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THE EXISTENCE AND LIMIT BEHAVIOR OF THE SHOCK LAYER FOR 1D STATIONARY COMPRESSIBLE NON-NEWTONIAN FLUIDS*

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Abstract. In this paper, we first define the shock layer to a class of stationary compressible non-Newtonian fluids in one dimension. Then the existence and uniqueness of the shock layer are established. In addition, the limit behavior of the shock layer is analyzed. It is shown that, as the viscosity coefficient and the heat conductivity coefficient vanish, the shock layer to the non-Newtonian fluids tends to a shock wave of the corresponding Euler equations. It is also shown that, as the viscosity coefficient tends to zero, the shock layer goes to a non-viscous shock layer to the non-Newtonian fluids, while as heat-conductivity coefficient tends to zero, the shock layer converges to a thermally non-conducting shock layer to the non-Newtonian fluids.

Keywords. Non-Newtonian fluids; Navier-Stokes equations; Euler equations; Shock layer; Shock wave.

AMS subject classifications. 76A05; 74J40; 35L67.

1. Introduction

The one-dimensional stationary compressible non-Newtonian fluids are described by the principle of conservations of mass, momentum and energy, which read as

$$\begin{cases} \rho u = c_1, \\ \rho u^2 + p - \mu(u_x)u_x = c_2, \\ \rho u\Big(\dfrac{1}{2}u^2 + e + \dfrac{p}{\rho}\Big) - \mu(u_x)uu_x - \lambda\theta_x = c_3, \end{cases} \tag{1.1}$$

where ρ, u, θ, $p=p(\tau,\theta)$, $e=e(\tau,\theta)$ and $\tau=1/\rho$ represent the density, velocity, temperature, pressure, internal energy and specific volume, respectively, $\mu=\mu(u_x)>0$ and $\lambda=\lambda(\tau,\theta)>0$ are the viscosity coefficient and the heat conductivity coefficient, respectively, c_1,c_2,c_3 are constants. In this paper, the viscosity coefficient is taken to be

$$\mu(u_x) = \mu_0|u_x|^{q-2},\ q>1 \text{ and } q\neq 2, \tag{1.2}$$

where $\mu_0=\mu_0(\tau,\theta)>0$. We also assume that λ, μ_0, p and e are sufficiently smooth functions of τ and θ.

In many fields, such as chemistry, study of biological fluids like blood, geology and glaciology, a large number of problems may arise with non-Newtonian fluids, which has sparked the increasing interest in the study of non-Newtonian fluids, see [2,19,20,23,25,

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29, 30], etc. Especially, Ladyzhenskaya [19] proposed a special model for incompressible fluids

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(\Gamma) + \nabla p = \rho f, \\ \mathrm{div} u = 0 \end{cases} \tag{1.3}$$

where f is an external force, Γ denotes the viscous stress tensor and

$$\Gamma_{ij} = (\mu_1 + \mu_2 |E(\nabla u)|^{q-2}) E_{ij}(\nabla u), \tag{1.4}$$

$E_{ij}(\nabla u) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$ is the rate of strain. For $\mu_1 = 0$ and $\mu_2 > 0$, if $q < 2$, it is a pseudo-plastic fluid, and if $q > 2$, then it is a dilatant fluid [2]. In the view of physics, the model captures the shear thinning fluid for the case of $1 < q < 2$, and captures the shear thickening fluid for the case of $q > 2$. Mamontov [22] established the global existence of sufficiently regular solutions to two-dimensional and three-dimensional equations of compressible non-Newtonian fluids. Yuan and his collaborators [29, 30] obtained the existence and uniqueness of local and global solutions for one-dimensional initial boundary value problem. Fang and Guo [4] gave the blow-up criterion for local strong solutions, constructed analytical solutions to a class of compressible non-Newtonian fluids with free boundaries in [5], and considered the existence and uniqueness of global classical solution for the initial boundary problem [7]. For weak solutions to the non-Newtonian fluids, Zhikov and Pastukhova [32] obtained the existence of weak solutions of initial boundary value problem for multidimensional cases. Guo and Zhu [11] investigated the partial regularity of the suitable weak solutions. Feireisl, Liao and Málek [9] studied mathematical properties of unsteady problem for three dimensional compressible non-Newtonian fluids in bounded domains and have shown the long-time and large-data existence result of weak solutions with strictly positive density. Fang, Kong and Liu [8] obtained the existence of weak solutions to one-dimensional full compressible non-Newtonian fluids. Besides, recently, Shi, Wang and Zhang [23], Fang and Guo [6], Guo, Dong and Liu [12] investigated the stability of rarefaction waves and boundary layer solutions for the compressible non-Newtonian fluids, respectively.

In the theory of compressible fluids, the asymptotic behavior of the compressible Navier-Stokes equations in the vanishing viscosity limit is one of the important, long-standing problems. Formally, as the viscosity coefficient and the heat conductivity coefficient vanish, i.e., $\lambda, \mu_0 \to 0$ in (1.1), the limit system of (1.1) becomes the Euler equations. It is expected that the solution to (1.1) should converge strongly to the solution to the corresponding Euler equations as dissipation vanishes. The first rigorous convergence analysis of vanishing physical viscosity from the Navier-Stokes equations to the Euler equations was made by Gilbarg [10], in which he established the mathematical existence and vanishing viscosity limit of the shock layer for the system (1.1), (1.2) with $q = 2$ under the following thermodynamical assumptions proposed by Weyl [24]:

- (I) $d\tau/dp|_{S=\text{const.}} < 0$, $S = $ entropy.
- (Ia) $S_p(\tau, p) > 0$, $\theta_p(\tau, p) > 0$.
- (II) $d^2\tau/dp^2|_{S=\text{const.}} > 0$.
- (III) In the continuous process of adiabatic compression one can raise pressure arbitrarily high.
- (IV) The thermodynamic state Z is uniquely specified by pressure p and specific volume τ, and the points (τ, p) representing the possible states Z in a (τ, p) diagram form a convex region.

(V) The thermodynamic relation should be satisfied: $de=\theta dS-pd\tau$.

In the past decades, many important results have been obtained for the problem of zero dissipation limit for the compressible Naiver-Stokes equations. For the compressible isentropic Naiver-Stokes equations, in the case of the basic wave patterns involving shock waves and rarefaction waves, we refer to Hoff and Liu [13], Huang et al. [17] and Zhang et al. [31] for shock waves, and [16, 27] for rarefaction waves. When the far field of the initial values of the isentropic Euler system has no vacuums, Chen and Perepelitsa [3] proved the vanishing viscosity limit of the isentropic Naiver-Stokes equations by compensated compactness method. For the compressible nonisentropic Naiver-Stokes equations, we refer to Jiang, Ni and Sun [18], Xin and Zeng [28] for the rarefaction wave, Wang [26] for the shock wave, Ma [21] for the contact discontinuity, and Huang, Wang and Yang [14, 15] for the superposition of two rarefaction waves and a contact discontinuity and the superposition of a shock wave and a rarefaction wave.

The shock phenomena can be identified in the non-Newtonian fluids, for instance, the strong pressure pulses with steep wave fronts in blood vessels [1]. However, because of the nonlinear constitutive relation between viscous stress tensor and rate of strain in the non-Newtonian fluids, the nonlinear waves may exhibit some properties which are different from those in the Newtonian fluids. In this paper, we establish the existence and uniqueness of the shock layer of the non-Newtonian fluids (1.1) which follow the thermodynamical assumptions (I)-(V) and investigate the limit behavior of the shock layer as the viscosity coefficient and the heat conductivity coefficient vanish.

Firstly, we give the definition of the shock layer for the system (1.1). Eliminating u from system (1.1), we have

$$\begin{cases}\lambda\dfrac{d\theta}{dx}=L(\tau,\theta),\\[2ex] \mu\dfrac{d\tau}{dx}=M(\tau,\theta),\end{cases}\tag{1.5}$$

where $M=\frac{1}{b}\big(p(\tau,\theta)+b^2(\tau-a)\big)$, $L=b\big(e(\tau,\theta)-\frac{1}{2}b^2(\tau-a)^2-c\big)$, and the constants $a=c_2/c_1^2$, $b=c_1$, $c=c_3/c_1-c_2^2/(2c_1^2)$. Furthermore, by the thermodynamical equation $de=\theta dS-pd\tau$, one can check that $c_1>0$.

Since $\mu=\mu_0|u_x|^{q-2}=\mu_0c_1^{q-2}|\tau_x|^{q-2}$, from $(1.5)_2$, we have

$$\mu_0|\tau_x|^{q-2}\tau_x=\frac{1}{c_1^{q-2}}M(\tau,\theta).\tag{1.6}$$

It is noticed from (1.6) that τ_x and $M(\tau,\theta)$ have the same sign. Thus, if $M(\tau,\theta)\geqslant 0$, (1.6) has the form $\mu_0^{\frac{1}{q-1}}\tau_x=c_1^{-\frac{q-2}{q-1}}\big(M(\tau,\theta)\big)^{\frac{1}{q-1}}$, and if $M(\tau,\theta)<0$, (1.6) becomes $\mu_0^{\frac{1}{q-1}}\tau_x=-c_1^{-\frac{q-2}{q-1}}\big(-M(\tau,\theta)\big)^{\frac{1}{q-1}}$. Setting

$$\tilde{\mu}_0:=\mu_0^{\frac{1}{q-1}},\tag{1.7}$$

and

$$\tilde{M}(\tau,\theta):=\begin{cases}c_1^{-\frac{q-2}{q-1}}\big(M(\tau,\theta)\big)^{\frac{1}{q-1}}, & M(\tau,\theta)\geqslant 0,\\ -c_1^{-\frac{q-2}{q-1}}\big(-M(\tau,\theta)\big)^{\frac{1}{q-1}}, & M(\tau,\theta)<0,\end{cases}\tag{1.8}$$

we thus can deduce from (1.5) that

$$\begin{cases} \lambda \dfrac{d\theta}{dx} = L(\tau,\theta), \\ \tilde{\mu}_0 \dfrac{d\tau}{dx} = \tilde{M}(\tau,\theta). \end{cases} \tag{1.9}$$

In the Z-plane consisting of a set of points $Z=(\tau,\theta)$ in the first quadrant (Figure 1.1), we enumerate four conditions on the functions $L(\tau,\theta)$ and $\tilde{M}(\tau,\theta)$:

(A) $L_\theta > 0$, $M_\theta > 0$. $\tilde{M}(\tau,\theta)$ is a sufficiently smooth (e.g. twice differentiable) function of τ and θ in the neighbourhood of Z_0 and Z_1, and $\tilde{M}_\theta(Z_0) > 0, \tilde{M}_\theta(Z_1) > 0$.

(B) Two curves L and $\tilde{M}$ determined by equation $L(\tau,\theta)=0$ and $\tilde{M}(\tau,\theta)=0$, respectively, intersect in two points $Z_0=(\tau_0,\theta_0)$ and $Z_1=(\tau_1,\theta_1)$, $(\tau_0>\tau_1)$. Z_0 and Z_1 are the only simultaneous solutions of $L(\tau,\theta)=0$ and $\tilde{M}(\tau,\theta)=0$.

(C) On curve L, $L_\tau > 0$, for $\tau_1 \le \tau \le \tau_0$.

(D) $L_\tau/L_\theta < \tilde{M}_\tau/\tilde{M}_\theta$ at Z_0; $L_\tau/L_\theta > \tilde{M}_\tau/\tilde{M}_\theta$ at Z_1.

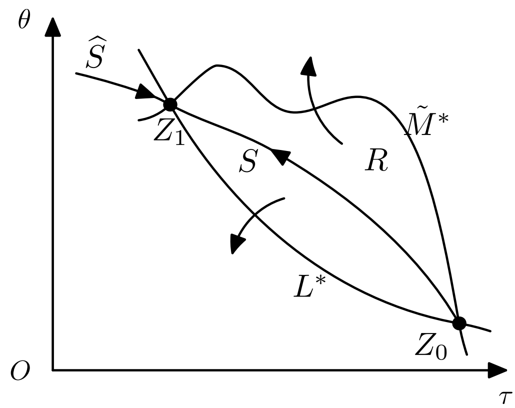

FIG. 1.1.

We can get from (A) that L and $\tilde{M}$ can be represented as single-valued functions of τ, namely, $\theta=l(\tau)$, $\theta=m(\tau)$, respectively. And from (C), we can see that $l(\tau)$ is monotonically decreasing in the interval $\tau_1 \leqslant \tau \leqslant \tau_0$, and thus $\theta_1 > \theta_0$. By (B) and (D), one has $m(\tau) > l(\tau)$ for $\tau_1 < \tau < \tau_0$.

We denote the arcs of L and $\tilde{M}$ between the two points $Z_0(\tau_0,\theta_0)$ and $Z_1(\tau_1,\theta_1)$ by L^* and $\tilde{M}^*$, respectively. The closed curve formed by L^* and $\tilde{M}^*$ bounds a simply connected region R of the Z-plane (Figure 1.1). And by virtue of (A) and (B), one can check that $L(\tau,\theta)>0$ and $\tilde{M}(\tau,\theta)<0$ in R.

For the points $Z_0(\tau_0,\theta_0)$ and $Z_1(\tau_1,\theta_1)$, since the shock conditions

$$p_0 + b^2\tau_0 = p_1 + b^2\tau_1 = b^2 a, \tag{1.10}$$

$$e_0 - \frac{1}{2}b^2(\tau_0 - a)^2 = e_1 - \frac{1}{2}b^2(\tau_1 - a)^2 = c, \tag{1.11}$$

hold, Z_0 and Z_1 represent possible initial and final states, respectively, of a normal shock wave of an ideal fluid which has the same equations of state as the given fluid.

Now, we can give the definition of the shock layer for the non-Newtonian fluids (1.1).

DEFINITION 1.1. *For given $\lambda(\tau,\theta)$ and $\mu_0(\tau,\theta)$, a solution $S(x)=(\tau(x),\theta(x)), x\in\mathbf{R}$, of Equations (1.9) is called a shock layer if*

$$\lim_{x\to-\infty} S(x)=Z_0, \quad \lim_{x\to+\infty} S(x)=Z_1.$$

In the Z-plane, the corresponding shock layer curve is the integral curve represented by the set of equivalent shock layers, $S(x+k)(k=constant)$.

It is pointed out that a shock layer is called parametrized if a particular representative of this class is designated, and equivalent shock layers are considered identical.

Moreover, we can get two degenerate systems from the system (1.9):

$$\begin{cases} \tilde{\mu}_0 \dfrac{d\tau}{dx} = \tilde{M}(\tau,\theta), \\ \qquad 0 = L(\tau,\theta), \end{cases} \tag{1.12}$$

and

$$\begin{cases} \lambda \dfrac{d\theta}{dx} = L(\tau,\theta), \\ \qquad 0 = \tilde{M}(\tau,\theta). \end{cases} \tag{1.13}$$

We call any solution of (1.12), for $\tau\in[\tau_1,\tau_0]$ and for all $x\in(-\infty,+\infty)$, a thermally non-conducting shock layer to the non-Newtonian flow (1.1), and any solution of (1.13), for $\tau\in[\tau_1,\tau_0]$ and for all $x\in(-\infty,+\infty)$, a non-viscous shock layer to the non-Newtonian flow (1.1).

Secondly, we prove that under the conditions (A)-(D), there exists a unique shock layer joining Z_0 and Z_1 to the system (1.1).

Finally, we study the limit behavior of shock layer when the viscosity coefficient and the heat conductivity coefficient vanish. Note that $\mu_0\to 0$ is equivalent to $\tilde{\mu}_0\to 0$. It is shown that, as $\lambda,\tilde{\mu}_0\to 0$, the shock layer of (1.1) converges to the shock wave of the corresponding Euler equations. It is also shown that, as $\tilde{\mu}_0\to 0$ (with λ fixed), the shock layer goes to a non-viscous shock layer; while for $\lambda\to 0$ (with μ_0 fixed), the shock layer converges to a thermally non-conducting shock layer.

We state our main result as follows.

THEOREM 1.1 (Existence and uniqueness). *Let $Z_0(\tau_0,\theta_0)$ and $Z_1(\tau_1,\theta_1)$ be initial and final states of a fluid which satisfy the shock conditions (1.10) and (1.11). Then, for any $\tilde{\mu}_0$ and λ, there exists a unique shock layer $S(x;\lambda,\tilde{\mu}_0)$ joining Z_0 to Z_1 of (1.1) subject to the conditions (A)-(D).*

THEOREM 1.2 (Limit behavior). *Let $Z_0(\tau_0,\theta_0)$ and $Z_1(\tau_1,\theta_1)$ be initial and final states of the shock wave, $Z=Z_0$, $-\infty<x<\xi$, $Z=Z_1$, $\xi<x<+\infty$. Then, the shock layer $S(x;\lambda,\tilde{\mu}_0)$ can be parameterized such that*

$$\lim_{\substack{\lambda\to 0\\ \tilde{\mu}_0\to 0}} S(x;\lambda,\tilde{\mu}_0) = \lim_{\lambda\to 0}(\lim_{\tilde{\mu}_0\to 0} S(x;\lambda,\tilde{\mu}_0)) = \lim_{\tilde{\mu}_0\to 0}(\lim_{\lambda\to 0} S(x;\lambda,\tilde{\mu}_0)) = \begin{cases} Z_0, & -\infty<x<\xi, \\ Z_1, & \xi<x<+\infty. \end{cases}$$

Remark 1.1. It can be proved that conditions (A)-(D) are satisfied for the fluids which follow the thermodynamical assumptions (I)-(V). The arguments are almost same as in [10], so, we omit the details. In particular, one can check from Section 2 that the prototypical pressure function, $p(\tau,\theta)=\beta\theta/\tau$, can not be taken in the non-Newtonian flow (1.1).

Remark 1.2. Based on this work, it is interesting to study the zero dissipation limit of shock wave for the non-stationary compressible non-Newtonian fluids. This is left to the forthcoming paper.

The rest of the paper is organized as follows. In Section 2, we investigate the existence and uniqueness of the shock layer. Section 3 studies the limit behavior of the shock layer.

2. Existence and uniqueness of the shock layer

In this section, we consider the existence and uniqueness of the shock layer to the system (1.1) for arbitrary $\tilde{\mu}_0=\tilde{\mu}_0(\tau,\theta)$, $\lambda=\lambda(\tau,\theta)$ and give the proof for Theorem 1.1.

According to condition (B), Z_0 and Z_1 are the only exclusive singular points of the system (1.9). Using Taylor's formula for the functions $L(\tau,\theta)$ and $\tilde{M}(\tau,\theta)$ at Z_0 and Z_1, we can obtain the characteristic equation of the system (1.9)

$$\begin{aligned}0&=\begin{vmatrix}L_\theta/\lambda-\kappa & L_\tau/\lambda\\ \tilde{M}_\theta/\tilde{\mu}_0 & \tilde{M}_\tau/\tilde{\mu}_0-\kappa\end{vmatrix}\\&=\kappa^2-\Big(\frac{\tilde{M}_\tau}{\tilde{\mu}_0}+\frac{L_\theta}{\lambda}\Big)\kappa+\Big(\frac{\tilde{M}_\theta L_\theta}{\lambda\tilde{\mu}_0}\Big)\Big(\frac{\tilde{M}_\tau}{\tilde{M}_\theta}-\frac{L_\tau}{L_\theta}\Big),\end{aligned}\tag{2.1}$$

where the values λ, $\tilde{\mu}_0$, L_τ, L_θ, $\tilde{M}_\tau$ and $\tilde{M}_\theta$ are to be taken at Z_0 and Z_1, κ is the characteristic root. Here, we notice that

$$\tilde{M}_\theta=\frac{c_1^{-\frac{q-2}{q-1}}}{q-1}\big|M(\tau,\theta)\big|^{\frac{2-q}{q-1}}M_\theta(\tau,\theta).$$

In the case $1<q<2$, at the points Z_0 and Z_1, since $M(Z_0)=M(Z_1)=0$, we require $M_\theta(Z_0)\to+\infty$ and $M_\theta(Z_1)\to+\infty$ such that $\tilde{M}_\theta(Z_0)$ and $\tilde{M}_\theta(Z_1)$ are given positive constants. For example, if $q=\frac{4}{3}$, then $M(\tau,\theta)$ can be taken as $M(\tau,\theta)=(\theta-m(\tau))^{\frac{1}{3}}$, where $m(\tau)$ is a smooth function of τ. In the case $q>2$, the arguments are similar, for instance, if $q=\frac{8}{3}$, then we can take $M(\tau,\theta)=(\theta-m(\tau))^{\frac{5}{3}}$. A similar analysis applies to $\tilde{M}_\tau$.

Thus, the discriminant of this equation is

$$\begin{aligned}\Delta|_{Z=Z_0,Z_1}&=\Big(\frac{\tilde{M}_\tau}{\tilde{\mu}_0}+\frac{L_\theta}{\lambda}\Big)^2-4\frac{\tilde{M}_\theta L_\theta}{\lambda\tilde{\mu}_0}\Big(\frac{\tilde{M}_\tau}{\tilde{M}_\theta}-\frac{L_\tau}{L_\theta}\Big)\\&=\Big(\frac{\tilde{M}_\tau}{\tilde{\mu}_0}\Big)^2+\Big(\frac{L_\theta}{\lambda}\Big)^2+2\frac{\tilde{M}_\tau L_\theta}{\lambda\tilde{\mu}_0}-4\frac{\tilde{M}_\tau L_\theta}{\lambda\tilde{\mu}_0}+4\frac{\tilde{M}_\theta L_\tau}{\lambda\tilde{\mu}_0}\\&=\bigg(\Big(\frac{\tilde{M}_\tau}{\tilde{\mu}_0}-\frac{L_\theta}{\lambda}\Big)^2+4\frac{\tilde{M}_\theta L_\tau}{\lambda\tilde{\mu}_0}\bigg)_{Z=Z_0,Z_1}.\end{aligned}\tag{2.2}$$

Note that $\tilde{M}_\theta>0$ and $L_\tau>0$ from conditions (A) and (C). We thus have $\Delta|_{Z=Z_0,Z_1}>0$ and the roots of (2.1) are real. By conditions (A) and (D), the constant terms of (2.1) are positive at Z_0, and negative at Z_1. Then we can conclude that Z_0 is a node and Z_1 is a saddle point. Moreover, since $\tilde{M}_\tau>0$ and $L_\theta>0$, Z_0 is an unstable node.

Thus, according to the property of saddle point, there are two integral curves of the system (1.9) which approach the saddle point Z_1 as $x \to +\infty$, and two which approach it as $x \to -\infty$, these pairs correspond to the negative and positive roots, respectively, of the characteristic Equation (2.1). The two members of each pair have the same slope at Z_1, but approach it from opposite directions. The slopes are given by

$$-\frac{L_\tau}{L_\theta - \kappa\lambda} = -\frac{\tilde{M}_\tau - \kappa\tilde{\mu}_0}{\tilde{M}_\theta}.$$

Especially, when $\kappa < 0$ at Z_1, it is negative.

Furthermore, we notice from (1.9) that, in the region R, the slopes of all the integral curves are

$$\frac{d\theta}{d\tau} = \frac{\tilde{\mu}_0}{\lambda}\frac{L(\tau,\theta)}{\tilde{M}(\tau,\theta)}. \tag{2.3}$$

Since $L(\tau,\theta)/\tilde{M}(\tau,\theta) < 0$ in the region R, all the integral curves have negative slopes. Therefore, one can get that one of the solutions tending to Z_1 as $x \to +\infty$ approaches it from the region R, in which the ratio is negative. We denote this solution by $S(x)$. Now, we can prove Theorem 1.1.

Proof. **(Proof of Theorem 1.1.)** We prove that $S(x)$ is a shock layer. Consider those integral curves of (1.9) passing through the points of $\tilde{M}^*$ and L^* in the Z-plane (Figure 1.1). On $\tilde{M}^*$, $\tilde{M}(\tau,\theta) = 0$, $L(\tau,\theta) > 0$, thus by (2.3), we find that all integral curves have vertical tangent vectors directing outwards from R for increasing x. On L^*, a similar consideration shows that all integral curves have zero slope and are directed outwards from R for increasing x. Thus, one can get that, for decreasing x, all integral curves of (1.9) which pass through L^* and $\tilde{M}^*$ are directed into R for decreasing x. Since all the integral curves have negative slopes and these integral curves are traversed for decreasing x in the direction of increasing τ and decreasing θ in the region R, thus $S(x)$ can not intersect with $\tilde{M}^*$ and L^* between Z_0 and Z_1. In addition, the system (1.9) has no other singular points and $S(x)$ is monotonic in the region R. Therefore, integral curves can not terminate in R and will go to the unstable node Z_0 as $x \to -\infty$. This proves the existence of the shock layer $S(x) = (\tau(x), \theta(x))$.

Next, we prove the uniqueness of the corresponding shock layer curve S. Assume $\widehat{S}$ is other shock layer curve which joins Z_0 and Z_1, and enters Z_1 as $x \to +\infty$, Z_0 as $x \to -\infty$. Then the arcs S and $\widehat{S}$ form a closed curve bounding a simply connected region F in the Z-plane. Since Z_1 is a saddle point, one of the two integral curves of (1.9), which approach Z_1 as $x \to -\infty$, enters this region. This integral curve can neither terminate in F, nor can it approach asymptotically a limit cycle in F. And Z_0 is an unstable node, the integral curves of (1.9) tend to Z_0 only when $x \to -\infty$. Thus, the only possibility remaining is that the curve intersects S or $\widehat{S}$. However, this contradicts the uniqueness of the integral curves of (1.9), thus $\widehat{S}$ can not be a shock layer curve. This completes the proof of the uniqueness for the shock layer. □

3. Limit behavior of the shock layer

In this section, we investigate the limit behavior of the shock layer in the Z-plane and prove Theorem 1.2. For given $\lambda(Z)$, $\tilde{\mu}_0(Z)$, we assume that these coefficients depend on parameters but are independent of τ and θ, in such a way that λ and $\tilde{\mu}_0$ can independently be made arbitrarily small in R. We also assume that the shock layers can be so parameterized that the appropriate limits exist under this parametrization.

In the following, we designate the shock layer by $S(x;\lambda,\tilde{\mu}_0)=(\tau(x;\lambda,\tilde{\mu}_0),\theta(x;\lambda,\tilde{\mu}_0))$ and the associated curve by $S(\lambda,\tilde{\mu}_0)$ and study three limits of the shock layer: (1) $\lambda,\tilde{\mu}_0\to 0$; (2) $\lambda\to 0$, then $\tilde{\mu}_0\to 0$; (3) $\tilde{\mu}_0\to 0$, then $\lambda\to 0$.

3.1. Double limit of the shock layer. We first consider the double limit of the shock layer as $\lambda,\tilde{\mu}_0\to 0$.

LEMMA 3.1. *Assume Z_0 and Z_1 are initial and final states of the shock wave,*

$$Z=Z_0,-\infty<x<\xi;\;\; Z=Z_1,\xi<x<+\infty.$$

Then the corresponding family of shock layers which is parameterized suitably converges to the shock wave as $\lambda,\tilde{\mu}_0\to 0$ independently, that is,

$$\lim_{\substack{\lambda\to 0\\ \tilde{\mu}_0\to 0}} S(x;\lambda,\tilde{\mu}_0)=\begin{cases} Z_0, & -\infty<x<\xi,\\ Z_1, & \xi<x<+\infty,\end{cases} \tag{3.1}$$

and the convergence is uniform in every closed interval not containing $x=\xi$.

Proof. For any $\epsilon>0$, let $R(\epsilon)$ be the subregion of R outside the circles of radius ϵ about Z_0, Z_1. In the region R, using (1.9), we can take the difference

$$\frac{d(\theta-\tau)}{dx}=\frac{1}{\lambda}L(\tau,\theta)-\frac{1}{\tilde{\mu}_0}\tilde{M}(\tau,\theta)=\frac{L(\tau,\theta)}{\lambda}+\frac{|\tilde{M}(\tau,\theta)|}{\tilde{\mu}_0}.$$

Since $L(\tau,\theta)>0$, $|\tilde{M}(\tau,\theta)|>0$ in $R(\epsilon)$, there exists $C(\epsilon)>0$ such that

$$L(\tau,\theta)+|\tilde{M}(\tau,\theta)|\geqslant C(\epsilon)>0,\quad (\tau,\theta)\in R(\epsilon). \tag{3.2}$$

Thus, for any integral curve in region $R(\epsilon)$, we have

$$\frac{d(\theta-\tau)}{dx}>\frac{C(\epsilon)}{\eta}>0, \tag{3.3}$$

where $\eta=\mathrm{Max}(\lambda,\tilde{\mu}_0)$.

For the shock layer $S(x;\lambda,\tilde{\mu}_0)$, let (τ_M,θ_M) designate the value that $S(x;\lambda,\tilde{\mu}_0)$ intersects the ϵ circle about Z_1, and (τ_m,θ_m) the value of $S(x;\lambda,\tilde{\mu}_0)$ where it intersects the ϵ circle about Z_0, with $S(x_1;\lambda,\tilde{\mu}_0)=(\tau_M,\theta_M)$ and $S(x_0;\lambda,\tilde{\mu}_0)=(\tau_m,\theta_m)$. Thus, for $x_0\leqslant x\leqslant x_1$, we have

$$\begin{aligned}\tau_0>\tau_m\geqslant\tau(x;\lambda,\tilde{\mu}_0)\geqslant\tau_M>\tau_1,\\ \theta_0<\theta_m\leqslant\theta(x;\lambda,\tilde{\mu}_0)\leqslant\theta_M<\theta_1.\end{aligned}$$

Then, for the shock layer $S(x;\lambda,\tilde{\mu}_0)$, one can get from (3.3) that

$$x_1-x_0\leqslant\frac{\eta}{C(\epsilon)}\big(\theta_M-\tau_M-(\theta_m-\tau_m)\big)<\frac{\eta}{C(\epsilon)}\big(\theta_1-\theta_0+\tau_0-\tau_1\big). \tag{3.4}$$

Therefore, given ϵ, δ and $S(x;\lambda,\tilde{\mu}_0)$ for which

$$\eta=\mathrm{Max}(\lambda,\tilde{\mu}_0)\leqslant\frac{\delta C(\epsilon)}{\theta_1-\theta_0+\tau_0-\tau_1},$$

we can obtain that the simultaneous inequalities

$$\begin{cases}|\tau(x;\lambda,\tilde{\mu}_0)-\tau_{0,1}|>\epsilon,\\ |\theta(x;\lambda,\tilde{\mu}_0)-\theta_{0,1}|>\epsilon,\end{cases} \tag{3.5}$$

hold if and only if the values of $S(x;\lambda,\tilde{\mu}_0)$ lie in $R(\epsilon)$, and thus in the interval of length less that $x_1-x_0<\delta$. This indicates that (3.1) holds as $\lambda,\tilde{\mu}_0\to 0$. In addition, it is noticed that the solutions $S(x;\lambda,\tilde{\mu}_0)$ can be parameterized such that the point $x=\xi$ is in the interval (x_0,x_1), then the convergence of the $S(x;\lambda,\tilde{\mu}_0)$ to the shock wave is uniform outside of every open interval containing $x=\xi$. □

3.2. Iterated limit of the shock layer. In this subsection, we study the iterated limit of the shock layer for Case (2) $\lambda\to 0$, then $\tilde{\mu}_0\to 0$; and Case (3) $\tilde{\mu}_0\to 0$, then $\lambda\to 0$.

3.2.1. Limit of the shock layer as $\lambda\to 0$. Fixing μ_0 in (1.1), we analyze the single limit of the shock layer as $\lambda\to 0$.

LEMMA 3.2. *Let G be any open neighborhood of the closed arc L^*. Then for $\lambda/\tilde{\mu}_0$ small enough, all shock layer curves $S(\lambda,\tilde{\mu}_0)$ lie entirely in G.*

Proof. As shown in Figure 3.1, assume that $\bar{L}$ is an arc of bounded negative slope with endpoints on $\tilde{M}^*$, and lies so close to L^*. Then we can obtain a subregion G of R contained between L^* and $\bar{L}$. Noting the monotonicity of L^* between Z_0 and Z_1, one can check that such a curve $\bar{L}$ exists. Let D be the subregion of R bounded by $\bar{L}$ and $\tilde{M}^*$, which contains the complement of G in R.

On $\bar{L}$, $L(\tau,\theta)\geqslant k_1>0$, $|\tilde{M}(\tau,\theta)|\leqslant k_2$ and $|\text{slope } \bar{L}|\leqslant N$, where k_1,k_2, and N are suitable positive constants. Without loss of generality, let $\tilde{\mu}_0/\lambda>(k_2/k_1)N$. Thus, on $\bar{L}$, the slopes of the corresponding integral curves of (1.9) satisfy

$$-\frac{d\theta}{d\tau}=-\frac{\tilde{\mu}_0}{\lambda}\frac{L(\tau,\theta)}{\tilde{M}(\tau,\theta)}\geqslant\frac{\tilde{\mu}_0 k_1}{\lambda k_2}>N,$$

from which we can conclude that these integral curves must be directed into D for increasing x. Therefore, under the condition $\tilde{\mu}_0/\lambda>(k_2/k_1)N$, any integral curve of (1.9) which contains a point of D can not intersect $\bar{L}$ beyond this point for increasing x, and hence can not pass through Z_1. Consequently, if $\lambda/\tilde{\mu}_0<k_1/(k_2N)$, $S(\lambda,\mu)$ must lie entirely in the region G. □

For fixed μ_0 in (1.1), recalling that any solution of the reduced system (1.12), for $\tau\in[\tau_1,\tau_0]$ and for all $x\in(-\infty,+\infty)$, is a thermally non-conducting shock layer, we have,

LEMMA 3.3. *Let $\bar{S}(x;\tilde{\bar{\mu}}_0)$ be a thermally non-conducting shock layer with viscosity $\bar{\mu}=\bar{\mu}_0|u_x|^{q-2}$, $(q>1,\ q\neq 2)$ and $\tilde{\bar{\mu}}_0:=(\bar{\mu}_0)^{\frac{1}{q-1}}$. Then, the family of shock layers $S(x;\lambda,\tilde{\bar{\mu}}_0)$, if suitably parameterized, approaches $\bar{S}(x;\tilde{\bar{\mu}}_0)$ as $\lambda\to 0$, that is,*

$$\lim_{\lambda\to 0}S(x;\lambda,\tilde{\bar{\mu}}_0)=\bar{S}(x;\tilde{\bar{\mu}}_0),$$

the convergence being uniform in x in every finite interval.

Proof. Let $\zeta=\bar{S}(\bar{x};\tilde{\bar{\mu}}_0)=(\bar{\tau}(\bar{x},\tilde{\bar{\mu}}_0),\bar{\theta}(\bar{x},\tilde{\bar{\mu}}_0))$ be a point on L^*, and take x_λ such that $x_\lambda\to\bar{x}$ as $\lambda\to 0$. By virtue of Lemma 3.2, we can parameterize shock layers $S(x;\lambda,\tilde{\bar{\mu}}_0)=(\tau(x;\lambda,\tilde{\bar{\mu}}_0),\theta(x;\lambda,\tilde{\bar{\mu}}_0))$ such that $\lim_{\lambda\to 0}S(x_\lambda;\lambda,\tilde{\bar{\mu}}_0)=\zeta$. Also from Lemma 3.2, we have for every shock layer,

$$\theta=l(\tau)+\epsilon(\tau,\lambda), \tag{3.6}$$

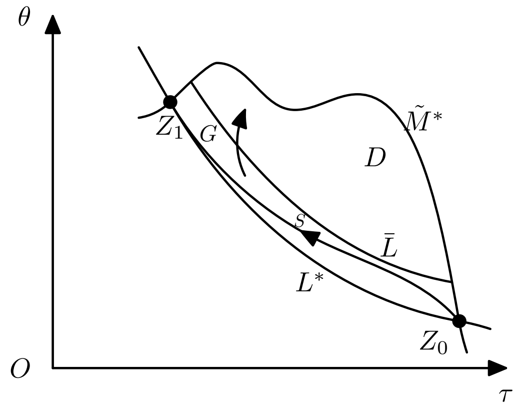

FIG. 3.1.

where $\lim_{\lambda\to 0}\epsilon(\tau,\lambda)=0$ uniformly in $\tau_1\leqslant\tau\leqslant\tau_0$, and therefore in $-\infty<x<+\infty$. Thus,

$$\tilde{\bar{\mu}}_0\frac{d\tau(x;\lambda,\tilde{\bar{\mu}}_0)}{dx}=\tilde{M}\big(\tau,l(\tau)+\epsilon(\tau,\lambda)\big). \tag{3.7}$$

Because of the choice for $\tau(x;\lambda,\tilde{\bar{\mu}}_0)$, it follows from (3.7) that $\lim_{\lambda\to 0}\tau(x;\lambda,\tilde{\bar{\mu}}_0)$ exists uniformly in every finite interval of x and satisfies

$$\tilde{\bar{\mu}}_0\frac{d\tau}{dx}=\tilde{M}(\tau,l(\tau)),$$

which means that $\lim_{\lambda\to 0}\tau(x;\lambda,\tilde{\bar{\mu}}_0)$ satisfies the reduced system (1.12). Since there is a unique solution of (1.12) passing through the point ζ at $x=\bar{x}$, we have $\lim_{\lambda\to 0}\tau(x;\lambda,\tilde{\bar{\mu}}_0)=\bar{\tau}(x;\tilde{\bar{\mu}}_0)$, and from (3.6), $\lim_{\lambda\to 0}\theta(x;\lambda,\tilde{\bar{\mu}}_0)=\bar{\theta}(x;\tilde{\bar{\mu}}_0)$. ∎

REMARK 3.1. The above analysis shows that if the shock layers are so parameterized that $\lim_{\lambda\to 0}S(x;\lambda,\tilde{\bar{\mu}}_0)$ exists for a single value of x, then it exists for all $x\in(-\infty,+\infty)$ and defines a thermally non-conducting shock layer.

3.2.2. Limit of the shock layer as $\tilde{\mu}_0\to 0$. In this subsection, fixing λ in (1.1), we turn to the single limit of the shock layer as $\tilde{\mu}_0\to 0$.

Consider the case when $\tilde{\mu}_0/\lambda$ is small. At this moment, one should pay attention to the possibility that the arc $\tilde{M}^*$ is not monotonic between Z_0 and Z_1. If the arc $\tilde{M}^*$ is monotonic, a similar argument as Lemma 3.2 applies. However, if the arc $\tilde{M}^*$ is not monotonic, we consider the arc $\bar{\bar{M}}^*$ defined by the monotonic function

$$\theta=\bar{\bar{m}}(\tau)=\operatorname*{Min}_{\tau_1\leqslant t\leqslant\tau}m(t),\qquad \tau_1\leqslant\tau\leqslant\tau_0. \tag{3.8}$$

This arc joining Z_0 and Z_1 encloses between it and L^* a subregion of R, one can check that all shock layer curves lie in this subregion (Figure 3.2). In fact, if $\theta=h(\tau)$ is the equation of any shock layer, we can get

$$h(\tau)\leqslant h(t)\leqslant m(t)=\bar{\bar{m}}(\tau),\quad \tau\in[\tau_1,\tau_0],\quad t\in[\tau_1,\tau],$$

from which we have $h(\tau)<\bar{\bar{m}}(\tau)$, $\tau\neq\tau_0,\tau_1$. Therefore, all shock layer curves must lie below $\bar{\bar{M}}^*$ in R.

Replacing $\tilde{M}^*$ by $\bar{\bar{M}}^*$, similar to the case when $\tilde{M}^*$ is monotonic, we can get that for sufficiently small $\tilde{\mu}_0/\lambda$, all shock layer curves $S(\lambda,\tilde{\mu}_0)$ lie entirely in any preassigned neighborhood of $\bar{\bar{M}}^*$. Thus we directly obtain the following,

LEMMA 3.4. *Let G be any open neighborhood of the closed arc $\bar{\bar{M}}^*$ which is defined in (3.8). Then for $\tilde{\mu}_0/\lambda$ small enough, all shock layer curves $S(\lambda,\mu)$ lie entirely in G.*

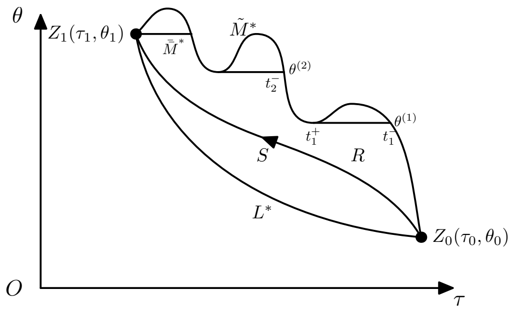

FIG. 3.2.

For λ fixed and $\tilde{\mu}_0\to 0$, similar to the case of Newtonian fluids shown in [10], the limit solution of the reduced system (1.13) is no longer continuous in general. Then, we should enlarge the notion of solution of the reduced system (1.13) by admitting discontinuous solutions, that is, functions $\bar{\tau}(x)$, $\bar{\theta}(x)$ satisfying (1.13) for $\bar{\tau}\in[\tau_1,\tau_0]$, except at points $x=x_i$, $i=1,\cdots,n$, where $\bar{\tau}(x_i-0)=t_i^-,\bar{\tau}(x_i+0)=t_i^+$; these solutions are uniquely determined in $-\infty<x<+\infty$ up to an x-translation. Thus, any solution of (1.13), for $\tau\in[\tau_1,\tau_0]$ and for all $x\in(-\infty,+\infty)$, whether continuous, or discontinuous in the above sense, is a non-viscous shock layer.

To simplify considerations, we assume that n is a finite number, which means that $\tilde{M}^*$ has only a finite number of minima, so that the arc $\bar{\bar{M}}^*$ contains at most a finite number of intervals on which θ is constant. We also note that Z_0 can not lie in such an interval, while Z_1 may or may not. As pictured in Figure 3.2, let the function $\theta=\bar{\bar{m}}(\tau)$, $\tau_1\leqslant\tau\leqslant\tau_0$, be constant on the intervals $[t_i^+,t_i^-]$, $(i=1,\cdots,n)$, which are ordered so that $t_i^->t_i^+>t_{i+1}^-$. Now, we can get the following result.

LEMMA 3.5. *Let $\bar{S}(x;\bar{\lambda})$ be a non-viscous shock layer with heat conductivity $\bar{\lambda}$. Then, the family of shock layers $S(x;\bar{\lambda},\tilde{\mu}_0)$, if suitably parameterized, approaches $\bar{S}(x;\bar{\lambda})$ as $\tilde{\mu}_0\to 0$, that is,*

$$\lim_{\tilde{\mu}_0\to 0} S(x;\bar{\lambda},\tilde{\mu}_0)=\bar{S}(x;\bar{\lambda}),$$

the convergence being uniform in x in any closed interval not containing a point of discontinuity of $\bar{S}(x;\bar{\lambda})$.

Proof. If $\tilde{M}^*$ is strictly monotonic, the proof is similar to that of Lemma 3.3. However, if $\tilde{M}^*$ is not monotonic, denoting $\bar{S}(x;\bar{\lambda})=(\bar{\tau}(x;\bar{\lambda}),\bar{\theta}(x;\bar{\lambda}))$ and assuming that $\bar{x}_1$ is the first point of discontinuity of $\bar{\tau}(x;\bar{\lambda})$, we proceed as follows.

Step 1. We first prove the convergence for $x<\bar{x}_1$. Let $\zeta=(\bar{\tau}(\bar{x};\bar{\lambda}),\bar{\theta}(\bar{x};\bar{\lambda}))$, $\bar{x}<\bar{x}_1$, be a point on $\tilde{M}^*$. For any set x_μ satisfying $x_\mu\to\bar{x}$ as $\tilde{\mu}_0\to 0$, parameterize the shock layers $S(x;\bar{\lambda},\tilde{\mu}_0)=(\tau(x;\bar{\lambda},\tilde{\mu}_0),\theta(x;\bar{\lambda},\tilde{\mu}_0))$ such that $\lim\limits_{\tilde{\mu}_0\to 0} S(x_\mu;\bar{\lambda},\tilde{\mu}_0)=\zeta$. Then, applying Lemma 3.4 yields

$$\tau=m^{-1}(\theta)+\epsilon(\theta,\tilde{\mu}_0), \tag{3.9}$$

where $m^{-1}(\theta)$ is the inverse of $m(\tau)$ for $t_1^- \leqslant \tau \leqslant \tau_0$, $\theta_0 \leqslant \theta \leqslant \theta^{(1)} = m(t_1^-)$, and $\lim_{\tilde{\mu}_0 \to 0} \epsilon(\theta, \tilde{\mu}_0) = 0$ uniformly in every closed interval of $\theta_0 \leqslant \theta < \theta^{(1)}$. Thus, we can obtain

$$\bar{\lambda} \frac{d\theta(x;\bar{\lambda},\tilde{\mu}_0)}{dx} = L\big(m^{-1}(\theta) + \epsilon(\theta,\tilde{\mu}_0), \theta\big). \tag{3.10}$$

Since $\theta(x_\mu;\bar{\lambda},\tilde{\mu}_0) \to \bar{\theta}(\bar{x};\bar{\lambda})$ and $x_\mu \to \bar{x}$ as $\tilde{\mu}_0 \to 0$, and $\bar{\tau} = m^{-1}(\bar{\theta})$ for $\theta_0 \leqslant \theta < \theta^{(1)}$, we can deduce that $\lim_{\tilde{\mu}_0 \to 0} \theta(x;\bar{\lambda},\tilde{\mu}_0) = \bar{\theta}(x;\bar{\lambda}), -\infty < x < \bar{x}_1$, and from (3.9), $\lim_{\tilde{\mu}_0 \to 0} \tau(x;\bar{\lambda},\tilde{\mu}_0) = \bar{\tau}(x;\bar{\lambda})$. Furthermore, the convergence in both cases is uniform in every closed interval to the left of $x = \bar{x}_1$. By the preceding, there exists value $x_\mu^{(1)} < \bar{x}_1$ such that $x_\mu^{(1)} \to \bar{x}_1, \tau(x_\mu^{(1)};\bar{\lambda},\tilde{\mu}_0) \to t_1^-$, and $\theta(x_\mu^{(1)};\bar{\lambda},\tilde{\mu}_0) \to \theta^{(1)}$, as $\tilde{\mu}_0 \to 0$.

Step 2. Then we consider the convergence for $x > \bar{x}_1$. Assume $\tau_1 \neq t_1^+$. In an analogous way as that in Step 1 and with the same parametrization for the $S(x;\bar{\lambda},\tilde{\mu}_0)$, let $\xi_\mu, (\xi_\mu > \bar{x}_1)$, be values satisfying $\theta(\xi_\mu;\bar{\lambda},\tilde{\mu}_0) = \theta^{(1)}$. Then by Lemma 3.4, we have $\tau(\xi_\mu;\bar{\lambda},\tilde{\mu}_0) \to t_1^+$ as $\tilde{\mu}_0 \to 0$.

Next, we prove that $|\xi_\mu - x_\mu^{(1)}| \to 0$ as $\tilde{\mu}_0 \to 0$. In the neighborhood of the segment $\theta = \theta^{(1)}$ of $\bar{\bar{M}}^*$, there exists a constant d such that $L(\tau,\theta) > d > 0$ in such a neighborhood. Then, by Lemma 3.4, if $\tilde{\mu}_0$ is small enough, the arcs of $S(x;\bar{\lambda},\tilde{\mu}_0)$ for $x_\mu^{(1)} \leqslant x \leqslant \xi_\mu$ lie in this neighborhood. Thus, integrating Equation $(1.9)_1$ yields

$$|\xi_\mu - x_\mu^{(1)}| \leqslant \frac{\bar{\lambda}}{d}\big(\theta(\xi_\mu;\bar{\lambda},\tilde{\mu}_0) - \theta(x_\mu^{(1)};\bar{\lambda},\tilde{\mu}_0)\big) = \frac{\bar{\lambda}}{d}\big(\theta^{(1)} - \theta(x_\mu^{(1)};\bar{\lambda},\tilde{\mu}_0)\big), \tag{3.11}$$

which tends to zero as $\tilde{\mu}_0 \to 0$. Hence, we conclude that $\xi_\mu \to \bar{x}_1$ as $\tilde{\mu}_0 \to 0$.

Step 3. We turn back to Equations (3.9) and (3.10), where now $m^{-1}(\theta)$ is the inverse of $\theta = m(\tau)$ for $t_2^- \leqslant \tau \leqslant t_1^+$, $(\theta^{(1)} \leqslant \theta \leqslant \theta^{(2)} = m(t_2^-))$, and $\lim_{\tilde{\mu}_0 \to 0} \epsilon(\theta,\tilde{\mu}_0) = 0$ uniformly in every closed interval of $\theta^{(1)} \leqslant \theta < \theta^{(2)}$. Since $\theta(\xi_\mu;\bar{\lambda},\tilde{\mu}_0) \to \theta^{(1)} = \bar{\theta}(\bar{x}_1;\bar{\lambda})$ as $\xi_\mu \to \bar{x}_1$, and $\bar{\tau} = m^{-1}(\bar{\theta})$ for $\theta^{(1)} \leqslant \bar{\theta} \leqslant \theta^{(2)}$, and the solution of

$$\bar{\lambda}\frac{d\theta}{dx} = L(m^{-1}(\theta),\theta)$$

is unique for $\theta(\bar{x}_1;\bar{\lambda}) = \theta^{(1)}$, $(\theta^{(1)} \leqslant \theta \leqslant \theta^{(2)})$, we have $\lim_{\tilde{\mu}_0 \to 0} \theta(x;\bar{\lambda},\tilde{\mu}_0) = \bar{\theta}(x;\bar{\lambda})$ and $\lim_{\tilde{\mu}_0 \to 0} \tau(x;\bar{\lambda},\tilde{\mu}_0) = \bar{\tau}(x;\bar{\lambda})$ for $\bar{x}_1 \leqslant x < \bar{x}_2$, which are uniform convergence in the closed interval of $(\bar{x}_1,\bar{x}_2)$.

We repeat the above procedure until the arc of $\bar{\bar{M}}^*$ including Z_1 is reached. If Z_1 is not contained in one of the intervals of $\bar{\bar{M}}^*$, by the analysis as above, we can obtain the convergence of $S(x;\bar{\lambda},\tilde{\mu}_0)$ to $\bar{S}(x;\bar{\lambda})$ in every closed subintervals of the half line $\bar{x}_n < x < +\infty$. If Z_1 is contained in one of the intervals of $\bar{\bar{M}}^*$, that is, $\theta = \theta_1$, then it indicates that $\tau_1 = t_n^+$ and $Z_1 = (\tau_1,\theta_1) = (\bar{\tau}(x;\bar{\lambda}),\bar{\theta}(x;\bar{\lambda})), \bar{x}_n < x < +\infty$. At this moment, for any small $\delta > 0$, let $R(\delta)$ be the subregion of R outside the circle of radius δ about Z_1. Assume that $L(\tau,\theta) \geqslant k(\delta) > 0$ in a neighborhood of $\theta = \theta_1$ in $R(\delta)$. We take $x_\mu^{(n)} < \bar{x}_n$ such that $x_\mu^{(n)} \to \bar{x}_n$, $\tau(x_\mu^{(n)};\bar{\lambda},\tilde{\mu}_0) \to t_n^-$, and $\theta(x_\mu^{(n)};\bar{\lambda},\tilde{\mu}_0) \to \theta^{(n)} = \theta_1$, as $\tilde{\mu}_0 \to 0$. Then, for any $\xi_\mu > x_\mu^{(n)}$ such that $S(\xi_\mu;\bar{\lambda},\tilde{\mu}_0) \in R(\delta)$, we can deduce from $(1.9)_1$ that as $\tilde{\mu}_0 \to 0$,

$$|\xi_\mu - x_\mu^{(n)}| \leqslant \frac{\bar{\lambda}}{k(\delta)}\Big(\theta_1 - \theta(x_\mu^{(n)};\bar{\lambda},\tilde{\mu}_0)\Big) \to 0,$$

which means that those $x>\bar{x}_n$ for which $\tau(x;\bar{\lambda},\tilde{\mu}_0)-\tau_1>\delta$ lie in an interval about $\bar{x}_n$ which grows arbitrarily small as $\tilde{\mu}_0\to 0$. Since δ is arbitrary, we can find that as $\tilde{\mu}_0\to 0$, the limits $\tau(x;\bar{\lambda},\tilde{\mu}_0)\to\tau_1$ and $\theta(x;\bar{\lambda},\tilde{\mu}_0)\to\theta_1$ are uniform convergence in any closed half line of $\bar{x}_n<x<\infty$. This completes the proof of Lemma 3.5. □

Remark 3.2. The above analysis indicates that if the shock layers are so parameterized that $\lim_{\tilde{\mu}_0\to 0} S(x;\bar{\lambda},\tilde{\mu}_0)$ exists for a single value of x, then it exists for all x and defines a non-viscous shock layer.

Now, we are in the position to establish the existence of the iterated limits, and then show that the iterated limits are equal to the double limit for the shock layer of (1.1). It suffices to parameterize the shock layer such that both the double limit and the two iterated limits exist simultaneously. To do this, we take a circle of sufficiently small radius ϵ about Z_0. This circle intersects shock layer curve $S(\lambda,\tilde{\mu}_0)$ at exactly one point $\zeta_{\lambda\mu}$, and let ζ_λ,ζ_μ represent the point of intersection with $\tilde{M}^*$ and L^*, respectively. By Lemma 3.2 and Lemma 3.4, one can get that $\lim_{\lambda\to 0}\zeta_{\lambda\mu}=\zeta_\mu$ and $\lim_{\tilde{\mu}_0\to 0}\zeta_{\lambda\mu}=\zeta_\lambda$. For fixed ξ, let

$$x_{\lambda\mu}=\xi-\frac{1}{2}\delta_{\lambda\mu},\quad \bar{x}_\mu=\xi-\frac{1}{2}\delta_\mu,\quad \bar{x}_\lambda=\xi-\frac{1}{2}\delta_\lambda,$$

where

$$\begin{aligned}\delta_{\lambda\mu}&=(\eta/C(\epsilon))(\theta_1-\theta_0+\tau_0-\tau_1),\quad \eta=\mathrm{Max}(\lambda,\tilde{\mu}_0),\\ \delta_\mu&=(\mathrm{Max}\,\tilde{\mu}_0/C(\epsilon))(\theta_1-\theta_0+\tau_0-\tau_1),\\ \delta_\lambda&=(\mathrm{Max}\,\lambda/C(\epsilon))(\theta_1-\theta_0+\tau_0-\tau_1),\end{aligned}$$

$C(\epsilon)$ is defined in inequality (3.2) of Lemma 3.1. Then, the desired parametrization of the shock layer can be obtained by making the assignment $S(x_{\lambda\mu};\lambda,\tilde{\mu}_0)=\zeta_{\lambda\mu}$. Under this parametrization, since ξ is contained in the interval $(x_{\lambda\mu},x_{\lambda\mu}+\delta_{\lambda\mu})$ as $\eta\to 0$, one can check from the proof of Lemma 3.1 that

$$\lim_{\substack{\lambda\to 0\\ \tilde{\mu}_0\to 0}} S(x;\lambda,\tilde{\mu}_0)=\begin{cases} Z_0, & -\infty<x<\xi,\\ Z_1, & \xi<x<+\infty.\end{cases}$$

For fixed $\tilde{\mu}_0$, since

$$\lim_{\lambda\to 0}\delta_{\lambda\mu}=\big(\mathrm{Max}\,\tilde{\mu}_0/C(\epsilon)\big)\big(\theta_1-\theta_0+\tau_0-\tau_1\big)=\delta_\mu,$$

we have $\lim_{\lambda\to 0}x_{\lambda\mu}=\xi-\frac{1}{2}\delta_\mu=\bar{x}_\mu$ and $\lim_{\lambda\to 0}S(x_{\lambda\mu};\lambda,\tilde{\mu}_0)=\zeta_\mu$. Therefore, as in the proof of Lemma 3.3, we obtain $\lim_{\lambda\to 0}S(x;\lambda,\tilde{\mu}_0)=(\bar{\tau}(x;\tilde{\mu}_0),\bar{\theta}(x;\tilde{\mu}_0))$, where $(\bar{\tau}(x;\tilde{\mu}_0),\bar{\theta}(x;\tilde{\mu}_0))$ is the solution of (1.12) which satisfies $(\bar{\tau}(\bar{x}_\mu;\tilde{\mu}_0),\bar{\theta}(\bar{x}_\mu;\tilde{\mu}_0))=\zeta_\mu$. Combining this with the existence of the double limit, we can establish the existence of the iterated limit $\lim_{\tilde{\mu}_0\to 0}(\lim_{\lambda\to 0}S(x;\lambda,\tilde{\mu}_0))$, and its equality with the double limit. We can investigate $\lim_{\lambda\to 0}(\lim_{\tilde{\mu}_0\to 0}S(x;\lambda,\tilde{\mu}_0))$ in a similar consideration. Thus, we can give the proof of Theorem 1.2.

Proof. **(Proof of Theorem 1.2.)** Combining Lemma 3.1, Lemma 3.3 with Lemma 3.5, we can obtain Theorem 1.2. □

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