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LOCAL SOLVABILITY FOR A QUASILINEAR WAVE EQUATION WITH THE FAR FIELD DEGENERACY: 1D CASE*

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Abstract. We study the Cauchy problem for the quasilinear wave equation $u_{tt}=(u^{2a}\partial_x u)_x+F(u)u_x$ with $a\geq 0$ and show a result for the local-in-time existence under new conditions. In the previous results, it is assumed that $u(0,x)\geq c_0>0$ for some constant c_0 to prove the existence and the uniqueness. This assumption ensures that the equation does not degenerate. In this paper, we allow the equation to degenerate at spatial infinity. Namely we consider the local well-posedness under the assumption that $u(0,x)>0$ and $u(0,x)\to 0$ as $|x|\to\infty$. Furthermore, to prove the local well-posedness, we find that the so-called Levi condition appears. Our proof is based on the method of characteristics and the contraction mapping principle via weighted L^∞ estimates.

Keywords. Quasilinear hyperbolic equation; First-order hyperbolic systems; Levi condition.

AMS subject classifications. 35L05; 35L80; 35L60.

1. Introduction

In this paper, we consider the following Cauchy problem of the model quasilinear wave equation in $\mathbb{R}$:

$$\begin{cases} u_{tt}=(u^{2a}u_x)_x+F(u)u_x, & (t,x)\in(0,T]\times\mathbb{R}, \\ u(0,x)=u_0(x), & x\in\mathbb{R}, \\ \partial_t u(0,x)=u_1(x), & x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where F is a given function and $a\geq 0$. The purpose of this paper is to show the local existence and the uniqueness under new conditions. The existence of solutions to the more general quasi-linear wave equations has been widely known since the 1970s. Kato [12] and Hughes, Kato and Marsden [9] have shown an abstract theorem about the well-posedness of the system of general quasi-linear wave equations in L^2 Sobolev space. In one-dimensional case, the well-posedness in C_b^1 class for first-order hyperbolic equations has been studied by Douglis [5] and Hartman and Winter [7] (see also Majda [14] and Courant and Lax [3]), where C_b^1 is a set of continuous and bounded functions whose derivatives are also bounded. In order to apply these results to the existence problem of (1.1), the following assumption is required:

$$u_0(x)\geq c_0>0 \tag{1.2}$$

for a constant c_0. This condition ensures that the equation in (1.1) is the strictly hyperbolic type near $t=0$. This paper relaxes this condition. We show the local existence and the uniqueness of solutions of (1.1) under the assumption that the equation degenerates at spatial infinity. Namely we weaken (1.2) by $u(0,x)>0$ and allow that $u(0,x)$ can decay to 0 as $|x|\to\infty$ (more precise assumptions are given later). To the best of my knowledge, the well-posedness has never been studied under these types of assumptions.

1.1. Known results. Let us review some results on the solvability for degenerate wave equations (weakly hyperbolic equations). The existence, nonexistence and

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regularity of solutions to the following type of linear weakly hyperbolic equations have been studied by many authors (e.g. Oleinik [18], Colombini and Spagnolo [2], Ivrii and Petkov [11] and Taniguchi and Tozaki [22]),

$$\partial_t^2 u - \sum_{i,j=1}^{n} a_{i,j}(t,x) u_{x_i x_j} + \sum_{j=1}^{n} b_j(t,x) u_{x_j} = 0, \tag{1.3}$$

where $a_{i,j}$ and b_j are smooth functions and $\sum_{i,j=1}^{n} a_{i,j}(t,x)\xi_i\xi_j \geq 0$ is assumed for $(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$. We note that $\sum_{i,j=1}^{n} a_{i,j}(t,x)\xi_i\xi_j = 0$ corresponds to the degeneracy. In Oleinik [18], (1.3) have been solved under the so-called Levi condition:

$$C_1\left(\sum_{j=1}^{n} b_j\xi_j\right)^2 \leq C_2\left(\sum_{i,j=1}^{n} a_{i,j}\xi_i\xi_j + \partial_t a_{i,j}\xi_i\xi_j\right). \tag{1.4}$$

Even if the Levi condition is assumed, we can only obtain the following energy estimate with the regularity loss for weakly hyperbolic equations:

$$\|u\|_{H^s} + \|u_t\|_{H^{s-1}} \leq C(\|u_0\|_{H^{s+r_1}} + \|u_1\|_{H^{s-1+r_2}}), \tag{1.5}$$

where s is an arbitrary real number and r_1 and r_2 are non-negative numbers. It is known that this estimate is optimal in the sense of the regularity by observing some explicit solution to some special linear weakly hyperbolic equations. Ivrii and Petkov in [11] have treated the following model of the $1D$ weakly hyperbolic equation:

$$u_{tt} - t^{2l} u_{xx} + t^k u_x = 0.$$

They have shown that the Levi condition ($k \geq l-1$) is necessary for the Cauchy problem of this equation to be C^∞ well-posed. Colombini and Spagnolo in [2] have given an example of a C^∞ function $a(t)\geq 0$ such that

$$u_{tt} - a(t)u_{xx} = 0$$

is not well-posed in C^∞. Roughly speaking, highly oscillatory behaviors of $a(t)$ near the point where $a(t)=0$ causes the ill-posedness. In [10], Han has derived an energy inequality with a regularity loss for the linear weakly hyperbolic equation:

$$u_{tt} - a(t,x)u_{xx} = 0,$$

where $a(t,x) = t^m + a_1(x)t^{m-1} + a_2(x)t^{m-2} + \cdots + a_{m-1}(x)t + a_m(x)$.

Manfrin in [17] has established the local existence and the uniqueness for following $1D$ degenerate quasilinear wave equations with $u_0, u_1 \in C_0^\infty(\mathbb{R}^n)$:

$$u_{tt} = a(u)\Delta u,$$

where $a(u)$ is an analytic function satisfying $a(0)=0$. This result can be extended to more general degenerate wave equations (see also Manfrin [15, 16]). In Dreher's paper [6], he also has shown the local solvability for $\partial_t^2 u = \partial_x(|\partial_x u|^{p-2}\partial_x u)$ with $p>5$ and under the initial condition that $u_0, u_1 \in C_0^k(\mathbb{R}^n)$ for a large natural number k. Since their proof is based on the Nash-Moser implicit function theorem and the argument in Oleinik's paper [18], the compactness of the support of initial data is essentially used.

Hence it does not seem difficult to extend Manfrin's method to the case that initial data are not compactly supported.

In [8], Hu and Wang have shown the local existence and uniqueness of solutions to the following variational wave equation:

$$u_{tt} = c(u,x)(c(u,x)u_x)_x \tag{1.6}$$

with initial data and the function $c(u,x)$ satisfying

$$\begin{aligned} &c(u(0,x),x) = 0,\\ &u_t(0,x) \geq c_0 > 0,\\ &c_u(u,x) \geq c_1 > 0 \end{aligned}$$

for some constants c_0 and c_1. The choice of initial data implies that the equation degenerates at $t=0$ and that $c(u,x)$ becomes positive uniformly and immediately after $t=0$. The method in [8] is inspired by Zhang and Zheng's paper [24] which studies the existence of solutions to Euler type equation in gas dynamics. In [24] and [8], they use method of characteristics for a new dependent variable and the fixed-point theorem in a special metric space.

1.2. Assumptions and main theorem. Before stating the main theorem of this paper, we introduce assumptions on initial data and the function F. We set $\gamma = \gamma(a,\alpha)$ as below:

$$\gamma = \begin{cases} 0, & a \geq 1,\\ (1-a)\alpha, & \text{otherwise.} \end{cases}$$

For initial data $u_0 \in C^2(\mathbb{R})$ and $u_1 \in C^1_b(\mathbb{R})$, we assume that

$$c_1 \langle x \rangle^{-\alpha} \leq u_0(x) \leq c_2, \tag{1.7}$$

$$|u_1(x) \pm u_0^a u_0'(x)| \leq c_3 \langle x \rangle^{-\beta}, \tag{1.8}$$

$$\left| \frac{d}{dx}(u_1(x) \pm u_0^a u_0'(x)) \right| \leq c_4 \langle x \rangle^{-\gamma} \tag{1.9}$$

with conditions on $\alpha, \beta, a \geq 0$ that

$$\alpha \leq \beta, \tag{1.10}$$

$$\alpha(a+1) \leq 2\beta, \tag{1.11}$$

where $u_0' = du_0/dx$ and $\langle x \rangle$ is defined by $\langle x \rangle = (1+x^2)^{1/2}$ and c_1, c_2, c_3, c_4 are positive constants. The Assumption (1.7) indicates that the equation in (1.1) can degenerate at spatial infinity. For the function $F \in C([0,\infty)) \cap C^1((0,\infty))$, we assume that

$$|F(\theta)| \leq C_K \theta^a, \tag{1.12}$$

$$|F'(\theta)| \leq C_K \theta^{a-1} \tag{1.13}$$

for $\theta \in (0,K]$ and C_K is a positive constant. Typical example of F is $F(\theta) = \theta^b$ with $b \geq a$. This condition, appearing in the study of weakly hyperbolic equations, is called a sufficient Levi condition (e.g. Manfrin's paper [17]). The main theorem of this paper is as follows.

Theorem 1.1. *Let $u_0 \in C^2(\mathbb{R})$ and $u_1 \in C_b^1(\mathbb{R})$. Suppose that the conditions (1.7)-(1.13) hold. Then there exists a number $T>0$ depending on the constants in (1.7)-(1.13) such that the Cauchy problem (1.1) has a unique local solution $u \in C^2([0,T]\times\mathbb{R})$ satisfying that for all $(t,x)\in[0,T]\times\mathbb{R}$*

$$C_1 \langle x\rangle^{-\alpha} \le u(t,x) \le C_2, \tag{1.14}$$

$$|(u_t \pm u^a u_x)(t,x)| \le C_3 \langle x\rangle^{-\beta}, \tag{1.15}$$

$$|(u_t \pm u^a u_x)_t(t,x)| + |(u_t \pm u^a u_x)_x(t,x)| \le C_4 \langle x\rangle^{-\gamma}, \tag{1.16}$$

where C_1, C_2, C_3, C_4 are positive constants.

Theorem 1.1 asserts the local existence and uniqueness of solutions of (1.1) under the Levi type condition without the regularity loss. In our problem, the equation in (1.1) is strictly hyperbolic locally for space. Thus, for arbitrarily fixed $x\in\mathbb{R}$, by applying previous results, we can solve the problem (1.1) in a triangle region with (T,x) as the vertex, if T is small. However, the smallness of T depends on x from the loss of strict hyperbolicity at spatial infinity. To show the existence of solution of (1.1) in $\mathbb{R}$, we need to show some nonlocal estimates for space. Our proof is based on the method of characteristics and the contraction mapping principle via weighted L^∞ estimates. In contrast to previous results on the existence for strictly hyperbolic equations, $1/u$ and u_x are not bounded (see (2.2) and the definitions of N_1 and N_2). To avoid this crux, we use the spatial decay of $u_t \pm u^a u_x$. In particular, this property helps to show the boundedness of the derivative of characteristic curves $x_\pm(t)$ with initial position (see Lemma 2.2). We also remark that our approach is applicable to various types of 1D quasilinear wave equations (e.g. the equation like (1.6) under suitable condition on $c(u,x)$).

Remark 1.1. Suppose that initial data is $(u_0(x),u_1(x)) = (\langle x\rangle^{-\alpha_1}, \langle x\rangle^{-\alpha_2})$ with $\alpha_1,\alpha_2 \ge 0$. If $\alpha_1 \le \alpha_2$ and $\alpha_1(a+1)\le 2\alpha_2$ are satisfied, then all assumptions (1.7)-(1.11) on initial data are satisfied. We also remark that the condition (1.11) is not necessary in the case that $a\le 1$, since (1.10) implies (1.11). While if $a \ge 1$, (1.10) is not necessary.

1.3. Notation and plan of the paper. For a domain $\Omega\subset\mathbb{R}^n$, we define $C_b^m(\Omega)$ with $m\in\mathbb{N}$ as follows

$$C_b^m(\Omega) = \{ f \in C^m(\Omega) \mid \sum_{|\alpha|\le m} \sup_{x\in\Omega} |\partial_x^\alpha f(x)| < \infty \}.$$

We write $C_b(\Omega) = C_b^0(\Omega)$ and denote the Lebesgue space for $1\le p\le\infty$ on $\mathbb{R}^n$ by L^p with the norm $\|\cdot\|_{L^p}$. For a Banach space X, $1\le p\le\infty$ and $T>0$, we denote the set of all X-valued L^p functions with $t\in[0,T]$ by $L^p([0,T];X)$. For convenience, we denote $L^p([0,T];X)$ by L_T^pX. The norm of L_T^pX is denoted by $\|f\|_{L_T^pX}$. Various constants are simply denoted by C or C_j for $j\in\mathbb{N}$. We denote that $\langle x\rangle = (1+x^2)^{1/2}$.

The remainder of the present paper is organized as follows. In Section 2, we review several formulas for the unknown variable $R = u_t + u^a u_x$ and $S = u_t - u^a u_x$, which are called Riemann invariants in the study of the 1D hyperbolic conservation law, and give some estimates for characteristic curves. In Section 3, we show Theorem 1.1 by using the method of characteristics, weighted L^∞ estimates and the contraction mapping principle. Concluding remarks are given in Section 4.

2. Preliminaries

2.1. Basic formulation for unknown variables R and S. We set $R(t,x)$ and $S(t,x)$ as follows

$$\begin{cases} R = u_t + u^a u_x, \\ S = u_t - u^a u_x. \end{cases} \tag{2.1}$$

By (1.1), R and S are solutions to the system of the following first-order equations:

$$\begin{cases} R_t - u^a R_x = N_1(u,R,S) + L(u,R,S), \\ u_x = \dfrac{1}{2u^a}(R-S), \\ S_t + u^a S_x = N_2(u,R,S) + L(u,R,S), \end{cases} \tag{2.2}$$

where we set

$$L(u,R,S) = \frac{F(u)(R-S)}{2u^a},$$

$$N_1(u,R,S) = \frac{a}{2u}(R^2 - RS)$$

and

$$N_2(u,R,S) = \frac{a}{2u}(S^2 - RS).$$

Let $x_\pm(t)$ be characteristic curves on the first and third equations of (2.2) respectively. That is, $x_+(t)$ and $x_-(t)$ are solutions to the following differential equations respectively:

$$\frac{d}{dt}x_\pm(t) = \pm u^a(t, x_\pm(t)). \tag{2.3}$$

When we emphasize the characteristic curves go through (s,y), we denote $x_\pm(t)$ by $x_\pm(t;s,y)$. That is, $x_\pm(t;s,y)$ satisfies that

$$x_\pm(t;s,y) = y \pm \int_s^t u^a(\tau, x_\pm(\tau;s,y))d\tau. \tag{2.4}$$

On the characteristic curves, R and S satisfy that

$$\begin{cases} \dfrac{d}{dt}R(t,x_-(t)) = N_1(u,R,S)(t,x_-(t)) + L(u,R,S)(t,x_-(t)), \\ \dfrac{d}{dt}S(t,x_+(t)) = N_2(u,R,S)(t,x_+(t)) + L(u,R,S)(t,x_+(t)). \end{cases} \tag{2.5}$$

2.2. Some estimates of characteristic curves. We prepare some estimates for characteristic curves for $u \in C^1([0,T]\times\mathbb{R})$ satisfying for $\alpha \geq 0$

$$A_0 \langle x \rangle^{-\alpha} \leq u(t,x) \leq A_1, \tag{2.6}$$

where A_0 and A_1 are positive constants. In addition, we assume that

$$0 \leq u^a |u_x(t,x)| \leq A_2 \langle x \rangle^{-\alpha} \tag{2.7}$$

for a constant A_2. The boundedness of u and (2.4) implies the following estimate with $s,t\in[0,T]$:

$$x-A_1^a|t-s|\le x_\pm(s;t,x)\le x+A_1^a|t-s|. \tag{2.8}$$

Next we show a lemma ensures uniform Lipschitz continuity of $x_\pm(t;s,y)$. This lemma helps to show that a sequence of characteristic curves satisfies an assumption of the Arzelá-Ascoli theorem.

LEMMA 2.1. *Let $u\in C^1([0,T]\times\mathbb{R})$. Suppose that (2.6) and (2.7) hold. Then the characteristic curves fulfill that for $x_1,x_2\in\mathbb{R}$ and $t_1,t_2,t_3,t_4\in[0,T]$*

$$|x_\pm(t_3;t_1,x_1)-x_\pm(t_4;t_2,x_2)|\le 3(1+A_1^a)(|x_1-x_2|+|t_1-t_2|+|t_3-t_4|), \tag{2.9}$$

if $T>0$ is sufficiently small.

Proof. First we show the case that $t_3=t_4=t\in[0,T]$ and $t\ge t_1,t_2$. From (2.4), we can easily compute that

$$\begin{aligned}
&|x_\pm(t;t_1,x_1)-x_\pm(t;t_2,x_2)|\\
&\le|x_1-x_2|+\left|\int_{t_1}^t u^a(\tau,x_\pm(\tau;t_1,x_1))d\tau-\int_{t_2}^t u^a(\tau,x_\pm(\tau;t_2,x_2))d\tau\right|\\
&\le|x_1-x_2|+\left|\int_{t_2}^{t_1} u^a(\tau,x_\pm(\tau;t_2,x_2))d\tau\right|\\
&\quad+\int_0^t |u^a(\tau,x_\pm(\tau;t_1,x_1))-u^a(\tau,x_\pm(\tau;t_2,x_2))|d\tau.
\end{aligned} \tag{2.10}$$

From (2.6) and (2.7), we have that $u^{a-1}u_x$ is bounded. Thus we have for the third term of the right-hand side in (2.10) that

$$\begin{aligned}
|u^a(\tau,x_\pm(\tau;t_1,x_1))-u^a(\tau,x_\pm(\tau;t_2,x_2))|&\le\left|\int_{x_\pm(\tau;t_2,x_2)}^{x_\pm(\tau;t_1,x_1)} u^{a-1}u_x(\tau,y)dy\right|\\
&\le C|x_\pm(\tau;t_1,x_1)-x_\pm(\tau;t_2,x_2)|.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&|x_\pm(t;t_1,x_1)-x_\pm(t;t_2,x_2)|\\
&\le|x_1-x_2|+A_1^\alpha|t_1-t_2|+C\int_0^t |x_\pm(\tau;t_1,x_1)-x_\pm(\tau;t_2,x_2)|d\tau.\\
&\le(1+A_1^a)(|x_1-x_2|+|t_1-t_2|)+C\int_0^t |x_\pm(\tau;t_1,x_1)-x_\pm(\tau;t_2,x_2)|d\tau.
\end{aligned}$$

Thus we have from the Gronwall inequality

$$|x_\pm(t;t_1,x_1)-x_\pm(t;t_2,x_2)|\le(1+A_1^a)(|x_1-x_2|+|t_1-t_2|)e^{Ct}.$$

Hence if T is small, then we have that with $t_1\ge t_2$

$$|x_\pm(t;t_1,x_1)-x_\pm(t;t_2,x_2)|\le 2(1+A_1^a)(|x_1-x_2|+|t_1-t_2|). \tag{2.11}$$

In the same way as above, we can show (2.11) with the case that $t<t_1$ or $t<t_2$. We omit the proof of this case. Next we show (2.9). The left-hand side of (2.9) is written by

$$|x_\pm(t_3;t_1,x_1)-x_\pm(t_4;t_2,x_2)|\le|x_\pm(t_3;t_1,x_1)-x_\pm(t_3;t_2,x_2)|+|x_\pm(t_3;t_2,x_2)-x_\pm(t_4;t_2,x_2)|$$

From (2.11), the first term of the right-hand side is estimated by $2(1+A_1^a)(|x_1-x_2|+|t_1-t_2|)$. From (2.4) and (2.6), the second term is estimated by $A_1^a|t_3-t_4|$. Therefore, we have the desired inequality. ∎

Following lemma is used to show the boundedness of the derivatives of R and S.

LEMMA 2.2. *Let $u\in C^1([0,T]\times\mathbb{R})$. Suppose that (2.6) and (2.7) hold. Then the characteristic curves $x_\pm(t;s,x)$ are differentiable with x and $\partial_x x_\pm(t;s,x)$ satisfies that with $(t,x)\in[0,T]\times\mathbb{R}$ and $s\in[0,T]$ for small $T>0$*

$$\begin{cases} \dfrac{d}{ds}\partial_x x_\pm(s;t,x)=\pm a u^{a-1}u_x(t,x_\pm(s;t,x))\partial_x x_\pm(s;t,x), \\ \partial_x x_\pm(t;t,x)=1 \end{cases} \tag{2.12}$$

and

$$|\partial_x x_\pm(s;t,x)|\le e^{C|t-s|}, \tag{2.13}$$

where the positive constant C depends on A_0,A_1 and A_2.

Proof. The differentiability of $x_\pm(s;t,x)$ and (2.12) are well-known as a basic fact (e.g., see textbook of Sideris [19]). We estimate $\partial_x x_\pm(s;t,x)$. We only show (2.13) with the case that $t\ge s$. From (2.6) and the boundedness of $\langle x\rangle^\alpha u^a u_x$, we obtain that

$$\begin{aligned}|\partial_x x_\pm(s;t,x)|&\le 1+a\int_s^t|u^{a-1}u_x||\partial_x x_\pm(\tau;t,x)|d\tau\\ &\le 1+C\int_s^t|\partial_x x_\pm(\tau;t,x)|d\tau.\end{aligned}$$

Hence, from the Gronwall inequality, we obtain (2.13) for small T. ∎

3. Proof of the main theorem

As in the Introduction, we set

$$\gamma=\begin{cases}0, & a\ge 1,\\ (1-a)\alpha, & \text{otherwise.}\end{cases}$$

We treat functions satisfying the following conditions for $\alpha,\beta\ge 0$ with $\alpha\le\beta$ such that

$$A_0\langle x\rangle^{-\alpha}\le f(t,x)\le A_1 \tag{3.1}$$

and

$$f^a(t,x)|f_x(t,x)|\le A_2\langle x\rangle^{-\beta}, \tag{3.2}$$

$$|f_t(t,x)|\le A_3\langle x\rangle^{-\beta} \tag{3.3}$$

or

$$|f(t,x)|\le A_3\langle x\rangle^{-\beta}, \tag{3.4}$$

and

$$|f_x(t,x)| \le A_4 \langle x\rangle^{-\gamma}, \tag{3.5}$$
$$|f_t(t,x)| \le A_5 \langle x\rangle^{-\gamma}, \tag{3.6}$$

where A_j are positive constants with $j=1,\cdots,5$. We define sets of C^1 functions X_α, $Y_{\beta,1}$, $Y_{\beta,2}$ as follows:

$$X_\alpha = \{f \in C_b^1 \mid f(0,x)=u_0(x) \text{ and } (3.1), (3.2) \text{ and } (3.3) \text{ hold.}\},$$
$$Y_{\beta,1} = \{f \in C_b^1 \mid f(0,x)=R_0(x) \text{ and } (3.4), (3.5) \text{ and } (3.6) \text{ hold.}\},$$
$$Y_{\beta,2} = \{f \in C_b^1 \mid f(0,x)=S_0(x) \text{ and } (3.4), (3.5) \text{ and } (3.6) \text{ hold.}\},$$

where given functions (u_0,R_0,S_0) belong to $C_b^1\times C_b^1\times C_b^1$. For given functions $(v,\bar R,\bar S)\in X_\alpha\times Y_{\beta,1}\times Y_{\beta,2}$, we consider the first-order linear hyperbolic equation:

$$\begin{cases} R_t - v^a R_x = N_1(v,\bar R,\bar S)+L(v,\bar R,\bar S), \\ S_t + v^a S_x = N_2(v,\bar R,\bar S)+L(v,\bar R,\bar S) \end{cases} \tag{3.7}$$

with initial condition $(R(0,x),S(0,x))=(R_0,S_0)\in C_b^1\times C_b^1$. We set

$$u = u_0(x)+\int_0^t \frac{R+S}{2}(s,x)ds. \tag{3.8}$$

We find that (3.7) with C^1 initial data has unique and time-global solutions such that $R,S\in C^1([0,T]\times\mathbb{R})$ with arbitrary fixed $T>0$ from the method of characteristics. From (3.8), it holds that $u\in C^1$. Namely we can define the map

$$\Phi : X_\alpha\times Y_{\beta,1}\times Y_{\beta,2}\to C^1\times C^1\times C^1$$

such that $\Phi(v,\bar R,\bar S)=(u,R,S)$. We take four positive numbers A_0,A_1,A_3,A_4 satisfying that

$$2A_0\langle x\rangle^{-\alpha}\le u_0(x)\le \frac{A_1}{2}, \tag{3.9}$$
$$\|\langle x\rangle^{\beta} R_0\|_{L^\infty}+\|\langle x\rangle^{\beta} S_0\|_{L^\infty}\le \frac{A_3}{4}, \tag{3.10}$$
$$\|\langle x\rangle^{\gamma} R_0'\|+\|\langle x\rangle^{\gamma} S_0'\|\le \frac{A_4}{8}. \tag{3.11}$$

The constants A_2 and A_5 in (3.6) will be taken later. Moreover, we assume that

$$\|\langle x\rangle^{\beta} u_0^a u_0'\|_{L^\infty}\le B_1 \tag{3.12}$$

for a positive constant B_1. In the following, we show that $(u,R,S)\in X_\alpha\times Y_{\beta,1}\times Y_{\beta,2}$ and Φ is a contraction mapping in the topology of L^∞ for sufficiently small T. X_α and $Y_{\beta,j}$ with $j=1,2$ are not a closed set of L^∞ space. Nevertheless it is possible to show that the fixed point belongs to $X_\alpha\times Y_{\beta,1}\times Y_{\beta,2}$. Furthermore we will show that the regularity is improved as $u\in C^2$. First we show the following proposition.

Proposition 3.1. *Let $(u_0,R_0,S_0)\in C^1\times C^1\times C^1$ satisfying (3.9)-(3.11) and (3.12). Suppose that $v,\bar R,\bar S\in X_\alpha\times Y_{\beta,1}\times Y_{\beta,2}$. Then $\Phi(v,\bar R,\bar S)=(u,R,S)\in X_\alpha\times Y_{\beta,1}\times Y_{\beta,2}$ for sufficiently small $T>0$.*

Proof. From the method of characteristics, we can see that the solution of (3.7) can be written by

$$\begin{cases} R(t,x)=R(0,x_-(0))+\int_0^t N_1(v,\bar{R},\bar{S})(s,x_-(s))+L(v,\bar{R},\bar{S})(s,x_-(s))ds, \\ S(t,x)=S(0,x_+(0))+\int_0^t N_2(v,\bar{R},\bar{S})(s,x_+(s))+L(v,\bar{R},\bar{S})(s,x_+(s))ds, \end{cases} \tag{3.13}$$

where the characteristic curves for the linear equation (3.7) are defined as follows:

$$\frac{d}{dt}x_\pm(t)=\pm v^a(t,x_\pm(t))$$

with initial data $x_\pm(t)=x$. From this expression, we have that $(u,R,S)\in C^1\times C^1\times C^1$. Now we estimate $\|\langle x\rangle^\beta R\|_{L^\infty_T L^\infty}$. From (2.8), if T is small, we have

$$\frac{\langle x\rangle^\beta}{2}\le\langle x_-(s;t,x)\rangle^\beta\le 2\langle x\rangle^\beta. \tag{3.14}$$

From (3.13), (3.14) and (3.1), we have that if T is small

$$\begin{aligned} |\langle x\rangle^\beta R(t,x)|\le&2\|\langle x\rangle^\beta R(0,\cdot)\|_{L^\infty}+\int_0^t\frac{\langle x_-(s)\rangle^\beta}{v}|\bar{R}^2-\bar{R}\bar{S}|+\langle x_-(s)\rangle^\beta\frac{F(v)}{v^a}|\bar{R}-\bar{S}|ds \\ \le&\frac{A_3}{2}+\int_0^t C\langle x_-(s)\rangle^{\alpha+\beta}|\bar{R}^2-\bar{R}\bar{S}|+C\langle x_-(s)\rangle^\beta|\bar{R}-\bar{S}|ds. \end{aligned}$$

Noting that $\langle x_-(s)\rangle^\alpha\le\langle x_-(s)\rangle^\beta$ from $\alpha\le\beta$, by using (3.4) and (1.12), we have that

$$\begin{aligned} &\int_0^t\int_0^t C\langle x_-(s)\rangle^{\alpha+\beta}|\bar{R}^2-\bar{R}\bar{S}|+C\langle x_-(s)\rangle^\beta|\bar{R}-\bar{S}|ds \\ \le&CT\left(\|\langle x\rangle^\beta\bar{R}\|^2_{L^\infty_T L^\infty}+\|\langle x\rangle^\beta\bar{R}\|_{L^\infty_T L^\infty}\|\langle x\rangle^\beta\bar{S}(s)\|_{L^\infty_T L^\infty}\right) \\ &\quad+CT(\|\langle x\rangle^\beta\bar{R}\|_{L^\infty_T L^\infty}+\|\langle x\rangle^\beta\bar{S}\|_{L^\infty_T L^\infty}) \\ \le&CT, \end{aligned}$$

where C is a positive constant depending on A_0,A_1,A_3. Hence we obtain that for sufficiently small T

$$\|\langle x\rangle^\beta R\|_{L^\infty_T L^\infty}\le A_3. \tag{3.15}$$

Similarly for S, we have that

$$\|\langle x\rangle^\beta S\|_{L^\infty_T L^\infty}\le A_3. \tag{3.16}$$

Next we estimate $\|\langle x\rangle^\gamma R_x\|_{L^\infty_T L^\infty}$ and $\|\langle x\rangle^\gamma S_x\|_{L^\infty_T L^\infty}$. Differentiating both sides of equations (3.7) with x, we can obtain integral equations for R_x and S_x as follows:

$$\begin{aligned} V(t,x)=&V_0(x_-(0;t,x))\partial_x x_-(0;t,x) \\ &+\int_0^t\partial_x x_-(s;t,x)\left(N_{1u}v_x+N_{1R}\bar{V}+N_{1S}\bar{W}\right)(t,x_-(s;t,x))ds \\ &+\int_0^t\partial_x x_-(s;t,x)\left(L_u v_x+L_R\bar{V}+L_S\bar{W}\right)(t,x_-(s;t,x))ds \end{aligned} \tag{3.17}$$

and

$$\begin{aligned}W(t,x)=&W_0(x_+(0;t,x))\partial_x x_+(0;t,x)\\&+\int_0^t \partial_x x_+(s;t,x)\left(N_{2u}v_x+N_{2R}\bar{W}+N_{2S}\bar{V}\right)(t,x_+(s;t,x))ds\\&+\int_0^t \partial_x x_+(s;t,x)\left(L_u v_x+L_R\bar{V}+L_S\bar{W}\right)(t,x_+(s;t,x))ds,\end{aligned}\tag{3.18}$$

where we denote $\bar{V}=\bar{R}_x$ and $\bar{W}=\bar{S}_x$ and $(V_0,W_0)=(R_0'(\cdot),S_0'(\cdot))$ and N_{ju},N_{jS},N_{jR} $(j=1,2)$ are partial derivatives of $N_j=N_j(u,R,S)$ with u,S,R respectively (the same manners are also used for L). From Lemma 2.2, we obtain that $|\partial_x x_+(0;t,x)|$ is bounded by 2, if T is small (note that smallness of T depends on A_0,A_1 and A_2). Hence we have that

$$\begin{aligned}|V(t,x)|\le&\frac{A_4\langle x\rangle^{-\gamma}}{2}+2\int_0^t\left|N_{1u}v_x+N_{1R}\bar{V}+N_{1S}\bar{W}\right|(t,x_-(s;t,x))ds\\&+2\int_0^t|L_u v_x+L_R\bar{V}+L_S\bar{W}|(t,x_-(s;t,x))ds.\end{aligned}\tag{3.19}$$

From (3.1)-(3.6), (1.10) and (1.11), $|N_{1R}\bar{V}|$ and $|N_{1S}\bar{W}|$ are trivially estimated as

$$\begin{aligned}|N_{1R}\bar{V}|+|N_{1S}\bar{W}|\le&\frac{C}{v}(|\bar{R}|+|\bar{S}|)(|\bar{V}|+|\bar{W}|)\\\le&C\langle x\rangle^{\alpha-\beta-\gamma}\\\le&C\langle x\rangle^{-\gamma}.\end{aligned}$$

Noting that $(2+a)\alpha-3\beta=(1+a)\alpha-2\beta+\alpha-\beta\le-\gamma$ from (1.10) and (1.11), we also have that for $|N_{1u}v_x|$

$$\begin{aligned}|N_{1u}v_x|\le&\frac{Cv^a|v_x|}{v^{2+a}}(|\bar{R}|^2+|\bar{S}|^2)\\\le&\frac{C}{v^{2+a}}(|\bar{R}|^3+|\bar{S}|^3)\\\le&C\langle x\rangle^{(2+a)\alpha-3\beta}\\\le&C\langle x\rangle^{-\gamma}.\end{aligned}$$

From (1.12) and (1.13), we estimate $L_u v_x$ as

$$\begin{aligned}|L_u v_x|\le&C\left(\frac{|F(v)|}{v^{a+1}}+\frac{|F'(v)|}{v^a}\right)(|\bar{R}|+|\bar{S}|)|v_x|\\\le&\frac{C(|R|^2+|S|^2)}{v^{a+1}}\\\le&C\langle x\rangle^{\alpha(a+1)-2\beta}\\\le&C\langle x\rangle^{-\gamma}.\end{aligned}$$

Similarly we have that

$$|L_R\bar{V}|+|L_S\bar{W}|\le C\langle x\rangle^{-\gamma}.$$

Applying these estimates to (3.19), we obtain that with small T

$$\|\langle x\rangle^{\gamma} V\|_{L^\infty_T L^\infty} \leq A_4. \tag{3.20}$$

Similarly

$$\|\langle x\rangle^{\gamma} W\|_{L^\infty_T L^\infty} \leq A_4. \tag{3.21}$$

Estimates of R_t and S_t are obtained from (3.7). In fact, we have that

$$\begin{aligned}
|R_t(t,x)|+|S_t(t,x)| \leq & |v^a R_x|+|v^a S_x|+|N_1(v,\bar{R},\bar{S})|+|N_2(v,\bar{R},\bar{S})|+2|L(v,\bar{R},\bar{S})| \\
\leq & C\langle x\rangle^{-\gamma}+C\langle x\rangle^{-\beta} \\
\leq & C_A\langle x\rangle^{-\gamma},
\end{aligned} \tag{3.22}$$

where C_A is a positive constant depending on A_1, A_3 and A_4 (independent of A_5). Here we choose A_5 as $A_5=C_A$. From the above estimates, we have that $(R,S)\in Y_{\beta,1}\times Y_{\beta,2}$. Next we show that $u\in X_\alpha$. From (3.8), (3.15) and (3.16), it follows for sufficiently small T that

$$\begin{aligned}
\langle x\rangle^{\alpha} u(t) \geq & \langle x\rangle^{\alpha} u_0(x) - \int_0^t \frac{\langle x\rangle^{\alpha}(|R|+|S|)}{2} ds \\
\geq & 2A_0 - \int_0^t \frac{\langle x\rangle^{\beta}(|R|+|S|)}{2} ds \\
\geq & 2A_0 - TA_3 \\
\geq & A_0.
\end{aligned} \tag{3.23}$$

Similarly we can easily check that $\|u\|_{L^\infty_T L^\infty} \leq A_1$, if T is small. Next we show that

$$\|\langle x\rangle^{\beta} u^a u_x\|_{L^\infty_T L^\infty} \leq A_2. \tag{3.24}$$

(3.8) directly implies that

$$u^a u_x = \left(u_0 + \int_0^t \frac{R+S}{2} ds\right)^a u_0' + \left(u_0 + \int_0^t \frac{R+S}{2} ds\right)^a \int_0^t \frac{R_x+S_x}{2} ds. \tag{3.25}$$

From (3.9) and the boundedness of $\langle x\rangle^{\beta} u_0^a |u_0'|$, the first term of (3.25) is estimated as

$$\begin{aligned}
\left(u_0 + \int_0^t \frac{R+S}{2} ds\right)^a |u_0'| \leq & 2^a (u_0^a + CT^a \langle x\rangle^{-a\beta})|u_0'| \\
\leq & C(1+T^a) u_0^a |u_0'| \\
\leq & C_{1,A}\langle x\rangle^{-\beta},
\end{aligned}$$

where we note that the positive constant $C_{1,A}$ does not depend on A_2. Deducing R_x+S_x via (3.7), we also obtain

$$\begin{aligned}
& \left(u_0 + \int_0^t \frac{R+S}{2} ds\right)^a \left|\int_0^t \frac{R_x+S_x}{2} ds\right| \\
= & \left(u_0 + \int_0^t \frac{R+S}{2} ds\right)^a \left|\int_0^t \frac{1}{2v^a}(R_t - S_t + N_2 - N_1) ds\right|
\end{aligned}$$

$$\le 2^a (u_0^a + T^a \langle x\rangle^{-a\beta}) \left| \int_0^t \frac{1}{2v^a}(R_t - S_t + N_2 - N_1) ds \right|.$$

Since the spatial decay of R_t and S_t is not enough to show (3.24), we need to change this term. From the integration by parts and the property of X_α that $v_0 = u_0$, we obtain that

$$\int_0^t \frac{R_t - S_t}{2v^a} ds = \frac{R-S}{2v^a} - \frac{R_0 - S_0}{2u_0^a} + \int_0^t \frac{a(R-S)v_t}{2v^{a+1}} ds. \tag{3.26}$$

While, from (3.3), we have that

$$u_0 - A_3 T \langle x\rangle^{-\beta} \le v \le u_0 + A_3 T \langle x\rangle^{-\beta}.$$

Hence, with the help of (3.1), it holds that

$$\frac{1}{2} \le 1 - A_3 A_0 T \le \frac{v}{u_0} \le 1 + A_3 A_0 T \le 2 \tag{3.27}$$

for small T. The third term of the right-hand side in (3.26) is estimated as

$$\begin{aligned}\int_0^t \frac{a|R-S||v_t|}{2v^{a+1}} ds &\le C \int_0^t \frac{\langle x\rangle^{\alpha} |R-S||v_t|}{v^a} ds \\ &\le C \langle x\rangle^{-\beta} \int_0^t v^{-a} ds.\end{aligned} \tag{3.28}$$

Applying (3.27) and (3.28) to (3.26), we have

$$2^a (u_0^a + T^a \langle x\rangle^{-a\beta}) \left| \int_0^t \frac{1}{2v^a}(R_t - S_t) ds \right| \le C_{2,A} \langle x\rangle^{-\beta}$$

and similarly

$$2^a (u_0^a + T^a \langle x\rangle^{-a\beta}) \left| \int_0^t \frac{1}{2v^a}(N_2 - N_1) ds \right| \le C_{3,A} \langle x\rangle^{-\beta},$$

where positive constants $C_{2,A}$ and $C_{3,A}$ are independent of A_2. Thus we have

$$\| \langle x\rangle^{\beta} u^a u_x \|_{L_T^\infty L^\infty} \le C_{1,A} + C_{2,A} + C_{3,A}.$$

Taking $A_2 = C_{1,A} + C_{2,A} + C_{3,A}$, we obtain (3.24). (3.8) and (3.4) directly yield that

$$\begin{aligned}\| \langle x\rangle^{\beta} u_t \|_{L_T^\infty L^\infty} &\le \left\| \frac{\langle x\rangle^{\beta} (R+S)}{2} \right\|_{L_T^\infty L^\infty} \\ &\le A_3.\end{aligned}$$

Therefore we have that $(u,R,S) \in X_\alpha \times Y_{\beta,1} \times Y_{\beta,2}$. In the end of the proof, we additionally show that (u,R,S) is (locally) Lipschitz continuous. From (3.20), (3.21) and (3.22), we can obviously check that R and S satisfies the following uniform Lipschitz estimate:

$$|R(t,x) - R(s,y)| + |S(t,x) - S(s,y)| \le 2(A_4 + A_5)(|t-s| + |x-y|). \tag{3.29}$$

Next we check that u is locally Lipschitz continuous. From the boundedness of $u_0^a u_0'$ and (3.1), we estimate $|u_0(x)-u_0(y)|$ with $x,y\in[-K,K]$ as

$$\begin{aligned}|u_0(x)-u_0(y)|\le&\left|\int_y^x |u_x(t,z)|dz\right|\\ \le&C\left|\int_y^x \langle z\rangle^{a\alpha}u^a|u_x|(t,z)dz\right|\\ \le&C\langle K\rangle^{a\alpha}|x-y|.\end{aligned}$$

Since R and S are uniformly Lipschitz continuous, we have from (3.8) that u is locally Lipschitz continuous such that for any $t_1,t_2\in[0,T]$, $x_1,x_2\in[-K,K]$ with $K\ge1$

$$|u(t_1,x_1)-u(t_2,x_2)|\le C(|t_1-t_2|+\langle K\rangle^{a\alpha}|x_1-x_2|), \tag{3.30}$$

where C is a positive constant depending on A_0,A_3,A_4 and A_5. These additional properties are used to show that the fixed point of Φ satisfies integral equations. ∎

Proposition 3.2. *Under the same assumptions on (3.1), Φ is a contraction mapping in the topology of L^∞-norm with small $T>0$. Namely, if $T>0$ is small, then there exists a constant $c\in(0,1)$ such that Φ satisfies that*

$$\begin{aligned}&\|u_1-u_2\|_{L_T^\infty L^\infty}+\|R_1-R_2\|_{L_T^\infty L^\infty}+\|S_1-S_2\|_{L_T^\infty L^\infty}\\ \le& c\left(\|v_1-v_2\|_{L_T^\infty L^\infty}+\|\bar R_1-\bar R_2\|_{L_T^\infty L^\infty}+\|\bar S_1-\bar S_2\|_{L_T^\infty L^\infty}\right),\end{aligned}$$

where $(u_j,R_j,S_j)=\Phi(v_j,\bar R_j,\bar S_j)$ with $j=1,2$.

Proof. Put $\tilde u=u_1-u_2$, $\tilde R=R_1-R_2$, $\tilde S=S_1-S_2$. From (3.7), we have

$$\begin{aligned}\tilde R_t-v_1^a\tilde R_x=&N_1(v_1,\bar R_1,\bar S_1)-N_1(v_2,\bar R_2,\bar S_2)\\ &+L(v_1,\bar R_1,\bar S_1)-L(v_2,\bar R_2,\bar S_2)\\ &+(v_1^a-v_2^a)R_{2x}.\end{aligned}$$

From the method of characteristics, we have that

$$\begin{aligned}\tilde R(t,x)=&\int_0^t(N_1(v_1,\bar R_1,\bar S_1)-N_1(v_2,\bar R_2,\bar S_2))ds\\ &+\int_0^t(L(v_1,\bar R_1,\bar S_1)-L(v_2,\bar R_2,\bar S_2))ds\\ &+\int_0^t(v_1^a-v_2^a)R_{2x}ds.\end{aligned}\tag{3.31}$$

The second term of the right-hand side in (3.31) can be written as

$$\begin{aligned}\int_0^t(L(v_1,\bar R_1,\bar S_1)-L(v_2,\bar R_2,\bar S_2))ds=&\int_0^t(G(v_1)-G(v_2))(\bar R_1-\bar S_1)ds\\ &+\int_0^t G(v_2)(\bar R_1-\bar S_1-\bar R_2+\bar S_2)ds,\end{aligned}\tag{3.32}$$

where we set $G(\theta)=F(\theta)/2\theta^a$. Using (1.12), (1.13), we obtain that

$$|G(v_1)-G(v_2)|\le\int_0^1|G'(\theta v_1+(1-\theta)v_2)|d\theta|v_1-v_2|\le\int_0^1\frac{Cd\theta}{\theta v_1+(1-\theta)v_2}|v_1-v_2|.$$

Applying (3.1) to the last term, we estimate that

$$\begin{aligned}\int_0^1 \frac{C d\theta}{\theta v_1+(1-\theta)v_2}|v_1-v_2| &\le \int_0^{1/2} \frac{C d\theta}{(1-\theta)v_2}|v_1-v_2|+\int_{1/2}^1 \frac{C d\theta}{\theta v_1}|v_1-v_2| \\ &\le C\langle x\rangle^{\alpha}|v_1-v_2|,\end{aligned}$$

which implies that the first term of the right-hand side in (3.32) is estimated as

$$\int_0^t |G(v_1)-G(v_2)||\bar{R}_1-\bar{S}_1|ds \le CTA_3\|v_1-v_2\|_{L^\infty_T L^\infty}.$$

From (1.12), the second term is estimated by

$$\int_0^t |G(v_2)||\bar{R}_1-\bar{S}_1-\bar{R}_2+\bar{S}_2|ds \le CT(\|\bar{R}_1-\bar{R}_2\|_{L^\infty_T L^\infty}+\|\bar{S}_1-\bar{S}_2\|_{L^\infty_T L^\infty}).$$

Setting $N_1(v,\bar{R},\bar{S})=\frac{a}{2v}Q(\bar{R},\bar{S})$ and $Q(\bar{R},\bar{S})=(\bar{R}^2-\bar{R}\bar{S})$, we change the first term of the right-hand side in (3.32) to

$$\begin{aligned}&\int_0^t (N_1(v_1,\bar{R}_1,\bar{S}_1)-N_1(v_2,\bar{R}_2,\bar{S}_2))ds \\ &=\int_0^t a\left(\frac{v_2-v_1}{2v_1v_2}\right)Q(\bar{R}_1,\bar{S}_1)ds+\int_0^t \frac{a}{2v_2}\left(Q(\bar{R}_1,\bar{S}_1)-Q(\bar{R}_2,\bar{S}_2)\right)ds.\end{aligned} \tag{3.33}$$

The first term of the right-hand side in (3.33) is estimated as

$$\begin{aligned}&\left|\int_0^t a\left(\frac{v_2-v_1}{2v_1v_2}\right)Q(\bar{R}_1,\bar{S}_1)ds\right| \\ &\le \int_0^t a\frac{|v_1-v_2|}{2\langle x\rangle^{2\alpha}|v_1v_2|}\langle x\rangle^{2\alpha}|Q(\bar{R}_1,\bar{S}_1)|ds \\ &\le \int_0^t a\frac{2|v_1-v_2|}{A_0^2}\langle x\rangle^{2\beta}|Q(\bar{R}_1,\bar{S}_1)|ds \\ &\le CT\|v_1-v_2\|_{L^\infty_T L^\infty}.\end{aligned}$$

The second term can be estimated as

$$\begin{aligned}&\left|\int_0^t \left(\frac{a}{2v_2}\right)\left(Q(\bar{R}_1,\bar{S}_1)-Q(\bar{R}_2,\bar{S}_2)\right)ds\right| \\ &\le \int_0^t \frac{a}{2A_0}\langle x\rangle^{\alpha}\left|Q(\bar{R}_1,\bar{S}_1)-Q(\bar{R}_2,\bar{S}_2)\right|ds \\ &\le \int_0^t \frac{a}{A_0}\langle x\rangle^{\beta}\left|Q(\bar{R}_1,\bar{S}_1)-Q(\bar{R}_2,\bar{S}_2)\right|ds \\ &\le CT\left(\|\bar{R}_1-\bar{R}_2\|_{L^\infty_T L^\infty}+\|\bar{S}_1-\bar{S}_2\|_{L^\infty_T L^\infty}\right).\end{aligned}$$

Next we estimate the third term of the right-hand side in (3.31). When $a\ge 1$, from the mean-value theorem for $|v_1^a-v_2^a|$ and the boundedness of R_x, we obtain that

$$|(v_1^a-v_2^a)R_{2x}|\le C\|v_1-v_2\|_{L^\infty_T L^\infty}.$$

While, $a<1$, by using the boundedness of $\langle x\rangle^\gamma R_x$, we have that

$$\begin{aligned}|(v_1^a-v_2^a)R_{2x}| &\le a\int_0^1(\theta v_1+(1-\theta)v_2)^{a-1}|v_1-v_2||R_{2x}|d\theta\\ &\le C\langle x\rangle^\gamma|v_1-v_2||R_{2x}|\\ &\le C\|v_1-v_2\|_{L_T^\infty L^\infty}.\end{aligned}$$

Therefore, we obtain that for sufficiently small T

$$\|\tilde{R}\|_{L_T^\infty L^\infty}\le\frac{1}{6}\left(\|v_1-v_2\|_{L_T^\infty L^\infty}+\|\bar{R}_1-\bar{R}_2\|_{L_T^\infty L^\infty}+\|\bar{S}_1-\bar{S}_2\|_{L_T^\infty L^\infty}\right).$$

In the same way as in the estimate of $\tilde{R}$, we have that

$$\|\tilde{S}\|_{L_T^\infty L^\infty}\le\frac{1}{6}\left(\|v_1-v_2\|_{L_T^\infty L^\infty}+\|\bar{R}_1-\bar{R}_2\|_{L_T^\infty L^\infty}+\|\bar{S}_1-\bar{S}_2\|_{L_T^\infty L^\infty}\right).$$

From (3.8), the above two estimates on $\tilde{R}$ and $\tilde{S}$ imply that for sufficiently small T

$$\begin{aligned}\|\tilde{u}\|_{L_T^\infty L^\infty} &\le\frac{T}{2}\left(\|\tilde{R}\|_{L_T^\infty L^\infty}+\|\tilde{S}\|_{L_T^\infty L^\infty}\right)\\ &\le\frac{1}{6}\left(\|v_1-v_2\|_{L_T^\infty L^\infty}+\|\bar{R}_1-\bar{R}_2\|_{L_T^\infty L^\infty}+\|\bar{S}_1-\bar{S}_2\|_{L_T^\infty L^\infty}\right).\end{aligned}$$

Therefore, we find that Φ is a contraction mapping for sufficiently small $T>0$. ∎

Next we construct a unique solution (u,R,S) of the nonlinear problem and the characteristic curves $x_\pm(\cdot;t,x)$.

PROPOSITION 3.3. *Under the same assumptions as in Proposition 3.2, if T is small, then there uniquely exist $(u,R,S)\in X_\alpha\times Y_{\beta,1}\times Y_{\beta,2}$ and $x_\pm(s)=x_\pm(s;t,x)$ satisfying that*

$$\begin{cases}R(t,x)=R(0,x_-(0))+\int_0^t N_1(u,R,S)(s,x_-(s))+L(u,R,S)(s,x_-(s))ds,\\ S(t,x)=S(0,x_+(0))+\int_0^t N_2(v,R,S)(s,x_+(s))+L(u,R,S)(s,x_+(s))ds\end{cases}\tag{3.34}$$

and

$$u(t,x)=u_0(x)+\int_0^t\frac{R+S}{2}ds\tag{3.35}$$

and

$$x_\pm(s;t,x)=x\pm\int_t^s u^a(\tau,x_\pm(\tau;t,x))d\tau.\tag{3.36}$$

Proof. We fix $K\ge1$ arbitrarily. From Proposition 3.1, we can define a sequence $\{u_n,R_n,S_n\}_{n\in\mathbb{N}}$ in $X_\alpha\times Y_{\beta,1}\times Y_{\beta,2}$ such that

$$(u_{n+1},R_{n+1},S_{n+1})=\Phi(u_n,R_n,S_n)$$

with initial term (u_0,S_0,R_0). By Proposition 3.2, (u_n,R_n,S_n) converges to the fixed point (u,R,S) in the topology of L^∞. While we can define a sequence of the characteristic curves $\{x_{\pm,n}(\cdot;t,x)\}_{n\in\mathbb{N}}$, we note that the characteristic curves can be defined uniquely on $[0,T]$ with arbitrarily fixed (t,x) by the local Lipschitz continuity and the boundedness of u_n^a. For arbitrarily fixed $K\geq 1$, we see that $\{x_{\pm,n}(\cdot;t,x)\}_{n\in\mathbb{N}}$ is uniform equicontinuous and uniform bounded on $[0,T]\times[0,T]\times[-K,K]$ from Lemma 2.1 and (2.8). Thus the Arzelá-Ascoli theorem implies that there exists a subsequence of $\{x_{\pm,n}(\cdot;t,x)\}_{n\in\mathbb{N}}$ (we use the same suffix as in the original sequence) such that $x_{\pm,n}(\cdot)$ converges $x_\pm(\cdot)$ uniformly on $[0,T]\times[0,T]\times[-K,K]$ as $n\to\infty$. Note that this choice of the subsequence depends on K. However, from Cantor's diagonal argument, we can reselect a subsequence independently of K such that the convergence holds on $[0,T]\times[0,T]\times[-K',K']$ with any $K'\geq 1$. From (3.29) and (3.30), we see that as $n\to\infty$

$$(u_n(t,x_{\pm,n}(t)),R_n(t,x_{\pm,n}(t)),S_n(t,x_{\pm,n}(t)))\to(u(t,x_\pm(t)),R(t,x_\pm(t)),S(t,x_\pm(t))).$$

Hence (3.34) and (3.36) are satisfied. Now we check that $(u,R,S)\in X_\alpha\times Y_{\beta,1}\times Y_{\beta,2}$. It is obvious that the properties (3.1) and (3.4) are satisfied. From the local Lipschitz continuity, u,R,S are differentiable almost everywhere. In the same way as in the proof of Proposition 3.1, we can obtain the boundedness of $\langle x\rangle^\beta u^a u_x$ and $\langle x\rangle^\beta u_t$, since the constant A_2 is taken independently of A_4 and A_5. Thus we have $u\in X_\alpha$. To show the boundedness of $\langle x\rangle^\gamma R_x$ and $\langle x\rangle^\gamma S_x$, differentiating both sides of (3.34) with x, we obtain that

$$\begin{aligned}V(t,x)=&V_0(x_-(0;t,x))\partial_x x_-(0;t,x)\\&+\int_0^t \partial_x x_-(s;t,x)(N_{1u}u_x+N_{1R}V+N_{1S}W)(t,x_-(s;t,x))ds\\&+\int_0^t \partial_x x_-(s;t,x)(L_u u_x+L_R V+L_S W)(t,x_-(s;t,x))ds\end{aligned}\tag{3.37}$$

and

$$\begin{aligned}W(t,x)=&W_0(x_+(0;t,x))\partial_x x_+(0;t,x)\\&+\int_0^t \partial_x x_+(s;t,x)(N_{2u}u_x+N_{2R}W+N_{2S}V)(t,x_+(s;t,x))ds\\&+\int_0^t \partial_x x_+(s;t,x)(L_u u_x+L_R V+L_S W)(t,x_+(s;t,x))ds.\end{aligned}\tag{3.38}$$

In the same way as in the proof of (3.20) and (3.21), we achieve the boundedness of $\langle x\rangle^\gamma W$ and $\langle x\rangle^\gamma V$. The estimates of $\langle x\rangle^\gamma R_t$ and $\langle x\rangle^\gamma S_t$ can be shown similarly as those in (3.22). Thus we have $(R,S)\in Y_{\beta,1}\times Y_{\beta,2}$. The uniqueness can be shown in the same way as in the proof of Proposition 3.2. ∎

In the discussions so far, we do not assume any relations between u_0 and R_0,S_0. To show u in (3.35) is a solution of (1.1), we assume that

$$\begin{cases}u_0'=\frac{R_0-S_0}{2u_0^a},\\ u_1=\frac{R_0+S_0}{2}.\end{cases}\tag{3.39}$$

Moreover, we improve the regularity of the solution if $(u_0,u_1)\in C^2\times C^1$. The following proposition completes the proof of Theorem 1.1.

Proposition 3.4. *In addition to the assumption of Proposition 3.3, we assume for $(u_0,u_1)\in C^2\times C_b^1$ that (3.39) is satisfied. Then the function u defined in (3.35) is C^2 on $[0,T]\times\mathbb{R}$ and is the classical solution of (1.1).*

Proof. From the Lipschitz continuity of R,S, these are differentiable almost everywhere and satisfy that

$$\begin{cases} R_t - u^a R_x = N_1(u,R,S) + L(u,R,S), \\ S_t + u^a S_x = N_2(u,R,S) + L(u,R,S). \end{cases} \tag{3.40}$$

Since u is also differentiable almost everywhere, differentiating (3.35), we have that

$$u_x = u_0'(x) + \int_0^t \frac{R_x + S_x}{2} ds. \tag{3.41}$$

From the first and second equations of (3.40), we have that

$$\begin{aligned} \int_0^t R_x + S_x ds &= \int_0^t \frac{1}{u^a} (N_2(u,R,S) - N_1(u,R,S) + R_t - S_t) ds \\ &= \int_0^t \frac{1}{u^a} \left(\frac{a}{2u}(S^2 - R^2) + R_t - S_t \right) ds. \end{aligned} \tag{3.42}$$

From the integration by parts, $u_t = \frac{R+S}{2}$ and (3.39), it follows that

$$\begin{aligned} \int_0^t \frac{1}{u^a} (R_t - S_t) ds &= -2u_0'(x) + \frac{R-S}{u^a} + \int_0^t \frac{a u_t}{u^{a+1}} (R-S) ds \\ &= -2u_0'(x) + \frac{R-S}{u^a} + \int_0^t \frac{a}{2u^{a+1}} (R^2 - S^2) ds. \end{aligned} \tag{3.43}$$

From (3.41), (3.42) and (3.43), we have that

$$u_x = \frac{R-S}{2u^a}. \tag{3.44}$$

Combining (3.40), (3.35) and (3.44), we have that the function u satisfies (1.1). Lastly, applying Theorem 4 in Douglis [5], we obtain the continuity of the $W = R_x$ and $V = S_x$. From the equations of R,S, we see that R_t and S_t are also continuous. Hence we have the continuity of u_{xx}, u_{tx}, u_{tt}. Therefore we have that $u \in C^2([0,T] \times \mathbb{R})$. □

4. Concluding remarks

4.1. Physical background.

We set a function G as a primitive function of F such that $G(0) = 0$. Integrating with x over $[-\infty, x]$, we formally obtain that

$$\int_{-\infty}^x u_{tt} dx = \left(\frac{u^{a+1}}{a+1} \right)_x + G(u).$$

Setting $v = \int_{-\infty}^x u_t dx$ and $\sigma(u) = u^{a+1}/a+1$, we have the following first-order hyperbolic equation:

$$\begin{cases} u_t - v_x = 0, \\ v_t - \sigma(u)_x = G(u). \end{cases} \tag{4.1}$$

These equations govern the motion for one-dimensional elastic waves with the case that the density of material is equal to 1. Unknown functions u and v describe the differentiations of the displacement X with x and t respectively. Namely $u = X_x(t,x)$ and $v = X_t(t,x)$. The first equation means the relation $u_t = X_{xt} = X_{tx} = v_x$. The second equation is Newton's second law since v_t is the acceleration. From the definition of u, u is the strain (more precisely, (1,1) component of the stain matrix) and $\sigma(u)$ is so-called stress-strain relation. G is a external force term depending only on the strain. The detailed derivation with $G \equiv 0$ is given in Cristescu's book [4].

4.2. On the generalization of the main theorem. Our existence theorem is also applicable to $u_{tt}=(c(u)^2u_x)_x+F(u)u_x$ under the following assumptions on $c(\cdot)\in C([0,\infty))\cap C^2((0,\infty))$ and $F\in C([0,\infty))\cap C^1((0,\infty))$

$$C_{1.K}\theta^a\leq c(\theta)\leq C_{2,K}, \tag{4.2}$$

$$|c'(\theta)|\leq C_{3,K}\theta^{a-1}, \tag{4.3}$$

$$|c''(\theta)|\leq C_{4,K}\theta^{a-2}, \tag{4.4}$$

and

$$|F(\theta)|\leq C_{5,K}\theta^a, \tag{4.5}$$

$$|F'(\theta)|\leq C_{6,K}\theta^{a-1}, \tag{4.6}$$

where $a\geq 0$, $\theta\in[0,K]$ for $K>0$ and $C_{j,K}$ are positive constants depending on K for $j=1,\dots,6$. For this equation, the unknown valuable R and S are defined by

$$R=u_t+c(u)u_x,$$

$$S=u_t-c(u)u_x$$

and R and S satisfy that

$$\begin{cases} R_t-u^aR_x=\frac{c'}{2c}(RS-S^2)+F(u)\frac{R-S}{2c}, \\ S_t+u^aS_x=\frac{c'}{2c}(RS-R^2)+F(u)\frac{R-S}{2c}. \end{cases} \tag{4.7}$$

Since we have that $\frac{|c'(u)|}{c(u)}\leq C\langle x\rangle^\alpha$ from the assumption on c and initial data, we can obtain weighted L^∞ estimates for R and S. The Assumption (4.4) is used in the proof of the construction of the contraction mapping.

4.3. Finite time blow-up or degeneracy. We define T^* as the maximal existence time of the solution constructed by Theorem 1.1. When $T^*<\infty$, we have the following criterion of the break-down:

$$\limsup_{t\to T^*}\|\langle x\rangle^\beta u_t\|_{L^\infty}+\|\langle x\rangle^\beta u_x\|_{L^\infty}=\infty \tag{4.8}$$

or

$$\liminf_{t\to T^*}\inf_{\mathbb{R}}\langle x\rangle^\alpha u(t,x)=0. \tag{4.9}$$

We call (4.8) and (4.9) the blow-up and the degeneracy respectively. In the case that $F\equiv 0$, we can obtain that the non-trivial solutions blow up in finite time, if $R(0,x)$ and $S(0,x)$ are non-negative. In fact, we can show that the non-negativity of $R(0,x)$ and $S(0,x)$ preserves as time goes by, from which we have $u_t(t,x)\geq 0$. Thus we find that (4.9) does not occur in finite time. Therefore, using the method of Lax [13] or [23] (see also Chen [1]), we have the conclusion. While, in the case that $F\equiv 0$, we can apply main theorems to the equation in (1.1) and find that (4.9) can occur in finite time for non-trivial solutions, if $R(0,x)$ and $S(0,x)$ are non-positive. Sufficient conditions for the occurrence of (4.9) have been studied in the author's papers [20, 21].

4.4. Multi-dimensional case. The multi-dimensional version of the equation in (1.1) is

$$\partial_t^2 u = u^{2a} \Delta u + F(u) \cdot \nabla u = 0.$$

The method of characteristics (and Riemann invariants) does not work, even with radial initial data. In the forthcoming paper, we deal this problem via a local-energy argument.

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