\Omega}|x|$, $|s-t^\ell|\ll 1$.

For *Type III* bounce $\ell(t^\ell,x^\ell,v^\ell)$, we choose $\mathbf{p}^\ell = (z^\ell,w^\ell)$ with $|z^\ell - x^\ell|\leq\sqrt{\delta}$ and we choose arbitrary $w^\ell\in\mathbb{S}^2$ satisfying $n(z^\ell)\cdot w^\ell = 0$. Note that unlike *Type I*, this $\mathbf{p}^\ell-$spherical coordinate might not be defined for $s\in[t^{\ell+1},t^\ell]$ but only defined near the boundary.

Whenever the moving frame is defined (for all $\tau\in(t^{\ell+1},t^\ell]$ when ℓ is *Type I* or *Type II*, and $|\tau - t^\ell|\ll 1$ when ℓ is *Type III*) we denote, by (4.4),

$$(\mathbf{X}_\ell(\tau),\mathbf{V}_\ell(\tau)) = (\mathbf{x}_{\perp_\ell}(\tau),\mathbf{x}_{\|_\ell}(\tau),\mathbf{v}_{\perp_\ell}(\tau),\mathbf{v}_{\|_\ell}(\tau)) := \Phi^{-1}_{\mathbf{p}^\ell}(X_{\mathbf{cl}}(\tau),V_{\mathbf{cl}}(\tau)).$$

Especially at the boundary we denote

$$(\mathbf{x}^\ell_{\perp_\ell},\mathbf{x}^\ell_{\|_\ell},\mathbf{v}^\ell_{\perp_\ell},\mathbf{v}^\ell_{\|_\ell}) := \lim_{\tau\uparrow t^\ell}(\mathbf{X}_\ell(\tau),\mathbf{V}_\ell(\tau)), \quad \text{with } \mathbf{x}^\ell_{\perp_\ell} = 0,\ \mathbf{v}^\ell_{\perp_\ell}\geq 0.$$

Then we define

$$(\mathbf{x}^{\ell+1}_{\perp_\ell},\mathbf{x}^{\ell+1}_{\|_\ell},\mathbf{v}^{\ell+1}_{\|_\ell}) = \lim_{\tau\downarrow t^{\ell+1}}(\mathbf{x}_{\perp_\ell}(\tau),\mathbf{x}_{\|_\ell}(\tau),\mathbf{v}_{\|_\ell}(\tau)),$$

and

$$\mathbf{v}^{\ell+1}_{\perp_\ell} := -\lim_{\tau\downarrow t^{\ell+1}}\mathbf{v}_{\perp_\ell}(\tau). \tag{5.22}$$

Now we regroup the indices of the specular cycles, without order changing, as

$$\{0,1,2,\cdots,\ell_*-1,\ell_*\}=\{0\}\cup\mathcal{G}_1\cup\mathcal{G}_2\cup\cdots\cup\mathcal{G}_{[\frac{|t-s||v|}{L_\xi}]}\cup\mathcal{G}_{[\frac{|t-s||v|}{L_\xi}]+1},$$

where $[a]\in\mathbb{N}$ is the greatest integer less than or equal to a. Each group is

$$\begin{aligned}\mathcal{G}_1&=\{1,\cdots,\ell_1-1,\ell_1\},\\ \mathcal{G}_2&=\{\ell_1,\ell_1+1,\cdots,\ell_2-1,\ell_2\},\\ &\vdots\\ \mathcal{G}_{[\frac{|t-s||v|}{L_\xi}]}&=\{\ell_{[\frac{|t-s||v|}{L_\xi}]-1},\ell_{[\frac{|t-s||v|}{L_\xi}]-1}+1,\cdots,\ell_{[\frac{|t-s||v|}{L_\xi}]}-1,\ell_{[\frac{|t-s||v|}{L_\xi}]}\},\\ \mathcal{G}_{[\frac{|t-s||v|}{L_\xi}]+1}&=\{\ell_{[\frac{|t-s||v|}{L_\xi}]},\ell_{[\frac{|t-s||v|}{L_\xi}]}+1,\cdots,\ell_*\},\end{aligned} \tag{5.23}$$

where $\ell_1=\inf\{\ell\in\mathbb{N}:|v|\times|t^0-t^{\ell_1}|\geq L_\xi\}$ and inductively

$$\ell_i=\inf\{\ell\in\mathbb{N}:|v|\times|t^{\ell_i}-t^{\ell_{i+1}}|\geq L_\xi\}, \tag{5.24}$$

and we have denoted $\ell_*=\ell_{[\frac{|t-s||v|}{L_\xi}]+1}$.

Our analysis is carried out in each group G_i. We note that within each G_i, $|t^{\ell_i}-t^{\ell_{i+1}}||v|<L_\xi$ by our design, so from the velocity lemma, r_{ℓ_i} is comparable to each other, so is $|v^\ell|$. By the chain rule, with the assigned $\mathbf{p}^\ell-$spherical coordinate (moving frame), we have for fixed $0\leq s\leq t$ and $s\in(t^{\ell_*+1},t^{\ell_*})$

$$\begin{aligned}&\frac{\partial(X_{\mathbf{cl}}(s;t,x,v),V_{\mathbf{cl}}(s;t,x,v))}{\partial(t,x,v)}\\ =&\underbrace{\frac{\partial(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))}{\partial(t^{\ell_*},\mathbf{x}^{\ell_*}_{\|_{\ell_*}},\mathbf{v}^{\ell_*}_{\perp_{\ell_*}},\mathbf{v}^{\ell_*}_{\|_{\ell_*}})}}_{\text{from the last bounce to the } s-\text{plane}}\\ &\underbrace{\times\prod_{i=1}^{[\frac{|t-s||v|}{L_*}]}\underbrace{\frac{\partial(t^{\ell_{i+1}},\mathbf{x}^{\ell_{i+1}}_{\|_{\ell_{i+1}}},\mathbf{v}^{\ell_{i+1}}_{\perp_{\ell_{i+1}}},\mathbf{v}^{\ell_{i+1}}_{\|_{\ell_{i+1}}})}{\partial(t^{\ell_{i+1}-1},\mathbf{x}^{\ell_{i+1}-1}_{\|_{\ell_{i+1}-1}},\mathbf{v}^{\ell_{i+1}-1}_{\perp_{\ell_{i+1}-1}},\mathbf{v}^{\ell_{i+1}-1}_{\|_{\ell_{i+1}-1}})}\times\cdots\times\frac{\partial(t^{\ell_i+1},\mathbf{x}^{\ell_i+1}_{\|_{\ell_i+1}},\mathbf{v}^{\ell_i+1}_{\perp_{\ell_i+1}},\mathbf{v}^{\ell_i+1}_{\|_{\ell_i+1}})}{\partial(t^{\ell_i},\mathbf{x}^{\ell_i}_{\|_{\ell_i}},\mathbf{v}^{\ell_i}_{\perp_{\ell_i}},\mathbf{v}^{\ell_i}_{\|_{\ell_i}})}}_{i-\text{th intermediate group}}}_{\text{whole intermediate groups}}\\ &\times\underbrace{\frac{\partial(t^1,\mathbf{x}^1_{\|_1},\mathbf{v}^1_{\perp_1},\mathbf{v}^1_{\|_1})}{\partial(t,x,v)}}_{\text{from the } t-\text{plane to the first bounce}}.\end{aligned} \tag{5.25}$$

Before we start to calculate the matrix for any bounces, we first prove a claim that will be used later: there exists a constant $C=C(\xi)$ such that for any bounce ℓ and any $t^{\ell+1}<s<t^\ell$ we have

$$\left|\frac{\partial F_\perp(s)}{\partial\tau}+\frac{\partial F_\perp(s)}{\partial t^\ell}\right|+\left|\frac{\partial F_\|(s)}{\partial\tau}+\frac{\partial F_\|(s)}{\partial t^\ell}\right|<C\|\partial_t E\|_{L^\infty_{t,x}}. \tag{5.26}$$

By direct computation we have

$$\begin{aligned}
\frac{\partial \mathbf{x}_\perp(s)}{\partial t^\ell} &= -\mathbf{v}_\perp(s) + \int_s^{t^\ell}\int_\tau^{t^\ell} (\partial_{\tau'} F_\perp(\tau') + \partial_{t^\ell} F_\perp(\tau'))\,d\tau' d\tau,\\
\frac{\partial \mathbf{x}_\parallel(s)}{\partial t^\ell} &= -\mathbf{v}_\parallel(s) + \int_s^{t^\ell}\int_\tau^{t^\ell} \left(\partial_{\tau'} F_\parallel(\tau') + \partial_{t^\ell} F_\parallel(\tau')\right) d\tau' d\tau,\\
\frac{\partial \mathbf{v}_\parallel(s)}{\partial t^\ell} &= -F_\parallel(s) - \int_s^{t^\ell} \left(\partial_\tau F_\parallel(\tau) + \partial_{t^\ell} F_\parallel(\tau)\right) d\tau,\\
\frac{\partial \mathbf{v}_\parallel(s)}{\partial t^\ell} &= -F_\perp(s) - \int_s^{t^\ell} (\partial_\tau F_\perp(\tau) + \partial_{t^\ell} F_\perp(\tau))\,d\tau,
\end{aligned} \tag{5.27}$$

and

$$\begin{aligned}
&\left|\frac{\partial F_\perp(s)}{\partial \tau} + \frac{\partial F_\perp(s)}{\partial t^\ell}\right| + \left|\frac{\partial F_\parallel(s)}{\partial \tau} + \frac{\partial F_\parallel(s)}{\partial t^\ell}\right|\\
&= \Bigg|\nabla_{\mathbf{x}_\perp} F_\perp \cdot \left(\mathbf{v}_\perp(s) + \frac{\partial \mathbf{x}_\perp(s)}{\partial t^\ell}\right) + \nabla_{\mathbf{x}_\parallel} F_\perp \cdot \left(\mathbf{v}_\parallel(s) + \frac{\partial \mathbf{x}_\parallel(s)}{\partial t^\ell}\right)\\
&\quad + \nabla_{\mathbf{v}_\parallel} F_\perp \cdot \left(F_\parallel(s) + \frac{\partial \mathbf{v}_\parallel(s)}{\partial t^\ell}\right) - \partial_s E \cdot \mathbf{n}(\mathbf{x}_\parallel)\Bigg|\\
&\quad + \Bigg|\nabla_{\mathbf{x}_\perp} F_\parallel \cdot \left(\mathbf{v}_\perp(s) + \frac{\partial \mathbf{x}_\perp(s)}{\partial t^\ell}\right) + \nabla_{\mathbf{x}_\parallel} F_\parallel \cdot \left(\mathbf{v}_\parallel(s) + \frac{\partial \mathbf{x}_\parallel(s)}{\partial t^\ell}\right)\\
&\quad + \nabla_{\mathbf{v}_\perp} F_\parallel \cdot \left(F_\perp(s) + \frac{\partial \mathbf{v}_\perp(s)}{\partial t^\ell}\right) + \nabla_{\mathbf{v}_\parallel} F_\parallel \cdot \left(F_\parallel(s) + \frac{\partial \mathbf{v}_\parallel(s)}{\partial t^\ell}\right)\\
&\quad - \sum_{i=1,2} G_{\mathbf{p},ij}(\mathbf{x}_{\perp_\mathbf{p}}, \mathbf{x}_{\parallel_\mathbf{p}}) \frac{(-1)^i \partial_s E(s, -\mathbf{x}_\perp \mathbf{n}(\mathbf{x}_\parallel) + \eta(\mathbf{x}_\parallel)) \cdot \{\mathbf{n}_\mathbf{p}(\mathbf{x}_{\parallel_\mathbf{p}}) \times \partial_{i+1}\eta_\mathbf{p}(\mathbf{x}_{\parallel_\mathbf{p}})\}}{\mathbf{n}_\mathbf{p}(\mathbf{x}_{\parallel_\mathbf{p}}) \cdot (\partial_1 \eta_\mathbf{p}(\mathbf{x}_{\parallel_\mathbf{p}}) \times \partial_2 \eta_\mathbf{p}(\mathbf{x}_{\parallel_\mathbf{p}}))}\Bigg|.
\end{aligned} \tag{5.28}$$

Then from (5.27), (5.28), and using the fact that $\|\nabla_{\mathbf{x}_\parallel,\mathbf{x}_\perp} F_\parallel\|_\infty + \|\nabla_{\mathbf{x}_\parallel,\mathbf{x}_\perp} F_\perp\|_\infty \lesssim |v|^2+1$, $\|\nabla_{\mathbf{v}_\parallel} F_\perp\|_\infty + \|\nabla_{\mathbf{v}_\perp,\mathbf{v}_\parallel} F_\parallel\|_\infty \lesssim |v|+1$, we have

$$\begin{aligned}
&\left|\frac{\partial F_\perp(s)}{\partial \tau} + \frac{\partial F_\perp(s)}{\partial t^\ell}\right| + \left|\frac{\partial F_\parallel(s)}{\partial \tau} + \frac{\partial F_\parallel(s)}{\partial t^\ell}\right|\\
&\lesssim (|v|^2+1)\int_s^{t^\ell}\int_\tau^{t^\ell} \left(|\partial_{\tau'} F_\perp(\tau') + \partial_{t^\ell} F_\perp(\tau')| + |\partial_{\tau'} F_\parallel(\tau') + \partial_{t^\ell} F_\parallel(\tau')|\right) d\tau' d\tau\\
&\quad + (|v|+1)\int_s^{t^\ell} \left(|\partial_{\tau'} F_\perp(\tau') + \partial_{t^\ell} F_\perp(\tau')| + |\partial_{\tau'} F_\parallel(\tau') + \partial_{t^\ell} F_\parallel(\tau')|\right) d\tau' + \|\partial_s E\|_\infty\\
&\lesssim (|v|^2+1)\int_s^{t^\ell} (\tau'-s)\left(|\partial_{\tau'} F_\perp(\tau') + \partial_{t^\ell} F_\perp(\tau')| + |\partial_{\tau'} F_\parallel(\tau') + \partial_{t^\ell} F_\parallel(\tau')|\right) d\tau'\\
&\quad + (|v|+1)\int_s^{t^\ell} \left(|\partial_{\tau'} F_\perp(\tau') + \partial_{t^\ell} F_\perp(\tau')| + |\partial_{\tau'} F_\parallel(\tau') + \partial_{t^\ell} F_\parallel(\tau')|\right) d\tau' + \|\partial_s E\|_\infty\\
&\lesssim (|v|+1)\int_s^{t^\ell} \left(|\partial_{\tau'} F_\perp(\tau') + \partial_{t^\ell} F_\perp(\tau')| + |\partial_{\tau'} F_\parallel(\tau') + \partial_{t^\ell} F_\parallel(\tau')|\right) d\tau' + \|\partial_s E\|_\infty,
\end{aligned} \tag{5.29}$$

where for the second inequality we switch the order of integration $\int_s^\ell \int_\tau^{t^\ell} 1 d\tau' d\tau = \int_s^\ell \int_s^{\tau'} 1 d\tau d\tau' = \int_s^\ell (\tau'-s) d\tau'$, and for the third inequality we use $|v|(t^\ell - s) \lesssim 1$.

Therefore from (5.28) and Gronwall's inequality we get

$$\left|\frac{\partial F_\perp(s)}{\partial \tau} + \frac{\partial F_\perp(s)}{\partial t^\ell}\right| + \left|\frac{\partial F_\parallel(s)}{\partial \tau} + \frac{\partial F_\parallel(s)}{\partial t^\ell}\right| \lesssim \|\partial_s E\|_{L^\infty_{t,x}} e^{\int_s^{t^\ell} (|v|+1) d\tau'} \lesssim \|\partial_s E\|_{L^\infty_{t,x}},$$

and this proves (5.26).

Step 2. From the last bounce ℓ_* to the $s-$plane

We choose $s^{\ell_*}\in(\frac{t^{\ell_*}+s}{2},t^{\ell_*})\subset(s,t^{\ell_*})$ such that $|v||t^{\ell_*}-s^{\ell_*}|\ll 1$ and the ℓ_*-spherical coordinate $(\mathbf{X}_{\ell_*}(s^{\ell_*}),\mathbf{V}_{\ell_*}(s^{\ell_*}))$ is well-defined regardless of types of ℓ_* in (5.19). Notice that s^{ℓ_*} is independent of t^{ℓ_*} and s so that $\frac{\partial s^{\ell_*}}{\partial t^{\ell_*}}=0=\frac{\partial s^{\ell_*}}{\partial s}$.

We first follow the flow in (x,v) co-ordinate to near the boundary at $t=s^{\ell_*}$, change to the chart to (X,V), then follow the flow in (X,V). Regarding s^{ℓ_*} as a free variable, by the chain rule,

$$\begin{aligned}
&\frac{\partial(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))}{\partial(t^{\ell_*},\mathbf{x}^{\ell_*}_{\|_{\ell_*}},\mathbf{v}^{\ell_*}_{\perp_{\ell_*}},\mathbf{v}^{\ell_*}_{\|_{\ell_*}})}\\
&=\frac{\partial(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))}{\partial(s^{\ell_*},\mathbf{X}_{\ell_*}(s^{\ell_*}),\mathbf{V}_{\ell_*}(s^{\ell_*}))}\frac{\partial(s^{\ell_*},\mathbf{x}_{\perp_{\ell_*}}(s^{\ell_*}),\mathbf{x}_{\|_{\ell_*}}(s^{\ell_*}),\mathbf{v}_{\perp_{\ell_*}}(s^{\ell_*}),\mathbf{v}_{\|_{\ell_*}}(s^{\ell_*}))}{\partial(t^{\ell_*},\mathbf{x}^{\ell_*}_{\|_{\ell_*}},\mathbf{v}^{\ell_*}_{\perp_{\ell_*}},\mathbf{v}^{\ell_*}_{\|_{\ell_*}})}\\
&=\frac{\partial(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))}{\partial(s^{\ell_*},X_{\mathbf{cl}}(s^{\ell_*}),V_{\mathbf{cl}}(s^{\ell_*}))}\frac{\partial(s^{\ell_*},X_{\mathbf{cl}}(s^{\ell_*}),V_{\mathbf{cl}}(s^{\ell_*}))}{\partial(s^{\ell_*},\mathbf{X}_{\ell_*}(s^{\ell_*}),\mathbf{V}_{\ell_*}(s^{\ell_*}))}\\
&\quad\times\frac{\partial(s^{\ell_*},\mathbf{x}_{\perp_{\ell_*}}(s^{\ell_*}),\mathbf{x}_{\|_{\ell_*}}(s^{\ell_*}),\mathbf{v}_{\perp_{\ell_*}}(s^{\ell_*}),\mathbf{v}_{\|}(s^{\ell_*}))}{\partial(t^{\ell_*},\mathbf{x}^{\ell_*}_{\|_{\ell_*}},\mathbf{v}^{\ell_*}_{\perp_{\ell_*}},\mathbf{v}^{\ell_*}_{\|_{\ell_*}})}.
\end{aligned}$$

Firstly, we claim

$$\begin{aligned}
&\frac{\partial(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))}{\partial(s^{\ell_*},\mathbf{X}_{\ell_*}(s^{\ell_*}),\mathbf{V}_{\ell_*}(s^{\ell_*}))}\\
&=\begin{bmatrix}-V_{\mathbf{cl}}(s^{\ell_*})+O(1)|s^{\ell_*}-s| & O_\xi(1)(1+|v||s^{\ell_*}-s|) & O_\xi(1)|s^{\ell_*}-s|\\ -E-O(1)(s^{\ell_*}-s)|v| & O_\xi(1)(|v|+|s^{\ell_*}-s|) & O_\xi(1)(1+|s^{\ell_*}-s|)\end{bmatrix}. \qquad (5.30)
\end{aligned}$$

Since

$$\begin{aligned}
X_{\mathbf{cl}}(s)&=X_{\mathbf{cl}}(s^{\ell_*})-\int_s^{s^{\ell_*}}V_{\mathbf{cl}}(\tau)\,d\tau=X_{\mathbf{cl}}(s^{\ell_*})-(s^{\ell_*}-s)V_{\mathbf{cl}}(s^{\ell_*})\\
&\quad+\int_s^{s^{\ell_*}}\int_\tau^{s^{\ell_*}}E(\tau',X_{\mathbf{cl}}(\tau'))\,d\tau'd\tau,\\
V_{\mathbf{cl}}(s)&=V_{\mathbf{cl}}(s^{\ell_*})-\int_s^{s^{\ell_*}}E(\tau,X_{\mathbf{cl}}(\tau))\,d\tau, \qquad (5.31)
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{\partial X_{\mathbf{cl}}(s)}{\partial s^{\ell_*}}\\
&=-V_{\mathbf{cl}}(s^{\ell_*})+\int_s^{s^{\ell_*}}\left[E(s^{\ell_*},X_{\mathbf{cl}}(s^{\ell_*}))+\int_\tau^{s^{\ell_*}}\nabla_xE(\tau',X_{\mathbf{cl}}(\tau'))\frac{\partial X_{\mathbf{cl}}(\tau')}{\partial s^{\ell_*}}\,d\tau'\right]d\tau\\
&=-V_{\mathbf{cl}}(s^{\ell_*})+(s^{\ell_*}-s)E(s^{\ell_*},X_{\mathbf{cl}}(s^{\ell_*}))+\int_s^{s^{\ell_*}}\int_s^{\tau'}\nabla_xE(\tau',X_{\mathbf{cl}}(\tau'))\frac{\partial X_{\mathbf{cl}}(\tau')}{\partial s^{\ell_*}}\,d\tau d\tau'\\
&=-V_{\mathbf{cl}}(s^{\ell_*})+(s^{\ell_*}-s)E(s^{\ell_*},X_{\mathbf{cl}}(s^{\ell_*}))+\int_s^{s^{\ell_*}}(\tau'-s)\nabla_xE(\tau',X_{\mathbf{cl}}(\tau'))\frac{\partial X_{\mathbf{cl}}(\tau')}{\partial s^{\ell_*}}\,d\tau'. \qquad (5.32)
\end{aligned}$$

By Gronwall's inequality we have

$$|\frac{\partial X_{\mathbf{cl}}(s)}{\partial s^{\ell_*}}| \lesssim (|V_{\mathbf{cl}}(s^{\ell_*})| + (s^{\ell_*}-s)|E|)e^{\int_s^{s^{\ell_*}}(s^{\ell_*}-s)|\nabla_x E|\,d\tau'} \lesssim |V_{\mathbf{cl}}(s^{\ell_*})| + |s^{\ell_*}-s|. \tag{5.33}$$

Plug (5.33) into (5.32) to get

$$\frac{\partial X_{\mathbf{cl}}(s)}{\partial s^{\ell_*}} = -V_{\mathbf{cl}}(s^{\ell_*}) + O_{\xi,\|\nabla E\|_{L^\infty_{t,x}}}(1)|s^{\ell_*}-s|. \tag{5.34}$$

Similarly we have

$$\begin{aligned}\frac{\partial V_{\mathbf{cl}}(s)}{\partial s^{\ell_*}} &= -E(s^{\ell_*}, X_{\mathbf{cl}}(s^{\ell_*})) - \int_s^{s^{\ell_*}} \nabla_x E(\tau, X_{\mathbf{cl}}(\tau)) \frac{\partial X_{\mathbf{cl}}(\tau)}{\partial s^{\ell_*}}\,d\tau \\ &= -E(s^{\ell_*}, X_{\mathbf{cl}}(s^{\ell_*})) - O_{\xi,\|\nabla E\|_{L^\infty_{t,x}}}|(1)(s^{\ell_*}-s)|v|. \end{aligned} \tag{5.35}$$

Also, using the fact that for $\partial = [\frac{\partial}{X_{\mathbf{cl}}(s^{\ell_*})}, \frac{\partial}{V_{\mathbf{cl}}(s^{\ell_*})}]$, $|\partial X_{\mathbf{cl}}(s)| + |\partial V_{\mathbf{cl}}(s)| \lesssim 1$, we can combine (5.31), (5.33), and (5.35) to get

$$\begin{aligned}&\frac{\partial(X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))}{\partial(s^{\ell_*}, X_{\mathbf{cl}}(s^{\ell_*}), V_{\mathbf{cl}}(s^{\ell_*}))} \\ &= \begin{bmatrix} -V_{\mathbf{cl}}(s^{\ell_*}) + O(1)|s^{\ell_*}-s| & \mathbf{Id}_{3,3} + O(1)|s^{\ell_*}-s| & -(s^{\ell_*}-s)\mathbf{Id}_{3,3} + O(1)|s^{\ell_*}-s| \\ -E - O(1)(s^{\ell_*}-s)|v| & \mathbf{0}_{3,3} + O(1)|s^{\ell_*}-s| & \mathbf{Id}_{3,3} + O(1)|s^{\ell_*}-s| \end{bmatrix}.\end{aligned}$$

Furthermore due to Lemma 4.1, we conclude

$$\begin{aligned}&\frac{\partial(s^{\ell_*}, X_{\mathbf{cl}}(s^{\ell_*}), V_{\mathbf{cl}}(s^{\ell_*}))}{\partial(s^{\ell_*}, \mathbf{X}_{\ell_*}(s^{\ell_*}), \mathbf{V}_{\ell_*}(s^{\ell_*}))} \\ &= \left[\begin{array}{c|ccc|ccc} 1 & & \mathbf{0}_{1,3} & & & \mathbf{0}_{1,3} & \\ \hline \mathbf{0}_{3,1} & -\mathbf{n}_{\ell_*} & \begin{matrix}\partial_1\eta_{\ell_*} \\ -\mathbf{x}_{\perp_{\ell_*}}\partial_1\mathbf{n}_{\ell_*}\end{matrix} & \begin{matrix}\partial_2\eta_{\ell_*} \\ -\mathbf{x}_{\perp_{\ell_*}}\partial_2\mathbf{n}_{\ell_*}\end{matrix} & & \mathbf{0}_{3,3} & \\ \hline \mathbf{0}_{3,1} & -\mathbf{v}_{\|_{\ell_*}}\cdot\nabla\mathbf{x}_{\|_{\ell_*}}\mathbf{n}_{\ell_*} & \begin{matrix}\mathbf{v}_{\|_{\ell_*}}\cdot\nabla\partial_1\eta_{\ell_*} \\ -\mathbf{v}_{\perp_{\ell_*}}\partial_1\mathbf{n}_{\ell_*} \\ -\mathbf{x}_{\perp_{\ell_*}}\mathbf{v}_{\|_{\ell_*}}\cdot\nabla\partial_1\mathbf{n}_{\ell_*}\end{matrix} & \begin{matrix}\mathbf{v}_{\|_{\ell_*}}\cdot\nabla\partial_2\eta_{\ell_*} \\ -\mathbf{v}_{\perp_{\ell_*}}\partial_2\mathbf{n}\ell_* \\ -\mathbf{x}_{\perp_{\ell_*}}\mathbf{v}_{\|_{\ell_*}}\cdot\nabla\partial_2\mathbf{n}_{\ell_*}\end{matrix} & -\mathbf{n}_{\ell_*} & \begin{matrix}\partial_1\eta_{\ell_*} \\ -\mathbf{x}_{\perp_{\ell_*}}\partial_1\mathbf{n}_{\ell_*}\end{matrix} & \begin{matrix}\partial_2\eta_{\ell_*} \\ -\mathbf{x}_{\perp_{\ell_*}}\partial_2\mathbf{n}_{\ell_*}\end{matrix} \end{array}\right],\end{aligned}$$

where all entries are evaluated at $(\mathbf{X}_{\ell_*}(s^{\ell_*}), \mathbf{V}_{\ell_*}(s^{\ell_*}))$. The multiplication of above two matrices gives (5.30).

Secondly, we claim that whenever $\mathbf{p}^\ell-$spherical coordinate is defined for all $\tau \in [s^\ell, t^\ell]$, we have following 7×6 matrix

$$\begin{aligned}&\frac{\partial(s^\ell, \mathbf{x}_{\perp_\ell}(s^\ell), \mathbf{x}_{\|_\ell}(s^\ell), \mathbf{v}_{\perp_\ell}(s^\ell), \mathbf{v}_{\|_\ell}(s^\ell))}{\partial(t^\ell, \mathbf{x}^\ell_{\|_\ell}, \mathbf{v}^\ell_{\perp_\ell}, \mathbf{v}^\ell_{\|_\ell})} \\ &= \left[\begin{array}{c|c|c|c} 0 & \mathbf{0}_{1,2} & 0 & \mathbf{0}_{1,2} \\ \hline -\mathbf{v}_\perp(s^\ell) + O(1)|t^\ell-s^\ell|^2 & O_\xi(1)|v|^2|t^\ell-s^\ell|^2 & O_\xi(1)|t^\ell-s^\ell| & O_\xi(1)|v||t^\ell-s^\ell|^2 \\ -\mathbf{v}_\|(s^\ell) + O(1)|t^\ell-s^\ell|^2 & \mathbf{Id}_{2,2} + O_\xi(1)|v|^2|t^\ell-s^\ell|^2 & O_\xi(1)|v||t^\ell-s^\ell|^2 & O_\xi(1)|t^\ell-s^\ell|(\mathbf{Id}_{2,2} + |v||t^\ell-s^\ell|) \\ \hline O(1)(|v|^2+1) & (O(1)+|v|^2)|t^\ell-s^\ell| & 1 + O_\xi(1)|v||t^\ell-s^\ell| & O_\xi(1)|v||t^\ell-s^\ell| \\ O(1)(|v|^2+1) & (O(1)+|v|^2)|t^\ell-s^\ell| & O_\xi(1)|v||t^\ell-s^\ell| & \mathbf{Id}_{2,2} + O_\xi(1)|v||t^\ell-s^\ell| \end{array}\right].\end{aligned} \tag{5.36}$$

In this step we just need (5.36) for $\ell = \ell_*$ but we need (5.36) for general ℓ in Step 8.

Clearly the first row is identically zero since s^ℓ is chosen to be independent of $(t^\ell, \mathbf{x}^\ell_{\|_\ell}, \mathbf{v}^\ell_{\perp_\ell}, \mathbf{v}^\ell_{\|_\ell})$. By directly taking $\frac{\partial}{\partial t^\ell}$ derivative to $\mathbf{v}_{\perp,\|}(s^\ell) = \mathbf{v}^\ell_{\perp,\|} - \int_{s^\ell}^{t^\ell} F_{\perp,\|}(\tau)d\tau$ and $\mathbf{x}_{\perp,\|}(s^\ell) = \mathbf{x}^\ell_{\perp,\|} - \int_{s^\ell}^{t^\ell}\mathbf{v}_{\perp,\|}(\tau)d\tau$ we have

$$\frac{\partial \mathbf{v}_{\perp,\|}(s)}{\partial t^\ell} = -F_{\perp,\|}(t^\ell) - \int_{s^\ell}^{t^\ell} \partial_{t^\ell}F_{\perp,\|}(\tau)d\tau = -F_{\perp,\|}(s) - \int_{s^\ell}^{t^\ell}\left(\partial_{t^\ell}F_{\perp,\|}(\tau) + \partial_\tau F_{\perp,\|}(\tau)\right)d\tau, \tag{5.37}$$

and

$$\begin{aligned}\frac{\partial \mathbf{x}_{\perp,\|}(s^\ell)}{\partial t^\ell} &= -\mathbf{v}_{\perp,\|}(t^\ell) - \int_{s^\ell}^{t^\ell} \partial_{t^\ell}\mathbf{v}_{\perp,\|}(\tau)d\tau \\ &= -\mathbf{v}_{\perp,\|}(s^\ell) - \int_{s^\ell}^{t^\ell} F_{\perp,\|}(s)ds - \int_{s^\ell}^{t^\ell} \partial_{t^\ell}\mathbf{v}_{\perp,\|}(\tau) \\ &= -\mathbf{v}_{\perp,\|}(s^\ell) + \int_{s^\ell}^{t^\ell}\int_{\tau}^{t^\ell} \big(\partial_{t^\ell}F_{\perp,\|}(\tau') + \partial_\tau F_{\perp,\|}(\tau')\big)\, d\tau' d\tau. \end{aligned} \tag{5.38}$$

Then from (5.26) we get the desired estimate for the first column of (5.36).

Now we turn to other entries in (5.36). From the characteristics ODE, (4.7) in the $\mathbf{p}^\ell-$spherical coordinate, (4.10), (4.11), and (4.12), we deduce (5.36) for $|v||s^\ell - t^\ell| \lesssim 1$.

Step 3. From $t-$plane to the first bounce.

We choose $s^1 \in (t^1, \frac{t^1+t}{2}) \subset (t^1, t)$ such that $|v||t^1 - s^1| \ll 1$ and the polar coordinate $(\mathbf{X}_1(s^1), \mathbf{V}_1(s^1))$ is well-defined. More precisely we choose $0 < \Delta$ such that $|v||t - \Delta - t^1| \ll 1$ and define

$$s^1 := t - \Delta. \tag{5.39}$$

We first follow the flow in the cartesian coordinate to near the boundary at s^1, change to the chart to $\mathbf{p}^\ell-$spherical coordinate, then follow the flow in that coordinate.

Then, by the chain rule,

$$\begin{aligned}&\frac{\partial(t^1, \mathbf{x}^1_{\|_1}, \mathbf{v}^1_{\perp_1}, \mathbf{v}^1_{\|_1})}{\partial(t,x,v)} \\ &= \frac{\partial(t^1, \mathbf{x}^1_{\|_1}, \mathbf{v}^1_{\perp_1}, \mathbf{v}^1_{\|_1})}{\partial(s^1, X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1))}\frac{\partial(s^1, X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1))}{\partial(t,x,v)} \\ &= \frac{\partial(t^1, \mathbf{x}^1_{\|_1}, \mathbf{v}^1_{\perp_1}, \mathbf{v}^1_{\|_1})}{\partial(s^1, \mathbf{x}_{\perp_1}(s^1), \mathbf{x}_{\|_1}(s^1), \mathbf{v}_{\perp_1}(s^1), \mathbf{v}_{\|_1}(s^1))}\frac{\partial(s^1, \mathbf{X}_1(s^1), \mathbf{V}_1(s^1))}{\partial(s^1, X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1))}\frac{\partial(s^1, X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1))}{\partial(t,x,v)}.\end{aligned}$$

We fix $\mathbf{p}^1-$spherical coordinate and drop the index of the chart.

Firstly, we claim

$$\frac{\partial(t^1, \mathbf{x}^1_{\|}, \mathbf{v}^1_{\perp}, \mathbf{v}^1_{\|})}{\partial(s^1, \mathbf{x}_\perp(s^1), \mathbf{x}_\|(s^1), \mathbf{v}_\perp(s^1), \mathbf{v}_\|(s^1))} \lesssim_\Omega \left[\begin{array}{c|cc|cc} \frac{|v|}{|v^1_\perp|} + \frac{O(1)|s^1-t^1|}{|v^1_\perp|} & \frac{1}{|\mathbf{v}^1_\perp|} & \frac{(|v|^2+O(1))|s^1-t^1|^2}{|\mathbf{v}^1_\perp|} & \frac{|s^1-t^1|}{|\mathbf{v}^1_\perp|} & \frac{(|v|+O(1))|s^1-t^1|^2}{|\mathbf{v}^1_\perp|} \\ \hline |v| + \frac{|v|^2+O(1)}{|v^1_\perp|} & \frac{|v|}{|\mathbf{v}^1_\perp|} + (|v|^2+O(1))|s^1-t^1|^2 & \mathbf{Id}_{2,2} + (|v|+O(1))|s^1-t^1| & \frac{|s^1-t^1||v|}{|\mathbf{v}^1_\perp|} + |s^1-t^1|^2|v| & |s^1-t^1| \\ \hline |v|^2+O(1) + \frac{|v|^3+O(1)}{|v^1_\perp|} & \frac{|v|^2+O(1)}{|\mathbf{v}^1_\perp|} + (|v|^2+O(1))|s^1-t^1| & (|v|^2+O(1))|s^1-t^1| & 1+|v||s^1-t^1| & (|v|+O(1))|s^1-t^1| \\ |v|^2+O(1) + \frac{|v|^3+O(1)}{|v^1_\perp|} & \frac{|v|^2+O(1)}{|\mathbf{v}^1_\perp|} + (|v|^2+O(1))|s^1-t^1| & (|v|^2+O(1))|s^1-t^1| & 1+|v||s^1-t^1| & \mathbf{Id}_2 + (|v|+O(1))|s^1-t^1| \end{array}\right]. \tag{5.40}$$

The t^1 is determined via $\mathbf{x}_\perp(t^1) = 0$, i.e.

$$0 = \mathbf{x}_\perp(s^1) - \mathbf{v}_\perp(s^1)(s^1 - t^1) + \int_{t^1}^{s^1}\int_s^{s^1} F_\perp(\mathbf{X}(\tau), \mathbf{V}_{\mathbf{cl}}(\tau))\mathrm{d}\tau \mathrm{d}s, \tag{5.41}$$

where

$$\mathbf{X}(\tau)=\mathbf{X}(\tau;s^1,\mathbf{X}(s^1;t,x,v),\mathbf{V}(s^1;t,x,v)),\mathbf{V}(\tau)=\mathbf{V}(\tau;s^1,\mathbf{X}(s^1;t,x,v),\mathbf{V}(s^1;t,x,v)).$$

For $\partial\in\{\partial_{\mathbf{x}_\perp(s^1)},\partial_{\mathbf{x}_\parallel(s^1)},\partial_{\mathbf{v}_\perp(s^1)},\partial_{\mathbf{v}_\parallel(s^1)}\}$,

$$\begin{aligned}&\mathbf{v}_\perp(s^1)\partial t^1-\partial t^1\int_{t^1}^{s^1}F_\perp(\mathbf{X}(\tau),\mathbf{V}(\tau))\mathrm{d}\tau+\partial\mathbf{x}_\perp(s^1)-\partial\mathbf{v}_\perp(s^1)(s^1-t^1)\\&+\int_{t^1}^{s^1}\int_s^{s^1}\{\partial\mathbf{X}(\tau)\cdot\nabla_{\mathbf{X}}F_\perp+\partial\mathbf{V}(\tau)\cdot\nabla_{\mathbf{V}}F_\perp\}(\mathbf{X}(\tau),\mathbf{V}(\tau))\mathrm{d}\tau\mathrm{d}s=0.\end{aligned}\tag{5.42}$$

But $\mathbf{v}_\perp^1=-\lim_{s\downarrow t^1}\mathbf{v}_\perp(s)=-\mathbf{v}_\perp(s^1)+\int_{t^1}^{s^1}F_\perp(\mathbf{X}(\tau),\mathbf{V}(\tau))\mathrm{d}\tau$, we apply Lemma 4.2 and $|s^1-t^1|\lesssim_\xi\min\{\frac{|\mathbf{v}_\perp^1|}{|v|^2},t\}$ and (5.2),

$$\begin{bmatrix}\frac{\partial t^1}{\partial\mathbf{x}_\perp(s^1)}\\\frac{\partial t^1}{\partial\mathbf{x}_\parallel(s^1)}\\\frac{\partial t^1}{\partial\mathbf{v}_\perp(s^1)}\\\frac{\partial t^1}{\partial\mathbf{v}_\parallel(s^1)}\end{bmatrix}=\begin{bmatrix}\frac{1}{\mathbf{v}_\perp^1}\left\{1+\int_{t^1}^{s^1}\int_s^{s^1}\frac{\partial}{\partial\mathbf{x}_\perp(s^1)}F_\perp(\mathbf{X}(\tau),\mathbf{V}(\tau))\mathrm{d}\tau\mathrm{d}s\right\}\\\frac{1}{\mathbf{v}_\perp^1}\int_{t^1}^{s^1}\int_s^{s^1}\frac{\partial}{\partial\mathbf{x}_\parallel(s^1)}F_\perp(\mathbf{X}(\tau),\mathbf{V}(\tau))\mathrm{d}\tau\mathrm{d}s\\\frac{1}{\mathbf{v}_\perp^1}\left\{(t^1-s^1)+\int_{t^1}^{s^1}\int_s^{s^1}\frac{\partial}{\partial\mathbf{v}_\perp(s^1)}F_\perp(\mathbf{X}(\tau),\mathbf{V}(\tau))\mathrm{d}\tau\mathrm{d}s\right\}\\\frac{1}{\mathbf{v}_\perp^1}\int_{t^1}^{s_1}\int_s^{s_1}\frac{\partial}{\partial\mathbf{v}_\parallel(s_1)}F_\perp(\mathbf{X}(\tau),\mathbf{V}(\tau))\mathrm{d}\tau\mathrm{d}s\end{bmatrix}\lesssim_{\xi,t}\begin{bmatrix}\frac{1}{|\mathbf{v}_\perp^1|}\\\frac{(|v|^2+O(1))|s^1-t^1|^2}{|\mathbf{v}_\perp^1|}\\\frac{|s^1-t^1|}{|\mathbf{v}_\perp^1|}\\\frac{(|v|+O(1))|s^1-t^1|^2}{|\mathbf{v}_\perp^1|}\end{bmatrix},$$

Taking $(\mathbf{x}(s^1),\mathbf{v}(s^1))$ derivatives of the characteristic equations

$$\begin{aligned}&\mathbf{v}_\perp^1=-\lim_{s\downarrow t^1}\mathbf{v}_\perp(s)=-\mathbf{v}_\perp(s^1)+\int_{t^1}^{s^1}F_\perp(\mathbf{X}_{\mathbf{cl}}(\tau),\mathbf{V}_{\mathbf{cl}}(\tau))\mathrm{d}\tau,\\&\mathbf{x}_\parallel^1=\mathbf{x}_\parallel(s^1)-\int_{t^1}^{s^1}\mathbf{v}_\parallel(s)\mathrm{d}s,\\&\mathbf{v}_\parallel^1=\mathbf{v}_\parallel(s^1)-\int_{t^1}^{s^1}F_\parallel(\mathbf{X}_{\mathbf{cl}}(\tau),\mathbf{V}_{\mathbf{cl}}(\tau))\mathrm{d}\tau.\end{aligned}$$

and using the above estimates and (5.42) and Lemma 4.2 yields

$$\begin{bmatrix}\frac{\partial\mathbf{x}_\parallel^1}{\partial\mathbf{x}_\perp(s^1)}\\\frac{\partial\mathbf{x}_\parallel^1}{\partial\mathbf{x}_\parallel(s^1)}\\\frac{\partial\mathbf{x}_\parallel^1}{\partial\mathbf{v}_\perp(s^1)}\\\frac{\partial\mathbf{x}_\parallel^1}{\partial\mathbf{v}_\parallel(s^1)}\end{bmatrix}\lesssim_{\xi,t}\begin{bmatrix}\frac{|v|}{|\mathbf{v}_\perp^1|}+(|v|^2+O(1))|s^1-t^1|^2\\\mathbf{Id}_{2,2}+(|v|+O(1))|s^1-t^1|\\\frac{|s^1-t^1||v|}{|\mathbf{v}_\perp^1|}+|s^1-t^1|^2|v|\\|s^1-t^1|\end{bmatrix},$$

$$\begin{bmatrix}\frac{\partial\mathbf{v}_\perp^1}{\partial\mathbf{x}_\perp(s^1)}\\\frac{\partial\mathbf{v}_\perp^1}{\partial\mathbf{x}_\parallel(s^1)}\\\frac{\partial\mathbf{v}_\perp^1}{\partial\mathbf{v}_\perp(s^1)}\\\frac{\partial\mathbf{v}_\perp^1}{\partial\mathbf{v}_\parallel(s^1)}\end{bmatrix}\lesssim_{\xi,t}\begin{bmatrix}\frac{|v|^2+O(1)}{|\mathbf{v}_\perp^1|}+(|v|^2+O(1))|s^1-t^1|\\(|v|^2+O(1))|s^1-t^1|\\1+|v||s^1-t^1|\\(|v|+O(1))|s^1-t^1|\end{bmatrix},$$

and

$$\begin{bmatrix}\frac{\partial\mathbf{v}_\parallel^1}{\partial\mathbf{x}_\perp(s^1)}\\\frac{\partial\mathbf{v}_\parallel^1}{\partial\mathbf{x}_\parallel(s^1)}\\\frac{\partial\mathbf{v}_\parallel^1}{\partial\mathbf{v}_\perp(s^1)}\\\frac{\partial\mathbf{v}_\parallel^1}{\partial\mathbf{v}_\parallel(s^1)}\end{bmatrix}\lesssim_{\xi,t}\begin{bmatrix}\frac{(|v|^2+O(1))}{|\mathbf{v}_\perp^1|}+(|v|^2+O(1))|s^1-t^1|\\(|v|^2+O(1))|s^1-t^1|\\1+|v||s^1-t^1|\\\mathbf{Id}_{2,2}+(|v|+O(1))|s^1-t^1|\end{bmatrix}.$$

Secondly, we claim

$$\frac{\partial(\mathbf{X}_1(s^1),\mathbf{V}_1(s^1))}{\partial(t,x,v)}=\frac{\partial(\mathbf{X}_1(s^1),\mathbf{V}_1(s^1))}{\partial(X_{\mathbf{cl}}(s^1),V_{\mathbf{cl}}(s^1))}\frac{\partial(X_{\mathbf{cl}}(s^1),V_{\mathbf{cl}}(s^1))}{\partial(t,x,v)}$$
$$=\left[\begin{array}{c|c|c} & O(1)+O_\xi(|v||t^1-s^1|^2) & O_\xi(|t-s^1|) \\ |t-s^1|^2 & O(1)+O_\xi(|v||t^1-s^1|^2) & O_\xi(|t-s^1|) \\ & O(1)+O_\xi(|v||t^1-s^1|^2) & O_\xi(|t-s^1|) \\ \hline & O_\xi(|v|) & O(1)+O_\xi(|v||t-s^1|) \\ |t-s^1| & O_\xi(|v|) & O(1)+O_\xi(|v||t-s^1|) \\ & O_\xi(|v|) & O(1)+O_\xi(|v||t-s^1|) \end{array}\right]_{6\times 7}, \tag{5.43}$$

where the entries are evaluated at $(\mathbf{X}_1(s^1),\mathbf{V}_1(s^1))$. Note that $|v||t^1-s^1|\lesssim_\xi 1$.

From (4.5)

$$\frac{\partial(X_{\mathbf{cl}}(s^1),V_{\mathbf{cl}}(s^1))}{\partial(\mathbf{X}(s^1),\mathbf{V}(s^1))}=\frac{\partial\Phi(\mathbf{X}(s^1),\mathbf{V}(s))}{\partial(\mathbf{X}(s^1),\mathbf{V}(s))}:=\left[\begin{array}{c|c} A & \mathbf{0}_{3,3} \\ B & A\end{array}\right]+\mathbf{x}_\perp\left[\begin{array}{c|c} \mathbf{0}_{3,3} & \mathbf{0}_{3,3} \\ D & \mathbf{0}_{3,3}\end{array}\right].$$

From direct computation and (4.3),

$$\det(A)=\det\left[-\mathbf{n}(\mathbf{x}_\parallel)\quad \begin{array}{c}\frac{\partial\eta}{\partial\mathbf{x}_{\parallel,1}}(\mathbf{x}_\parallel)\\ +\mathbf{x}_\perp[-\frac{\partial\mathbf{n}}{\partial\mathbf{x}_{\parallel,1}}(\mathbf{x}_\parallel)]\end{array}\quad \begin{array}{c}\frac{\partial\eta}{\partial\mathbf{x}_{\parallel,2}}(\mathbf{x}_\parallel)\\ +\mathbf{x}_\perp[-\frac{\partial\mathbf{n}}{\partial\mathbf{x}_{\parallel,2}}(\mathbf{x}_\parallel)]\end{array}\right]$$
$$=-\mathbf{n}(\mathbf{x}_\parallel)\cdot\left(\frac{\partial\eta}{\partial\mathbf{x}_{\parallel,1}}(\mathbf{x}_\parallel)\times\frac{\partial\eta}{\partial\mathbf{x}_{\parallel,2}}(\mathbf{x}_\parallel)\right)+O_\xi(|\mathbf{x}_\perp|)\neq 0,$$
$$A^{-1}=\frac{1}{[-\mathbf{n}]\cdot(\partial_{\mathbf{x}_{\parallel,1}}\eta\times\partial_{\mathbf{x}_{\parallel,2}}\eta)+O(1)|\mathbf{x}_\perp|}$$
$$\times\left[(1-\mathbf{x}_\perp)^2(\partial_{\mathbf{x}_{\parallel,1}}\eta\times\partial_{\mathbf{x}_{\parallel,2}}\eta)^T,(1-\mathbf{x}_\perp)(\partial_{\mathbf{x}_{\parallel,2}}\eta\times[-\mathbf{n}])^T,(1-\mathbf{x}_\perp)([-\mathbf{n}]\times\partial_{\mathbf{x}_{\parallel,1}}\eta)^T\right].$$

From basic linear algebra

$$\det\left(\frac{\partial(X_{\mathbf{cl}}(s^1),V_{\mathbf{cl}}(s^1))}{\partial(\mathbf{X}_{\mathbf{cl}}(s^1),\mathbf{V}_{\mathbf{cl}}(s^1))}\right)=\det\left[\begin{array}{c|c} A & \mathbf{0}_{3,3} \\ B+\mathbf{x}_\perp D & A\end{array}\right]=\{\det(A)\}^2$$
$$=\left\{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)+O(1)|\mathbf{x}_\perp|\right\}^2,$$

and $\left(\frac{\partial(X_{\mathbf{cl}}(s^1),V_{\mathbf{cl}}(s^1))}{\partial(\mathbf{X}_{\mathbf{cl}}(s^1),\mathbf{V}_{\mathbf{cl}}(s^1))}\right)$ is invertible. By the basic linear algebra

$$\frac{\partial(\mathbf{X}_{\mathbf{cl}}(s^1),\mathbf{V}_{\mathbf{cl}}(s^1))}{\partial(X_{\mathbf{cl}}(s^1),V_{\mathbf{cl}}(s^1))}=\left[\frac{\partial(X_{\mathbf{cl}}(s^1),V_{\mathbf{cl}}(s^1))}{\partial(\mathbf{X}_{\mathbf{cl}}(s^1),\mathbf{V}_{\mathbf{cl}}(s^1))}\right]^{-1}=\left[\begin{array}{c|c} A & \mathbf{0}_{3,3} \\ B+\mathbf{x}_\perp D & A\end{array}\right]^{-1}$$
$$=\left[\begin{array}{c|c} A^{-1} & \mathbf{0}_{3,3} \\ -A^{-1}(B+\mathbf{x}_\perp D)A^{-1} & A^{-1}\end{array}\right]=\left[\begin{array}{cc} A^{-1}(\mathbf{x}_\parallel) & \mathbf{0}_{3,3} \\ |v|+O_\xi(\mathbf{x}_\perp) & A^{-1}(\mathbf{x}_\parallel)\end{array}\right], \tag{5.44}$$

and we obtain

$$\frac{\partial(\mathbf{X_{cl}}(s^1),\mathbf{V_{cl}}(s^1))}{\partial(X_{\mathbf{cl}}(s^1),V_{\mathbf{cl}}(s^1))}=\left[\begin{array}{c|c} \frac{(1-\mathbf{x}_\perp)^2(\partial_1\eta\times\partial_2\eta)^T}{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)+O(1)|\mathbf{x}_\perp|} & \\ \frac{(1-\mathbf{x}_\perp)(\partial_2\eta\times[-\mathbf{n}])^T}{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)+O(1)|\mathbf{x}_\perp|} & \mathbf{0}_{3,3} \\ \frac{(1-\mathbf{x}_\perp)([-\mathbf{n}]\times\partial_1\eta)^T}{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)+O(1)|\mathbf{x}_\perp|} & \\ \hline O_\xi(1)(|v|) & \frac{(1-\mathbf{x}_\perp)^2(\partial_1\eta\times\partial_2\eta)^T}{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)+O(1)|\mathbf{x}_\perp|} \\ O_\xi(1)(|v|) & \frac{(1-\mathbf{x}_\perp)(\partial_2\eta\times[-\mathbf{n}])^T}{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)+O(1)|\mathbf{x}_\perp|} \\ O_\xi(1)(|v|) & \frac{(1-\mathbf{x}_\perp)([-\mathbf{n}]\times\partial_1\eta)^T}{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)+O(1)|\mathbf{x}_\perp|} \end{array}\right].$$

From $X_{\mathbf{cl}}(s^1;t,x,v)=x-(t-s^1)v+\int_{s^1}^t\int_s^t E(\tau)d\tau ds=x-\Delta\times v+\int_{t-\Delta}^t\int_s^t E(\tau)d\tau ds$, and $V_{\mathbf{cl}}(s^1;t,x,v)=v-\int_{t-\Delta}^t E(s)ds$, we have

$$\begin{aligned}\frac{X_{\mathbf{cl}}(s^1)}{\partial t}&=-\int_{t-\Delta}^t E(s)ds+\int_{t-\Delta}^t E(t)ds+\int_{t-\Delta}^t\int_s^t\partial_t E(\tau)d\tau ds\\&=-\int_{t-\Delta}^t\int_s^t\left(\frac{\partial E(\tau)}{\partial\tau}+\frac{\partial E(\tau)}{\partial t}\right)d\tau ds\\&=-\int_{t-\Delta}^t\int_s^t(\partial_\tau E+\nabla E\cdot\nabla_x X)\,d\tau ds=O(1)|t-s^1|^2,\\ \frac{V_{\mathbf{cl}}(s^1)}{\partial t}&=-E(t)+E(t-\Delta)-\int_{t-\Delta}^t\frac{\partial E(s)}{\partial t}ds\\&=-\int_{t-\Delta}^t\left(\frac{\partial E(s)}{\partial s}+\frac{\partial E(s)}{\partial t}\right)ds\\&=-\int_{t-\Delta}^t(\partial_s E+\nabla E\cdot\nabla_x X)\,ds=O(1)|t-s^1|.\end{aligned}$$

And using $|\nabla_{x,v}X_{\mathbf{cl}}(s^1)|+|\nabla_{x,v}V_{\mathbf{cl}}(s^1)|\lesssim 1$,

$$\frac{\partial(X_{\mathbf{cl}}(s_1),V_{\mathbf{cl}}(s_1))}{\partial(t,x,v)}=\begin{bmatrix}O(1)|t-s^1|^2 & \mathbf{Id}_{3,3}+O(1)|t-s^1|^2 & -(t-s^1)\mathbf{Id}_{3,3}+O(1)|t-s^1|^2\\ O(1)|t-s^1| & O(1)|t-s^1| & \mathbf{Id}_{3,3}+O(1)|t-s^1|\end{bmatrix}.$$

Finally we multiply above two matrices and use $|\mathbf{x}_\perp(s^1)|\lesssim|v||t^1-s^1|$ to conclude the second claim (5.43).

Step 4. Estimate of $\partial(t^{\ell+1},\mathbf{x}^{\ell+1}_{\|_{\ell+1}},\mathbf{v}^{\ell+1}_{\perp_{\ell+1}},\mathbf{v}^{\ell+1}_{\|_{\ell+1}})/\partial(t^\ell,\mathbf{x}^\ell_{\|_\ell},\mathbf{v}^\ell_{\perp_\ell},\mathbf{v}^\ell_{\|_\ell})$.

Recall $\mathbf{r}^\ell$ from (5.18). We show that for $0<T\ll 1$ small enough, there exists $0<\delta_1\ll 1$, $M=M_{\xi,t}\gg 1$, such that for all $\ell\in\mathbb{N}$ and $0\le t^{\ell+1}\le t^\ell\le t$, if ℓ is *Type II* or *Type III*,

$$J_\ell^{\ell+1}:=\frac{\partial(t^{\ell+1},\mathbf{x}^{\ell+1}_{\|_{\ell+1}},\mathbf{v}^{\ell+1}_{\perp_{\ell+1}},\mathbf{v}^{\ell+1}_{\|_{\ell+1}})}{\partial(t^\ell,\mathbf{x}^\ell_{\|_\ell},\mathbf{v}^\ell_{\perp_\ell},\mathbf{v}^\ell_{\|_\ell})}$$

$$\le\left[\begin{array}{c|ccc|cc} 1+M|t^\ell-t^{\ell+1}| & \frac{M}{|v|}\mathbf{r}^{\ell+1} & \frac{M}{|v|}\mathbf{r}^{\ell+1} & \frac{M}{|v|^2} & \frac{M}{|v|^2}\mathbf{r}^{\ell+1} & \frac{M}{|v|^2}\mathbf{r}^{\ell+1}\\ \hline M|t^\ell-t^{\ell+1}| & 1+M\mathbf{r}^{\ell+1} & M\mathbf{r}^{\ell+1} & \frac{M}{|v|} & \frac{M}{|v|}\mathbf{r}^{\ell+1} & \frac{M}{|v|}\mathbf{r}^{\ell+1}\\ M|t^\ell-t^{\ell+1}| & M\mathbf{r}^{\ell+1} & 1+M\mathbf{r}^{\ell+1} & \frac{M}{|v|} & \frac{M}{|v|}\mathbf{r}^{\ell+1} & \frac{M}{|v|}\mathbf{r}^{\ell+1}\\ \hline M|t^\ell-t^{\ell+1}|^2|v| & M|v|(\mathbf{r}^{\ell+1})^2 & M|v|(\mathbf{r}^{\ell+1})^2 & 1+M\mathbf{r}^{\ell+1} & M(\mathbf{r}^{\ell+1})^2 & M(\mathbf{r}^{\ell+1})^2\\ M|t^\ell-t^{\ell+1}| & M|v|\mathbf{r}^{\ell+1} & M|v|\mathbf{r}^{\ell+1} & M & 1+M\mathbf{r}^{\ell+1} & M\mathbf{r}^{\ell+1}\\ M|t^\ell-t^{\ell+1}| & M|v|\mathbf{r}^{\ell+1} & M|v|\mathbf{r}^{\ell+1} & M & M\mathbf{r}^{\ell+1} & 1+M\mathbf{r}^{\ell+1}\end{array}\right]$$

$$
\begin{aligned}
&\leq \left[\begin{array}{c|ccc|cc}
1+5M\mathbf{r}^{\ell+1} & \frac{M}{|v|}\mathbf{r}^{\ell+1} & \frac{M}{|v|}\mathbf{r}^{\ell+1} & \frac{M}{|v|^2} & \frac{M}{|v|^2}\mathbf{r}^{\ell+1} & \frac{M}{|v|^2}\mathbf{r}^{\ell+1} \\ \hline
5M\mathbf{r}^{\ell+1}|v| & 1+M\mathbf{r}^{\ell+1} & M\mathbf{r}^{\ell+1} & \frac{M}{|v|} & \frac{M}{|v|}\mathbf{r}^{\ell+1} & \frac{M}{|v|}\mathbf{r}^{\ell+1} \\
5M\mathbf{r}^{\ell+1}|v| & M\mathbf{r}^{\ell+1} & 1+M\mathbf{r}^{\ell+1} & \frac{M}{|v|} & \frac{M}{|v|}\mathbf{r}^{\ell+1} & \frac{M}{|v|}\mathbf{r}^{\ell+1} \\ \hline
5M(\mathbf{r}^{\ell+1})^2|v|^2 & M|v|(\mathbf{r}^{\ell+1})^2 & M|v|(\mathbf{r}^{\ell+1})^2 & 1+M\mathbf{r}^{\ell+1} & M(\mathbf{r}^{\ell+1})^2 & M(\mathbf{r}^{\ell+1})^2 \\
5M\mathbf{r}^{\ell+1}|v|^2 & M|v|\mathbf{r}^{\ell+1} & M|v|\mathbf{r}^{\ell+1} & M & 1+M\mathbf{r}^{\ell+1} & M\mathbf{r}^{\ell+1} \\
5M\mathbf{r}^{\ell+1}|v|^2 & M|v|\mathbf{r}^{\ell+1} & M|v|\mathbf{r}^{\ell+1} & M & M\mathbf{r}^{\ell+1} & 1+M\mathbf{r}^{\ell+1}
\end{array}\right] \\
&:= \underbrace{J(\mathbf{r}^{\ell+1})}_{\text{Definition of } J(\mathbf{r}^{\ell+1})} . \qquad (5.45)
\end{aligned}
$$

And if ℓ is *Type I*, then

$$
\begin{aligned}
J_\ell^{\ell+1} &:= \frac{\partial(t^{\ell+1},\mathbf{x}^{\ell+1}_{\|_{\ell+1}},\mathbf{v}^{\ell+1}_{\perp_{\ell+1}},\mathbf{v}^{\ell+1}_{\|_{\ell+1}})}{\partial(t^{\ell},\mathbf{x}^{\ell}_{\|_{\ell}},\mathbf{v}^{\ell}_{\perp_{\ell}},\mathbf{v}^{\ell}_{\|_{\ell}})} \\
&\leq \left[\begin{array}{c|ccc|cc}
1+M|t^\ell-t^{\ell+1}| & M\mathbf{v}_\perp^{\ell+1} & M\mathbf{v}_\perp^{\ell+1} & M & M\mathbf{v}_\perp^{\ell+1} & M\mathbf{v}_\perp^{\ell+1} \\ \hline
M|t^\ell-t^{\ell+1}| & 1+M\mathbf{v}_\perp^{\ell+1} & M\mathbf{v}_\perp^{\ell+1} & M & M\mathbf{v}_\perp^{\ell+1} & M\mathbf{v}_\perp^{\ell+1} \\
M|t^\ell-t^{\ell+1}| & M\mathbf{v}_\perp^{\ell+1} & 1+M\mathbf{v}_\perp^{\ell+1} & M & M\mathbf{v}_\perp^{\ell+1} & M\mathbf{v}_\perp^{\ell+1} \\ \hline
M|t^\ell-t^{\ell+1}|^2 & M(\mathbf{v}_\perp^{\ell+1})^2 & M(\mathbf{v}_\perp^{\ell+1})^2 & 1+M\mathbf{v}_\perp^{\ell+1} & M(\mathbf{v}_\perp^{\ell+1})^2 & M(\mathbf{v}_\perp^{\ell+1})^2 \\
M|t^\ell-t^{\ell+1}| & M\mathbf{v}_\perp^{\ell+1} & M\mathbf{v}_\perp^{\ell+1} & M & 1+M\mathbf{v}_\perp^{\ell+1} & M\mathbf{v}_\perp^{\ell+1} \\
M|t^\ell-t^{\ell+1}| & M\mathbf{v}_\perp^{\ell+1} & M\mathbf{v}_\perp^{\ell+1} & M & M\mathbf{v}_\perp^{\ell+1} & 1+M\mathbf{v}_\perp^{\ell+1}
\end{array}\right] \\
&\leq \left[\begin{array}{c|ccc|cc}
1+5M\mathbf{v}_\perp^{\ell+1} & M\mathbf{v}_\perp^{\ell+1} & M\mathbf{v}_\perp^{\ell+1} & M & M\mathbf{v}_\perp^{\ell+1} & M\mathbf{v}_\perp^{\ell+1} \\ \hline
5M\mathbf{v}_\perp^{\ell+1} & 1+M\mathbf{v}_\perp^{\ell+1} & M\mathbf{v}_\perp^{\ell+1} & M & M\mathbf{v}_\perp^{\ell+1} & M\mathbf{v}_\perp^{\ell+1} \\
5M\mathbf{v}_\perp^{\ell+1} & M\mathbf{v}_\perp^{\ell+1} & 1+M\mathbf{v}_\perp^{\ell+1} & M & M\mathbf{v}_\perp^{\ell+1} & M\mathbf{v}_\perp^{\ell+1} \\ \hline
5M(\mathbf{v}_\perp^{\ell+1})^2 & M(\mathbf{v}_\perp^{\ell+1})^2 & M(\mathbf{v}_\perp^{\ell+1})^2 & 1+M\mathbf{v}_\perp^{\ell+1} & M(\mathbf{v}_\perp^{\ell+1})^2 & M(\mathbf{v}_\perp^{\ell+1})^2 \\
5M\mathbf{v}_\perp^{\ell+1} & M\mathbf{v}_\perp^{\ell+1} & M\mathbf{v}_\perp^{\ell+1} & M & 1+M\mathbf{v}_\perp^{\ell+1} & M\mathbf{v}_\perp^{\ell+1} \\
5M\mathbf{v}_\perp^{\ell+1} & M\mathbf{v}_\perp^{\ell+1} & M\mathbf{v}_\perp^{\ell+1} & M & M\mathbf{v}_\perp^{\ell+1} & 1+M\mathbf{v}_\perp^{\ell+1}
\end{array}\right] \\
&:= \underbrace{J(\mathbf{v}_\perp^{\ell+1})}_{\text{Definition of } J(\mathbf{v}_\perp^{\ell+1})} . \qquad (5.46)
\end{aligned}
$$

We also denote the Jacobian matrix within a single $\mathbf{p}^\ell-$ spherical coordinate:

$$
\tilde{J}_\ell^{\ell+1} := \frac{\partial(t^{\ell+1},\mathbf{x}^{\ell+1}_{\|_{\ell}},\mathbf{v}^{\ell+1}_{\perp_{\ell}},\mathbf{v}^{\ell+1}_{\|_{\ell}})}{\partial(t^{\ell},\mathbf{x}^{\ell}_{\|_{\ell}},\mathbf{v}^{\ell}_{\perp_{\ell}},\mathbf{v}^{\ell}_{\|_{\ell}})}.
$$

We split the proof for each *Type*:

Proof of (5.46) (Type I), and (5.45) when ℓ is Type II: Note that $\mathbf{p}^\ell-$spherical coordinate is well-defined of all $\tau \in [t^{\ell+1}, t^\ell]$ for those cases. Due to the chart changing

$$
\frac{\partial(t^{\ell+1},\mathbf{x}^{\ell+1}_{\|_{\ell+1}},\mathbf{v}^{\ell+1}_{\perp_{\ell+1}},\mathbf{v}^{\ell+1}_{\|_{\ell+1}})}{\partial(t^{\ell},\mathbf{x}^{\ell}_{\|_{\ell}},\mathbf{v}^{\ell}_{\perp_{\ell}},\mathbf{v}^{\ell}_{\|_{\ell}})} = \left[\begin{array}{c|c}
1 & \mathbf{0}_{1,5} \\ \hline
\mathbf{0}_{5,1} & \frac{\partial(\mathbf{x}^{\ell+1}_{\|_{\ell+1}},\mathbf{v}^{\ell+1}_{\perp_{\ell+1}},\mathbf{v}^{\ell+1}_{\|_{\ell+1}})}{\partial(\mathbf{x}^{\ell}_{\|_{\ell}},\mathbf{v}^{\ell}_{\perp_{\ell}},\mathbf{v}^{\ell}_{\|_{\ell}})}
\end{array}\right] \underbrace{\frac{\partial(t^{\ell+1},0,\mathbf{x}^{\ell+1}_{\|_{\ell}},\mathbf{v}^{\ell+1}_{\perp_{\ell}},\mathbf{v}^{\ell+1}_{\|_{\ell}})}{\partial(t^{\ell},0,\mathbf{x}^{\ell}_{\|_{\ell}},\mathbf{v}^{\ell}_{\perp_{\ell}},\mathbf{v}^{\ell}_{\|_{\ell}})}}_{=\tilde{J}_\ell^{\ell+1}}.
$$

where $\frac{\partial(\mathbf{x}^{\ell+1}_{\|_{\ell+1}},\mathbf{v}^{\ell+1}_{\perp_{\ell+1}},\mathbf{v}^{\ell+1}_{\|_{\ell+1}})}{\partial(\mathbf{x}^{\ell}_{\|_{\ell}},\mathbf{v}^{\ell}_{\perp_{\ell}},\mathbf{v}^{\ell}_{\|_{\ell}})}$ is the 5×5 right lower submatrix of (4.6).

Note that $|\mathbf{p}^\ell-\mathbf{p}^{\ell+1}|\lesssim_\xi\sqrt{\delta}$ from (5.20). In order to show (5.45) and (5.46) it suffices to show that $\tilde{J}_\ell^{\ell+1}$ is bounded:

$$\begin{aligned}\tilde{J}_\ell^{\ell+1} \;&\leq\; J(\mathbf{r}^{\ell+1}), \text{ if } \ell \text{ is } \textit{Type II} \text{ or } \textit{Type III},\\ \tilde{J}_\ell^{\ell+1} \;&\leq\; J(\mathbf{v}_\perp^{\ell+1}), \text{ if } \ell \text{ is } \textit{Type I}.\end{aligned} \tag{5.47}$$

This is due to the following matrix multiplication

$$\left[\begin{array}{c|c} 1 & \mathbf{0}_{1,5} \\ \hline \mathbf{0}_{5,1} & \frac{\partial(\mathbf{x}^{\ell+1}_{\|_{\ell+1}},\mathbf{v}^{\ell+1}_{\perp_{\ell+1}},\mathbf{v}^{\ell+1}_{\|_{\ell+1}})}{\partial(\mathbf{x}^{\ell}_{\|_\ell},\mathbf{v}^{\ell}_{\perp_\ell},\mathbf{v}^{\ell}_{\|_\ell})} \end{array}\right]\tilde{J}_\ell^{\ell+1}$$

$$\leq \left[\begin{array}{c|cc|ccc} 1 & \mathbf{0}_{1,2} & & \mathbf{0}_{1,3} & & \\ \hline \mathbf{0}_{2,1} & 1+C\mathbf{r}^{\ell+1} & C\mathbf{r}^{\ell+1} & \mathbf{0}_{3,3} & & \\ & C\mathbf{r}^{\ell+1} & 1+C\mathbf{r}^{\ell+1} & & & \\ \hline & 0 & 0 & 1 & 0 & 0 \\ \mathbf{0}_{3,1} & C\mathbf{r}^{\ell+1}|v| & C\mathbf{r}^{\ell+1}|v| & 0 & 1+C\mathbf{r}^{\ell+1} & C\mathbf{r}^{\ell+1} \\ & C\mathbf{r}^{\ell+1}|v| & C\mathbf{r}^{\ell+1}|v| & 0 & C\mathbf{r}^{\ell+1} & 1+C\mathbf{r}^{\ell+1} \end{array}\right] J(\mathbf{r}^{\ell+1})\leq J(C\mathbf{r}^{\ell+1}), \text{ if } |v|>\delta,$$

$$\left[\begin{array}{c|c} 1 & \mathbf{0}_{1,5} \\ \hline \mathbf{0}_{5,1} & \frac{\partial(\mathbf{x}^{\ell+1}_{\|_{\ell+1}},\mathbf{v}^{\ell+1}_{\perp_{\ell+1}},\mathbf{v}^{\ell+1}_{\|_{\ell+1}})}{\partial(\mathbf{x}^{\ell}_{\|_\ell},\mathbf{v}^{\ell}_{\perp_\ell},\mathbf{v}^{\ell}_{\|_\ell})} \end{array}\right]\tilde{J}_\ell^{\ell+1}$$

$$\leq \left[\begin{array}{c|cc|ccc} 1 & \mathbf{0}_{1,2} & & \mathbf{0}_{1,3} & & \\ \hline \mathbf{0}_{2,1} & 1+C\mathbf{r}^{\ell+1} & C\mathbf{r}^{\ell+1} & \mathbf{0}_{3,3} & & \\ & C\mathbf{r}^{\ell+1} & 1+C\mathbf{r}^{\ell+1} & & & \\ \hline & 0 & 0 & 1 & 0 & 0 \\ \mathbf{0}_{3,1} & C\mathbf{r}^{\ell+1}|v| & C\mathbf{r}^{\ell+1}|v| & 0 & 1+C\mathbf{r}^{\ell+1} & C\mathbf{r}^{\ell+1} \\ & C\mathbf{r}^{\ell+1}|v| & C\mathbf{r}^{\ell+1}|v| & 0 & C\mathbf{r}^{\ell+1} & 1+C\mathbf{r}^{\ell+1} \end{array}\right] J(\mathbf{v}_\perp^{\ell+1})\leq J(C\mathbf{v}_\perp^{\ell+1}), \text{ if } |v|\leq\delta,$$

where we used (4.6) with an adjusted constant $C>0$.

Now we prove the claim (5.47). We fix the $\mathbf{p}^\ell-$spherical coordinate and drop the index ℓ for the chart.

If $\mathbf{v}_\perp^\ell=0$ then $t^{\ell+1}=t^\ell$. Otherwise if $\mathbf{v}_\perp^\ell\neq 0$ then $t^{\ell+1}$ is determined through

$$0=\mathbf{v}_\perp^\ell(t^{\ell+1}-t^\ell)+\int_{t^{\ell+1}}^{t^\ell}\int_s^{t^\ell}F_\perp(\mathbf{X}_\ell(\tau;t^\ell,x^\ell,v^\ell),\mathbf{V}_\ell(\tau;t^\ell,x^\ell,v^\ell))\mathrm{d}\tau\mathrm{d}s. \tag{5.48}$$

We first consider the $\frac{\partial}{\partial t^\ell}$ derivatives.

Using the trajectory in the standard coordinates we have

$$0=\xi(x^{\ell+1})=\xi\left(x^\ell-(t^\ell-t^{\ell+1})v^\ell+\int_{t^{\ell+1}}^{t^\ell}\int_s^{t^\ell}E(\tau,X(\tau))d\tau ds\right). \tag{5.49}$$

Taking the $\frac{\partial}{\partial t^\ell}$ derivative we get

$$\begin{aligned}0=\nabla\xi(x^{\ell+1})\cdot\Bigg[&-(1-\frac{\partial t^{\ell+1}}{\partial t^\ell})v^\ell-\frac{\partial t^{\ell+1}}{\partial t^\ell}\int_{t^{\ell+1}}^{t^\ell}E(\tau,X(\tau))d\tau+\int_{t^{\ell+1}}^{t^\ell}E(t^\ell,x^\ell)ds\\ &+\int_{t^{\ell+1}}^{t^\ell}\int_s^{t^\ell}\frac{\partial E(\tau,X(\tau))}{\partial t^\ell}d\tau ds\Bigg]\end{aligned}$$

$$
\begin{aligned}
&= \nabla\xi(x^{\ell+1})\cdot\left[-v^{\ell}+\frac{\partial t^{\ell+1}}{\partial t^{\ell}}v^{\ell+1}+\int_{t^{\ell+1}}^{t^{\ell}}E(s,X(s))ds+\int_{t^{\ell+1}}^{t^{\ell}}\big(E(t^{\ell},x^{\ell})-E(s,X(s))\big)\,ds\right.\\
&\qquad\left.+\int_{t^{\ell+1}}^{t^{\ell}}\int_{s}^{t^{\ell}}\frac{\partial E(\tau,X(\tau))}{\partial t^{\ell}}d\tau ds\right]\\
&= \nabla\xi(x^{\ell+1})\cdot\left[-v^{\ell+1}+\frac{\partial t^{\ell+1}}{\partial t^{\ell}}v^{\ell+1}+\int_{t^{\ell+1}}^{t^{\ell}}\int_{s}^{t^{\ell}}\left(\frac{\partial E(\tau,X(\tau))}{\partial \tau}+\frac{\partial E(\tau,X(\tau))}{\partial t^{\ell}}\right)d\tau ds\right].
\end{aligned}
\tag{5.50}
$$

Thus

$$
\frac{\partial t^{\ell+1}}{\partial t^{\ell}}=1-\frac{\nabla\xi(x^{\ell+1})}{\nabla\xi(x^{\ell+1})\cdot v^{\ell+1}}\cdot\int_{t^{\ell+1}}^{t^{\ell}}\int_{s}^{t^{\ell}}\left(\frac{\partial E(\tau,X(\tau))}{\partial \tau}+\frac{\partial E(\tau,X(\tau))}{\partial t^{\ell}}\right)d\tau ds. \tag{5.51}
$$

By (5.34) we have

$$
\begin{aligned}
\left|\frac{\partial E(\tau,X(\tau))}{\partial \tau}+\frac{\partial E(\tau,X(\tau))}{\partial t^{\ell}}\right| &= |\partial_s E(\tau,X(\tau))_\infty+\nabla_x E\cdot\big(V(\tau)-V(\tau)+O(1)|t^{\ell}-t^{\ell+1}|\big)|\\
&= |\partial_s E(\tau,X(\tau))_\infty+O(1)\nabla_x E(\tau,X(\tau))(t^{\ell}-t^{\ell+1})|\\
&\lesssim \|\partial_t E\|_{L^\infty_{t,x}}+\|\nabla_x E\|_{L^\infty_{t,x}}|t^{\ell}-t^{\ell+1}|.
\end{aligned}
\tag{5.52}
$$

Thus from (5.51), (5.52), and (5.2) we have

$$
\frac{\partial t^{\ell+1}}{\partial t^{\ell}}=1-O_{\xi,E}(1)\frac{|t^{\ell}-t^{\ell+1}|^2}{|v_\perp^{\ell+1}|}=1-O_{\xi,E}(1)|t^{\ell}-t^{\ell+1}|. \tag{5.53}
$$

Now by directly computing $\frac{\partial \mathbf{v}_\perp^{\ell+1}}{\partial t^{\ell}}$ we would have

$$
\begin{aligned}
\frac{\partial \mathbf{v}_\perp^{\ell+1}}{\partial t^{\ell}} &= \frac{-F_\perp(t^{\ell+1})}{\mathbf{v}_\perp^{\ell+1}}\int_{t^{\ell+1}}^{t^{\ell}}\int_{s}^{t^{\ell}}\left(\frac{\partial F_\perp(\tau)}{\partial \tau}+\frac{\partial F_\perp(\tau)}{\partial t^{\ell}}\right)d\tau ds\\
&\quad+\int_{t^{\ell+1}}^{t^{\ell}}\left(\frac{\partial F_\perp(s)}{\partial s}+\frac{\partial F_\perp(s)}{\partial t^{\ell}}\right)ds.
\end{aligned}
\tag{5.54}
$$

Recall

$$
\begin{aligned}
F_\perp &= F_\perp(\mathbf{x}_\perp,\mathbf{x}_\parallel,\mathbf{v}_\parallel)\\
&= \sum_{j,k=1}^{2}\mathbf{v}_{\parallel,k}\mathbf{v}_{\parallel,j}\;\partial_j\partial_k\eta(\mathbf{x}_\parallel)\cdot\mathbf{n}(\mathbf{x}_\parallel)-\mathbf{x}_\perp\sum_{k=1}^{2}\mathbf{v}_{\parallel,k}(\mathbf{v}_\parallel\cdot\nabla)\partial_k\mathbf{n}(\mathbf{x}_\parallel)\cdot\mathbf{n}(\mathbf{x}_\parallel)\\
&\quad-E(s,-\mathbf{x}_\perp\mathbf{n}(\mathbf{x}_\parallel)+\eta(\mathbf{x}_\parallel))\cdot\mathbf{n}(\mathbf{x}_\parallel).
\end{aligned}
\tag{5.55}
$$

So by direct computation

$$
\dot F_\perp(\tau):=\frac{\partial F_\perp(\tau)}{\partial \tau}=\mathbf{v}_\perp\nabla_{\mathbf{x}_\perp}F_\perp+\mathbf{v}_\parallel\nabla_{\mathbf{x}_\parallel}F_\perp+F_\parallel\nabla_{\mathbf{v}_\parallel}F_\perp-\partial_s E\cdot n(\mathbf{x}_\parallel), \tag{5.56}
$$

so $\|\nabla_{\mathbf{x}_\perp}\dot F_\perp\|_\infty+\|\nabla_{\mathbf{x}_\parallel}\dot F_\perp\|_\infty\lesssim|v|^3+1$, and $\|\nabla_{\mathbf{v}_\perp}\dot F_\perp\|_\infty+\|\nabla_{\mathbf{v}_\parallel}\dot F_\perp\|_\infty\lesssim|v|^2+1$. Thus together with (5.27) we have

$$
\left|\frac{d}{d\tau}\left(\frac{\partial F_\perp(\tau)}{\partial \tau}+\frac{\partial F_\perp(\tau)}{\partial t^{\ell}}\right)\right|
$$

$$
\begin{aligned}
&=\left|\frac{\partial \dot F_\perp(\tau)}{\partial \tau}+\frac{\partial \dot F_\perp(\tau)}{\partial t^\ell}\right|\\
&=\Bigg|\nabla_{\mathbf{x}_\perp}\dot F_\perp\cdot\left(\mathbf{v}_\perp(s)+\frac{\partial \mathbf{x}_\perp(s)}{\partial t^\ell}\right)+\nabla_{\mathbf{x}_\parallel}\dot F_\perp\cdot\left(\mathbf{v}_\parallel(s)+\frac{\partial \mathbf{x}_\parallel(s)}{\partial t^\ell}\right)\\
&\quad+\nabla_{\mathbf{v}_\parallel}\dot F_\perp\cdot\left(F_\parallel(s)+\frac{\partial \mathbf{v}_\parallel(s)}{\partial t^\ell}\right)+\nabla_{\mathbf{v}_\perp}\dot F_\perp\cdot\left(F_\perp(s)+\frac{\partial \mathbf{v}_\perp(s)}{\partial t^\ell}\right)-\partial_s^2E\cdot\mathbf{n}(\mathbf{x}_\parallel)\Bigg|\\
&\lesssim(|v|^3+1)\int_s^{t^\ell}\int_\tau^{t^\ell}\left(|\partial_{\tau'}F_\perp(\tau')+\partial_{t^\ell}F_\perp(\tau')|+\left|\partial_{\tau'}F_\parallel(\tau')+\partial_{t^\ell}F_\parallel(\tau')\right|\right)d\tau'd\tau\\
&\quad+(|v|^2+1)\int_s^{t^\ell}\left(|\partial_{\tau'}F_\perp(\tau')+\partial_{t^\ell}F_\perp(\tau')|+\left|\partial_{\tau'}F_\parallel(\tau')+\partial_{t^\ell}F_\parallel(\tau')\right|\right)d\tau'+\|\partial_t^2E\|_{L^\infty_{t,x}}\\
&\lesssim\|\partial_t^2E\|_{L^\infty_{t,x}}+(|v|^3+1)(t^\ell-t^{\ell+1})^2+(|v|^2+1)(t^\ell-t^{\ell+1})\\
&\lesssim\|\partial_t^2E\|_{L^\infty_{t,x}}+|v|+1.
\end{aligned}\tag{5.57}
$$

Combining (5.26), (5.54), (5.57), and expanding $\frac{\partial F_\perp(\tau)}{\partial\tau}+\frac{\partial F_\perp(\tau)}{\partial t^\ell}$ at t^ℓ we get

$$
\begin{aligned}
\frac{\partial \mathbf{v}_\perp^{\ell+1}}{\partial t^\ell}=&\left(\frac{\partial F_\perp(t^\ell)}{\partial\tau}+\frac{\partial F_\perp(t^\ell)}{\partial t^\ell}\right)\left(\frac{F_\perp(t^{\ell+1})}{\mathbf{v}_\perp^{\ell+1}}\frac{|t^\ell-t^{\ell+1}|^2}{2}-|t^\ell-t^{\ell+1}|\right)\\
&+O_{\|E\|_{C^2},\Omega}(1)|t^\ell-t^{\ell+1}|^2(|v|+1).
\end{aligned}\tag{5.58}
$$

Now since we have

$$
\begin{aligned}
0=\mathbf{x}_\perp^\ell=&\mathbf{x}_\perp^{\ell+1}+\int_{t^{\ell+1}}^{t^\ell}\mathbf{v}_\perp(s)ds\\
&=\int_{t^{\ell+1}}^{t^\ell}\left(-\mathbf{v}_\perp^{\ell+1}+\int_{t^{\ell+1}}^{s}F_\perp(\tau)d\tau\right)ds\\
&=(t^\ell-t^{\ell+1})(-\mathbf{v}_\perp^{\ell+1})+\int_{t^{\ell+1}}^{t^\ell}\int_{t^{\ell+1}}^{s}F_\perp(\tau)d\tau ds\\
&=(t^\ell-t^{\ell+1})(-\mathbf{v}_\perp^{\ell+1})+\frac{|t^\ell-t^{\ell+1}|^2}{2}F_\perp(t^{\ell+1})+O(1)(\|\partial_tE\|_{L^\infty_{t,x}}+|v|^3)|t^\ell-t^{\ell+1}|^3,
\end{aligned}\tag{5.59}
$$

we get the following important cancellation identity:

$$
\frac{F_\perp(t^{\ell+1})}{\mathbf{v}_\perp^{\ell+1}}\frac{|t^\ell-t^{\ell+1}|^2}{2}-|t^\ell-t^{\ell+1}|=O(1)(\|\partial_tE\|_{L^\infty_{t,x}}+|v|^3)\frac{|t^\ell-t^{\ell+1}|^3}{\mathbf{v}_\perp^{\ell+1}}.\tag{5.60}
$$

By (5.58) and (5.60) we get

$$
\left|\frac{\partial \mathbf{v}_\perp^{\ell+1}}{\partial t^\ell}\right|\lesssim\left(\|\partial_tE\|^2_{L^\infty_{t,x}}+\|\partial_t^2E\|_{L^\infty_{t,x}}+1\right)\left(|v||t^\ell-t^{\ell+1}|^2+|t^\ell-t^{\ell+1}|^2\right).\tag{5.61}
$$

Next, taking $\frac{\partial}{\partial t^\ell}$ derivative to $\mathbf{v}_\parallel^{\ell+1}=\mathbf{v}_\parallel^\ell-\int_{t^{\ell+1}}^{t^\ell}F_\parallel(s)ds$, and $\mathbf{x}_\parallel^{\ell+1}=\mathbf{x}_\parallel^\ell-(t^\ell-t^{\ell+1})\mathbf{v}_\parallel^\ell+$

$\int_{t^{\ell+1}}^{t^\ell}\int_s^{t^\ell} F_\|(\tau)d\tau ds$ we get

$$\begin{aligned}\frac{\partial \mathbf{v}_\|^{\ell+1}}{\partial t^\ell} &= -F_\|(t^\ell)+\frac{\partial t^{\ell+1}}{\partial t^\ell}F_\|(t^\ell)-\int_{t^{\ell+1}}^{t^\ell}\partial_{t^\ell}F_\|(s)ds\\ &= F_\|(t^{\ell+1})-F_\|(t^\ell)+O(1)\frac{|t^\ell-t^{\ell+1}|^2}{|\mathbf{v}_\perp^{\ell+1}|}-\int_{t^{\ell+1}}^{t^\ell}\partial_{t^\ell}F_\|(s)ds\\ &= O(1)\frac{|t^\ell-t^{\ell+1}|^2}{|\mathbf{v}_\perp^{\ell+1}|}-\int_{t^{\ell+1}}^{t^\ell}\left(\partial_s F_\|(s)+\partial_{t^\ell}F_\|(s)\right)ds \lesssim |t^\ell-t^{\ell+1}|,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \mathbf{x}_\|^{\ell+1}}{\partial t^\ell} &= -\mathbf{v}_\|^\ell+\frac{\partial t^{\ell+1}}{\partial t^\ell}\mathbf{v}_\|^{\ell+1}+\int_{t^{\ell+1}}^{t^\ell}F_\|(t^\ell)ds+\int_{t^{\ell+1}}^{t^\ell}\int_s^{t^\ell}\partial_{t^\ell}F_\|(\tau)d\tau ds\\ &= \mathbf{v}_\|^{\ell+1}-\mathbf{v}_\|^\ell-O(1)\frac{|t^\ell-t^{\ell+1}|^2}{|\mathbf{v}_\perp^{\ell+1}|}\mathbf{v}_\|^{\ell+1}+\int_{t^{\ell+1}}^{t^\ell}F_\|(t^\ell)ds+\int_{t^{\ell+1}}^{t^\ell}\int_s^{t^\ell}\partial_{t^\ell}F_\|(\tau)d\tau ds\\ &= \int_{t^{\ell+1}}^{t^\ell}\left(F_\|(t^\ell)-F_\|(s)\right)ds+\int_{t^{\ell+1}}^{t^\ell}\int_s^{t^\ell}\partial_{t^\ell}F_\|(\tau)d\tau ds-O(1)\frac{|t^\ell-t^{\ell+1}|^2}{|\mathbf{v}_\perp^{\ell+1}|}\mathbf{v}_\|^{\ell+1}\\ &= \int_{t^{\ell+1}}^{t^\ell}\int_s^{t^\ell}\left(\partial_s F_\|(\tau)+\partial_{t^\ell}F_\|(\tau)\right)d\tau ds+O(1)|t^\ell-t^{\ell+1}|\lesssim |t^\ell-t^{\ell+1}|.\end{aligned}$$

Where we've used (5.26) and (5.53). This proves the first column of (5.45) and (5.46).

Taking derivatives of (5.48) as before and using $|t^\ell-t^{\ell+1}|\lesssim_{\xi,t}\min\{\frac{|\mathbf{v}_\perp^{\ell+1}|}{|v|^2},1\}$ and Lemma 4.2,

$$\begin{aligned}\begin{bmatrix}\frac{\partial t^{\ell+1}}{\partial \mathbf{x}_\|^\ell}\\ \frac{\partial t^{\ell+1}}{\partial \mathbf{v}_\perp^\ell}\\ \frac{\partial t^{\ell+1}}{\partial \mathbf{v}_\|^\ell}\end{bmatrix} &= \begin{bmatrix}\frac{1}{\mathbf{v}_\perp^{\ell+1}}\int_{t^\ell}^{t^{\ell+1}}\int_{t^\ell}^s\frac{\partial}{\partial \mathbf{x}_\|^\ell}F_\perp(\mathbf{X}_\ell(\tau),\mathbf{V}_\ell(\tau))\mathrm{d}\tau\mathrm{d}s\\ \frac{1}{\mathbf{v}_\perp^{\ell+1}}\left\{(t^{\ell+1}-t^\ell)+\int_{t^\ell}^{t^{\ell+1}}\int_{t^\ell}^s\frac{\partial}{\partial \mathbf{v}_\perp^\ell}F_\perp(\mathbf{X}_\ell(\tau),\mathbf{V}_\ell(\tau))\mathrm{d}\tau\mathrm{d}s\right\}\\ \frac{1}{\mathbf{v}_\perp^{\ell+1}}\int_{t^{\ell+1}}^{t^\ell}\int_s^{t^\ell}\frac{\partial}{\partial \mathbf{v}_\|^\ell}F_\perp(\mathbf{X}_\ell(\tau),\mathbf{V}_\ell(\tau))\mathrm{d}\tau\mathrm{d}s\end{bmatrix}\\ &\lesssim_{\xi,t}\begin{bmatrix}\frac{|t^\ell-t^{\ell+1}|^2}{|\mathbf{v}_\perp^{\ell+1}|}(|v|^2+O(1))\\ \frac{|t^\ell-t^{\ell+1}|}{|\mathbf{v}_\perp^{\ell+1}|}+\frac{|t^\ell-t^{\ell+1}|^2}{|\mathbf{v}_\perp^{\ell+1}|}(|v|+O(1))\\ \frac{|t^\ell-t^{\ell+1}|^2}{|\mathbf{v}_\perp^{\ell+1}|}(|v|+O(1))\end{bmatrix}.\end{aligned} \tag{5.62}$$

Thus from (5.2) we have

$$\begin{bmatrix}\frac{\partial t^{\ell+1}}{\partial \mathbf{x}_\|^\ell}\\ \frac{\partial t^{\ell+1}}{\partial \mathbf{v}_\perp^\ell}\\ \frac{\partial t^{\ell+1}}{\partial \mathbf{v}_\|^\ell}\end{bmatrix}\lesssim\begin{bmatrix}\frac{1}{|v|}\frac{|\mathbf{v}_\perp^{\ell+1}|}{|v|}\\ \frac{1}{|v|^2}\\ \frac{1}{|v|^2}\frac{|\mathbf{v}_\perp^{\ell+1}|}{|v|}\end{bmatrix},\ \text{for } |v|>\delta.\quad \begin{bmatrix}\frac{\partial t^{\ell+1}}{\partial \mathbf{x}_\|^\ell}\\ \frac{\partial t^{\ell+1}}{\partial \mathbf{v}_\perp^\ell}\\ \frac{\partial t^{\ell+1}}{\partial \mathbf{v}_\|^\ell}\end{bmatrix}\lesssim\begin{bmatrix}|\mathbf{v}_\perp^{\ell+1}|\\ O(1)\\ |\mathbf{v}_\perp^{\ell+1}|\end{bmatrix},\ \text{for } |v|\le\delta. \tag{5.63}$$

Taking $(\mathbf{x}(t^\ell),\mathbf{v}(t^\ell))$ derivatives of the characteristic equations

$$\mathbf{x}_\|^{\ell+1}=\mathbf{x}_\|^\ell-\int_{t^{\ell+1}}^{t^\ell}\mathbf{v}_\|(s;t^\ell x^\ell,v^\ell)ds,$$

by Lemma 4.2 and (5.62), we estimate directly

$$\begin{bmatrix} \frac{\partial \mathbf{x}_{\|}^{\ell+1}}{\partial \mathbf{x}_{\|}^{\ell}} \\ \frac{\partial \mathbf{x}_{\|}^{\ell+1}}{\partial \mathbf{v}_{\perp}^{\ell}} \\ \frac{\partial \mathbf{x}_{\|}^{\ell+1}}{\partial \mathbf{v}_{\|}^{\ell}} \end{bmatrix} \lesssim_{\xi,t} \begin{bmatrix} \mathbf{Id}_{2,2} + \frac{|t^\ell - t^{\ell+1}|^2 |v|^3}{|\mathbf{v}_{\perp}^{\ell+1}|} + O(1)\frac{|t^\ell - t^{\ell+1}|^2|v|}{|\mathbf{v}_{\perp}^{\ell+1}|} \\ O(1)\frac{|t^\ell - t^{\ell+1}||v|}{|\mathbf{v}_{\perp}^{\ell+1}|} + |t^\ell - t^{\ell+1}| \\ \frac{|t^\ell - t^{\ell+1}|^2}{|\mathbf{v}_{\perp}^{\ell+1}|}(|v|^2 + O(1)|v|) + |t^\ell - t^{\ell+1}| \end{bmatrix}.$$

Thus from (5.2) we have

$$\begin{bmatrix} \frac{\partial \mathbf{x}_{\|}^{\ell+1}}{\partial \mathbf{x}_{\|}^{\ell}} \\ \frac{\partial \mathbf{x}_{\|}^{\ell+1}}{\partial \mathbf{v}_{\perp}^{\ell}} \\ \frac{\partial \mathbf{x}_{\|}^{\ell+1}}{\partial \mathbf{v}_{\|}^{\ell}} \end{bmatrix} \lesssim_{\xi,t} \begin{bmatrix} \mathbf{Id}_{2,2} + \frac{|\mathbf{v}_{\perp}^{\ell}|}{|v|} \\ \frac{1}{|v|} \\ \frac{1}{|v|}\frac{|\mathbf{v}_{\perp}^{\ell+1}|}{|v|} \end{bmatrix}, \text{ for } |v|>\delta. \quad \begin{bmatrix} \frac{\partial \mathbf{x}_{\|}^{\ell+1}}{\partial \mathbf{x}_{\|}^{\ell}} \\ \frac{\partial \mathbf{x}_{\|}^{\ell+1}}{\partial \mathbf{v}_{\perp}^{\ell}} \\ \frac{\partial \mathbf{x}_{\|}^{\ell+1}}{\partial \mathbf{v}_{\|}^{\ell}} \end{bmatrix} \lesssim_{\xi,t} \begin{bmatrix} \mathbf{Id}_{2,2} + |\mathbf{v}_{\perp}^{\ell}| \\ O(1) \\ |\mathbf{v}_{\perp}^{\ell}| \end{bmatrix}, \text{ for } |v|\le\delta.$$

Also,

$$\begin{bmatrix} \frac{\partial \mathbf{v}_{\|}^{\ell+1}}{\partial \mathbf{x}_{\|}^{\ell}} \\ \frac{\partial \mathbf{v}_{\|}^{\ell+1}}{\partial \mathbf{v}_{\perp}^{\ell}} \\ \frac{\partial \mathbf{v}_{\|}^{\ell+1}}{\partial \mathbf{v}_{\|}^{\ell}} \end{bmatrix} \lesssim_{\xi,t} \begin{bmatrix} \frac{|t^\ell - t^{\ell+1}|^2}{|\mathbf{v}_{\perp}^{\ell+1}|}(|v|^2 + O(1)|v|)^2 + |t^\ell - t^{\ell+1}|(|v|^2 + O(1)) \\ (|v|^2 + O(1))\left(\frac{|t^\ell - t^{\ell+1}|}{|\mathbf{v}_{\perp}^{\ell+1}|} + \frac{|t^\ell - t^{\ell+1}|^2(|v|^2+O(1)|v|)}{|\mathbf{v}_{\perp}^{\ell+1}|}\right) + |t^\ell - t^{\ell+1}|\langle v\rangle \\ \mathbf{Id}_{2,2} + (|v|^2 + O(1)|v|)\frac{|t^\ell - t^{\ell+1}|^2}{|\mathbf{v}_{\perp}^{\ell+1}|}(|v| + O(1)) + |t^\ell - t^{\ell+1}|(|v| + O(1)) \end{bmatrix}.$$

Thus from (5.2) we have

$$\begin{bmatrix} \frac{\partial \mathbf{v}_{\|}^{\ell+1}}{\partial \mathbf{x}_{\|}^{\ell}} \\ \frac{\partial \mathbf{v}_{\|}^{\ell+1}}{\partial \mathbf{v}_{\perp}^{\ell}} \\ \frac{\partial \mathbf{v}_{\|}^{\ell+1}}{\partial \mathbf{v}_{\|}^{\ell}} \end{bmatrix} \lesssim_{\xi,t} \begin{bmatrix} |\mathbf{v}_{\perp}^{\ell+1}| \\ 1 + \frac{|\mathbf{v}_{\perp}^{\ell+1}|}{|v^\ell|} \\ \mathbf{Id}_{2,2} + \frac{|\mathbf{v}_{\perp}^{\ell+1}|}{|v^\ell|} \end{bmatrix}, \text{ for } |v|>\delta. \quad \begin{bmatrix} \frac{\partial \mathbf{v}_{\|}^{\ell+1}}{\partial \mathbf{x}_{\|}^{\ell}} \\ \frac{\partial \mathbf{v}_{\|}^{\ell+1}}{\partial \mathbf{v}_{\perp}^{\ell}} \\ \frac{\partial \mathbf{v}_{\|}^{\ell+1}}{\partial \mathbf{v}_{\|}^{\ell}} \end{bmatrix} \lesssim_{\xi,t} \begin{bmatrix} |\mathbf{v}_{\perp}^{\ell+1}| \\ 1 + |\mathbf{v}_{\perp}^{\ell+1}| \\ \mathbf{Id}_{2,2} + |\mathbf{v}_{\perp}^{\ell+1}| \end{bmatrix}, \text{ for } |v|\le\delta.$$

Now we move to $D\mathbf{v}_{\perp}^{\ell+1}$ estimates.

Taking derivatives in (5.77), from the extra cancellation in terms of order of $t^\ell - t^{\ell+1}$ in (5.60), by (5.62), and plugging the expansion

$$\frac{\partial}{\partial \mathbf{x}_{\|}^{\ell}} F_\perp(\mathbf{X}_\ell(\tau), \mathbf{V}_\ell(\tau)) = \frac{\partial}{\partial \mathbf{x}_{\|}^{\ell}} F_\perp(\mathbf{x}^\ell, \mathbf{v}^\ell) - \int_\tau^{t^\ell} \frac{d}{d\tau'}\left(\frac{\partial}{\partial \mathbf{x}_{\|}^{\ell}} F_\perp(\mathbf{X}_\ell(\tau'), \mathbf{V}_\ell(\tau'))\right) d\tau'$$

into

$$\begin{aligned} \frac{\partial \mathbf{v}_{\perp}^{\ell+1}}{\partial \mathbf{x}_{\|}^{\ell}} &= \frac{-F_\perp(\mathbf{x}^{\ell+1}, \mathbf{v}^{\ell+1})}{\mathbf{v}_{\perp}^{\ell+1}} \int_{t^{\ell+1}}^{t^\ell} \int_s^{t^\ell} \frac{\partial}{\partial \mathbf{x}_{\|}^{\ell}} F_\perp(\mathbf{X}_\ell(\tau), \mathbf{V}_\ell(\tau)) \mathrm{d}\tau \mathrm{d}s \\ &\quad + \int_{t^{\ell+1}}^{t^\ell} \frac{\partial}{\partial \mathbf{x}_{\|}^{\ell}} F_\perp(\mathbf{X}_\ell(\tau), \mathbf{V}_\ell(\tau)) \mathrm{d}\tau, \end{aligned}$$

and using the cancellation (5.60) we obtain

$$\frac{\partial \mathbf{v}_{\perp}^{\ell+1}}{\partial \mathbf{x}_{\|}^{\ell}} = \left\{\frac{(t^\ell - t^{\ell+1})F_\perp(\mathbf{x}^{\ell+1}, \mathbf{v}^{\ell+1})}{-2\mathbf{v}_{\perp}^{\ell+1}} + 1\right\}(t^\ell - t^{\ell+1})\frac{\partial}{\partial \mathbf{x}_{\|}^{\ell}} F_\perp(\mathbf{x}^\ell, \mathbf{v}^\ell)$$

$$
\begin{aligned}
&+\frac{F_\perp(\mathbf{x}^{\ell+1},\mathbf{v}^{\ell+1})}{\mathbf{v}_\perp^{\ell+1}}\int_{t^{\ell+1}}^{t^\ell}\int_s^{t^\ell}\int_\tau^{t^\ell}\frac{d}{d\tau'}\left(\frac{\partial}{\partial \mathbf{x}_\parallel^\ell}F_\perp(\mathbf{X}_\ell(\tau'),\mathbf{V}_\ell(\tau'))\right)d\tau'\mathrm{d}\tau\mathrm{d}s\\
&+\int_{t^{\ell+1}}^{t^\ell}\int_\tau^{t^\ell}\frac{d}{d\tau'}\left(\frac{\partial}{\partial \mathbf{x}_\parallel^\ell}F_\perp(\mathbf{X}_\ell(\tau'),\mathbf{V}_\ell(\tau'))\right)d\tau'\mathrm{d}\tau\\
\lesssim&\Big\{-1+O_\xi(1)\frac{|t^\ell-t^{\ell+1}|^2(|v^\ell|^3+1)}{|\mathbf{v}_\perp^{\ell+1}|}+1\Big\}|t^\ell-t^{\ell+1}|(|v^\ell|^2+1)\\
&+\frac{F_\perp(\mathbf{x}^{\ell+1},\mathbf{v}^{\ell+1})}{\mathbf{v}_\perp^{\ell+1}}\int_{t^{\ell+1}}^{t^\ell}\int_s^{t^\ell}\int_\tau^{t^\ell}\frac{d}{d\tau'}\left(\frac{\partial}{\partial \mathbf{x}_\parallel^\ell}F_\perp(\mathbf{X}_\ell(\tau'),\mathbf{V}_\ell(\tau'))\right)d\tau'\mathrm{d}\tau\mathrm{d}s\\
&+\int_{t^{\ell+1}}^{t^\ell}\int_\tau^{t^\ell}\frac{d}{d\tau'}\left(\frac{\partial}{\partial \mathbf{x}_\parallel^\ell}F_\perp(\mathbf{X}_\ell(\tau'),\mathbf{V}_\ell(\tau'))\right)d\tau'\mathrm{d}\tau. \qquad (5.64)
\end{aligned}
$$

Now since

$$
\begin{aligned}
&\frac{d}{d\tau'}\left(\frac{\partial}{\partial \mathbf{x}_\parallel^\ell}F_\perp(\mathbf{X}_\ell,\mathbf{V}_\ell)\right)\\
&\lesssim|v^\ell|^3+|\frac{d}{d\tau'}\frac{\partial}{\partial \mathbf{x}_\parallel^\ell}\left(E(\tau',\mathbf{X}_\ell)\cdot\mathbf{n}(\mathbf{X}_\ell)\right)|\\
&\lesssim|v^\ell|^3+|\frac{d}{d\tau'}\left(\mathbf{n}(\mathbf{X}_\ell)\cdot\nabla_xE(\tau',\mathbf{X}_\ell)\cdot\frac{\partial\mathbf{X}_\ell}{\partial\mathbf{x}_\parallel^\ell}+E(\tau',\mathbf{X}_\ell)\cdot\nabla_x\mathbf{n}(\mathbf{X}_\ell)\cdot\frac{\partial\mathbf{X}_\ell}{\partial\mathbf{x}_\parallel^\ell}\right)|\\
&\lesssim|v^\ell|^3+\Bigg|\mathbf{n}(\mathbf{X}_\ell)\cdot\nabla_xE(\tau',\mathbf{X}_\ell)\cdot\left(\frac{d}{d\tau'}\frac{\partial\mathbf{X}_\ell}{\partial\mathbf{x}_\parallel^\ell}\right)+\left(\frac{d}{d\tau'}\mathbf{n}(\mathbf{X}_\ell)\right)\cdot\nabla_xE(\tau',\mathbf{X}_\ell)\cdot\frac{\partial\mathbf{X}_\ell}{\partial\mathbf{x}_\parallel^\ell}\\
&\qquad+\mathbf{n}(\mathbf{X}_\ell)\cdot\partial_t\nabla_xE(\tau',\mathbf{X}_\ell)\cdot\frac{\partial\mathbf{X}_\ell}{\partial\mathbf{x}_\parallel^\ell}+\sum_{1\le i,j,k\le3}\mathbf{n}^i(\mathbf{X}_\ell)\partial_{x_j}\partial_{x_k}E^i(\tau',\mathbf{X}_\ell)\frac{\partial\mathbf{X}_\ell^j}{\partial\mathbf{x}_\parallel^\ell}\mathbf{V}_\ell^k(\tau')\\
&\qquad+(\partial_tE(\tau',\mathbf{X}_\ell)+\nabla_xE(\tau',\mathbf{X}_\ell)\cdot\mathbf{V}_\ell(\tau'))\cdot\nabla_x\mathbf{n}(\mathbf{X}_\ell)\cdot\frac{\partial\mathbf{X}_\ell}{\partial\mathbf{x}_\parallel^\ell}\\
&\qquad+E(\tau',\mathbf{X}_\ell)\cdot\left(\frac{d}{d\tau'}\nabla_x\mathbf{n}(\mathbf{X}_\ell)\right)\cdot\frac{\partial\mathbf{X}_\ell}{\partial\mathbf{x}_\parallel^\ell}+E(\tau',\mathbf{X}_\ell)\cdot\nabla_x\mathbf{n}(\mathbf{X}_\ell)\cdot\left(\frac{d}{d\tau'}\frac{\partial\mathbf{X}_\ell}{\partial\mathbf{x}_\parallel^\ell}\right)\Bigg|\\
&\lesssim|v^\ell|^3+\Bigg|\mathbf{n}(\mathbf{X}_\ell)\cdot\nabla_xE(\tau',\mathbf{X}_\ell)\cdot\frac{\partial\mathbf{V}_\ell}{\partial\mathbf{x}_\parallel^\ell}+(\nabla_x\mathbf{n}(\mathbf{X}_\ell)\cdot\mathbf{V}_\ell(\tau'))\cdot\nabla_xE(\tau',\mathbf{X}_\ell)\cdot\frac{\partial\mathbf{X}_\ell}{\partial\mathbf{x}_\parallel^\ell}\\
&\qquad+\mathbf{n}(\mathbf{X}_\ell)\cdot\partial_t\nabla_xE(\tau',\mathbf{X}_\ell)\cdot\frac{\partial\mathbf{X}_\ell}{\partial\mathbf{x}_\parallel^\ell}+\sum_{1\le i,j,k\le3}\mathbf{n}^i(\mathbf{X}_\ell)\partial_{x_j}\partial_{x_k}E^i(\tau',\mathbf{X}_\ell)\frac{\partial\mathbf{X}_\ell^j}{\partial\mathbf{x}_\parallel^\ell}\mathbf{V}_\ell^k(\tau')\\
&\qquad+(\partial_tE(\tau',\mathbf{X}_\ell)+\nabla_xE(\tau',\mathbf{X}_\ell)\cdot\mathbf{V}_\ell(\tau'))\cdot\nabla_x\mathbf{n}(\mathbf{X}_\ell)\cdot\frac{\partial\mathbf{X}_\ell}{\partial\mathbf{x}_\parallel^\ell}\\
&\qquad+E(\tau',\mathbf{X}_\ell)\cdot\left(\nabla_x^2\mathbf{n}(\mathbf{X}_\ell)\cdot\mathbf{V}_\ell(\tau')\right)\cdot\frac{\partial\mathbf{X}_\ell}{\partial\mathbf{x}_\parallel^\ell}+E(\tau',\mathbf{X}_\ell)\cdot\nabla_x\mathbf{n}(\mathbf{X}_\ell)\cdot\frac{\partial\mathbf{V}_\ell}{\partial\mathbf{x}_\parallel^\ell}\Bigg|\\
&\lesssim|v^\ell|^3+|v^\ell|\|\nabla_x^2E\|_{L^\infty_{t,x}}+\|\partial_t\nabla_xE\|_{L^\infty_{t,x}}, \qquad (5.65)
\end{aligned}
$$

where we use the bounds from (4.10). We have

$$\begin{aligned}
&\frac{\partial \mathbf{v}_\perp^{\ell+1}}{\partial \mathbf{x}_\parallel^\ell}\\
&\lesssim \Big(\frac{|t^\ell-t^{\ell+1}|(|v^\ell|^2+1)}{|\mathbf{v}_\perp^{\ell+1}|}\Big)\Big(|t^\ell-t^{\ell+1}|^2(|v^\ell|^3+1)+|v^\ell|^3+|v^\ell|\|\nabla_x^2 E\|_{L^\infty_{t,x}}+\|\partial_t\nabla_x E\|_{L^\infty_{t,x}}\Big)\\
&\lesssim_{\xi,t}\min\{\frac{|\mathbf{v}_\perp^{\ell+1}|^2}{|v^\ell|},|\mathbf{v}_\perp^{\ell+1}|^2\}
\end{aligned}\tag{5.66}$$

as long as $\|\nabla_x^2 E\|_{L^\infty_{t,x}}+\|\partial_t\nabla_x E\|_{L^\infty_{t,x}}<\infty$. Similarly,

$$\begin{aligned}
&\frac{\partial \mathbf{v}_\perp^{\ell+1}}{\partial \mathbf{v}_\perp^\ell}=-1-\frac{\partial t^{\ell+1}}{\partial \mathbf{v}_\perp^\ell}F_\perp(\mathbf{x}^{\ell+1},\mathbf{v}^{\ell+1})+\int_{t^{\ell+1}}^{t^\ell}\frac{\partial}{\partial \mathbf{v}_\perp^\ell}F_\perp(\mathbf{X}_\ell(\tau),\mathbf{V}_\ell(\tau))\mathrm{d}\tau\\
&=-1+\frac{F_\perp(\mathbf{x}^{\ell+1},\mathbf{v}^{\ell+1})}{\mathbf{v}_\perp^{\ell+1}}(t^\ell-t^{\ell+1})\\
&\quad-\frac{F_\perp(\mathbf{x}^{\ell+1},\mathbf{v}^{\ell+1})}{\mathbf{v}_\perp^{\ell+1}}\int_{t^\ell}^{t^{\ell+1}}\int_{t^\ell}^{s}\frac{\partial}{\partial \mathbf{v}_\perp^\ell}F_\perp(\mathbf{X}_\ell(\tau),\mathbf{V}_\ell(\tau))\mathrm{d}\tau\mathrm{d}s\\
&\quad+\int_{t^{\ell+1}}^{t^\ell}\frac{\partial}{\partial \mathbf{v}_\perp^\ell}F_\perp(\mathbf{X}_\ell(\tau),\mathbf{V}_\ell(\tau))\mathrm{d}\tau\\
&=-1+2+O_\xi(1)\frac{|t^\ell-t^{\ell+1}|^2(|v^\ell|^3+1)}{\mathbf{v}_\perp^{\ell+1}}\\
&\quad-\frac{F_\perp(\mathbf{x}^\ell,\mathbf{v}^\ell)}{\mathbf{v}_\perp^{\ell+1}}\frac{(t^\ell-t^{\ell+1})^2}{2}\Big\{\lim_{s\uparrow t^\ell}\frac{\partial}{\partial \mathbf{v}_\perp^\ell}F_\perp(\mathbf{X}_\ell(\tau),\mathbf{V}_\ell(\tau))+O_\xi(1)|t^\ell-t^{\ell+1}|(|v^\ell|^2+1)\Big\}\\
&\quad+(t^\ell-t^{\ell+1})\Big\{\lim_{s\uparrow t^\ell}\frac{\partial}{\partial \mathbf{v}_\perp^\ell}F_\perp(\mathbf{X}_\ell(\tau),\mathbf{V}_\ell(\tau))+O_\xi(1)|t^\ell-t^{\ell+1}|(|v^\ell|^2+1)\Big\}\\
&=1+O_\xi(1)\Big\{\frac{|t^\ell-t^{\ell+1}|^2(|v^\ell|^3+1)}{|\mathbf{v}_\perp^{\ell+1}|}\\
&\quad+\frac{|t^\ell-t^{\ell+1}|^3}{|\mathbf{v}_\perp^{\ell+1}|}(|v^\ell|^3+1)\Big|\lim_{s\uparrow t^\ell}\frac{\partial}{\partial \mathbf{v}_\perp^\ell}F_\perp(\mathbf{X}_\ell(\tau),\mathbf{V}_\ell(\tau))\Big|+|t^\ell-t^{\ell+1}|^2(|v^\ell|^2+1)\Big\}\\
&\lesssim 1+|t^\ell-t^{\ell+1}|^2(|v^\ell|^2+1)\Big\{1+\frac{|v^\ell|+1}{|\mathbf{v}_\perp^{\ell+1}|}+\frac{|t^\ell-t^{\ell+1}|(|v^\ell|^2+1)}{|\mathbf{v}_\perp^{\ell+1}|}\Big\}\\
&\lesssim_{\xi,t}1+\min\{\frac{|\mathbf{v}_\perp^{\ell+1}|}{|v^\ell|},|\mathbf{v}_\perp^{\ell+1}|\},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathbf{v}_\perp^{\ell+1}}{\partial \mathbf{v}_\parallel^\ell}&=\frac{-F_\perp(\mathbf{x}^{\ell+1},v^{\ell+1})}{\mathbf{v}_\perp^{\ell+1}}\int_{t^{\ell+1}}^{t^\ell}\int_{s}^{t^\ell}\frac{\partial}{\partial \mathbf{v}_\parallel^\ell}F_\perp(\mathbf{X}_\ell(\tau),\mathbf{V}_\ell(\tau))\mathrm{d}\tau\mathrm{d}s\\
&\quad-\int_{t^{\ell+1}}^{t^\ell}\frac{\partial}{\partial \mathbf{v}_\parallel^\ell}F_\perp(\mathbf{X}_\ell(\tau),\mathbf{V}_\ell(\tau))\mathrm{d}\tau\\
&=\Big\{\frac{(t^\ell-t^{\ell+1})F_\perp(\mathbf{x}^{\ell+1},\mathbf{v}^{\ell+1})}{-2\mathbf{v}_\perp^{\ell+1}}+1\Big\}(t^\ell-t^{\ell+1})\frac{\partial}{\partial \mathbf{v}_\parallel^\ell}F_\perp(\mathbf{x}^\ell,\mathbf{v}^\ell)\\
&\quad+O_\xi(1)|t^\ell-t^{\ell+1}|^2(|v^\ell|^2+1)\Big\{\frac{|F_\perp(\mathbf{x}^{\ell+1},\mathbf{v}^{\ell+1})||t^\ell-t^{\ell+1}|}{|\mathbf{v}_\perp^{\ell+1}|}+1\Big\}
\end{aligned}$$

$$\lesssim_{\xi} |t^{\ell}-t^{\ell+1}|^2(|v^{\ell}|^2+1)\Big\{1+\frac{|t^{\ell}-t^{\ell+1}|(|v^{\ell}|^2+1)}{|\mathbf{v}_{\perp}^{\ell+1}|}\Big\} \lesssim_{\xi,t} \min\{\frac{|\mathbf{v}_{\perp}^{\ell+1}|^2}{|v^{\ell}|^2},|\mathbf{v}_{\perp}^{\ell+1}|^2\}. \tag{5.67}$$

These estimates complete the proof of the claims (5.46), and of (5.45) when ℓ is *Type II*.

Proof of (5.45) when ℓ is Type III: Recall that we chose a $\mathbf{p}^{\ell}-$spherical coordinate as $\mathbf{p}^{\ell}=(z^{\ell},w^{\ell})$ with $|z^{\ell}-x^{\ell}|\leq\sqrt{\delta}$ and any $w^{\ell}\in\mathbb{S}^2$ with $n(z^{\ell})\cdot w^{\ell}=0$.

Fix ℓ. Let us choose fixed numbers $\Delta_1,\Delta_2>0$ such that $|v|\Delta_1\ll 1$ and $|v||t^{\ell+1}-(t^{\ell}-\Delta_1-\Delta_2)|\ll 1$ so that

$$s^{\ell}\equiv t^{\ell}-\Delta_1, \quad s^{\ell+1}\equiv s^{\ell}-\Delta_2=t^{\ell}-\Delta_1-\Delta_2,$$

satisfying $|v||t^{\ell+1}-s^{\ell+1}|=|v||t^{\ell+1}-(t^{\ell}-\Delta_1-\Delta_2)|\ll 1$ and $|v||t^{\ell}-s^{\ell}|=|v||\Delta_1|\ll 1$ so that the spherical coordinates are well-defined for $s\in[t^{\ell+1},s^{\ell+1}]$ and $s\in[s^{\ell},t^{\ell}]$.

Notice that

$$\frac{\partial s^{\ell+1}}{\partial s^{\ell}}=\frac{\partial(s^{\ell}-\Delta_1)}{\partial s^{\ell}}=1, \quad \frac{\partial s^{\ell}}{\partial t^{\ell}}=\frac{\partial(t^{\ell}-\Delta_1)}{\partial t^{\ell}}=1.$$

We first follow the flow in $\mathbf{p}^{\ell}-$spherical coordinate, then change to the Euclidian coordinate to near the boundary at s^{ℓ}, follow the flow until $s^{\ell+1}$, and then change to the chart to $\mathbf{p}^{\ell+1}-$spherical coordinate. By the chain rule,

$$\begin{aligned}
&\frac{\partial(t^{\ell+1},\mathbf{x}_{\|_{\ell+1}}^{\ell+1},\mathbf{v}_{\perp_{\ell+1}}^{\ell+1},\mathbf{v}_{\|_{\ell+1}}^{\ell+1})}{\partial(t^{\ell},\mathbf{x}_{\|_{\ell}}^{\ell},\mathbf{v}_{\perp_{\ell}}^{\ell},\mathbf{v}_{\|_{\ell}}^{\ell})}\\
&=\frac{\partial(t^{\ell+1},\mathbf{x}_{\|_{\ell+1}}^{\ell+1},\mathbf{v}_{\perp_{\ell+1}}^{\ell+1},\mathbf{v}_{\|_{\ell+1}}^{\ell+1})}{\partial(s^{\ell+1},\mathbf{x}_{\perp_{\ell+1}}(s^{\ell+1}),\mathbf{x}_{\|_{\ell+1}}(s^{\ell+1}),\mathbf{v}_{\perp_{\ell+1}}(s^{\ell+1}),\mathbf{v}_{\|_{\ell+1}}(s^{\ell+1}))}\\
&\quad\times\frac{\partial(s^{\ell+1},\mathbf{X}_{\mathbf{p}^{\ell+1}}(s^{\ell+1}),\mathbf{V}_{\mathbf{p}^{\ell+1}}(s^{\ell+1}))}{\partial(s^{\ell+1},X_{\mathbf{cl}}(s^{\ell+1}),V_{\mathbf{cl}}(s^{\ell+1}))}\frac{\partial(s^{\ell+1},X_{\mathbf{cl}}(s^{\ell+1}),V_{\mathbf{cl}}(s^{\ell+1}))}{\partial(s^{\ell},X_{\mathbf{cl}}(s^{\ell}),V_{\mathbf{cl}}(s^{\ell}))}\\
&\quad\times\frac{\partial(s^{\ell},X_{\mathbf{cl}}(s^{\ell}),V_{\mathbf{cl}}(s^{\ell}))}{\partial(s^{\ell},\mathbf{X}_{\mathbf{p}^{\ell}}(s^{\ell}),\mathbf{V}_{\mathbf{p}^{\ell}}(s^{\ell}))}\frac{\partial(s^{\ell},\mathbf{x}_{\perp_{\ell}}(s^{\ell}),\mathbf{x}_{\|_{\ell}}(s^{\ell}),\mathbf{v}_{\perp_{\ell}}(s^{\ell}),\mathbf{v}_{\|_{\ell}}(s^{\ell}))}{\partial(t^{\ell},\mathbf{x}_{\|_{\ell}}^{\ell},\mathbf{v}_{\perp_{\ell}}^{\ell},\mathbf{v}_{\|_{\ell}}^{\ell})}.
\end{aligned}$$

We can express that $t^{\ell+1}=t^{\ell}-t_{\mathbf{b}}(x^{\ell},v^{\ell})=s^{\ell+1}+\Delta_1+\Delta_2-t_{\mathbf{b}}(x^{\ell},v^{\ell})$. Let us regard $t^{\ell+1}$ as t^1 and $s^{\ell+1}$ as s^1 and $\Delta_1+\Delta_2$ as Δ in (5.39). Then we use (5.40) and (5.2) to have

$$\begin{aligned}
&\frac{\partial(t^{\ell+1},\mathbf{x}_{\|}^{\ell+1},\mathbf{v}_{\perp}^{\ell+1},\mathbf{v}_{\|}^{\ell+1})}{\partial(s^{\ell+1},\mathbf{x}_{\perp}(s^{\ell+1}),\mathbf{x}_{\|}(s^{\ell+1}),\mathbf{v}_{\perp}(s^{\ell+1}),\mathbf{v}_{\|}(s^{\ell+1}))}\\
&\leq\left[\begin{array}{c|c|c}
1+O(1)|t^{\ell}-t^{\ell+1}| & O_{\delta,\xi}(1)\frac{1}{|v|} & O_{\delta,\xi}(1)\frac{1}{|v|^2}\\ \hline
O(1)|t^{\ell}-t^{\ell+1}| & O_{\delta,\xi}(1) & O_{\delta,\xi}(1)\frac{1}{|v|}\\ \hline
O(1)|t^{\ell}-t^{\ell+1}| & O_{\delta,\xi}(1)(|v|+\frac{1}{|\mathbf{v}_{\perp_{\ell+1}}|}) & O_{\delta,\xi}(1)
\end{array}\right]\\
&\leq\left[\begin{array}{c|c|c}
1+O(1)|t^{\ell}-t^{\ell+1}| & O_{\delta,\xi}(1)\frac{1}{|v|} & O_{\delta,\xi}(1)\frac{1}{|v|^2}\\ \hline
O(1)|t^{\ell}-t^{\ell+1}| & O_{\delta,\xi}(1) & O_{\delta,\xi}(1)\frac{1}{|v|}\\ \hline
O(1)|t^{\ell}-t^{\ell+1}| & O_{\delta,\delta',\xi}(1)|v| & O_{\delta,\xi}(1)
\end{array}\right],
\end{aligned}$$

where we have used from (5.44)

$$\frac{\partial(s^{\ell+1},\mathbf{X}_{\mathbf{p}^{\ell+1}}(s^{\ell+1}),\mathbf{V}_{\mathbf{p}^{\ell+1}}(s^{\ell+1}))}{\partial(s^{\ell+1},X_{\mathbf{cl}}(s^{\ell+1}),V_{\mathbf{cl}}(s^{\ell+1}))}\lesssim_\xi \left[\begin{array}{c|c|c} 1 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \hline \mathbf{0}_{3,1} & O_\xi(1) & \mathbf{0}_{3,3} \\ \hline \mathbf{0}_{3,1} & O_\xi(1)|v| & O_\xi(1) \end{array}\right],$$

and from $s^{\ell+1}=s^\ell-\Delta_2,\ X_{\mathbf{cl}}(s^{\ell+1})=X_{\mathbf{cl}}(s^\ell)-(s^{\ell+1}-s^\ell)V_{\mathbf{cl}}(s^\ell),\ V_{\mathbf{cl}}(s^{\ell+1})=V_{\mathbf{cl}}(s^\ell)$,

$$\frac{\partial(s^{\ell+1},X_{\mathbf{cl}}(s^{\ell+1}),V_{\mathbf{cl}}(s^{\ell+1}))}{\partial(s^{\ell},X_{\mathbf{cl}}(s^{\ell}),V_{\mathbf{cl}}(s^{\ell}))}\lesssim_\xi \begin{bmatrix} 1 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \mathbf{0}_{3,1} & \mathbf{Id}_{3,3} & |s_1-s_2|\mathbf{Id}_{3,3} \\ \mathbf{0}_{3,1} & \mathbf{0}_{3,3} & \mathbf{Id}_{3,3} \end{bmatrix},$$

and from (4.5)

$$\frac{\partial(s^{\ell},X_{\mathbf{cl}}(s^{\ell}),V_{\mathbf{cl}}(s^{\ell}))}{\partial(s^{\ell},\mathbf{X}_{\mathbf{p}^{\ell}}(s^{\ell}),\mathbf{V}_{\mathbf{p}^{\ell}}(s^{\ell}))}\lesssim_\xi \begin{bmatrix} 1 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \mathbf{0}_{3,1} & O_\xi(1) & \mathbf{0}_{3,3} \\ \mathbf{0}_{3,1} & |v| & O_\xi(1) \end{bmatrix}.$$

Recalling (5.36), we have

$$\frac{\partial(s^\ell,\mathbf{x}_{\perp_\ell}(s^\ell),\mathbf{x}_{\|_\ell}(s^\ell),\mathbf{v}_{\perp_\ell}(s^\ell),\mathbf{v}_{\|_\ell}(s^\ell))}{\partial(t^\ell,\mathbf{x}^\ell_{\|_\ell},\mathbf{v}^\ell_{\perp_\ell},\mathbf{v}^\ell_{\|_\ell})}\lesssim_\xi \left[\begin{array}{c|c|c} 1 & \mathbf{0}_{1,2} & \mathbf{0}_{1,3} \\ \hline O_\xi(1)|v| & O_\xi(1) & O_\xi(1)|t^\ell-s_1| \\ \hline O_\xi(1)|v|^2 & O_\xi(1)|v| & O_\xi(1) \end{array}\right].$$

By direct matrix multiplication

$$\frac{\partial(t^{\ell+1},\mathbf{x}^{\ell+1}_{\|_{\ell+1}},\mathbf{v}^{\ell+1}_{\perp_{\ell+1}},\mathbf{v}^{\ell+1}_{\|_{\ell+1}})}{\partial(t^\ell,\mathbf{x}^\ell_{\|_\ell},\mathbf{v}^\ell_{\perp_\ell},\mathbf{v}^\ell_{\|_\ell})}\lesssim_{t,\xi} \left[\begin{array}{c|c|c} 1 & \frac{1}{|v|} & \frac{1}{|v|^2} \\ \hline \mathbf{0}_{2,1} & 1 & \frac{1}{|v|} \\ \hline \mathbf{0}_{3,1} & |v| & 1 \end{array}\right].$$

Note that for *Type III* we have $\mathbf{r}^{\ell+1}\gtrsim\sqrt{\delta}$ so that from (5.45)

$$J(\mathbf{r}^{\ell+1})\gtrsim \left[\begin{array}{c|c|c} 1 & \frac{M}{|v|}\sqrt{\delta} & \frac{M}{|v|^2}\min\{1,\sqrt{\delta}\} \\ \hline \mathbf{0}_{2,1} & M\sqrt{\delta} & \frac{M}{|v|}\min\{1,\sqrt{\delta}\} \\ \hline \mathbf{0}_{3,1} & M|v|\min\{\delta,\sqrt{\delta}\} & M\min\{\delta,\sqrt{\delta}\} \end{array}\right]\gtrsim_{\delta,t,\xi}\frac{\partial(t^{\ell+1},\mathbf{x}^{\ell+1}_{\|_{\ell+1}},\mathbf{v}^{\ell+1}_{\perp_{\ell+1}},\mathbf{v}^{\ell+1}_{\|_{\ell+1}})}{\partial(t^\ell,\mathbf{x}^\ell_{\|_\ell},\mathbf{v}^\ell_{\perp_\ell},\mathbf{v}^\ell_{\|_\ell})}.$$

This proves our claim (5.45) for *Type III.*

Step 5. Eigenvalues and diagonalization of (5.45).

We consider the case when ℓ is *Type II* or *Type III.* By a basic linear algebra (row and column operations), the characteristic polynomial of (5.45) equals, with $\mathbf{r}=\mathbf{r}^{\ell+1}$,

$$\det\begin{bmatrix} 1+5M\mathbf{r}-\lambda & \frac{M}{|v|}\mathbf{r} & \frac{M}{|v|}\mathbf{r} & \frac{M}{|v|^2} & \frac{M}{|v|^2}\mathbf{r} & \frac{M}{|v|^2}\mathbf{r} \\ 5M\mathbf{r}|v| & 1+M\mathbf{r}-\lambda & M\mathbf{r} & \frac{M}{|v|} & \frac{M}{|v|}\mathbf{r} & \frac{M}{|v|}\mathbf{r} \\ 5M\mathbf{r}|v| & M\mathbf{r} & 1+M\mathbf{r}-\lambda & \frac{M}{|v|} & \frac{M}{|v|}\mathbf{r} & \frac{M}{|v|}\mathbf{r} \\ 5M\mathbf{r}^2|v|^2 & M|v|\mathbf{r}^2 & M|v|\mathbf{r}^2 & 1+M\mathbf{r}-\lambda & M\mathbf{r}^2 & M\mathbf{r}^2 \\ 5M\mathbf{r}|v|^2 & M|v|\mathbf{r} & M|v|\mathbf{r} & M & 1+M\mathbf{r}-\lambda & M\mathbf{r} \\ 5M\mathbf{r}|v|^2 & M|v|\mathbf{r} & M|v|\mathbf{r} & M & M\mathbf{r} & 1+M\mathbf{r}-\lambda \end{bmatrix}$$
$$=(\lambda-1)^5(\lambda-(10M\mathbf{r}+1)).$$

Therefore eigenvalues are

$$\lambda_1=\lambda_2=\lambda_3=\lambda_4=\lambda_5=1,\lambda_6=1+10M\mathbf{r}. \tag{5.68}$$

Corresponding eigenvectors are

$$\begin{pmatrix} -\frac{1}{5|v|} \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{5|v|} \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{5|v|^2\mathbf{r}} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{5|v|^2} \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{5|v|^2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{|v|^2} \\ \frac{1}{|v|} \\ \frac{1}{|v|} \\ \mathbf{r} \\ 1 \\ 1 \end{pmatrix}.$$

Write $P = P(\mathbf{r}^\ell)$ as a block matrix of above column eigenvectors. Then

$$\mathcal{P} = \begin{bmatrix} -\frac{1}{5|v|} & -\frac{1}{5|v|} & -\frac{1}{5|v|^2\mathbf{r}} & -\frac{1}{5|v|^2} & -\frac{1}{5|v|^2} & \frac{1}{|v|^2} \\ 1 & 0 & 0 & 0 & 0 & \frac{1}{|v|} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{|v|} \\ 0 & 0 & 1 & 0 & 0 & \mathbf{r} \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

$$\mathcal{P}^{-1} = \begin{bmatrix} -\frac{|v|}{2} & \frac{9}{10} & -\frac{1}{10} & -\frac{1}{10|v|\mathbf{r}} & -\frac{1}{10|v|} & -\frac{1}{10|v|} \\ -\frac{|v|}{2} & -\frac{1}{10} & \frac{9}{10} & -\frac{1}{10|v|\mathbf{r}} & -\frac{1}{10|v|} & -\frac{1}{10|v|} \\ -\frac{|v|^2\mathbf{r}}{2} & -\frac{|v|\mathbf{r}}{10} & -\frac{|v|\mathbf{r}}{10} & \frac{9}{10} & -\frac{\mathbf{r}}{10} & -\frac{\mathbf{r}}{10} \\ -\frac{|v|^2}{2} & -\frac{|v|}{10} & -\frac{|v|}{10} & -\frac{1}{10\mathbf{r}} & \frac{9}{10} & -\frac{1}{10} \\ -\frac{|v|^2}{2} & -\frac{|v|}{10} & -\frac{|v|}{10} & -\frac{1}{10\mathbf{r}} & -\frac{1}{10} & \frac{9}{10} \\ \frac{|v|^2}{2} & \frac{|v|}{10} & \frac{|v|}{10} & \frac{1}{10\mathbf{r}} & \frac{1}{10} & \frac{1}{10} \end{bmatrix}. \tag{5.69}$$

Therefore

$$J(\mathbf{r}) = \mathcal{P}(\mathbf{r})\Lambda(\mathbf{r})\mathcal{P}^{-1}(\mathbf{r}),$$

and

$$\Lambda(\mathbf{r}) := \mathrm{diag}\Big[1,1,1,1,1,1+10M\mathbf{r}\Big],$$

where the notation $\mathrm{diag}[a_1,\cdots,a_m]$ is a $m\times m-$matrix with $a_{ii} = a_i$ and $a_{ij} = 0$ for all $i \neq j$.

Similarly for the case when ℓ is *Type I*, the eigenvalues of the matrix (5.46) are (with $\mathbf{v}_\perp = \mathbf{v}_\perp^{\ell+1}$)

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 1, \lambda_6 = 1 + 10M\mathbf{v}_\perp. \tag{5.70}$$

Corresponding eigenvectors are

$$\begin{pmatrix} -\frac{1}{5} \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{5} \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{5\mathbf{v}_\perp} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{5} \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{5} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ \mathbf{v}_\perp \\ 1 \\ 1 \end{pmatrix}.$$

Write $P = P(\mathbf{v}^\ell_\perp)$ as a block matrix of above column eigenvectors. Then

$$\mathcal{P} = \begin{bmatrix} -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5\mathbf{v}_\perp} & -\frac{1}{5} & -\frac{1}{5} & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & \mathbf{v}_\perp \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \ \mathcal{P}^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{9}{10} & -\frac{1}{10} & -\frac{1}{10\mathbf{v}_\perp} & -\frac{1}{10} & -\frac{1}{10} \\ -\frac{1}{2} & -\frac{1}{10} & \frac{9}{10} & -\frac{1}{10\mathbf{v}_\perp} & -\frac{1}{10} & -\frac{1}{10} \\ -\frac{\mathbf{v}_\perp}{2} & -\frac{\mathbf{v}_\perp}{10} & -\frac{\mathbf{v}_\perp}{10} & \frac{9}{10} & -\frac{\mathbf{v}_\perp}{10} & -\frac{\mathbf{v}_\perp}{10} \\ -\frac{1}{2} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10\mathbf{v}_\perp} & \frac{9}{10} & -\frac{1}{10} \\ -\frac{1}{2} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10\mathbf{v}_\perp} & -\frac{1}{10} & \frac{9}{10} \\ \frac{1}{2} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10\mathbf{v}_\perp} & \frac{1}{10} & \frac{1}{10} \end{bmatrix}. \tag{5.71}$$

Therefore

$$J(\mathbf{v}_\perp) = \mathcal{P}(\mathbf{v}_\perp)\Lambda(\mathbf{v}_\perp)\mathcal{P}^{-1}(\mathbf{v}_\perp),$$

and

$$\Lambda(\mathbf{v}_\perp) := \mathrm{diag}\Big[1,1,1,1,1,1+10M\mathbf{v}_\perp\Big],$$

Step 6. The $i-th$ intermediate group.

If ℓ is *Type II* or *Type III*, we claim that, for $i = 1,2,\cdots,[\frac{|t-s||v|}{L_\xi}]$,

$$\begin{aligned} & J^{\ell_{i+1}}_{\ell_{i+1}-1} \times \cdots \times J^{\ell_i+1}_{\ell_i} \\ &= \frac{\partial(t^{\ell_{i+1}}, \mathbf{x}^{\ell_{i+1}}_{\|_{\ell_{i+1}}}, \mathbf{v}^{\ell_{i+1}}_{\perp_{\ell_{i+1}}}, \mathbf{v}^{\ell_{i+1}}_{\|_{\ell_{i+1}}})}{\partial(t^{\ell_{i+1}-1}, \mathbf{x}^{\ell_{i+1}-1}_{\|_{\ell_{i+1}-1}}, \mathbf{v}^{\ell_{i+1}-1}_{\perp_{\ell_{i+1}-1}}, \mathbf{v}^{\ell_{i+1}-1}_{\|_{\ell_{i+1}-1}})} \times \cdots \times \frac{\partial(t^{\ell_i+1}, \mathbf{x}^{\ell_i+1}_{\|_{\ell_i+1}}, \mathbf{v}^{\ell_i+1}_{\perp_{\ell_i+1}}, \mathbf{v}^{\ell_i+1}_{\|_{\ell_i+1}})}{\partial(t^{\ell_i}, \mathbf{x}^{\ell_i}_{\|_{\ell_i}}, \mathbf{v}^{\ell_i}_{\perp_{\ell_i}}, \mathbf{v}^{\ell_i}_{\|_{\ell_i}})} \\ &\le \mathcal{P}(\mathbf{r}_i)(\Lambda(\mathbf{r}_i))^{\frac{C_\xi}{\mathbf{r}_i}}\mathcal{P}^{-1}(\mathbf{r}_i). \end{aligned} \tag{5.72}$$

By the definition of the group, $L_\xi \le |v||t^{\ell_i} - t^{\ell_{i+1}}| \le C_1 < +\infty$ for all i. By the Velocity lemma (Lemma 3.1),

$$\begin{aligned} & \frac{1}{\mathcal{C}_1} e^{-\frac{\mathcal{C}}{2}C_1}\mathbf{r}^{\ell_i} \le \mathbf{r}^{\ell_{i+1}} \equiv \frac{|\mathbf{v}^{\ell_{i+1}}_\perp|}{|v^{\ell_{i+1}}|}, \mathbf{r}^{\ell_{i+1}-1} \equiv \frac{|\mathbf{v}^{\ell_{i+1}-1}_\perp|}{|v^{\ell_{i+1}-1}|}, \\ & \cdots, \mathbf{r}^{\ell_i+1} \equiv \frac{|\mathbf{v}^{\ell_i+1}_\perp|}{|v^{\ell_i+1}|}, \mathbf{r}^{\ell_i} \equiv \frac{|\mathbf{v}^{\ell_i}_\perp|}{|v^{\ell_i}|} \le \mathcal{C}_1 e^{\frac{\mathcal{C}}{2}C_1}\mathbf{r}^{\ell_i}, \end{aligned}$$

and define

$$\mathbf{r}_i \equiv \mathcal{C}_1 e^{\frac{\mathcal{C}}{2}C_1}\mathbf{r}^{\ell_i}.$$

Then we have

$$\frac{1}{(\mathcal{C}_1)^2} e^{-\mathcal{C}C_1}\mathbf{r}_i \ \le\ \mathbf{r}^j \ \le\ \mathbf{r}_i \quad \text{for all} \ \ \ell_{i+1} \le j \le \ell_i. \tag{5.73}$$

From (5.45), we have a uniform bound for all $\ell_{i+1} \le j \le \ell_i$

$$J^{j+1}_j \lesssim J(\mathbf{r}_i) = \mathcal{P}(\mathbf{r}_i)\Lambda(\mathbf{r}_i)\mathcal{P}^{-1}(\mathbf{r}_i).$$

Therefore

$$J^{\ell_{i+1}}_{\ell_{i+1}-1} \times \cdots \times J^{\ell_i+1}_{\ell_i} \le \mathcal{P}(\mathbf{r}_i)[\Lambda(\mathbf{r}_i)]^{|\ell_{i+1}-\ell_i|}\mathcal{P}^{-1}(\mathbf{r}_i).$$

Now we have only left to prove $|\ell_{i+1}-\ell_i|\lesssim_\Omega \frac{1}{\mathbf{r}_i}$: For any $\ell_{i+1}\le j\le \ell_i$, we have $\xi(x^j)=0=\xi(x^{j+1})=\xi(x^j-(t^j-t^{j+1})v^j)$. We expand $\xi(x^j-(t^j-t^{j+1})v^j)$ in time to have

$$\begin{aligned}\xi(x^{j+1})&=\xi(x^j)+\int_{t^j}^{t^{j+1}}\frac{d}{ds}\xi(X_{\mathbf{cl}}(s))\mathrm{d}s\\&=\xi(x^j)+(v^j\cdot\nabla\xi(x^j))(t^{j+1}-t^j)+\int_{t^j}^{t^{j+1}}\int_{t^j}^{s}\frac{d^2}{d\tau^2}\xi(X_{\mathbf{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s,\end{aligned}$$

and

$$\begin{aligned}0=&(v^j\cdot\nabla\xi(x^j))(t^{j+1}-t^j)+\frac{(t^j-t^{j+1})^2}{2}(V_{\mathbf{cl}}(\tau_*)\cdot\nabla^2\xi(X_{\mathbf{cl}}(\tau_*))\cdot V_{\mathbf{cl}}(\tau_*)\\&+E(\tau,X_{\mathbf{cl}}(\tau_*))\cdot\nabla\xi(X_{\mathbf{cl}}(\tau_*))),\end{aligned}$$

for some $\tau_*\in[t^{j+1},t^j]$. Therefore

$$\frac{v^j\cdot\nabla\xi(x^j)}{|v|}=(t^j-t^{j+1})|v|\frac{V_{\mathbf{cl}}(\tau_*)\cdot\nabla^2\xi(X_{\mathbf{cl}}(\tau_*))\cdot V_{\mathbf{cl}}(\tau_*)+E(\tau,X_{\mathbf{cl}}(\tau_*))\cdot\nabla\xi(X_{\mathbf{cl}}(\tau_*))}{2|v|^2}.$$

Thus there exists $C_2(\delta,\xi,E)\gg 1$

$$\frac{|v^j\cdot\nabla\xi(x^j)|}{|v|}\le C_2|t^j-t^{j+1}||v|. \tag{5.74}$$

Therefore we have a lower bound of $|v||t^j-t^{j+1}|$: $|v||t^j-t^{j+1}|\ge\frac{1}{C_2}|\mathbf{r}^j|\ge\frac{1}{(\mathcal{C}_1)^2C_2}e^{-\mathcal{C}C_1}\mathbf{r}_i$, where we have used (5.73). Finally, using the definition of one group($1\le|v||t^{\ell_i}-t^{\ell_{i+1}}|\le C_1$), we have the following upper bound of the number of bounces in this one group($i-$th intermediate group)

$$|\ell_i-\ell_{i+1}|\le\frac{|v||t^{\ell_i}-t^{\ell_{i+1}}|}{\min_{\ell_i\le j\le\ell_{i+1}}|v||t^j-t^{j+1}|}\le\frac{C_1}{\frac{1}{(\mathcal{C}_1)^2C_2}e^{-\mathcal{C}C_1}\mathbf{r}_i}\lesssim_\xi\frac{1}{\mathbf{r}_i},$$

and this completes our claim (5.72).

Let's consider the whole intermediate groups

$$J_{\ell_*-1}^{\ell_*}\times\cdots\times J_\ell^{\ell+1}\times J_{\ell-1}^{\ell}\times\cdots\times J_1^2\le J(\mathbf{r}^{\ell_*})\times\cdots\times J(\mathbf{r}^{\ell+1})\times J(\mathbf{r}^\ell)\times\cdots J(\mathbf{r}^2). \tag{5.75}$$

We have from (5.69) that

$$J(\mathbf{r}^{\ell+1})\times J(\mathbf{r}^\ell)=\mathcal{P}(\mathbf{r}^{\ell+1})\Lambda(\mathbf{r}^{\ell+1})\mathcal{P}^{-1}(\mathbf{r}^{\ell+1})\mathcal{P}(\mathbf{r}^\ell)\Lambda(\mathbf{r}^\ell)\mathcal{P}^{-1}(\mathbf{r}^\ell),$$

and by direct computation

$$\mathcal{P}^{-1}(\mathbf{r}^{\ell+1})\mathcal{P}(\mathbf{r}^\ell)=\begin{bmatrix}-\frac{|v|}{2}&\frac{9}{10}&-\frac{1}{10}&-\frac{1}{10|v|\mathbf{r}^{\ell+1}}&-\frac{1}{10|v|}&-\frac{1}{10|v|}\\-\frac{|v|}{2}&-\frac{1}{10}&\frac{9}{10}&-\frac{1}{10|v|\mathbf{r}^{\ell+1}}&-\frac{1}{10|v|}&-\frac{1}{10|v|}\\-\frac{|v|^2\mathbf{r}^{\ell+1}}{2}&-\frac{|v|\mathbf{r}^{\ell+1}}{10}&-\frac{|v|\mathbf{r}^{\ell+1}}{10}&\frac{9}{10}&-\frac{\mathbf{r}^{\ell+1}}{10}&-\frac{\mathbf{r}^{\ell+1}}{10}\\-\frac{|v|^2}{2}&-\frac{|v|}{10}&-\frac{|v|}{10}&-\frac{1}{10\mathbf{r}^{\ell+1}}&\frac{9}{10}&-\frac{1}{10}\\-\frac{|v|^2}{2}&-\frac{|v|}{10}&-\frac{|v|}{10}&-\frac{1}{10\mathbf{r}^{\ell+1}}&-\frac{1}{10}&\frac{9}{10}\\\frac{|v|^2}{2}&\frac{|v|}{10}&\frac{|v|}{10}&\frac{1}{10\mathbf{r}^{\ell+1}}&\frac{1}{10}&\frac{1}{10}\end{bmatrix}$$

$$\times\begin{bmatrix} -\frac{1}{5|v|} & -\frac{1}{5|v|} & -\frac{1}{5|v|^2\mathbf{r}^\ell} & -\frac{1}{5|v|^2} & -\frac{1}{5|v|^2} & \frac{1}{|v|^2} \\ 1 & 0 & 0 & 0 & 0 & \frac{1}{|v|} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{|v|} \\ 0 & 0 & 1 & 0 & 0 & \mathbf{r}^\ell \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{1}{10|v|}(\frac{1}{\mathbf{r}^\ell}-\frac{1}{\mathbf{r}^{\ell+1}}) & 0 & 0 & \frac{1}{10|v|}(1-\frac{\mathbf{r}^\ell}{\mathbf{r}^{\ell+1}}) \\ 0 & 1 & \frac{1}{10|v|}(\frac{1}{\mathbf{r}^\ell}-\frac{1}{\mathbf{r}^{\ell+1}}) & 0 & 0 & \frac{1}{10|v|}(1-\frac{\mathbf{r}^\ell}{\mathbf{r}^{\ell+1}}) \\ 0 & 0 & 1+\frac{1}{10}(\frac{\mathbf{r}^{\ell+1}}{\mathbf{r}^\ell}-1) & 0 & 0 & \frac{9}{10}(\mathbf{r}^\ell-\mathbf{r}^{\ell+1}) \\ 0 & 0 & \frac{1}{10}(\frac{1}{\mathbf{r}^\ell}-\frac{1}{\mathbf{r}^{\ell+1}}) & 1 & 0 & \frac{1}{10}(1-\frac{\mathbf{r}^\ell}{\mathbf{r}^{\ell+1}}) \\ 0 & 0 & \frac{1}{10}(\frac{1}{\mathbf{r}^\ell}-\frac{1}{\mathbf{r}^{\ell+1}}) & 0 & 1 & \frac{1}{10}(1-\frac{\mathbf{r}^\ell}{\mathbf{r}^{\ell+1}}) \\ 0 & 0 & -\frac{1}{10}(\frac{1}{\mathbf{r}^\ell}-\frac{1}{\mathbf{r}^{\ell+1}}) & 0 & 0 & 1+\frac{1}{10}(\frac{\mathbf{r}^\ell}{\mathbf{r}^{\ell+1}}-1) \end{bmatrix}. \tag{5.76}$$

Since from the definition of $\mathbf{v}_\perp^\ell$, and (5.60) we have

$$\begin{aligned} \mathbf{v}_\perp^{\ell+1} = -\lim_{s\downarrow t^{\ell+1}}\mathbf{v}_\perp(s) = -\mathbf{v}_\perp^\ell + \int_{t^{\ell+1}}^{t^\ell} F_\perp(\mathbf{X}(\tau;t,x,v),\mathbf{V}(\tau;t,x,v))\mathrm{d}\tau \\ = -\mathbf{v}_\perp^\ell + (t^\ell - t^{\ell+1})F_\perp(t^\ell) + O(1)|t^\ell - t^{\ell+1}|^2(|v|^3+1) \\ = -\mathbf{v}_\perp^\ell + 2\mathbf{v}_\perp^{\ell+1} + O(1)|t^\ell - t^{\ell+1}|^2(|v|^3+1). \end{aligned} \tag{5.77}$$

This implies $\mathbf{v}_\perp^\ell - \mathbf{v}_\perp^{\ell+1} = O(1)|t^\ell - t^{\ell+1}|^2(|v|^3+1)$. Similarly by plugging in

$$(t^\ell - t^{\ell+1})F_\perp(t^\ell) = 2\mathbf{v}_\perp^\ell + O(1)|t^\ell - t^{\ell+1}|^2(|v|^3+1),$$

(5.77) becomes

$$\begin{aligned} \mathbf{v}_\perp^{\ell+1} &= -\mathbf{v}_\perp^\ell + (t^\ell - t^{\ell+1})F_\perp(t^\ell) + O(1)|t^\ell - t^{\ell+1}|^2(|v|^3+1) \\ &= \mathbf{v}_\perp^\ell + O(1)|t^\ell - t^{\ell+1}|^2(|v|^3+1). \end{aligned}$$

Thus $\mathbf{v}_\perp^{\ell+1} - \mathbf{v}_\perp^\ell = O(1)|t^\ell - t^{\ell+1}|^2(|v|^3+1)$, therefore

$$|\mathbf{v}_\perp^{\ell+1} - \mathbf{v}_\perp^\ell| = O(1)|t^\ell - t^{\ell+1}|^2(|v|^3+1). \tag{5.78}$$

From (5.78) we have

$$\Big|\frac{1}{\mathbf{r}^\ell} - \frac{1}{\mathbf{r}^{\ell+1}}\Big| = \frac{1}{|v|}\frac{|\mathbf{v}_\perp^{\ell+1} - \mathbf{v}_\perp^\ell|}{|\mathbf{v}_\perp^{\ell+1}||\mathbf{v}_\perp^\ell|} \lesssim \frac{|t^\ell - t^{\ell+1}|^2(|v|^2+1)}{|\mathbf{v}_\perp^{\ell+1}||\mathbf{v}_\perp^\ell|} \lesssim 1, \tag{5.79}$$

and

$$\Big|1 - \frac{\mathbf{r}^\ell}{\mathbf{r}^{\ell+1}}\Big| = \frac{|\mathbf{v}_\perp^{\ell+1} - \mathbf{v}_\perp^\ell|}{\mathbf{v}_\perp^{\ell+1}} \lesssim \frac{|t^\ell - t^{\ell+1}|^2(|v|^2+1)}{|\mathbf{v}_\perp^{\ell+1}|} \lesssim \mathbf{r}^\ell. \tag{5.80}$$

Thus

$$|\mathcal{P}^{-1}(\mathbf{r}^{\ell+1})\mathcal{P}(\mathbf{r}^\ell)| \le \begin{bmatrix} 1 & 0 & \frac{M}{|v|} & 0 & 0 & \frac{M}{|v|}\mathbf{r}^\ell \\ 0 & 1 & \frac{M}{|v|} & 0 & 0 & \frac{M}{|v|}\mathbf{r}^\ell \\ 0 & 0 & 1+M\mathbf{r}^\ell & 0 & 0 & M(\mathbf{r}^\ell)^2 \\ 0 & 0 & M & 1 & 0 & M\mathbf{r}^\ell \\ 0 & 0 & M & 0 & 1 & M\mathbf{r}^\ell \\ 0 & 0 & M & 0 & 0 & 1+M\mathbf{r}^\ell \end{bmatrix} := \mathcal{Q}(\mathbf{r}^\ell).$$

Now we have

$$\begin{aligned}
&J(\mathbf{r}^{\ell_*})\times\cdots\times J(\mathbf{r}^{\ell+1})\times J(\mathbf{r}^{\ell})\times\cdots J(\mathbf{r}^2)\\
\leq&\widetilde{\mathcal{P}(\mathbf{r}^{\ell_*})}\Lambda(\mathbf{r}^{\ell_*})\mathcal{Q}(\mathbf{r}^{\ell_*-1})\Lambda(\mathbf{r}^{\ell_*-1})\cdots\mathcal{Q}(\mathbf{r}^{\ell})\Lambda(\mathbf{r}^{\ell})\cdots\mathcal{Q}(\mathbf{r}^2)\Lambda(\mathbf{r}^2)\widetilde{\mathcal{P}^{-1}(\mathbf{r}^2)}\\
\leq&\prod_{j=2}^{\ell_*}(1+10M\mathbf{r}^j)\widetilde{\mathcal{P}(\mathbf{r}^{\ell_*})}\mathcal{Q}(\mathbf{r}^{\ell_*-1})\cdots\mathcal{Q}(\mathbf{r}^2)\widetilde{\mathcal{P}^{-1}(\mathbf{r}^2)}\\
\leq&C^{C(t-s)|v|}\widetilde{\mathcal{P}(\mathbf{r}^{\ell_*})}\mathcal{Q}(\mathbf{r}^{\ell_*-1})\cdots\mathcal{Q}(\mathbf{r}^2)\widetilde{\mathcal{P}^{-1}(\mathbf{r}^2)},
\end{aligned}\tag{5.81}$$

where we have used $\Lambda(\mathbf{r}^j)\leq(1+10M\mathbf{r}^j)\mathbf{Id}_{6,6}$, and

$$\prod_{j=2}^{\ell_*}(1+10M\mathbf{r}^j)\leq\prod_{i=1}^{[\frac{\tilde{t}|v|}{L_\xi}]}\prod_{j=\ell_{i-1}}^{\ell_i}(1+10M\mathbf{r}^j)\lesssim\prod_{i=1}^{[\frac{\tilde{t}|v|}{L_\xi}]}(1+10M\mathbf{r}_i)^{\frac{C_\xi}{\mathbf{r}_i}}\lesssim C^{C(t-s)|v|}.$$

Next we estimate $\mathcal{Q}(\mathbf{r}^{\ell_*-1})\cdots\mathcal{Q}(\mathbf{r}^2)$. First, by diagonalization we have

$$\begin{aligned}
&\mathcal{Q}(\mathbf{r})=\mathcal{R}(\mathbf{r})\mathcal{B}(\mathbf{r})\mathcal{R}^{-1}(\mathbf{r})\\
&=\begin{bmatrix}1&0&0&0&0&\frac{1}{|v|}\\0&1&0&0&0&\frac{1}{|v|}\\0&0&1&0&-\mathbf{r}&\mathbf{r}\\0&0&0&1&0&1\\0&0&0&0&0&1\\0&0&0&0&1&1\end{bmatrix}\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1+2M\mathbf{r}\end{bmatrix}\begin{bmatrix}1&0&-\frac{1}{2|v|\mathbf{r}}&0&0&-\frac{1}{2|v|}\\0&1&-\frac{1}{2|v|\mathbf{r}}&0&0&-\frac{1}{2|v|}\\0&0&-\frac{1}{2\mathbf{r}}&0&0&-\frac{1}{2}\\0&0&-\frac{1}{2\mathbf{r}}&1&0&-\frac{1}{2}\\0&0&-\frac{1}{2\mathbf{r}}&0&1&\frac{1}{2}\\0&0&\frac{1}{2\mathbf{r}}&0&0&\frac{1}{2}\end{bmatrix}.
\end{aligned}\tag{5.82}$$

Thus

$$\prod_{j=2}^{\ell_*}\mathcal{Q}(\mathbf{r}^j)\leq\prod_{i=1}^{[\frac{\tilde{t}|v|}{L_\xi}]}\prod_{j=\ell_{i-1}}^{\ell_i}\mathcal{Q}(\mathbf{r}^j)\lesssim\prod_{i=1}^{[\frac{\tilde{t}|v|}{L_\xi}]}[\mathcal{Q}(\mathbf{r}_i)]^{\ell_i-\ell_{i-1}}\leq\prod_{i=1}^{[\frac{\tilde{t}|v|}{L_\xi}]}\mathcal{R}(\mathbf{r}_i)[\mathcal{B}(\mathbf{r}_i)]^{\ell_i-\ell_{i-1}}\mathcal{R}^{-1}(\mathbf{r}_i),$$

note that for some $C\gg1$

$$[\mathcal{B}(\mathbf{r}_i)]^{\ell_i-\ell_{i-1}}\lesssim[\mathcal{B}(\mathbf{r}_i)]^{\frac{C_\xi}{\mathbf{r}_i}}\lesssim\operatorname{diag}\Big[1,1,1,1,1,C\Big].\tag{5.83}$$

Next we have again by explicit computation and using $|\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}|\lesssim C_\xi$

$$\begin{aligned}
\mathcal{R}^{-1}(\mathbf{r}_{i+1})\mathcal{R}(\mathbf{r}_i)&=\begin{bmatrix}1&0&0&0&\frac{1}{2|v|}(\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}-1)&-\frac{1}{2|v|}(\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}-1)\\0&1&0&0&\frac{1}{2|v|}(\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}-1)&-\frac{1}{2|v|}(\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}-1)\\0&0&1&0&\frac{1}{2}(\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}-1)&-\frac{1}{2}(\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}-1)\\0&0&0&1&\frac{1}{2}(\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}-1)&-\frac{1}{2}(\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}-1)\\0&0&0&0&\frac{1}{2}(\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}+1)&-\frac{1}{2}(\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}-1)\\0&0&0&0&-\frac{1}{2}(\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}-1)&\frac{1}{2}(\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}+1)\end{bmatrix}\\
&\lesssim\begin{bmatrix}1&0&0&0&\frac{C_\xi}{|v|}&\frac{C_\xi}{|v|}\\0&1&0&0&\frac{C_\xi}{|v|}&\frac{C_\xi}{|v|}\\0&0&1&0&C_\xi&C_\xi\\0&0&0&1&C_\xi&C_\xi\\0&0&0&0&C_\xi&C_\xi\\0&0&0&0&C_\xi&C_\xi\end{bmatrix}:=\mathcal{S}.
\end{aligned}\tag{5.84}$$

Again we diagonalize $\mathcal{S}$ as

$$\mathcal{S}=\mathcal{F}\mathcal{A}\mathcal{F}^{-1}$$
$$:=\begin{bmatrix} 0 & 1\,0\,0\,0 & \frac{2C_\xi}{|v|(2C_\xi-1)} \\ 0 & 0\,1\,0\,0 & \frac{2C_\xi}{|v|(2C_\xi-1)} \\ 0 & 0\,0\,1\,0 & \frac{2C_\xi}{2C_\xi-1} \\ 0 & 0\,0\,0\,1 & \frac{2C_\xi}{2C_\xi-1} \\ -1 & 0\,0\,0\,0 & 1 \\ 1 & 0\,0\,0\,0 & 1 \end{bmatrix}\begin{bmatrix} 0\,0\,0\,0\,0 & 0 \\ 0\,1\,0\,0\,0 & 0 \\ 0\,0\,1\,0\,0 & 0 \\ 0\,0\,0\,1\,0 & 0 \\ 0\,0\,0\,0\,1 & 0 \\ 0\,0\,0\,0\,0 & 2C_\xi \end{bmatrix}\begin{bmatrix} 0\,0\,0\,0 & -\frac{1}{2} & \frac{1}{2} \\ 1\,0\,0\,0 & -\frac{C_\xi}{|v|(2C_\xi-1)} & -\frac{C_\xi}{|v|(2C_\xi-1)} \\ 0\,1\,0\,0 & -\frac{C_\xi}{|v|(2C_\xi-1)} & -\frac{C_\xi}{|v|(2C_\xi-1)} \\ 0\,0\,1\,0 & -\frac{C_\xi}{(2C_\xi-1)} & -\frac{C_\xi}{(2C_\xi-1)} \\ 0\,0\,0\,1 & -\frac{C_\xi}{(2C_\xi-1)} & -\frac{C_\xi}{(2C_\xi-1)} \\ 0\,0\,0\,0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \tag{5.85}$$

and directly

$$\begin{aligned}\mathcal{S}^{[\frac{\tilde{t}|v|}{L_\xi}]}=&\mathcal{F}\mathcal{A}^{[\frac{\tilde{t}|v|}{L_\xi}]}\mathcal{F}^{-1}\\ =&\mathcal{F}\mathrm{diag}\Big[0,1,1,1,1,(2C_\xi)^{[\frac{\tilde{t}|v|}{L_\xi}]}\Big]\mathcal{F}^{-1}\\ =&\begin{bmatrix} 1\,0\,0\,0 & \frac{1}{|v|}\frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{\tilde{t}|v|}{L_\xi}]}-1) & \frac{1}{|v|}\frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{\tilde{t}|v|}{L_\xi}]}-1) \\ 0\,1\,0\,0 & \frac{1}{|v|}\frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{\tilde{t}|v|}{L_\xi}]}-1) & \frac{1}{|v|}\frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{\tilde{t}|v|}{L_\xi}]}-1) \\ 0\,0\,1\,0 & \frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{\tilde{t}|v|}{L_\xi}]}-1) & \frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{\tilde{t}|v|}{L_\xi}]}-1) \\ 0\,0\,0\,1 & \frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{\tilde{t}|v|}{L_\xi}]}-1) & \frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{\tilde{t}|v|}{L_\xi}]}-1) \\ 0\,0\,0\,0 & \frac{(2C_\xi)^{[\frac{\tilde{t}|v|}{L_\xi}]}}{2} & \frac{(2C_\xi)^{[\frac{\tilde{t}|v|}{L_\xi}]}}{2} \\ 0\,0\,0\,0 & \frac{(2C_\xi)^{[\frac{\tilde{t}|v|}{L_\xi}]}}{2} & \frac{(2C_\xi)^{[\frac{\tilde{t}|v|}{L_\xi}]}}{2} \end{bmatrix}\\ \leq&\begin{bmatrix} 1\,0\,0\,0 & \frac{1}{|v|}C^{[\frac{\tilde{t}|v|}{L_\xi}]} & \frac{1}{|v|}C^{[\frac{\tilde{t}|v|}{L_\xi}]} \\ 0\,1\,0\,0 & \frac{1}{|v|}C^{[\frac{\tilde{t}|v|}{L_\xi}]} & \frac{1}{|v|}C^{[\frac{\tilde{t}|v|}{L_\xi}]} \\ 0\,0\,1\,0 & C^{[\frac{\tilde{t}|v|}{L_\xi}]} & C^{[\frac{\tilde{t}|v|}{L_\xi}]} \\ 0\,0\,0\,1 & C^{[\frac{\tilde{t}|v|}{L_\xi}]} & C^{[\frac{\tilde{t}|v|}{L_\xi}]} \\ 0\,0\,0\,0 & C^{[\frac{\tilde{t}|v|}{L_\xi}]} & C^{[\frac{\tilde{t}|v|}{L_\xi}]} \\ 0\,0\,0\,0 & C^{[\frac{\tilde{t}|v|}{L_\xi}]} & C^{[\frac{\tilde{t}|v|}{L_\xi}]} \end{bmatrix}:=\mathcal{D}.\end{aligned} \tag{5.86}$$

Therefore from (5.83) and (5.86) we have for some $C_1\gg 1$,

$$\prod_{j=2}^{\ell_*}\mathcal{Q}(\mathbf{r}^j)\leq C_1^{[\frac{\tilde{t}|v|}{L_\xi}]}\widetilde{\mathcal{R}(\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]})}\mathcal{F}\mathcal{A}^{[\frac{\tilde{t}|v|}{L_\xi}]}\mathcal{F}^{-1}\widetilde{\mathcal{R}^{-1}(\mathbf{r}_1)}\leq C_1^{[\frac{\tilde{t}|v|}{L_\xi}]}\widetilde{\mathcal{R}(\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]})}\mathcal{D}\widetilde{\mathcal{R}^{-1}(\mathbf{r}_1)}. \tag{5.87}$$

Finally using $\mathbf{r}_1\sim\mathbf{r}^2\sim\mathbf{r}^1$, and $\mathbf{r}^{\ell_*}\sim\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]}$. Putting everything together we have from (5.81), for $C_2\gg 1$,

$$\begin{aligned}&J(\mathbf{r}^{\ell_*})\times\cdots\times J(\mathbf{r}^{\ell+1})\times J(\mathbf{r}^{\ell})\times\cdots J(\mathbf{r}^2)\\ \leq& C^{C(t-s)|v|}\widetilde{\mathcal{P}(\mathbf{r}^{\ell_*})}\prod_{j=2}^{\ell_*}\mathcal{Q}(\mathbf{r}^j)\widetilde{\mathcal{P}^{-1}(\mathbf{r}^2)}\end{aligned}$$

$$
\begin{aligned}
&\le C_2^{C_2(t-s)|v|}\widetilde{\mathcal{P}(\mathbf{r}^{\ell_*})}\widetilde{\mathcal{R}(\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]})}\mathcal{D}\widetilde{\mathcal{R}^{-1}(\mathbf{r}_1)}\widetilde{\mathcal{P}^{-1}(\mathbf{r}^2)}\\
&\lesssim C_2^{C_2(t-s)|v|}\widetilde{\mathcal{P}(\mathbf{r}^{\ell_*})}\widetilde{\mathcal{R}(\mathbf{r}^{\ell_*})}\mathcal{D}\widetilde{\mathcal{R}^{-1}(\mathbf{r}^1)}\widetilde{\mathcal{P}^{-1}(\mathbf{r}^1)}\\
&= C_2^{C_2(t-s)|v|}\\
&\quad\times\begin{bmatrix}
\frac{1}{5|v|} & \frac{1}{5|v|} & \frac{1}{5|v|^2} & \frac{1}{5|v|^2} & \frac{6}{5|v|^2} & \frac{2}{|v|^2}\\
1 & 0 & 0 & 0 & \frac{1}{|v|} & \frac{2}{|v|}\\
0 & 1 & 0 & 0 & \frac{1}{|v|} & \frac{2}{|v|}\\
0 & 0 & 0 & 0 & \frac{2}{\mathbf{r}^{\ell_*}} & \frac{2}{\mathbf{r}^{\ell_*}}\\
0 & 0 & 1 & 0 & 1 & 2\\
0 & 0 & 0 & 1 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
1\,0\,0\,0 & \frac{1}{|v|}C^{[\frac{\tilde{t}|v|}{L_\xi}]} & \frac{1}{|v|}C^{[\frac{\tilde{t}|v|}{L_\xi}]}\\
0\,1\,0\,0 & \frac{1}{|v|}C^{[\frac{\tilde{t}|v|}{L_\xi}]} & \frac{1}{|v|}C^{[\frac{\tilde{t}|v|}{L_\xi}]}\\
0\,0\,1\,0 & C^{[\frac{\tilde{t}|v|}{L_\xi}]} & C^{[\frac{\tilde{t}|v|}{L_\xi}]}\\
0\,0\,0\,1 & C^{[\frac{\tilde{t}|v|}{L_\xi}]} & C^{[\frac{\tilde{t}|v|}{L_\xi}]}\\
0\,0\,0\,0 & C^{[\frac{\tilde{t}|v|}{L_\xi}]} & C^{[\frac{\tilde{t}|v|}{L_\xi}]}\\
0\,0\,0\,0 & C^{[\frac{\tilde{t}|v|}{L_\xi}]} & C^{[\frac{\tilde{t}|v|}{L_\xi}]}
\end{bmatrix}\\
&\quad\times\begin{bmatrix}
|v| & 1 & \frac{1}{5} & \frac{3}{5\mathbf{r}^1|v|} & \frac{1}{5|v|} & \frac{1}{5|v|}\\
|v| & \frac{1}{5} & 1 & \frac{3}{5\mathbf{r}^1|v|} & \frac{1}{5|v|} & \frac{1}{5|v|}\\
|v|^2 & \frac{|v|}{5} & \frac{|v|}{5} & \frac{3}{5\mathbf{r}^1} & 1 & \frac{1}{5}\\
|v|^2 & \frac{|v|}{5} & \frac{|v|}{5} & \frac{3}{5\mathbf{r}^1} & \frac{1}{5} & 1\\
\frac{|v|^2}{2} & \frac{|v|}{10} & \frac{|v|}{10} & \frac{1}{2\mathbf{r}^1} & \frac{1}{10} & \frac{1}{10}\\
\frac{|v|^2}{2} & \frac{|v|}{10} & \frac{|v|}{10} & \frac{1}{2\mathbf{r}^1} & \frac{1}{10} & \frac{1}{10}
\end{bmatrix}\\
&= C_2^{C_2(t-s)|v|}\\
&\quad\times\begin{bmatrix}
4C^{[\frac{\tilde{t}|v|}{L_\xi}]}+\frac{4}{5} & \frac{20C^{[\frac{\tilde{t}|v|}{L_\xi}]}+8}{25|v|} & \frac{20C^{[\frac{\tilde{t}|v|}{L_\xi}]}+8}{25|v|} & \frac{4(25C^{[\frac{\tilde{t}|v|}{L_\xi}]}+3)}{25\mathbf{r}^1|v|^2} & \frac{4(5C^{[\frac{\tilde{t}|v|}{L_\xi}]}+2)}{25|v|^2} & \frac{4(5C^{[\frac{\tilde{t}|v|}{L_\xi}]}+2)}{25|v|^2}\\
|v|(4C^{[\frac{\tilde{t}|v|}{L_\xi}]}+1) & \frac{4C^{[\frac{\tilde{t}|v|}{L_\xi}]}}{5}+1 & \frac{4C^{[\frac{\tilde{t}|v|}{L_\xi}]}}{5}+\frac{1}{5} & \frac{20C^{[\frac{\tilde{t}|v|}{L_\xi}]}+3}{5\mathbf{r}^1|v|} & \frac{4C^{[\frac{\tilde{t}|v|}{L_\xi}]}+1}{5|v|} & \frac{4C^{[\frac{\tilde{t}|v|}{L_\xi}]}+1}{5|v|}\\
|v|(4C^{[\frac{\tilde{t}|v|}{L_\xi}]}+1) & \frac{4C^{[\frac{\tilde{t}|v|}{L_\xi}]}}{5}+\frac{1}{5} & \frac{4C^{[\frac{\tilde{t}|v|}{L_\xi}]}}{5}+1 & \frac{20C^{[\frac{\tilde{t}|v|}{L_\xi}]}+3}{5\mathbf{r}^1|v|} & \frac{4C^{[\frac{\tilde{t}|v|}{L_\xi}]}+1}{5|v|} & \frac{4C^{[\frac{\tilde{t}|v|}{L_\xi}]}+1}{5|v|}\\
4C^{[\frac{\tilde{t}|v|}{L_\xi}]}\mathbf{r}^{\ell_*}|v|^2 & \frac{4}{5}C^{[\frac{\tilde{t}|v|}{L_\xi}]}\mathbf{r}^{\ell_*}|v| & \frac{4}{5}C^{[\frac{\tilde{t}|v|}{L_\xi}]}\mathbf{r}^{\ell_*}|v| & \frac{4C^{[\frac{\tilde{t}|v|}{L_\xi}]}\mathbf{r}^{\ell_*}}{\mathbf{r}^1} & \frac{4}{5}C^{[\frac{\tilde{t}|v|}{L_\xi}]}\mathbf{r}^{\ell_*} & \frac{4}{5}C^{[\frac{\tilde{t}|v|}{L_\xi}]}\mathbf{r}^{\ell_*}\\
|v|^2(4C^{[\frac{\tilde{t}|v|}{L_\xi}]}+1) & \frac{|v|}{5}(4C^{[\frac{\tilde{t}|v|}{L_\xi}]}+1) & \frac{|v|}{5}(4C^{[\frac{\tilde{t}|v|}{L_\xi}]}+1) & \frac{20C^{[\frac{\tilde{t}|v|}{L_\xi}]}+3}{5\mathbf{r}^1} & \frac{4C^{[\frac{\tilde{t}|v|}{L_\xi}]}}{5}+1 & \frac{4C^{[\frac{\tilde{t}|v|}{L_\xi}]}}{5}+\frac{1}{5}\\
|v|^2(4C^{[\frac{\tilde{t}|v|}{L_\xi}]}+1) & \frac{|v|}{5}(4C^{[\frac{\tilde{t}|v|}{L_\xi}]}+1) & \frac{|v|}{5}(4C^{[\frac{\tilde{t}|v|}{L_\xi}]}+1) & \frac{20C^{[\frac{\tilde{t}|v|}{L_\xi}]}+3}{5\mathbf{r}^1} & \frac{4C^{[\frac{\tilde{t}|v|}{L_\xi}]}}{5}+\frac{1}{5} & \frac{4C^{[\frac{\tilde{t}|v|}{L_\xi}]}}{5}+1
\end{bmatrix}\\
&\lesssim C^{C|t-s||v|}\left[\begin{array}{c|c|c|c}
O_\xi(1) & \frac{1}{|v|} & \frac{1}{|v||\mathbf{v}_\perp^1|} & \frac{1}{|v|^2}\\ \hline
|v| & O_\xi(1) & \frac{1}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|}\\ \hline
|\mathbf{v}_\perp^1||v| & |\mathbf{v}_\perp^1| & O_\xi(1) & \frac{|\mathbf{v}_\perp^1|}{|v|}\\ \hline
|v|^2 & |v| & \frac{|v|}{|\mathbf{v}_\perp^1|} & O_\xi(1)
\end{array}\right]_{6\times 6},
\end{aligned}
\tag{5.88}
$$

where we have used (5.74) and the Velocity lemma (Lemma 3.1) and (5.2) and

$$
\mathbf{r}_i = \mathcal{C}_1 e^{\frac{\mathcal{C}}{2}C_1}\mathbf{r}^i \lesssim e^{C|t-s||v|}\frac{|\mathbf{v}_\perp^1|}{|v|},\quad\text{and}\quad \frac{\mathbf{r}_{[\frac{|t-s||v|}{L_\xi}]}}{\mathbf{r}_1} = \frac{\mathbf{r}^{[\frac{|t-s||v|}{L_\xi}]}}{\mathbf{r}^1} = \frac{\left|\mathbf{v}_\perp^{[\frac{|t-s||v|}{L_\xi}]}\right|}{|\mathbf{v}_\perp^1|} \le \mathcal{C}_1 e^{\frac{\mathcal{C}}{2}|v||t-s|}.
$$

The case where ℓ is *Type I* is easier, we first claim

$$
J_{\ell_*-1}^{\ell_*}\times\cdots\times J_1^2 = \frac{\partial(t^{\ell_*},\mathbf{x}_{\|_{\ell_*}}^{\ell_*},\mathbf{v}_{\perp_{\ell_*}}^{\ell_*},\mathbf{v}_{\|_{\ell_*}}^{\ell_*})}{\partial(t^{\ell_*-1},\mathbf{x}_{\|_{\ell_*-1}}^{\ell_*-1},\mathbf{v}_{\perp_{\ell_*-1}}^{\ell_*-1},\mathbf{v}_{\|_{\ell_*-1}}^{\ell_*-1})}\times\cdots\times\frac{\partial(t^2,\mathbf{x}_{\|_2}^2,\mathbf{v}_{\perp_2}^2,\mathbf{v}_{\|_2}^2)}{\partial(t^1,\mathbf{x}_{\|_1}^1,\mathbf{v}_{\perp_1}^1,\mathbf{v}_{\|_1}^1)}
$$

$$\leq \mathcal{P}(\mathbf{v}_\perp^1)(\Lambda(\mathbf{v}_\perp^1))^{\frac{C_\xi}{\mathbf{v}_\perp^1}}\mathcal{P}^{-1}(\mathbf{v}_\perp^1). \tag{5.89}$$

From the same arguments between (5.72) and (5.73), we have

$$\frac{1}{(\mathcal{C}_1)^2}e^{-\mathcal{C}C_1}\mathbf{v}_\perp^1 \leq \mathbf{v}_\perp^j \leq \mathbf{v}_\perp^1 \quad \text{for all} \quad 1\leq j\leq \ell_*. \tag{5.90}$$

Therefore

$$J_{\ell_*-1}^{\ell_*}\times\cdots\times J_1^2 \leq \mathcal{P}(\mathbf{v}_\perp^1)(\Lambda(\mathbf{v}_\perp^1))^{|\ell_*|}\mathcal{P}^{-1}(\mathbf{v}_\perp^1).$$

Now we have only left to prove $|\ell_*|\lesssim_\Omega \frac{1}{\mathbf{v}_\perp^1}$: For any $1\leq j\leq \ell_*$, we have $\xi(x^j)=0=\xi(x^{j+1})=\xi(x^j-(t^j-t^{j+1})v^j)$. We expand $\xi(x^j-(t^j-t^{j+1})v^j)$ in time to have

$$\begin{aligned}\xi(x^{j+1}) &= \xi(x^j)+\int_{t^j}^{t^{j+1}}\frac{d}{ds}\xi(X_{\mathbf{cl}}(s))\mathrm{d}s\\ &= \xi(x^j)+(v^j\cdot\nabla\xi(x^j))(t^{j+1}-t^j)+\int_{t^j}^{t^{j+1}}\int_{t^j}^{s}\frac{d^2}{d\tau^2}\xi(X_{\mathbf{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s,\end{aligned}$$

and

$$\begin{aligned}0=&(v^j\cdot\nabla\xi(x^j))(t^{j+1}-t^j)+\frac{(t^j-t^{j+1})^2}{2}(V_{\mathbf{cl}}(\tau_*)\cdot\nabla^2\xi(X_{\mathbf{cl}}(\tau_*))\cdot V_{\mathbf{cl}}(\tau_*)\\ &+E(\tau,X_{\mathbf{cl}}(\tau_*))\cdot\nabla\xi(X_{\mathbf{cl}}(\tau_*))),\end{aligned}$$

for some $\tau_*\in[t^{j+1},t^j]$. Therefore there exists $C_2(\delta,\xi,E)\gg 1$ such that

$$|t^j-t^{j+1}|=\frac{2v^j\cdot\nabla\xi(x^j)}{V_{\mathbf{cl}}(\tau_*)\cdot\nabla^2\xi(X_{\mathbf{cl}}(\tau_*))\cdot V_{\mathbf{cl}}(\tau_*)+E(\tau,X_{\mathbf{cl}}(\tau_*))\cdot\nabla\xi(X_{\mathbf{cl}}(\tau_*))}\geq\frac{1}{C_2}\mathbf{v}_\perp^j\gtrsim\mathbf{v}_\perp^1.$$

Thus

$$|\ell_*|\leq\frac{T}{\min_j|t^j-t^{j+1}|}\lesssim\frac{T}{\mathbf{v}_\perp^1},$$

and this completes our claim (5.89).

Then directly from (5.89) we have for some $C\gg 1$,

$$\begin{aligned}J_{\ell_*-1}^{\ell_*}\times\cdots\times J_1^2 &\leq \widetilde{\mathcal{P}(\mathbf{v}_\perp^1)}(\Lambda(\mathbf{v}_\perp^1))^{\frac{C_\xi}{\mathbf{v}_\perp^1}}\widetilde{\mathcal{P}^{-1}(\mathbf{v}_\perp^1)}\\ &\leq \widetilde{\mathcal{P}(\mathbf{v}_\perp^1)}((1+M\mathbf{v}_\perp^1)^{\frac{C_\xi}{\mathbf{v}_\perp^1}}\mathbf{Id}_{6,6})\widetilde{\mathcal{P}^{-1}(\mathbf{v}_\perp^1)}\\ &\leq C\widetilde{\mathcal{P}(\mathbf{v}_\perp^1)}\widetilde{\mathcal{P}^{-1}(\mathbf{v}_\perp^1)}\\ &\leq C\left[\begin{array}{c|cc|c|cc} 1 & \frac{9}{25} & \frac{9}{25} & \frac{9}{25|\mathbf{v}_\perp^1|} & \frac{9}{25} & \frac{9}{25}\\ \hline 1 & 1 & \frac{1}{5} & \frac{1}{5|\mathbf{v}_\perp^1|} & \frac{1}{5} & \frac{1}{5}\\ 1 & \frac{1}{5} & 1 & \frac{1}{5|\mathbf{v}_\perp^1|} & \frac{1}{5} & \frac{1}{5}\\ \hline |\mathbf{v}_\perp^1| & \frac{|\mathbf{v}_\perp^1|}{5} & \frac{|\mathbf{v}_\perp^1|}{5} & 1 & \frac{|\mathbf{v}_\perp^1|}{5} & \frac{|\mathbf{v}_\perp^1|}{5}\\ \hline 1 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5|\mathbf{v}_\perp^1|} & 1 & \frac{1}{5}\\ 1 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5|\mathbf{v}_\perp^1|} & \frac{1}{5} & 1 \end{array}\right].\end{aligned} \tag{5.91}$$

Step 8. Intermediate summary for the matrix method and the final estimate for Type III.

Recall from (5.25) and (5.36), (5.88), (5.40),

$$
\begin{aligned}
&\frac{\partial(s^{\ell_*},\mathbf{X}_{\ell_*}(s^{\ell_*}),\mathbf{V}_{\ell_*}(s^{\ell_*}))}{\partial(s^1,\mathbf{X}_1(s^1),\mathbf{V}_1(s^1))} = \frac{\partial(s^{\ell_*},\mathbf{x}_{\perp_{\ell_*}}(s^{\ell_*}),\mathbf{x}_{\|_{\ell_*}}(s^{\ell_*}),\mathbf{v}_{\perp_{\ell_*}}(s^{\ell_*}),\mathbf{v}_{\|_{\ell_*}}(s^{\ell_*}))}{\partial(s^1,\mathbf{x}_{\perp_1}(s^1),\mathbf{x}_{\|_1}(s^1),\mathbf{v}_{\perp_1}(s^1),\mathbf{v}_{\|_1}(s^1))}\\
&= \frac{\partial(s^{\ell_*},\mathbf{x}_{\perp_{\ell_*}}(s^{\ell_*}),\mathbf{x}_{\|_{\ell_*}}(s^{\ell_*}),\mathbf{v}_{\perp_{\ell_*}}(s^{\ell_*}),\mathbf{v}_{\|_{\ell_*}}(s^{\ell_*}))}{\partial(t^{\ell_*},\mathbf{x}^{\ell_*}_{\|_{\ell_*}},\mathbf{v}^{\ell_*}_{\perp_{\ell_*}},\mathbf{v}^{\ell_*}_{\|_{\ell_*}})}\\
&\times \prod_{i=1}^{[\frac{|t-s||v|}{L_\xi}]} \frac{\partial(t^{\ell_{i+1}},\mathbf{x}^{\ell_{i+1}}_{\|_{\ell_{i+1}}},\mathbf{v}^{\ell_{i+1}}_{\perp_{\ell_{i+1}}},\mathbf{v}^{\ell_{i+1}}_{\|_{\ell_{i+1}}})}{\partial(t^{\ell_{i+1}-1},\mathbf{x}^{\ell_{i+1}-1}_{\|_{\ell_{i+1}-1}},\mathbf{v}^{\ell_{i+1}-1}_{\perp_{\ell_{i+1}-1}},\mathbf{v}^{\ell_{i+1}-1}_{\|_{\ell_{i+1}-1}})} \times\cdots\times \frac{\partial(t^{\ell_i+1},\mathbf{x}^{\ell_i+1}_{\|_{\ell_i+1}},\mathbf{v}^{\ell_i+1}_{\perp_{\ell_i+1}},\mathbf{v}^{\ell_i+1}_{\|_{\ell_i+1}})}{\partial(t^{\ell_i},\mathbf{x}^{\ell_i}_{\|_{\ell_i}},\mathbf{v}^{\ell_i}_{\perp_{\ell_i}},\mathbf{v}^{\ell_i}_{\|_{\ell_i}})}\\
&\times \frac{\partial(t^1,\mathbf{x}^1_{\|_1},\mathbf{v}^1_{\perp_1},\mathbf{v}^1_{\|_1})}{\partial(s^1,\mathbf{x}_{\perp_1}(s^1),\mathbf{x}_{\|_1}(s^1),\mathbf{v}_{\perp_1}(s^1),\mathbf{v}_{\|_1}(s^1))}\\
&\le (5.36)\times(5.88)\times(5.40).
\end{aligned}
$$

Then directly since $|v|>\delta$, we bound it by

$$
\begin{aligned}
&\le (5.36)\times C^{C|t-s||v|}\\
&\times \left[\begin{array}{c|cc|cc}
\frac{|v|^2}{|\mathbf{v}^1_\perp|^2} & \frac{1}{|\mathbf{v}^1_\perp|}+\frac{|v|}{|\mathbf{v}^1_\perp|^2}+|t^1-s^1| & \frac{1}{|v|} & \frac{1}{|v||\mathbf{v}^1_\perp|}+|s^1-t^1|^2 & \frac{1}{|v|^2}+\frac{|s^1-t^1|}{|v|}\\ \hline
\frac{|v|^3}{|\mathbf{v}^1_\perp|}+\frac{|v|^2}{|\mathbf{v}^1_\perp|^2} & \frac{|v|^2}{|\mathbf{v}^1_\perp|^2}+\frac{|v|}{|\mathbf{v}^1_\perp|}+|v||s^1-t^1| & O_\xi(1) & \frac{1}{|\mathbf{v}^1_\perp|}+|s^1-t^1| & \frac{1}{|v|}\\ \hline
\frac{|v|^3}{|\mathbf{v}^1_\perp|} & \frac{|v|^2}{|\mathbf{v}^1_\perp|}+|v| & |\mathbf{v}^1_\perp| & O_\xi(1) & \frac{|\mathbf{v}^1_\perp|}{|v|}\\
\frac{|v|^4}{|\mathbf{v}^1_\perp|^2} & \frac{|v|^3}{|\mathbf{v}^1_\perp|^2}+\frac{|v|^2}{|\mathbf{v}^1_\perp|}+|v|^2|s^1-t^1| & |v| & \frac{|v|}{|\mathbf{v}^1_\perp|}+|v||s^1-t^1| & O_\xi(1)
\end{array}\right],
\end{aligned}
\tag{5.92}
$$

where we have used the Velocity lemma (Lemma 3.1) and (5.74), (5.2), and

$$
|v||t^1-s^1| \le \lesssim_\Omega \min\{\frac{|\mathbf{v}^1_\perp|}{|v|},(t-s)|v|\} \lesssim_\Omega C^{C|t-s||v|}\min\{\frac{|\mathbf{v}^1_\perp|}{|v|},1\}.
$$

Again we use the Velocity lemma (Lemma 3.1), (5.2), and

$$
\begin{aligned}
|v||t^{\ell_*}-s^{\ell_*}| \le \min\{|v||t^{\ell_*}-t^{\ell_*+1}|,|t-s||v|\} &\lesssim_\Omega \min\{\frac{|\mathbf{v}^{\ell_*}_\perp|}{|v|},|t-s||v|\}\\
&\lesssim_\Omega C^{C|t-s||v|}\min\{\frac{|\mathbf{v}^1_\perp|}{|v|},1\},
\end{aligned}
$$

and $|\mathbf{v}_\perp(s^{\ell_*})|\lesssim_\Omega C^{C|v|(t-s)}|\mathbf{v}^1_\perp|$ to have, from (5.92)

$$
\frac{\partial(s^{\ell_*},\mathbf{X}_{\ell_*}(s^{\ell_*}),\mathbf{V}_{\ell_*}(s^{\ell_*}))}{\partial(s^1,\mathbf{X}_1(s^1),\mathbf{V}_1(s^1))} \lesssim C^{C|t-s||v|}
\left[\begin{array}{c|cc|cc}
0 & 0 & \mathbf{0}_{1,2} & 0 & \mathbf{0}_{1,2}\\ \hline
\frac{|v|^2}{|\mathbf{v}^1_\perp|} & \frac{|v|}{|\mathbf{v}^1_\perp|} & \frac{|\mathbf{v}^1_\perp|}{|v|} & \frac{1}{|v|} & \frac{1}{|v|}\\
\frac{|v|^3}{|\mathbf{v}^1_\perp|^2} & \frac{|v|^2}{|\mathbf{v}^1_\perp|^2} & 1 & \frac{1}{|\mathbf{v}^1_\perp|} & \frac{1}{|v|}\\ \hline
\frac{|v|^4}{|\mathbf{v}^1_\perp|^2} & \frac{|v|^3}{|\mathbf{v}^1_\perp|^2} & |v| & \frac{|v|}{|\mathbf{v}^1_\perp|} & O_\xi(1)
\end{array}\right]_{7\times 7}. \tag{5.93}
$$

We consider the following case:

$$\text{There exists } \ell \in [\ell_*(s;t,x,v),0] \text{ such that } \mathbf{r}^\ell \geq \sqrt{\delta}. \tag{5.94}$$

Therefore ℓ is *Type III* in (5.19). Equivalently $\tau \in [t^{\ell+1}, t^\ell]$ for some $\ell_* \leq \ell \leq 0$ and $|\xi(X_{\mathbf{cl}}(\tau;t,x,v))| \geq C\delta$. By the Velocity lemma (Lemma 3.1), for all $1 \leq i \leq \ell_*(s;t,x,v)$,

$$|\mathbf{r}^i| = \frac{|\mathbf{v}_\perp^i|}{|v|} \gtrsim_\xi e^{-C_\xi |v||t^i - t^\ell|} |\mathbf{r}^\ell| \gtrsim_\xi e^{-C_\xi |v|(t-s)} \sqrt{\delta}.$$

Especially, for all $1 \leq i \leq \ell_*(s;t,x,v)$,

$$|\mathbf{r}^1| \gtrsim_\xi e^{-C_\xi |v|(t-s)} \sqrt{\delta}, \quad \frac{1}{|\mathbf{r}^i|} = \frac{|v|}{|\mathbf{v}_\perp^i|} \lesssim_\xi \frac{e^{C_\xi |v|(t-s)}}{\sqrt{\delta}}.$$

Note that $\ell_*(s;t,x,v) \lesssim \max_i \frac{|v||t-s|}{\mathbf{r}^i} \lesssim_\delta C^{C|v||t-s|}$.

Therefore in the case of (5.94), from (5.93),

$$\frac{\partial(s^{\ell_*}, \mathbf{X}_{\ell_*}(s^{\ell_*}), \mathbf{V}_{\ell_*}(s^{\ell_*}))}{\partial(s^1, \mathbf{X}_1(s^1), \mathbf{V}_1(s^1))} \lesssim C^{C(t-s)|v|} \left[\begin{array}{c|cc|cc} 0 & 0 & \mathbf{0}_{1,2} & 0 & \mathbf{0}_{1,2} \\ \hline |v| & \frac{1}{\sqrt{\delta}} & \frac{1}{\sqrt{\delta}} & \frac{1}{|v|} & \frac{1}{|v|} \\ |v| & \frac{1}{\delta} & \frac{1}{\delta} & \frac{1}{|v|}\frac{1}{\sqrt{\delta}} & \frac{1}{|v|} \\ \hline |v|^2 & |v|\frac{1}{\delta} & |v|\frac{1}{\delta} & \frac{1}{\sqrt{\delta}} & 1 \end{array}\right]$$

$$\lesssim_\delta C^{C|v|(t-s)} \left[\begin{array}{c|c|c} 0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \hline |v| & 1 & \frac{1}{|v|} \\ \hline |v|^2 & |v| & 1 \end{array}\right].$$

From (5.30) and (5.43) we conclude

$$\begin{aligned} &\frac{\partial(X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v))}{\partial(t,x,v)} \\ &\lesssim_{\delta,\xi} C^{C|v|(t-s)} \frac{\partial(X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))}{\partial(s^{\ell_*}, \mathbf{X}_{\ell_*}(s^{\ell_*}), \mathbf{V}_{\ell_*}(s^{\ell_*}))} \left[\begin{array}{c|c|c} 0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \hline |v| & 1 & \frac{1}{|v|} \\ \hline |v|^2 & |v| & 1 \end{array}\right] \\ &\qquad \times \frac{\partial(s^1, \mathbf{x}_{\perp_1}(s^1), \mathbf{x}_{\parallel_1}(s^1), \mathbf{v}_{\perp_1}(s^1), \mathbf{v}_{\parallel_1}(s^1))}{\partial(t,x,v)} \\ &\lesssim_{\delta,\xi} C^{C|v|(t-s)} \begin{bmatrix} |v| & 1 & |s^{\ell_*} - s| \\ 1 & |v| & 1 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ |v| & 1 & \frac{1}{|v|} \\ |v|^2 & |v| & 1 \end{bmatrix} \left[\begin{array}{c|c|c} 1 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ |t-s^1|^2 & 1 & |t-s^1| \\ |t-s^1| & |v| & 1 \end{array}\right] \\ &\lesssim_{\delta,\xi} C^{C|v|(t-s)} \begin{bmatrix} |v|+1 & 1 & \frac{1}{|v|} \\ |v|^2+1 & |v| & 1 \end{bmatrix}_{6\times 7}. \end{aligned} \tag{5.95}$$

Now for $|v| < \delta$, we have

$$\begin{aligned} &\frac{\partial(s^{\ell_*}, \mathbf{X}_{\ell_*}(s^{\ell_*}), \mathbf{V}_{\ell_*}(s^{\ell_*}))}{\partial(s^1, \mathbf{X}_1(s^1), \mathbf{V}_1(s^1))} \\ &\leq (5.36) \times (5.91) \times (5.40) \end{aligned}$$

$$
\begin{aligned}
&\lesssim \left[\begin{array}{c|c|c|c} 0 & \mathbf{0}_{1,3} & 0 & \mathbf{0}_{1,2} \\ \hline |\mathbf{v}_\perp^1| & |v|^2|\mathbf{v}_\perp^1| & |\mathbf{v}_\perp^1| & |v||\mathbf{v}_\perp^1|^2 \\ |v| & 1 & |v||\mathbf{v}_\perp^1|^2 & |\mathbf{v}_\perp^1| \\ \hline 1 & |\mathbf{v}_\perp^1| & 1 & |v||\mathbf{v}_\perp^1| \\ 1 & |\mathbf{v}_\perp^1| & |v||\mathbf{v}_\perp^1| & 1 \end{array}\right]_{7\times 6} \left[\begin{array}{c|c|c|c} 1 & 1 & \frac{1}{|\mathbf{v}_\perp^1|} & 1 \\ \hline 1 & 1 & \frac{1}{|\mathbf{v}_\perp^1|} & 1 \\ \hline |\mathbf{v}_\perp^1| & |\mathbf{v}_\perp^1| & 1 & |\mathbf{v}_\perp^1| \\ \hline 1 & 1 & \frac{1}{|\mathbf{v}_\perp^1|} & 1 \end{array}\right]_{6\times 6} \\
&\quad \times \left[\begin{array}{c|cc|cc} \frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{1}{|\mathbf{v}_\perp^1|} & |\mathbf{v}_\perp^1| & 1 & |\mathbf{v}_\perp^1| \\ \hline \frac{1}{|\mathbf{v}_\perp^1|} & \frac{|v|}{|\mathbf{v}_\perp^1|} & 1 & |v| & |\mathbf{v}_\perp^1| \\ \hline \frac{1}{|\mathbf{v}_\perp^1|} & \frac{1}{|\mathbf{v}_\perp^1|} & |\mathbf{v}_\perp^1| & 1 & |\mathbf{v}_\perp^1| \\ \frac{1}{|\mathbf{v}_\perp^1|} & \frac{1}{|\mathbf{v}_\perp^1|} & |\mathbf{v}_\perp^1| & 1 & 1 \end{array}\right]_{6\times 7} \\
&\lesssim \left[\begin{array}{c|c|c|c} 0 & \mathbf{0}_{1,3} & 0 & \mathbf{0}_{1,2} \\ \hline |\mathbf{v}_\perp^1| & |v|^2|\mathbf{v}_\perp^1| & |\mathbf{v}_\perp^1| & |v||\mathbf{v}_\perp^1|^2 \\ |v| & 1 & |v||\mathbf{v}_\perp^1|^2 & |\mathbf{v}_\perp^1| \\ \hline 1 & |\mathbf{v}_\perp^1| & 1 & |v||\mathbf{v}_\perp^1| \\ 1 & |\mathbf{v}_\perp^1| & |v||\mathbf{v}_\perp^1| & 1 \end{array}\right]_{7\times 6} \times \left[\begin{array}{c|cc|cc} \frac{1}{|\mathbf{v}_\perp^1|^2} & \frac{1}{|\mathbf{v}_\perp^1|^2} & 1 & \frac{1}{|\mathbf{v}_\perp^1|} & 1 \\ \hline \frac{1}{|\mathbf{v}_\perp^1|^2} & \frac{1}{|\mathbf{v}_\perp^1|^2} & 1 & \frac{1}{|\mathbf{v}_\perp^1|} & 1 \\ \hline \frac{1}{|\mathbf{v}_\perp^1|} & \frac{1}{|\mathbf{v}_\perp^1|} & |\mathbf{v}_\perp^1| & 1 & |\mathbf{v}_\perp^1| \\ \frac{1}{|\mathbf{v}_\perp^1|^2} & \frac{1}{|\mathbf{v}_\perp^1|^2} & 1 & \frac{1}{|\mathbf{v}_\perp^1|} & 1 \end{array}\right]_{6\times 7} \\
&\lesssim \left[\begin{array}{c|c|c|c|c} 0 & 0 & \mathbf{0}_{1,2} & 0 & \mathbf{0}_{1,2} \\ \hline \frac{1}{|\mathbf{v}_\perp^1|} & \frac{1}{|\mathbf{v}_\perp^1|} & |\mathbf{v}_\perp^1| & 1 & |\mathbf{v}_\perp^1| \\ \frac{|v|}{|\mathbf{v}_\perp^1|^2} & \frac{1}{|\mathbf{v}_\perp^1|^2} & 1 & \frac{1}{|\mathbf{v}_\perp^1|} & 1 \\ \hline \frac{1}{|\mathbf{v}_\perp^1|^2} & \frac{1}{|\mathbf{v}_\perp^1|^2} & 1 & \frac{1}{|\mathbf{v}_\perp^1|} & 1 \\ \frac{1}{|\mathbf{v}_\perp^1|^2} & \frac{1}{|\mathbf{v}_\perp^1|^2} & 1 & \frac{1}{|\mathbf{v}_\perp^1|} & 1 \end{array}\right]_{7\times 7}.
\end{aligned}
\tag{5.96}
$$

Now let's address the derivatives $\partial_x t^\ell$, and $\partial_v t^\ell$ for any $1\le \ell \le \ell^*$, as we will need them later. For $|v|>\delta$, we compute [the first row of (5.88) $\times$ (5.40)] $\cdot \frac{\partial(s^1,\mathbf{x}_\perp(s^1),\mathbf{x}_\parallel(s^1),\mathbf{v}_\perp(s^1),\mathbf{v}_\parallel(s^1))}{\partial(t,x,v)}$ and use (5.2) to get

$$
\begin{bmatrix} \partial_x t^\ell \\ \partial_v t^\ell \end{bmatrix} \lesssim \begin{bmatrix} \frac{|v|^2}{|\mathbf{v}_\perp^1|} & \frac{|v|}{|\mathbf{v}_\perp^1|^2} & \frac{1}{|v||\mathbf{v}_\perp^1|} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ 1 & |t-s^1| \\ |v| & 1 \end{bmatrix} \lesssim \begin{bmatrix} \frac{|v|}{|\mathbf{v}_\perp^1|^2} \\ \frac{1}{|v||\mathbf{v}_\perp^1|} \end{bmatrix}. \tag{5.97}
$$

And similarly, for $|v|\le\delta$, we compute [the first row of (5.91) $\times$ (5.40)] $\cdot \frac{\partial(s^1,\mathbf{x}_\perp(s^1),\mathbf{x}_\parallel(s^1),\mathbf{v}_\perp(s^1),\mathbf{v}_\parallel(s^1))}{\partial(t,x,v)}$ and use (5.2) to get

$$
\begin{bmatrix} \partial_x t^\ell \\ \partial_v t^\ell \end{bmatrix} \lesssim \begin{bmatrix} \frac{1}{|\mathbf{v}_\perp^1|} & \frac{1}{|\mathbf{v}_\perp^1|^2} & \frac{1}{|\mathbf{v}_\perp^1|} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ 1 & |t-s^1| \\ |v| & 1 \end{bmatrix} \lesssim \begin{bmatrix} \frac{1}{|\mathbf{v}_\perp^1|^2} \\ \frac{1}{|\mathbf{v}_\perp^1|} \end{bmatrix}. \tag{5.98}
$$

We remark $\partial \mathbf{x}_{\perp_{\ell_*}}$ and $\partial \mathbf{v}_{\perp_{\ell_*}}$ have desired bounds but $\partial \mathbf{x}_{\parallel_{\ell_*}}$ and $\partial \mathbf{v}_{\parallel_{\ell_*}}$ still have undesired bounds in (5.93), (5.96).

We only need to consider the remaining cases, i.e. ℓ is *Type I* or *Type II*. Note that in either case the moving frame ($\mathbf{p}^\ell-$spherical coordinate) is well-defined for all $\tau\in[s,t]$. In next two steps we use the ODE method to refine the submatrix of (5.93) and (5.96):

$$
\frac{\partial(\mathbf{x}_{\parallel_{\ell_*}}(s^{\ell_*}),\mathbf{v}_{\parallel_{\ell_*}}(s^{\ell_*}))}{\partial(\mathbf{x}_{\perp_1}(s^1),\mathbf{x}_{\parallel_1}(s^1),\mathbf{v}_{\perp_1}(s^1),\mathbf{v}_{\parallel_1}(s^1))} = \begin{bmatrix} \frac{\partial \mathbf{x}_{\parallel_{\ell_*}}(s^{\ell_*})}{\partial \mathbf{x}_{\perp_1}(s^1)} & \frac{\partial \mathbf{x}_{\parallel_{\ell_*}}(s^{\ell_*})}{\partial \mathbf{x}_{\parallel_1}(s^1)} & \frac{\partial \mathbf{x}_{\parallel_{\ell_*}}(s^{\ell_*})}{\partial \mathbf{v}_{\perp_1}(s^1)} & \frac{\partial \mathbf{x}_{\parallel_{\ell_*}}(s^{\ell_*})}{\partial \mathbf{v}_{\parallel_1}(s^1)} \\ \frac{\partial \mathbf{v}_{\parallel_{\ell_*}}(s^{\ell_*})}{\partial \mathbf{x}_{\perp_1}(s^1)} & \frac{\partial \mathbf{v}_{\parallel_{\ell_*}}(s^{\ell_*})}{\partial \mathbf{x}_{\parallel_1}(s^1)} & \frac{\partial \mathbf{v}_{\parallel_{\ell_*}}(s^{\ell_*})}{\partial \mathbf{v}_{\perp_1}(s^1)} & \frac{\partial \mathbf{v}_{\parallel_{\ell_*}}(s^{\ell_*})}{\partial \mathbf{v}_{\parallel_1}(s^1)} \end{bmatrix}_{4\times 6}.
$$

Step 9. ODE method within the time scale $|t-s||v|\simeq L_\xi$.

Recall the end points (time) of intermediate groups from (5.23):

$$s<\underbrace{t^{\ell_*}<t^{\ell_{[\frac{|t-s||v|}{L_\xi}]+1}}}_{[\frac{|t-s||v|}{L_\xi}]+1}<\underbrace{t^{\ell_{[\frac{|t-s||v|}{L_\xi}]}}<t^{\ell_{[\frac{|t-s||v|}{L_\xi}]-1}+1}}_{[\frac{|t-s||v|}{L_\xi}]}<\cdots<\underbrace{t^{\ell_i}<t^{\ell_{i-1}+1}}_{i}<\cdots<\underbrace{t^{\ell_1}<t^1}_{1}<t,$$

where the underbraced numbering indicates the index of the intermediate group. We further choose points independently on (t,x,v) for all $i=1,2,\cdots,[\frac{|t-s||v|}{L_\xi}]$:

$$\begin{gathered}
t^{\ell_1+1}<s^2<t^{\ell_1},\\
t^{\ell_2+1}<s^3<t^{\ell_2},\\
\vdots\\
t^{\ell_i+1}<s^{i+1}<\underbrace{t^{\ell_i}<\cdots\cdots<t^{\ell_{i-1}+1}}_{i-\text{intermediate group}}<s^i<t^{\ell_{i-1}},\\
\vdots\\
t^{\ell_{[\frac{|t-s||v|}{L_\xi}]+1}}<s^{\ell_{[\frac{|t-s||v|}{L_\xi}]+1}}<t^{\ell_{[\frac{|t-s||v|}{L_\xi}]}}.
\end{gathered}$$

We claim the following estimate at s^{i+1} via s^i. Within the $i-$th intermediate group, we fix $\mathbf{p}^{\ell_i}-$spherical coordinate in *Step 9*. The goal is to estimate derivatives with respect to initial $(\mathbf{x}_1,\mathbf{v}_1)$ at s^{i+1} in terms of s^i. This is different from previous steps.

$$\begin{aligned}
&\begin{bmatrix} |\frac{\partial \mathbf{x}_{\|_{\ell_i}}(s^{i+1})}{\partial \mathbf{x}_{\perp_1}(s^1)}| & |\frac{\partial \mathbf{x}_{\|_{\ell_i}}(s^{i+1})}{\partial \mathbf{x}_{\|_1}(s^1)}| \\ |\frac{\partial \mathbf{v}_{\|_{\ell_i}}(s^{i+1})}{\partial \mathbf{x}_{\perp_1}(s^1)}| & |\frac{\partial \mathbf{v}_{\|_{\ell_i}}(s^{i+1})}{\partial \mathbf{x}_{\|_1}(s^1)}| \end{bmatrix}\\
&\lesssim_{\delta,\xi} \begin{bmatrix} 1 & \frac{1}{|v|} \\ |v| & 1 \end{bmatrix} \begin{bmatrix} |\frac{\partial \mathbf{x}_{\|_{\ell_i}}(s^{i})}{\partial \mathbf{x}_{\perp_1}(s^1)}| & |\frac{\partial \mathbf{x}_{\|_{\ell_i}}(s^{i})}{\partial \mathbf{x}_{\|_1}(s^1)}| \\ |\frac{\partial \mathbf{v}_{\|_{\ell_i}}(s^{i})}{\partial \mathbf{x}_{\perp_1}(s^1)}| & |\frac{\partial \mathbf{v}_{\|_{\ell_i}}(s^{i})}{\partial \mathbf{x}_{\|_1}(s^1)}| \end{bmatrix}\\
&\qquad + e^{C|v||t-s^i|}\begin{bmatrix} 1 & \frac{1}{|v|} \\ |v| & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ |v|\Big(1+\frac{|v|}{|\mathbf{v}_\perp^1|}\Big) & |v|\Big(1+\frac{|v|}{|\mathbf{v}_\perp^1|}\Big) \end{bmatrix},
\end{aligned} \tag{5.99}$$

$$\begin{aligned}
&\begin{bmatrix} |\frac{\partial \mathbf{x}_{\|_{\ell_i}}(s^{i+1})}{\partial \mathbf{v}_{\perp_1}(s^1)}| & |\frac{\partial \mathbf{x}_{\|_{\ell_i}}(s^{i+1})}{\partial \mathbf{v}_{\|_1}(s^1)}| \\ |\frac{\partial \mathbf{v}_{\|_{\ell_i}}(s^{i+1})}{\partial \mathbf{v}_{\perp_1}(s^1)}| & |\frac{\partial \mathbf{v}_{\|_{\ell_i}}(s^{i+1})}{\partial \mathbf{v}_{\|_1}(s^1)}| \end{bmatrix}\\
&\lesssim_{\delta,\xi} \begin{bmatrix} 1 & \frac{1}{|v|} \\ |v| & 1 \end{bmatrix} \begin{bmatrix} |\frac{\partial \mathbf{x}_{\|_{\ell_i}}(s^{i})}{\partial \mathbf{v}_{\perp_1}(s^1)}| & |\frac{\partial \mathbf{x}_{\|_{\ell_i}}(s^{i})}{\partial \mathbf{v}_{\|_1}(s^1)}| \\ |\frac{\partial \mathbf{v}_{\|_{\ell_i}}(s^{i})}{\partial \mathbf{v}_{\perp_1}(s^1)}| & |\frac{\partial \mathbf{v}_{\|_{\ell_i}}(s^{i})}{\partial \mathbf{v}_{\|_1}(s^1)}| \end{bmatrix} + e^{C|v||t-s^i|}\begin{bmatrix} 1 & \frac{1}{|v|} \\ |v| & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.
\end{aligned}$$

For the sake of simplicity we drop the index ℓ_i.

Denote, from (4.9),

$$F_{\|}(\mathbf{x}_\perp,\mathbf{x}_\|,\mathbf{v}_\perp,\mathbf{v}_\|):=D(\mathbf{x}_\perp,\mathbf{x}_\|,\mathbf{v}_\|)+H(\mathbf{x}_\perp,\mathbf{x}_\|,\mathbf{v}_\|)\mathbf{v}_\perp, \tag{5.100}$$

where D is a $\mathbf{r}^3$-vector-valued function and H is a 3×3 matrix-valued function:

$$\begin{aligned}&D(\mathbf{x}_\perp,\mathbf{x}_\parallel,\mathbf{v}_\parallel)\\&=\sum_i G_{ij}(\mathbf{x}_\perp,\mathbf{x}_\parallel)\frac{(-1)^{i+1}}{-\mathbf{n}(\mathbf{x}_\parallel)\cdot(\partial_1\eta(\mathbf{x}_\parallel)\times\partial_2\eta(\mathbf{x}_\parallel))}(-\mathbf{n}(\mathbf{x}_\parallel)\times\partial_{i+1}\eta(\mathbf{x}_\parallel))\\&\quad\cdot\Big\{\mathbf{v}_\parallel\cdot\nabla^2\eta(\mathbf{x}_\parallel)\cdot\mathbf{v}_\parallel-\mathbf{x}_\perp\mathbf{v}_\parallel\cdot\nabla^2\mathbf{n}(\mathbf{x}_\parallel)\cdot\mathbf{v}_\parallel-E(s,-\mathbf{x}_\perp\mathbf{n}(\mathbf{x}_\parallel)+\eta(\mathbf{x}_\parallel))\Big\},\end{aligned}$$

and

$$\begin{aligned}&H(\mathbf{x}_\perp,\mathbf{x}_\parallel,\mathbf{v}_\parallel)\\&=\sum_i G_{ij}(\mathbf{x}_\perp,\mathbf{x}_\parallel)\frac{(-1)^{i+1}}{-\mathbf{n}(\mathbf{x}_\parallel)\cdot(\partial_1\eta(\mathbf{x}_\parallel)\times\partial_2\eta(\mathbf{x}_\parallel))}2\mathbf{v}_\parallel\cdot\nabla\mathbf{n}(\mathbf{x}_\parallel)\cdot(-\mathbf{n}(\mathbf{x}_\parallel)\times\partial_{i+1}\eta(\mathbf{x}_\parallel)).\end{aligned}$$

Note that H is linear in $\mathbf{v}_\parallel$. Here $G_{ij}(\cdot,\cdot)$ is a smooth bounded function defined in (4.16) and we used the notational convention $i\equiv i \bmod 2$.

From Lemma 4.1 we take the time integration of (4.7) along the characteristics to have

$$\begin{aligned}&\mathbf{x}_\parallel(s^{i+1})=\mathbf{x}_\parallel(s^i)-\int_{s^{i+1}}^{s^i}\mathbf{v}_\parallel(\tau)\mathrm{d}\tau,\\&\mathbf{v}_\parallel(s^{i+1})=\mathbf{v}_\parallel(s^i)-\int_{s^{i+1}}^{s^i}\{H(\mathbf{x}_\perp(\tau),\mathbf{x}_\parallel(\tau),\mathbf{v}_\parallel(\tau))\mathbf{v}_\perp(\tau)+D(\mathbf{x}_\perp(\tau),\mathbf{x}_\parallel(\tau),\mathbf{v}_\parallel(\tau))\}\,\mathrm{d}\tau.\end{aligned}$$

Note that $\mathbf{v}_\perp(\tau)$ is not continuous with respect to the time τ. Using (4.7) we rewrite this time integration as

$$\int_{s^{i+1}}^{s^i}H(\mathbf{x}_\perp(\tau),\mathbf{x}_\parallel(\tau),\mathbf{v}_\parallel(\tau))\mathbf{v}_\perp(\tau)\mathrm{d}\tau=\int_{t^{\ell_{i-1}+1}}^{s^i}+\sum_{\ell=\ell_i-1}^{\ell_{i-1}+1}\int_{t^{\ell+1}}^{t^\ell}+\int_{s^{i+1}}^{t^{\ell_i}},$$

then we use $\mathbf{v}_\perp(\tau)=\dot{\mathbf{x}}_\perp(\tau)$ and the integration by parts to have

$$\begin{aligned}&\int_{t^{\ell_{i-1}+1}}^{s^i}H(\mathbf{x}_\perp(\tau),\mathbf{x}_\parallel(\tau),\mathbf{v}_\parallel(\tau))\dot{\mathbf{x}}_\perp(\tau)\mathrm{d}\tau+\sum_{\ell=\ell_i-1}^{\ell_{i-1}+1}\int_{t^{\ell+1}}^{t^\ell}H(\mathbf{x}_\perp(\tau),\mathbf{x}_\parallel(\tau),\mathbf{v}_\parallel(\tau))\dot{\mathbf{x}}_\perp(\tau)\mathrm{d}\tau\\&\quad+\int_{s^{i+1}}^{t^{\ell_i}}H(\mathbf{x}_\perp(\tau),\mathbf{x}_\parallel(\tau),\mathbf{v}_\parallel(\tau))\dot{\mathbf{x}}_\perp(\tau)\mathrm{d}\tau\\&=H(s^i)\mathbf{x}_\perp(s^i)-H(t^{\ell_{i-1}+1})\underbrace{\mathbf{x}_\perp(t^{\ell_{i-1}+1})}_{=0}-\int_{t^{\ell_{i-1}+1}}^{s^i}\big[\mathbf{v}_\perp(\tau),\mathbf{v}_\parallel(\tau),F_\parallel(\tau)\big]\cdot\nabla H(\tau)\mathbf{x}_\perp(\tau)\mathrm{d}\tau\\&\quad+\sum_{\ell=\ell_i-1}^{\ell_{i-1}+1}\Big\{H(t^\ell)\underbrace{\mathbf{x}_\perp(t^\ell)}_{=0}-H(t^{\ell+1})\underbrace{\mathbf{x}_\perp(t^{\ell+1})}_{=0}\\&\quad-\int_{t^{\ell+1}}^{t^\ell}\big[\mathbf{v}_\perp(\tau),\mathbf{v}_\parallel(\tau),F_\parallel(\tau)\big]\cdot\nabla H(\tau)\mathbf{x}_\perp(\tau)\mathrm{d}\tau\Big\}\\&\quad+H(t^{\ell_i})\underbrace{\mathbf{x}_\perp(t^{\ell_i})}_{=0}-H(s^{i+1})\mathbf{x}_\perp(s^{i+1})-\int_{s^{i+1}}^{t^{\ell_i}}\big[\mathbf{v}_\perp(\tau),\mathbf{v}_\parallel(\tau),F_\parallel(\tau)\big]\cdot\nabla H(\tau)\mathbf{x}_\perp(\tau)\mathrm{d}\tau\end{aligned}$$

$$=H(\mathbf{x}_\perp,\mathbf{x}_\|,\mathbf{v}_\|)(s^i)\mathbf{x}_\perp(s^i)-H(s^{i+1})\mathbf{x}_\perp(s^{i+1}) \\ -\int_{s^i}^{s^{i+1}}\big[\mathbf{v}_\perp(\tau),\mathbf{v}_\|(\tau),F_\|(\tau)\big]\cdot\nabla H(\tau)\mathbf{x}_\perp(\tau)\mathrm{d}\tau,$$

where we have used the fact $X_{\mathbf{cl}}(t^\ell)\in\partial\Omega$(therefore $\mathbf{x}_\perp(t^\ell)=0$) and the notation $H(\tau)=H(\mathbf{x}_\perp(\tau),\mathbf{x}_\|(\tau),\mathbf{v}_\|(\tau))$, $D(\tau)=D(\mathbf{x}_\perp(\tau),\mathbf{x}_\|(\tau),\mathbf{v}_\|(\tau))$, $F_\|(\tau)=F_\|(\mathbf{x}_\perp(\tau),\mathbf{x}_\|(\tau),\mathbf{v}_\perp(\tau),\mathbf{v}_\|(\tau))$.

Overall we have

$$\begin{aligned}&\mathbf{x}_\|(s^{i+1})=\mathbf{x}_\|(s^i)-\int_{s^{i+1}}^{s^i}\mathbf{v}_\|(\tau)\mathrm{d}\tau,\\ &\mathbf{v}_\|(s^{i+1})=\mathbf{v}_\|(s^i)-H(s^i)\mathbf{x}_\perp(s^i)+H(s^{i+1})\mathbf{x}_\perp(s^{i+1})\\ &\qquad+\int_{s^{i+1}}^{s^i}\big[\mathbf{v}_\perp(\tau),\mathbf{v}_\|(\tau),F_\|(\tau)\big]\cdot\nabla H(\tau)\mathbf{x}_\perp(\tau)\mathrm{d}\tau-\int_{s^{i+1}}^{s^i}D(\tau)\mathrm{d}\tau.\end{aligned}\tag{5.101}$$

Denote

$$\partial=[\partial_{\mathbf{x}_\perp(s^1)},\partial_{\mathbf{x}_\|(s^1)},\partial_{\mathbf{v}_\perp(s^1)},\partial_{\mathbf{v}_\|(s^1)}]=[\frac{\partial}{\partial\mathbf{x}_\perp(s^1)},\frac{\partial}{\partial\mathbf{x}_\|(s^1)},\frac{\partial}{\partial\mathbf{v}_\perp(s^1)},\frac{\partial}{\partial\mathbf{v}_\|(s^1)}].$$

We claim that, in a sense of distribution on $(s^1,\mathbf{x}_\perp(s^1),\mathbf{x}_\|(s^1),\mathbf{v}_\perp(s^1),\mathbf{v}_\|(s^1))\in[0,\infty)\times(0,C_\xi)\times(0,2\pi]\times(\delta,\pi-\delta)\times\mathbb{R}\times\mathbb{R}^2$,

$$\begin{aligned}&\big[\partial\mathbf{x}_\perp(s^{i+1};s^1,\mathbf{x}(s^1),\mathbf{v}(s^1)),\partial\mathbf{x}_\|(s^{i+1};s^1,\mathbf{x}(s^1),\mathbf{v}(s^1)),\ \partial\mathbf{v}_\|(s^{i+1};s^1,\mathbf{x}(s^1),\mathbf{v}(s^1))\big]\\ &=\sum_\ell\mathbf{1}_{[t^{\ell+1},t^\ell)}(s^{i+1})[\partial\mathbf{x}_\perp,\partial\mathbf{x}_\|,\partial\mathbf{v}_\|],\\ &\partial\big[\mathbf{v}_\perp(s^{i+1};s^1,\mathbf{x}(s^1),\mathbf{v}(s^1))\mathbf{x}_\perp(s^{i+1};s^1,\mathbf{x}(s^1),\mathbf{v}(s^1))\big]\\ &=\sum_\ell\mathbf{1}_{[t^{\ell+1},t^\ell)}(s^{i+1})\big\{\partial\mathbf{v}_\perp\mathbf{x}_\perp+\mathbf{v}_\perp\partial\mathbf{x}_\perp\big\},\end{aligned}\tag{5.102}$$

i.e. the distributional derivatives of $[\mathbf{x}_\perp,\mathbf{x}_\|,\mathbf{v}_\|]$ and $\mathbf{v}_\perp\mathbf{x}_\perp$ equal the piecewise derivatives.

Proof. **(Proof of (5.102).)** Let $\phi(\tau',\mathbf{x}_\perp,\mathbf{x}_\|,\mathbf{v}_\perp,\mathbf{v}_\|)\in C_c^\infty([0,\infty)\times(0,C_\xi)\times\mathbb{S}^2\times\mathbb{R}\times\mathbb{R}^2)$. Therefore $\phi\equiv0$ when $\mathbf{x}_\perp<\delta$, $|v|>\frac{1}{\delta}$. For $\mathbf{x}_\perp\geq\delta$ we use the proof of Lemma 4.1: For $x=\eta(\mathbf{x}_\|)+\mathbf{x}_\perp[-\mathbf{n}(\mathbf{x}_\|)]$,

$$|\mathbf{x}_\perp|\ \lesssim_\xi\ \xi(x)=\xi(\eta(\mathbf{x}_\|)+\mathbf{x}_\perp[-\mathbf{n}(\mathbf{x}_\|)])\ \lesssim_\xi\ |\mathbf{x}_\perp|,$$

and therefore $\xi(x)\gtrsim_\xi\delta$ and $\alpha(t,x,v)\gtrsim_{\xi,E}\sqrt{|\xi(x)|}\gtrsim_{\xi,E}\sqrt{\delta}$. By the Velocity lemma, for $(x,v)\in\text{supp}(\phi)$

$$\alpha(x^\ell,v^\ell)\gtrsim_\xi e^{-C(|v|+1)|t^1-t^\ell|}\alpha(t,x,v)\gtrsim_\xi e^{-\frac{C}{\delta}(t-s)}\sqrt{\delta}\gtrsim_{\xi,E,|t-s|,\delta,\phi}1>0,$$

where we used the fact that ϕ vanishes away from a compact subset $\text{supp}(\phi)$. Therefore $t^\ell(t,x,v)=t^\ell(t,\mathbf{x}_\perp,\mathbf{x}_\|,\mathbf{v}_\perp,\mathbf{v}_\|)$ is C^1 with respect to $\mathbf{x}_\perp,\mathbf{x}_\|,\mathbf{v}_\perp,\mathbf{v}_\|$ locally on $\text{supp}(\phi)$ and therefore $\mathcal{M}=\{(\tau',\mathbf{x},\mathbf{v})\in\text{supp}(\phi):\tau'=t^\ell(t,\mathbf{x},\mathbf{v})\}$ is a C^1 manifold.

It suffices to consider the case $|\tau' - t^\ell(t,x,v)| \ll 1$. Denote $\partial_{\mathbf{e}} \in \{\partial_{\mathbf{x}_\perp}, \partial_{\mathbf{x}_{\|,1}}, \partial_{\mathbf{x}_{\|,2}}, \partial_{\mathbf{v}_\perp}, \partial_{\mathbf{v}_{\|,1}}, \partial_{\mathbf{v}_{\|,2}}\}$ and $n_{\mathcal{M}} = e_1$ to have

$$
\begin{aligned}
&\int_{\{(\tau',\mathbf{x},\mathbf{v})\in \mathrm{supp}(\phi)\}} [\partial_{\mathbf{e}}\mathbf{x}_\perp(\tau';t,\mathbf{x},\mathbf{v}), \partial_{\mathbf{e}}\mathbf{x}_\|(\tau';t,\mathbf{x},\mathbf{v}), \partial_{\mathbf{e}}\mathbf{v}_\|(\tau';t,\mathbf{x},\mathbf{v})]\phi(\tau',\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}\mathrm{d}\tau' \\
&= \int_{\tau'<t^\ell} + \int_{\tau'\geq t^\ell} \\
&= \int_{\mathcal{M}} \Big(\lim_{\tau'\uparrow t^\ell}[\mathbf{x}_\perp(\tau'),\mathbf{x}_\|(\tau'),\mathbf{v}_\|(\tau')] - \lim_{\tau'\downarrow t^\ell}[\mathbf{x}_\perp(\tau'),\mathbf{x}_\|(\tau'),\mathbf{v}_\|(\tau')]\Big)\phi(\tau',\mathbf{x},\mathbf{v})\{\mathbf{e}\cdot n_{\mathcal{M}}\}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} \\
&\quad - \int_{\{\tau'\neq t^\ell(t,\mathbf{x},\mathbf{v})\}} [\mathbf{x}_\perp(\tau'),\mathbf{x}_\|(\tau'),\mathbf{v}_\|(\tau')]\partial_{\mathbf{e}}\phi(\tau',\mathbf{x},\mathbf{v})\mathrm{d}\tau'\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} \\
&= - \int_{\{\tau'\neq t^\ell(t,\mathbf{x},\mathbf{v})\}} [\mathbf{x}_\perp(\tau'),\mathbf{x}_\|(\tau'),\mathbf{v}_\|(\tau')]\partial_{\mathbf{e}}\phi(\tau',\mathbf{x},\mathbf{v})\mathrm{d}\tau'\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x},
\end{aligned}
$$

where we used the continuity of $[\mathbf{x}_\perp(\tau';t,\mathbf{x},\mathbf{v}), \mathbf{x}_\|(\tau';t,\mathbf{x},\mathbf{v}), \mathbf{v}_\|(\tau';t,\mathbf{x},\mathbf{v})]$ in terms of τ' near $t^\ell(t,\mathbf{x},\mathbf{v})$.

Note that $\mathbf{v}_\perp(\tau';t,\mathbf{x},\mathbf{v})$ is discontinuous around $|\tau'-t^\ell| \ll 1$ ($\lim_{\tau'\downarrow t^\ell}\mathbf{v}_\perp(\tau') = -\lim_{\tau'\uparrow t^\ell}\mathbf{v}_\perp(\tau')$). However with crucial $\mathbf{x}_\perp(\tau')-$multiplication we have $\mathbf{x}_\perp(t^\ell)\mathbf{v}_\perp(t^\ell) = 0$ and therefore

$$
\begin{aligned}
&\int_{\{(\tau',\mathbf{x},\mathbf{v})\in \mathrm{supp}(\phi)\}} \partial_{\mathbf{e}}[\mathbf{x}_\perp(\tau';t,\mathbf{x},\mathbf{v})\mathbf{v}_\perp(\tau';t,\mathbf{x},\mathbf{v})]\phi(\tau',\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}\mathrm{d}\tau' \\
&= \int_{\tau'<t^\ell} + \int_{\tau'\geq t^\ell} \\
&= \int_{\mathcal{M}} \Big(\lim_{\tau'\uparrow t^\ell}[\mathbf{x}_\perp(\tau')\mathbf{v}_\perp(\tau')] - \lim_{\tau'\downarrow t^\ell}[\mathbf{x}_\perp(\tau')\mathbf{v}_\perp(\tau')]\Big)\phi(\tau',\mathbf{x},\mathbf{v})\{\mathbf{e}\cdot n_{\mathcal{M}}\}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} \\
&\quad - \int_{\{\tau'\neq t^\ell(t,\mathbf{x},\mathbf{v})\}} [\mathbf{x}_\perp(\tau')\mathbf{v}_\perp(\tau')]\partial_{\mathbf{e}}\phi(\tau',\mathbf{x},\mathbf{v})\mathrm{d}\tau'\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} \\
&= - \int_{\{\tau'\neq t^\ell(t,\mathbf{x},\mathbf{v})\}} [\mathbf{x}_\perp(\tau';t,\mathbf{x},\mathbf{v})\mathbf{v}_\perp(\tau';t,\mathbf{x},\mathbf{v})]\partial_{\mathbf{e}}\phi(\tau',\mathbf{x},\mathbf{v})\mathrm{d}\tau'\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x}.
\end{aligned}
$$

This completes the proof of (5.102). ☐

Since $\mathbf{v}_\perp$ always is multiplied with $\mathbf{x}_\perp$ in (5.101), we may apply (5.102) and take derivative inside each $\int_{s^{i+1}}^{s^i}$ of (5.101), separating the main terms with $\partial_{\mathbf{e}}\mathbf{x}_{\|}$ and $\partial_{\mathbf{e}}\mathbf{v}_{\|}$, and treating the rest (underbraced terms) as forcing terms to be obtained, for $\partial_{\mathbf{e}} \in \{\partial_{\mathbf{x}_\perp}, \partial_{\mathbf{x}_{\|,1}}, \partial_{\mathbf{x}_{\|,2}}, \partial_{\mathbf{v}_\perp}, \partial_{\mathbf{v}_{\|,1}}, \partial_{\mathbf{v}_{\|,2}}\}$,

$$
\begin{aligned}
&\partial_{\mathbf{e}}\mathbf{x}_\|(s^{i+1}) = \partial_{\mathbf{e}}\mathbf{x}_\|(s^i) - \int_{s^{i+1}}^{s^i} \partial_{\mathbf{e}}\mathbf{v}_\|(\tau)\mathrm{d}\tau, \\
&\partial\mathbf{v}_\|(s^{i+1}) = \partial_{\mathbf{e}}H(s^{i+1})\mathbf{x}_\perp(s^{i+1}) + H(s^{i+1})\underbrace{\partial_{\mathbf{e}}\mathbf{x}_\perp(s^{i+1})} + \partial_{\mathbf{e}}\mathbf{v}_\|(s^i) - \partial_{\mathbf{e}}[H(\mathbf{x}_\perp,\mathbf{x}_\|,\mathbf{v}_\|)\mathbf{x}_\perp](s^{i+1}) \\
&\quad + \int_{s^{i+1}}^{s^i} \underbrace{\partial_{\mathbf{e}}\mathbf{v}_\perp(\tau)}\partial_{\mathbf{x}_\perp}H(\tau)\mathbf{x}_\perp(\tau) + \partial_{\mathbf{e}}\mathbf{v}_\|(\tau)\cdot\nabla_{\mathbf{x}_\|}H(\tau)\mathbf{x}_\perp(\tau)\mathrm{d}\tau \\
&\quad + \int_{s^{i+1}}^{s^i} \Big\{\Big[\underbrace{\partial_{\mathbf{e}}\mathbf{x}_\perp(\tau)}\partial_{\mathbf{x}_\perp}H(\tau) + \partial_{\mathbf{e}}\mathbf{x}_\|(\tau)\cdot\nabla_{\mathbf{x}_\|}H(\tau) + \partial_{\mathbf{e}}\mathbf{v}_\|(\tau)\cdot\nabla_{\mathbf{v}_\|}H(\tau)\Big]\mathbf{v}_\perp(\tau) \\
&\quad + H(\tau)\underbrace{\partial_{\mathbf{e}}\mathbf{v}_\perp(\tau)} + \underbrace{\partial_{\mathbf{e}}\mathbf{x}_\perp(\tau)}\partial_{\mathbf{x}_\perp}D(\tau) + \partial_{\mathbf{e}}\mathbf{x}_\|(\tau)\cdot\nabla_{\mathbf{x}_\|}D(\tau) + \partial_{\mathbf{e}}\mathbf{v}_\|(\tau)\nabla_{\mathbf{v}_\|}D(\tau)\Big\}\cdot\nabla_{\mathbf{v}_\|}H(\tau)\mathbf{x}_\perp(\tau)\mathrm{d}\tau
\end{aligned}
$$

$$+\int_{s^{i+1}}^{s^i}\Big\{\mathbf{v}_\perp(\tau)[\underbrace{\partial_{\mathbf{e}}\mathbf{x}_\perp(\tau)},\partial_{\mathbf{e}}\mathbf{x}_\parallel(\tau),\partial_{\mathbf{e}}\mathbf{v}_\parallel(\tau)]\cdot\nabla\partial_{\mathbf{x}_\perp}H(\tau)+\mathbf{v}_\parallel(\tau)\cdot[\underbrace{\partial_{\mathbf{e}}\mathbf{x}_\perp(\tau)},\partial_{\mathbf{e}}\mathbf{x}_\parallel(\tau),\partial_{\mathbf{e}}\mathbf{v}_\parallel(\tau)]\cdot\nabla\nabla_{\mathbf{x}_\parallel}H(\tau)$$
$$+F_\parallel(\tau)\cdot[\underbrace{\partial_{\mathbf{e}}\mathbf{x}_\perp(\tau)},\partial_{\mathbf{e}}\mathbf{x}_\parallel(\tau),\partial_{\mathbf{e}}\mathbf{v}_\parallel(\tau)]\cdot\nabla\nabla_{\mathbf{v}_\parallel}H(\tau)\Big\}\mathbf{x}_\perp(\tau)\mathrm{d}\tau$$
$$+\int_{s^{i+1}}^{s^i}\{\mathbf{v}_\perp(\tau)\partial_{\mathbf{x}_\perp}H(\tau)+\mathbf{v}_\parallel(\tau)\cdot\nabla_{\mathbf{x}_\parallel}H(\tau)+F_\parallel(\tau)\cdot\nabla_{\mathbf{v}_\parallel}H(\tau)\}\underbrace{\partial_{\mathbf{e}}\mathbf{x}_\perp(\tau)}\mathrm{d}\tau$$
$$-\int_{s^{i+1}}^{s^i}\big[\underbrace{\partial_{\mathbf{e}}\mathbf{x}_\perp(\tau)},\partial_{\mathbf{e}}\mathbf{x}_\parallel(\tau),\partial_{\mathbf{e}}\mathbf{v}_\parallel(\tau)\big]\cdot\nabla D(\tau)\mathrm{d}\tau. \tag{5.103}$$

Now we use (5.93) to control the underbraced term of (5.103). Notice that we cannot directly use (5.93) since now we fix the chart for whole i−th intermediate group but the estimate (5.93) is for the moving frame (for clarity, we write the index for the chart for this part). Note the times of bounces within the i−th intermediate group ($|t^{\ell_{i-1}}-t^{\ell_i}||v|\simeq L_\xi$) are

$$t^{\ell_i+1}<s^{i+1}<t^{\ell_i}<t^{\ell_i-1}<\cdots\cdots<t^{\ell_{i-1}+2}<t^{\ell_{i-1}+1}<s^i<t^{\ell_{i-1}}.$$

Now we apply (4.6) and (5.93) to bound, for $\tau\in(s^{i+1},s^i)$ and $\ell\in\{\ell_i,\ell_i-1,\cdots,\ell_{i-1}+2,\ell_{i-1}+1,\ell_{i-1}\}$

$$\frac{\partial(\mathbf{x}_{\perp_\ell}(\tau),\mathbf{x}_{\parallel_\ell}(\tau),\mathbf{v}_{\perp_\ell}(\tau),\mathbf{v}_{\parallel_\ell}(\tau))}{\partial(\mathbf{x}_{\perp_1}(s^1),\mathbf{x}_{\parallel_1}(s^1),\mathbf{v}_{\perp_1}(s^1),\mathbf{v}_{\parallel_1}(s^1))}$$
$$=\frac{\partial(\mathbf{x}_{\perp_\ell}(\tau),\mathbf{x}_{\parallel_\ell}(\tau),\mathbf{v}_{\perp_\ell}(\tau),\mathbf{v}_{\parallel_\ell}(\tau))}{\partial(\mathbf{x}_{\perp_{\ell_i}}(\tau),\mathbf{x}_{\parallel_{\ell_i}}(\tau),\mathbf{v}_{\perp_{\ell_i}}(\tau),\mathbf{v}_{\parallel_{\ell_i}}(\tau))}\frac{\partial(\mathbf{x}_{\perp_{\ell_i}}(\tau),\mathbf{x}_{\parallel_{\ell_i}}(\tau),\mathbf{v}_{\perp_{\ell_i}}(\tau),\mathbf{v}_{\parallel_{\ell_i}}(\tau))}{\partial(\mathbf{x}_{\perp_1}(s^1),\mathbf{x}_{\parallel_1}(s^1),\mathbf{v}_{\perp_1}(s^1),\mathbf{v}_{\parallel_1}(s^1))}$$
$$\lesssim e^{C|t-s||v|}\left\{\mathbf{Id}_{6,6}+O_\xi(|\mathbf{p}^\ell-\mathbf{p}^{\ell_i}|)\left[\begin{array}{ccc|ccc}0&0&0&&&\\0&1&1&&\mathbf{0}_{3,3}&\\0&1&1&&&\\\hline 0&0&0&0&0&0\\0&|v|&|v|&0&1&1\\0&|v|&|v|&0&1&1\end{array}\right]\right\}$$
$$\times\begin{bmatrix}\frac{|v|+1}{|\mathbf{v}_\perp^1|}&\frac{|v|+1}{|\mathbf{v}_\perp^1|}&\frac{|v|+1}{|\mathbf{v}_\perp^1|}&\min\{\frac{1}{|v|},1\}&\min\{\frac{1}{|v|},1\}&\min\{\frac{1}{|v|},1\}\\\frac{|v|^2+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|^2+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|^2+1}{|\mathbf{v}_\perp^1|^2}&\frac{1}{|\mathbf{v}_\perp^1|}&\min\{\frac{1}{|v|},1\}&\min\{\frac{1}{|v|},1\}\\\frac{|v|^2+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|^2+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|^2+1}{|\mathbf{v}_\perp^1|^2}&\frac{1}{|\mathbf{v}_\perp^1|}&\min\{\frac{1}{|v|},1\}&\min\{\frac{1}{|v|},1\}\\\frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|+1}{|\mathbf{v}_\perp^1|}&1&1\\\frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|+1}{|\mathbf{v}_\perp^1|}&1&1\\\frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|+1}{|\mathbf{v}_\perp^1|}&1&1\end{bmatrix}$$
$$\lesssim e^{C|t-s||v|}\begin{bmatrix}\frac{|v|+1}{|\mathbf{v}_\perp^1|}&\frac{|v|+1}{|\mathbf{v}_\perp^1|}&\frac{|v|+1}{|\mathbf{v}_\perp^1|}&\min\{\frac{1}{|v|},1\}&\min\{\frac{1}{|v|},1\}&\min\{\frac{1}{|v|},1\}\\\frac{|v|^2+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|^2+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|^2+1}{|\mathbf{v}_\perp^1|^2}&\frac{1}{|\mathbf{v}_\perp^1|}&\min\{\frac{1}{|v|},1\}&\min\{\frac{1}{|v|},1\}\\\frac{|v|^2+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|^2+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|^2+1}{|\mathbf{v}_\perp^1|^2}&\frac{1}{|\mathbf{v}_\perp^1|}&\min\{\frac{1}{|v|},1\}&\min\{\frac{1}{|v|},1\}\\\frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|+1}{|\mathbf{v}_\perp^1|}&1&1\\\frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|+1}{|\mathbf{v}_\perp^1|}&1&1\\\frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2}&\frac{|v|+1}{|\mathbf{v}_\perp^1|}&1&1\end{bmatrix}, \tag{5.104}$$

where we have used $|\mathbf{p}^{\ell}-\mathbf{p}^{\ell_i}|\lesssim 1$.

We plug in (5.103) with (5.104) respectively with

$$|\partial_{\mathbf{x}_{\perp_1}}\mathbf{x}_\perp(\tau)|\lesssim \frac{|v|+1}{|\mathbf{v}_\perp^1|}, |\partial_{\mathbf{x}_{\perp_1}}\mathbf{v}_\perp(\tau)|\lesssim \frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2},$$
$$|\partial_{\mathbf{v}_{\perp_1}}\mathbf{x}_\perp(\tau)|\lesssim \min\{\frac{1}{|v|},1\}, |\partial_{\mathbf{v}_{\perp_1}}\mathbf{v}_\perp(\tau)|\lesssim 1,$$

and

$$|\nabla_{\mathbf{v}_\parallel}H(\tau)|\lesssim 1, |\nabla_{\mathbf{x}_\parallel,\mathbf{x}_\perp}H(\tau)|\lesssim |v|+1, |\nabla_{\mathbf{v}_\parallel}D(\tau)|\lesssim |v|+1, |\nabla_{\mathbf{x}_\parallel,\mathbf{x}_\perp}D(\tau)|\lesssim |v|^2+1,$$

and use the fact that $|s^i-s^{i+1}|\lesssim \frac{1}{|v|+1}$ by the way we define s^i. Collecting terms with tedious but straightforward bounds, we summarize the results as: for $s\in[s^{i+1},s^i]$

$$\begin{aligned}
\begin{bmatrix} |\frac{\partial \mathbf{x}_\parallel(s)}{\partial \mathbf{x}_\perp}| \\ |\frac{\partial \mathbf{v}_\parallel(s)}{\partial \mathbf{x}_\perp}| \end{bmatrix} &\lesssim_\xi \begin{bmatrix} |\frac{\partial \mathbf{x}_\parallel(s^i)}{\partial \mathbf{x}_\perp}| \\ |\frac{\partial \mathbf{v}_\parallel(s^i)}{\partial \mathbf{x}_\perp}|+|v||\frac{\partial \mathbf{x}_\parallel(s^i)}{\partial \mathbf{x}_\perp}| \end{bmatrix} + \begin{bmatrix} \int_s^{s^i}|\frac{\partial \mathbf{v}_\parallel}{\partial \mathbf{x}_\perp}| \\ \int_s^{s^i}(|v|^2+1)|\frac{\partial \mathbf{x}_\parallel}{\partial \mathbf{x}_\perp}|+(|v|+1)|\frac{\partial \mathbf{v}_\parallel}{\partial \mathbf{x}_\perp}| \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 \\ e^{C|v||t-s|}\frac{|v|^2+1}{|\mathbf{v}_\perp^1|} \end{bmatrix}, \\
\begin{bmatrix} |\frac{\partial \mathbf{x}_\parallel(s)}{\partial \mathbf{v}_\perp}| \\ |\frac{\partial \mathbf{v}_\parallel(s)}{\partial \mathbf{v}_\perp}| \end{bmatrix} &\lesssim_\xi \begin{bmatrix} |\frac{\partial \mathbf{x}_\parallel(s^i)}{\partial \mathbf{v}_\perp}| \\ |\frac{\partial \mathbf{v}_\parallel(s^i)}{\partial \mathbf{v}_\perp}|+|v||\frac{\partial \mathbf{x}_\parallel(s^i)}{\partial \mathbf{v}_\perp}| \end{bmatrix} + \begin{bmatrix} \int_s^{s^i}|\frac{\partial \mathbf{v}_\parallel}{\partial \mathbf{v}_\perp}| \\ \int_s^{s^i}(|v|^2+1)|\frac{\partial \mathbf{x}_\parallel}{\partial \mathbf{v}_\perp}|+(|v|+1)|\frac{\partial \mathbf{v}_\parallel}{\partial \mathbf{v}_\perp}| \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 \\ e^{C|v||t-s|} \end{bmatrix}.
\end{aligned} \tag{5.105}$$

From (5.105) we have

$$\begin{aligned}
\langle v\rangle|\frac{\partial \mathbf{x}_\parallel(s)}{\partial \mathbf{x}_\perp}|+|\frac{\partial \mathbf{v}_\parallel(s)}{\partial \mathbf{x}_\perp}(s)| &\lesssim |\frac{\partial \mathbf{v}_\parallel(s^i)}{\partial \mathbf{x}_\perp}|+\langle v\rangle|\frac{\partial \mathbf{x}_\parallel(s^i)}{\partial \mathbf{x}_\perp}|+e^{C|v||t-s|}\frac{|v|^2+1}{|\mathbf{v}_\perp^1|} \\
&\quad + \int_s^{s^i}\langle v\rangle\left(\langle v\rangle|\frac{\partial \mathbf{x}_\parallel}{\partial \mathbf{x}_\perp}|+|\frac{\partial \mathbf{v}_\parallel}{\partial \mathbf{x}_\perp}|\right),
\end{aligned} \tag{5.106}$$

from the Gronwall inequality we get

$$\begin{aligned}
\langle v\rangle|\frac{\partial \mathbf{x}_\parallel(s)}{\partial \mathbf{x}_\perp}|+|\frac{\partial \mathbf{v}_\parallel(s)}{\partial \mathbf{x}_\perp}(s)| &\le C_\xi' e^{\int_s^{s^i}\langle v\rangle d\tau}\left(|\frac{\partial \mathbf{v}_\parallel(s^i)}{\partial \mathbf{x}_\perp}|+\langle v\rangle|\frac{\partial \mathbf{x}_\parallel(s^i)}{\partial \mathbf{x}_\perp}|+e^{C|v||t-s|}\frac{|v|^2+1}{|\mathbf{v}_\perp^1|}\right) \\
&\le C(\xi)\left(|\frac{\partial \mathbf{v}_\parallel(s^i)}{\partial \mathbf{x}_\perp}|+\langle v\rangle|\frac{\partial \mathbf{x}_\parallel(s^i)}{\partial \mathbf{x}_\perp}|+e^{C|v||t-s|}\frac{|v|^2+1}{|\mathbf{v}_\perp^1|}\right).
\end{aligned} \tag{5.107}$$

Iterating (5.107) we get

$$\begin{aligned}
&\langle v\rangle|\frac{\partial \mathbf{x}_\parallel(s)}{\partial \mathbf{x}_\perp}|+|\frac{\partial \mathbf{v}_\parallel(s)}{\partial \mathbf{x}_\perp}(s)| \\
\le& C^2\left(|\frac{\partial \mathbf{v}_\parallel(s^{i-1})}{\partial \mathbf{x}_\perp}|+\langle v\rangle|\frac{\partial \mathbf{x}_\parallel(s^{i-1})}{\partial \mathbf{x}_\perp}|\right)+(C^2+C)e^{C|v||t-s|}\frac{|v|^2+1}{|\mathbf{v}_\perp^1|} \\
&\vdots
\end{aligned}$$

$$
\begin{aligned}
&\leq C^{[\frac{|t-s||v|}{L_\xi}]}\left(|\frac{\partial \mathbf{v}_\parallel(s^1)}{\partial \mathbf{x}_\perp}|+\langle v\rangle|\frac{\partial \mathbf{x}_\parallel(s^1)}{\partial \mathbf{x}_\perp}|\right)+(C^{[\frac{|t-s||v|}{L_\xi}]}+\cdots+C)e^{C|v||t-s|}\frac{|v|^2+1}{|\mathbf{v}_\perp^1|}\\
&\leq C^{[\frac{|t-s||v|}{L_\xi}]}\left(|\frac{\partial \mathbf{v}_\parallel(s^1)}{\partial \mathbf{x}_\perp}|+\langle v\rangle|\frac{\partial \mathbf{x}_\parallel(s^1)}{\partial \mathbf{x}_\perp}|\right)+C^{2[\frac{|t-s||v|}{L_\xi}]}e^{C|v||t-s|}\frac{|v|^2+1}{|\mathbf{v}_\perp^1|}\\
&\leq C^{C|t-s||v|}\left(|\frac{\partial \mathbf{v}_\parallel(s^1)}{\partial \mathbf{x}_\perp}|+\langle v\rangle|\frac{\partial \mathbf{x}_\parallel(s^1)}{\partial \mathbf{x}_\perp}|+\frac{|v|^2+1}{|\mathbf{v}_\perp^1|}\right)\\
&\leq C^{C|t-s||v|}\frac{\langle v\rangle^2}{|\mathbf{v}_\perp^1|}.
\end{aligned}\tag{5.108}
$$

And by the same argument as (5.106) - (5.108), we get from (5.105) that

$$
\langle v\rangle|\frac{\partial \mathbf{x}_\parallel(s)}{\partial \mathbf{v}_\perp}|+|\frac{\partial \mathbf{v}_\parallel(s)}{\partial \mathbf{v}_\perp}(s)|\leq C^{C|t-s||v|}.\tag{5.109}
$$

Therefore, from (5.108) and (5.109) we get

$$
\begin{bmatrix}|\frac{\partial \mathbf{x}_\parallel(s)}{\partial \mathbf{x}_{\perp 1}}|\\ |\frac{\partial \mathbf{x}_\parallel(s)}{\partial \mathbf{v}_{\perp 1}}|\end{bmatrix}\lesssim C^{C|t-s||v|}\begin{bmatrix}\frac{\langle v\rangle}{|\mathbf{v}_\perp^1|}\\ \frac{1}{\langle v\rangle}\end{bmatrix}.\tag{5.110}
$$

With the estimate (5.110), we refine (5.93) and (5.96) to give a final estimate for the case that some ℓ is *Type I* or *Type II* :

$$
\begin{aligned}
&\frac{\partial(s^{\ell_*},\mathbf{x}_\perp(s^{\ell_*}),\mathbf{x}_\parallel(s^{\ell_*}),\mathbf{v}_\perp(s^{\ell_*}),\mathbf{v}_\parallel(s^{\ell_*}))}{\partial(s^1,\mathbf{x}_\perp(s^1),\mathbf{x}_\parallel(s^1),\mathbf{v}_\perp(s^1),\mathbf{v}_\parallel(s^1))}\\
&\lesssim C^{C|v|(t-s)}\left[\begin{array}{c|cc|cc}
0 & 0 & \mathbf{0}_{1,2} & 0 & \mathbf{0}_{1,2}\\ \hline
\frac{|v|^2+1}{|\mathbf{v}_\perp^1|} & \frac{|v|+1}{|\mathbf{v}_\perp^1|} & \min\{|\mathbf{v}_\perp^1|,\frac{|\mathbf{v}_\perp^1|}{\langle v\rangle}\} & \frac{1}{\langle v\rangle} & \frac{1}{\langle v\rangle}\\
\frac{|v|^3+|v|}{|\mathbf{v}_\perp^1|^2} & \frac{|v|+1}{|\mathbf{v}_\perp^1|} & 1 & \frac{1}{\langle v\rangle} & \frac{1}{\langle v\rangle}\\ \hline
\frac{|v|^4+1}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2} & |v|+1 & \frac{|v|}{|\mathbf{v}_\perp^1|} & O_\xi(1)\\
\frac{|v|^4+1}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^2+1}{|\mathbf{v}_\perp^1|} & |v|+1 & O_\xi(1) & O_\xi(1)
\end{array}\right],
\end{aligned}\tag{5.111}
$$

and from (5.30) and (5.43)

$$
\begin{aligned}
&\frac{\partial(X_{\mathbf{cl}}(s;t,x,v),V_{\mathbf{cl}}(s;t,x,v))}{\partial(t,x,v)}\\
&\lesssim C^{C|v|(t-s)}\frac{\partial(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))}{\partial(s^{\ell_*},\mathbf{X}_{\mathbf{cl}}(s^{\ell_*}),\mathbf{V}_{\mathbf{cl}}(s^{\ell_*}))}\left[\begin{array}{c|c|c}
0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3}\\ \hline
\frac{|v|^3+|v|}{|\mathbf{v}_\perp^1|^2} & \frac{|v|+1}{|\mathbf{v}_\perp^1|} & \frac{1}{\langle v\rangle}\\ \hline
\frac{|v|^4+1}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2} & \frac{|v|+1}{|\mathbf{v}_\perp^1|}
\end{array}\right]\\
&\qquad\times\frac{\partial(s^1,\mathbf{x}_\perp(s^1),\mathbf{x}_\parallel(s^1),\mathbf{v}_\perp(s^1),\mathbf{v}_\parallel(s^1))}{\partial(t,x,v)}\\
&\lesssim C^{C|v|(t-s)}\begin{bmatrix}|v|+|s^{\ell_*}-s| & 1 & |s^{\ell_*}-s|\\ 1 & |v| & 1\end{bmatrix}\left[\begin{array}{c|c|c}
0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3}\\ \hline
\frac{|v|^3+|v|}{|\mathbf{v}_\perp^1|^2} & \frac{|v|+1}{|\mathbf{v}_\perp^1|} & \frac{1}{\langle v\rangle}\\ \hline
\frac{|v|^4+1}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2} & \frac{|v|+1}{|\mathbf{v}_\perp^1|}
\end{array}\right]\begin{bmatrix}1 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3}\\ |t-s^1|^2 & 1 & |t-s^1|\\ |t-s^1| & |v| & 1\end{bmatrix}\\
&\lesssim C^{C|v|(t-s)}\left[\begin{array}{c|c|c}
\frac{|v|^3+|v|}{|\mathbf{v}_\perp^1|^2} & \frac{|v|+1}{|\mathbf{v}_\perp^1|} & \frac{1}{\langle v\rangle}\\ \hline
\frac{|v|^4+1}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^3+1}{|\mathbf{v}_\perp^1|^2} & \frac{|v|+1}{|\mathbf{v}_\perp^1|}
\end{array}\right].
\end{aligned}\tag{5.112}
$$

Finally from (5.95) and (5.112) we conclude, for all $\tau \in [s,t]$

$$\frac{\partial(X_{\mathbf{cl}}(s;t,x,v),V_{\mathbf{cl}}(s;t,x,v))}{\partial(t,x,v)} \leq Ce^{C|v|(t-s)} \left[\begin{array}{c|c|c} \frac{|v|^3+|v|}{|\mathbf{v}_\perp^1|^2} & \frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{1}{\langle v\rangle} \\ \hline \frac{|v|^4+1}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|}{|\mathbf{v}_\perp^1|} \end{array}\right]_{6\times 7}.$$

From the Velocity lemma (Lemma 3.1),

$$\begin{aligned} |\mathbf{v}_\perp^1| &= |v^1\cdot[-n(x^1)]| = |V_{\mathbf{cl}}(t^1;t,x,v)\cdot n(X_{\mathbf{cl}}(t^1;t,x,v))| \\ &= \sqrt{\alpha(X_{\mathbf{cl}}(t^1),V_{\mathbf{cl}}(t^1))} \geq e^{\mathcal{C}|v||t-t^1|}\alpha(t,x,v) \gtrsim \alpha(t,x,v), \end{aligned}$$

and this completes the proof. ∎

6. Weighted C^1 estimate

In this section, we put together all the results we got in previous sections and prove our main theorem.

Proof. **(Proof of Theorem 1.1.)** We use the approximation sequence (2.5) with (2.6). Due to (2.7) we have

$$\sup_m \sup_{0\leq t\leq T} ||e^{\theta|v|^2}f^m(t)||_\infty \lesssim_{\xi,T} P(||e^{\theta'|v|^2}f_0||_\infty).$$

Now we claim that the distributional derivatives coincide with the piecewise derivatives. This is due to Proposition 2.1 with an invariant property of $\Gamma(f,f) = \Gamma_{\text{gain}}(f,f) - \nu(\sqrt{\mu}f)f$: *Assume $f^m(v) = f^{m-1}(\mathcal{O}v)$ holds for some orthonormal matrix. Then*

$$\Gamma(f^m,f^m)(v) = \Gamma(f^{m-1},f^{m-1})(\mathcal{O}v). \tag{6.1}$$

Denote

$$\begin{aligned} \nu^{m-\ell}(s) &:= \nu^{m-\ell}(s,X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s)) \\ &:= \nu(\sqrt{\mu}f^{m-\ell})(s,X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s)) - \frac{V_{\mathbf{cl}}(s))}{2}\cdot E(s,X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s)). \end{aligned} \tag{6.2}$$

Using (6.1), we apply Proposition 2.1 to have

$$\begin{aligned} &f^m(t,x,v) \\ &= e^{-\int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^\ell)}(s)\nu^{m-\ell}(s)\mathrm{d}s} f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0)) \\ &+ \int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^\ell)}(s) e^{-\int_s^t \sum_{j=0}^{\ell_*(s)} \mathbf{1}_{[t^{j+1},t^j)}\nu^{m-j}\mathrm{d}\tau} \Gamma_{\text{gain}}(f^{m-\ell},f^{m-\ell})(s,X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))\mathrm{d}s. \end{aligned}$$

Now we consider the spatial and velocity derivatives. In the sense of distributions, we have for $\partial_{\mathbf{e}} \in \{\nabla_x,\nabla_v\}$

$$\partial_{\mathbf{e}} f^m(t,x,v) = \mathrm{I}_{\mathbf{e}} + \mathrm{II}_{\mathbf{e}} + \mathrm{III}_{\mathbf{e}}. \tag{6.3}$$

Here

$$\mathrm{I}_{\mathbf{e}} = e^{-\int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^\ell)}(s)\nu^{m-\ell}(s)\mathrm{d}s}\, \partial_{\mathbf{e}}[X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0)]\cdot\nabla_{x,v} f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0)),$$

and

$$\begin{aligned}\mathrm{II}_{\mathbf{e}} = &\int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^{\ell})}(s) e^{-\int_s^t \sum_{j=0}^{\ell_*(s)} \mathbf{1}_{[t^{j+1},t^j)}(\tau)\nu^{m-j}(\tau)\mathrm{d}\tau} \\
&\times \partial_{\mathbf{e}}\big[\Gamma_{\text{gain}}(f^{m-\ell},f^{m-\ell})(s,X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))\big]\mathrm{d}s \\
&- \int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^{\ell})}(s) e^{-\int_s^t \sum_j \mathbf{1}_{[t^{j+1},t^j)}(\tau)\nu^{m-j}(\tau)\mathrm{d}\tau} \\
&\times \int_s^t \sum_{j=0}^{\ell_*(s)} \mathbf{1}_{[t^{j+1},t^j)}(\tau)\partial_{\mathbf{e}}[\nu^{m-j}(\tau,X_{\mathbf{cl}}(\tau),V_{\mathbf{cl}}(\tau))]\mathrm{d}\tau \\
&\times \Gamma_{\text{gain}}(f^{m-\ell},f^{m-\ell})(s,X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))\mathrm{d}s - e^{-\int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^{\ell})}(s)\nu^{m-\ell}(s)\mathrm{d}s} \\
&\times f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0)) \int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^{\ell})}(s)\partial_{\mathbf{e}}\big[\nu^{m-\ell}(s,X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))\big]\mathrm{d}s,\end{aligned}$$

and

$$\begin{aligned}&\mathbf{III}_{\mathbf{e}} \\
&= \sum_{\ell=0}^{\ell_*(0)} \big[-\partial_{\mathbf{e}} t^{\ell} \lim_{s\uparrow t^{\ell}} \nu^{m-\ell}(s) + \partial_{\mathbf{e}} t^{\ell+1} \lim_{s\downarrow t^{\ell+1}} \nu^{m-\ell}(s)\big] \times e^{-\int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^{\ell})}(s)\nu^{m-\ell}(s)} \\
&\quad + \sum_{\ell=0}^{\ell_*(0)} \Big[\lim_{s\uparrow t^{\ell}} e^{-\int_s^t \sum_j \mathbf{1}_{[t^{j+1},t^j)}(\tau)\nu^{m-j}(\tau)\mathrm{d}\tau}\Gamma_{\text{gain}}(f^{m-\ell},f^{m-\ell})(s,X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s)) \\
&\quad - \lim_{s\downarrow t^{\ell+1}} e^{-\int_s^t \sum_j \mathbf{1}_{[t^{j+1},t^j)}(\tau)\nu^{m-j}(\tau)\mathrm{d}\tau}\Gamma_{\text{gain}}(f^{m-\ell},f^{m-\ell})(s,X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s)) \\
&\quad + \int_0^t \sum_{\ell} \mathbf{1}_{[t^{\ell+1},t^{\ell})}(s) \sum_{j=0}^{\ell_*(s)} \big[-\lim_{\tau\downarrow t^j} \nu^{m-j}(\tau,X_{\mathbf{cl}}(\tau),V_{\mathbf{cl}}(\tau)) + \lim_{\tau\uparrow t^{j+1}} \nu^{m-j}(\tau,X_{\mathbf{cl}}(\tau),V_{\mathbf{cl}}(\tau))\big] \\
&\quad \times e^{-\int_s^t \sum_{j=0}^{\ell_*(s)} \mathbf{1}_{[t^{j+1},t^j)}(\tau)\nu^{m-j}(\tau)\mathrm{d}\tau}\Gamma_{\text{gain}}(f^{m-\ell},f^{m-\ell})(s,X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s)).\end{aligned}$$

For $\mathbf{III}_{\mathbf{e}}$ we rearrange the summation and use (5.22), (6.2) and apply (6.1) to get

$$\begin{aligned}&\mathbf{III}_{\mathbf{e}} \\
&= \sum_{\ell=0}^{\ell_*(0)} \Big[-\nu^{m-\ell}(t^{\ell},x^{\ell},v^{\ell}) + \nu^{m-\ell+1}(t^{\ell},x^{\ell},R_{x^{\ell}}v^{\ell})\Big]\partial_{\mathbf{e}} t^{\ell} e^{-\int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^{\ell})}(s)\nu^{m-\ell}(s)} \\
&\quad + \sum_{\ell=0}^{\ell_*(0)} e^{-\int_{t^{\ell}}^t \sum_j \mathbf{1}_{[t^{j+1},t^j)}(\tau)\nu(\sqrt{\mu}f^{m-j})(\tau)\mathrm{d}\tau} \\
&\quad \times \Big[\Gamma_{\text{gain}}(f^{m-\ell},f^{m-\ell})(t^{\ell},x^{\ell},v^{\ell}) - \Gamma_{\text{gain}}(f^{m-\ell+1},f^{m-\ell+1})(t^{\ell},x^{\ell},R_{x^{\ell}}v^{\ell})\Big] \\
&\quad + \int_0^t \sum_{\ell} \mathbf{1}_{[t^{\ell+1},t^{\ell})}(s)^{-\int_s^t \sum_{j=0}^{\ell_*(s)} \mathbf{1}_{[t^{j+1},t^j)}(\tau)\nu^{m-j}(\tau)\mathrm{d}\tau}\Gamma_{\text{gain}}(f^{m-\ell},f^{m-\ell})(s,X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s)) \\
&\quad \times \sum_{\ell=0}^{\ell_*(s)} \Big[-\nu^{m-\ell}(t^{\ell},x^{\ell},v^{\ell}) + \nu^{m-\ell+1}(t^{\ell},x^{\ell},R_{x^{\ell}}v^{\ell})\Big] \\
&= \sum_{\ell=0}^{\ell_*(0)} \left[\frac{R_{x^{\ell}}v^{\ell} - v^{\ell}}{2} \cdot E(t^{\ell},x^{\ell})\right] \partial_{\mathbf{e}} t^{\ell} e^{-\int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^{\ell})}(s)\nu^{m-\ell}(s)}\end{aligned}$$

$$+\int_0^t \sum_\ell \mathbf{1}_{[t^{\ell+1},t^\ell)}(s)^{-\int_s^t \sum_{j=0}^{\ell_*(s)} \mathbf{1}_{[t^{j+1},t^j)}(\tau)\nu^{m-j}(\tau)\mathrm{d}\tau}\Gamma_{\mathrm{gain}}(f^{m-\ell},f^{m-\ell})(s,X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))$$
$$\times \sum_{\ell=0}^{\ell_*(s)}\Big[\frac{R_{x^\ell}v^\ell - v^\ell}{2}\cdot E(t^\ell,x^\ell)\Big]. \tag{6.4}$$

Proof. **(Proof of (6.1).)** The proof is due to the change of variables

$$\tilde u = \mathcal{O}u,\quad \tilde\omega=\mathcal{O}\omega,\quad \mathrm{d}\tilde u=\mathrm{d}u,\quad \mathrm{d}\tilde\omega = \mathrm{d}\omega.$$

Note

$$\begin{aligned}
&\Gamma(f^m,f^m)(v)\\
&=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-u|^\kappa q_0(\frac{v-u}{|v-u|}\cdot\omega)\sqrt{\mu(u)}\\
&\qquad\times\Big\{f^m(u-[(u-v)\cdot\omega]\omega)f^m(v+[(u-v)\cdot\omega]\omega)-f^m(u)f^m(v)\Big\}\mathrm{d}\omega\mathrm{d}u\\
&=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|\mathcal{O}v-\mathcal{O}u|^\kappa q_0(\frac{\mathcal{O}v-\mathcal{O}u}{|\mathcal{O}v-\mathcal{O}u|}\cdot\mathcal{O}\omega)\sqrt{\mu(\mathcal{O}u)}\\
&\qquad\times\Big\{f^{m-1}(\mathcal{O}u-[(\mathcal{O}u-\mathcal{O}v)\cdot\mathcal{O}\omega]\mathcal{O}\omega)f^{m-1}(\mathcal{O}v+[(\mathcal{O}u-\mathcal{O}v)\cdot\mathcal{O}\omega]\mathcal{O}\omega)\\
&\qquad-f^{m-1}(\mathcal{O}u)f^{m-1}(\mathcal{O}v)\Big\}\mathrm{d}\omega\mathrm{d}u\\
&=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|\mathcal{O}v-\tilde u|^\kappa q_0(\frac{\mathcal{O}v-\tilde u}{|\mathcal{O}v-\tilde u|}\cdot\tilde\omega)\sqrt{\mu(\tilde u)}\\
&\qquad\times\Big\{f^{m-1}(\tilde u-[(\tilde u-\mathcal{O}v)\cdot\tilde\omega]\tilde\omega)f^{m-1}(\mathcal{O}v+[(\tilde u-\mathcal{O}v)]\cdot\tilde\omega)\tilde\omega\\
&\qquad-f^{m-1}(\tilde u)f^{m-1}(\mathcal{O}v)\Big\}\mathrm{d}\tilde\omega\mathrm{d}\tilde u\\
&=\Gamma(f^{m-1},f^{m-1})(\mathcal{O}v).
\end{aligned}$$

This proves (6.1). Especially we can apply (6.1) for the specular reflection BC (2.4) with $\mathcal{O}v=R_xv$. ∎

Using Lemma 2.2 and (2.7), we obtain for $\partial_{\mathbf{e}}\in\{\nabla_x,\nabla_v\}$

$$\begin{aligned}
\mathrm{II}_{\mathbf{e}}\lesssim{}& P(\|e^{\theta|v|^2}f_0\|_\infty)\int_0^t\sum_{\ell=0}^{\ell_*(0)}\mathbf{1}_{[t^{\ell+1},t^\ell)}(s)|\partial_{\mathbf{e}}X_{\mathbf{cl}}(s)|\\
&\times\int_{\mathbb{R}^3}\frac{e^{-C_\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}|\nabla_x f^{m-\ell}(s,X_{\mathbf{cl}}(s),u)|\mathrm{d}u\mathrm{d}s\\
&+P(\|e^{\theta|v|^2}f_0\|_\infty)\int_0^t\sum_{\ell=0}^{\ell_*(0)}\mathbf{1}_{[t^{\ell+1},t^\ell)}(s)|\partial_{\mathbf{e}}V_{\mathbf{cl}}(s)|\\
&\times\int_{\mathbb{R}^3}\frac{e^{-C_\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}|\nabla_v f^{m-\ell}(s,X_{\mathbf{cl}}(s),u)|\mathrm{d}u\mathrm{d}s\\
&+tP(\|e^{\theta|v|^2}f_0\|_\infty)\langle v\rangle^\kappa e^{-\theta|v|^2}\Big(\|E\|_{L^\infty_{t,x}}+\|\nabla_xE\|_{L^\infty_{t,x}}\Big)\\
&\times\Big(\sup_{0\le s\le t}|\partial_{\mathbf{e}}V(s;t,x,v)|+\sup_{0\le s\le t}|\partial_{\mathbf{e}}X(s;t,x,v)|\Big).
\end{aligned}$$

We shall estimate the following:

$$e^{-\varpi\langle v\rangle t}\frac{[\alpha(t,x,v)]^{\beta}}{\langle v\rangle^{b+1}}|\partial_x f(t,x,v)|,\quad e^{-\varpi\langle v\rangle t}\frac{[\alpha(t,x,v)]^{\beta-1}}{\langle v\rangle^{b-1}}|\partial_v f(t,x,v)|.$$

From (5.1), the Velocity lemma (Lemma 3.1), Lemma 2.1, and $F^m\geq 0$ for all m, with $\varpi\gg 1$

$$\begin{aligned}
&e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{b+1}}[\alpha(t,x,v)]^{\beta}\ \mathrm{I}_{\mathbf{x}}\\
&\lesssim_{\xi,t} e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{b+1}}[\alpha(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))]^{\beta}e^{2\mathcal{C}|v|t}\\
&\quad\times\Big\{\frac{\langle v\rangle}{\alpha(t,x,v)}|\partial_x f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))|+\frac{\langle v\rangle^3}{\alpha^2(t,x,v)}|\partial_v f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))|\Big\}\\
&\lesssim_{\xi,t}||\ \frac{\langle v\rangle}{\langle v\rangle^{b+1}}\alpha^{\beta-1}\partial_x f_0\ ||_\infty+||\ \frac{\langle v\rangle^3}{\langle v\rangle^{b+1}}\alpha^{\beta-2}\partial_v f_0\ ||_\infty\\
&\lesssim_{\xi,t}||\ \frac{\alpha^{\beta-1}}{\langle v\rangle^{b}}\partial_x f_0\ ||_\infty+||\ \frac{\alpha^{\beta-2}}{\langle v\rangle^{b-2}}\partial_v f_0\ ||_\infty,
\end{aligned}$$

and

$$\begin{aligned}
&e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{b-1}}[\alpha(t,x,v)]^{\beta-1}\ \mathrm{I}_{\mathbf{v}}\\
&\lesssim_{\xi,t}\ e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{b-1}}[\alpha(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))]^{\beta-1}e^{2\mathcal{C}|v|t}\\
&\quad\times\Big\{\frac{1}{\langle v\rangle}|\partial_x f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))|+\frac{\langle v\rangle}{\alpha(t,x,v)}|\partial_v f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))|\Big\}\\
&\lesssim_{\xi,t}\ ||\ \frac{\alpha^{\beta-1}}{\langle v\rangle^{b}}\partial_x f_0\ ||_\infty+||\ \frac{1}{\langle v\rangle^{b-2}}\alpha^{\beta-2}\partial_v f_0\ ||_\infty,
\end{aligned}$$

where we have used $\alpha(t,x,v)\lesssim_\xi |v|^2$ and the choice of $\varpi\gg 1$.

From Lemma 2.1 and Lemma 2.2,

$$\begin{aligned}
\mathrm{II}_{\mathbf{e}}\lesssim_t P(||e^{\theta|v|^2}f_0||_\infty)\int_0^t \mathrm{d}s\sum_{\ell=0}^{\ell_*(0)}\mathbf{1}_{[t^{\ell+1},t^{\ell})}(s)\int_{\mathbb{R}^3}\mathrm{d}u\frac{e^{-C_\theta|u-V_{\mathbf{cl}}(s)|^2}}{|u-V_{\mathbf{cl}}(s)|^{2-\kappa}}\\
\times\Big\{|\partial_{\mathbf{e}}X_{\mathbf{cl}}(s)||\partial_x f^{m-j}(s,X_{\mathbf{cl}}(s),u)|+|\partial_{\mathbf{e}}V_{\mathbf{cl}}(s)|\big(1+|\partial_v f^{m-j}(s,X_{\mathbf{cl}}(s),u)|\big)\Big\}.
\end{aligned}$$

Now we use (5.1) to have

$$\begin{aligned}
&e^{-\varpi\langle v\rangle t}\frac{[\alpha(t,x,v)]^{\beta}}{\langle v\rangle^{b+1}}\ \mathrm{II}_{\mathbf{x}}\lesssim_{t,\xi}P(||e^{\theta|v|^2}f_0||_\infty)\\
&quad\times\Big\{\int_0^t\int_{\mathbb{R}^3}\frac{e^{-C_\theta|V_{\mathbf{cl}}(s)-u|^2}}{|u-V_{\mathbf{cl}}(s)|^{2-\kappa}}e^{-\varpi\langle v\rangle t}e^{\varpi\langle u\rangle s}e^{C|v||t-s|}\frac{|v|[\alpha(t,x,v)]^{\beta-\frac12}}{[\alpha(s,X_{\mathbf{cl}}(s),u)]^{\beta}}\frac{\langle u\rangle^{b+1}}{\langle v\rangle^{b+1}}\mathrm{d}u\mathrm{d}s\\
&\quad\times\sup_m\sup_{0\leq s\leq t}||e^{-\varpi\langle u\rangle s}\frac{[\alpha(s,X_{\mathbf{cl}}(s),u)]^{\beta}}{\langle u\rangle^{b+1}}\partial_x f^{m-j}(s,X_{\mathbf{cl}}(s),u)||_\infty\\
&\quad+\int_0^t\int_{\mathbb{R}^3}\frac{e^{-C_\theta|V_{\mathbf{cl}}(s)-u|^2}}{|u-V_{\mathbf{cl}}(s)|^{2-\kappa}}e^{-\varpi\langle v\rangle t}e^{\varpi\langle u\rangle s}e^{C|v||t-s|}\frac{\langle u\rangle^b}{\langle v\rangle^b}\frac{|v|^2[\alpha(t,x,v)]^{\beta-1}}{|u|[\alpha(s,X_{\mathbf{cl}}(s),u)]^{\beta-\frac12}}\\
&\qquad\times\sup_m\sup_{0\leq s\leq t}||e^{-\varpi\langle u\rangle s}\frac{|u|[\alpha(s,X_{\mathbf{cl}}(s),u)]^{\beta-\frac12}}{\langle u\rangle^{b}}\partial_v f^{m-j}(s,X_{\mathbf{cl}}(s),u)||_\infty\Big\}.
\end{aligned}$$

We first claim that

$$e^{-\varpi\langle v\rangle t}e^{\varpi\langle u\rangle s}e^{C|v|(t-s)}e^{-C'|v-u|^2}\lesssim e^{-\frac{\varpi\langle v\rangle}{2}(t-s)}e^{C''(s+s^2)}e^{-C''|v-u|^2}. \tag{6.5}$$

Using $\langle u\rangle\leq 1+|u|\leq 1+|v|+|u-v|\leq 1+\langle v\rangle+|v-u|$, we bound the first three exponents as

$$-(\varpi-C)\langle v\rangle(t-s)-\varpi(\langle v\rangle-\langle u\rangle)s\leq-(\varpi-C)\langle v\rangle(t-s)+\varpi|v-u|s+\varpi s.$$

Then we use a complete square trick, for $0<\sigma\ll 1$

$$\varpi|v-u|s=\frac{\sigma\varpi^2}{2}|v-u|^2+\frac{s^2}{2\sigma}-\frac{1}{2\sigma}\big[s-\sigma\varpi|v-u|\big]^2\leq\frac{\sigma\varpi^2}{2}|v-u|^2+\frac{s^2}{2\sigma},$$

to bound the whole exponents of (6.5) by

$$\begin{aligned}&-(\varpi-C)\langle v\rangle(t-s)+\varpi|v-u|s-C'|v-u|^2+\varpi s\\ &\leq-(\varpi-C)\langle v\rangle(t-s)-(C-\frac{\sigma\varpi^2}{2})|v-u|^2+\frac{s^2}{2\sigma}+\varpi s\\ &\leq-(\varpi-C)\langle v\rangle(t-s)-C_{\sigma,\varpi}|v-u|^2+C'_{\sigma,\varpi}\{s^2+s\}.\end{aligned}$$

Hence we prove the claim (6.5) for $\varpi\gg 1$.

Now we use (6.5) to bound

$$\begin{aligned}&e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{b+1}}[\alpha(t,x,v)]^{\beta}\ \mathrm{II}_{\mathbf{x}}\\ &\lesssim_{t,\xi}P(||e^{\theta|v|^2}f_0||_\infty)\\ &\quad\times\Big\{\underbrace{\int_0^t\int_{\mathbb{R}^3}e^{-\frac{\varpi\langle v\rangle}{2}(t-s)}\frac{e^{-C'_\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}\frac{\langle u\rangle^{b+1}}{\langle v\rangle^{b+1}}\frac{\langle v\rangle[\alpha(t,x,v)]^{\beta-1}}{[\alpha(s,X_{\mathbf{cl}}(s),u)]^{\beta}}\mathrm{d}u\mathrm{d}s}_{(\mathbf{A})}\\ &\quad\times\sup_m\sup_{0\leq s\leq t}\Big|\Big|e^{-\varpi\langle v\rangle s}\frac{\alpha^\beta}{\langle v\rangle^{b+1}}\partial_x f^m(s)\Big|\Big|_\infty\\ &\quad+\underbrace{\int_0^t\int_{\mathbb{R}^3}e^{-\frac{\varpi\langle v\rangle}{2}(t-s)}\frac{e^{-C'_\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}\frac{\langle u\rangle^{b-1}}{\langle v\rangle^{b+1}}\frac{\langle v\rangle^3[\alpha(t,x,v)]^{\beta-2}}{[\alpha(s,X_{\mathbf{cl}}(s),u)]^{\beta-1}}\mathrm{d}u\mathrm{d}s}_{(\mathbf{B})}\\ &\quad\times\sup_m\sup_{0\leq s\leq t}\Big|\Big|e^{-\varpi\langle v\rangle s}\frac{\alpha^{\beta-1}}{\langle v\rangle^{b-1}}\partial_v f^m(s)\Big|\Big|_\infty\Big\}.\end{aligned} \tag{6.6}$$

For (**A**) we use (3.7) with $Z=\langle v\rangle\big[\alpha(t,x,v)\big]^{\beta-1}$ and $l=\frac{\varpi}{2}$ and $r=b+1$. For (**B**) we use (3.7) with $\beta\mapsto\beta-1$ and $Z=\langle v\rangle\big[\alpha(t,x,v)\big]^{\beta-2}$ and $l=\frac{\varpi}{2}$ and $r=b-1$. Then

$$(\mathbf{A}),\ (\mathbf{B})\ \ll\ 1.$$

Similarly, but with a different weight $e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{b-1}}[\alpha(t,x,v)]^{\beta-1}$, we use (5.1) to have

$$
\begin{aligned}
&e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{b-1}}[\alpha(t,x,v)]^{\beta-1}\ \mathrm{II}_{\mathbf{v}}\\
&\lesssim_{t,\xi} P(||e^{\theta|v|^2}f_0||_\infty)\\
&\quad\times\Big\{\int_0^t\int_{\mathbb{R}^3}\frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|u-V_{\mathbf{cl}}(s)|^{2-\kappa}}e^{-\varpi\langle v\rangle t}e^{\varpi\langle u\rangle s}e^{C|v||t-s|}\frac{[\alpha(t,x,v)]^{\beta-1}}{[\alpha(s,X_{\mathbf{cl}}(s),u)]^{\beta}}\frac{\langle u\rangle^{b+1}}{\langle v\rangle^{b}}\mathrm{d}u\mathrm{d}s\\
&\quad\times\sup_m\sup_{0\le s\le t}||e^{-\varpi\langle u\rangle s}\frac{[\alpha(s,X_{\mathbf{cl}}(s),u)]^{\beta}}{\langle u\rangle^{b+1}}\partial_x f^m(s,X_{\mathbf{cl}}(s),u)||_\infty\\
&\quad+\int_0^t\int_{\mathbb{R}^3}\frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|u-V_{\mathbf{cl}}(s)|^{2-\kappa}}e^{-\varpi\langle v\rangle t}e^{\varpi\langle u\rangle s}e^{C|v||t-s|}\frac{\langle u\rangle^{b-1}}{\langle v\rangle^{b-1}}\frac{\langle v\rangle[\alpha(t,x,v)]^{\beta-2}}{[\alpha(s,X_{\mathbf{cl}}(s),u)]^{\beta-1}}\\
&\quad\times\sup_m\sup_{0\le s\le t}||e^{-\varpi\langle u\rangle s}\frac{[\alpha(s,X_{\mathbf{cl}}(s),u)]^{\beta-1}}{\langle u\rangle^{b-1}}\partial_v f^m(s,X_{\mathbf{cl}}(s),u)||_\infty\Big\}.
\end{aligned}
$$

Again we use (6.5) and (3.7) exactly as (6.6). Therefore for $0<\delta=\delta(||e^{\theta|v|^2}f_0||_\infty)\ll 1$

$$
\begin{aligned}
&e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{b+1}}[\alpha(t,x,v)]^{\beta}\mathrm{II}_{\mathbf{x}}+e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^{b}}[\alpha(t,x,v)]^{\beta-1}\mathrm{II}_{\mathbf{v}}\\
&\lesssim\delta\ \{\sup_m\sup_{0\le s\le t}||e^{-\varpi\langle v\rangle s}\frac{\alpha^\beta}{\langle v\rangle^{b+1}}\partial_x f^m(s)||_\infty+\sup_m\sup_{0\le s\le t}||e^{-\varpi\langle v\rangle s}\frac{\alpha^{\beta-1}}{\langle v\rangle^{b-1}}\partial_v f^m(s)||_\infty\}.
\end{aligned}
$$

Finally using $\frac{R_{x^\ell}v^\ell-v^\ell}{2}=\mathbf{v}_\perp^\ell$, the bound on $\partial_{\mathbf{e}}t^\ell$ in (5.97) and (5.98), from (5.1), the Velocity lemma (Lemma 3.1)

$$
\begin{aligned}
e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{b+1}}[\alpha(t,x,v)]^{\beta}\ \mathrm{III}_{\mathbf{x}}&\lesssim e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{b+1}}[\alpha(t,x,v)]^{\beta}\|E\|_{L^\infty_{t,x}}e^{C\langle v\rangle t}\alpha(t,x,v)\ell_*(0)\\
&\quad\times\sup_{0\le\ell\le\ell_*(0)}|\partial_x t^\ell|+tP(||e^{\theta|v|^2}f_0||_\infty)\|E\|_{L^\infty_{t,x}}\\
&\lesssim\|E\|_{L^\infty_{t,x}}\alpha(t,x.v)^{\beta-2}+tP(||e^{\theta|v|^2}f_0||_\infty)\|E\|_{L^\infty_{t,x}},
\end{aligned}
$$

for $\varpi\gg 1$. And similarly

$$
\begin{aligned}
e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{b-1}}[\alpha(t,x,v)]^{\beta-1}\ \mathrm{III}_{\mathbf{v}}&\lesssim e^{-\varpi\langle v\rangle t}\tfrac{1}{\langle v\rangle^{b-1}}[\alpha(t,x,v)]^{\beta-1}\|E\|_{L^\infty_{t,x}}e^{C\langle v\rangle t}\alpha(t,x,v)\ell_*(0)\\
&\quad\times\sup_{0\le\ell\le\ell_*(0)}|\partial_v t^\ell|+tP(||e^{\theta|v|^2}f_0||_\infty)\|E\|_{L^\infty_{t,x}}\\
&\lesssim\|E\|_{L^\infty_{t,x}}\alpha(t,x.v)^{\beta-2}+tP(||e^{\theta|v|^2}f_0||_\infty)\|E\|_{L^\infty_{t,x}}.
\end{aligned}
$$

Collecting all the terms, for $2<\beta<3$ and $b>1$ with $\varpi\gg 1$ and $0<\delta\ll 1$, we get

$$
\begin{aligned}
&\sup_m\sup_{0\le s\le t}||e^{-\varpi\langle v\rangle s}\frac{\alpha^\beta}{\langle v\rangle^{b+1}}\partial_x f^m(s)||_\infty+\sup_m\sup_{0\le s\le t}||e^{-\varpi\langle v\rangle s}\frac{\alpha^{\beta-1}}{\langle v\rangle^{b-1}}\partial_v f^m(s)||_\infty\\
&\lesssim||\frac{\alpha^{\beta-1}}{\langle v\rangle^{b}}\partial_x f_0||_\infty+||\frac{\alpha^{\beta-2}}{\langle v\rangle^{b-2}}\partial_v f_0||_\infty+P(||e^{\theta|v|^2}f_0||_\infty).
\end{aligned}
$$

We remark that this sequence f^m is Cauchy in $L^\infty([0,T]\times\bar{\Omega}\times\mathbb{R}^3)$ for $0<T\ll 1$. Therefore the limit function f is a solution of the Boltzmann equation satisfying the

specular reflection BC. On the other hand, due to the weak lower semi-continuity of L^p, $p>1$, we pass a limit $\partial f^m \rightharpoonup \partial f$ weakly in the weighted $L^\infty-$norm.

Now we consider the continuity of $e^{-\varpi\langle v\rangle t}\langle v\rangle^{-b-1}\alpha^\beta\partial_x f$ and $e^{-\varpi\langle v\rangle t}\langle v\rangle^{-b+1}\alpha^{\beta-1}\partial_v f$. Remark that $e^{-\varpi\langle v\rangle t}\lambda anglev\rangle^{-b-1}\alpha^\beta\partial_x f^m$ and $e^{-\varpi\langle v\rangle t}\langle v\rangle^{-b+1}\alpha^{\beta-1}\partial_v f^m$ satisfy all the conditions of Proposition 2.1. Therefore we conclude

$$e^{-\varpi\langle v\rangle t}\langle v\rangle^{-b-1}\alpha^\beta\partial_x f^m \in C^0([0,T]\times\bar{\Omega}\times\mathbb{R}^3),$$
$$e^{-\varpi\langle v\rangle t}\langle v\rangle^{-b+1}\alpha^{\beta-1}\partial_v f^m \in C^0([0,T]\times\bar{\Omega}\times\mathbb{R}^3).$$

Now we follow $W^{1,\infty}$ estimate proof for $e^{-\varpi\langle v\rangle t}\langle v\rangle^{-b-1}\alpha^\beta[\partial_x f^{m+1}-\partial_x f^m]$ and $e^{-\varpi\langle v\rangle t}\langle v\rangle^{-b+1}\alpha^{\beta-1}[\partial_v f^{m+1}-\partial f^m]$ to show that $e^{-\varpi\langle v\rangle t}\langle v\rangle^{-b-1}\alpha^\beta\partial_x f^m$ and $e^{-\varpi\langle v\rangle t}\langle v\rangle^{-b+1}\alpha^{\beta-1}\partial_v f^m$ are Cauchy in L^∞. Then we pass a limit $e^{-\varpi\langle v\rangle t}\langle v\rangle^{-b-1}\alpha^\beta\partial_x f^m \to e^{-\varpi\langle v\rangle t}\langle v\rangle^{-b-1}\alpha^\beta\partial_x f$ and $e^{-\varpi\langle v\rangle t}\langle v\rangle^{-b+1}\alpha^{\beta-1}\partial_v f^m \to e^{-\varpi\langle v\rangle t}\langle v\rangle^{-b+1}\alpha^{\beta-1}\partial_v f$ strongly in L^∞ so that $e^{-\varpi\langle v\rangle t}\langle v\rangle^{-b-1}\alpha^\beta\partial_x f \in C^0([0,T^*]\times\bar{\Omega}\times\mathbb{R}^3)$ and $e^{-\varpi\langle v\rangle t}\langle v\rangle^{-b+1}\alpha^{\beta-1}\partial_v f \in C^0([0,T^*]\times\bar{\Omega}\times\mathbb{R}^3)$. ∎

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