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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE UNIPOLAR HYDRODYNAMIC MODEL OF SEMICONDUCTORS WITH TIME-DEPENDENT DAMPING IN BOUNDED DOMAIN*

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Abstract. This paper concerns asymptotic behavior of solutions to the initial boundary-value problem for one-dimensional unipolar hydrodynamic model of semiconductors with time-dependent damping $-\frac{\rho u}{(1+t)^\lambda}$ for $\lambda\in(0,1)$. The damping effect is time-gradually-degenerate when $\lambda\in(0,1)$. We prove that the system admits a unique global smooth solution and the solution time-asymptotically converges to the constant steady-state in the sub-exponential form when the doping profile is completely flat. The adopted method of the proof is the elementary energy estimates but with some technical development.

Keywords. Unipolar hydrodynamic model; semiconductor; time-dependent damping; initial boundary-value problem; convergence; steady-state.

AMS subject classifications. 35B40; 35L50; 35L60; 35L65.

1. Introduction

In this paper, we consider the one-dimensional unipolar hydrodynamic model of semiconductors with time-dependent damping, represented by the following Euler-Poisson equations

$$\begin{cases}\rho_t+(\rho u)_x=0,\\ (\rho u)_t+(\rho u^2+p(\rho))_x=\rho\phi_x-\dfrac{\rho u}{(1+t)^\lambda},\\ \phi_{xx}=\rho-\mathcal{C}(x),\end{cases} \tag{1.1}$$

where $(x,t)\in(0,1)\times\mathbb{R}_+$, $\rho(x,t)>0$, $u(x,t)$ and $\phi(x,t)$ represent the electron density, the electron velocity and the electrostatic potential, respectively. The function $p=p(\rho)$ is the pressure density relation and $\mathcal{C}(x)>0$ is the doping profile which stands for the density of impurities in semiconductor device. The term $-\frac{\rho u}{(1+t)^\lambda}$ represents the damping effect with a parameter $\lambda\in(-\infty,+\infty)$, the damping effect is time-gradually-enhancing for $\lambda<0$ and time-gradually-degenerate for $\lambda>0$. Let $j=\rho u$ be the current density, then the system (1.1) becomes

$$\begin{cases}\rho_t+j_x=0,\\ j_t+\left(\dfrac{j^2}{\rho}+p(\rho)\right)_x=\rho\phi_x-\dfrac{j}{(1+t)^\lambda}, \quad (x,t)\in(0,1)\times\mathbb{R}_+,\\ \phi_{xx}=\rho-\mathcal{C}(x).\end{cases} \tag{1.2}$$

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Our target in this paper is to investigate the asymptotic behavior of solutions to the initial boundary-value problem (denoted IBVP for simplicity) for the system (1.2) with $\lambda \in (0,1)$, supplemented with the initial value

$$(\rho,j)(x,0)=(\rho_0,j_0)(x), \quad x\in(0,1), \tag{1.3}$$

and the Dirichlet boundary conditions, the so-called Ohmic contact boundary conditions,

$$\begin{cases} \rho(0,t)=\rho_1, \quad \rho(1,t)=\rho_2, \quad t\geq 0, \\ \phi(0,t)=\phi_0, \quad \phi(1,t)=\phi_1, \quad t\geq 0, \end{cases} \tag{1.4}$$

where $\rho_1,\rho_2>0,\phi_0$ and ϕ_1 are constants. Throughout this paper, we assume that the pressure function p satisfies

$$p(\cdot)\in C^3(0,+\infty) \text{ and } p'(s)>0 \text{ for } s>0. \tag{1.5}$$

The condition $p'(s)>0$ is physical, and the example is the Gamma-Law $p(s)=ks^\gamma$ for $k>0$ and $\gamma\geq 1$.

The hydrodynamic model of semiconductors is usually used to describe the dynamic phenomena of charged particles, such as positively and negatively charged ions in plasma [37] or electrons and holes in semiconductor devices [2,17,25]. These models can be derived from kinetic transport equation and are applied to physics and engineering, we can refer to [16,24,25,27,33,34,36] for details. The theoretical study and numerical computations on the hydrodynamic model of semiconductors have been one of the hot spots in mathematical physics. For the steady-state system, the existence and uniqueness of subsonic solutions in one dimension for isentropic flow was studied in [4] and the three-dimensional irrotational case in [5]. The existence of subsonic solutions in two dimensions was obtained in [26]. For the corresponding investigations on supersonic and transonic solutions, see [1,7,8,19,20,32,35] for details.

The damping in system (1.2) is reduced to the regular case when $\lambda=0$. There are also many results about the asymptotic behavior of the solutions to the unipolar hydrodynamic model of semiconductors with the regular damping, see [10–13,18,29,38]. Among them, Li-Markowich-Mei [18] proved that for the IBVP to the system (1.2) with $\lambda=0$, there exists a global smooth solution and the solution tends exponentially to the steady-state. In [38], Sun-Mei-Zhang discussed the case that the system (1.2) for $\lambda=0$ on the half line with two different boundary effects, and obtained that the solutions of original IBVP converge to their corresponding asymptotic profiles by using energy method. The Cauchy problem to (1.2) with $\lambda=0$ was investigated in [12], where Huang-Mei-Wang-Yu [12] showed that the solutions decay exponentially to the stationary solutions. Regarding the study on the asymptotic behavior of the solutions to the bipolar hydrodynamic model of semiconductors with the regular damping, we refer to [6,14,15,28]. However, the structure of the solutions to the system (1.2) will be more complicated when $\lambda\neq 0$.

What we are interested in is how the time-dependent damping affects the structure of solutions. The one-dimensional compressible Euler equations with time-dependent damping in Lagrangian coordinates can be written as

$$\begin{cases} v_t-u_x=0, \\ u_t+p(v)_x=-\dfrac{\mu}{(1+t)^\lambda}u. \end{cases} \tag{1.6}$$

Pan [30, 31] proved that the solutions of the system (1.6) globally exist for small initial perturbation when $0 \le \lambda < 1$, $\mu > 0$ or $\lambda = 1$, $\mu > 2$, and will blow up in finite time for some large data when $\lambda > 1$, $\mu \ge 0$ or $\lambda = 1$, $0 \le \mu \le 2$. When $\lambda \in (0,1)$, Cui-Yin-Zhang-Zhu [3] showed that the system (1.6) possess a unique global solution and the solution converges to the corresponding diffusion wave at the algebraic rates. Clearly, the time-gradually-degenerate damping plays the crucial role which causes the system (1.6) to behave like a degenerate parabolic system with diffusion phenomena when $\lambda \in (0,1)$. For the critical case, i.e. $\lambda = 1$ with $\mu > 2$, Geng-Lin-Mei [9] observed that the damping effect and the hyperbolicity both play key roles and cannot be ignored. They artfully constructed the asymptotic profile which is the solution of the linear wave equation with damping and proved that the solution converges to the asymptotic profile at the algebraic rates related to μ. For the case of $\lambda > 1$, the damping effect is too weak and can be ignored, which causes the system (1.6) to behave like a hyperbolic system, such that the shock waves will form.

Compared with the Euler system with time-dependent damping, the studies on the asymptotic behavior of the solutions to the hydrodynamic model of semiconductors with time-dependent damping are quite limited. In [21], Li-Li-Mei-Zhang first investigated the Cauchy problem to the one-dimensional bipolar hydrodynamic model of semiconductors with time-dependent damping for $\lambda \in (-1,1)$. Note that the doping profile is restricted to be $\mathcal{C}(x) = 0$ in [21]. By using the time-weighted energy method, they proved that the bipolar system possess a unique global solution and the solution converges to the corresponding diffusion wave at the algebraic rates. Under the assumption of $\mathcal{C}(x) = 0$, the asymptotic profiles of the bipolar system are reduced to the diffusion waves, which causes that the a prior estimates can be smoothly established. For the case of $\lambda = 1$ with $\mu > 2$ (critical case), Luan-Mei-Rubino-Zhu [23] showed that the solutions to the Cauchy problem for the bipolar hydrodynamic model of semiconductors with time-dependent damping converge to the constant steady-states. Particularly, Sun-Mei-Zhang [39] discussed the Cauchy problem to the one-dimensional unipolar hydrodynamic model of semiconductors with time-dependent damping for $\lambda \in (-1,0) \cup (0,1)$. They proved that the system admits a unique global solution which converges to the steady-state at the sub-exponential rates when $\lambda \in (-1,0)$, and converges to the constant steady-state at the sub-exponential rates when $\lambda \in (0,1)$.

The main issue in this paper is to study asymptotic behavior of solutions for IBVP to the one-dimensional unipolar hydrodynamic model of semiconductors with time-dependent damping for $\lambda \in (0,1)$. Different from the study on the IBVP to the unipolar hydrodynamic model of semiconductors with regular damping [18], since the non-trivial doping profile will cause some essential difficulty in establishing the a prior estimates for $\lambda \in (0,1)$, we have to assume that the doping profile is completely flat, i.e. $\mathcal{C}(x) \equiv \text{constant} > 0$. For technical reason, we explain it in the Appendix to the paper. And different from the Cauchy problem to the unipolar hydrodynamic model of semiconductors with time-dependent damping [39], because of the boundary effect, we cannot directly get the decay rates of the solutions for all $t \ge 0$. In order to overcome this difficulty, we first obtain that the solutions of the system globally exist with algebraic convergence rates by using the time-weighted energy method, and then enhance the algebraic rates to the sub-exponential rates when t is large enough. Besides, the establishment of the a prior estimates will be more complicated compared with the Cauchy problem due to the boundary effect. The study on the asymptotic behavior of the solutions to IBVP for the unipolar hydrodynamic model of semiconductors with time-dependent damping for $\lambda \in (-1,0)$ or $\lambda = 1$ will be our targets in future.

We state our main result as follow:

Because we consider the case that $\lambda\in(0,1)$, we have to restrict $\mathcal{C}(x)\equiv\text{constant}:=\hat{C}$ for technical reason, and the expected steady-state is reduced to the constant steady-state $(\hat{C},0,\phi_0)$. We prove that the system (1.2)–(1.4) with $\lambda\in(0,1)$ admits a unique global solution (ρ,j,ϕ) and for all $t\geq 0$, the solution satisfies

$$\|j(t)\|+(1+t)^{(\theta+\lambda)/2}\big(\|\rho(t)-\hat{C}\|_2+\|j_x(t)\|_1+\|\phi(t)-\phi_0\|_2\big)\leq C_\theta\Psi_0.$$

Furthermore, the solution converges to the constant steady-state $(\hat{C},0,\phi_0)$ in the following form as $t\to+\infty$,

$$\begin{cases}\|\rho(t)-\hat{C}\|_2+\|j_x(t)\|_1+\|\phi(t)-\phi_0\|_2\leq C_\theta\Psi_0(1+t)^{-(\theta+\lambda)/2}\mathrm{e}^{-\sigma(1+t)^{1-\lambda}},\\ \|j(t)\|\leq C_\theta\Psi_0\mathrm{e}^{-\sigma(1+t)^{1-\lambda}},\end{cases}$$

with some positive constant σ, provided that the initial perturbation Ψ_0 is sufficiently small. Here $\theta\in[\lambda,+\infty)$ is related to the initial perturbation Ψ_0 and could be large enough as the initial perturbation reduces to zero.

The rest of this paper is organized as follows. In Section 2, we formulate the perturbation system and state the main results of this paper. In Section 3, we derive the a prior estimates of the solution to the perturbation system. Section 4 is devoted to the proof of the sub-exponential decay rates of the solution to the perturbation system. We will show the details of why we have to restrict the doping profile $\mathcal{C}(x)$ as a constant in the Appendix.

Notations: Throughout this paper, $C>0$ always denotes a generic constant which may be different in different lines. $L^2((0,1))$ is the square-integrable real-valued functional space defined on $[0,1]$ with the norm $\|f\|:=\|f\|_{L^2((0,1))}$, and $H^m((0,1))\,(m\geq 0)$ is the usual Sobolev space with the norm $\|f\|_m:=\sum_{i=0}^{m}\|\partial_x^i f\|$. For simplicity, we denote $\|(f,g)\|^2:=\|f\|^2+\|g\|^2$.

2. Main results and formulation of the perturbation system

In this section, we will firstly find the asymptotic profiles of the solutions to IBVP (1.2)–(1.4), and then formulate the perturbation system, and state the main results at the end of this section.

Since the doping profile $\mathcal{C}(x)\neq 0$, the expected asymptotic profiles of the solutions to IBVP (1.2)–(1.4) will be the steady-states $(\bar{\rho},\bar{j},\bar{\phi})$ and $(\bar{\rho},\bar{j},\bar{\phi})$ satisfying the following stationary system (see [12, 18, 22] for details)

$$\begin{cases}\bar{j}=0,\\ p(\bar{\rho})_x=\bar{\rho}\bar{\phi}_x, \quad x\in(0,1),\\ \bar{\phi}_{xx}=\bar{\rho}-\mathcal{C}(x),\end{cases}\tag{2.1}$$

with the Dirichlet boundary conditions

$$\begin{cases}\bar{\rho}(0)=\rho_1, \quad \bar{\rho}(1)=\rho_2,\\ \bar{\phi}(0)=\phi_0, \quad \bar{\phi}(1)=\phi_1.\end{cases}\tag{2.2}$$

Because we consider the case that λ belongs to $(0,1)$, we have to restrict $\mathcal{C}(x)=\hat{C}$ for a constant $\hat{C}>0$, and the expected steady-state is reduced to the constant steady-state $(\hat{C},0,\phi_0)$. We will explain why we have to add the restriction $\mathcal{C}(x)=\hat{C}$ in the Appendix to this paper.

Let $\mathcal{C}(x)=\hat{C}$, then the system (1.2) becomes

$$\begin{cases}\rho_t+j_x=0,\\ j_t+\left(\dfrac{j^2}{\rho}+p(\rho)\right)_x=\rho\phi_x-\dfrac{j}{(1+t)^\lambda}, \quad (x,t)\in(0,1)\times\mathbb{R}_+,\\ \phi_{xx}=\rho-\hat{C},\end{cases} \tag{2.3}$$

with the initial value and boundary value conditions

$$\begin{cases}(\rho,j)(x,0)=(\rho_0,j_0)(x), \quad x\in(0,1),\\ \rho(0,t)=\rho(1,t)=\hat{C}, \quad \phi(0,t)=\phi(1,t)=\phi_0, \quad t\ge 0.\end{cases} \tag{2.4}$$

Set

$$\psi=\rho-\hat{C}, \quad \eta=j-0=j, \quad e=\phi-\phi_0,$$

then from the system (2.3)–(2.4), we know that (ψ,η,e) satisfy the following perturbation system

$$\begin{cases}\psi_t+\eta_x=0,\\ \eta_t+\left(\dfrac{\eta^2}{\psi+\hat{C}}\right)_x+p(\psi+\hat{C})_x=(\psi+\hat{C})e_x-\dfrac{\eta}{(1+t)^\lambda}, \quad (x,t)\in(0,1)\times\mathbb{R}_+,\\ e_{xx}=\psi,\end{cases} \tag{2.5}$$

with the initial value and boundary value conditions

$$\begin{cases}\psi_0(x):=\psi(x,0)=\rho_0(x)-\hat{C}, \quad \eta_0(x):=\eta(x,0)=j_0(x), \quad x\in(0,1),\\ \psi(0,t)=\psi(1,t)=0, \quad e(0,t)=e(1,t)=0, \quad t\ge 0.\end{cases} \tag{2.6}$$

Differentiating $(2.5)_2$ with respect to x and using $(2.5)_1$ and $(2.5)_3$, one can get

$$\begin{cases}\psi_{tt}+(1+t)^{-\lambda}\psi_t+\hat{C}\psi-p(\psi+\hat{C})_{xx}=-\psi^2-\psi_x e_x+\left(\dfrac{\eta^2}{\psi+\hat{C}}\right)_{xx},\\ \psi(0,t)=\psi(1,t)=0, \quad t\ge 0,\\ \psi(x,0)=\psi_0(x), \quad \psi_t(x,0)=-\eta_0'(x), \quad x\in(0,1).\end{cases} \tag{2.7}$$

We are now ready to state the main result of this paper.

THEOREM 2.1. *Assume that (1.5) holds, $(\psi_0,\eta_0)\in H^2((0,1))$ and $\Psi_0:=\|(\psi_0,\eta_0)\|_2$ is sufficiently small. Then, the IBVP (2.3)–(2.4) admits a unique global smooth solution (ρ,j,ϕ) and for all $t\ge 0$, the solution satisfies*

$$\|j(t)\|+(1+t)^{(\theta+\lambda)/2}\big(\|\rho(t)-\hat{C}\|_2+\|j_x(t)\|_1+\|\phi(t)-\phi_0\|_2\big)\le C_\theta\Psi_0. \tag{2.8}$$

Furthermore, the solution converges to the constant steady-state $(\hat{C},0,\phi_0)$ in the following form as $t\to+\infty$,

$$\begin{cases}\|\rho(t)-\hat{C}\|_2+\|j_x(t)\|_1+\|\phi(t)-\phi_0\|_2\le C_\theta\Psi_0(1+t)^{-(\theta+\lambda)/2}\mathrm{e}^{-\sigma(1+t)^{1-\lambda}},\\ \|j(t)\|\le C_\theta\Psi_0\mathrm{e}^{-\sigma(1+t)^{1-\lambda}},\end{cases} \tag{2.9}$$

with some positive constant σ, where $\theta\in[\lambda,+\infty)$, the positive constant C_θ depends on θ, λ and $\hat{C}$ and satisfies $C_\theta\Psi_0<+\infty$.

REMARK 2.1. In fact, the constant $\theta\in[\lambda,+\infty)$ is related to the initial perturbation and could be large enough as the initial perturbation reduces to zero.

REMARK 2.2. The non-trivial doping profile will cause some essential difficulty in establishing the a prior estimates when $\lambda\in(0,1)$, in order to overcome this difficulty, we have to restrict the doping profile $\mathcal{C}(x)$ as a constant. We expect to develop a new technique to remove this restriction in future.

Next, we state the a prior estimates of the solution to the perturbation system (2.5)–(2.6).

PROPOSITION 2.1. *Under the conditions of Theorem 2.1, for the constant $\theta\in[\lambda,+\infty)$, let β satisfy*

$$\beta\geq\max\{(\theta+\lambda)/\lambda,\ 2\theta/((1-\lambda)\hat{C})\}^{1/(1-\lambda)}. \tag{2.10}$$

There exists a constant $\varepsilon_1>0$ sufficiently small such that, for given $T>0$, if the solution to the system (2.5)–(2.6) on $[0,T]$ satisfies

$$\sup_{0\leq t\leq T}\{\|\eta(t)\|+(1+t)^{(\theta+\lambda)/2}(\|\psi(t)\|_2+\|\psi_t(t)\|_1)\}\leq\frac{\varepsilon_1}{\beta^{(\theta+\lambda)/2}}, \tag{2.11}$$

then for any $t\in[0,T]$,

$$\|\eta(t)\|+(1+t)^{(\theta+\lambda)/2}\big(\|\psi(t)\|_2+\|\psi_t(t)\|_1+\|e(t)\|_2\big)\leq C\beta^{(\theta+2\lambda)/2}\Psi_0. \tag{2.12}$$

We can derive the sub-exponential decay estimates based on the estimate (2.12).

PROPOSITION 2.2. *Under the conditions of Theorem 2.1, if the system (2.5)–(2.6) possesses a global solution and the estimate (2.12) holds for all $t\geq0$, then as $t\to+\infty$,*

$$\begin{cases}\|\psi(t)\|_2+\|\psi_t(t)\|_1+\|e(t)\|_2\leq C_\theta\Psi_0(1+t)^{-(\theta+\lambda)/2}\mathrm{e}^{-\sigma(1+t)^{1-\lambda}},\\ \|\eta(t)\|\leq C_\theta\Psi_0\mathrm{e}^{-\sigma(1+t)^{1-\lambda}},\end{cases} \tag{2.13}$$

with some positive constant σ, where $\theta\in[\lambda,+\infty)$ and C_θ is a positive constant depending on θ, λ and $\hat{C}$ and satisfying $C_\theta\Psi_0<+\infty$.

Proof. **(Proof of Theorem 2.1.)** The local existence of the solution to the IBVP (2.5)–(2.6) can be obtained by the standard iteration method. Then, by using the a prior estimate (2.12) in Proposition 2.1 and the usual continuity arguments, we can extend the local solution to the global solution. Equivalently, we have proved that the IBVP (2.3)–(2.4) possesses a global solution. The estimates (2.8) and (2.9) can be derived from (2.12) and (2.13), respectively. ∎

For $T>0$, we denote the basic solution space for the IBVP (2.5)–(2.6) as

$$X(T)=\{(\psi,\eta,e)\in H^2((0,1)),\ 0\leq t\leq T\},$$

where its norm is given by

$$N(0,T)=\sup_{0\leq t\leq T}\{\|\eta(t)\|+(1+t)^{(\theta+\lambda)/2}(\|\psi(t)\|_2+\|\eta_x(t)\|_1+\|e(t)\|_2)\}.$$

3. The a prior estimates

In this section, we derive the a prior estimates of the solution to the perturbation system (2.5)–(2.6), namely, we devote to prove Proposition 2.1.

From the a prior assumption (2.11), it is easy to verify that

$$0<\hat{C}/2\le\psi+\hat{C}\le 2\hat{C},\qquad |\eta|\le\bar{C}, \tag{3.1}$$

with a positive constant $\bar{C}$.

In order to prove Proposition 2.1, we need to establish the following estimates based on the a prior assumption (2.11) and (3.1).

Lemma 3.1. *For $(\psi,\eta,e)\in X(T)$, under the conditions of Proposition 2.1, it holds that*

$$\int_0^1 e_x^2dx\le C\int_0^1\psi^2dx,\qquad e_x^2\le C\int_0^1\psi^2dx, \tag{3.2}$$

$$\int_0^1 e_{xt}^2dx\le C\int_0^1\psi_t^2dx,\qquad e_{xt}^2\le C\int_0^1\psi_t^2dx, \tag{3.3}$$

$$\begin{aligned}\int_0^1\eta^2dx\le C\Big(&\mathrm{e}^{-\alpha(1+t)^{1-\lambda}}\int_0^1\eta_0^2dx\\&+\int_0^t(1+s)^\lambda\mathrm{e}^{\alpha[(1+s)^{1-\lambda}-(1+t)^{1-\lambda}]}\int_0^1(\psi_t^2+\psi^2)dxds\Big),\end{aligned} \tag{3.4}$$

$$\begin{aligned}\int_0^1\eta_t^2dx\le C\int_0^1(\psi_t^2+\psi_x^2+\psi^2)dx+C(1+t)^{-2\lambda}\Big(&\mathrm{e}^{-\alpha(1+t)^{1-\lambda}}\int_0^1\eta_0^2dx\\&+\int_0^t(1+s)^\lambda\mathrm{e}^{\alpha[(1+s)^{1-\lambda}-(1+t)^{1-\lambda}]}\int_0^1(\psi_t^2+\psi^2)dxds\Big),\end{aligned} \tag{3.5}$$

where $\alpha=1/(1-\lambda)$.

Proof. For the proofs of the inequalities (3.2) and (3.3), we can refer to [18] for details. Now we focus on estimating (3.4) and (3.5).

Multiplying $(2.5)_2$ by η and integrating the resulting equation with respect to x over (0,1), one has

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1\eta^2dx+\int_0^1(1+t)^{-\lambda}\eta^2dx=&\int_0^1(\psi+\hat{C})e_x\eta dx-\int_0^1p(\psi+\hat{C})_x\eta dx\\&-\int_0^1\left(\frac{\eta^2}{\psi+\hat{C}}\right)_x\eta dx=:K_1+K_2+K_3.\end{aligned} \tag{3.6}$$

By Hölder inequality and (3.2), K_1 can be estimated as

$$\begin{aligned}K_1&\le\frac{1}{4}\int_0^1(1+t)^{-\lambda}\eta^2dx+C\int_0^1(1+t)^\lambda e_x^2dx\\&\le\frac{1}{4}\int_0^1(1+t)^{-\lambda}\eta^2dx+C\int_0^1(1+t)^\lambda\psi^2dx.\end{aligned} \tag{3.7}$$

From Hölder inequality, Taylor's formula and $(2.5)_1$, we can estimate K_2 as

$$K_2=-\int_0^1(p(\psi+\hat{C})-p(\hat{C}))_x\eta dx$$

$$
\begin{aligned}
&= -\int_0^1 (p(\psi+\hat{C})-p(\hat{C}))\psi_t dx \\
&\le C\int_0^1 (\psi_t^2+\psi^2)dx. \qquad (3.8)
\end{aligned}
$$

It follows from $(2.5)_1$ and Hölder inequality that

$$
\begin{aligned}
K_3 &= -\frac{\eta^3}{\psi+\hat{C}}\Big|_0^1 - \int_0^1 \frac{\eta^2}{\psi+\hat{C}}\psi_t dx \\
&= \int_0^1 \left(\frac{3\eta^2\psi_t}{\hat{C}} - \frac{\eta^2\psi_t}{\psi+\hat{C}}\right)dx \\
&\le \frac{1}{4}\int_0^1 (1+t)^{-\lambda}\eta^2 dx + C\int_0^1 (1+t)^{\lambda}\psi_t^2 dx. \qquad (3.9)
\end{aligned}
$$

Substituting (3.7)–(3.9) into (3.6) leads to

$$
\frac{d}{dt}\int_0^1 \eta^2 dx + \int_0^1 (1+t)^{-\lambda}\eta^2 dx \le C(1+t)^{\lambda}\int_0^1 (\psi_t^2+\psi^2)dx. \qquad (3.10)
$$

Multiplying (3.10) with $\mathrm{e}^{\alpha(1+t)^{1-\lambda}}$ yields

$$
\frac{d}{dt}\left(\mathrm{e}^{\alpha(1+t)^{1-\lambda}}\int_0^1 \eta^2 dx\right) \le C(1+t)^{\lambda}\mathrm{e}^{\alpha(1+t)^{1-\lambda}}\int_0^1 (\psi_t^2+\psi^2)dx,
$$

where $\alpha = 1/(1-\lambda)$. Integrating above inequality over $(0,t)$, we obtain the desired estimate (3.4).

It follows from $(2.5)_2$ that

$$
\begin{aligned}
\eta_t^2 &\le C\left((1+t)^{-2\lambda}\eta^2 + (\psi+\hat{C})^2 e_x^2 + p(\psi+\hat{C})_x^2 + \left(\frac{\eta^2}{\psi+\hat{C}}\right)_x^2\right) \\
&\le C\left((1+t)^{-2\lambda}\eta^2 + e_x^2 + \psi_x^2 + \psi_t^2\right),
\end{aligned}
$$

then the desired estimate (3.5) can be derived from (3.2) and (3.4). ∎

LEMMA 3.2. *For $(\psi,\eta,e) \in X(T)$, under the conditions of Proposition 2.1, it holds that*

$$
\begin{aligned}
&\int_0^1 (\beta+t)^{\theta+\lambda}(\psi_t^2+\psi_x^2+\psi^2)dx \\
&\qquad + \int_0^t\int_0^1 (\beta+s)^{\theta}(\psi_t^2+\psi_x^2+\psi^2)dxds \le C\beta^{\theta+\lambda}\|(\psi_0,\eta_0)\|_1^2, \qquad (3.11)
\end{aligned}
$$

$$
\int_0^1 (\beta+t)^{\theta+\lambda}(e_x^2+e_{xx}^2+e_{xt}^2)dx \le C\beta^{\theta+\lambda}\|(\psi_0,\eta_0)\|_1^2, \qquad (3.12)
$$

$$
\int_0^1 \eta^2 dx \le C\beta^{\theta+\lambda}\|(\psi_0,\eta_0)\|_1^2. \qquad (3.13)
$$

Proof. Owing to $0<\lambda<1$ and β satisfying (2.10), we have the following inequalities:

$$
(\beta+t)^{\lambda}\hat{C} - \frac{\theta}{2}(\beta+t)^{-1} \ge \frac{1}{2}(\beta+t)^{\lambda}\hat{C}, \qquad (3.14)
$$

$$(\beta+t)^{\lambda}(1+t)^{-\lambda}\geq 1, \tag{3.15}$$

$$(\theta+\lambda)(\beta+t)^{\lambda-1}\leq \lambda, \tag{3.16}$$

$$(1-\lambda)\hat{C}+\frac{\theta(\theta-1)}{2}(\beta+t)^{-2}-\frac{\theta}{2}(\beta+t)^{-1}(1+t)^{-\lambda}\geq \frac{1}{2}(1-\lambda)\hat{C}. \tag{3.17}$$

Next, we focus on establishing the a prior estimates of the solution to (2.5)–(2.6) based on (3.14)–(3.17).

Multiplying $(2.7)_1$ by $\left[2(\beta+t)^{\theta+\lambda}\psi_t+(\beta+t)^{\theta}\psi\right]$ and integrating it with respect to x over (0,1), we get

$$\begin{aligned}
&\frac{d}{dt}\int_0^1\left[(\beta+t)^{\theta}\psi\psi_t+(\beta+t)^{\theta+\lambda}(\psi_t^2+\hat{C}\psi^2)+\frac{1}{2}(\beta+t)^{\theta}(1+t)^{-\lambda}\psi^2-\frac{\theta}{2}(\beta+t)^{\theta-1}\psi^2\right]dx\\
&\quad+\int_0^1\left[2(\beta+t)^{\lambda}(1+t)^{-\lambda}-1-(\theta+\lambda)(\beta+t)^{\lambda-1}\right](\beta+t)^{\theta}\psi_t^2dx\\
&\quad+\int_0^1\Big[\hat{C}\left(1-(\theta+\lambda)(\beta+t)^{\lambda-1}\right)+\frac{\theta(\theta-1)}{2}(\beta+t)^{-2}-\frac{\theta}{2}(\beta+t)^{-1}(1+t)^{-\lambda}\\
&\quad+\frac{\lambda}{2}(1+t)^{-\lambda-1}\Big](\beta+t)^{\theta}\psi^2dx+\underbrace{\int_0^1 p(\psi+\hat{C})_x\left[2(\beta+t)^{\theta+\lambda}\psi_{xt}+(\beta+t)^{\theta}\psi_x\right]dx}_{I_1}\\
&=\underbrace{-\int_0^1\psi^2\left[2(\beta+t)^{\theta+\lambda}\psi_t+(\beta+t)^{\theta}\psi\right]dx}_{I_2}\underbrace{-\int_0^1\psi_xe_x\left[2(\beta+t)^{\theta+\lambda}\psi_t+(\beta+t)^{\theta}\psi\right]dx}_{I_3}\\
&\quad\underbrace{-\int_0^1\left(\frac{\eta^2}{\psi+\hat{C}}\right)_x\left[2(\beta+t)^{\theta+\lambda}\psi_{xt}+(\beta+t)^{\theta}\psi_x\right]dx}_{I_4}.
\end{aligned}\tag{3.18}$$

Here, I_1, I_2, I_3 and I_4 can be estimated as

$$\begin{aligned}
I_1=&\int_0^1 p'(\psi+\hat{C})\psi_x\left[2(\beta+t)^{\theta+\lambda}\psi_{xt}+(\beta+t)^{\theta}\psi_x\right]dx\\
=&\frac{d}{dt}\int_0^1(\beta+t)^{\theta+\lambda}p'(\psi+\hat{C})\psi_x^2dx-(\theta+\lambda)\int_0^1(\beta+t)^{\theta+\lambda-1}p'(\psi+\hat{C})\psi_x^2dx\\
&\quad-\int_0^1(\beta+t)^{\theta+\lambda}p''(\psi+\hat{C})\psi_t\psi_x^2dx+\int_0^1(\beta+t)^{\theta}p'(\psi+\hat{C})\psi_x^2dx\\
\geq&\frac{d}{dt}\int_0^1(\beta+t)^{\theta+\lambda}p'(\psi+\hat{C})\psi_x^2dx+\int_0^1\left[1-(\theta+\lambda)(\beta+t)^{\lambda-1}\right](\beta+t)^{\theta}p'(\psi+\hat{C})\psi_x^2dx\\
&\quad-C\varepsilon_1\int_0^1(\beta+t)^{\theta}\psi_x^2dx.
\end{aligned}\tag{3.19}$$

It is easy to see that

$$I_2\leq C\varepsilon_1\int_0^1(\beta+t)^{\theta}(\psi_t^2+\psi^2)dx. \tag{3.20}$$

From Hölder inequality and (3.2), we can estimate I_3 as

$$I_3\leq C\varepsilon_1\int_0^1(\beta+t)^{\theta}(\psi_x^2+e_x^2)dx\leq C\varepsilon_1\int_0^1(\beta+t)^{\theta}(\psi_x^2+\psi^2)dx. \tag{3.21}$$

By $(2.5)_1$, (3.4), (3.5) and Hölder inequality, I_4 can be estimated as

$$
\begin{aligned}
I_4 =& \int_0^1 \left[\frac{2\eta\psi_t}{\psi+\hat{C}} + \frac{\eta^2\psi_x}{(\psi+\hat{C})^2}\right]\left[2(\beta+t)^{\theta+\lambda}\psi_{xt} + (\beta+t)^\theta\psi_x\right]dx \\
=& \frac{d}{dt}\int_0^1 (\beta+t)^{\theta+\lambda}\frac{\eta^2\psi_x^2}{(\psi+\hat{C})^2}dx - (\theta+\lambda)\int_0^1(\beta+t)^{\theta+\lambda-1}\frac{\eta^2\psi_x^2}{(\psi+\hat{C})^2}dx \\
& - 2\int_0^1(\beta+t)^{\theta+\lambda}\frac{\eta\eta_t\psi_x^2}{(\psi+\hat{C})^2}dx + 2\int_0^1(\beta+t)^{\theta+\lambda}\frac{\eta^2\psi_t\psi_x^2}{(\psi+\hat{C})^3}dx \\
& + 2\int_0^1(\beta+t)^{\theta+\lambda}\frac{\psi_t^3}{\psi+\hat{C}}dx + 2\int_0^1(\beta+t)^{\theta+\lambda}\frac{\eta\psi_t^2\psi_x}{(\psi+\hat{C})^2}dx \\
& + 2\int_0^1(\beta+t)^\theta\frac{\eta\psi_x\psi_t}{\psi+\hat{C}}dx + \int_0^1(\beta+t)^\theta\frac{\eta^2\psi_x^2}{(\psi+\hat{C})^2}dx \\
\le& \frac{d}{dt}\int_0^1(\beta+t)^{\theta+\lambda}\frac{\eta^2\psi_x^2}{(\psi+\hat{C})^2}dx + C\varepsilon_1\int_0^1(\beta+t)^\theta(\psi_t^2+\psi_x^2)dx \\
& + C\varepsilon_1\int_0^1(\beta+t)^{-\lambda}\eta^2dx + C\varepsilon_1\int_0^1(\beta+t)^\lambda\eta_t^2dx \\
\le& \frac{d}{dt}\int_0^1(\beta+t)^{\theta+\lambda}\frac{\eta^2\psi_x^2}{(\psi+\hat{C})^2}dx + C\varepsilon_1\int_0^1(\beta+t)^\theta(\psi_t^2+\psi_x^2+\psi^2)dx \\
& + C\varepsilon_1(\beta+t)^\lambda(1+t)^{-2\lambda}\int_0^t(1+s)^\lambda e^{\alpha[(1+s)^{1-\lambda}-(1+t)^{1-\lambda}]}\int_0^1(\psi_t^2+\psi^2)dxds \\
& + C\beta^\lambda(1+t)^{-\lambda}e^{-\alpha(1+t)^{1-\lambda}}\int_0^1\eta_0^2dx. \qquad (3.22)
\end{aligned}
$$

Putting (3.19)–(3.22) into (3.18), we obtain

$$
\begin{aligned}
& \frac{d}{dt}\int_0^1\left[(\beta+t)^\theta\psi\psi_t + (\beta+t)^{\theta+\lambda}\left(\psi_t^2+\hat{C}\psi^2+\left(p'(\psi+\hat{C})-\frac{\eta^2}{(\psi+\hat{C})^2}\right)\psi_x^2\right)\right. \\
& \left. + \frac{1}{2}(\beta+t)^\theta(1+t)^{-\lambda}\psi^2 - \frac{\theta}{2}(\beta+t)^{\theta-1}\psi^2\right]dx + \int_0^1\left[2(\beta+t)^\lambda(1+t)^{-\lambda} - 1\right. \\
& \left. -(\theta+\lambda)(\beta+t)^{\lambda-1}\right](\beta+t)^\theta\psi_t^2dx + \int_0^1\left[\left(1-(\theta+\lambda)(\beta+t)^{\lambda-1}\right)\hat{C}\right. \\
& \left. + \frac{\theta(\theta-1)}{2}(\beta+t)^{-2} - \frac{\theta}{2}(\beta+t)^{-1}(1+t)^{-\lambda} + \frac{\lambda}{2}(1+t)^{-\lambda-1}\right](\beta+t)^\theta\psi^2dx \\
& + \int_0^1\left[1-(\theta+\lambda)(\beta+t)^{\lambda-1}\right](\beta+t)^\theta p'(\psi+\hat{C})\psi_x^2dx \\
\le& C\varepsilon_1\int_0^1(\beta+t)^\theta(\psi_t^2+\psi_x^2+\psi^2)dx + C\beta^\lambda(1+t)^{-\lambda}e^{-\alpha(1+t)^{1-\lambda}}\int_0^1\eta_0^2dx \\
& + C\varepsilon_1(\beta+t)^\lambda(1+t)^{-2\lambda}\int_0^t(1+s)^\lambda e^{\alpha[(1+s)^{1-\lambda}-(1+t)^{1-\lambda}]}\int_0^1(\psi_t^2+\psi^2)dxds.
\end{aligned}
$$

Thus, in view of the inequalities (3.14)–(3.17), we have

$$
\frac{d}{dt}\int_0^1(\beta+t)^{\theta+\lambda}(\psi_t^2+\psi_x^2+\psi^2)dx + \int_0^1(\beta+t)^\theta(\psi_t^2+\psi_x^2+\psi^2)dx
$$

$$\leq C\beta^{\lambda}(1+t)^{-\lambda}\mathrm{e}^{-\alpha(1+t)^{1-\lambda}}\int_0^1\eta_0^2dx$$
$$+C\varepsilon_1(\beta+t)^{\lambda}(1+t)^{-2\lambda}\int_0^t(1+s)^{\lambda}\mathrm{e}^{\alpha[(1+s)^{1-\lambda}-(1+t)^{1-\lambda}]}\int_0^1(\psi_t^2+\psi^2)dxds. \quad (3.23)$$

Integrating the inequality (3.23) over $(0,t)$ yields

$$\int_0^1(\beta+t)^{\theta+\lambda}(\psi_t^2+\psi_x^2+\psi^2)dx+\int_0^t\int_0^1(\beta+s)^{\theta}(\psi_t^2+\psi_x^2+\psi^2)dxds$$
$$\leq\beta^{\theta+\lambda}\int_0^1(\psi_t^2+\psi_x^2+\psi^2)(x,0)dx+C\beta^{\lambda}\int_0^t(1+s)^{-\lambda}\mathrm{e}^{-\alpha(1+s)^{1-\lambda}}ds\int_0^1\eta_0^2dx$$
$$+C\varepsilon_1\int_0^t(\beta+s)^{\lambda}(1+s)^{-2\lambda}\int_0^s(1+\tau)^{\lambda}\mathrm{e}^{\alpha[(1+\tau)^{1-\lambda}-(1+s)^{1-\lambda}]}\int_0^1(\psi_t^2+\psi^2)dxd\tau ds$$
$$\leq C\beta^{\theta+\lambda}\|(\psi_0,\eta_0)\|_1^2$$
$$+C\varepsilon_1\int_0^t(1+\tau)^{\lambda}\mathrm{e}^{\alpha(1+\tau)^{1-\lambda}}\int_0^1(\psi_t^2+\psi^2)dx\int_\tau^t(\beta+s)^{\lambda}(1+s)^{-2\lambda}\mathrm{e}^{-\alpha(1+s)^{1-\lambda}}dsd\tau$$
$$\leq C\beta^{\theta+\lambda}\|(\psi_0,\eta_0)\|_1^2+C\beta^{\lambda}\varepsilon_1\int_0^t(1+\tau)^{\lambda}\int_0^1(\psi_t^2+\psi^2)dxd\tau,$$

where we used the fact that

$$\int_0^t(1+s)^{-\lambda}\mathrm{e}^{-\alpha(1+s)^{1-\lambda}}ds=-\mathrm{e}^{-\alpha(1+s)^{1-\lambda}}\Big|_0^t=\mathrm{e}^{-\alpha}-\mathrm{e}^{-\alpha(1+t)^{1-\lambda}}.$$

Then, this implies the estimate (3.11). Furthermore, combining (3.11) and Lemma 3.1, we can obtain the estimates (3.12) and (3.13). □

LEMMA 3.3. *For $(\psi,\eta,e)\in X(T)$, under the conditions of Proposition 2.1, it holds that*

$$\int_0^1(\beta+t)^{\theta+\lambda}(\psi_{tt}^2+\psi_{xt}^2+\psi_{xx}^2)dx$$
$$+\int_0^t\int_0^1(\beta+s)^{\theta}(\psi_{tt}^2+\psi_{xt}^2)dxds\leq C\beta^{\theta+2\lambda}\|(\psi_0,\eta_0)\|_2^2. \quad (3.24)$$

Proof. Firstly, we estimate $\int_0^1\eta_{tt}^2dx$. Differentiating $(2.5)_2$ in t gives

$$\eta_{tt}=-(1+t)^{-\lambda}\eta_t+\lambda(1+t)^{-\lambda-1}\eta+(\psi+\hat{C})e_{xt}+\psi_te_x$$
$$-(p'(\psi+\hat{C})\psi_t)_x-\left(\frac{\eta^2}{\psi+\hat{C}}\right)_{xt}, \quad (3.25)$$

thus,

$$\eta_{tt}^2\leq C\Big[(1+t)^{-2\lambda}\eta_t^2+(1+t)^{-2\lambda-2}\eta^2+(\psi+\hat{C})^2e_{xt}^2$$
$$+\psi_t^2e_x^2+(p'(\psi+\hat{C})\psi_t)_x^2+\left(\frac{\eta^2}{\psi+\hat{C}}\right)_{xt}^2\Big]. \quad (3.26)$$

Integrating (3.26) with respect to x over $(0,1)$ and using Lemma 3.1, we have

$$\int_0^1 \eta_{tt}^2 dx \le C(1+t)^{-4\lambda} e^{-\alpha(1+t)^{1-\lambda}} \int_0^1 \eta_0^2 dx + C\int_0^1 (\psi_{tt}^2+\psi_{xt}^2+\psi_t^2+\psi_x^2+\psi^2)dx$$
$$+C(1+t)^{-4\lambda}\int_0^t (1+s)^{\lambda} e^{\alpha[(1+s)^{1-\lambda}-(1+t)^{1-\lambda}]}\int_0^1(\psi_t^2+\psi^2)dxds. \quad (3.27)$$

Now we deal with (3.24). Differentiating $(2.7)_1$ with respect to t, one can get

$$\psi_{ttt}+(1+t)^{-\lambda}\psi_{tt}-\lambda(1+t)^{-\lambda-1}\psi_t+\hat{C}\psi_t-(p'(\psi+\hat{C})\psi_t)_{xx}$$
$$=-2\psi\psi_t-\psi_{xt}e_x-\psi_x e_{xt}+\left(\frac{\eta^2}{\psi+\hat{C}}\right)_{xxt}. \quad (3.28)$$

We multiply (3.28) by $\left[2(\beta+t)^{\theta+\lambda}\psi_{tt}+(\beta+t)^{\theta}\psi_t\right]$ and integrate the resulting equation with respect to x over $(0,1)$ to get

$$\frac{d}{dt}\int_0^1 \Big[(\beta+t)^{\theta}\psi_t\psi_{tt}+(\beta+t)^{\theta+\lambda}(\psi_{tt}^2+\hat{C}\psi_t^2)+\frac{1}{2}(\beta+t)^{\theta}(1+t)^{-\lambda}\psi_t^2$$
$$-\frac{\theta}{2}(\beta+t)^{\theta-1}\psi_t^2-\lambda(\beta+t)^{\theta+\lambda}(1+t)^{-\lambda-1}\psi_t^2\Big]dx$$
$$+\int_0^1\left[2(\beta+t)^{\lambda}(1+t)^{-\lambda}-1-(\theta+\lambda)(\beta+t)^{\lambda-1}\right](\beta+t)^{\theta}\psi_{tt}^2dx$$
$$+\int_0^1\Big[\left(1-(\theta+\lambda)(\beta+t)^{\lambda-1}\right)\hat{C}+\frac{\theta(\theta-1)}{2}(\beta+t)^{-2}$$
$$-\frac{\theta}{2}(\beta+t)^{-1}(1+t)^{-\lambda}-\lambda(\lambda+1)(\beta+t)^{\lambda}(1+t)^{-\lambda-2}$$
$$-\frac{\lambda}{2}(1+t)^{-\lambda-1}+\lambda(\theta+\lambda)(\beta+t)^{\lambda-1}(1+t)^{-\lambda-1}\Big](\beta+t)^{\theta}\psi_t^2dx$$
$$+\underbrace{\int_0^1 (p'(\psi+\hat{C})\psi_t)_x\left[2(\beta+t)^{\theta+\lambda}\psi_{xtt}+(\beta+t)^{\theta}\psi_{xt}\right]dx}_{I_5}$$
$$=\underbrace{-\int_0^1 2\psi\psi_t\left[2(\beta+t)^{\theta+\lambda}\psi_{tt}+(\beta+t)^{\theta}\psi_t\right]dx}_{I_6}$$
$$\underbrace{-\int_0^1(\psi_{xt}e_x+\psi_x e_{xt})\left[2(\beta+t)^{\theta+\lambda}\psi_{tt}+(\beta+t)^{\theta}\psi_t\right]dx}_{I_7}$$
$$\underbrace{-\int_0^1\left(\frac{\eta^2}{\psi+\hat{C}}\right)_{xt}\left[2(\beta+t)^{\theta+\lambda}\psi_{xtt}+(\beta+t)^{\theta}\psi_{xt}\right]dx}_{I_8}. \quad (3.29)$$

Next, we focus on estimating I_5, I_6, I_7 and I_8. By Hölder inequality and (3.16), one has

$$I_5=\int_0^1\left[p'(\psi+\hat{C})\psi_{xt}+p''(\psi+\hat{C})\psi_x\psi_t\right]\left[2(\beta+t)^{\theta+\lambda}\psi_{xtt}+(\beta+t)^{\theta}\psi_{xt}\right]dx$$

$$
\begin{aligned}
&= \frac{d}{dt}\int_0^1 (\beta+t)^{\theta+\lambda} p'(\psi+\hat{C})\psi_{xt}^2 dx - (\theta+\lambda)\int_0^1 (\beta+t)^{\theta+\lambda-1} p'(\psi+\hat{C})\psi_{xt}^2 dx \\
&\quad - 3\int_0^1 (\beta+t)^{\theta+\lambda} p''(\psi+\hat{C})\psi_t \psi_{xt}^2 dx + 2\frac{d}{dt}\int_0^1 (\beta+t)^{\theta+\lambda} p''(\psi+\hat{C})\psi_x\psi_t\psi_{xt} dx \\
&\quad - 2(\theta+\lambda)\int_0^1 (\beta+t)^{\theta+\lambda-1} p''(\psi+\hat{C})\psi_x\psi_t\psi_{xt} dx \\
&\quad - 2\int_0^1 (\beta+t)^{\theta+\lambda} p'''(\psi+\hat{C})\psi_x\psi_t^2\psi_{xt} dx - 2\int_0^1 (\beta+t)^{\theta+\lambda} p''(\psi+\hat{C})\psi_x\psi_{tt}\psi_{xt} dx \\
&\quad + \int_0^1 (\beta+t)^{\theta} p'(\psi+\hat{C})\psi_{xt}^2 dx + \int_0^1 (\beta+t)^{\theta} p''(\psi+\hat{C})\psi_x\psi_t\psi_{xt} dx \\
&\geq \frac{d}{dt}\int_0^1 (\beta+t)^{\theta+\lambda} p'(\psi+\hat{C})\psi_{xt}^2 dx + 2\frac{d}{dt}\int_0^1 (\beta+t)^{\theta+\lambda} p''(\psi+\hat{C})\psi_x\psi_t\psi_{xt} dx \\
&\quad + \int_0^1 \left[1-(\theta+\lambda)(\beta+t)^{\lambda-1}\right](\beta+t)^{\theta} p'(\psi+\hat{C})\psi_{xt}^2 dx \\
&\quad - C\varepsilon_1 \int_0^1 (\beta+t)^{\theta}(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2) dx.
\end{aligned}
\tag{3.30}
$$

It is easy to verify that

$$
I_6 \leq C\varepsilon_1 \int_0^1 (\beta+t)^{\theta}(\psi_{tt}^2+\psi_t^2) dx. \tag{3.31}
$$

From Hölder inequality and (3.3), we can estimate I_7 as

$$
\begin{aligned}
I_7 &\leq C\varepsilon_1 \int_0^1 (\beta+t)^{\theta}(\psi_{tt}^2+\psi_{xt}^2+e_{xt}^2+\psi_t^2) dx \\
&\leq C\varepsilon_1 \int_0^1 (\beta+t)^{\theta}(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2) dx.
\end{aligned}
\tag{3.32}
$$

By Hölder inequality, (3.16) and (3.27), I_8 can be estimated as

$$
\begin{aligned}
I_8 &= \int_0^1 \left[\frac{2\eta_t\psi_t}{\psi+\hat{C}} + \frac{2\eta\psi_{tt}}{\psi+\hat{C}} - \frac{2\eta\psi_t^2}{(\psi+\hat{C})^2} + \frac{2\eta\eta_t\psi_x}{(\psi+\hat{C})^2} + \frac{\eta^2\psi_{xt}}{(\psi+\hat{C})^2} - \frac{2\eta^2\psi_x\psi_t}{(\psi+\hat{C})^3}\right] \\
&\quad \cdot \left[2(\beta+t)^{\theta+\lambda}\psi_{xtt} + (\beta+t)^{\theta}\psi_{xt}\right] dx \\
&= \frac{d}{dt}\int_0^1 (\beta+t)^{\theta+\lambda} \frac{\eta^2\psi_{xt}^2}{(\psi+\hat{C})^2} dx - \int_0^1 \left[(\beta+t)^{\theta+\lambda}\frac{\eta^2}{(\psi+\hat{C})^2}\right]_t \psi_{xt}^2 dx \\
&\quad - 2\int_0^1 (\beta+t)^{\theta+\lambda}\left(\frac{\eta}{\psi+\hat{C}}\right)_x \psi_{tt}^2 dx + 4\frac{d}{dt}\int_0^1 (\beta+t)^{\theta+\lambda}\frac{\eta\eta_t\psi_x\psi_{xt}}{(\psi+\hat{C})^2} dx \\
&\quad - 4\int_0^1 \left[(\beta+t)^{\theta+\lambda}\frac{\eta\eta_t\psi_x}{(\psi+\hat{C})^2}\right]_t \psi_{xt} dx - 4\frac{d}{dt}\int_0^1 (\beta+t)^{\theta+\lambda}\frac{\eta^2\psi_t\psi_x\psi_{xt}}{(\psi+\hat{C})^3} dx \\
&\quad + 4\int_0^1 \left[(\beta+t)^{\theta+\lambda}\frac{\eta^2\psi_t\psi_x}{(\psi+\hat{C})^3}\right]_t \psi_{xt} dx - 4\int_0^1 (\beta+t)^{\theta+\lambda}\left[\frac{\eta_t\psi_t}{\psi+\hat{C}}\right]_x \psi_{tt} dx \\
&\quad + 4\int_0^1 (\beta+t)^{\theta+\lambda}\left[\frac{\eta\psi_t^2}{(\psi+\hat{C})^2}\right]_x \psi_{tt} dx + \int_0^1 \left[\frac{2\eta_t\psi_t}{\psi+\hat{C}} + \frac{2\eta\psi_{tt}}{\psi+\hat{C}} - \frac{2\eta\psi_t^2}{(\psi+\hat{C})^2}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2\eta\eta_t\psi_x}{(\psi+\hat{C})^2}+\frac{\eta^2\psi_{xt}}{(\psi+\hat{C})^2}-\frac{2\eta^2\psi_x\psi_t}{(\psi+\hat{C})^3}\Big](\beta+t)^\theta\psi_{xt}dx\\
\le&\frac{d}{dt}\int_0^1(\beta+t)^{\theta+\lambda}\frac{\eta^2\psi_{xt}^2}{(\psi+\hat{C})^2}dx+4\frac{d}{dt}\int_0^1(\beta+t)^{\theta+\lambda}\frac{\eta\eta_t\psi_x\psi_{xt}}{(\psi+\hat{C})^2}dx\\
&-4\frac{d}{dt}\int_0^1(\beta+t)^{\theta+\lambda}\frac{\eta^2\psi_t\psi_x\psi_{xt}}{(\psi+\hat{C})^3}dx+C\varepsilon_1(\beta+t)^\lambda\int_0^1\eta_{tt}^2dx\\
&+C\beta^\lambda\varepsilon_1\int_0^1(\beta+t)^\theta(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2+\psi_x^2)dx\\
\le&\frac{d}{dt}\int_0^1(\beta+t)^{\theta+\lambda}\frac{\eta^2\psi_{xt}^2}{(\psi+\hat{C})^2}dx+4\frac{d}{dt}\int_0^1(\beta+t)^{\theta+\lambda}\frac{\eta\eta_t\psi_x\psi_{xt}}{(\psi+\hat{C})^2}dx\\
&-4\frac{d}{dt}\int_0^1(\beta+t)^{\theta+\lambda}\frac{\eta^2\psi_t\psi_x\psi_{xt}}{(\psi+\hat{C})^3}dx+C\beta^\lambda\varepsilon_1\int_0^1(\beta+t)^\theta(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2)dx\\
&+C\beta^\lambda(1+t)^{-3\lambda}\mathrm{e}^{-\alpha(1+t)^{1-\lambda}}\int_0^1\eta_0^2dx+C\beta^\lambda\int_0^1(\beta+t)^\theta(\psi_x^2+\psi^2)dx\\
&+C\varepsilon_1(\beta+t)^\lambda(1+t)^{-4\lambda}\int_0^t(1+s)^\lambda\mathrm{e}^{\alpha[(1+s)^{1-\lambda}-(1+t)^{1-\lambda}]}\int_0^1(\psi_t^2+\psi^2)dxds. \quad (3.33)
\end{aligned}
$$

Substituting (3.30)–(3.33) into (3.29) gives

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1\Big[(\beta+t)^\theta\psi_t\psi_{tt}+(\beta+t)^{\theta+\lambda}\Big(\psi_{tt}^2+\hat{C}\psi_t^2+\Big(p'(\psi+\hat{C})-\frac{\eta^2}{(\psi+\hat{C})^2}\Big)\psi_{xt}^2\Big)\\
&\quad+\frac{1}{2}(\beta+t)^\theta(1+t)^{-\lambda}\psi_t^2-\frac{\theta}{2}(\beta+t)^{\theta-1}\psi_t^2-\lambda(\beta+t)^{\theta+\lambda}(1+t)^{-\lambda-1}\psi_t^2\Big]dx\\
&\quad+2\frac{d}{dt}\int_0^1(\beta+t)^{\theta+\lambda}p''(\psi+\hat{C})\psi_x\psi_t\psi_{xt}dx-4\frac{d}{dt}\int_0^1(\beta+t)^{\theta+\lambda}\frac{\eta\eta_t\psi_x\psi_{xt}}{(\psi+\hat{C})^2}dx\\
&\quad+4\frac{d}{dt}\int_0^1(\beta+t)^{\theta+\lambda}\frac{\eta^2\psi_t\psi_x\psi_{xt}}{(\psi+\hat{C})^3}dx+\int_0^1\Big[\big(1-(\theta+\lambda)(\beta+t)^{\lambda-1}\big)\hat{C}\\
&\quad+\frac{\theta(\theta-1)}{2}(\beta+t)^{-2}-\frac{\theta}{2}(\beta+t)^{-1}(1+t)^{-\lambda}-\lambda(\lambda+1)(\beta+t)^\lambda(1+t)^{-\lambda-2}\\
&\quad-\frac{\lambda}{2}(1+t)^{-\lambda-1}+\lambda(\theta+\lambda)(\beta+t)^{\lambda-1}(1+t)^{-\lambda-1}\Big](\beta+t)^\theta\psi_t^2dx\\
&\quad+\int_0^1\big[2(\beta+t)^\lambda(1+t)^{-\lambda}-1-(\theta+\lambda)(\beta+t)^{\lambda-1}\big](\beta+t)^\theta\psi_{tt}^2dx\\
&\quad+\int_0^1\big[1-(\theta+\lambda)(\beta+t)^{\lambda-1}\big](\beta+t)^\theta p'(\psi+\hat{C})\psi_{xt}^2dx\\
\le&C\beta^\lambda\varepsilon_1\int_0^1(\beta+t)^\theta(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2)dx+C\beta^\lambda\int_0^1(\beta+t)^\theta(\psi_x^2+\psi^2)dx\\
&\quad+C\beta^\lambda(1+t)^{-3\lambda}\mathrm{e}^{-\alpha(1+t)^{1-\lambda}}\int_0^1\eta_0^2dx\\
&\quad+C\varepsilon_1(\beta+t)^\lambda(1+t)^{-4\lambda}\int_0^t(1+s)^\lambda\mathrm{e}^{\alpha[(1+s)^{1-\lambda}-(1+t)^{1-\lambda}]}\int_0^1(\psi_t^2+\psi^2)dxds.
\end{aligned}
$$

Let

$$G(t):=\int_0^1\Big[(\beta+t)^\theta\psi_t\psi_{tt}+(\beta+t)^{\theta+\lambda}\Big(\psi_{tt}^2+\hat{C}\psi_t^2+\Big(p'(\psi+\hat{C})-\frac{\eta^2}{(\psi+\hat{C})^2}\Big)\psi_{xt}^2\Big)$$
$$+\frac{1}{2}(\beta+t)^\theta(1+t)^{-\lambda}\psi_t^2-\frac{\theta}{2}(\beta+t)^{\theta-1}\psi_t^2-\lambda(\beta+t)^{\theta+\lambda}(1+t)^{-\lambda-1}\psi_t^2$$
$$+2(\beta+t)^{\theta+\lambda}p''(\psi+\hat{C})\psi_x\psi_t\psi_{xt}-4(\beta+t)^{\theta+\lambda}\frac{\eta\eta_t\psi_x\psi_{xt}}{(\psi+\hat{C})^2}$$
$$+4(\beta+t)^{\theta+\lambda}\frac{\eta^2\psi_t\psi_x\psi_{xt}}{(\psi+\hat{C})^3}\Big]dx.$$

Thanks to the inequalities (3.14)–(3.17), we obtain

$$\frac{d}{dt}G(t)+C\int_0^1(\beta+t)^\theta(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2)dx$$
$$\le C\beta^\lambda(1+t)^{-3\lambda}\mathrm{e}^{-\alpha(1+t)^{1-\lambda}}\int_0^1\eta_0^2dx+C\beta^\lambda\int_0^1(\beta+t)^\theta(\psi_t^2+\psi_x^2+\psi^2)dx$$
$$+C(\beta+t)^\lambda(1+t)^{-4\lambda}\int_0^t(1+s)^\lambda\mathrm{e}^{\alpha[(1+s)^{1-\lambda}-(1+t)^{1-\lambda}]}\int_0^1(\psi_t^2+\psi^2)dxds, \tag{3.34}$$

and

$$\begin{cases}G(t)\ge C_1\int_0^1(\beta+t)^{\theta+\lambda}(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2)dx-C_2\int_0^1(\beta+t)^{\theta+\lambda}(\psi_t^2+\psi_x^2)dx,\\ G(t)\le C_3\int_0^1(\beta+t)^{\theta+\lambda}(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2+\psi_x^2)dx,\end{cases} \tag{3.35}$$

where C_1, C_2 and C_3 are some positive constants. Integrating the inequality (3.34) over $(0,t)$ and by (3.35), one has

$$\int_0^1(\beta+t)^{\theta+\lambda}(\psi_{tt}^2+\psi_{xt}^2)dx+\int_0^t\int_0^1(\beta+s)^\theta(\psi_{tt}^2+\psi_{xt}^2)dxds$$
$$\le C\beta^{\theta+\lambda}\int_0^1(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2+\psi_x^2)(x,0)dx+C\int_0^1(\beta+t)^{\theta+\lambda}(\psi_t^2+\psi_x^2)dx$$
$$+C\beta^\lambda\int_0^t(1+s)^{-3\lambda}\mathrm{e}^{-\alpha(1+s)^{1-\lambda}}ds\int_0^1\eta_0^2dx+C\beta^\lambda\int_0^t\int_0^1(\beta+s)^\theta(\psi_t^2+\psi_x^2+\psi^2)dxds$$
$$+C\int_0^t(\beta+s)^\lambda(1+s)^{-4\lambda}\int_0^s(1+\tau)^\lambda\mathrm{e}^{\alpha[(1+\tau)^{1-\lambda}-(1+s)^{1-\lambda}]}\int_0^1(\psi_t^2+\psi^2)dxd\tau ds$$
$$\le C\beta^{\theta+\lambda}\int_0^1(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2+\psi_x^2)(x,0)dx+C\beta^\lambda\int_0^1\eta_0^2dx+C\int_0^1(\beta+t)^{\theta+\lambda}(\psi_t^2+\psi_x^2)dx$$
$$+C\beta^\lambda\int_0^t\int_0^1(\beta+s)^\theta(\psi_t^2+\psi_x^2+\psi^2)dxds$$
$$\le C\beta^{\theta+2\lambda}\|(\psi_0,\eta_0)\|_2^2, \tag{3.36}$$

where we used the fact that

$$\int_0^1\psi_{tt}^2(x,0)dx\le C\int_0^1(\psi^2+\psi_x^2+\psi_t^2+\psi_{xx}^2+\psi_{xt}^2)(x,0)dx.$$

Finally, we estimate $\int_0^1 \psi_{xx}^2 dx$. It follows from (2.7)$_1$ that

$$\left(p'(\psi+\hat{C})-\frac{\eta^2}{(\psi+\hat{C})^2}\right)\psi_{xx}$$
$$=-\frac{2\psi_t^2}{\psi+\hat{C}}+\frac{2\eta\psi_{xt}}{\psi+\hat{C}}-\frac{4\eta\psi_t\psi_x}{(\psi+\hat{C})^2}-\frac{2\eta^2\psi_x^2}{(\psi+\hat{C})^3}$$
$$-p''(\psi+\hat{C})\psi_x^2+\psi_{tt}+(1+t)^{-\lambda}\psi_t+\hat{C}\psi+\psi^2+\psi_x e_x.$$

Multiplying above equation by ψ_{xx} and integrating it with respect to x over $(0,1)$, we have

$$\int_0^1\left(p'(\psi+\hat{C})-\frac{\eta^2}{(\psi+\hat{C})^2}\right)\psi_{xx}^2 dx$$
$$=\int_0^1\Bigg[-\frac{2\psi_t^2}{\psi+\hat{C}}+\frac{2\eta\psi_{xt}}{\psi+\hat{C}}-\frac{4\eta\psi_t\psi_x}{(\psi+\hat{C})^2}-\frac{2\eta^2\psi_x^2}{(\psi+\hat{C})^3}$$
$$-p''(\psi+\hat{C})\psi_x^2+\psi_{tt}+(1+t)^{-\lambda}\psi_t+\hat{C}\psi+\psi^2+\psi_x e_x\Bigg]\psi_{xx}dx,$$

then by Hölder inequality, (3.11) and (3.36), we get

$$\int_0^1 \psi_{xx}^2 dx \le C\int_0^1(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2+\psi_x^2+\psi^2)dx$$
$$\le C\beta^{\theta+2\lambda}(\beta+t)^{-(\theta+\lambda)}\|(\psi_0,\eta_0)\|_2^2. \tag{3.37}$$

The desired estimate (3.24) follows from (3.36) and (3.37). ∎

Proof. **(Proof of Proposition 2.1.)** Lemmas 3.2 and 3.3 imply Proposition 2.1. ∎

4. Sub-exponential decay rates

In this section, we prove the sub-exponential decay rates of the solution to the system (2.5)–(2.6), namely, the proof of Proposition 2.2.

When the estimate (2.12) in Proposition 2.1 holds for all $t\ge 0$, we can get

$$\sup_{t\ge 0}\{\|\eta(t)\|+(1+t)^{(\theta+\lambda)/2}\big(\|\psi(t)\|_2+\|\psi_t(t)\|_1\big)\}\le C\beta^{(\theta+2\lambda)/2}\Psi_0:=\varepsilon_2, \tag{4.1}$$

in fact, we have $\varepsilon_2\ll 1$. Thanks to (4.1), (3.1) still holds in this section and we can prove the following lemma in a similar way as Lemma 3.1.

Lemma 4.1. *For $(\psi,\eta,e)\in X(T)$, under the conditions of Proposition 2.1, if (4.1) holds for all $t\ge 0$, then for any $t\ge T_0$, we have*

$$\int_0^1 e_x^2 dx\le C\int_0^1\psi^2 dx,\qquad \int_0^1 e_{xt}^2 dx\le C\int_0^1\psi_t^2 dx, \tag{4.2}$$

$$\int_0^1\eta^2 dx\le C\mathrm{e}^{-\alpha(1+t)^{1-\lambda}}\mathrm{e}^{\alpha(1+T_0)^{1-\lambda}}\beta^{\theta+2\lambda}\Psi_0^2$$
$$+C\int_{T_0}^t(1+s)^\lambda\mathrm{e}^{\alpha[(1+s)^{1-\lambda}-(1+t)^{1-\lambda}]}\int_0^1(\psi_t^2+\psi^2)dxds, \tag{4.3}$$

$$\int_0^1 \eta_t^2 dx \le C\int_0^1 (\psi_t^2+\psi_x^2+\psi^2)dx + C(1+t)^{-2\lambda}\mathrm{e}^{-\alpha(1+t)^{1-\lambda}}\mathrm{e}^{\alpha(1+T_0)^{1-\lambda}}\beta^{\theta+2\lambda}\Psi_0^2$$
$$+C(1+t)^{-2\lambda}\int_{T_0}^t (1+s)^\lambda \mathrm{e}^{\alpha[(1+s)^{1-\lambda}-(1+t)^{1-\lambda}]}\int_0^1(\psi_t^2+\psi^2)dxds, \quad (4.4)$$

where $T_0=\max\{(\theta+\lambda)/\lambda,\ 2\theta/((1-\lambda)\hat{C}),\ 2(2\theta+3\lambda)/(3(1-\lambda)\hat{C})\}^{1/(1-\lambda)}-1$ *and* $\alpha = 1/(1-\lambda)$.

LEMMA 4.2. *For* $(\psi,\eta,e)\in X(T)$, *under the conditions of Proposition 2.1, if* (4.1) *holds for all* $t\ge 0$, *then for any* $t\ge T_0$, *we have*

$$\int_0^1 (\psi_t^2+\psi_x^2+\psi^2)dx \le C\beta^{\theta+2\lambda}\mathrm{e}^{\gamma(1+T_0)^{1-\lambda}}(1+t)^{-(\theta+\lambda)}\mathrm{e}^{-\gamma(1+t)^{1-\lambda}}\Psi_0^2, \quad (4.5)$$

$$\int_0^1 (e_x^2+e_{xx}^2+e_{xt}^2)dx \le C\beta^{\theta+2\lambda}\mathrm{e}^{\gamma(1+T_0)^{1-\lambda}}(1+t)^{-(\theta+\lambda)}\mathrm{e}^{-\gamma(1+t)^{1-\lambda}}\Psi_0^2, \quad (4.6)$$

$$\int_0^1 \eta^2 dx \le C\beta^{\theta+2\lambda}\mathrm{e}^{\mu_1(1+T_0)^{1-\lambda}}\mathrm{e}^{-\mu_2(1+t)^{1-\lambda}}\Psi_0^2, \quad (4.7)$$

$$\int_0^1 \eta_t^2 dx \le C\beta^{\theta+2\lambda}\mathrm{e}^{\mu_1(1+T_0)^{1-\lambda}}(1+t)^{-2\lambda}\mathrm{e}^{-\mu_2(1+t)^{1-\lambda}}\Psi_0^2, \quad (4.8)$$

for some positive constant γ, *and* $\mu_1=\max\{\alpha,\gamma\}$, $\mu_2=\min\{\alpha,\gamma\}$.

Proof. Since $0<\lambda<1$ and $t\ge T_0$, we can obtain the following inequalities:

$$(1+t)^\lambda\hat{C}-\frac{\theta}{2}(1+t)^{-1}\ge\frac{1}{2}(1+t)^\lambda\hat{C}, \quad (4.9)$$

$$(\theta+\lambda)(1+t)^{\lambda-1}\le\lambda, \quad (4.10)$$

$$(1-\lambda)\hat{C}-\frac{\theta}{2}(1+t)^{-2}-\frac{\theta}{2}(1+t)^{-\lambda-1}\ge\frac{1}{2}(1-\lambda)\hat{C}. \quad (4.11)$$

Multiplying $(2.7)_1$ by $\left[2(1+t)^{\theta+\lambda}\psi_t+(1+t)^\theta\psi\right]$ and integrating the resulting equation with respect to x over $(0,1)$ to obtain

$$\frac{d}{dt}\int_0^1\left[(1+t)^\theta\psi\psi_t+(1+t)^{\theta+\lambda}(\psi_t^2+\hat{C}\psi^2)+\frac{1}{2}(1+t)^{\theta-\lambda}\psi^2-\frac{\theta}{2}(1+t)^{\theta-1}\psi^2\right]dx$$
$$+\int_0^1[1-(\theta+\lambda)(1+t)^{\lambda-1}](1+t)^\theta\psi_t^2dx+\int_0^1\Big[(1-(\theta+\lambda)(1+t)^{\lambda-1})\hat{C}$$
$$+\frac{\theta(\theta-1)}{2}(1+t)^{-2}-\frac{\theta-\lambda}{2}(1+t)^{-\lambda-1}\Big](1+t)^\theta\psi^2dx$$
$$+\underbrace{\int_0^1 p(\psi+\hat{C})_x\left[2(1+t)^{\theta+\lambda}\psi_{xt}+(1+t)^\theta\psi_x\right]dx}_{J_1}$$
$$=\underbrace{-\int_0^1\psi^2\left[2(1+t)^{\theta+\lambda}\psi_t+(1+t)^\theta\psi\right]dx}_{J_2}\underbrace{-\int_0^1\psi_xe_x\left[2(1+t)^{\theta+\lambda}\psi_t+(1+t)^\theta\psi\right]dx}_{J_3}$$
$$\underbrace{-\int_0^1\left(\frac{\eta^2}{\psi+\hat{C}}\right)_x\left[2(1+t)^{\theta+\lambda}\psi_{xt}+(1+t)^\theta\psi_x\right]dx}_{J_4}. \quad (4.12)$$

Furthermore, $J_1,\ J_2,\ J_3$ and J_4 can be estimated as follows.

$$\begin{aligned}J_1=&\int_0^1 p'(\psi+\hat{C})\psi_x\big[2(1+t)^{\theta+\lambda}\psi_{xt}+(1+t)^{\theta}\psi_x\big]dx\\ \geq&\frac{d}{dt}\int_0^1(1+t)^{\theta+\lambda}p'(\psi+\hat{C})\psi_x^2dx+\int_0^1[1-(\theta+\lambda)(1+t)^{\lambda-1}](1+t)^{\theta}p'(\psi+\hat{C})\psi_x^2dx\\ &-C\varepsilon_2\int_0^1(1+t)^{\theta}\psi_x^2dx,\end{aligned}\tag{4.13}$$

and

$$J_2\leq C\varepsilon_2\int_0^1(1+t)^{\theta}(\psi_t^2+\psi^2)dx.\tag{4.14}$$

For any $t\geq T_0$, from (4.2) and Hölder inequality, one can get

$$J_3\leq C\varepsilon_2\int_0^1(1+t)^{\theta}(\psi_x^2+\psi^2)dx.\tag{4.15}$$

By $(2.5)_1$ and Hölder inequality, for any $t\geq T_0$, we can estimate J_4 as

$$\begin{aligned}J_4=&\int_0^1\left[\frac{2\eta\psi_t}{\psi+\hat{C}}+\frac{\eta^2\psi_x}{(\psi+\hat{C})^2}\right][2(1+t)^{\theta+\lambda}\psi_{xt}+(1+t)^{\theta}\psi_x]dx\\ \leq&\frac{d}{dt}\int_0^1(1+t)^{\theta+\lambda}\frac{\eta^2\psi_x^2}{(\psi+\hat{C})^2}dx+C\varepsilon_2\int_0^1(1+t)^{\theta}(\psi_t^2+\psi_x^2)dx.\end{aligned}\tag{4.16}$$

Putting (4.13)–(4.16) into (4.12), for any $t\geq T_0$, we have

$$\begin{aligned}&\frac{d}{dt}\int_0^1\left[(1+t)^{\theta}\psi\psi_t+(1+t)^{\theta+\lambda}\left(\psi_t^2+\hat{C}\psi^2+\left(p'(\psi+\hat{C})-\frac{\eta^2}{(\psi+\hat{C})^2}\right)\psi_x^2\right)\right.\\ &\quad\left.+\frac{1}{2}(1+t)^{\theta-\lambda}\psi^2-\frac{\theta}{2}(1+t)^{\theta-1}\psi^2\right]dx+\int_0^1\big[1-(\theta+\lambda)(1+t)^{\lambda-1}\big](1+t)^{\theta}\psi_t^2dx\\ &\quad+\int_0^1[1-(\theta+\lambda)(1+t)^{\lambda-1}](1+t)^{\theta}p'(\psi+\hat{C})\psi_x^2dx+\int_0^1\left[(1-(\theta+\lambda)(1+t)^{\lambda-1})\hat{C}\right.\\ &\quad\left.+\frac{\theta(\theta-1)}{2}(1+t)^{-2}-\frac{\theta-\lambda}{2}(1+t)^{-\lambda-1}\right](1+t)^{\theta}\psi^2dx\\ \leq&C\varepsilon_2\int_0^1(1+t)^{\theta}(\psi_t^2+\psi_x^2+\psi^2)dx.\end{aligned}\tag{4.17}$$

Let

$$\begin{aligned}F_1(x,t):=&(1+t)^{\theta}\psi\psi_t+(1+t)^{\theta+\lambda}\left(\psi_t^2+\hat{C}\psi^2+\left(p'(\psi+\hat{C})-\frac{\eta^2}{(\psi+\hat{C})^2}\right)\psi_x^2\right)\\ &+\frac{1}{2}(1+t)^{\theta-\lambda}\psi^2-\frac{\theta}{2}(1+t)^{\theta-1}\psi^2.\end{aligned}$$

Applying (4.10) and (4.11) to (4.17), one has

$$\frac{d}{dt}\int_0^1F_1(x,t)dx+c_1(1+t)^{\theta}\int_0^1(\psi_t^2+\psi_x^2+\psi^2)dx\leq 0,\quad t\geq T_0,$$

for some constant $c_1>0$. It follows from (4.1) and (4.9) that there exist two positive constants c_2 and c_3 such that

$$c_2(1+t)^{\theta+\lambda}(\psi_t^2+\psi_x^2+\psi^2)\le F_1(x,t)\le c_3(1+t)^{\theta+\lambda}(\psi_t^2+\psi_x^2+\psi^2),\quad t\ge T_0. \tag{4.18}$$

Thus, for any $t\ge T_0$, it holds

$$\frac{d}{dt}\int_0^1 F_1(x,t)dx+c_4(1+t)^{-\lambda}\int_0^1 F_1(x,t)dx\le 0.$$

Multiplying above inequality with $\mathrm{e}^{\gamma(1+t)^{1-\lambda}}$ leads to

$$\frac{d}{dt}\left(\mathrm{e}^{\gamma(1+t)^{1-\lambda}}\int_0^1 F_1(x,t)dx\right)\le 0,\qquad t\ge T_0, \tag{4.19}$$

where $\gamma=c_4/(1-\lambda)$. Integrating (4.19) over (T_0,t), we have

$$\begin{aligned}\int_0^1 F_1(x,t)dx\le&\mathrm{e}^{\gamma(1+T_0)^{1-\lambda}}\mathrm{e}^{-\gamma(1+t)^{1-\lambda}}\int_0^1 F_1(x,T_0)dx\\ \le& C\mathrm{e}^{\gamma(1+T_0)^{1-\lambda}}\mathrm{e}^{-\gamma(1+t)^{1-\lambda}}\beta^{\theta+2\lambda}\Psi_0^2,\end{aligned} \tag{4.20}$$

where we used (4.18) and (4.1). Then, (4.5) follows from (4.18) and (4.20). Furthermore, (4.6)–(4.8) can be derived from (4.5) and Lemma 4.1. ∎

LEMMA 4.3. *For $(\psi,\eta,e)\in X(T)$, under the conditions of Proposition 2.1, if (4.1) holds for all $t\ge 0$, then for any $t\ge T_0$, we have*

$$\int_0^1(\psi_{tt}^2+\psi_{xt}^2+\psi_{xx}^2+\psi_t^2)dx\le C\beta^{\theta+2\lambda}\mathrm{e}^{\nu_1(1+T_0)^{1-\lambda}}(1+t)^{-(\theta+\lambda)}\mathrm{e}^{-\nu_2(1+t)^{1-\lambda}}\Psi_0^2, \tag{4.21}$$

for some positive constants ν_1 and ν_2.

Proof. Since $0<\lambda<1$ and $t\ge T_0$, we can obtain the following inequalities:

$$(1+t)^{\lambda}\hat{C}-\frac{\theta+2\lambda}{2}(1+t)^{-1}\ge\frac14(1+t)^{\lambda}\hat{C}, \tag{4.22}$$

$$(\theta+\lambda)(1+t)^{\lambda-1}\le\lambda, \tag{4.23}$$

$$(1-\lambda)\hat{C}-\frac{\theta+2\lambda}{2}(1+t)^{-2}-\frac{\theta+\lambda}{2}(1+t)^{-\lambda-1}\ge\frac14(1-\lambda)\hat{C}. \tag{4.24}$$

Multiplying (3.28) by $\left[2(1+t)^{\theta+\lambda}\psi_{tt}+(1+t)^{\theta}\psi_t\right]$ and integrating the resulting equation with respect to x over $(0,1)$, we have

$$\begin{aligned}&\frac{d}{dt}\int_0^1\Big[(1+t)^{\theta}\psi_t\psi_{tt}+(1+t)^{\theta+\lambda}(\psi_{tt}^2+\hat{C}\psi_t^2)+\frac12(1+t)^{\theta-\lambda}\psi_t^2\\&-\frac{\theta+2\lambda}{2}(1+t)^{\theta-1}\psi_t^2\Big]dx+\int_0^1\Big[\Big(1-(\theta+\lambda)(1+t)^{\lambda-1}\Big)\hat{C}\\&+(\theta-1)\frac{\theta+2\lambda}{2}(1+t)^{-2}-\frac{\theta+\lambda}{2}(1+t)^{-\lambda-1}\Big](1+t)^{\theta}\psi_t^2dx\\&+\int_0^1\big[1-(\theta+\lambda)(1+t)^{\lambda-1}\big](1+t)^{\theta}\psi_{tt}^2dx\end{aligned}$$

$$
\begin{aligned}
&+\underbrace{\int_0^1 (p'(\psi+\hat{C})\psi_t)_x\big[2(1+t)^{\theta+\lambda}\psi_{xtt}+(1+t)^{\theta}\psi_{xt}\big]dx}_{J_5}\\
=&-\underbrace{\int_0^1 2\psi\psi_t\big[2(1+t)^{\theta+\lambda}\psi_{tt}+(1+t)^{\theta}\psi_t\big]dx}_{J_6}\\
&-\underbrace{\int_0^1 (\psi_{xt}e_x+\psi_x e_{xt})\big[2(1+t)^{\theta+\lambda}\psi_{tt}+(1+t)^{\theta}\psi_t\big]dx}_{J_7}\\
&-\underbrace{\int_0^1 \left(\frac{\eta^2}{\psi+\hat{C}}\right)_{xt}\big[2(1+t)^{\theta+\lambda}\psi_{xtt}+(1+t)^{\theta}\psi_{xt}\big]dx}_{J_8}. \qquad (4.25)
\end{aligned}
$$

Now we focus on estimating J_5, J_6, J_7 and J_8. By (4.23) and Hölder inequality, we can estimate J_5 as

$$
\begin{aligned}
J_5=&\int_0^1 (p'(\psi+\hat{C})\psi_{xt}+p''(\psi+\hat{C})\psi_x\psi_t)\big[2(1+t)^{\theta+\lambda}\psi_{xtt}+(1+t)^{\theta}\psi_{xt}\big]dx\\
\geq&\frac{d}{dt}\int_0^1 (1+t)^{\theta+\lambda}p'(\psi+\hat{C})\psi_{xt}^2dx+2\frac{d}{dt}\int_0^1 (1+t)^{\theta+\lambda}p''(\psi+\hat{C})\psi_x\psi_t\psi_{xt}dx\\
&+\int_0^1 \big[1-(\theta+\lambda)(1+t)^{\lambda-1}\big](1+t)^{\theta}p'(\psi+\hat{C})\psi_{xt}^2dx\\
&-C\varepsilon_2\int_0^1 (1+t)^{\theta}(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2)dx. \qquad (4.26)
\end{aligned}
$$

For any $t\geq T_0$, it follows from Hölder inequality and (4.2) that

$$
J_6\leq C\varepsilon_2\int_0^1 (1+t)^{\theta}(\psi_{tt}^2+\psi_t^2)dx, \qquad (4.27)
$$

$$
J_7\leq C\varepsilon_2\int_0^1 (1+t)^{\theta}(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2)dx. \qquad (4.28)
$$

For any $t\geq T_0$, J_8 can be estimated as

$$
\begin{aligned}
J_8=&\int_0^1 \left[\frac{2\eta_t\psi_t}{\psi+\hat{C}}+\frac{2\eta\psi_{tt}}{\psi+\hat{C}}-\frac{2\eta\psi_t^2}{(\psi+\hat{C})^2}+\frac{2\eta\eta_t\psi_x}{(\psi+\hat{C})^2}+\frac{\eta^2\psi_{xt}}{(\psi+\hat{C})^2}-\frac{2\eta^2\psi_x\psi_t}{(\psi+\hat{C})^3}\right]\\
&\cdot\big[2(1+t)^{\theta+\lambda}\psi_{xtt}+(1+t)^{\theta}\psi_{xt}\big]dx\\
\leq&\frac{d}{dt}\int_0^1 (1+t)^{\theta+\lambda}\frac{\eta^2\psi_{xt}^2}{(\psi+\hat{C})^2}dx+4\frac{d}{dt}\int_0^1 (1+t)^{\theta+\lambda}\frac{\eta\eta_t\psi_x\psi_{xt}}{(\psi+\hat{C})^2}dx\\
&-4\frac{d}{dt}\int_0^1 (1+t)^{\theta+\lambda}\frac{\eta^2\psi_t\psi_x\psi_{xt}}{(\psi+\hat{C})^3}dx+C\varepsilon_2\int_0^1 (1+t)^{\lambda}\eta_{tt}^2dx\\
&+C\varepsilon_2\int_0^1 (1+t)^{\theta}(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2+\psi_x^2)dx\\
\leq&\frac{d}{dt}\int_0^1 (1+t)^{\theta+\lambda}\frac{\eta^2\psi_{xt}^2}{(\psi+\hat{C})^2}dx+4\frac{d}{dt}\int_0^1 (1+t)^{\theta+\lambda}\frac{\eta\eta_t\psi_x\psi_{xt}}{(\psi+\hat{C})^2}dx
\end{aligned}
$$

$$
\begin{aligned}
&-4\frac{d}{dt}\int_0^1(1+t)^{\theta+\lambda}\frac{\eta^2\psi_t\psi_x\psi_{xt}}{(\psi+\hat{C})^3}dx+C\varepsilon_2\int_0^1(1+t)^{\theta}(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2)dx\\
&+C\varepsilon_2(1+t)^{\theta}\int_0^1(\psi_x^2+\psi^2)dx+C\varepsilon_2(1+t)^{-\lambda}\int_0^1\eta_t^2dx+C\varepsilon_2(1+t)^{-3\lambda}\int_0^1\eta^2dx\\
\leq&\frac{d}{dt}\int_0^1(1+t)^{\theta+\lambda}\frac{\eta^2\psi_{xt}^2}{(\psi+\hat{C})^2}dx+4\frac{d}{dt}\int_0^1(1+t)^{\theta+\lambda}\frac{\eta\eta_t\psi_x\psi_{xt}}{(\psi+\hat{C})^2}dx\\
&-4\frac{d}{dt}\int_0^1(1+t)^{\theta+\lambda}\frac{\eta^2\psi_t\psi_x\psi_{xt}}{(\psi+\hat{C})^3}dx+C\varepsilon_2\int_0^1(1+t)^{\theta}(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2)dx\\
&+C\varepsilon_2\beta^{\theta+2\lambda}\mathrm{e}^{\mu_1(1+T_0)^{1-\lambda}}\Psi_0^2(1+t)^{-\lambda}\mathrm{e}^{-\mu_2(1+t)^{1-\lambda}},
\end{aligned}\tag{4.29}
$$

where we used the fact that

$$
\begin{aligned}
\int_0^1\eta_{tt}^2dx\leq&C\int_0^1(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2+\psi_x^2+\psi^2)dx\\
&+C(1+t)^{-2\lambda}\int_0^1\eta_t^2dx+C(1+t)^{-4\lambda}\int_0^1\eta^2dx.
\end{aligned}\tag{4.30}
$$

Here, the proof of the inequality (4.30) is similar to (3.27) in Lemma 3.3, and is omitted.

Substituting (4.26)–(4.29) into (4.25), for any $t\geq T_0$, one gets

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1\Bigg[(1+t)^{\theta}\psi_t\psi_{tt}+(1+t)^{\theta+\lambda}\Bigg(\psi_{tt}^2+\hat{C}\psi_t^2+\Bigg(p'(\psi+\hat{C})-\frac{\eta^2}{(\psi+\hat{C})^2}\Bigg)\psi_{xt}^2\Bigg)\\
&+\frac{1}{2}(1+t)^{\theta-\lambda}\psi_t^2-\frac{\theta+2\lambda}{2}(1+t)^{\theta-1}\psi_t^2\Bigg]dx+4\frac{d}{dt}\int_0^1(1+t)^{\theta+\lambda}\frac{\eta^2\psi_t\psi_x\psi_{xt}}{(\psi+\hat{C})^3}dx\\
&-4\frac{d}{dt}\int_0^1(1+t)^{\theta+\lambda}\frac{\eta\eta_t\psi_x\psi_{xt}}{(\psi+\hat{C})^2}dx+2\frac{d}{dt}\int_0^1(1+t)^{\theta+\lambda}p''(\psi+\hat{C})\psi_x\psi_t\psi_{xt}dx\\
&+\int_0^1\big[1-(\theta+\lambda)(1+t)^{\lambda-1}\big](1+t)^{\theta}\psi_{tt}^2dx+\int_0^1\Bigg[\big(1-(\theta+\lambda)(1+t)^{\lambda-1}\big)\hat{C}\\
&+(\theta-1)\frac{\theta+2\lambda}{2}(1+t)^{-2}-\frac{\theta+\lambda}{2}(1+t)^{-\lambda-1}\Bigg](1+t)^{\theta}\psi_t^2dx\\
&+\int_0^1\big[1-(\theta+\lambda)(1+t)^{\lambda-1}\big](1+t)^{\theta}p'(\psi+\hat{C})\psi_{xt}^2dx\\
\leq&C\varepsilon_2\int_0^1(1+t)^{\theta}(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2)dx\\
&+C\varepsilon_2\beta^{\theta+2\lambda}\mathrm{e}^{\mu_1(1+T_0)^{1-\lambda}}(1+t)^{-\lambda}\mathrm{e}^{-\mu_2(1+t)^{1-\lambda}}\Psi_0^2.
\end{aligned}\tag{4.31}
$$

Next, we multiply (3.25) by $2\eta_t$ and integrate the resulting equation with respect to x over $(0,1)$ to obtain, for any $t\geq T_0$,

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1\eta_t^2dx+2\int_0^1(1+t)^{-\lambda}\eta_t^2dx\\
=&2\lambda\int_0^1(1+t)^{-\lambda-1}\eta\eta_tdx+2\int_0^1(\psi+\hat{C})e_{xt}\eta_tdx+2\int_0^1\psi_te_x\eta_tdx\\
&-2\int_0^1(p'(\psi+\hat{C})\psi_t)_x\eta_tdx-2\int_0^1\left(\frac{\eta^2}{\psi+\hat{C}}\right)_{xt}\eta_tdx
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1-\lambda}{8}\int_0^1(1+t)^\lambda p'(\psi+\hat{C})\psi_{xt}^2dx+C\varepsilon_2\int_0^1(1+t)^\lambda(\psi_{tt}^2+\psi_{xt}^2)dx\\
&\quad +C\int_0^1(1+t)^\lambda(\psi_t^2+\psi_x^2)dx+C\int_0^1(1+t)^{-\lambda}\eta_t^2dx+C\int_0^1(1+t)^{-\lambda-2}\eta^2dx\\
&\leq \frac{1-\lambda}{8}\int_0^1(1+t)^\theta p'(\psi+\hat{C})\psi_{xt}^2dx+C\varepsilon_2\int_0^1(1+t)^\theta(\psi_{tt}^2+\psi_{xt}^2)dx\\
&\quad +C\beta^{\theta+2\lambda}\mathrm{e}^{\mu_1(1+T_0)^{1-\lambda}}(1+t)^{-\lambda}\mathrm{e}^{-\mu_2(1+t)^{1-\lambda}}\Psi_0^2.
\end{aligned}\tag{4.32}
$$

Adding (4.32) to (4.31), for any $t\geq T_0$, one gets

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1\Big[(1+t)^\theta\psi_t\psi_{tt}+(1+t)^{\theta+\lambda}\Big(\psi_{tt}^2+\hat{C}\psi_t^2+\Big(p'(\psi+\hat{C})-\frac{\eta^2}{(\psi+\hat{C})^2}\Big)\psi_{xt}^2\Big)\\
&\quad+\frac{1}{2}(1+t)^{\theta-\lambda}\psi_t^2-\frac{\theta+2\lambda}{2}(1+t)^{\theta-1}\psi_t^2\Big]dx+4\frac{d}{dt}\int_0^1(1+t)^{\theta+\lambda}\frac{\eta^2\psi_t\psi_x\psi_{xt}}{(\psi+\hat{C})^3}dx\\
&\quad-4\frac{d}{dt}\int_0^1(1+t)^{\theta+\lambda}\frac{\eta\eta_t\psi_x\psi_{xt}}{(\psi+\hat{C})^2}dx+2\frac{d}{dt}\int_0^1(1+t)^{\theta+\lambda}p''(\psi+\hat{C})\psi_x\psi_t\psi_{xt}dx\\
&\quad+\frac{d}{dt}\int_0^1\eta_t^2dx+2\int_0^1(1+t)^{-\lambda}\eta_t^2dx+\int_0^1\big[1-(\theta+\lambda)(1+t)^{\lambda-1}\big](1+t)^\theta\psi_{tt}^2dx\\
&\quad+\int_0^1\big[1-(\theta+\lambda)(1+t)^{\lambda-1}\big](1+t)^\theta p'(\psi+\hat{C})\psi_{xt}^2dx+\int_0^1\Big[\big(1-(\theta+\lambda)(1+t)^{\lambda-1}\big)\hat{C}\\
&\quad+(\theta-1)\frac{\theta+2\lambda}{2}(1+t)^{-2}-\frac{\theta+\lambda}{2}(1+t)^{-\lambda-1}\Big](1+t)^\theta\psi_t^2dx\\
&\leq\frac{1-\lambda}{8}\int_0^1(1+t)^\theta p'(\psi+\hat{C})\psi_{xt}^2dx+C\varepsilon_2\int_0^1(1+t)^\theta(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2)dx\\
&\quad+C\beta^{\theta+2\lambda}\mathrm{e}^{\mu_1(1+T_0)^{1-\lambda}}(1+t)^{-\lambda}\mathrm{e}^{-\mu_2(1+t)^{1-\lambda}}\Psi_0^2.
\end{aligned}\tag{4.33}
$$

Let

$$
\begin{aligned}
F_2(x,t):=&(1+t)^\theta\psi_t\psi_{tt}+(1+t)^{\theta+\lambda}\Big(\psi_{tt}^2+\hat{C}\psi_t^2+\Big(p'(\psi+\hat{C})-\frac{\eta^2}{(\psi+\hat{C})^2}\Big)\psi_{xt}^2\Big)\\
&+\frac{1}{2}(1+t)^{\theta-\lambda}\psi_t^2-\frac{\theta+2\lambda}{2}(1+t)^{\theta-1}\psi_t^2+4(1+t)^{\theta+\lambda}\frac{\eta^2\psi_t\psi_x\psi_{xt}}{(\psi+\hat{C})^3}\\
&-4(1+t)^{\theta+\lambda}\frac{\eta\eta_t\psi_x\psi_{xt}}{(\psi+\hat{C})^2}+2(1+t)^{\theta+\lambda}p''(\psi+\hat{C})\psi_x\psi_t\psi_{xt}+\eta_t^2.
\end{aligned}
$$

Applying (4.23) and (4.24) to (4.33) yields

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1F_2(x,t)dx+c_5\int_0^1\big[(1+t)^\theta(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2)+(1+t)^{-\lambda}\eta_t^2\big]dx\\
&\leq C\beta^{\theta+2\lambda}\mathrm{e}^{\mu_1(1+T_0)^{1-\lambda}}(1+t)^{-\lambda}\mathrm{e}^{-\mu_2(1+t)^{1-\lambda}}\Psi_0^2,\quad t\geq T_0,
\end{aligned}
$$

for some positive constant c_5. Thanks to (4.1) and (4.22), there exist two positive constants c_6 and c_7 such that, for any $t\geq T_0$,

$$
c_6\big[(1+t)^{\theta+\lambda}(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2)+\eta_t^2\big]\leq F_2(x,t)\leq c_7\big[(1+t)^{\theta+\lambda}(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2)+\eta_t^2\big].\tag{4.34}
$$

Then, for any $t \geq T_0$, it holds

$$\frac{d}{dt}\int_0^1 F_2(x,t)dx + c_8(1+t)^{-\lambda}\int_0^1 F_2(x,t)dx \leq C\beta^{\theta+2\lambda}\mathrm{e}^{\mu_1(1+T_0)^{1-\lambda}}(1+t)^{-\lambda}\mathrm{e}^{-\mu_2(1+t)^{1-\lambda}}\Psi_0^2.$$

Multiplying above inequality by $\mathrm{e}^{\delta(1+t)^{1-\lambda}}$ gives

$$\frac{d}{dt}\left(\mathrm{e}^{\delta(1+t)^{1-\lambda}}\int_0^1 F_2(x,t)dx\right) \leq C\beta^{\theta+2\lambda}\mathrm{e}^{\mu_1(1+T_0)^{1-\lambda}}(1+t)^{-\lambda}\mathrm{e}^{(\delta-\mu_2)(1+t)^{1-\lambda}}\Psi_0^2,\ t \geq T_0,$$

where $\delta = c_8/(1-\lambda)$. Integrating the above inequality over (T_0,t), one has

$$\begin{aligned}\int_0^1 F_2(x,t)dx \leq & \mathrm{e}^{\delta(1+T_0)^{1-\lambda}}\mathrm{e}^{-\delta(1+t)^{1-\lambda}}\int_0^1 F_2(x,T_0)dx \\ & + C\beta^{\theta+2\lambda}\mathrm{e}^{\mu_1(1+T_0)^{1-\lambda}}\mathrm{e}^{-\delta(1+t)^{1-\lambda}}\int_{T_0}^t (1+s)^{-\lambda}\mathrm{e}^{(\delta-\mu_2)(1+s)^{1-\lambda}}ds\Psi_0^2 \\ \leq & C\mathrm{e}^{\delta(1+T_0)^{1-\lambda}}\mathrm{e}^{-\delta(1+t)^{1-\lambda}}\int_0^1 \big[(1+T_0)^{\theta+\lambda}(\psi_{tt}^2+\psi_{xt}^2+\psi_t^2)(x,T_0) \\ & + \eta_t^2(x,T_0)\big]dx + C\beta^{\theta+2\lambda}\mathrm{e}^{\mu_1(1+T_0)^{1-\lambda}}\mathrm{e}^{-\nu_2(1+t)^{1-\lambda}}\Psi_0^2 \\ \leq & C\beta^{\theta+2\lambda}\mathrm{e}^{\nu_1(1+T_0)^{1-\lambda}}\mathrm{e}^{-\nu_2(1+t)^{1-\lambda}}\Psi_0^2,\end{aligned} \tag{4.35}$$

where $\nu_1 = \max\{\delta,\mu_1\}$, $\nu_2 = \min\{\delta,\mu_2\}$ and we used (4.34), (4.1) and

$$\int_0^1 \psi_{tt}^2(x,T_0)dx \leq C\int_0^1 (\psi_{xx}^2+\psi_{xt}^2+\psi_t^2+\psi_x^2+\psi^2)(x,T_0)dx.$$

Then, it follows from (4.34) and (4.35) that

$$\int_0^1 (\psi_{tt}^2+\psi_{xt}^2+\psi_t^2)dx \leq C\beta^{\theta+2\lambda}\mathrm{e}^{\nu_1(1+T_0)^{1-\lambda}}(1+t)^{-(\theta+\lambda)}\mathrm{e}^{-\nu_2(1+t)^{1-\lambda}}\Psi_0^2,\ t \geq T_0. \tag{4.36}$$

Analogous to Lemma 3.3, we can estimate $\int_0^1 \psi_{xx}^2 dx$ by using (4.5) and (4.36):

$$\int_0^1 \psi_{xx}^2 dx \leq C\beta^{\theta+2\lambda}\mathrm{e}^{\nu_1(1+T_0)^{1-\lambda}}(1+t)^{-(\theta+\lambda)}\mathrm{e}^{-\nu_2(1+t)^{1-\lambda}}\Psi_0^2, \quad t \geq T_0. \tag{4.37}$$

We finally get the estimate (4.21) by combining (4.36) and (4.37). □

Proof. **(Proof of Proposition 2.2.)** Lemmas 4.2 and 4.3 imply Proposition 2.2. □

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Appendix. This section is devoted to showing the reason for the restriction of $\mathcal{C}(x)=\hat{C}$.

Set

$$\tilde{\psi}=\rho-\bar{\rho},\quad \tilde{\eta}=j-\bar{j}=j,\quad \tilde{e}=\phi-\bar{\phi},$$

then from (1.2)–(1.4) and (2.1)–(2.2), we can deduce that $(\tilde{\psi},\tilde{\eta},\tilde{e})$ satisfy the following perturbation system

$$\begin{cases}\tilde{\psi}_t+\tilde{\eta}_x=0,\\ \tilde{\eta}_t+\left(\dfrac{\tilde{\eta}^2}{\tilde{\psi}+\bar{\rho}}\right)_x+(p(\tilde{\psi}+\bar{\rho})-p(\bar{\rho}))_x=(\tilde{\psi}+\bar{\rho})\tilde{e}_x+\tilde{\psi}\bar{\phi}_x-\dfrac{\tilde{\eta}}{(1+t)^{\lambda}},\\ \tilde{e}_{xx}=\tilde{\psi},\end{cases}\tag{A.1}$$

with the initial value and boundary value conditions

$$\begin{cases}\tilde{\psi}_0(x):=\tilde{\psi}(x,0)=\rho_0(x)-\bar{\rho}(x),\quad \tilde{\eta}_0(x):=\tilde{\eta}(x,0)=j_0(x),\quad x\in(0,1),\\ \tilde{\psi}(0,t)=\tilde{\psi}(1,t)=0,\quad \tilde{e}(0,t)=\tilde{e}(1,t)=0,\quad t\geq 0.\end{cases}\tag{A.2}$$

Differentiating $(A.1)_2$ with respect to x and by $(A.1)_1$ and $(A.1)_3$, one has

$$\begin{aligned}&\tilde{\psi}_{tt}+(1+t)^{-\lambda}\tilde{\psi}_t+\bar{\rho}\tilde{\psi}-(p(\tilde{\psi}+\bar{\rho})-p(\bar{\rho}))_{xx}\\ &=-(\bar{\phi}_{xx}+\tilde{\psi})\tilde{\psi}-\bar{\phi}_x\tilde{\psi}_x-(\tilde{\psi}+\bar{\rho})_x\tilde{e}_x+\left(\frac{\tilde{\eta}^2}{\tilde{\psi}+\bar{\rho}}\right)_{xx}.\end{aligned}\tag{A.3}$$

Multiplying (A.3) by $\left[2(1+t)^{\alpha}\tilde{\psi}_t+(1+t)^{\beta}\tilde{\psi}\right]$ for two constants $\alpha,\beta>0$, and integrating the resulting equation with respect to x over $(0,1)$ by parts, we get

$$\begin{aligned}&\frac{d}{dt}\int_0^1\left[(1+t)^{\beta}\tilde{\psi}\tilde{\psi}_t+(1+t)^{\alpha}(\tilde{\psi}_t^2+\bar{\rho}\tilde{\psi}^2)+\frac{1}{2}(1+t)^{\beta-\lambda}\tilde{\psi}^2-\frac{\beta}{2}(1+t)^{\beta-1}\tilde{\psi}^2\right]dx\\ &\quad+\int_0^1\left[(1+t)^{\beta}\bar{\rho}-\alpha(1+t)^{\alpha-1}\bar{\rho}-\frac{1}{2}(\beta-\lambda)(1+t)^{\beta-\lambda-1}+\frac{\beta}{2}(\beta-1)(1+t)^{\beta-2}\right]\tilde{\psi}^2dx\\ &\quad+\int_0^1\left[2(1+t)^{\alpha-\lambda}-(1+t)^{\beta}-\alpha(1+t)^{\alpha-1}\right]\tilde{\psi}_t^2dx\\ &\quad+\int_0^1(p(\tilde{\psi}+\bar{\rho})-p(\bar{\rho}))_x\left[2(1+t)^{\alpha}\tilde{\psi}_{xt}+(1+t)^{\beta}\tilde{\psi}_x\right]dx\\ &=-\int_0^1\left(\frac{\tilde{\eta}^2}{\tilde{\psi}+\bar{\rho}}\right)_x\left[2(1+t)^{\alpha}\tilde{\psi}_{xt}+(1+t)^{\beta}\tilde{\psi}_x\right]dx\\ &\quad-\int_0^1(1+t)^{\beta}\tilde{\psi}\left[(\bar{\phi}_{xx}+\tilde{\psi})\tilde{\psi}+\bar{\phi}_x\tilde{\psi}_x+(\tilde{\psi}+\bar{\rho})_x\tilde{e}_x\right]dx\\ &\quad-\int_0^1 2(1+t)^{\alpha}\tilde{\psi}_t\left[(\bar{\phi}_{xx}+\tilde{\psi})\tilde{\psi}+\bar{\phi}_x\tilde{\psi}_x+(\tilde{\psi}+\bar{\rho})_x\tilde{e}_x\right]dx,\end{aligned}$$

the term $\int_0^1(p(\tilde{\psi}+\bar{\rho})-p(\bar{\rho}))_x[2(1+t)^{\alpha}\tilde{\psi}_{xt}+(1+t)^{\beta}\tilde{\psi}_x]dx$ can be written as

$$\int_0^1(p(\tilde{\psi}+\bar{\rho})-p(\bar{\rho}))_x[2(1+t)^{\alpha}\tilde{\psi}_{xt}+(1+t)^{\beta}\tilde{\psi}_x]dx$$

$$
\begin{aligned}
&= \frac{d}{dt}\int_0^1 (1+t)^\alpha p'(\tilde{\psi}+\bar{\rho})\tilde{\psi}_x^2 dx + 2\frac{d}{dt}\int_0^1 (1+t)^\alpha (p'(\tilde{\psi}+\bar{\rho})-p'(\bar{\rho}))\bar{\rho}_x\tilde{\psi}_x dx \\
&\quad + \int_0^1 [(1+t)^\beta - \alpha(1+t)^{\alpha-1}]p'(\tilde{\psi}+\bar{\rho})\tilde{\psi}_x^2 dx - \int_0^1 (1+t)^\alpha p''(\tilde{\psi}+\bar{\rho})\tilde{\psi}_t\tilde{\psi}_x(\tilde{\psi}_x+2\bar{\rho}_x)dx \\
&\quad + \int_0^1 [(1+t)^\beta - 2\alpha(1+t)^{\alpha-1}](p'(\tilde{\psi}+\bar{\rho})-p'(\bar{\rho}))\bar{\rho}_x\tilde{\psi}_x dx.
\end{aligned}
$$

We need to choose α and β satisfying $\beta+\lambda<\alpha<\beta+1$ to ensure that the following inequalities hold for all $t\geq 0$,

$$
\begin{cases}
2(1+t)^{\alpha-\lambda} - (1+t)^\beta - \alpha(1+t)^{\alpha-1} \geq C_0(1+t)^{\alpha-\lambda}, \\
(1+t)^\beta - \alpha(1+t)^{\alpha-1} \geq C_0(1+t)^\beta,
\end{cases}
$$

where $C_0>0$ is some constant. However, in order to estimate $\int_0^1(1+t)^\alpha\bar{\phi}_x\tilde{\psi}_t\tilde{\psi}_x dx$, we need to choose α and β satisfying $\alpha\leq\beta-\lambda$, which contradicts with $\beta+\lambda<\alpha<\beta+1$ due to $0<\lambda<1$. Thus, we expect that $\bar{\phi}_x=0$ which can be deduced by $\mathcal{C}(x)\equiv$constant.

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