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LARGE, MODERATE DEVIATIONS PRINCIPLE AND α-LIMIT FOR THE 2D STOCHASTIC LANS-α*

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Abstract. In this paper we consider the Lagrangian Averaged Navier-Stokes Equations, also known as, LANS-α Navier-Stokes model on the two dimensional torus. We assume that the noise is a cylindrical Wiener process and its coefficient is multiplied by $\sqrt{\alpha}$. We then study through the lenses of the large and moderate deviations principles the behaviour of the trajectories of the solutions of the stochastic system as α goes to 0. Instead of giving two separate proofs of the two deviations principles we present a unifying approach to the proof of the LDP and MDP and express the rate function in term of the unique solution of the Navier-Stokes equations. Our proof is based on the weak convergence approach to large deviations principle. As a by-product of our analysis we also prove that the solutions of the stochastic LANS-α model converge in probability to the solutions of the deterministic Navier-Stokes equations.

Keywords. LANS-α model; Camassa-Holm equations; large deviation principle; Stochastic Navier-Stokes Equations.

AMS subject classifications. 35R60; 60F10; 76D05.

1. Introduction

The Navier-Stokes system is the most used model in turbulence theory, but its numerical simulation is computationally expensive. In order to overcome this issue, regularisation models of Navier-Stokes equations such as the Navier-Stokes-α (also known as Lans-α), Leray-α, modified Leray-α, Clark-α to name a few, were introduced as subgrid scale models of the Navier-Stokes equations (NSEs) in recent years. See, for instance, [6, 7, 9–11, 20]. Numerical analyses in [12, 21, 23, 25, 28, 29, 32, 33] seem to confirm that the previous examples of α-models can capture remarkably well the physical phenomenon of turbulence in fluid flows at a lower computational cost. It is worth mentioning that while many of the regularisations of the NSEs mentioned above do not satisfy Kelvin's circulation theorem, the LANS-α, for $\alpha > 0$, regularisation model does. This particularity overcomes some of the physical limitations present in the other regularisations. Furthermore, the derivation of the LANS-α was based on substituting in Hamilton's principle the decomposition of the Lagrangian fluid-parcel trajectory into its mean and fluctuating components. This was followed by truncating a Taylor series approximation and averaging at constant Lagrangian coordinate, before taking variations. This derivation has more "physical" flavour than the derivations of other regularisation models. We refer to [10] for more detail on the derivation of the LANS-α.

Another tool used to tackle the closure problem in turbulent flows is to introduce a stochastic forcing. These forcings were introduced in order to better understand the situation of small variations or perturbations at small scale present in fluid flows. Similar to the derivation of the LANS-α, this probabilistic approach, which is motivated

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by Reynolds' work which stipulates that hydrodynamic turbulence is composed of slow (deterministic) and fast (stochastic) components, was used in [31] to derive stochastic Navier-Stokes equations with gradient and nonlinear diffusion coefficient. It is also worth emphasising that the presence of the stochastic term (noise) in the model often leads to qualitatively new types of behaviour, which are very helpful in understanding real processes and is also often more realistic. In particular, for the 2D Navier-Stokes equations, some ergodic properties are proved when adding a random perturbation.

Motivated by these facts, in this paper we will consider a stochastic version of the LANS-α in non-dimensionalised form. More precisely we let $\mathcal{O}=[0,2\pi]^2$ be the $2D$ torus, we fix an arbitrary time horizon $T\in(0,\infty)$ and we consider the following system

$$\begin{cases} d\mathbf{v}^\alpha+[-\Delta\mathbf{v}^\alpha+\mathbf{u}^\alpha\cdot\nabla\mathbf{v}^\alpha+\sum_{j=1}^2\mathbf{v}_j^\alpha\nabla\mathbf{u}^\alpha+\nabla\mathbf{p}^\alpha]dt=\alpha^{\frac12}G(\mathbf{u}^\alpha)dW\\ \mathbf{v}^\alpha=\mathbf{u}^\alpha-\alpha^2\Delta\mathbf{u}^\alpha\\ \operatorname{div}\mathbf{u}^\alpha=0\\ \int_{\mathcal{O}}\mathbf{u}^\alpha(x)dx=0\\ \mathbf{u}^\alpha(t=0)=\xi, \end{cases} \tag{1.1}$$

where $\mathbf{u}^\alpha,\mathbf{p}^\alpha$ are the fluid velocity and fluid pressure, respectively. The symbol W represents the cylindrical Wiener process evolving on a given separable Hilbert space K. The noise coefficient is a nonlinear map defined to take values on Hilbert spaces that will be given later. The symbol α denotes a small positive parameter.

We should note that the choice of the form of the stochastic perturbation $\alpha^{\frac12}G(\mathbf{u}^\alpha)dW$ appearing in (1.1) is for mathematical convenience. In fact, in some parts of this paper we rely on previous results on existence and uniqueness of solutions of the stochastic LANS-α proved in [8] and [17]. Hence, we chose the form of and assumptions on the stochastic perturbation in our system in such a way that we can use the results in [8] and [17]. In view of the derivation of the LANS-α model, see [9–11], or the approach in [31] the stochastic perturbation will probably take the form $\alpha^2F(\mathbf{u}^\alpha,\mathbf{v}^\alpha)dW$ for some (possibly) mapping F. Here we chose the pre-factor $\alpha^{\frac12}$ having in mind the time dimensionality of noise and as it is customary in the study of LDP for SPDEs. We postpone the investigation of this issue of the noise of the form $\alpha^2F(\mathbf{u}^\alpha,\mathbf{v}^\alpha)dW$ to the future.

We also observe that when $\alpha=0$ the above system reduces to the deterministic 2D Navier-Stokes equations (NSEs):

$$\begin{cases} d\mathbf{u}+[-\Delta\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla\mathbf{p}]dt=0\\ \operatorname{div}\mathbf{u}=0\\ \int_{\mathcal{O}}\mathbf{u}(x)dx=0\\ \mathbf{u}(t=0)=\xi. \end{cases} \tag{1.2}$$

Thus, we expect that a sequence of solutions to the system (1.1) will converge in appropriate sense to a solution to (1.2) as $\alpha\to0$. For the deterministic case, *i.e.*, when $G\equiv0$, it is known from [20] that then as $\alpha\to0$ a weak solution to the deterministic $3D$ LANS-α (1.1) model converges to $3D$ Navier-Stokes equations. In [7], when $G\equiv0$ the rate of convergence of the unique solution to (1.1) to the unique solution to the $2D$ Navier-Stokes equations was studied. For the stochastic models, it was proved in [8] that the stochastic $3D$ (1.1) has a unique strong solution when the noise coefficient G is globally Lipschitz. When G is only continuous, it was proved in [17] that the stochastic $3D$ (1.1) has global weak (or martingale) solutions. Furthermore, it is shown in [16] that when $\alpha\to0$ a sequence of weak (or martingale) solutions of the stochastic $3D$ (1.1) model converges in distribution to a weak solution (or martingale) of the $3D$ Navier-Stokes equations. In the above references, the coefficient of the noise is not allowed to converge to 0 as $\alpha \to 0$.

Our main goal in this paper is to study the behaviour of the solutions $\mathbf{u}^\alpha$ to the system (1.1) as $\alpha \to 0$ through the lenses of the Large and Moderate Deviations Principles (LDP and MDP). For this purpose we assume that the coefficient of the noise is multiplied by the square root of α, *i.e.*, of the form $\alpha^{\frac{1}{2}} G(\mathbf{u}^\alpha)$. We then analyse the asymptotic behaviour, as $\alpha \to 0$, of the family of trajectories of $(\mathbf{u}^\alpha)_{\alpha \in (0,1]}$ and $\left(\alpha^{-\frac{1}{2}} \lambda^{-1}(\alpha)[\mathbf{u}^\alpha - \mathbf{u}]\right)_{\alpha \in (0,1]}$ where $\lambda : (0,1] \to (0,\infty)$ is a function satisfying

$$\lambda(\alpha) \to \infty \text{ and } \alpha^{\frac{1}{2}} \lambda(\alpha) \to 0 \text{ as } \alpha \to 0, \tag{1.3}$$

and $\mathbf{u}$ is the solution to the deterministic NSEs with initial data ξ. Thus, our goal and results in the present paper are different from the results in [16] and from results from several papers dealing with the deviation principles of α-models of Navier-Stokes equations, see for instance [13] and [39].

Roughly speaking, in the study of the MDP one is interested in probabilities of deviations of lower speed than in the classical LDP. In small diffusion (the coefficient of the noise is usually multiplied by $\alpha^{\frac{1}{2}}$) the speed for the LDP is usually of order α. The speed for the MDP is of order $\lambda^2(\alpha)$ and is provided by an LDP result for $\left(\alpha^{-\frac{1}{2}} \lambda^{-1}(\alpha)[\mathbf{u}^\alpha - \mathbf{u}]\right)_{\alpha \in (0,1]}$. Observe that since $\lambda(\alpha)$ converges to ∞ as slow as desired, then the MDP bridges the gap between the Central Limit Theorem and the LDP. We refer, for instance, to [22] and [24] for more detailed explanation and historical account of the MDP. We refer, for instance, to [1–5, 13, 18, 26, 27, 37, 38, 40–42] and references therein for a small sample of results from the extensive literature devoted to MDP and LDP for stochastic differential equations with small noise.

In several papers about LDP and MDP for stochastic system, the authors usually present two separate proofs of the two deviations principles. In this present paper, instead of presenting two separate proofs of the LDP and MDP results we present a unifying approach for these deviation principles for the LANS-α model. A similar approach was introduced in [35] for the vanishing viscosity limit of the second grade fluid. To be precise, we fix $\delta \in \{0,1\}$ and consider the following problem

$$\begin{cases} d\mathbf{y}^{\alpha,\delta} + \Big[\mathrm{A}\mathbf{y}^{\alpha,\delta} + \lambda_\delta(\alpha)\tilde{B}_\alpha(\mathbf{y}^{\alpha,\delta}, \mathbf{z}^{\alpha,\delta}) + \delta\Big[\tilde{B}_\alpha(\mathbf{u}, \mathbf{z}^{\alpha,\delta}) + \tilde{B}_\alpha(\mathbf{y}^{\alpha,\delta}, J_\alpha^{-1}\mathbf{u})\Big]\Big] \\ \quad = -\lambda_\delta^{-1}(\alpha)\delta[\tilde{B}_\alpha(\mathbf{u}, J_\alpha^{-1}\mathbf{u}) - B(\mathbf{u},\mathbf{u})]dt + \alpha^{\frac{1}{2}}\lambda_\delta^{-1}(\alpha)\mathbf{G}_\alpha(\delta\mathbf{u} + \lambda_\delta(\alpha)\mathbf{y}^{\alpha,\delta})dW, \\ \mathbf{z}^{\alpha,\delta} = \mathbf{y}^{\alpha,\delta} + \alpha^2 \mathrm{A}\mathbf{y}^{\alpha,\delta}, \\ \mathbf{y}^{\alpha,\delta}(t=0) = (1-\delta)\xi, \end{cases} \tag{1.4}$$

where

- $\mathbf{u}$ is the unique solution to the deterministic NSEs with initial data ξ;
- A is the Stokes operator, $J_\alpha = (I + \alpha^2 \mathrm{A})^{-1}$;
- $B(u,v)$ is roughly speaking the projection of $u \cdot \nabla v$ into the space of divergence-free functions;
- $\widetilde{B}(u,v)$ is the projection of $u \cdot \nabla v + \sum_{j=1}^2 v_j \nabla u_j$ into the space of divergence-free functions;
- finally, $\widetilde{B}_\alpha(u,v) = J_\alpha \widetilde{B}(u,v)$ and $\mathbf{G}_\alpha(u) = J_\alpha G(u)$.

The major part of the paper is devoted to the proof of LDP result for the system (1.4). Denoting by $\mathbf{H}$ the subspace of the Sobolev space $\mathbf{L}^2(\mathcal{O})$ consisting of periodic, divergence-free functions that have zero mean, by $\mathbf{V}$ the space $D(\mathrm{A}^{\frac{1}{2}})$ and by $\mathscr{L}_2(X,Y)$ the space of Hilbert-Schmidt operators from X onto Y, the main results in this paper can be roughly summarised in the following theorem.

THEOREM 1.1. *Let $\delta\in\{0,1\}$, $\xi\in D(\mathrm{A}^{\frac{1}{2}})$ and the map $G:\mathbf{H}\longrightarrow\mathscr{L}_2(\mathrm{K},\mathbf{H})\cap\mathscr{L}_2(\mathrm{K},\mathbf{V})$ has the following property: there exists $C>0$ such that for any $u,v\in\mathbf{H}$,*

$$\|G(u)-G(v)\|_{\mathscr{L}_2(\mathrm{K},\mathbf{V})}+\|G(u)-G(v)\|_{\mathscr{L}_2(\mathrm{K},\mathbf{H})}\le C|u-v|$$
$$\|G(u)\|_{\mathscr{L}_2(\mathrm{K},\mathbf{V})}+\|G(u)\|_{\mathscr{L}_2(\mathrm{K},\mathbf{H})}\le C(1+|u|).$$

Then, the family $(\mathbf{u}^{\alpha,\delta})_{\alpha\in(0,1]}$ of solutions to (1.4) satisfies an LDP on the space $C([0,T];\mathbf{H})\cap\mathbf{L}^2(0,T;D(\mathrm{A}^{\frac{1}{2}}))$ with speed $\alpha^{-1}\lambda_\delta^2(\alpha)$ and rate function I_δ given by

$$I_\delta(x)=\inf_{\{h\in\mathbf{L}^2(0,T;\mathrm{K}):x=\Gamma_\xi^{0,\delta}(\int_0^\cdot h(r)dr)\}}\left\{\frac{1}{2}\int_0^T\|h(r)\|_{\mathrm{K}}^2dr\right\}.$$

As usual, we understand that $\inf\emptyset=\infty$.

In the above theorem $\Gamma_\xi^{0,\delta}:C([0,T];\mathrm{K})\to C([0,T];\mathbf{H})\cap\mathbf{L}^2(0,T;D(\mathrm{A}^{\frac{1}{2}}))$ is the solution map of the skeleton equation associated to (1.4) which is obtained by replacing dW by an element $h\in\mathbf{L}^2(0,T;\mathrm{K})$ in (1.4), see Subsection 4.2 and Proposition 4.1 for more details. The precise definitions of all used notations and the formulation of the assumptions on our problem are presented in Section 3.

The above theorem, which will be restated and proved in Section 5.3, provides the LDP and MDP results for the LANS-α model (1.1). In fact, we observe that:

- when $\delta=0$, the unique solution to (1.4) is exactly the unique solution to the LANS-α (1.1). Thus, the LDP results for system (1.1) follow from the LDP result for the system (1.4) when $\delta=0$.
- When $\delta=1$, the unique solution to (1.4) is exactly $\alpha^{-\frac{1}{2}}\lambda^{-1}(\alpha)[\mathbf{u}^\alpha-\mathbf{u}]$ where $\mathbf{u}^\alpha$ and $\mathbf{u}$ are the unique solutions to (1.1) and the deterministic NSEs with initial data ξ, respectively. Hence, the MDP result for (1.1) follows from the LDP results for the system (1.4) when $\delta=1$.

The precise statement of the above result will be done in Theorem 5.1 whose proof is presented in Section 5.3 and based on weak convergence approach to LDP and Budhiraja-Dupuis' results on representation of functionals of Brownian motion, see [4] and [5]. Also, we closely follow the techniques presented in the recent paper [3]. Note however that our results do not fall into the framework of these papers or the results in [1,2,13,35,37,39]. The authors of the papers [13,37] and [39] study the LDP or MDP of the Navier-Stokes equations and other hydroddynamical models, but their physical parameters such as the viscosity in their equations are not allowed to vanish. The papers [1, 2] and [35] treat the LDP and zero viscosity limit of the shell models, the Navier-Stokes equations and the second grade fluids, respectively.

It is also worth pointing out that even though we rely on the abstract results in [4] and [5], our analysis is not trivial. Our results require the derivation of uniform estimates on the difference between the terms in the LANS-α model and the Navier-Stokes equations. Due to the unifying approach to the LDP and MDP we present in this paper, these crucial estimates are not available from previous works. We also note that as a by-product of our analysis we also show that the solution to (1.1) converges in probability to the unique solution to the deterministic NSEs with initial data ξ as $\alpha \to 0$, see Lemma 5.1 and Remark 5.1. Of course, since we are in the two-dimensional case this result is stronger than what was proved in [16].

To close this introduction we now outline the layout of the paper. We introduce the necessary notations and the basic model in Section 2. In the same section we also give several preliminary results which are crucial for the subsequent analysis. In Section 3, we introduce the standing assumption on the noise and state and prove a theorem on the existence and uniqueness of the solution to the problem 1.4. Section 4 is devoted to the study of auxiliary deterministic and stochastic controlled systems. The results in Section 4 are important not only for the description of the rate functions associated to the LDP and MDP results but also for their proofs. Section 5 contains the main results and their proofs. Therein, we show the convergence in probability of the solution to the stochastic LANS-α to the unique solution of the deterministic Navier-Stokes equations. By using the weak convergence approach we also prove in Section 5 that the solution of (1.4) satisfies the LDP on $C([0,T];\mathbf{H})\cap L^2(0,T;\mathbf{V})$. This LDP result for (1.4) provides the LDP and MDP results for the problem (1.1).

2. Notations, the basic problems and some key estimates

2.1. Notation and the basic problems. We introduce necessary definitions of functional spaces frequently used in this work.

For a topological vector space X we denote by X' its dual space and we denote by $\langle u,u*\rangle_{X'}$ the duality paring between $u\in X$ and $u*\in X'$.

Throughout this paper we denote by $L^p(\mathcal{O};\mathbb{R}^2)$ and $W^{m,p}(\mathcal{O};\mathbb{R}^2)$, $p\in[1,\infty]$, $m\in\mathbb{N}$, the Lebesgue and Sobolev spaces of functions defined on $\mathcal{O}$ and taking values in $\mathbb{R}^2$. The spaces of $u\in L^p(\mathcal{O};\mathbb{R}^2)$ and $W^{m,p}(\mathcal{O};\mathbb{R}^2)$ which are 2π-periodic in each direction x_i, $i=1,2$, see for example [14], are denoted by $\mathbf{L}^p(\mathcal{O})$ and $\mathbf{W}^{m,p}(\mathcal{O})$, respectively. We simply write $\mathbf{L}^p$ (resp. $\mathbf{W}^{m,p}$) instead of $\mathbf{L}^p(\mathcal{O})$ (resp. $\mathbf{W}^{m,p}(\mathcal{O})$) when there is no risk of ambiguity. We will also use the notation $\mathbf{H}^m:=\mathbf{W}^{m,2}$. For non-integer $r>0$ the Sobolev space $\mathbf{H}^r$ is defined by using classical interpolation method. The space $[\mathcal{C}^\infty_{\text{per}}(\mathbb{R}^2)]^2:=\mathcal{C}^\infty_{\text{per}}(\mathbb{R}^2,\mathbb{R}^2)$ denotes the space of functions which are infinitely differentiable and 2π-periodic in each direction x_i, $i=1,2$.

We also introduce the following spaces

$$\begin{aligned}\mathbf{H}&=\left\{u\in\mathbf{L}^2(\mathcal{O});\int_{\mathcal{O}}u(x)dx=0,\ \operatorname{div}u=0\right\},\\ \mathbf{V}&=\mathbf{H}^1\cap\mathbf{H}.\end{aligned}$$

It is well-known, see [36], that $\mathbf{H}$ and $\mathbf{V}$ are the closure of

$$\mathcal{V}=\left\{u\in[\mathcal{C}^\infty_{\text{per}}(\mathbb{R}^2)]^2;\int_{\mathcal{O}}u(x)dx=0,\ \operatorname{div}u=0\right\},$$

with respect to the $\mathbf{L}^2$ and $\mathbf{H}^1$ norms. We denote by $(\cdot,\cdot)$ and $|\cdot|$ the inner product and the norm induced by the inner product and the norm in $\mathbf{L}^2(\mathcal{O})$ on $\mathbf{H}$, respectively. Thanks to the Poincaré inequality we can endow the space $\mathbf{V}$ with the norm $\|u\|=|\nabla u|, u\in\mathbf{V}$.

Let $\Pi:\mathbf{L}^2(\mathcal{O})\to\mathbf{H}$ be the Helmholtz-Leray projection, and $\mathrm{A}=-\Pi\Delta$ be the Stokes operator with the domain $D(\mathrm{A})=\mathbf{H}^2(\mathcal{O})\cap\mathbf{H}$. It is well-known that A is a self-adjoint positive operator with compact inverse, see for instance [36, Chapter 1, Section 2.6].

Hence, it has an orthonormal sequence of eigenvectors $\{e_j; j\in\mathbb{N}\}$ with corresponding eigenvalues $0<\lambda_1<\lambda_2<\dots$ The domain of A^r, $r\in\mathbb{R}$ is characterised by

$$D(\mathrm{A}^r)=\mathbf{V}\cap\mathbf{H}^{2r},$$

see [14, page 43].

For $\alpha\in(0,1)$ we set

$$\|u\|_\alpha=\sqrt{|u|^2+\alpha^2|\mathrm{A}u|^2},\, u\in\mathbf{V}.$$

Then, we observe that $\|\cdot\|$, $\|\cdot\|_\alpha$, $\alpha\in(0,1)$, and $|\mathrm{A}^{\frac{1}{2}}\cdot|$ define three equivalent norms on $\mathbf{V}$.

For the time being we assume that the stochastic perturbation $G(\mathbf{u}^\alpha)dW$ is a divergence-free function. Then, when projecting the system (1.1) onto the space of divergence-free functions we obtain the following stochastic evolution equation

$$\begin{cases} d\mathbf{v}^\alpha+[\mathrm{A}\mathbf{v}^\alpha+\tilde{B}(\mathbf{u}^\alpha,\mathbf{v}^\alpha)]dt=\alpha^{\frac{1}{2}}G(\mathbf{u}^\alpha)dW\\ \mathbf{v}^\alpha=\mathbf{u}^\alpha+\alpha^2\mathrm{A}\mathbf{u}^\alpha\\ \mathbf{u}^\alpha(t=0)=\xi. \end{cases} \tag{2.1}$$

In a similar way, we can also write the $2D$ Navier-Stokes equations as the abstract evolution equation

$$\begin{cases} \frac{d}{dt}\mathbf{u}+\mathrm{A}\mathbf{u}+B(\mathbf{u},\mathbf{u})=0\\ \mathbf{u}(t=0)=\xi. \end{cases} \tag{2.2}$$

In (2.1) and (2.2) the nonlinear terms $\tilde{B}$ and B are roughly defined by

$$\begin{aligned} \tilde{B}(u,v)&=\Pi(u\cdot\nabla v+\sum_{j=1}^{2}v_j\nabla u_j)\\ B(u,v)&=\Pi(u\cdot\nabla v), \end{aligned}$$

respectively. These nonlinear maps satisfy several properties that will be recalled in the last subsection of this section.

By introducing the following nonlinear maps

$$\begin{aligned} \tilde{B}_\alpha(u,v)&=(I+\alpha^2\mathrm{A})^{-1}\tilde{B}(u,v),\\ \mathbf{G}_\alpha(u)&=(I+\alpha^2\mathrm{A})^{-1}G(u), \end{aligned}$$

the Equation (2.1) can be rewritten in the following form:

$$\begin{cases} d\mathbf{u}^\alpha+\mathrm{A}\mathbf{u}^\alpha+\tilde{B}_\alpha(\mathbf{u}^\alpha,\mathbf{v}^\alpha)=\alpha^{\frac{1}{2}}\mathbf{G}_\alpha(\mathbf{u}^\alpha)dW\\ \mathbf{u}^\alpha(t=0)=\xi. \end{cases}$$

In the next few lines we will introduce an abstract stochastic evolution equation which will enable us to give a unifying approach to the large and moderate deviations for the problem (2.1). For this purpose we fix a $\delta\in\{0,1\}$ and introduce the function $\lambda_\delta:(0,\infty)\to(0,\infty)$ defined by

$$\lambda_\delta(\alpha)=\begin{cases} 1, & \text{if } \delta=0,\\ \alpha^{\frac{1}{2}}\lambda(\alpha), & \text{if } \delta=1, \end{cases}$$

where $\lambda:(0,\infty)\to(0,\infty)$ is a function satisfying (1.3).

REMARK 2.1. In view of the definition of $\lambda_\delta(\alpha)$, we see that as $\alpha\to 0$

$$\lambda_\delta(\alpha)\to 1-\delta.$$

Observe also that for $\ell\in\{1,2\}$ and $k\geq\frac{\ell}{2}$

$$\alpha^k\lambda_\delta^{-\ell}(\alpha)\to 0 \text{ as } \alpha\to 0.$$

Hence, we can and will assume that for $\ell\in\{1,2\}$, $k\geq\frac{\ell}{2}$ and $\alpha\in(0,1)$

$$\alpha^k\lambda_\delta^{-\ell}(\alpha)\leq 2 \tag{2.3}$$

where $\lambda(\alpha)$ is the function considered in (1.3).

Before proceeding further we recall the following result on the 2D Navier-Stokes equations, see, for instance, [14] and [36] for its proof.

THEOREM 2.1. *Let $\xi\in\mathbf{V}$. Then, the problem (2.2) has a unique solution $\mathbf{u}\in C([0,T];\mathbf{V})\cap L^2(0,T;D(\mathrm{A}))$.*

Throughout this paper, the symbol $\mathbf{u}$ will denote the unique solution of the problem (2.2).

Now, we consider the following stochastic evolution equations.

$$\begin{cases} d\mathbf{y}^{\alpha,\delta}+\Big[\mathrm{A}\mathbf{y}^{\alpha,\delta}+\lambda_\delta(\alpha)\tilde{B}_\alpha(\mathbf{y}^{\alpha,\delta},\mathbf{z}^{\alpha,\delta})+\delta\Big[\tilde{B}_\alpha(\mathbf{u},\mathbf{z}^{\alpha,\delta})+\tilde{B}_\alpha(\mathbf{y}^{\alpha,\delta},J_\alpha^{-1}\mathbf{u})\Big]\Big] \\ \quad =-\lambda_\delta^{-1}(\alpha)\delta[\tilde{B}_\alpha(\mathbf{u},J_\alpha^{-1}\mathbf{u})-B(\mathbf{u},\mathbf{u})]dt+\alpha^{\frac{1}{2}}\lambda_\delta^{-1}(\alpha)\mathbf{G}_\alpha(\delta\mathbf{u}+\lambda_\delta(\alpha)\mathbf{y}^{\alpha,\delta})dW, \\ \mathbf{z}^{\alpha,\delta}=\mathbf{y}^{\alpha,\delta}+\alpha^2\mathrm{A}\mathbf{y}^{\alpha,\delta}, \\ \mathbf{y}^{\alpha,\delta}(t=0)=(1-\delta)\xi, \end{cases} \tag{2.4}$$

where $J_\alpha=(I+\alpha^2\mathrm{A})^{-1}$.

REMARK 2.2. Observe that, if one is able to prove a LDP result for (2.4) then one just proved LDP and MDP results for (2.1). In fact, (2.4) reduces to (2.1) when $\delta=0$. When $\delta=1$ then an LDP result for (2.4) yields an LDP for the process $\mathbf{y}^{\alpha,1}=\frac{\mathbf{u}^\alpha-\mathbf{u}}{\lambda_\delta(\alpha)}$. This is just an MDP result for (2.1).

2.2. Several key estimates. To close the present section, we recall and prove several well-known properties of the bilinear maps B and $\tilde{B}$. These properties will play an important role in the sequel.

We first recall the following lemma that was proved in [20].

LEMMA 2.1.

(1) *Let X be either B or $\tilde{B}$. Then, the operator X can be extended continuously from $\mathbf{V}\times\mathbf{V}$ with values in $\mathbf{V}'$ (the dual space of $\mathbf{V}$). In particular, for $u,v,w\in\mathbf{V}$,*

$$|\langle X(u,v),w\rangle_{\mathbf{V}'}|\leq c|u|^{1/2}||u||^{1/2}||v||||w||, \tag{2.5}$$

and

$$(\tilde{B}(u,v),w)=(B(u,v),w)-(B(w,v),u). \tag{2.6}$$

Moreover

$$(B(u,v),w) = -(B(u,w),v),\ u,v,w \in \mathbf{V}, \tag{2.7}$$

which in turn implies that

$$(B(u,v),v) = 0,\ u,v \in \mathbf{V}. \tag{2.8}$$

Also,

$$(\tilde{B}(u,v),w) = (B(u,v),w) - (B(w,v),u),\ u,v,w \in \mathbf{V}, \tag{2.9}$$

and hence

$$(\tilde{B}(u,v),u) = 0,\ u,v \in \mathbf{V}. \tag{2.10}$$

(2) *Furthermore, let* $u \in D(\mathrm{A}), v \in \mathbf{V}, w \in \mathbf{H}$ *and let* X *be either* B *or* $\tilde{B}$*, then*

$$|(X(u,v),w)| \le c|\mathrm{A}^{\frac{1}{2}}u|^{1/2}|\mathrm{A}u|^{1/2}|\mathrm{A}^{\frac{1}{2}}v||w|. \tag{2.11}$$

(3) *Let* $u \in \mathbf{V}, v \in D(\mathrm{A}), w \in \mathbf{H}$*, then*

$$|(B(u,v),w)| \le c|\mathrm{A}^{\frac{1}{2}}u||\mathrm{A}^{\frac{1}{2}}v|^{1/2}|\mathrm{A}v|^{1/2}|w|. \tag{2.12}$$

(4) *The operator* B *and* $\tilde{B}$ *can be also extended continuously from* $D(\mathrm{A}^{\frac{1}{2}}) \times \mathbf{H}$ *with values in* $D(\mathrm{A}^{-1})$*. In particular, if* $u \in D(\mathrm{A}^{\frac{1}{2}}), v \in \mathbf{H}, w \in D(\mathrm{A})$*, then*

$$\langle \tilde{B}(u,v),w\rangle \le C[|u|^{\frac{1}{2}}|\mathrm{A}^{\frac{1}{2}}u|^{\frac{1}{2}}|\mathrm{A}^{\frac{1}{2}}w|^{\frac{1}{2}}|\mathrm{A}w|^{\frac{1}{2}} + |\mathrm{A}w||\mathrm{A}^{\frac{1}{2}}u|]|v|, \tag{2.13}$$

hence the Poincaré inequality yields

$$|\langle \tilde{B}(u,v),w\rangle_{D(\mathrm{A})'}| \le C|\mathrm{A}^{\frac{1}{2}}u||v||\mathrm{A}w|. \tag{2.14}$$

Also, by symmetry we have for all $u \in D(\mathrm{A}), v \in \mathbf{H}, w \in D(\mathrm{A}^{\frac{1}{2}})$*,*

$$|\langle \tilde{B}(u,v),w\rangle_{D(\mathrm{A})'}| \le C|\mathrm{A}u||v||\mathrm{A}^{\frac{1}{2}}w|. \tag{2.15}$$

Remark 2.3. Observe that by using the Hölder and the Gagliardo-Nirenberg inequalities one can refine the estimate (2.12) as follows. Let $u \in \mathbf{V}, v \in D(\mathrm{A}), w \in \mathbf{H}$, then

$$|(B(u,v),w)| \le c|u|^{\frac{1}{2}}|\mathrm{A}^{\frac{1}{2}}u|^{\frac{1}{2}}|\mathrm{A}^{\frac{1}{2}}v|^{1/2}|\mathrm{A}v|^{1/2}|w|. \tag{2.16}$$

See [14] and [36] for the details.

Notice also that thanks to (2.10) we have

$$(\tilde{B}_\alpha(u,v), J_\alpha^{-1}u) = \langle \tilde{B}(u,v),u\rangle_{\mathbf{V}'} = 0 \text{ for all } u,v \in \mathbf{V}. \tag{2.17}$$

The following lemma, which was proved in [7], will be needed in several places later on.

Lemma 2.2. Let $\phi \in \mathbf{H}$, $w \in D(\mathrm{A}^{\frac{1}{2}})$. Then, for any $\alpha \in (0,1)$ we have

$$\langle \phi - J_\alpha \phi, w\rangle \le \frac{\alpha}{2}|\phi||\mathrm{A}^{\frac{1}{2}}w|. \tag{2.18}$$

We also need the following three lemmata.

LEMMA 2.3. *There exists a constant* $C>0$ *such that for any* $\alpha\in(0,1)$*, any* $y,u\in D(\mathrm{A})$ *we have*

$$(J_\alpha^{-1}y,B(u,u)-\tilde{B}_\alpha(u,J_\alpha^{-1}u))\le C\left[\frac{\alpha}{2}|\mathrm{A}^{\frac12}y|+\alpha^2|\mathrm{A}y|\right]\left(|B(u,u)|+|\mathrm{A}^{\frac12}u||\mathrm{A}u|\right), \quad (2.19)$$

$$(y,B(u,u)-\tilde{B}_\alpha(u,J_\alpha^{-1}u))\le C\frac{\alpha}{2}|\mathrm{A}^{\frac12}y||B(u,u)|+\frac{\alpha}{2}C|\mathrm{A}u|^2|y|. \quad (2.20)$$

Proof. Let $y,u\in D(\mathrm{A})$ and $\alpha\in(0,1)$. In order to simplify the notation we set $\phi=B(u,u)$. By the bilinearity of $\tilde{B}$ and B, and the fact $B(u,u)=\tilde{B}(u,u)$ we have

$$\begin{aligned}&(J_\alpha^{-1}y,B(u,u)-\tilde{B}_\alpha(u,J_\alpha^{-1}u))\\ &=(J_\alpha^{-1}y,B(u,u)-J_\alpha B(u,u))+\alpha^2(J_\alpha^{-1}y,B(u,u)-J_\alpha\tilde{B}(u,\mathrm{A}u)),\\ &=(y,B(u,u)-J_\alpha B(u,u))+\alpha^2(\mathrm{A}y,B(u,u)-J_\alpha\tilde{B}(u,\mathrm{A}u))+\alpha^2(J_\alpha^{-1}y,B(u,u)-J_\alpha\tilde{B}(u,\mathrm{A}u)).\end{aligned}$$

By using the last line, (2.18), the facts that

$$|\alpha^2\mathrm{A}(I+\alpha^2\mathrm{A})^{-1}|_{\mathscr{L}(H)}=\sup_{k\in\mathbb{N}}\frac{\alpha^2\lambda_k}{1+\alpha^2\lambda_k}\le 1, \quad (2.21)$$

$$|\alpha\mathrm{A}^{\frac12}(I+\alpha^2\mathrm{A})^{-1}|_{\mathscr{L}(H)}=\sup_{k\in\mathbb{N}}\frac{(\alpha^2\lambda_k)^{\frac12}}{1+\alpha^2\lambda_k}\le\frac12, \quad (2.22)$$

and the inequalities (2.13) we easily establish (2.19).

The second estimate (2.20) is proved in a similar way. □

LEMMA 2.4. *There exists a constant* $C>0$ *such that for any* $y,\in\mathbf{H},v,w\in D(\mathrm{A})$*, and any* $\alpha\in(0,1)$ *we have*

$$(\tilde{B}_\alpha(v,J_\alpha^{-1}w),y)\le C|\mathrm{A}v||y|\left(|\mathrm{A}^{\frac12}w|+\alpha|\mathrm{A}w|\right). \quad (2.23)$$

Proof. Throughout this proof C will denote a constant independent of α.
Let $y\in\mathbf{H},v,w\in D(\mathrm{A})$, and $\alpha\in(0,1)$. Observe that

$$(\tilde{B}_\alpha(v,J_\alpha^{-1}w),y)=(\tilde{B}(v,w),J_\alpha y)+\alpha^2\langle\tilde{B}(v,\mathrm{A}w),J_\alpha y\rangle_{D(\mathrm{A})'}.$$

Now, by applying the inequalities (2.9), (2.6), (2.13) and the Hölder inequality we find that there is a constant $C>0$ such that

$$(\tilde{B}_\alpha(v,J_\alpha^{-1}w),y)\le C|\mathrm{A}v||\mathrm{A}^{\frac12}w||J_\alpha y|+\alpha^2|\mathrm{A}^{\frac12}J_\alpha y||\mathrm{A}w||\mathrm{A}v|.$$

By using the fact $|(\alpha^2\mathrm{A})^{\frac12}(I+\alpha^2\mathrm{A})^{-1}|_{\mathscr{L}(\mathbf{H})}\le\frac12$ and $|J_\alpha|_{\mathscr{L}(\mathbf{H})}\le 1$ we see that

$$(\tilde{B}_\alpha(v,J_\alpha^{-1}w),y)\le C|\mathrm{A}v||\mathrm{A}^{\frac12}w||y|+\alpha|y||\mathrm{A}w||\mathrm{A}v|,$$

which completes the proof of the lemma. □

LEMMA 2.5. *There exists a constant* $C>0$ *such that for any* $\alpha\in(0,1)$ *and any* $u\in D(\mathrm{A}),y\in D(\mathrm{A})$*:*

$$\left|(\tilde{B}_\alpha(u,J_\alpha^{-1}y),J_\alpha^{-1}y)+(\tilde{B}_\alpha(y,J_\alpha^{-1}u),J_\alpha^{-1}y)\right|$$

$$\le \frac{1}{4}|y|^2 + \frac{\alpha^2}{2}|\mathrm{A}^{\frac{1}{2}}y|^2 + \frac{\alpha^4}{4}|\mathrm{A}y|^2 + \alpha^{-2}C|u|^2_{D(\mathrm{A})}\left[|y|^2 + \alpha^2|\mathrm{A}^{\frac{1}{2}}y|^2\right].$$

Proof. Let $u \in D(\mathrm{A}), y \in D(\mathrm{A})$ and $z = y + \alpha^2 \mathrm{A}y$. Using equation (4) on page 5 of [20] we obtain:

$$\begin{aligned}(\tilde{B}_\alpha(u,z),z) &= (J_\alpha \tilde{B}(u,z), J_\alpha^{-1}y)\\ &= \langle \tilde{B}(u,z),y\rangle_{\mathbf{V}'}\\ &= \langle B(u,z),y\rangle_{\mathbf{V}'} - \langle B(y,z),u\rangle_{\mathbf{V}'}.\end{aligned}$$

Using the well-known property

$$\langle B(u,z),y\rangle_{\mathbf{V}'} = -\langle B(u,y),z\rangle_{\mathbf{V}'},$$

we obtain

$$(\tilde{B}_\alpha(u,z),z) = -\langle B(u,y),z\rangle_{\mathbf{V}'} + \langle B(y,u),z\rangle_{\mathbf{V}'}.$$

Now the Hölder and Young inequalities along with the Sobolev embedding, $D(\mathrm{A}) \subset \mathbf{L}^\infty$ and $\mathbf{V} \subset \mathbf{L}^4$ we find that there exists $C > 0$ such that

$$\begin{aligned}(\tilde{B}_\alpha(u,z),z) &\le C|z|\left[|u|_{\mathbf{L}^\infty}\|y\| + \|y\||u|_{D(\mathrm{A})}\right]\\ &\le C|z||u|_{D(\mathrm{A})}\|y\|\\ &\le \frac{1}{4}|z|^2 + C|u|^2_{D(\mathrm{A})}\|y\|^2\\ &\le \frac{1}{4}|y+\alpha^2\mathrm{A}y|^2 + \alpha^{-2}C|u|^2_{D(\mathrm{A})}\alpha^2\|y\|^2\\ &\le \frac{1}{4}|y|^2 + \frac{\alpha^2}{2}|\mathrm{A}^{\frac{1}{2}}y|^2 + \frac{1}{4}|\mathrm{A}y|^2 + \alpha^{-2}C|u|^2_{D(\mathrm{A})}\left[|y|^2 + \alpha^2\|y\|^2\right].\end{aligned}$$

The last line of the inequality completes the proof of the lemma because

$$\begin{aligned}(\tilde{B}_\alpha(y,J_\alpha^{-1}u),z) &= (J_\alpha \tilde{B}(y,J_\alpha^{-1}u), J_\alpha^{-1}y)\\ &= \langle \tilde{B}(y,J_\alpha^{-1}u),y\rangle_{D(\mathrm{A})'} = 0.\end{aligned}$$

□

3. Assumptions on the noise coefficient and a well-posedness result

This section is devoted to the formulation of the standing assumption on the noise and the presentation of a well-posedness result.

3.1. Formulation of the assumptions on the noise. Throughout we fix a complete filtered probability space $\mathscr{U} := (\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ where the filtration $\mathbb{F} = \{\mathscr{F}_t;\ t \in [0,T]\}$ satisfies the usual conditions. We also fix two separable Hilbert spaces K and K_1 such that the canonical injection $\iota : \mathrm{K} \to \mathrm{K}_1$ is Hilbert-Schmidt. The operator $Q = \iota\iota^*$, where ι^* is the adjoint of ι, is symmetric, nonnegative. Since ι is Hilbert-Schmidt Q is also of trace class. Moreover, from [34, Corollary C.0.6] we infer that $\mathrm{K} = Q^{\frac{1}{2}}(\mathrm{K}_1)$. Now, let W be a cylindrical Wiener process evolving on K. It is well-known, see [15, Theorem 4.5], that W has the following series representation

$$W(t) = \sum_{j=1}^{\infty} \sqrt{q_j}\beta_j(t)h_j, \quad t \in [0,T],$$

where $\{\beta_j; j\in\mathbb{N}\}$ is a sequence of mutually independent and identically distributed standard Brownian motions, $\{h_j; j\in\mathbb{N}\}$ is an orthonormal basis of K consisting of eigenvectors of Q and $\{q_j; j\in\mathbb{N}\}$ is the family of eigenvalues of Q. It is also well-known, see [15, Section 4.1] and [34, Section 2.5.1], that W is a K_1-valued Wiener process with covariance Q.

Now, we recall few basic facts about stochastic integrals with respect to a cylindrical Wiener process evolving on K. For this purpose, let $\mathscr{H}$ be a separable Banach space, $\mathscr{L}(\mathrm{K},\mathscr{H})$ the space of all bounded linear $\mathscr{H}$-valued operators defined on K, and $\mathscr{M}^2_T(\mathscr{H}):=\mathscr{M}^2(\Omega\times[0,T];\mathscr{H})$ the space of all equivalence classes of $\mathbb{F}$-progressively measurable processes $\Psi:\Omega\times[0,T]\to\mathrm{K}$ satisfying

$$\mathbb{E}\int_0^T \|\Psi(r)\|^2_{\mathscr{H}}dr<\infty.$$

We denote by $\mathscr{L}_2(\mathrm{K},\mathscr{H})$ the Hilbert space of all operators $\Psi\in\mathscr{L}(\mathrm{K},\mathscr{H})$ satisfying

$$\|\Psi\|^2_{\mathscr{L}_2(\mathrm{K},\mathscr{H})}=\sum_{j=1}^\infty \|\Psi h_j\|^2_{\mathscr{H}}<\infty.$$

From the theory of stochastic integration on infinite dimensional Hilbert space, see [30, Chapter 5, Section 26] and [15, Chapter 4], for any $\Psi\in\mathscr{M}^2_T(\mathscr{L}_2(\mathrm{K},\mathscr{H}))$ the process M defined by

$$M(t)=\int_0^t \Psi(r)dW(r), t\in[0,T],$$

is an $\mathscr{H}$-valued martingale. Moreover, we have the following Itô isometry

$$\mathbb{E}\left(\left\|\int_0^t \Psi(r)dW(r)\right\|^2_{\mathscr{H}}\right)=\mathbb{E}\left(\int_0^t \|\Psi(r)\|^2_{\mathscr{L}_2(\mathrm{K},\mathscr{H})}dr\right),\quad \forall t\in[0,T], \tag{3.1}$$

and the Burkholder-Davis-Gundy's (BDG's) inequality

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le s\le t}\left\|\int_0^s \Psi(r)dW(r)\right\|^q\right)\\ &\le C_q\mathbb{E}\left(\int_0^t \|\Psi(r)\|^2_{\mathscr{L}_2(\mathrm{K},\mathscr{H})}dr\right)^{\frac{q}{2}}, \forall t\in[0,T], \forall q\in(0,\infty).\end{aligned} \tag{3.2}$$

The standing assumptions on G are given below.

Assumption 3.1. *The map $G:\mathbf{H}\longrightarrow\mathscr{L}_2(\mathrm{K},\mathbf{H})\cap\mathscr{L}_2(\mathrm{K},\mathbf{V})$ satisfies the following: there exists $C>0$ such that for any $u,v\in\mathbf{H}$*

$$\begin{aligned}&\|G(u)-G(v)\|_{\mathscr{L}_2(\mathrm{K},\mathbf{V})}+\|G(u)-G(v)\|_{\mathscr{L}_2(\mathrm{K},\mathbf{H})}\le C|u-v|\\ &\|G(u)\|_{\mathscr{L}_2(\mathrm{K},\mathbf{V})}+\|G(u)\|_{\mathscr{L}_2(\mathrm{K},\mathbf{H})}\le C(1+|u|).\end{aligned}$$

Remark 3.1. From the above assumption, we infer that there exists $C>0$ such that for any $u,v\in\mathbf{V}$

$$\|G(u)-G(v)\|_{\mathscr{L}_2(\mathrm{K},\mathbf{V})}\le\begin{cases}C|\mathrm{A}^{\frac12}(u-v)|\\ C|u-v| \;+\; \alpha^2|\mathrm{A}^{\frac12}(u-v)|,\end{cases}$$

and

$$\|G(u)\|_{\mathscr{L}_2(\mathrm{K},\mathbf{V})}\le\begin{cases}C(1+|u|)\\ C\left(1+|\mathrm{A}^{\frac12}u|\right)\\ C(1+|u|) \;+\; \alpha^2|\mathrm{A}^{\frac12}u|.\end{cases}$$

3.2. The well-posedness of the basic problem (2.4). In this subsection we state a well-posedness result of basic problem (2.4). Since there are several papers which deal with the existence of solutions of LANS-α model we give a rather sketchy proof of this well-posedness result.

We first give the concept of solutions to (2.4) that we adopt in this paper.

DEFINITION 3.1. *Given $\delta \in \{0,1\}$, $u_0 \in \mathbf{V} := D(\mathrm{A}^{\frac{1}{2}})$, a stochastic process $\mathbf{y}^{\alpha,\delta} : [0,T] \to \mathbf{V}$ is a strong solution to (2.4) if and only if*

- *$\mathbf{y}^{\alpha,\delta}$ is $\mathbb{F}-$adapted, i.e., for each t, $\mathbf{y}^{\alpha,\delta}(t)$ is $\mathscr{F}_t$-measurable,*
- *and $\mathbf{y}^{\alpha,\delta} \in C([0,T];\mathbf{V}) \cap \mathbf{L}^2([0,T];D(\mathrm{A}))$ with probability 1.*
- *Moreover, for all $t \in [0,T]$, $\mathbb{P}$.a.s.*

$$\begin{aligned}
&\langle \mathbf{y}^{\alpha,\delta}(t),\varphi\rangle + \int_0^t \langle \mathrm{A}\mathbf{y}^{\alpha,\delta} + \lambda_\delta(\alpha)\tilde{B}_\alpha(\mathbf{y}^{\alpha,\delta},\mathbf{z}^{\alpha,\delta}) + \delta[\tilde{B}_\alpha(\mathbf{u},\mathbf{z}^{\alpha,\delta}) \\
&\quad + \tilde{B}_\alpha(\mathbf{y}^{\alpha,\delta}, J_\alpha^{-1}\mathbf{u}),\varphi]\rangle ds \\
=&\langle \mathbf{y}^{\alpha,\delta}(0),\varphi\rangle + \int_0^t \langle \lambda_\delta^{-1}(\alpha)[B(\mathbf{u},\mathbf{u}) - \tilde{B}_\alpha(\mathbf{u},J_\alpha^{-1}\mathbf{u})],\varphi\rangle ds \\
&\quad + \alpha^{\frac{1}{2}}\lambda_\delta^{-1}(\alpha)\langle \int_0^t \mathbf{G}_\alpha(\delta\mathbf{u} + \lambda_\delta(\alpha)\mathbf{y}^{\alpha,\delta})dW,\varphi\rangle.
\end{aligned}$$

With this definition in mind we now give the following results which concerns the uniqueness of solutions to problem (2.4).

PROPOSITION 3.1. *Let $\delta \in \{0,1\}$ and $\xi \in \mathbf{V}$. Assume that G satisfies the Assumption 3.1. If the stochastic evolution Equation (2.4) has two strong solutions $\mathbf{y}_i^{\alpha,\delta}, i = 1,2$ in the sense of Definition 3.1 such that*

$$\mathbf{y}_i^{\alpha,\delta} \in \mathbf{L}^4(\Omega; C[0,T];\mathbf{V}) \cap \mathbf{L}^2([0,T];D(\mathrm{A})).$$

then with probability 1

$$\mathbf{y}_1^{\alpha,\delta}(t) = \mathbf{y}_2^{\alpha,\delta}(t) \text{ for all } t \in [0,T].$$

That is, the strong solution to the system (2.4) is pathwise unique.

Proof. The proof of the proposition follows the same lines as in [8], but for the sake of completeness we give the details.

Since the system reduces to the $2D$ Stochastic LANS-α model when $\delta = 0$ and the proof of the uniqueness result can be done as in [8]. Hence, we will focus on the case $\delta = 1$. Since the parameter α does not play an important role for the proof of uniqueness, we can and will assume $\alpha = 1$ and $\lambda_\delta(\alpha) = 1$. With these in mind, let $\mathbf{y}_1$ and $\mathbf{y}_2$ be two solutions to the system (2.4). We set

$$\begin{aligned}
\mathbf{y} &= \mathbf{y}_1 - \mathbf{y}_2 \\
\mathbf{z} &= \mathbf{y} + \mathrm{A}\mathbf{y} \\
\mathbf{z}_i &= \mathbf{y}_i + \mathrm{A}\mathbf{y}_i, \quad \text{for } i = 1,2.
\end{aligned}$$

Then, using the bilinearity of $\tilde{B}_\alpha, B$ we see that $\mathbf{y}$ satisfies:

$$\begin{cases}
d\mathbf{y} + \Big[\mathrm{A}\mathbf{y} + \tilde{B}_1(\mathbf{y},\mathbf{z}_1) + \tilde{B}_1(\mathbf{y}_2,\mathbf{z}) + \tilde{B}_1(\mathbf{u},\mathbf{z}) + \tilde{B}_1(\mathbf{y}, J_1^{-1}\mathbf{u})\Big] \\
\qquad\qquad = [\mathbf{G}_1(\mathbf{u}+\mathbf{y}_1) - \mathbf{G}_1(\mathbf{u}+\mathbf{y}_2)]dW \\
\mathbf{y}(t=0) = 0.
\end{cases}$$

Recall that $\tilde{B}_1(x,y) = (I+A)^{-1}\tilde{B}(x,y)$ and $\mathbf{G}_1(x) = (I+A)^{-1}G(x)$. Let $N>0$ and τ_N be the stopping time defined by

$$\tau_N = \inf\left\{t\in[0,T] : |\mathrm{A}^{\frac{1}{2}}\mathbf{y}_1(t)| > N\right\} \wedge \inf\left\{t\in[0,T] : |\mathrm{A}^{\frac{1}{2}}\mathbf{y}_2(t)| > N\right\}.$$

Let $t\in(0,T]$ be fixed.

Applying the Itô formula to $\mathbf{y}$ and the functional $\varphi(x) = |x|^2 + |\mathrm{A}^{\frac{1}{2}}x|^2$ for $x\in\mathbf{V}$, yields

$$\begin{aligned}&d\left(|\mathbf{y}|^2 + |\mathrm{A}^{\frac{1}{2}}\mathbf{y}|^2\right) + \left[|\mathrm{A}^{\frac{1}{2}}\mathbf{y}|^2 + |\mathrm{A}\mathbf{y}|^2 + (\mathbf{y},\tilde{B}(\mathbf{y}_2,\mathbf{z}) + \tilde{B}(\mathbf{u},\mathbf{z}))\right]dt\\ &= \|G(\mathbf{u}+\mathbf{y}_1) - G(\mathbf{u}+\mathbf{y}_2)\|^2_{\mathscr{L}_2(\mathrm{K},\mathbf{H})}dt + \langle \mathbf{y}, G(\mathbf{u}+\mathbf{y}_1) - G(\mathbf{u}+\mathbf{y}_2)dW\rangle,\end{aligned}$$

where we used the facts that for $x\in\mathbf{H}$

$$\begin{aligned}\varphi'(\mathbf{y})[(I+\mathrm{A}^{-1})x] &= 2(\mathbf{y}+\mathrm{A}\mathbf{y},(I+\mathrm{A})^{-1}x) = 2(\mathbf{y},x)\\ \frac{1}{2}\varphi''(\mathbf{y})[x,x] &= (x+\mathrm{A}x,(I+\mathrm{A})^{-1}x) = |x|^2,\end{aligned}$$

and the cancellation property (2.10). Note that thanks to (2.15), the continuous embedding $\mathbf{H}\subset D(\mathrm{A}^{\frac{1}{2}})$ and the Young inequality we deduce that there exists a constant $C_0>0$ such that

$$\begin{aligned}&2|(\tilde{B}(\mathbf{y}_2+\mathbf{u},\mathbf{z}),\mathbf{y})|\\ &\le \left[|\mathrm{A}^{\frac{1}{2}}\mathbf{y}|^2 + |\mathrm{A}\mathbf{y}|^2\right] + C_0\left[|\mathrm{A}\mathbf{y}_2|^2 + |\mathrm{A}\mathbf{u}|^2\right]\left[|\mathrm{A}^{\frac{1}{2}}\mathbf{y}_2|^2 + |\mathrm{A}\mathbf{u}|^2 + |\mathbf{y}|^2\right].\end{aligned} \tag{3.3}$$

Now, we let

$$\Psi(t) = e^{-C_0\int_0^t [|\mathrm{A}\mathbf{y}_2(s)|^2 + |\mathrm{A}\mathbf{u}(s)|^2]ds}, \quad t\in[0,T],$$

and apply the Itô formula to the real-valued process

$$x(t) = \Psi(t)\varphi(\mathbf{y}(t)) = \Psi(t)\left[|\mathbf{y}(t)|^2 + |\mathrm{A}^{\frac{1}{2}}\mathbf{y}(t)|^2\right], t\in[0,T].$$

This procedure, along with (3.3) and the Lipschitz continuity of G, yield

$$\begin{aligned}&x(t\wedge\tau_N) + 2\int_0^{t\wedge\tau_N}\psi(s)\left[|\mathrm{A}^{\frac{1}{2}}\mathbf{y}(s)|^2 + |\mathrm{A}\mathbf{y}(s)|^2\right]ds\\ &\le x(0) - C_0\int_0^{t\wedge\tau_N}\Psi(s)\left[|\mathrm{A}\mathbf{y}_2(s)|^2 + |\mathrm{A}\mathbf{u}(s)|^2\right]\left[|\mathrm{A}^{\frac{1}{2}}\mathbf{y}(s)|^2 + |\mathbf{y}(s)|^2\right]ds\\ &\quad + \int_0^{t\wedge\tau_N}\Psi(s)|(\tilde{B}(\mathbf{y}_2+\mathbf{u},\mathbf{z}),\mathbf{y})|ds + \int_0^{t\wedge\tau_N}\Psi(s)\|G(\mathbf{u}+\mathbf{y}_1) - G(\mathbf{u}+\mathbf{y}_2)\|^2_{\mathscr{L}(\mathrm{K},\mathbf{H})}ds\\ &\quad + \int_0^{t\wedge\tau_N}\Psi(s)\langle\mathbf{y}(s), G(\mathbf{u}+\mathbf{y}_1) - G(\mathbf{u}+\mathbf{y}_2)dW\rangle\\ &\le x(0) + C\int_0^{t\wedge\tau_N}\Psi(s)|\mathbf{y}_1 - \mathbf{y}_2|^2_{\mathbf{H}}ds + \int_0^{t\wedge\tau_N}\Psi(s)\left[|\mathrm{A}^{\frac{1}{2}}\mathbf{y}(s)|^2 + |\mathrm{A}\mathbf{y}(s)|^2\right]ds\\ &\quad + \int_0^{t\wedge\tau_N}\Psi(s)\langle\mathbf{y}(s), G(\mathbf{u}+\mathbf{y}_1) - G(\mathbf{u}+\mathbf{y}_2)dW\rangle.\end{aligned}$$

Observe that,

$$|\mathbf{y}_1-\mathbf{y}_2|^2=|\mathbf{y}|^2\le|\mathbf{y}|^2+|\mathrm{A}^{\frac{1}{2}}\mathbf{y}|^2.$$

Hence, by taking the mathematical expectation and using the fact that the stopped stochastic integral in the above inequalities are a zero mean martingale we obtain that

$$\begin{aligned}&\mathbb{E}x(t\wedge\tau_N)+\mathbb{E}\int_0^{t\wedge\tau_N}\Psi(s)\left[|\mathrm{A}^{\frac{1}{2}}\mathbf{y}(s)|^2+|\mathrm{A}\mathbf{y}(s)|^2\right]ds\\ \le&\mathbb{E}x(0)+C\int_0^{t\wedge\tau_N}\mathbb{E}x(s\wedge\tau_N)ds.\end{aligned}$$

By applying the Gronwall Lemma and the fact that $x(0)=0$, we see that

$$\mathbb{E}x(t\wedge\tau_N)=0,\quad t\in[0,T].$$

Since $x\ge0$ and $\Psi>0$ we see that for all $t\in[0,T]$ a.e.

$$\mathbf{y}_1(t)=\mathbf{y}_2(t)\text{ in }\mathbf{V}.$$

From the fact $\mathbf{y}_i\in C([0,T];D(\mathrm{A}^{\frac{1}{2}}))$ a.e. we finally conclude that a.e. for all $t\in[0,T]$

$$\mathbf{y}_1(t)=\mathbf{y}_2(t),$$

which completes the proof of the proposition. ∎

THEOREM 3.1. *Let $\delta\in\{0,1\}$ and $\xi\in\mathbf{V}$. Assume that G satisfies the Assumption 3.1. Then the stochastic evolution Equation (2.4) has a unique solution $\mathbf{y}^{\alpha,\delta}$ in the sense of Definition 3.1 such that*

$$\mathbf{y}^{\alpha,\delta}\in\mathbf{L}^p(\Omega;C[0,T];\mathbf{V})\cap\mathbf{L}^2([0,T];D(\mathrm{A}))\textit{ for all }p\in[1,\infty).$$

Proof. Observe that if $\delta=0$, then the problem (2.4) reduces to the stochastic system (2.1). Under the Assumption 3.1 it was proved in [8] that (2.1) has a unique solution $\mathbf{y}^{\alpha,0}$ satisfying $\mathbf{y}^{\alpha,0}\in\mathbf{L}^4(\Omega;C([0,T];\mathbf{V}))\cap\mathbf{L}^2(0,T;D(\mathrm{A}))$. The fact that $\mathbf{y}^{\alpha,0}\in\mathbf{L}^p(\Omega;C([0,T];\mathbf{V}))\cap\mathbf{L}^2(0,T;D(\mathrm{A}))$ for all $p\ge1$ is proved in [17].

Next, we recall that the deterministic evolution Equation (2.2) with initial data $\xi\in\mathbf{V}$ has a unique strong solution, $\mathbf{u}\in C([0,T];\mathbf{V})\cap\mathbf{L}^2(0,T;D(\mathrm{A}))$. Note that $\mathbf{u}$ is deterministic. If $\delta=1$, then, as discussed above, the stochastic process $\mathbf{y}^{\alpha,1}$ defined by

$$\mathbf{y}^{\alpha,1}=\frac{\mathbf{y}^{\alpha,0}-\mathbf{u}}{\lambda_\delta(\alpha)}\in\mathbf{L}^p(\Omega;C([0,T];\mathbf{V}))\cap\mathbf{L}^2(0,T;D(\mathrm{A})),$$

satisfies the problem (2.4). ∎

REMARK 3.2. The existence and uniqueness of a strong solution to (2.4) enables us to define a Borel measurable map $\Gamma_\xi^{\alpha,\delta}:C([0,T];\mathrm{K})\to C([0,T];\mathbf{H})\cap\mathbf{L}^2(0,T;D(\mathrm{A}^{\frac{1}{2}}))$ such that $\Gamma_\xi^{\alpha,\delta}(W)$ is the unique solution to (2.4) on the filtered probability space $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P})$ with the Wiener process W.

4. Analysis of the controlled evolution equations

In order to describe the rate functions associated to the LDP and MDP results, we also need to introduce a few additional notations and two auxiliary problems: the stochastic and deterministic controlled evolution equations.

For fixed $M>0$ we set

$$\mathcal{A}_M=\Big\{h\in L^2(0,T;\mathrm{K}):\int_0^T\|h(r)\|_{\mathrm{K}}^2dr\le M\Big\}.$$

The set $\mathcal{A}_M$, endowed with the weak topology

$$d_1(h,k)=\sum_{k\ge 1}\frac{1}{2^k}\Big|\int_0^T\big(h(r)-k(r),\tilde e_k(r)\big)_{\mathrm{K}}dr\Big|, \tag{4.1}$$

where $(\tilde e_k,k\ge 1)$ is an orthonormal basis for $L^2(0,T;\mathrm{K})$, is a Polish (complete separable metric) space, see [5].

We also introduce the class $\mathscr{A}$ as the set of K-valued $(\mathscr{F}_t)-$predictable stochastic processes h such that $\int_0^T\|h(r)\|_{\mathrm{K}}^2dr<\infty$, a.s. For $M>0$ we set

$$\mathscr{A}_M=\{h\in\mathscr{A}:h\in\mathcal{A}_M\ a.s.\}. \tag{4.2}$$

4.1. Analysis of the stochastic controlled evolution equations.

With the above notations at hand we now consider the stochastic controlled equation:

$$\begin{aligned}&d\mathbf{y}^{\alpha,\delta}+A\mathbf{y}^{\alpha,\delta}dt+\lambda_\delta(\alpha)\tilde B_\alpha(\mathbf{y}^{\alpha,\delta},\mathbf{z}^{\alpha,\delta})dt+\delta\tilde B_\alpha(\mathbf{u},\mathbf{z}^{\alpha,\delta})dt+\delta\tilde B_\alpha(\mathbf{y}^{\alpha,\delta},J_\alpha^{-1}\mathbf{u})dt\\&\quad+\delta\lambda_\delta^{-1}(\alpha)\Big[\tilde B_\alpha(\mathbf{u},J_\alpha^{-1}\mathbf{u})-B(\mathbf{u},\mathbf{u})\Big]dt\\&=\mathbf{G}_\alpha(\delta\mathbf{u}+\lambda_\delta(\alpha)\mathbf{y}^{\alpha,\delta})h(t)dt+\alpha^{\frac12}\lambda_\delta^{-1}(\alpha)\mathbf{G}_\alpha(\delta\mathbf{u}+\lambda_\delta(\alpha)\mathbf{y}^{\alpha,\delta})dW,\end{aligned} \tag{4.3}$$

where $h\in L^2(0,T;\mathrm{K})$.

We now need to prove the existence and uniqueness of (4.3) and derive uniform estimates for its solution. This will be the subject of the following theorem.

PROPOSITION 4.1. *Let $\delta\in\{0,1\}$, $\xi\in D(\mathrm{A}^{\frac12})$, $p\in[1,\infty)$. Let us also fix $M>0$ and $h\in\mathscr{A}_M$. If Assumption 3.1 is satisfied, then the stochastic controlled system (4.3) has a unique solution $\mathbf{y}_h^{\alpha,\delta}\in C([0,T];\mathbf{V})\cap\mathbf{L}^2(0,T;D(\mathrm{A}))$ such that*

$$\mathbf{y}_h^{\alpha,\delta}=\Gamma_\xi^{\alpha,\delta}\Big(W+\alpha^{-\frac12}\lambda_\delta(\alpha)\int_0^\cdot h(r)dr\Big).$$

Furthermore, then there exists a constant $C>0$ (which may depend on p) such that for any $\alpha\in(0,1)$ we have

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]}\|\mathbf{y}_h^{\alpha,\delta}(t)\|_\alpha^{2p}+\mathbb{E}\int_0^t\|\mathbf{y}_h^{\alpha,\delta}(t)\|_\alpha^{2p-2}\|\mathbf{y}_h^{\alpha,\delta}\|_{\mathbf{V}}^2ds\\&\le C\Big(1+|\mathrm{A}^{\frac12}\mathbf{y}_h^{\alpha,\delta}(0)|^2+CM^{\frac12}T^{\frac12}\Big(1+\delta^{2p}|\mathrm{A}^{\frac12}\mathbf{u}|^{2p}\Big)\Big)e^{\int_0^T\Phi_\delta(s)ds}\,\mathbb{P}\text{-}a.s.,\end{aligned} \tag{4.4}$$

where

$$\Phi_\delta:=1+\|h\|_{\mathrm{K}}+\delta^2[|\mathrm{A}\mathbf{u}|^2+|\mathbf{u}|^2]+|B(\mathbf{u},\mathbf{u})|^2+|\mathrm{A}\mathbf{u}|^2|\mathrm{A}^{\frac12}\mathbf{u}|^2.$$

Proof. Let $\delta\in\{0,1\}$, $\xi\in D(\mathrm{A}^{\frac{1}{2}})$, $p\in[1,\infty)$. Let us also fix $M>0$ and $h\in\mathscr{A}_M$. The proof of the theorem is divided into two parts.

Part I: Well-posedness of problem (4.3). Since $h\in\mathscr{A}_M$ we have

$$\mathbb{E}\exp\left(\frac{1}{2}\alpha^{-1}\lambda_\delta^2(\alpha)\int_0^T\|h(r)\|^2dr\right)<\infty.$$

Thus, by Girsanov's theorem there exists a probability measure $\mathbb{P}_h$ such that

$$\frac{d\mathbb{P}_h}{d\mathbb{P}}=\exp\left(\frac{1}{2}\alpha^{-1}\lambda_\delta(\alpha)^2\int_0^T\|h(r)\|^2_{\mathrm{K}}dr-\alpha^{-\frac{1}{2}}\lambda_\delta(\alpha)\int_0^T h(r)dW(r)\right),$$

and the stochastic process $\tilde{W}(\cdot):=W(\cdot)+\alpha^{-\frac{1}{2}}\lambda(\alpha)\int_0^\cdot h(r)dr$ defines a cylindrical Wiener process evolving on K and defined on the filtered probability space $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P}_h)$. Furthermore, the probability measure $\mathbb{P}$ is absolutely continuous with respect to the new probability measure $\mathbb{P}_h$. We also note that on $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P}_h)$ the problem (4.3) reads as

$$\begin{aligned}&d\mathbf{y}^{\alpha,\delta}+A\mathbf{y}^{\alpha,\delta}dt+\lambda_\delta(\alpha)\tilde{B}_\alpha(\mathbf{y}^{\alpha,\delta},\mathbf{z}^{\alpha,\delta})dt+\delta\tilde{B}_\alpha(\mathbf{u},\mathbf{z}^{\alpha,\delta})dt+\delta\tilde{B}_\alpha(\mathbf{y}^{\alpha,\delta},J_\alpha^{-1}\mathbf{u})dt\\&\quad+\delta\lambda_\delta^{-1}(\alpha)\left[\tilde{B}_\alpha(\mathbf{u},J_\alpha^{-1}\mathbf{u})-B(\mathbf{u},\mathbf{u})\right]dt=\alpha^{\frac{1}{2}}\lambda_\delta^{-1}(\alpha)\mathbf{G}_\alpha(\delta\mathbf{u}+\lambda_\delta(\alpha)\mathbf{y}^{\alpha,\delta})d\tilde{W}.\end{aligned}\tag{4.5}$$

Thus, similarly to the proof of Theorem 3.1 we can show that on $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P}_h)$ the problem (4.5) admits a unique strong solution $\mathbf{y}_h^{\alpha,\delta}$. In view of Remark 3.2 $\mathbf{y}_h^{\alpha,\delta}=\Gamma_\xi^{\alpha,\delta}(\tilde{W})$ on $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P}_h)$. Note that on $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P})$ we have

$$\mathbf{y}_h^{\alpha,\delta}=\Gamma_\xi^{\alpha,\delta}\left(W(\cdot)+\alpha^{-\frac{1}{2}}\lambda(\alpha)\int_0^\cdot h(r)dr\right).$$

That is, $\mathbf{y}_h^{\alpha,\delta}$ is the unique solution to (4.3).

Part II: Proof of the uniform estimates (4.4).

The proof of the estimate (4.4) relies on the application of the Itô formula to the functional $N_\alpha(x):=\|x\|_\alpha^2$ and the Itô process $\mathbf{y}^{\alpha,\delta}$. Before proceeding further let us observe that $N_\alpha(\cdot)$ is twice differentiable and its first and second derivatives satisfy

$$\begin{aligned}N_\alpha'(x)[h]&=2(x,h)+2\alpha^2(\mathrm{A}^{\frac{1}{2}}x,\mathrm{A}^{\frac{1}{2}}h)\ x,h\in D(\mathrm{A}^{\frac{1}{2}}),\\N_\alpha''(x)[h,k]&=2(h,k)+2\alpha^2(\mathrm{A}^{\frac{1}{2}}h,\mathrm{A}^{\frac{1}{2}}k)\ x,h,k\in D(\mathrm{A}^{\frac{1}{2}}).\end{aligned}$$

Also, if $x\in D(\mathrm{A})$ and $h\in D(\mathrm{A}^{\frac{1}{2}})$ then

$$N_\alpha'(x)[h]=2((I+\alpha^2\mathrm{A})x,h)=2(x,h)+\alpha^2(\mathrm{A}^{\frac{1}{2}}x,\mathrm{A}^{\frac{1}{2}}h)=:2\langle x,h\rangle_\alpha.$$

Now, by applying the Itô's formula to $N_\alpha(\mathbf{y}^{\alpha,\delta})$ and then to the function $\varphi(x)=x^p$ and $\|\mathbf{y}^{\alpha,\delta}\|_\alpha^2$ and using the property (2.17) we obtain

$$\begin{aligned}&\|\mathbf{y}_h^{\alpha,\delta}(t)\|_\alpha^{2p}+p\int_0^t\|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p-2}\|\mathrm{A}^{\frac{1}{2}}\mathbf{y}_h^{\alpha,\delta}\|^2_{\mathbf{H}}ds+p\alpha^2\int_0^t\|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p-2}\|\mathrm{A}\mathbf{y}_h^{\alpha,\delta}\|^2_{\mathbf{H}}ds\\&\quad-p\delta\int_0^t\|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p-2}\langle\mathbf{y}_h^{\alpha,\delta},\tilde{B}_\alpha(\mathbf{u},\mathbf{z}_h^{\alpha,\delta})\rangle_\alpha ds\end{aligned}$$

$$
\begin{aligned}
&+2\lambda_\delta^{-1}(\alpha)\int_0^t \|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p-2}\langle \tilde{B}_\alpha(\mathbf{u},J_\alpha^{-1}\mathbf{u})-B(\mathbf{u},\mathbf{u}),\mathbf{y}_h^{\alpha,\delta}\rangle_\alpha ds\\
\le&\|\mathbf{y}_h^{\alpha,\delta}(0)\|_\alpha^{2p}+p\alpha\lambda_\delta^{-2}(\alpha)\sum_{k\ge1}\int_0^t \|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p-2}\|G(\delta\mathbf{u}+\lambda_\delta(\alpha)\mathbf{y}_h^{\alpha,\delta})e_k\|_{\mathbf{L}^2}^2 ds\\
&+p(p-1)\alpha\lambda_\delta^{-2}(\alpha)\sum_{k\ge1}\int_0^t \|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p-4}\left(G(\delta\mathbf{u}+\lambda_\delta(\alpha)\mathbf{y}_h^{\alpha,\delta})e_k,\mathbf{y}^{\alpha,\delta}\right)^2 ds\\
&+\alpha^{\frac12}\lambda_\delta^{-1}(\alpha)p\int_0^t \|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p-2}\langle \mathbf{y}_h^{\alpha,\delta},\mathbf{G}_\alpha(\delta\mathbf{u}+\lambda_\delta(\alpha)\mathbf{y}_h^{\alpha,\delta})d\tilde{W}\rangle_\alpha. \qquad (4.6)
\end{aligned}
$$

We need to estimate the terms one by one in this relation. For doing this we start with the Itô correction terms. It is not difficult to see that there exists a constant $C(p)>0$ such that

$$
\begin{aligned}
I_1+I_2:=&p\alpha\lambda_\delta^{-2}(\alpha)\sum_{k\ge1}\int_0^t \|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p-2}\|G(\delta\mathbf{u}+\lambda_\delta(\alpha)\mathbf{y}_h^{\alpha,\delta})e_k\|_{\mathbf{L}^2}^2 ds\\
&+p(p-1)\alpha\lambda_\delta^{-2}(\alpha)\sum_{k\ge1}\int_0^t \|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p-4}\left(G(\delta\mathbf{u}+\lambda_\delta(\alpha)\mathbf{y}_h^{\alpha,\delta})e_k,\mathbf{y}^{\alpha,\delta}\right)^2 ds\\
\le& C(p)\alpha\lambda_\delta^{-2}(\alpha)\int_0^t \|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p-2}\|G(\delta\mathbf{u}+\lambda_\delta(\alpha)\mathbf{y}_h^{\alpha,\delta})\|_{\mathscr{L}_2(\mathrm{K},\mathbf{H})}^2 ds,
\end{aligned}
$$

from which, along with the Assumption 3.1 and Remark 3.1 and the Young inequality, we deduce that

$$
\begin{aligned}
I_1+I_2\le& C(p)\alpha\lambda_\delta^{-2}(\alpha)\int_0^t \left[1+\|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha\right]^{2p-2}\left(1+\delta^2|\mathbf{u}|^2+\lambda_\delta^2(\alpha)\|\mathbf{y}_h^{\alpha,\delta}\|_\alpha^2\right)ds\\
\le& C(p)\alpha\lambda_\delta^{-2}(\alpha)\int_0^t \|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p-2}\left[1+\delta^2|\mathbf{u}|^2\right]ds+C(p)\alpha\int_0^t \|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p}ds\\
\le& C(p)\alpha\lambda_\delta^{-2}(\alpha)\int_0^t \left[1+\|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p}\right]\left[1+\delta^2|\mathbf{u}|^2\right]ds+C(p)\alpha\int_0^t \|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p}ds. \qquad (4.7)
\end{aligned}
$$

Next, since $h\in\mathscr{A}_M$, applications of Cauchy-Schwarz's inequality, Assumption (3.1) and Remark 3.1, and Young's inequality imply that there exists a constant $C>0$ such that for any $\alpha\in(0,1)$

$$
\begin{aligned}
&\int_0^t \|\mathbf{y}^{\alpha,\delta}(r)\|_\alpha^{2p-2}(G(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{y}^{\alpha,\delta})h(r),\mathbf{y}^{\alpha,\delta}(r))dr\\
\le& C(1+\lambda_\delta(\alpha))\int_0^t \|\mathbf{y}^{\alpha,\delta}(r)\|_\alpha^{2p}\|h(r)\|_{\mathrm{K}}dr+CM^{\frac12}T^{\frac12}\left(1+\delta^{2p}\sup_{s\in[0,T]}|\mathbf{u}(s)|^{2p}\right).
\end{aligned}
$$

The perturbation term containing $\mathbf{u}$ can be estimated as follows. By applying (2.19) we infer that

$$
\begin{aligned}
&\lambda_\delta^{-1}(\alpha)\int_0^t \|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p-2}(\tilde{B}_\alpha(\mathbf{u},J_\alpha^{-1}\mathbf{u})-B(\mathbf{u},\mathbf{u}),(I+\alpha^2A)\mathbf{y}_h^{\alpha,\delta})ds\\
\le& C\lambda_\delta^{-1}(\alpha)\int_0^t \|\mathbf{y}_h^{\alpha,\delta}\|_\alpha^{2p-2}[\alpha^2|\mathbf{y}_h^{\alpha,\delta}|+\alpha|\mathrm{A}^{\frac12}\mathbf{y}_h^{\alpha,\delta}|+\alpha^2|\mathrm{A}\mathbf{y}_h^{\alpha,\delta}|]|B(\mathbf{u},\mathbf{u})|ds\\
&+C\alpha^2\lambda_\delta^{-1}(\alpha)\int_0^t \|\mathbf{y}_h^{\alpha,\delta}\|_\alpha^{2p-2}|\mathrm{A}\mathbf{y}_h^{\alpha,\delta}||\mathrm{A}\mathbf{u}||\mathrm{A}^{\frac12}\mathbf{u}|ds.
\end{aligned}
$$

Several applications of the Young inequality on the right-hand side of the last inequality imply

$$\begin{aligned}&\lambda_\delta^{-1}(\alpha)\int_0^t \|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p-2}(\tilde{B}_\alpha(\mathbf{u},J_\alpha^{-1}\mathbf{u})-B(\mathbf{u},\mathbf{u}),(I+\alpha^2\mathrm{A})\mathbf{y}_h^{\alpha,\delta})ds\\ \le& C(p)\alpha^2\lambda_\delta^{-2}(\alpha)\int_0^t\left(1+|B(\mathbf{u},\mathbf{u})|^2+|\mathrm{A}\mathbf{u}|^2|\mathrm{A}^{\frac12}\mathbf{u}|^2\right)\left[1+\|\mathbf{y}^{\alpha,\delta}\|_\alpha^{2p}\right]ds\\ &+\frac{p}{4}\int_0^t \|\mathbf{y}_h^{\alpha,\delta}\|_\alpha^{2p-2}\left(|\mathrm{A}^{\frac12}\mathbf{y}^{\alpha,\delta}|^2+\alpha^2|\mathrm{A}\mathbf{y}^{\alpha,\delta}|^2\right)ds \qquad (4.8)\end{aligned}$$

By using the estimate (2.15), the Young inequality yields and the fact $|y|\le\|y\|_\alpha$ we see that

$$\begin{aligned}&p\delta\int_0^t \|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p-2}\langle\mathbf{y}_h^{\alpha,\delta},\tilde{B}_\alpha(\mathbf{u},\mathbf{z}_h^{\alpha,\delta})\rangle_\alpha ds\\ =&p\delta\int_0^t \|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p-2}\langle\mathbf{y}_h^{\alpha,\delta},\tilde{B}(\mathbf{u},\mathbf{z}_h^{\alpha,\delta})\rangle ds\\ \le& Cp\delta\int_0^t \|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p-2}|\mathrm{A}^{\frac12}\mathbf{y}_h^{\alpha,\delta}||\mathrm{A}\mathbf{u}|\left(|\mathbf{y}_h^{\alpha,\delta}|+\alpha^2|\mathrm{A}\mathbf{y}_h^{\alpha,\delta}|\right)ds\\ \le& C(p)\delta^2\int_0^t \|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p}|\mathrm{A}\mathbf{u}(s)|^2ds\\ &+\frac{p}{4}\int_0^t \|\mathbf{y}_h^{\alpha,\delta}(s)\|_\alpha^{2p-2}\left(|\mathrm{A}^{\frac12}\mathbf{y}^{\alpha,\delta}|^2+\alpha^2|\mathrm{A}\mathbf{y}^{\alpha,\delta}|^2\right)ds. \qquad (4.9)\end{aligned}$$

The following estimates are obtained by applying the Burkholder-Davis-Gundy inequality (BDG) and the Young inequality

$$\begin{aligned}&\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s \|\mathbf{y}^{\alpha,\delta}(s)\|_\alpha^{2p-2}\langle\mathbf{G}_\alpha(\delta\mathbf{u}+\lambda_\delta(\alpha)\mathbf{y}^{\alpha,\delta})d\tilde{W},\mathbf{y}^{\alpha,\delta}\rangle_\alpha\right|\\ \le&\mathbb{E}\left[\int_0^t \|\mathbf{y}^{\alpha,\delta}(s)\|_\alpha^{2p-2}\|G(\delta\mathbf{u}+\lambda_\delta(\alpha)\mathbf{y}^{\alpha,\delta})\|^2_{\mathscr{L}_2(\mathrm{K},\mathbf{H})}\|\mathbf{y}^{\alpha,\delta}(s)\|_\alpha^{2p}ds\right]^{\frac12}\\ \le&\frac12\mathbb{E}\sup_{0\le s\le t}\|\mathbf{y}^{\alpha,\delta}\|_\alpha^{2p}+C\mathbb{E}\int_0^t \|\mathbf{y}^{\alpha,\delta}(s)\|_\alpha^{2p-2}\|G(\delta\mathbf{u}+\lambda_\delta(\alpha)\mathbf{y}^{\alpha,\delta})\|^2_{\mathscr{L}_2(\mathrm{K},\mathbf{H})}ds.\end{aligned}$$

Observe that second term of the right-hand side of the last inequality can be dealt with the same technique as used in the proof of (4.7). In particular, we see that

$$\begin{aligned}&\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s \|\mathbf{y}^{\alpha,\delta}(s)\|_\alpha^{2p-2}\langle\mathbf{G}_\alpha(\delta\mathbf{u}+\lambda_\delta(\alpha)\mathbf{y}^{\alpha,\delta})d\tilde{W},\mathbf{y}^{\alpha,\delta}\rangle_\alpha\right|\\ \le&\frac12\mathbb{E}\sup_{0\le s\le t}\|\mathbf{y}^{\alpha,\delta}\|_\alpha^{2p}+C\mathbb{E}\int_0^t\left[1+\|\mathbf{y}^{\alpha,\delta}(s)\|_\alpha^{2p}\right]\left[1+\delta^2|\mathbf{u}|^2+\lambda_\delta^2(\alpha)\right]ds. \qquad (4.10)\end{aligned}$$

By plugging the inequalities (4.7), (4.8), (4.9) and (4.10) into (4.6) and by taking into account the Remark 2.1 we find that there exists a constant $C(p)>0$ such that

$$\mathbb{E}\sup_{s\in[0,t]}\|\mathbf{y}_h^{\alpha,\delta}(t)\|_\alpha^{2p}+p\mathbb{E}\int_0^t \|\mathbf{y}_h^{\alpha,\delta}(t)\|_\alpha^{2p-2}\left[\|\mathrm{A}^{\frac12}\mathbf{y}_h^{\alpha,\delta}\|^2_{\mathbf{H}}+\alpha^2\|\mathrm{A}\mathbf{y}_h^{\alpha,\delta}\|^2_{\mathbf{H}}\right]ds$$

$$\leq C(p)\mathbb{E}\int_0^t \left[1+\|\mathbf{y}_h^{\alpha,\delta}(t)\|_\alpha\right]^{2p}\left(1+\|h\|_{\mathrm{K}}+\delta^2[|\mathrm{A}\mathbf{u}|^2+|\mathbf{u}|^2]\right)ds$$
$$+C(p)\mathbb{E}\int_0^t \left[1+\|\mathbf{y}_h^{\alpha,\delta}(t)\|_\alpha\right]^{2p}\left(|B(\mathbf{u},\mathbf{u})|^2+|\mathrm{A}\mathbf{u}|^2|\mathrm{A}^{\frac{1}{2}}\mathbf{u}|^2\right)ds$$
$$+CM^{\frac{1}{2}}T^{\frac{1}{2}}\left(1+\delta^{2p}|\mathrm{A}^{\frac{1}{2}}\mathbf{u}|^{2p}\right)+\mathbb{E}\|\mathbf{y}_h^{\alpha,\delta}(0)\|_\alpha^{2p}. \tag{4.11}$$

Applying the Gronwall lemma now yields that there exists a constant $C>0$ such that for any $\alpha\in(0,1)$ we have

$$\mathbb{E}\sup_{t\in[0,T]}\|\mathbf{y}_h^{\alpha,\delta}(t)\|_\alpha^{2p}+\mathbb{E}\int_0^t\|\mathbf{y}_h^{\alpha,\delta}(t)\|_\alpha^{2p-2}\|\mathbf{y}_h^{\alpha,\delta}\|_{\mathbf{V}}^2 ds$$
$$\leq C\left(1+\mathbb{E}|\mathrm{A}^{\frac{1}{2}}\mathbf{y}_h^{\alpha,\delta}(0)|^2+CM^{\frac{1}{2}}T^{\frac{1}{2}}\left(1+\delta^{2p}|\mathrm{A}^{\frac{1}{2}}\mathbf{u}|^{2p}\right)\right)\mathbb{E}e^{\int_0^T\Phi_\delta(s)ds}, \tag{4.12}$$

where

$$\Phi:=1+M+\delta^2[|\mathrm{A}\mathbf{u}|^2+|\mathbf{u}|^2]+|B(\mathbf{u},\mathbf{u})|^2+|\mathrm{A}\mathbf{u}|^2|\mathrm{A}^{\frac{1}{2}}\mathbf{u}|^2.$$

This completes the proof of Proposition 4.1. ∎

4.2. Analysis of the deterministic controlled Navier-Stokes-α. In this subsection we fix $h\in\mathbf{L}^2(0,T;\mathrm{K})$ and analyse the following deterministic controlled Navier-Stokes-α model:

$$d\mathbf{y}^\delta+\mathrm{A}\mathbf{y}^\delta dt+(1-\delta)B(\mathbf{y}^\delta,\mathbf{y}^\delta)dt+\delta B(\mathbf{u},\mathbf{y}^\delta)+\delta B(\mathbf{y}^\delta,\mathbf{u})=G(\delta\mathbf{u}+(1-\delta)\mathbf{y}^\delta)h, \tag{4.13a}$$
$$\mathbf{y}^\delta(0)=(1-\delta)\xi. \tag{4.13b}$$

The main result of this subsection is given in the following theorem.

THEOREM 4.1. *Let $h\in\mathbf{L}^2(0,T;\mathrm{K})$ and $\xi\in\mathbf{V}$. Then, (4.13) has a unique solution $\mathbf{y}_h^\delta\in C([0,T];\mathbf{H}^1)\cap\mathbf{L}^2(0,T;\mathbf{H}^2)$. Moreover, if $h\in\mathcal{A}_M, M>0$, then there exists a deterministic constant $C>0$, which depends only on M and $\|\xi\|$, such that*

$$\sup_{t\in[0,T]}(|\mathbf{y}_h^\delta(t)|^2+\|\mathbf{y}_h^\delta(t)\|^2)+\int_0^T\left(|\mathrm{A}\mathbf{y}_h^\delta(t)|^2+|\mathrm{A}\mathbf{y}_h^\delta(t)|^2\right)dt\leq C. \tag{4.14}$$

REMARK 4.1.

- Note that when $\delta=0$ and $h=0$ the above theorem provides also the following estimates for $\mathbf{u}$, the unique solution to (2.2):

$$\sup_{t\in[0,T]}\|\mathbf{u}(t)\|^2+\int_0^T|\mathrm{A}\mathbf{u}(t)|^2dt\leq C. \tag{4.15}$$

  This estimate could be found in many classical works for the Navier-Stokes equations such as [36] and [14].
- Note that when $h\in\mathscr{A}_M, M>0$, then (4.13) still has a unique solution $\mathbf{y}_h^\delta$ such that $\mathbf{y}_h^\delta\in C([0,T];\mathbf{H}^1)\cap\mathbf{L}^2(0,T;\mathbf{H}^2)$ with probability 1. Moreover, the estimate (4.14) holds with probability 1.

Proof. (**Proof of Proposition 4.1.**) Since the system (4.13) is the Navier-Stokes equations with the linear perturbations $\delta B(\mathbf{u},\mathbf{y}^\delta)+\delta B(\mathbf{y}^\delta,\mathbf{u})$ and the Lipschitz continuous perturbations $G(\delta\mathbf{u}+(1-\delta)\mathbf{y}^\delta)h$, we can prove the existence and uniqueness results in the above theorem by following the standard scheme of proof for the Navier-Stokes equations, see, for instance, [36]. Since this is now standard we only focus on deriving the crucial estimates for the solutions. For the sake of simplicity, we will just write $\mathbf{y}^\delta$ instead of $\mathbf{y}_h^\delta$. We will also suppress the dependence of $\mathbf{y}^\delta$ on the time variable.

By formally multiplying the first equation in (4.13) by $(I+\mathrm{A})\mathbf{y}^\delta(t)$ we find that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(|\mathbf{y}^\delta|^2+|\mathrm{A}^{\frac{1}{2}}\mathbf{y}^\delta|^2\right)+|\mathrm{A}\mathbf{y}^\delta|^2+|\mathrm{A}^{\frac{1}{2}}\mathbf{y}^\delta|^2+(1-\delta)((I+\mathrm{A})\mathbf{y}^\delta,B(\mathbf{y}^\delta,\mathbf{y}^\delta))\\&\quad+\delta((I+\mathrm{A})\mathbf{y}^\delta,B(\mathbf{u},\mathbf{y}^\delta)+B(\mathbf{y}^\delta,\mathbf{u}))=\big((I+\mathrm{A})\mathbf{y}^\delta,G(\delta\mathbf{u}+(1-\delta)\mathbf{y}^\delta)h(t)\big).\end{aligned}\tag{4.16}$$

Observe that since we are working on the torus

$$((I+\mathrm{A})\mathbf{y}^\delta,B(\mathbf{y}^\delta,\mathbf{y}^\delta))=0\quad\text{and}\quad(B(\mathbf{u},\mathbf{y}^\delta),\mathbf{y}^\delta)=0.$$

By the Hölder, the Gagliardo-Nirenberg and the Young inequalities we obtain

$$\begin{aligned}&\delta(B(\mathbf{u},\mathbf{y}^\delta)+B(\mathbf{y}^\delta,\mathbf{u}),(I+\mathrm{A})\mathbf{y}^\delta)\\&\le\delta|C(I+\mathrm{A})\mathbf{y}^\delta|[|\mathbf{u}|_{\mathbf{L}^\infty}|\mathrm{A}^{\frac{1}{2}}\mathbf{y}^\delta|+|\mathbf{y}^\delta|_{\mathbf{L}^2}|\nabla\mathbf{u}|_{\mathbf{L}^2}]\\&\le\delta C[|\mathbf{y}|^\delta+|\mathrm{A}\mathbf{y}^\delta|]|\mathrm{A}\mathbf{u}||\mathrm{A}^{\frac{1}{2}}\mathbf{y}^\delta|\\&\le\frac{1}{4}|\mathrm{A}\mathbf{y}^\delta|^2+C\delta[1+|\mathrm{A}\mathbf{u}|^2][|\mathbf{y}^\delta|^2+|\mathrm{A}^{\frac{1}{2}}\mathbf{y}^\delta|].\end{aligned}$$

We will now deal with the term containing G. By using the Cauchy-Schwartz, the Young inequalities and the Assumption 3.1 we see that

$$\begin{aligned}&(G(\delta\mathbf{u}+(1-\delta)\mathbf{y}^\delta)h,(I+\mathrm{A})\mathbf{y}^\delta)\\&\le G|(I+\mathrm{A})\mathbf{y}^\delta||h|_{\mathrm{K}}|G(\delta\mathbf{u}+(1-\delta)\mathbf{y}^\delta)|_{\mathscr{L}_2(\mathrm{K},\mathbf{H})}\\&\le\frac{1}{4}|\mathrm{A}\mathbf{y}^\delta|^2+C|\mathbf{y}^\delta|^2[|h|_{\mathrm{K}}^2+(1-\delta)^2]+C|h|_{\mathrm{K}}^2[1+\delta^2|\mathbf{u}|^2].\end{aligned}$$

Collecting these inequalities together implies

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}[|\mathbf{y}^\delta|^2+|\mathrm{A}^{\frac{1}{2}}\mathbf{y}^\delta|]+|\mathrm{A}^{\frac{1}{2}}\mathbf{y}^\delta|^2+\frac{1}{2}|\mathrm{A}\mathbf{y}^\delta|^2\\&\le C[|\mathbf{y}^\delta|^2+|\mathrm{A}^{\frac{1}{2}}\mathbf{y}^\delta|][\delta(1+|\mathrm{A}\mathbf{u}|^2)+|h|_{\mathrm{K}}^2+(1-\delta)^2]+C|h|_{\mathrm{K}}^2[1+\delta^2|\mathbf{u}|^2].\end{aligned}$$

Applying the Gronwall's inequality yields that there exists a constant $C>0$, depending only on M and $|\mathrm{A}\xi|$, such that

$$\sup_{t\in[0,T]}(|\mathbf{y}^\delta(t)|^2+|\mathrm{A}^{\frac{1}{2}}\mathbf{y}^\delta(t)|^2)+\int_0^T[|\mathrm{A}^{\frac{1}{2}}\mathbf{y}^\delta(t)|^2+|\mathrm{A}\mathbf{y}^\delta(t)]dt\le C,\tag{4.17}$$

which completes the proof of Proposition 4.1. ∎

REMARK 4.2. The above theorem enables us to define a map $\Gamma_\xi^{0,\delta}:C([0,T];\mathrm{K})\to C([0,T];\mathbf{H})\cap\mathbf{L}^2(0,T;D(\mathrm{A}^{\frac{1}{2}}))$ by setting

- $\Gamma_\xi^{0,\delta}(x)$ is the unique solution $\mathbf{u}_h^\delta$ to (4.13a) if $x=\int_0^\cdot h(r)dr$, $h\in\mathbf{L}^2(0,T;\mathrm{K})$;
- $\Gamma_\xi^{0,\delta}(x)=0$ otherwise.

We will see in the next theorem and Remark 4.3 that this map is in fact Borel measurable.

We now state and prove the following two important results.

PROPOSITION 4.2. *Let* δ *and* $\xi\in D(\mathrm{A}^{\frac{1}{2}})$. *Then, the set* $\{\Gamma^{0,\delta}_{\xi}(\int_0^{\cdot} h(s)ds): h\in \mathrm{A}_M\}$ *is a compact set of* $C([0,T]:\mathbf{H})\cap \mathbf{L}^2(0,T;D(\mathrm{A}^{\frac{1}{2}}))$.

REMARK 4.3. The above proposition amounts to saying that if $(h_n)_{n\in\mathbb{N}}\subset \mathcal{A}_M$, $M>0$, is a sequence that converges weakly to $h\in\mathcal{A}_M$, then $\Gamma^{0,\delta}_{\xi}\left(\int_0^{\cdot} h_n(r)dr\right)$ strongly converges to $\Gamma^{0,\delta}_{\xi}(\int_0^{\cdot} h(r)dr)$ in $C([0,T];\mathbf{H})\cap \mathbf{L}^2(0,T;D(\mathrm{A}^{\frac{1}{2}}))$. Consequently, the map

$$\mathcal{A}_M\ni h\mapsto \Gamma^{0,\delta}_{\xi}(\int_0^{\cdot} h(r)dr)\in C([0,T];\mathbf{H})\cap \mathbf{L}^2(0,T;D(\mathrm{A}^{\frac{1}{2}})),$$

is Borel measurable.

Proof. (**Proof of Proposition 4.2.**) Let $(h_n)_{n\in\mathbb{N}}\subset\mathcal{A}_M$ and $h\in\mathcal{A}_M$ such that

$$h_n\to h \qquad \text{weakly in} \qquad \mathbf{L}^2(0,T;\mathrm{K}).$$

Let us denote by $\mathbf{y}_n=\Gamma^{0,\delta}(\int_0^{\cdot} h_n(s)ds)$, $n\in\mathbb{N}$. Then by Proposition 4.1 there exists a constant $C>0$ such that for all $n\in\mathbb{N}$

$$\sup_{t\in[0,T]}\left(|\mathbf{y}_n(t)|^2+|\mathrm{A}^{\frac{1}{2}}\mathbf{y}_n(t)|^2\right)+\int_0^T\left(|\mathrm{A}^{\frac{1}{2}}\mathbf{y}_n|^2+|\mathrm{A}\mathbf{y}_n(t)|^2\right)ds<C. \tag{4.18}$$

Furthermore,

$$|\partial_t\mathbf{y}_n|\le|\mathrm{A}\mathbf{y}_n|+(1-\delta)|B(\mathbf{y}_n,\mathbf{y}_n)|+\delta[|B(\mathbf{u},\mathbf{y}_n)|+|B(\mathbf{y}_n,\mathbf{u})|]$$

Now, observe that by making use of the Hölder and the Gagliardo-Nirenberg inequalities we obtain that there exists $C>0$ such that for all $n\in\mathbb{N}$

$$\begin{aligned}&(1-\delta)|\mathbf{y}_n|_{\mathbf{L}^\infty}|\mathrm{A}^{\frac{1}{2}}\mathbf{y}_n|+\delta[|\mathbf{u}|_{\mathbf{L}^\infty}|\mathrm{A}^{\frac{1}{2}}\mathbf{y}_n|+|\mathbf{y}_n|_{\mathbf{L}^\infty}|\mathrm{A}^{\frac{1}{2}}\mathbf{u}|]\\ \le& C|\mathrm{A}\mathbf{y}_n|[(1-\delta)|\mathrm{A}^{\frac{1}{2}}\mathbf{y}_n|+\delta|\mathrm{A}^{\frac{1}{2}}\mathbf{u}|]+C\delta|\mathrm{A}^{\frac{1}{2}}\mathbf{y}_n||\mathrm{A}\mathbf{u}|.\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^T|\partial_t\mathbf{y}_n|^2dt\le& C[(1-\delta)\sup_t|\mathrm{A}^{\frac{1}{2}}\mathbf{y}_n|^2+\delta\sup_t|\mathrm{A}^{\frac{1}{2}}\mathbf{u}|^2]\int_0^T|\mathrm{A}\mathbf{y}_n|^2ds\\ &+C\delta\sup_t|\mathrm{A}^{\frac{1}{2}}\mathbf{y}_n|^2\int_0^T|\mathrm{A}\mathbf{u}|^2ds\\ \le& C[(1-\delta)+\delta\sup_t|\mathrm{A}^{\frac{1}{2}}\mathbf{u}|^2]+C\delta\int_0^T|\mathrm{A}\mathbf{u}|^2ds.\end{aligned} \tag{4.19}$$

The estimates (4.18) and (4.19) imply that

- $(\mathbf{y}_n)_n$ is uniformly bounded in $C([0,T];D(\mathrm{A}^{\frac{1}{2}}))\cap\mathbf{L}^2(0,T;D(\mathrm{A}))$.
- $(\partial_t\mathbf{y}_n)_n$ is uniformly bounded in $\mathbf{L}^2(0,T;\mathbf{H})$.

Hence, by the Banach-Alaoglu and the Aubin-Lions theorem there exists a subsequence, still denoted by $\mathbf{y}_n$, of $\mathbf{y}_n$ and $\mathbf{y}$ such that

$$\mathbf{y}_n\to\tilde{\mathbf{y}} \quad \text{weak-* in } \mathbf{L}^\infty(0,T;D(\mathrm{A}^{\frac{1}{2}}))$$

$$\mathbf{y}_n \to \tilde{\mathbf{y}} \quad \text{weak in } \mathbf{L}^2(0,T;D(\mathrm{A})) \tag{4.20}$$

$$\mathbf{y}_n \to \tilde{\mathbf{y}} \quad \text{strong in } \mathbf{L}^2(0,T;D(\mathrm{A}^{\frac{1}{2}})) \tag{4.21}$$

$$\partial_t \mathbf{y}_n \to \partial_t \tilde{\mathbf{y}} \quad \text{weak in } \mathbf{L}^2(0,T;\mathbf{H})$$

The last convergence and the first one imply that

$$\tilde{\mathbf{y}} \in C([0,T];\mathbf{H}) \cap C_w([0,T];D(\mathrm{A}^{\frac{1}{2}})),$$

where $C_w([0,T];D(\mathrm{A}^{\frac{1}{2}}))$ denotes the space of functions $f:[0,T]\to X$, X a given Banach space, that are weakly continuous. By passing to the limits we shall show that $\tilde{\mathbf{y}}$ is a solution to the system (4.13). In fact by arguing exactly as in [36] we see that

$$B(\mathbf{y}^n,\mathbf{y}^n) \to B(\tilde{\mathbf{y}},\tilde{\mathbf{y}}) \quad \text{in} \quad \mathbf{L}^2(0,T;\mathbf{H}).$$

Since $B(\mathbf{u},\mathbf{y}^n)$, $B(\mathbf{y}^n,\mathbf{u})$ and $\mathrm{A}\mathbf{y}^n$ are linear continuous $D(\mathrm{A}^{\frac{1}{2}}), D(\mathrm{A}^{\frac{1}{2}})$ and $D(\mathrm{A})$, respectively, by using the strong convergence (4.21) and the weak convergence (4.20) we obtain

- $B(\mathbf{u},\mathbf{y}_n)+B(\mathbf{y}_n,\mathbf{u}) \to B(\mathbf{u},\tilde{\mathbf{y}})+B(\tilde{\mathbf{y}},\mathbf{u})$ strong in $\mathbf{L}^2(0,T;\mathbf{H})$.
- $\mathrm{A}\mathbf{y}_n \to \mathrm{A}\tilde{\mathbf{y}}$ weak in $\mathbf{L}^2(0,T;\mathbf{H})$.

Note that the first convergence holds because $\mathbf{u} \in C([0,T],D(\mathrm{A}^{\frac{1}{2}})) \cap \mathbf{L}^2(0,T;D(\mathrm{A}))$.

What remains to prove is that

$$G(\delta\mathbf{u}+\lambda_\delta(\alpha)\mathbf{y}_n)h_n \to G(\delta\mathbf{u}+\lambda_\delta(\alpha)y)h \quad \text{in } \mathbf{L}^2(0,T;\mathbf{H}). \tag{4.22}$$

To prove this fact we first observe that the Assumption 3.1 and the strong convergence (4.21) imply

$$G(\delta\mathbf{u}+\lambda_\delta(\alpha)\mathbf{y}_n) \to G(\delta\mathbf{u}+\lambda_\delta(\alpha)\tilde{\mathbf{y}}) \quad \text{strongly in} \quad \mathbf{L}^2(0,T;\mathscr{L}_2(\mathrm{K},\mathbf{H})).$$

This, along with the assumption that $h_n \to h$ weak in $\mathbf{L}^2(0,T;\mathrm{K})$, implies that convergence (4.22) holds true.

By collecting all the above convergences, it is not difficult to see that $\tilde{\mathbf{y}}$ is a solution to (4.13). By uniqueness, $\tilde{\mathbf{y}} = \mathbf{y}_h^\delta = \Gamma^{0,\delta}(\int_0^\cdot h(s)ds)$ and the whole sequence $\mathbf{y}_n$ converges to $\mathbf{y}_h^\delta$. This completes the proof of Proposition 4.2. ∎

5. The deviation principles result and its proofs

5.1. Formulation of the main results. This section is the heart of this paper. We will state and prove our main results in this section, but before doing so we briefly recall a few definitions from the LDP theory.

Let $\mathcal{E}$ be a Polish space and $\mathcal{B}(\mathcal{E})$ its Borel σ-algebra.

DEFINITION 5.1. *A function $I:\mathcal{E}\to[0,\infty]$ is a (good) rate function if it is lower semicontinuous and the level sets $\{e\in\mathcal{E}; I(e)\le a\}$, $a\in[0,\infty)$, are compact subsets of $\mathcal{E}$.*

Next let ϱ be a real-valued map defined on $[0,\infty)$ such that

$$\varrho(\varepsilon)\to\infty \text{ as } \varepsilon\to\infty.$$

DEFINITION 5.2. *Let $(\Omega,\mathscr{F},\mathbb{P})$ be a complete probability space. An $\mathcal{E}$-valued random variable $(X_\varepsilon)_{\varepsilon\in(0,1]}$ satisfies the LDP on $\mathcal{E}$ with speed $\varrho(\varepsilon)$ and rate function I if and only if the following two conditions hold*

(a) *for any closed set* $F \subset \mathcal{E}$

$$\limsup_{\varepsilon\to 0} \varrho^{-1}(\varepsilon)\log \mathbb{P}(X_\varepsilon \in F) \leq -\inf_{x\in F} I(x);$$

(b) *for any open set* $O \subset \mathcal{E}$

$$\liminf_{\varepsilon\to 0} \varrho^{-1}(\varepsilon)\log \mathbb{P}(X_\varepsilon \in O) \geq -\inf_{x\in O} I(x).$$

We are now ready to state our main results.

THEOREM 5.1. *Let* $\delta \in \{0,1\}$*,* $\xi \in D(\mathrm{A}^{\frac{1}{2}})$ *and Assumption 3.1 holds. Then, the family* $(\mathbf{u}^{\alpha,\delta})_{\alpha\in(0,1]}$ *satisfies an LDP on* $C([0,T];\mathbf{H})\cap \mathbf{L}^2(0,T;D(\mathrm{A}^{\frac{1}{2}}))$ *with speed* $\alpha^{-1}\lambda_\delta^2(\alpha)$ *and rate function* I_δ *given by*

$$I_\delta(x) = \inf_{\{h\in \mathbf{L}^2(0,T;\mathrm{K}):x=\Gamma_\xi^{0,\delta}(\int_0^\cdot h(r)dr)\}} \left\{\frac{1}{2}\int_0^T \|h(r)\|_{\mathrm{K}}^2 dr\right\}.$$

As usual, we understand that $\inf\emptyset = \infty$.

Proof. The proof requires a few preparations and hence it will be postponed to Subsection 5.3. ∎

We can divide the result in the above theorem into two parts which will form the following two corollaries. They give the LDP and MDP on $C([0,T];\mathbf{H})\cap\mathbf{L}^2(0,T;D(\mathrm{A}^{\frac{1}{2}}))$ for the solution $\mathbf{u}^\alpha$ to (2.1).

COROLLARY 5.1. *Let* $\xi\in D(\mathrm{A}^{\frac{1}{2}})$ *and* G *satisfies Assumption 3.1. Then, the family of solutions* $(\mathbf{u}^\alpha)_{\alpha\in(0,1]}$ *to* (2.1) *satisfies an LDP on* $C([0,T];\mathbf{H})\cap\mathbf{L}^2(0,T;D(\mathrm{A}^{\frac{2}{2}}))$ *with speed* α^{-1} *and rate function* I_0 *given by*

$$I_0(x) = \inf_{\{h\in \mathbf{L}^2(0,T;\mathrm{K}):x=\Gamma_\xi^{0,0}(\int_0^\cdot h(r)dr)\}} \left\{\frac{1}{2}\int_0^T \|h(r)\|_{\mathrm{K}}^2 dr\right\}. \tag{5.1}$$

COROLLARY 5.2. *If* $\xi\in D(\mathrm{A}^{\frac{1}{2}})$ *and* G *satisfies Assumption 3.1, then* $\left(\alpha^{-\frac{1}{2}}\lambda^{-1}(\alpha)[\mathbf{u}^\alpha-\mathbf{u}]\right)_{\alpha\in(0,1]}$ *satisfies an LDP on* $C([0,T];\mathbf{H})\cap\mathbf{L}^2(0,T;D(\mathrm{A}^{\frac{1}{2}}))$ *with speed* $\lambda^2(\alpha)$ *and rate function* I_1 *given by*

$$I_1(x) = \inf_{\{h\in \mathbf{L}^2(0,T;\mathrm{K}):x=\Gamma_\xi^{0,1}(\int_0^\cdot h(r)dr)\}} \left\{\frac{1}{2}\int_0^T \|h(r)\|_{\mathrm{K}}^2 dr\right\}. \tag{5.2}$$

5.2. Intermediate results. In order to prove Theorem 5.1 we will use the weak convergence approach to LDP and Budhiraja-Dupuis' results on representation of functionals of Brownian motion, see [4] and [5]. These require a few intermediate results which are stated and proved below.

LEMMA 5.1. *Let* $M>0, (h_n)_n\subset\mathcal{A}_M$ *and* $(\alpha_n)_{n\in\mathbb{N}}$ *be a sequence such that* $\alpha_n\to 0$ *as* $n\to\infty$*. Let* $\mathbf{y}_n=\Gamma^{\alpha_n,\delta}\left(W+\alpha_n^{-\frac{1}{2}}\lambda_\delta(\alpha_n)\int_0^\cdot h_n(s)ds\right)$ *and* $\mathbf{z}_n=\Gamma^{0,\delta}\left(\int_0^\cdot h_n(s)ds\right)$*. Then, for any* $\varepsilon>0$

$$\lim_{n\to\infty}\mathbb{P}\left(\left[\sup_{t\in[0,T]}|\mathbf{y}_n(t)-\mathbf{z}_n(t)|^2+\int_0^T|\mathrm{A}^{\frac{1}{2}}(\mathbf{y}_n-\mathbf{z}_n)|^2ds\right]>\varepsilon\right)=0.$$

Before proving lemma we state the following important remark.

REMARK 5.1. Observe that if $h_n\equiv 0, \forall n\in\mathbb{N}$, and $\delta=0$, then the above lemma gives a result on the convergence in probability of the solutions of the stochastic LANS-α_n to the solution of the NSEs (2.2) as $\alpha_n\to 0$. In fact, $\mathbf{y}_n=\Gamma^{\alpha_n,0}(W)$ and $\Gamma^{0,0}(0)=\mathbf{u}$ are the unique solutions to the stochastic LANS-α_n corresponding to problem (2.1) and to the NSEs (2.2), respectively.

Proof. (**Proof of Lemma 5.1.**) Let $\mathbf{y}_n$ and $\mathbf{z}_n$ be as in the statement of the theorem. Let us put $\mathbf{w}_n=\mathbf{y}_n-\mathbf{z}_n$. Let $\tau_{n,N}$ be the stopping time defined

$$\tau_{n,N}=\inf\{t\in[0,T]:|\mathbf{y}_n(t)|\geq N\}\wedge T, N\geq 0.$$

For the sake of simplification we just write α instead of α_n throughout this proof. Also, we simply write τ_N in place of $\tau_{n,N}$.

Since $\mathbf{y}_n$ and $\mathbf{z}_n$ are the unique solutions to the stochastic controlled and deterministic controlled systems, respectively, it is not difficult to see that $\mathbf{w}_n$ satisfies

$$\begin{aligned}
&d\mathbf{w}_n+A\mathbf{w}_n+\lambda_\delta(\alpha)\tilde{B}_\alpha(\mathbf{y}_n,J_\alpha^{-1}\mathbf{y}_n)-(1-\delta)B(\mathbf{z}_n,\mathbf{z}_n)\\
&\quad+\delta[\tilde{B}_\alpha(\mathbf{u},J_\alpha^{-1}\mathbf{y}_n)-B(\mathbf{u},\mathbf{z}_n)+\tilde{B}_\alpha(\mathbf{u},J_\alpha^{-1}\mathbf{u})-B(\mathbf{z}_n,\mathbf{u})]\\
&=\delta\lambda_\delta^{-1}(\alpha)[B(\mathbf{u},\mathbf{u})-\tilde{B}_\alpha(\mathbf{u},J_\alpha^{-1}\mathbf{u})]\\
&\quad+\mathbf{G}_\alpha(\Psi_n)h_n-G(\Phi_n)h+\alpha^{\frac{1}{2}}\lambda_\delta^{-1}(\alpha)\mathbf{G}_\alpha(\Psi_n)dW,
\end{aligned}$$

where $\Psi_n=\delta\mathbf{u}+\lambda_\delta(\alpha)\mathbf{y}_n$ and $\Phi_n=\delta\mathbf{u}+(1-\delta)\mathbf{z}_n$.

Let

$$\begin{aligned}
&\mathbf{N}[\mathbf{y}_n,\mathbf{z}_n]=\lambda_\delta(\alpha)\tilde{B}_\alpha(\mathbf{y}_n,J_\alpha^{-1}\mathbf{y}_n)-(1-\delta)B(\mathbf{z}_n,\mathbf{z}_n)\quad\text{and}\\
&\mathbf{L}[\mathbf{y}_n,\mathbf{z}_n]=\delta\left[\tilde{B}_\alpha(\mathbf{u},J_\alpha^{-1}\mathbf{y}_n)-B(\mathbf{u},\mathbf{z}_n)+\tilde{B}_\alpha(\mathbf{y}_n,J_\alpha^{-1}\mathbf{u})-B(\mathbf{z}_n,\mathbf{u})\right].
\end{aligned}$$

By applying Itô's formula to $\varphi(x)=\|x\|_\alpha=|x|^2+\alpha^2\left|\mathrm{A}^{\frac{1}{2}}x\right|^2$ and to the process $\mathbf{w}_n$, taking the supremum and the mathematical expectation to the resulting equation we obtain

$$\begin{aligned}
&\|\mathbf{w}_n(t\wedge\tau_N)\|_\alpha^2+2\int_0^{t\wedge\tau_N}\left[\left|\mathrm{A}^{\frac{1}{2}}\mathbf{w}_n(s)\right|^2+\alpha^2|\mathrm{A}\mathbf{w}_n(s)|^2\right]ds\\
&\leq 2\int_0^{t\wedge\tau_N}|(\mathbf{N}[\mathbf{y}_n,\mathbf{z}_n]+\mathbf{L}[\mathbf{y}_n,\mathbf{z}_n],J_\alpha^{-1}\mathbf{w}_n)|ds\\
&\quad+2\delta\lambda_\delta^{-1}(\alpha)\int_0^{t\wedge\tau_N}|(B(\mathbf{u},\mathbf{u})-\tilde{B}_\alpha(\mathbf{u},J_\alpha^{-1}\mathbf{u}),J_\alpha^{-1}\mathbf{w}_n)|ds\\
&\quad+2\int_0^{t\wedge\tau_N}|(\mathbf{G}_\alpha(\Psi_n)h_n,\mathbf{G}(\Phi)h_n,J_\alpha^{-1}\mathbf{w}_n)|ds\\
&\quad+\alpha\lambda_\delta^{-2}(\alpha)\mathbb{E}\int_0^{t\wedge\tau_N}\|G(\Psi_n)\|^2_{\mathscr{L}(\mathrm{K},\mathbf{H})}ds+\alpha^{\frac{1}{2}}\lambda_\delta^{-1}(\alpha)\int_0^{t\wedge\tau_N}(\mathbf{w}_n,G(\Psi_n)dW).
\end{aligned}\tag{5.3}$$

Using (2.19) and Young's inequality we see that

$$\begin{aligned}
&2\delta\lambda_\delta^{-1}(\alpha)\int_0^{t\wedge\tau_N}(B(\mathbf{u},\mathbf{u})-\tilde{B}_\alpha(\mathbf{u},J_\alpha^{-1}\mathbf{u}),J_\alpha^{-1}\mathbf{w}_n)ds\\
&\leq 2\delta\lambda_\delta^{-1}(\alpha)\int_0^{t\wedge\tau_N}\left(\left[\frac{\alpha}{2}\left|\mathrm{A}^{\frac{1}{2}}\mathbf{w}_n\right|+\alpha^2|\mathrm{A}\mathbf{w}_n|^2\right]|B(\mathbf{u},\mathbf{u})|+C\alpha^2\left|\mathrm{A}^{\frac{1}{2}}\mathbf{w}_n\right|^2|\mathrm{A}\mathbf{u}|^2\right)ds
\end{aligned}$$

$$\leq \int_0^{t\wedge\tau_N}\left(\frac{1}{24}\left|\mathrm{A}^{\frac{1}{2}}\mathbf{w}_n\right|^2+\frac{\alpha^2}{24}|\mathrm{A}\mathbf{w}_n|^2+C\left[\alpha^2\lambda_\delta^{-1}(\alpha)\left|\mathrm{A}^{\frac{1}{2}}\mathbf{u}\right|^2+\alpha^2\right]|\mathrm{A}\mathbf{u}|^2\right)ds. \tag{5.4}$$

It follows from the bilinearity of B and $\tilde{B}$, and the Equations (2.8) and (2.10) that

$$\begin{aligned}&\lambda_\delta(\alpha)\left(\tilde{B}_\alpha(\mathbf{y}_n,J_\alpha^{-1}\mathbf{y}_n)-\tilde{B}_\alpha(\mathbf{z}_n,J_\alpha^{-1}\mathbf{z}_n),J_\alpha^{-1}\mathbf{w}_n\right)\\&=\lambda_\delta(\alpha)\left(\tilde{B}(\mathbf{w}_n,\mathbf{w}_n),\mathbf{z}_n\right)+\alpha^2\lambda_\delta(\alpha)\left(\tilde{B}(\mathbf{z}_n,A\mathbf{w}_n),\mathbf{w}_n\right).\end{aligned} \tag{5.5}$$

Thanks to the estimate (2.16) and the Young inequality we obtain

$$\begin{aligned}\lambda_\delta(\alpha)(B(\mathbf{w}_n,\mathbf{w}_n),\mathbf{z}_n)&=-\lambda_\delta(\alpha)(B(\mathbf{w}_n,\mathbf{z}_n),\mathbf{w}_n)\\&\leq\lambda_\delta(\alpha)|\mathbf{w}_n|\left|\mathrm{A}^{\frac{1}{2}}\mathbf{z}_n\right|^{\frac{1}{2}}|\mathrm{A}\mathbf{z}_n|^{\frac{1}{2}}|\mathbf{w}_n|^{\frac{1}{2}}\left|\mathrm{A}^{\frac{1}{2}}\mathbf{w}_n\right|^{\frac{1}{2}}\\&\leq\lambda_\delta(\alpha)|\mathbf{w}_n|^{\frac{3}{2}}\left|\mathrm{A}^{\frac{1}{2}}\mathbf{z}_n\right|^{\frac{1}{2}}|\mathrm{A}\mathbf{z}_n|^{\frac{1}{2}}\left|\mathrm{A}^{\frac{1}{2}}\mathbf{w}_n\right|^{\frac{1}{2}}\\&\leq\frac{1}{24}\left|\mathrm{A}^{\frac{1}{2}}\mathbf{w}_n\right|^2+\lambda_\delta^{\frac{4}{3}}(\alpha)|\mathbf{w}_n|^2\left|\mathrm{A}^{\frac{1}{2}}\mathbf{z}_n\right|^{\frac{2}{3}}|\mathrm{A}\mathbf{z}_n|^{\frac{2}{3}}.\end{aligned} \tag{5.6}$$

We now proceed in estimating the term $\alpha^2\lambda_\delta(\alpha)(\tilde{B}(\mathbf{z}_n,\mathrm{A}\mathbf{w}_n),\mathbf{w}_n)$. For doing so we utilise (2.15) and Young's inequality and find that

$$\begin{aligned}\alpha^2\lambda_\delta(\alpha)(\tilde{B}(\mathbf{z}_n,\mathrm{A}\mathbf{w}_n),\mathbf{w}_n)&\leq\alpha^2\lambda_\delta(\alpha)|\mathrm{A}\mathbf{w}_n||\mathrm{A}\mathbf{z}_n|\left|\mathrm{A}^{\frac{1}{2}}\mathbf{w}_n\right|\\&\leq\frac{\alpha^2}{24}|\mathrm{A}\mathbf{w}_n|^2+C\lambda_\delta^2(\alpha)\alpha^2\left|\mathrm{A}^{\frac{1}{2}}\mathbf{w}_n\right|^2|\mathrm{A}\mathbf{z}_n|^2.\end{aligned} \tag{5.7}$$

Thus,

$$\begin{aligned}&\lambda_\delta(\alpha)\left(\tilde{B}_\alpha(\mathbf{y}_n,J_\alpha^{-1}\mathbf{y}_n)-\tilde{B}_\alpha(\mathbf{z}_n,J_\alpha^{-1}\mathbf{z}_n),J_\alpha^{-1}\mathbf{w}_n\right)\\&\leq\frac{1}{24}\left|\mathrm{A}^{\frac{1}{2}}\mathbf{w}_n\right|^2+C\lambda_\delta^{\frac{4}{3}}(\alpha)|\mathbf{w}_n|^2\left|\mathrm{A}^{\frac{1}{2}}\mathbf{z}_n\right|^{\frac{2}{3}}|\mathrm{A}\mathbf{z}_n|^{\frac{2}{3}}\\&\quad+\frac{\alpha^2}{24}|\mathrm{A}\mathbf{w}_n|^2+C\lambda_\delta^2(\alpha)|\mathrm{A}\mathbf{z}_n|^2\left(\alpha^2\left|\mathrm{A}^{\frac{1}{2}}\mathbf{w}_n\right|^2+|\mathbf{w}_n|^2\right).\end{aligned} \tag{5.8}$$

By using the bilinearity of $\tilde{B}$ and B it is not difficult to see that

$$\begin{aligned}&(\lambda_\delta(\alpha)\tilde{B}_\alpha(\mathbf{z}_n,J_\alpha^{-1}\mathbf{z}_n)-(1-\delta)B(\mathbf{z}_n,\mathbf{z}_n),J_\alpha^{-1}\mathbf{w}_n)\\&=(\lambda_\delta(\alpha)\tilde{B}_\alpha(\mathbf{z}_n,\mathbf{z}_n)-(1-\delta)B(\mathbf{z}_n,\mathbf{z}_n),J_\alpha^{-1}\mathbf{w}_n)+\alpha^2\lambda_\delta(\alpha)(\tilde{B}_\alpha(\mathbf{z}_n,\mathrm{A}\mathbf{z}_n),J_\alpha^{-1}\mathbf{w}_n)\\&=([\lambda_\delta(\alpha)-(1-\delta)]B(\mathbf{z}_n,\mathbf{z}_n),\mathbf{w}_n\rangle+(1-\delta)\langle J_\alpha B(\mathbf{z}_n,\mathbf{z}_n)-B(\mathbf{z}_n,\mathbf{z}_n),J_\alpha^{-1}(\alpha)\mathbf{w}_n)\\&\quad+\alpha^2\lambda_\delta(\alpha)(\tilde{B}_\alpha(\mathbf{z}_n,A\mathbf{z}_n),\mathbf{w}_n)\\&=R_1+R_2+R_3.\end{aligned}$$

Owing to (2.12) and the Young inequality we get:

$$R_1\leq|\lambda_\delta(\alpha)-(1-\delta)|\left|\mathrm{A}^{\frac{1}{2}}\mathbf{z}_n\right|^{\frac{3}{2}}|\mathrm{A}\mathbf{z}_n|^{\frac{1}{2}}|\mathbf{w}_n|.$$

In a similar way,

$$R_2\leq(1-\delta)\alpha^2\langle B(\mathbf{z}_n,\mathbf{z}_n),\mathrm{A}\mathbf{w}_n\rangle$$

$$
\begin{aligned}
&\le (1-\delta)\alpha^2 |\mathbf{z}_n|^{\frac12} \left|\mathrm{A}^{\frac12}\mathbf{z}_n\right| |\mathrm{A}\mathbf{z}_n|^{\frac12} |\mathrm{A}\mathbf{w}_n| \\
&\le \frac{\alpha^2}{24}|\mathrm{A}\mathbf{w}_n|^2 + (1-\delta)^2\alpha^2|\mathbf{z}_n|\left|\mathrm{A}^{\frac12}\mathbf{z}_n\right|^2 |\mathrm{A}\mathbf{z}_n|.
\end{aligned}
$$

As for R_3 we use (2.14) and the Young inequality to obtain

$$
\begin{aligned}
R_3 &= \alpha^2\lambda_\delta(\alpha)\langle \tilde{B}(\mathbf{z}_n,\mathrm{A}\mathbf{z}_n),\mathbf{w}_n\rangle \\
&\le \alpha^2\lambda_\delta(\alpha)\left|\mathrm{A}^{\frac12}\mathbf{z}_n\right| |\mathrm{A}\mathbf{z}_n||\mathrm{A}\mathbf{w}_n| \\
&\le \frac{\alpha^2}{24}|\mathrm{A}\mathbf{w}_n|^2 + \alpha^2\lambda_\delta^2(\alpha)\left|\mathrm{A}^{\frac12}\mathbf{z}_n\right|^2 |\mathrm{A}\mathbf{z}_n|^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&(\lambda_\delta(\alpha)\tilde{B}_\alpha(\mathbf{z}_n, J_\alpha^{-1}\mathbf{z}_n) - (1-\delta)B(\mathbf{z}_n,\mathbf{z}_n), J_\alpha^{-1}\mathbf{w}_n) \\
&\le \frac{\alpha^2}{12}|\mathrm{A}\mathbf{w}_n|^2 + C\alpha^2[\lambda_\delta^2(\alpha)+(1-\delta)]\left|\mathrm{A}^{\frac12}\mathbf{z}_n\right|^2 |\mathrm{A}\mathbf{z}_n|[|\mathbf{z}_n|+|\mathrm{A}\mathbf{z}_n|] \\
&\quad + |\lambda_\delta(\alpha)-(1-\delta)|\left|\mathrm{A}^{\frac12}\mathbf{z}_n\right|^{\frac32} |\mathrm{A}\mathbf{z}_n|^{\frac12}|\mathbf{w}_n|. \qquad (5.9)
\end{aligned}
$$

Combining (5.8) and (5.9) we see that

$$
\begin{aligned}
&|(-\mathbf{N}[\mathbf{y}_n,\mathbf{z}_n], J_\alpha^{-1}\mathbf{w}_n)| \\
&\le C\lambda_\delta^{\frac43}(\alpha)\left|\mathrm{A}^{\frac12}\mathbf{z}_n\right|^{\frac23} |\mathrm{A}\mathbf{z}_n|^{\frac23}[|\mathbf{w}_n|^2+\alpha^2|\mathrm{A}^{\frac12}\mathbf{w}_n|^2] \\
&\quad + C\lambda_\delta^2(\alpha)|\mathrm{A}\mathbf{z}_n|^2\left(|\mathbf{w}_n|^2+\alpha^2\left|\mathrm{A}^{\frac12}\mathbf{w}_n\right|^2\right) \\
&\quad + |\lambda_\delta(\alpha)-(1-\delta)|\left(\left|\mathrm{A}^{\frac12}\mathbf{z}_n\right||\mathrm{A}\mathbf{z}_n| + |\mathrm{A}^{\frac12}\mathbf{z}_n|^2\left[|\mathbf{w}_n|^2+\alpha^2|\mathrm{A}^{\frac12}\mathbf{w}_n|^2\right]\right) \\
&\quad + C\alpha^2[\lambda_\delta^2(\alpha)+(1-\delta)]\left|\mathrm{A}^{\frac12}\mathbf{z}_n\right|^2|\mathrm{A}\mathbf{z}_n|[|\mathbf{z}_n|+|\mathrm{A}\mathbf{z}_n|] + \frac{3\alpha^2}{24}|\mathrm{A}\mathbf{w}_n|^2 + \frac{1}{24}\left|\mathrm{A}^{\frac12}\mathbf{w}_n\right|^2. \qquad (5.10)
\end{aligned}
$$

Our next task is to estimate

$$
\begin{aligned}
(\mathbf{L}[\mathbf{y}_n,\mathbf{z}_n], J_\alpha^{-1}\mathbf{w}_n) &= \delta(\tilde{B}_\alpha(\mathbf{u}, J_\alpha^{-1}\mathbf{y}_n) - B(\mathbf{u},\mathbf{z}_n), J_\alpha^{-1}\mathbf{w}_n) \\
&\quad + \delta(\tilde{B}_\alpha(\mathbf{y}_n, J_\alpha^{-1}\mathbf{u}) - B(\mathbf{z}_n,\mathbf{u}), J_\alpha^{-1}\mathbf{w}_n) \\
&=: I + L.
\end{aligned}
$$

Using the bilinearity of $\tilde{B}$ and (2.6) we see that

$$
\begin{aligned}
I &= \delta\langle \tilde{B}(\mathbf{u}, J_\alpha^{-1}\mathbf{w}_n),\mathbf{w}_n\rangle_{D(\mathrm{A})'} + \delta(\tilde{B}_\alpha(\mathbf{u}, J_\alpha^{-1}\mathbf{z}_n) - B(\mathbf{u},\mathbf{z}_n), J_\alpha^{-1}\mathbf{w}_n) \\
&= -\delta(\tilde{B}(\mathbf{w}_n,\mathbf{w}_n),\mathbf{u}) + \delta\alpha^2\langle \tilde{B}(\mathbf{u},\mathrm{A}\mathbf{w}_n),\mathbf{w}_n\rangle_{D(\mathrm{A})'} + \delta(\tilde{B}_\alpha(\mathbf{u}, J_\alpha^{-1}\mathbf{z}_n) - B(\mathbf{u},\mathbf{z}_n), J_\alpha^{-1}\mathbf{w}_n).
\end{aligned}
$$

By denoting I_1 and I_2 the first two terms on the right-hand side of the above equation and using the bilinearity of $\tilde{B}$ again and (2.9) we find that

$$
I = I_1 + I_2 + \delta\alpha^2\langle \tilde{B}(\mathbf{u},\mathrm{A}\mathbf{z}_n),\mathbf{w}_n\rangle_{D(\mathrm{A})'} - \delta\alpha^2\langle B(\mathbf{u},\mathbf{z}_n),\mathrm{A}\mathbf{w}_n\rangle_{D(\mathrm{A})'} - \delta(B(\mathbf{w}_n,\mathbf{z}_n),\mathbf{u}).
$$

In a similar way we can show that

$$
L = \delta(\tilde{B}_\alpha(\mathbf{y}_n, J_\alpha^{-1}\mathbf{u}) - B(\mathbf{z}_n,\mathbf{u}), J_\alpha^{-1}\mathbf{w}_n)
$$

$$=\delta\alpha^2\langle\tilde{B}(\mathbf{z}_n,\mathrm{A}\mathbf{u}),\mathbf{w}_n\rangle_{D(\mathrm{A})'}+\delta\alpha^2\langle B(\mathbf{z}_n,\mathbf{u}),\mathrm{A}\mathbf{w}_n\rangle_{D(\mathrm{A})'}+\delta(B(\mathbf{w}_n,\mathbf{z}_n),\mathbf{u}).$$

Hence

$$\begin{aligned}I+L&=I_1+I_2+\delta\alpha^2\langle\tilde{B}(\mathbf{u},\mathrm{A}\mathbf{z}_n),\mathbf{w}_n\rangle_{D(\mathrm{A})'}-\delta\alpha^2(B(\mathbf{u},\mathbf{z}_n),\mathrm{A}\mathbf{w}_n)\\&\quad+\delta\alpha^2\langle\tilde{B}(\mathbf{z}_n,\mathrm{A}\mathbf{u}),\mathbf{w}_n\rangle_{D(\mathrm{A})'}+\delta\alpha^2\langle B(\mathbf{z}_n,\mathbf{u}),\mathrm{A}\mathbf{w}_n\rangle_{D(\mathrm{A})'}\\&=:\sum_{i=1}^{6}I_i.\end{aligned}$$

In the next lines we will estimate I_i, $i=1,\dots,6$.

Using (2.7), (2.16) and the Young inequality we see that

$$\begin{aligned}I_1=\delta(B(\mathbf{w}_n,\mathbf{u}),\mathbf{w}_n)&\le\delta|\mathbf{w}_n|^{\frac12}\left|\mathrm{A}^{\frac12}\mathbf{w}_n\right|^{\frac12}\left|\mathrm{A}^{\frac12}\mathbf{u}\right|^{\frac12}|\mathrm{A}\mathbf{u}|^{\frac12}|\mathbf{w}_n|\\&\le\frac{1}{24}\left|\mathrm{A}^{\frac12}\mathbf{w}_n\right|^2+C\delta^2|\mathbf{w}_n|^2\left|\mathrm{A}^{\frac12}\mathbf{u}\right|^{\frac23}|\mathrm{A}\mathbf{u}|^{\frac23}.\end{aligned}$$

In a similar fashion we can show that

$$\begin{aligned}I_2&=\delta(B(\mathbf{u},\mathbf{w}_n),\mathrm{A}\mathbf{w}_n)+\delta\alpha^2(B(\mathbf{w}_n,\mathbf{u}),\mathrm{A}\mathbf{w}_n)\\&\le\delta\alpha^2|\mathrm{A}\mathbf{w}_n|\left[|\mathbf{u}|^{\frac12}\left|\mathrm{A}^{\frac12}\mathbf{u}\right|^{\frac12}\left|\mathrm{A}^{\frac12}\mathbf{w}_n\right|^{\frac12}|\mathrm{A}\mathbf{w}_n|^{\frac12}+|\mathbf{w}_n|^{\frac12}\left|\mathrm{A}^{\frac12}\mathbf{w}_n\right|^{\frac12}\left|\mathrm{A}^{\frac12}\mathbf{u}\right|^{\frac12}|\mathrm{A}\mathbf{u}|^{\frac12}\right]\\&\le\frac{\alpha^2}{96}|\mathrm{A}\mathbf{w}_n|^2+C\delta^2\alpha^2\left[|\mathbf{u}|^2\left|\mathrm{A}^{\frac12}\mathbf{u}\right|^2\left|\mathrm{A}^{\frac12}\mathbf{w}_n\right|^2+|\mathbf{w}_n|^2\left|\mathrm{A}^{\frac12}\mathbf{w}_n\right|\left|\mathrm{A}^{\frac12}\mathbf{u}\right||\mathrm{A}\mathbf{u}|\right]\\&\le\frac{\alpha^2}{96}|\mathrm{A}\mathbf{w}_n|^2+C\delta^2\alpha^2\left[\left(1+|\mathbf{u}|^2\left|\mathrm{A}^{\frac12}\mathbf{u}\right|^2\right)\left|\mathrm{A}^{\frac12}\mathbf{w}_n\right|^2+|\mathbf{w}_n|^2\left|\mathrm{A}^{\frac12}\mathbf{u}\right|^2|\mathrm{A}\mathbf{u}|^2\right].\end{aligned}$$

We now proceed to the estimate of I_3. By using (2.14) and Young's inequality

$$\begin{aligned}I_3&=\delta\alpha^2\langle\tilde{B}(\mathbf{u},\mathrm{A}\mathbf{z}_n),\mathbf{w}_n\rangle_{D(\mathrm{A})'}\\&\le C\delta\alpha^2|\mathrm{A}\mathbf{z}_n|\left|\mathrm{A}^{\frac12}\mathbf{u}\right||\mathrm{A}\mathbf{u}|\\&\le C\delta^2\alpha^2|\mathrm{A}\mathbf{z}_n|^2\left|\mathrm{A}^{\frac12}\mathbf{u}\right|+\frac{\alpha^2}{96}|\mathrm{A}\mathbf{w}_n|^2.\end{aligned}$$

In a similar way we show that

$$\begin{aligned}I_5&=\delta\alpha^2\langle\tilde{B}(\mathbf{z}_n,\mathrm{A}\mathbf{u}),\mathbf{w}_n\rangle_{D(\mathrm{A})'}\\&\le C\delta^2\alpha^2|\mathrm{A}\mathbf{u}|^2\left|\mathrm{A}^{\frac12}\mathbf{z}_n\right|+\frac{\alpha^2}{96}|\mathrm{A}\mathbf{w}_n|^2.\end{aligned}$$

Finally by using (2.11), (2.12) and the Young inequality, the term I_4+I_6 can be estimated as follows

$$\begin{aligned}I_4+I_6&\le\delta\alpha^2|\mathrm{A}\mathbf{w}_n|\left|\mathrm{A}^{\frac12}\mathbf{u}\right|^{\frac12}|\mathrm{A}\mathbf{u}|^{\frac12}\left|\mathrm{A}^{\frac12}\mathbf{z}_n\right|\\&\le\frac{\alpha^2}{96}|\mathrm{A}\mathbf{w}_n|^2+\left|\mathrm{A}^{\frac12}\mathbf{z}_n\right|^2\left|\mathrm{A}^{\frac12}\mathbf{u}\right||\mathrm{A}\mathbf{u}|.\end{aligned}$$

Thus,

$$I+L=\langle\mathbf{L}[\mathbf{y}_n,\mathbf{z}_n],J_\alpha^{-1}\mathbf{w}_n\rangle$$

$$\leq \frac{\alpha^2}{24}|\mathrm{A}\mathbf{w}_n|^2+\frac{1}{24}\left|\mathrm{A}^{\frac12}\mathbf{w}_n\right|^2+C\delta^2\left(|\mathbf{w}_n|^2+\alpha^2\left|\mathrm{A}^{\frac12}\mathbf{w}_n\right|^2\right)\left(1+\left|\mathrm{A}^{\frac12}\mathbf{u}\right|^2|\mathrm{A}\mathbf{u}|^2\right)$$
$$+C\delta^2\alpha^2\left[\left|\mathrm{A}^{\frac12}\mathbf{u}\right|^2\left(|\mathrm{A}\mathbf{z}_n|^2\left|\mathrm{A}^{\frac12}\mathbf{z}_n\right|^2\right)+|\mathrm{A}\mathbf{u}|^2\left|\mathrm{A}^{\frac12}\mathbf{z}_n\right|^2\right]. \quad (5.11)$$

We now deal with the control terms. It is not difficult to prove that

$$(\mathbf{G}_\alpha(\Phi_n)h_n-G(\Phi)h_n,J_\alpha^{-1}\mathbf{w}_n)$$
$$=(J_\alpha G(\Psi_n)h_n-J_\alpha G(\Phi_n)h_n,J_\alpha^{-1}\mathbf{w}_n)+(J_\alpha G(\Phi_n)h_n-G(\Phi)h_n,J_\alpha^{-1}\mathbf{w}_n)$$
$$\leq|G(\Psi_n)h_n-G(\Phi)h_n||\mathbf{w}_n|+\langle J_\alpha G(\Phi_n)h_n-G(\Phi_n)h_n,J_\alpha^{-1}\mathbf{w}_n\rangle.$$

Using the Assumption 3.1 and the definitions of Ψ_n and Φ_n yields

$$(\mathbf{G}_\alpha(\Phi_n)h_n-G(\Phi)h_n,J_\alpha^{-1}\mathbf{w}_n)$$
$$\leq|\Psi_n-\Phi_n|\|h_n\|_{\mathrm{K}}|\mathbf{w}_n|+|(J_\alpha G(\Phi_n)h_n-G(\Phi_n)h_n,J_\alpha^{-1}\mathbf{w}_n)|$$
$$\leq C[\lambda_\delta(\alpha)|\mathbf{w}_n|^2+|\lambda_\delta(\alpha)-(1-\delta|)|\mathbf{z}_n||\mathbf{w}_n|]\|h_n\|_{\mathrm{K}}+|(J_\alpha G(\Phi_n)h_n-G(\Phi_n)h_n,J_\alpha^{-1}\mathbf{w}_n)|.$$

Let us now deal with the second term on the right-hand side of the last inequality. Thanks to Assumption 3.1, inequality (2.22) and the Young inequality we have

$$|(J_\alpha G(\Phi_n)h_n-G(\Phi_n)h_n,J_\alpha^{-1}\mathbf{w}_n)|$$
$$\leq\alpha|(\alpha^2\mathrm{A})^{\frac12}J_\alpha\mathrm{A}^{\frac12}G(\Phi_n)h_n||J_\alpha^{-1}\mathbf{w}_n|$$
$$\leq C\alpha(1+|\Phi_n|)\|h_n\|_{\mathrm{K}}|J_\alpha^{-1}\mathbf{w}_n|$$
$$\leq C\alpha^2(1+\delta^2|\mathbf{u}|^2+(1-\delta)^2|\mathbf{z}_n|^2)\|h_n\|_{\mathrm{K}}^2+C\|h_n\|_{\mathrm{K}}^2|\mathbf{w}_n|^2+\frac{\alpha^2}{24}|\mathrm{A}\mathbf{w}_n|^2.$$

Thus,

$$(\mathbf{G}_\alpha(\Psi_n)h_n-G(\Phi_n)h_n,J_\alpha^{-1}\mathbf{w}_n)$$
$$\leq C(1+\lambda_\delta(\alpha))\|h_n\|_{\mathrm{K}}^2[|\mathbf{w}_n|^2+\alpha^2|\mathrm{A}^{\frac12}\mathbf{w}_n|^2]+\frac{\alpha^2}{24}|\mathrm{A}\mathbf{w}_n|^2$$
$$+C\alpha^2(1+\delta^2|\mathbf{u}|^2+(1-\delta)^2|\mathbf{z}_n|^2)\|h_n\|_{\mathrm{K}}^2+|\lambda_\delta(\alpha)-(1-\delta)|^2|\mathbf{z}_n|^2. \quad (5.12)$$

By using Assumption 3.1 and the definition of the stopping time τ_N, it is not difficult to show that

$$\alpha\lambda_\delta^{-2}(\alpha)\int_0^{t\wedge\tau_N}\|G(\Psi_n)\|_{\mathscr{L}(\mathrm{K},\mathbf{H})}^2ds\leq C\alpha\lambda_\delta^{-2}(\alpha)\int_0^{t\wedge\tau_N}(1+\delta^2|\mathbf{u}|^2+\lambda_\delta^2(\alpha)|\mathbf{y}_n|^2)ds$$
$$\leq C\alpha\lambda_\delta^{-2}(\alpha)T(1+\delta^2\sup_{s\in[0,T]}|\mathbf{u}(s)|^2+\lambda_\delta^2(\alpha)N). \quad (5.13)$$

Before proceeding further we set

$$\mathrm{Y}_n=\left(\lambda_\delta^2(\alpha)+|\lambda_\delta(\alpha)-(1-\delta)|+\lambda_\delta^{\frac43}(\alpha)\right)(1+|\mathrm{A}\mathbf{z}_n|^2)+\delta^2(1+|\mathrm{A}^{\frac12}\mathbf{u}|^2|\mathrm{A}\mathbf{u}|^2)$$
$$+(1+\lambda_\delta(\alpha))\|h_n\|_{\mathrm{K}}^2.$$

$$\mathrm{R}_n=\left(|\lambda_\delta(\alpha)-(1-\delta)|+\alpha^2[\lambda_\delta^2(\alpha)+(1-\delta)]|\mathrm{A}^{\frac12}\mathbf{z}_n|^2+\delta^2\alpha^2|\mathrm{A}^{\frac12}\mathbf{u}|^2|\mathrm{A}^{\frac12}\mathbf{z}_n|^2\right)|\mathrm{A}\mathbf{z}_n|^2$$
$$+\left(\alpha^2+\alpha^2\lambda_\delta^{-1}(\alpha)|\mathrm{A}^{\frac12}\mathbf{u}|^2+\delta^2\alpha^2|\mathrm{A}^{\frac12}\mathbf{z}_n|^2\right)+|\lambda_\delta(\alpha)-(1-\delta)|^2|\mathbf{z}_n|^2$$

$$+\alpha^2\left(1+\delta^2\sup_{s\in[0,T]}|\mathbf{u}(s)|^2+(1-\delta)^2\sup_{s\in[0,T]}|\mathbf{z}_n(s|^2\right)\|h_n\|_{\mathrm{K}}^2$$
$$+\alpha\lambda_\delta^{-2}(\alpha)T(1+\delta^2\sup_{s\in[0,T]}|\mathbf{u}(s)|^2+\lambda_\delta^2(\alpha)N).$$

Then, by plugging (5.4), (5.10), (5.11), (5.12), and (5.13) into (5.3) and invoking the Poincaré inequality we obtain that there exist constants $C_0,C_1>0$ such that with probability 1 and for all $n\in\mathbb{N}$

$$\|\mathbf{w}_n(t\wedge\tau_N)\|_\alpha^2+\int_0^{t\wedge\tau_N}\left[\frac{1}{2}\left|\mathrm{A}^{\frac{1}{2}}\mathbf{w}_n(s)\right|^2+\frac{\alpha^2}{2}|\mathrm{A}\mathbf{w}_n(s)|^2\right]ds$$
$$\leq\|\mathbf{w}_n(0)|_\alpha^2+C_0\int_0^{t\wedge\tau_N}\mathrm{Y}_n(s)\|\mathbf{w}_n(s)\|_\alpha^2ds+C_1\int_0^{t\wedge\tau_N}R_n(s)ds+\alpha^{\frac{1}{2}}\lambda_\delta^{-1}(\alpha)\mathscr{M}_n(t\wedge\tau_N),\tag{5.14}$$

where

$$\mathscr{M}_n(t)=\int_0^t(J_\alpha^{-1}\mathbf{w}_n,\mathbf{G}_\alpha(\Psi_n)dW),\,t\in[0,T].$$

We now deal with the stochastic term. By using the Burkholder-Davis-Gundy inequality, the Assumption 3.1 and the Young inequality we deduce that for any $\theta>0$ there exist two constant $C_2,c_2>0$ such that for all $n\in\mathbb{N}$

$$\alpha^{\frac{1}{2}}\lambda_\delta^{-1}(\alpha)\mathbb{E}\sup_{0\leq s\leq t}\left|\int_0^{s\wedge\tau_N}(J_\alpha^{-1}\mathbf{w}_n,\mathbf{G}_\alpha(\Psi_n)dW)\right|$$
$$=\alpha^{\frac{1}{2}}\lambda_\delta^{-1}(\alpha)\mathbb{E}\sup_{0\leq s\leq t}\left|\int_0^{s\wedge\tau_N}(\mathbf{w}_n,G(\Psi_n)dW)\right|$$
$$\leq c_2\alpha^{\frac{1}{2}}\lambda_\delta^{-1}(\alpha)\mathbb{E}\left[\int_0^{t\wedge\tau_N}\|\mathbf{w}_n\|_\alpha^2\|G(\Psi_n)\|^2_{\mathscr{L}(\mathrm{K},\mathbf{H})}ds\right]^{\frac{1}{2}}$$
$$\leq\theta\mathbb{E}\sup_{s\in[0,t]}\|\mathbf{w}_n(s\wedge\tau_N)\|_\alpha^2+C_2\alpha\lambda_\delta^{-2}(\alpha)\mathbb{E}\int_0^{t\wedge\tau_N}(1+\delta^2|\mathbf{u}|^2+\lambda_\delta^2(\alpha)|\mathbf{y}_n|^2)ds.$$

Using the definition of the stopping time τ_N yields

$$\alpha^{\frac{1}{2}}\lambda_\delta^{-1}(\alpha)\mathbb{E}\sup_{0\leq s\leq t}\left|\int_0^{s\wedge\tau_N}\langle J_\alpha^{-1}\mathbf{w}_n,\mathbf{G}_\alpha(\Psi_n)dW\rangle\right|$$
$$\leq\theta\mathbb{E}\sup_{s\in[0,t]}\|\mathbf{w}_n(s\wedge\tau_N)\|_\alpha^2+C_2\alpha\lambda_\delta^{-2}(\alpha)\mathbb{E}\int_0^{t\wedge\tau_N}(1+\delta^2|\mathbf{u}|^2+\lambda_\delta^2(\alpha)N)ds.\tag{5.15}$$

Next observe that thanks to the estimates (4.14)-(4.15), the fact $\int_0^T\|h_n\|_{\mathrm{K}}^2\leq M$ we see that there exists a deterministic constant $c_3>0$ such that with probability 1

$$e^{\int_0^T C_0\mathrm{Y}_n(s)ds}\leq e^{(T+c_3)\left(\lambda_\delta^2(\alpha)+|\lambda_\delta(\alpha)-(1-\delta)|+\lambda_\delta^{\frac{4}{3}}(\alpha)\right)+\delta^2c_3(1+c_3)M+\alpha^2\lambda_\delta(\alpha)T.}\tag{5.16}$$

In a similar way, we can show that there exists a deterministic constant $C_3>0$, which may depend on M, N and T, such that with probability 1

$$\int_0^{T\wedge\tau_n}\mathrm{R}_n(s)ds\leq C_3\Sigma_n,\tag{5.17}$$

where the sequence $\Sigma_n, n\in\mathbb{N}$ is defined by

$$\Sigma_n=|\lambda_\delta(\alpha_n)-(1-\delta)|+\alpha_n^2[\lambda_\delta^2(\alpha_n)+(1-\delta)]+\delta^2\alpha_n^2+\alpha_n^2\lambda_\delta^{-1}(\alpha_n)+\alpha_n\lambda_\delta^{-2}(\alpha_n)+\alpha_n^2+\alpha_n. \quad (5.18)$$

Next, since $\lambda_\delta(\alpha_n)\leq 1$, $|\lambda_\delta(\alpha_n)-(1-\delta)|\to 0$, and $\alpha_n^2\lambda_\delta(\alpha_n)\to 0$ as $n\to\infty$, we deduce that there exists a deterministic constant $C_4>0$ such that with probability 1

$$\sup_{n\in\mathbb{N}} e^{\int_0^T C_0 \Upsilon_n(s)ds}\leq C_4. \quad (5.19)$$

Thus, by choosing $\theta>0$ so that $2\theta C_4\leq 1$ and applying the version of Gronwall's lemma given in [13, Lemma A.1] we obtain that for all $t\in[0,T]$ and $n\in\mathbb{N}$

$$\mathbb{E}\sup_{s\in[0,t]}\|\mathbf{w}_n(t\wedge\tau_N)\|_\alpha^2+\mathbb{E}\int_0^{t\wedge\tau_N}\left[\frac{1}{2}\left|\mathrm{A}^{\frac{1}{2}}\mathbf{w}_n(s)\right|^2+\frac{\alpha^2}{2}|\mathrm{A}\mathbf{w}_n(s)|^2\right]ds\leq C_4C_3\Sigma_n,$$

which implies

$$\mathbb{E}\sup_{s\in[0,t]}|\mathbf{w}_n(t\wedge\tau_N)|^2+\mathbb{E}\int_0^{t\wedge\tau_N}\frac{1}{2}\left|\mathrm{A}^{\frac{1}{2}}\mathbf{w}_n(s)\right|^2ds\leq C_4C_3\Sigma_n.$$

Now, since $\lambda_\delta(\alpha_n)\leq 1$, $|\lambda_\delta(\alpha_n)-(1-\delta)|\to 0$ and $\alpha_n\lambda_\delta^{-\ell}(\alpha_n)\to 0$, $\ell\in\{1,2\}$, as $n\to\infty$ we infer that

$$\mathbb{E}\sup_{r\in[0,T]}|\mathbf{w}_n(r\wedge\tau_N)|^2+\mathbb{E}\int_0^{T\wedge\tau_N}|\mathrm{A}^{\frac{1}{2}}\mathbf{w}_n(s)|^2ds\to 0 \text{ as } n\to\infty. \quad (5.20)$$

Next, let $\gamma>0$ and $\varepsilon>0$ be arbitrary numbers. Let us set

$$X_n(T)=\sup_{r\in[0,T]}|\mathbf{w}_n(r)|^2+\int_0^T|\mathrm{A}^{\frac{1}{2}}\mathbf{w}_n(s)|^2ds.$$

Then, it is not difficult to check that

$$\begin{aligned}\mathbb{P}(X_n(T)\geq\varepsilon)&\leq\mathbb{P}(\sup_{r\in[0,T]}|X_n(T),\tau_N=T)+\mathbb{P}(\sup_{r\in[0,T]}|\mathbf{y}_n(r)|^2\geq N)\\&\leq\frac{1}{\varepsilon}\mathbb{E}X_n(T\wedge\tau_N)+\frac{1}{N}\mathbb{E}\sup_{r\in[0,T]}(|\mathbf{y}_n(r)|).\end{aligned} \quad (5.21)$$

Owing to estimate (4.4) one can find $N_0>0$ such that if $N\geq N_0$ then

$$\frac{1}{N}\mathbb{E}\sup_{r\in[0,T]}(|\mathbf{y}_n(r)|^2)<\frac{\gamma}{2}.$$

Thus, thanks to (5.20) and (5.21) we infer that there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$

$$\mathbb{P}\left(\left[\sup_{r\in[0,T]}|\mathbf{w}_n(r)|^2+\int_0^T|\mathrm{A}^{\frac{1}{2}}\mathbf{w}_n(s)|^2ds\right]\geq\varepsilon\right)<\gamma,$$

which completes the proof of Proposition 5.1. ∎

We will also need the following result.

LEMMA 5.2. *Let $M>0$, $(h_n)_{n\in\mathbb{N}}\subset\mathscr{A}_M$, $h\in\mathscr{A}_M$, and $(\alpha_n)_{n\in\mathbb{N}}\subset(0,1]$ be a sequence converging to 0. Also, let $\delta\in\{0,1\}$ and $\xi\in D(\mathrm{A}^{\frac{1}{2}})$. Let us assume that Assumption 3.1 holds. Let h_n be a sequence converging in distribution to h as $\mathcal{A}_M$-valued random variable.*

Then, the process $\Gamma_\xi^{0,\delta}\left(\int_0^\cdot h_n(r)dr\right)$ converges in distribution to $\Gamma_\xi^{0,\delta}\left(\int_0^\cdot h(r)dr\right)$ as $C([0,T];\mathbf{H})\cap\mathbf{L}^2(0,T;D(\mathrm{A}^{\frac{1}{2}}))$-valued random variables.

Proof. (**Proof of Lemma 5.2.**) Before diving into the depth of the proof we recall that $\mathcal{A}_M$ is a Polish space when endowed with the metric defined in (4.1). Now, since, by assumption, $h_n\to h$ in law as $\mathcal{A}_M$-valued random variables, we can infer from the Skorokhod's theorem that one can find a probability space $(\bar\Omega,\bar{\mathscr{F}},\bar{\mathbb{P}})$ on which there exist $\mathcal{A}_M$-valued random variables $\bar h_n$, $\bar h$ having the same laws as h_n and h, respectively, and satisfying

$$\bar h_n\to h \text{ in } \mathcal{A}_M, \bar{\mathbb{P}}-\text{a.s.}. \tag{5.22}$$

From the last property and Proposition 4.2 we derive that

$$\Gamma_\xi^{0,\delta}\left(\int_0^\cdot \bar h_n(r)dr\right)\to\Gamma_\xi^{0,\delta}\left(\int_0^\cdot \bar h(r)dr\right) \text{ in } C([0,T];\mathbf{H})\cap\mathbf{L}^2(0,T;D(\mathrm{A}^{\frac{1}{2}}))\ \bar{\mathbb{P}}-\text{a.s.}. \tag{5.23}$$

Observe that Proposition 4.2 implies in particular that $\Gamma_\xi^{0,\delta}:\mathcal{A}_M\to C([0,T];\mathbf{H})\cap\mathbf{L}^2(0,T;D(\mathrm{A}^{\frac{1}{2}}))$ is continuous. Hence, from the equality of the laws of h_n (resp. h) and $\bar h_n$ (resp. $\bar h$) we infer that the laws of $\Gamma_\xi^{0,\delta}\left(\int_0^\cdot \bar h_n(r)dr\right)$ and $\Gamma_\xi^{0,\delta}\left(\int_0^\cdot \bar h(r)dr\right)$ are equal to the laws of $\Gamma_\xi^{0,\delta}\left(\int_0^\cdot h_n(r)dr\right)$ and $\Gamma_\xi^{0,\delta}\left(\int_0^\cdot h(r)dr\right)$, respectively. This observation and the convergence (5.23) complete the proof of Lemma 5.2. ∎

The next result that we need is contained in the following theorem.

PROPOSITION 5.1. *Let $M>0$, $(h_n)_{n\in\mathbb{N}}\subset\mathscr{A}_M$, $h\in\mathscr{A}_M$, and $(\alpha_n)_{n\in\mathbb{N}}\subset(0,1]$ be a sequence converging to 0. Also, let $\delta\in\{0,1\}$ and $\xi\in D(\mathrm{A}^{\frac{1}{2}})$.*
If Assumption 3.1 holds and h_n is a sequence converging in distribution to h as $\mathcal{A}_M$-valued random variable, then the process $\Gamma_\xi^{\alpha_n,\delta}\left(W+\alpha_n^{-\frac{1}{2}}\lambda_\delta(\alpha_n)\int_0^\cdot h_n(r)dr\right)$ converges in distribution to $\Gamma_\xi^{0,\delta}\left(\int_0^\cdot h(r)dr\right)$ as $C([0,T];\mathbf{H})\cap\mathbf{L}^2(0,T;D(\mathrm{A}^{\frac{1}{2}}))$-valued random variables.

Proof. Proposition 5.1 readily follows from [19, Theorem 11.3.3], Lemmata 5.1 and 5.2. ∎

5.3. Proof of Theorem 5.1. In this subsection we will give the proof of our main results which are contained in Theorem 5.1. The proof relies on a LDP result which follows from [4, Theorem 3.6 and Theorem 4.4]. We first recall this LDP result.

Let K, K_1 be two separable Hilbert spaces and W a Wiener process as in Subsection 3. We recall that $\mathscr{A}$ is the set of all K-valued predictable processes h such that

$$\mathbb{P}\left(\int_0^T \|h(r)\|_{\mathrm{K}}^2 dr<\infty\right)=1. \tag{5.24}$$

We recall the following result which is exactly [4, Theorem 3.6].

THEOREM 5.2. *Let $\Gamma: C([0,T];\mathrm{K}) \to \mathbb{R}$ be a bounded, Borel measurable function. Then*

$$-\log \mathbb{E} e^{-\Gamma(W)} = \inf_{h \in \mathscr{A}} \mathbb{E}\left\{ \frac{1}{2} \int_0^T \|h(r)\|_{\mathrm{K}}^2 + \Gamma\left(W + \int_0^{\cdot} h(r) dr \right) \right\}. \tag{5.25}$$

Next, let $\mathcal{E}$ be a Polish space, $(\Psi^\varepsilon)_{\varepsilon \in (0,1]}$ a family of Borel measurable maps from $C([0,T];\mathrm{K})$ onto $\mathcal{E}$, and $(X^\varepsilon)_{\varepsilon \in (0,1]}$ a family of $\mathcal{E}$-valued random variables. We have the following result which can be proved by using Theorem 5.2 and the idea in the proof of [4, Theorem 4.4].

THEOREM 5.3. *Let ϱ be a real-valued function defined on $(0,\infty)$ such that*

$$\varrho(\varepsilon) \to \infty \text{ as } \varepsilon \to 0.$$

Assume that there exists a Borel measurable map $\Psi^0 : C([0,T];\mathrm{K}) \to \mathcal{E}$ such that the following hold:

(**A**1) *if $(h_\varepsilon)_{\varepsilon \in (0,1]} \subset \mathscr{A}_M$, $M > 0$, converges in distribution to $h \in \mathscr{S}_M$ as $\mathscr{A}_M$-valued random variables, then $\Psi^\varepsilon(W + \varrho(\varepsilon) \int_0^{\cdot} h_\varepsilon(r) dr)$ converges in distribution to $\Psi^0(\int_0^{\cdot} h(r) dr)$.*

(**A**2) *For every $M > 0$ the set $K_M = \{\Psi^0(\int_0^{\cdot} h(r) dr) : h \in \mathscr{A}_M\}$ is a compact subset of $\mathcal{E}$.*

Then, the family $(X^\varepsilon)_{\varepsilon \in (0,1]}$ satisfies an LDP with speed $\varrho^2(\varepsilon)$ and rate function I given by

$$I(x) = \inf_{\{h \in \mathbf{L}^2(0,T;\mathrm{K}) :\, x = \Psi^0(\int_0^{\cdot} h(r) dr)\}} \left\{ \frac{1}{2} \int_0^T \|h(r)\|_{\mathrm{K}}^2 \right\}. \tag{5.26}$$

Now, we are ready to give the promised proof of our main theorem.

Proof. (**Proof of Theorem 5.1.**) Owing to Propositions 4.2 and 5.1 the assumptions (**A**1) and (**A**2) of Theorem 5.3 are satisfied on $\mathcal{E} = C([0,T];\mathbf{H}) \cap \mathbf{L}^2(0,T;D(\mathrm{A}^{\frac{1}{2}}))$. Thus, we infer that for $\delta \in \{0,1\}$ the solution $\mathbf{u}^{\alpha,\delta}$ to (2.4) satisfies an LDP on $\mathcal{E}$ with speed $\alpha^{-1}\lambda_\delta^2(\alpha)$ and rate function I_δ. This completes the proof of Theorem 5.1. ∎

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