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RELAXATION IN A KELLER-SEGEL-CONSUMPTION SYSTEM INVOLVING SIGNAL-DEPENDENT MOTILITIES*

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Abstract. Two relaxation features of the migration-consumption chemotaxis system involving signal-dependent motilities,

$$\begin{cases} u_t = \Delta\big(u\phi(v)\big), \\ v_t = \Delta v - uv, \end{cases} \qquad (\star)$$

are studied in smoothly bounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 1$: It is shown that if $\phi \in C^0([0,\infty))$ is positive on $[0,\infty)$, then for any initial data (u_0,v_0) belonging to the space $(C^0(\overline{\Omega}))^\star \times L^\infty(\Omega)$ an associated no-flux type initial-boundary value problem admits a global very weak solution. Beyond this initial relaxation property, it is seen that under the additional hypotheses that $\phi \in C^1([0,\infty))$ and $n \leq 3$, each of these solutions stabilizes toward a semi-trivial spatially homogeneous steady state in the large time limit.

By thus applying to irregular and partially even measure-type initial data of arbitrary size, this firstly extends previous results on global solvability in $(\star)$ which have been restricted to initial data not only considerably more regular but also suitably small. Secondly, this reveals a significant difference between the large time behavior in $(\star)$ and that in related degenerate counterparts involving functions ϕ with $\phi(0)=0$, about which, namely, it is known that some solutions may asymptotically approach nonhomogeneous states.

Keywords. Chemotaxis; instantaneous regularization; large time behavior.

AMS subject classifications. 35B40 (primary); 35D30, 35K55, 35Q92, 92C17 (secondary).

1. Introduction

Chemotaxis systems accounting for local sensing mechanisms have attracted increased interest in the recent literature. When viewed in the context of general Keller-Segel type systems ([23]) of the form

$$\begin{cases} u_t = \nabla\cdot\big(D(u,v)\nabla u - \chi(u,v)u\nabla v\big), \\ v_t = \Delta v + K(u,v), \end{cases} \tag{1.1}$$

setting a corresponding focus amounts to assuming the diffusivity D and the cross-diffusion rate χ to be linked through the relations

$$D(u,v) = \phi(v) \qquad \text{and} \qquad \chi(u,v) = \phi'(v), \tag{1.2}$$

with some nonnegative function ϕ exhibiting suitable decay at large values of the signal concentration v in order to reflect the local character of sensing ([10, 26]); typical examples thus include

$$\phi(\xi) = \frac{1}{(\xi+a)^\alpha} \qquad \text{or also} \qquad \phi(\xi) = e^{-\beta\xi} \tag{1.3}$$

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for $\xi \ge 0$, with $\alpha > 0, a \ge 0$ and $\beta > 0$. For accordingly obtained versions of (1.1) that address situations in which the considered signal is produced by cells, remarkably comprehensive knowledge has been achieved in several respects. Namely, in the case when $K(u,v) = u - v$, throughout considerably large classes of the key ingredient ϕ to (1.1)-(1.2), the literature meanwhile provides not only far-reaching results on global solvability ([1, 5, 7, 13, 14, 37]; cf. also [18, 19, 21, 27–29, 38, 45] and [30] for corresponding studies on some close relatives), but also on large time asymptotics, and especially on the identification of situations in which either stabilization toward equilibria can be observed, or, alternatively, infinite-time blow-up occurs ([8, 11, 12, 14, 20, 37]). In particular, it has been found that within this class of chemotaxis-production systems, appropriate choices of the form in (1.3) lead to a substantial support of spatial structures in the sense either of large-time singularity formation ([12, 14, 20]), or at least of heterogeneous long-term aymptotics trivially exhibited by non-constant steady state solutions ([8]). These observations are quite in line with numerous findings on emergence and stabilization of singular structures ([4, 6, 17, 31, 33, 34]), and supplementary also of more subtle destabilization of spatial homogeneity ([40]), in various further examples among the chemotaxis-production versions of (1.1) with constituents more general than in (1.2).

In contrast to this, the present study now focuses on contexts in which chemotactic motion based on local sensing is directed by a cue that is consumed by individuals, rather than produced. By describing signal absorption in the apparently most standard functional form, a resulting chemotaxis-consumption version of (1.1)-(1.2) becomes

$$\begin{cases} u_t = \Delta\big(u\phi(v)\big), \\ v_t = \Delta v - uv, \end{cases} \tag{1.4}$$

and in stark difference to the situations discussed above, questions related to the evolution of structures in the above sense seem to have remained widely unaddressed in such frameworks. Indeed, the only result concerned with global solutions to a problem of this form in high-dimensional settings, as recently achieved in [24], relies on a smallness restriction on the initial data to assert global classical solvability and large time convergence to homogeneous states in non-degenerate cases in which ϕ is strictly positive throughout $[0,\infty)$. Large-data solutions appear to have been constructed only in spatially one- and two-dimensional domains in [44], where a focus has been on a degenerate version in which $\phi > 0$ on $(0,\infty)$ but $\phi(0) = 0$; in such cases, for suitably regular initial data some global smooth solutions are found to exist and to approach a steady state $(u_\infty, 0)$ in the large time limit, with u_∞ known to be nonconstant whenever $v|_{t=0}$ is appropriately small ([44]).

Main results. One purpose of the present study is to make sure that this latter observation, paralleling similar findings on the existence of non-constant large-time patterns in related taxis-consumption systems of the form (1.1) with signal-dependent motility degeneracies ([43]), cannot be made in any non-degenerate version of (1.4). Beyond asserting this absence of structure support on large time scales, however, a second goal will consist in identifying a second pattern-counteracting feature of (1.4) which will become manifest in a result on instantaneous relaxation of solutions emanating from considerably irregular initial data.

More precisely, in a smoothly bounded domain $\Omega \subset \mathbb{R}^n$, $n \ge 1$, let us consider the initial-boundary value problem

$$\begin{cases} u_t = \Delta\big(u\phi(v)\big), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \\ \nabla\big(u\phi(v)\big) \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.5}$$

and first concentrate on the latter question on initial relaxation by deriving a result on global solvability under mild requirements on data regularity which inter alia even allow for measure-valued first components of the initial distributions. In formulating this and throughout the sequel, we let $C^0_{w-\star}([0,\infty);(C^0(\overline{\Omega}))^\star)$ and $C^0_{w-\star}([0,\infty);L^\infty(\Omega))$ denote the spaces of functions which are continuous on $[0,\infty)$ with respect to the weak-$\star$ topology in $(C^0(\overline{\Omega}))^\star$ and $L^\infty(\Omega)$, respectively.

Specifically, the first of our main results indeed reveals that even such singular initial settings undergo an immediate relaxation into globally existing solutions to (1.5) which belong to $L^2(\Omega)\times W^{2,2}(\Omega)$ at a.e. positive time, under the mere assumption that ϕ be continuous and positive:

THEOREM 1.1. *Let $n\ge 1$ and $\Omega\subset\mathbb{R}^n$ be a bounded domain with smooth boundary, and suppose that*

$$\phi\in C^0([0,\infty)) \quad \textit{is such that} \quad \phi(\xi)>0 \textit{ for all } \xi\ge 0, \tag{1.6}$$

and that

$$u_0\in (C^0(\overline{\Omega}))^\star \textit{ as well as } v_0\in L^\infty(\Omega) \quad \textit{are nonnegative.} \tag{1.7}$$

Then there exist nonnegative functions

$$\begin{cases} u\in C^0_{w-\star}([0,\infty);(C^0(\overline{\Omega}))^\star)\cap L^\infty((0,\infty);L^1(\Omega))\cap L^2_{loc}(\overline{\Omega}\times(0,\infty)) \qquad \textit{and} \\ v\in C^0_{w-\star}([0,\infty;L^\infty(\Omega))\cap L^\infty(\Omega\times(0,\infty)) \\ \qquad\qquad \cap L^2_{loc}((0,\infty);W^{2,2}(\Omega))\cap L^4_{loc}((0,\infty);W^{1,4}(\Omega))\cap L^\infty_{loc}((0,\infty);W^{1,2}(\Omega)) \end{cases} \tag{1.8}$$

which are such that (u,v) forms a global very weak solution of (1.5) in the sense of Definition 2.1.

REMARK 1.1.

(i) In Keller-Segel-production systems of the form (1.1) with $D\equiv 1$ and $\chi\equiv 1$ as well as $K(u,v)=u-v$, available existence results including measure-type initial data seem restricted to one-dimensional and certain subcritical-mass two-dimensional settings ([3,32,42]; cf. also [16]). Only under the influence of certain additional superlinear zero order degradation mechanisms of logistic type, results on instantaneous smoothing of comparably strong singularities seem to have been established in the literature ([22,41]).

(ii) Even for the classical chemotaxis-consumption version of (1.1) given by

$$\begin{cases} u_t=\Delta u-\nabla\cdot(u\nabla v), \\ v_t=\Delta v-uv, \end{cases} \tag{1.9}$$

the apparently only existence result covering large measures as initial population distributions is limited to planar and radially symmetric frameworks ([39]).

(iii) Upon imposing further restrictions on the regularity of ϕ and the initial data, higher regularity features of the obtained solutions can be derived. Pursuing this in detail would go beyond the scope of the present study, however, and will be addressed in [25].

Now in the presence of slightly more regular coefficient functions ϕ, a second relaxation effect, unconditional with respect to the initial data from the above class and especially independent of their size, can be observed on large time scales. In addressing this second main objective of this study, we let A denote the realization of $-\Delta$

under homogeneous Neumann boundary conditions in $L^2_\perp(\Omega):=\{\varphi\in L^2(\Omega)\mid \int_\Omega\varphi=0\}$, with its domain given by $W^{2,2}_N(\Omega)\cap L^2_\perp(\Omega)$, where for $p\in[1,\infty]$ we set $W^{2,p}_N(\Omega):=\{\varphi\in W^{2,p}(\Omega)\mid \frac{\partial\varphi}{\partial\nu}=0$ on $\partial\Omega\}$; furthermore, with regard to spatial averages we will refer to the notation $\overline{\varphi}:=\frac{1}{|\Omega|}\varphi(\mathbf{1}_\Omega)$ for $\varphi\in(C^0(\overline{\Omega}))^\star$, which reduces to the identity $\overline{\varphi}=\frac{1}{|\Omega|}\int_\Omega\varphi$ whenever $\varphi\in L^1(\Omega)$.

The second of our main results can thereby be formulated as a statement on long-term stabilization toward homogeneous states in an appropriate topological setting:

THEOREM 1.2. *Let $n\in\{1,2,3\}$ and $\Omega\subset\mathbb{R}^n$ be a smoothly bounded domain, and assume that*

$$\phi\in C^1([0,\infty)) \quad \textit{is such that} \quad \phi(\xi)>0 \textit{ for all } \xi\geq 0, \tag{1.10}$$

and that (1.7) holds. Then one can find a global very weak solution (u,v) of (1.5) which satisfies (1.8) and for which there exists a null set $N\subset(0,\infty)$ such that $A^{-\frac12}\big(u(\cdot,t)-\overline{u}_0\big)\in L^2(\Omega)$ for all $t\in(0,\infty)\setminus N$, and that

$$A^{-\frac12}\big(u(\cdot,t)-\overline{u}_0\big)\to 0 \quad \textit{in } L^2(\Omega) \qquad \textit{as } (0,\infty)\setminus N\ni t\to\infty \tag{1.11}$$

and

$$v(\cdot,t)\to 0 \quad \textit{in } L^\infty(\Omega) \qquad \textit{as } (0,\infty)\setminus N\ni t\to\infty. \tag{1.12}$$

REMARK 1.2.

(i) While marking a sharp contrast to the mentioned findings on persistently non-homogeneous behavior in corresponding chemotaxis-production versions of (1.1)-(1.2) ([8, 12, 14])), the outcome of Theorem 1.2 rather parallels known results on large time convergence to constant equilibria in the Keller-Segel consumption system (1.9) ([35]). How far asymptotic homogenization properties of this style indeed constitute a common feature of taxis-absorption interaction involving non-degenerate diffusion, however, is to be addressed in forthcoming studies.

Main ideas. In line with the particular structure distinguishing (1.4) from (1.9), at the core of our reasoning will be a duality-based argument related to the behavior of

$$\int_\Omega\left|A^{-\frac12}(u-\overline{u}_0)\right|^2 \tag{1.13}$$

along trajectories in suitably regularized variants of (1.5) (cf. (2.6)). Here, in a fundamental inequality describing the evolution thereof (Lemma 3.3), in the general setting of Theorem 1.1 a fairly rough estimation of a corresponding exciting contribution is sufficient to turn this into a quasi-energy inequality (Lemma 3.4). Again thanks to the favorable structure of (1.4), the a priori information thereby obtained in time intervals of the form (τ,∞) for arbitrary $\tau>0$ can be combined with a time-independent $(W^{2,\infty}_N(\Omega))^\star$-valued boundedness feature of both time derivatives in (1.5) (Lemma 3.5), allowing for suitable control of the solution behavior near $t=0$, to establish Theorem 1.1 in Section 3.

Section 4 will thereafter reveal that in the low-dimensional and slightly more regular context of Theorem 1.2, the forcing contribution to the evolution of the functional in (1.13) can actually even be controlled in terms of suitably dissipated quantities (Lemmata 4.1 and 4.2). An accordingly discovered energy feature will hence form the basis for our derivation of both stabilization statements from Theorem 1.2 (Lemmata 4.3 and 4.4).

2. Preliminaries

To begin with, let us specify the concept of solvability to be pursued in this paper.

DEFINITION 2.1. *Let $\phi \in C^0([0,\infty))$ be nonnegative, and assume that $u_0 \in (C^0(\overline{\Omega}))^\star$ and $v_0 \in L^\infty(\Omega)$ are nonnegative. Then a pair (u,v) of nonnegative functions*

$$\begin{cases} u \in C^0_{w-\star}([0,\infty);(C^0(\overline{\Omega}))^\star) \cap L^1_{loc}(\overline{\Omega}\times(0,\infty)) \qquad \text{and} \\ v \in C^0_{w-\star}([0,\infty);L^\infty(\Omega)) \end{cases} \tag{2.1}$$

will be called a global very weak solution *of (1.5) if $u(\cdot,0)=u_0$ in $(C^0(\overline{\Omega}))^\star$ and $v(\cdot,0)=v_0$ in $L^\infty(\Omega)$, and if for each $\varphi \in C_0^\infty(\overline{\Omega}\times(0,\infty))$ fulfilling $\frac{\partial\varphi}{\partial\nu}=0$ on $\partial\Omega\times(0,\infty)$ we have*

$$-\int_0^\infty \int_\Omega u\varphi_t = \int_0^\infty \int_\Omega u\phi(v)\Delta\varphi \tag{2.2}$$

and

$$-\int_0^\infty \int_\Omega v\varphi_t = \int_0^\infty \int_\Omega v\Delta\varphi - \int_0^\infty \int_\Omega uv\varphi. \tag{2.3}$$

In order to construct such solutions to (1.5) as limits of solutions to suitably regularized problems, we approximate the motility function ϕ as well as initial data (u_0,v_0) in (1.5) by introducing families of functions $(\phi_\varepsilon)_{\varepsilon\in(0,1)}$, $(u_{0\varepsilon})_{\varepsilon\in(0,1)}$ and $(v_{0\varepsilon})_{\varepsilon\in(0,1)}$ with the properties that

$$\begin{cases} (\phi_\varepsilon)_{\varepsilon\in(0,1)} \subset C^3([0,\infty)) \quad \text{is such that} \\ \phi_\varepsilon \ge \phi \text{ on } [0,\infty) \text{ for all } \varepsilon\in(0,1), \quad \text{and that} \\ \phi_\varepsilon \to \phi \text{ in } C^0_{loc}([0,\infty)) \quad \text{as } \varepsilon\searrow 0, \end{cases} \tag{2.4}$$

and that moreover

$$\begin{cases} (u_{0\varepsilon})_{\varepsilon\in(0,1)} \subset W^{1,\infty}(\Omega) \text{ and } (v_{0\varepsilon})_{\varepsilon\in(0,1)} \subset W^{1,\infty}(\Omega) \quad \text{are such that} \\ u_{0\varepsilon}\ge 0 \quad \text{and} \quad v_{0\varepsilon}>0 \quad \text{in } \overline{\Omega} \qquad \text{for all } \varepsilon\in(0,1), \qquad \text{that} \\ \int_\Omega u_{0\varepsilon} = \overline{u}_0|\Omega| \quad \text{and } \|v_{0\varepsilon}\|_{L^\infty(\Omega)} \le \|v_0\|_{L^\infty(\Omega)}+1 \quad \text{for all } \varepsilon\in(0,1), \quad \text{and that} \\ u_{0\varepsilon} \overset{\star}{\rightharpoonup} u_0 \text{ in } (C^0(\overline{\Omega}))^\star \quad \text{and} \quad v_{0\varepsilon} \overset{\star}{\rightharpoonup} v_0 \text{ in } L^\infty(\Omega) \quad \text{as } \varepsilon\searrow 0. \end{cases} \tag{2.5}$$

For $\varepsilon\in(0,1)$, we then consider the regularized variant of (1.5) given by

$$\begin{cases} u_{\varepsilon t} = \Delta\big(u_\varepsilon\phi_\varepsilon(v_\varepsilon)\big), & x\in\Omega,\ t>0, \\ v_{\varepsilon t} = \Delta v_\varepsilon - \frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon}, & x\in\Omega,\ t>0, \\ \frac{\partial u_\varepsilon}{\partial\nu} = \frac{\partial v_\varepsilon}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0, \\ u_\varepsilon(x,0)=u_{0\varepsilon}(x), \quad v_\varepsilon(x,0)=v_{0\varepsilon}(x), & x\in\Omega, \end{cases} \tag{2.6}$$

which is globally solved in the classical sense:

LEMMA 2.1. *For each $\varepsilon\in(0,1)$ there exist*

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)) \qquad \text{and} \\ v_\varepsilon \in \bigcap_{q>2} C^0([0,\infty);W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)) \end{cases} \tag{2.7}$$

such that $u_\varepsilon \ge 0$ and $v_\varepsilon > 0$ in $\overline{\Omega} \times [0,\infty)$, and that $(u_\varepsilon, v_\varepsilon)$ solves (2.6) in the classical sense. Furthermore, the solution satisfies

$$\int_\Omega u_\varepsilon(\cdot,t) = \int_\Omega u_0 \qquad \textit{for all } t>0 \textit{ and } \varepsilon \in (0,1), \tag{2.8}$$

and

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|v_\varepsilon(\cdot,t_0)\|_{L^\infty(\Omega)} \qquad \textit{for all } t_0 \ge 0,\ t>t_0 \textit{ and } \varepsilon\in(0,1). \tag{2.9}$$

Proof. We start by asserting the local classical solvability for (2.6) by means of the well-established parabolic theory from [2]. To this end, we fix $\delta_0 > 0$, and for $\varepsilon \in (0,1)$ introducing $D_0 := (0,\infty)\times(-\delta_0,\infty)$ as well as

$$A_\varepsilon\begin{pmatrix}\eta\\ \xi\end{pmatrix} := \begin{pmatrix}1 & 0\\ \xi\phi_\varepsilon'(\eta) & \phi_\varepsilon(\eta)\end{pmatrix} \quad \text{and} \quad f_\varepsilon\begin{pmatrix}\eta\\ \xi\end{pmatrix} := \begin{pmatrix}-\frac{\xi\eta}{1+\varepsilon\xi}\\ 0\end{pmatrix} \quad \text{for} \quad \begin{pmatrix}\eta\\ \xi\end{pmatrix}\in D_0,$$

we may recast (2.6) as the quasilinear problem

$$\begin{cases} Z_{\varepsilon t} = \nabla\cdot\big(A_\varepsilon(Z_\varepsilon)\nabla Z_\varepsilon\big) + f_\varepsilon(Z_\varepsilon), & x\in\Omega,\ t>0,\\ \frac{\partial Z_\varepsilon}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ Z_\varepsilon(x,0) = \begin{pmatrix} v_{0\varepsilon}(x)\\ u_{0\varepsilon}(x)\end{pmatrix}, & x\in\Omega,\end{cases} \tag{2.10}$$

where $Z_\varepsilon = (v_\varepsilon, u_\varepsilon)$. Using (1.6) and (2.4), we observe that for each $U\in D_0$, $A_\varepsilon(U)$ is a positive definite matrix of lower triangular form, so that from [2, Theorem 1], in line with (2.5) we deduce that there exists $T_{max,\varepsilon}\in(0,\infty]$ such that (2.6) possesses a classical solution $(u_\varepsilon, v_\varepsilon)$ which is such that $v_\varepsilon>0$ in $\overline{\Omega}\times[0,T_{max,\varepsilon})$, and that

$$\text{if } T_{max,\varepsilon}<\infty, \quad \textit{then} \quad \limsup_{t\nearrow T_{max,\varepsilon}} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{2.11}$$

Moreover, taking into account $u_{0\varepsilon}\ge 0$ in (2.5), we obtain $u_\varepsilon \ge 0$ in $\overline{\Omega}\times[0,T_{max,\varepsilon})$ through a simple comparison argument. Next after an integration performed in the first equation in (2.6), we find that

$$\int_\Omega u_\varepsilon(\cdot,t) = \int_\Omega u_0 \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \text{ and } \varepsilon\in(0,1), \tag{2.12}$$

whereas an application of the maximum principle to the second equation therein shows that

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|v_\varepsilon(\cdot,t_0)\|_{L^\infty(\Omega)} \tag{2.13}$$

for all $t_0\in[0,T_{max,\varepsilon})$, $t\in(t_0,T_{max,\varepsilon})$ and $\varepsilon\in(0,1)$. To finally prove that $T_{max,\varepsilon}=\infty$, assuming on the contrary that $T_{max,\varepsilon}<\infty$ for some $\varepsilon\in(0,1)$, we use (2.13) to see that $\left\|\frac{u_\varepsilon(\cdot,t)v_\varepsilon(\cdot,t)}{1+\varepsilon u_\varepsilon(\cdot,t)}\right\|_{L^\infty(\Omega)} \le \frac{\|v_{0\varepsilon}\|_{L^\infty(\Omega)}}{\varepsilon}$ for all $t\in(0,T_{max,\varepsilon})$, and can then rely on standard parabolic regularity theory applied to the second equation in (2.6) to find $c_1(\varepsilon)>0$ such that

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_1(\varepsilon) \qquad \text{for all } t\in(0,T_{max,\varepsilon}). \tag{2.14}$$

In addition, again thanks to (2.13), (1.6) and (2.4) warrant the existence of positive constants c_2 and $c_3(\varepsilon)$ fulfilling

$$\phi_\varepsilon(v_\varepsilon) \ge c_2 \quad \text{and} \quad \frac{(\phi'_\varepsilon(v_\varepsilon))^2}{\phi_\varepsilon(v_\varepsilon)} \le c_3(\varepsilon) \qquad \text{in } \Omega \times (0,T_{max,\varepsilon}). \tag{2.15}$$

Therefore, integrating by parts in the first equation from (2.6) and using the Cauchy-Schwarz inequality and (2.14), we see that whenever $p>1$,

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_\varepsilon^p &= -p(p-1)\int_\Omega \phi_\varepsilon(v_\varepsilon)u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 - p(p-1)\int_\Omega \phi'_\varepsilon(v_\varepsilon)u_\varepsilon^{p-1}\nabla u_\varepsilon\cdot\nabla v_\varepsilon \\ &\le -\frac{p(p-1)}{2}\int_\Omega \phi_\varepsilon(v_\varepsilon)u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 + \frac{p(p-1)}{2}\int_\Omega \frac{(\phi'_\varepsilon(v_\varepsilon))^2}{\phi_\varepsilon(v_\varepsilon)}u_\varepsilon^p|\nabla v_\varepsilon|^2 \\ &\le -\frac{2c_2(p-1)}{p}\int_\Omega |\nabla u_\varepsilon^{\frac{p}{2}}|^2 + \frac{c_1^2(\varepsilon)c_3(\varepsilon)p(p-1)}{2}\int_\Omega u_\varepsilon^p \end{aligned}\tag{2.16}$$

for all $t\in(0,T_{max,\varepsilon})$. According to an Ehrling type inequality associated with the compactness of the embedding $W^{1,2}(\Omega)\hookrightarrow L^2(\Omega)$, from (2.12) it follows that for any such p, with some $c_4(p,\varepsilon)>0$ we have

$$\begin{aligned}\frac{c_1^2(\varepsilon)c_3(\varepsilon)p(p-1)}{2}\int_\Omega u_\varepsilon^p &= \frac{c_1^2(\varepsilon)c_3(\varepsilon)p(p-1)}{2}\|u_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \\ &\le \frac{2c_2(p-1)}{p}\|\nabla u_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^2 + c_4(p,\varepsilon)\|u_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^2 \\ &= \frac{2c_2(p-1)}{p}\|\nabla u_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^2 + c_4(p,\varepsilon)\Big\{\int_\Omega u_0\Big\}^p\end{aligned}$$

for all $t\in(0,T_{max,\varepsilon})$, which, when substituted back into (2.16), upon an integration in time entails that

$$\begin{aligned}\int_\Omega u_\varepsilon^p &\le c_4(p,\varepsilon)\Big\{\int_\Omega u_0\Big\}^p T_{max,\varepsilon} + \int_\Omega u_{0\varepsilon}^p \\ &\le c_4(p,\varepsilon)\Big\{\int_\Omega u_0\Big\}^p T_{max,\varepsilon} + \|u_{0\varepsilon}\|_{W^{1,\infty}(\Omega)}^p|\Omega| \qquad \text{for all } t\in(0,T_{max,\varepsilon}).\end{aligned}$$

Since $p>1$ was arbitrary here, in view of a standard Moser-type iteration ([36]) we may find $c_5(\varepsilon)>0$ such that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_5(\varepsilon) \qquad \text{for all } t\in(0,T_{max,\varepsilon}),$$

which contradicts (2.11) and thus confirms that, indeed, $T_{max,\varepsilon}=\infty$. □

3. Instantaneous relaxation. Proof of Theorem 1.1

The following basic properties of solutions to (2.6) can be established in a straightforward manner.

Lemma 3.1. Let $n\ge1$, and assume (2.5) and (2.4). Then

$$\int_0^\infty\int_\Omega \frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon} \le |\Omega|\cdot\big(\|v_0\|_{L^\infty(\Omega)}+1\big) \qquad \text{for all } \varepsilon\in(0,1) \tag{3.1}$$

and

$$\int_0^\infty\int_\Omega |\nabla v_\varepsilon|^2 \le \frac{1}{2}|\Omega|\cdot\big(\|v_0\|_{L^\infty(\Omega)}+1\big)^2 \qquad \text{for all } \varepsilon\in(0,1). \tag{3.2}$$

Proof. For $p \ge 1$, we integrate using the second equation in (2.6) to see that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega v_\varepsilon^p + (p-1)\int_\Omega v_\varepsilon^{p-2}|\nabla v_\varepsilon|^2 + \int_\Omega \frac{u_\varepsilon v_\varepsilon^p}{1+\varepsilon u_\varepsilon} = 0 \qquad \text{for all } t>0 \text{ and } \varepsilon \in (0,1). \quad (3.3)$$

When specified to the case $p=1$, upon a time integration this implies that due to (2.5),

$$\int_0^T \int_\Omega \frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon} = -\int_\Omega v_\varepsilon(\cdot,T) + \int_\Omega v_{0\varepsilon} \le |\Omega| \cdot \big(\|v_0\|_{L^\infty(\Omega)} + 1\big) \qquad \text{for all } T>0 \text{ and } \varepsilon \in (0,1),$$

and that thus (3.1) holds. Secondly, in the case when $p=2$, an integration of (3.3) shows that for all $T>0$ and $\varepsilon \in (0,1)$,

$$\int_0^T \int_\Omega |\nabla v_\varepsilon|^2 = -\int_0^T \int_\Omega \frac{u_\varepsilon v_\varepsilon^2}{1+\varepsilon u_\varepsilon} - \frac{1}{2}\int_\Omega v_\varepsilon^2(\cdot,T) + \frac{1}{2}\int_\Omega v_{0\varepsilon}^2 \le \frac{1}{2}|\Omega|\big(\|v_0\|_{L^\infty(\Omega)} + 1\big)^2,$$

and thereby establishes (3.2). □

The following outcome of an essentially straightforward testing procedure is formulated in such a way that it can not only serve as an ingredient in our derivation of boundedness features in the general setting of Theorem 1.1, but also be used in the course of our asymptotic analysis in the low-dimensional situations addressed by Theorem 1.2.

LEMMA 3.2. *If $n \ge 1$, and if (2.5) and (2.4) hold, then one can find $\Gamma_1 > 0$ such that for all $t>0$ and $\varepsilon \in (0,1)$,*

$$\frac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\Delta v_\varepsilon|^2 + \frac{1}{\Gamma_1}\int_\Omega |\nabla v_\varepsilon|^4 \le \Gamma_1 \int_\Omega (u_\varepsilon - \overline{u}_0)^2. \quad (3.4)$$

Proof. Let $F_\varepsilon(\xi) := \frac{\xi}{1+\varepsilon\xi}$ for $\xi \ge 0$ and $\varepsilon \in (0,1)$. Then since $0 \le F_\varepsilon' \le 1$ for all $\varepsilon \in (0,1)$, from the mean value theorem it follows that

$$\left| \frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} - \frac{\overline{u}_0}{1+\varepsilon \overline{u}_0} \right| = \big|F_\varepsilon(u_\varepsilon) - F_\varepsilon(\overline{u}_0)\big| \le |u_\varepsilon - \overline{u}_0| \quad \text{in } \Omega \times (0,\infty) \qquad \text{for all } \varepsilon \in (0,1).$$

Therefore, if we multiply the second equation in (2.6) by $-\Delta v_\varepsilon$ and integrate by parts and using Young's inequality, (2.9) and (2.5), we see that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega |\Delta v_\varepsilon|^2 &= \int_\Omega \frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon}\Delta v_\varepsilon \\
&\le \frac{1}{2}\int_\Omega |\Delta v_\varepsilon|^2 + \frac{1}{2}\int_\Omega \Big(\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} - \frac{\overline{u}_0}{1+\varepsilon \overline{u}_0}\Big)^2 v_\varepsilon^2 \\
&\le \frac{1}{2}\int_\Omega |\Delta v_\varepsilon|^2 + \frac{1}{2}\cdot\big(\|v_0\|_{L^\infty(\Omega)} + 1\big)^2 \int_\Omega (u_\varepsilon - \overline{u}_0)^2 \quad (3.5)
\end{aligned}$$

for all $t>0$ and $\varepsilon \in (0,1)$, because

$$\int_\Omega v_\varepsilon \Delta v_\varepsilon = -\int_\Omega |\nabla v_\varepsilon|^2 \le 0 \qquad \text{for all } t>0 \text{ and } \varepsilon \in (0,1).$$

Now from elliptic regularity theory we obtain that with some $c_1 > 0$ we have

$$\int_\Omega |D^2\varphi|^2 \le c_1 \int_\Omega |\Delta\varphi|^2 \qquad \text{for all } \varphi \in C^2(\overline{\Omega}) \text{ fulfilling } \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega,$$

so that another integration by parts together with the Cauchy-Schwarz inequality shows that

$$\begin{aligned}\int_\Omega |\nabla v_\varepsilon|^4 &= -\int_\Omega v_\varepsilon \cdot \Big\{ 2\nabla v_\varepsilon \cdot (D^2 v_\varepsilon \cdot \nabla v_\varepsilon) + |\nabla v_\varepsilon|^2 \Delta v_\varepsilon \Big\} \\ &\le 2\int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 |D^2 v_\varepsilon| + \int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 |\Delta v_\varepsilon| \\ &\le \|v_\varepsilon\|_{L^\infty(\Omega)} \cdot \Big\{ \int_\Omega |\nabla v_\varepsilon|^4 \Big\}^{\frac12} \cdot \Big\{ 2\cdot \Big\{ \int_\Omega |D^2 v_\varepsilon|^2 \Big\}^{\frac12} + \Big\{ \int_\Omega |\Delta v_\varepsilon|^2 \Big\}^{\frac12} \Big\} \\ &\le \|v_\varepsilon\|_{L^\infty(\Omega)} \cdot \Big\{ \int_\Omega |\nabla v_\varepsilon|^4 \Big\}^{\frac12} \cdot (2\sqrt{c_1}+1) \cdot \Big\{ \int_\Omega |\Delta v_\varepsilon|^2 \Big\}^{\frac12} \quad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),\end{aligned}$$

and thus, by (2.9) and (2.5),

$$\int_\Omega |\nabla v_\varepsilon|^4 \le \|v_\varepsilon\|_{L^\infty(\Omega)}^2 (2\sqrt{c_1}+1)^2 \int_\Omega |\Delta v_\varepsilon|^2 \le c_2 \int_\Omega |\Delta v_\varepsilon|^2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1)$$

with $c_2 := (2\sqrt{c_1}+1)^2 (\|v_0\|_{L^\infty(\Omega)}+1)^2$. Therefore, (3.5) entails that

$$\begin{aligned}\frac12 \frac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^2 + \frac14 \int_\Omega |\Delta v_\varepsilon|^2 + \frac{1}{4c_2}\int_\Omega |\nabla v_\varepsilon|^4 &\le \frac12 \frac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^2 + \frac12 \int_\Omega |\Delta v_\varepsilon|^2 \\ &\le \frac12 \big(\|v_0\|_{L^\infty(\Omega)} + 1\big)^2 \int_\Omega (u_\varepsilon - \overline{u}_0)^2\end{aligned}$$

for all $t>0$ and $\varepsilon\in(0,1)$, and that hence (3.4) holds with $\Gamma_1 := \max\{2c_2, (\|v_0\|_{L^\infty(\Omega)} + 1)^2\}$. □

Also our first information on an evolution property of the first solution components, acting in an H^{-1} framework reminiscent of that already resorted to in [37], is at this stage kept general enough so as to remain compatible with our analysis of both Theorem 1.1 and Theorem 1.2.

Lemma 3.3. *Suppose that $n\ge1$, and that (2.5) and (2.4) are satisfied. Then there exists $\Gamma_2>0$ with the property that*

$$\frac{d}{dt}\int_\Omega \Big| A^{-\frac12}(u_\varepsilon - \overline{u}_0)\Big|^2 + \frac{1}{\Gamma_2}\int_\Omega (u_\varepsilon - \overline{u}_0)^2 \le \Gamma_2 \int_\Omega \Big| \overline{u}_0 \phi_\varepsilon(v_\varepsilon) - \overline{u_\varepsilon(\cdot,t)\phi_\varepsilon\big(v_\varepsilon(\cdot,t)\big)}\Big|^2 \tag{3.6}$$

for all $t>0$ and $\varepsilon\in(0,1)$.

Proof. According to (2.6), we have

$$\partial_t (u_\varepsilon - \overline{u}_0) = -A\big(u_\varepsilon\phi_\varepsilon(v_\varepsilon) - \overline{u_\varepsilon\phi_\varepsilon(v_\varepsilon)}\big) \quad \text{in } \Omega\times(0,\infty) \qquad \text{for all } \varepsilon\in(0,1),$$

which when tested against $A^{-1}(u_\varepsilon - \overline{u}_0)$ implies that since both $A^{-\frac12}$ and A^{-1} are self-adjoint,

$$\begin{aligned}\frac12\frac{d}{dt}\int_\Omega \Big| A^{-\frac12}(u_\varepsilon - \overline{u}_0)\Big|^2 &= -\int_\Omega \big(u_\varepsilon\phi_\varepsilon(v_\varepsilon) - \overline{u_\varepsilon\phi_\varepsilon(v_\varepsilon)}\big)\cdot(u_\varepsilon - \overline{u}_0) \\ &= -\int_\Omega \big(u_\varepsilon\phi_\varepsilon(v_\varepsilon) - \overline{u}_0\phi_\varepsilon(v_\varepsilon) + \overline{u}_0\phi_\varepsilon(v_\varepsilon) - \overline{u_\varepsilon\phi_\varepsilon(v_\varepsilon)}\big)\cdot(u_\varepsilon - \overline{u}_0) \\ &= -\int_\Omega (u_\varepsilon - \overline{u}_0)^2 \phi_\varepsilon(v_\varepsilon) - \int_\Omega \big(\overline{u}_0\phi_\varepsilon(v_\varepsilon) - \overline{u_\varepsilon\phi_\varepsilon(v_\varepsilon)}\big)\cdot(u_\varepsilon - \overline{u}_0)\end{aligned}$$

for all $t>0$ and $\varepsilon\in(0,1)$. Since (2.4) together with (1.6), (2.9) and (2.5) entails the existence of $c_1>0$ such that $\phi_\varepsilon(v_\varepsilon)\ge c_1$ in $\Omega\times(0,\infty)$ for all $\varepsilon\in(0,1)$, and since

$$-\int_\Omega \left(\overline{u}_0\phi_\varepsilon(v_\varepsilon)-\overline{u_\varepsilon\phi_\varepsilon(v_\varepsilon)}\right)\cdot(u_\varepsilon-\overline{u}_0)\le \frac{c_1}{2}\int_\Omega(u_\varepsilon-\overline{u}_0)^2+\frac{1}{2c_1}\int_\Omega\left|\overline{u}_0\phi_\varepsilon(v_\varepsilon)-\overline{u_\varepsilon\phi_\varepsilon(v_\varepsilon)}\right|^2$$

for all $t>0$ and $\varepsilon\in(0,1)$ thanks to Young's inequality, this already leads to (3.6) if we let $\Gamma_2:=\frac{1}{c_1}$, for instance. □

In our first application of Lemma 3.2 and Lemma 3.3, we may estimate the expression on the right of (3.6) in a fairly rough manner. Actually focusing especially on initial relaxation features here, we can thereby, after all, make sure that a functional of the form in (1.13) satisfies an ODI containing a superlinear absorptive term, an appropriate exploitation of which yields a priori information within time intervals of the form (τ,∞) for arbitrary $\tau>0$:

LEMMA 3.4. *Let $n\ge1$, and assume (2.5) and (2.4). Then for all $\tau>0$ there exists $C(\tau)>0$ such that*

$$\int_\Omega\left|A^{-\frac12}\left(u_\varepsilon(\cdot,t)-\overline{u}_0\right)\right|^2\le C(\tau)\qquad \textit{for all } t>\tau \textit{ and } \varepsilon\in(0,1), \tag{3.7}$$

that

$$\int_\Omega\left|\nabla v_\varepsilon(\cdot,t)\right|^2\le C(\tau)\qquad \textit{for all } t>\tau \textit{ and } \varepsilon\in(0,1), \tag{3.8}$$

that

$$\int_t^{t+1}\int_\Omega u_\varepsilon^2\le C(\tau)\qquad \textit{for all } t>\tau \textit{ and } \varepsilon\in(0,1), \tag{3.9}$$

that

$$\int_t^{t+1}\int_\Omega|\Delta v_\varepsilon|^2\le C(\tau)\qquad \textit{for all } t>\tau \textit{ and } \varepsilon\in(0,1), \tag{3.10}$$

and that

$$\int_t^{t+1}\int_\Omega|\nabla v_\varepsilon|^4\le C(\tau)\qquad \textit{for all } t>\tau \textit{ and } \varepsilon\in(0,1). \tag{3.11}$$

Proof. As a consequence of (2.4), (1.6), (2.9) and (2.5), we can find $c_1>0$ such that $\phi_\varepsilon(v_\varepsilon)\le c_1$ in $\Omega\times(0,\infty)$ for all $\varepsilon\in(0,1)$. Since thus especially

$$\overline{u_\varepsilon\phi_\varepsilon(v_\varepsilon)}=\frac{1}{|\Omega|}\int_\Omega u_\varepsilon\phi_\varepsilon(v_\varepsilon)\le\frac{c_1}{|\Omega|}\int_\Omega u_\varepsilon=c_1\overline{u}_0\qquad\text{for all } t>0 \text{ and } \varepsilon\in(0,1)$$

by (2.8), on the right-hand side of (3.6) we can estimate

$$\Gamma_2\int_\Omega\left|\overline{u}_0\phi_\varepsilon(v_\varepsilon)-\overline{u_\varepsilon(\cdot,t)\phi_\varepsilon\left(v_\varepsilon(\cdot,t)\right)}\right|^2\le\Gamma_2\int_\Omega|c_1\overline{u}_0+c_1\overline{u}_0|^2=c_2:=4c_1^2\Gamma_2\overline{u}_0^2|\Omega| \tag{3.12}$$

for all $t>0$ and $\varepsilon\in(0,1)$. From Lemma 3.3 and Lemma 3.2 we therefore obtain that if we let $a:=\frac{1}{4\Gamma_1\Gamma_2}$, with Γ_1 taken from Lemma 3.2, and define

$$y_\varepsilon(t):=\int_\Omega\left|A^{-\frac12}\left(u_\varepsilon(\cdot,t)-\overline{u}_0\right)\right|^2+a\int_\Omega\left|\nabla v_\varepsilon(\cdot,t)\right|^2,\qquad t>0,\ \varepsilon\in(0,1), \tag{3.13}$$

as well as

$$g_\varepsilon(t) := \frac{1}{2\Gamma_2}\int_\Omega \big(u_\varepsilon(\cdot,t)-\overline{u}_0\big)^2 + \frac{a}{2}\int_\Omega \big|\Delta v_\varepsilon(\cdot,t)\big|^2 + \frac{a}{2\Gamma_1}\int_\Omega \big|\nabla v_\varepsilon(\cdot,t)\big|^4, \quad t>0,\ \varepsilon\in(0,1), \tag{3.14}$$

then

$$\begin{aligned} & y_\varepsilon'(t) + g_\varepsilon(t) + \frac{1}{4\Gamma_2}\int_\Omega (u_\varepsilon-\overline{u}_0)^2 + \frac{a}{2\Gamma_1}\int_\Omega |\nabla v_\varepsilon|^4 \\ \le & \Big\{ -\frac{1}{4\Gamma_2}\int_\Omega (u_\varepsilon-\overline{u}_0)^2 + c_2 \Big\} + a\cdot\Gamma_1\int_\Omega (u_\varepsilon-\overline{u}_0)^2 \\ = & c_2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1). \end{aligned} \tag{3.15}$$

In order to turn this into a superlinearly damped ODI for $(y_\varepsilon)_{\varepsilon\in(0,1)}$, we pick any $\beta>\frac{1}{2}$ such that $\beta>\frac{n}{4}$ and note that then the inequalities $0<\frac{1}{2}<\beta$ enable us to invoke a standard interpolation inequality ([9, Theorem 14.1]) to find $c_3>0$ fulfilling

$$\|A^{-\frac{1}{2}}\varphi\|_{L^2(\Omega)} \le c_3\|\varphi\|_{L^2(\Omega)}^{\theta}\|A^{-\beta}\varphi\|_{L^2(\Omega)}^{1-\theta} \qquad \text{for all } \varphi\in L^2_\perp(\Omega)$$

with $\theta:=\frac{2\beta-1}{2\beta}\in(0,1)$. Since the restriction $\beta>\frac{n}{4}$ warrants that for the corresponding domains of definition we have $D(A^\beta)\hookrightarrow L^\infty(\Omega)$ and hence $L^1(\Omega)\hookrightarrow D(A^{-\beta})$ ([15]), we may combine this with (2.8) to see that with some $c_4>0$ we have

$$\Big\{\int_\Omega \Big|A^{-\frac{1}{2}}(u_\varepsilon-\overline{u}_0)\Big|^2\Big\}^{\frac{1}{\theta}} \le c_4\int_\Omega (u_\varepsilon-\overline{u}_0)^2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).$$

Writing $\kappa:=\min\{\frac{1}{\theta},2\}>1$ and

$$c_5 := \min\Big\{\frac{1}{2^{\kappa+1}c_4\Gamma_2}, \frac{1}{2^\kappa\Gamma_1 a^{\kappa-1}|\Omega|}\Big\}, \tag{3.16}$$

we thus infer that in line with (3.13) and thanks to the Cauchy-Schwarz inequality,

$$\begin{aligned} c_5 y_\varepsilon^\kappa(t) &\le 2^{\kappa-1}c_5\cdot\Big\{\int_\Omega \Big|A^{-\frac{1}{2}}(u_\varepsilon-\overline{u}_0)\Big|^2\Big\}^\kappa + 2^{\kappa-1}a^\kappa c_5\cdot\Big\{\int_\Omega|\nabla v_\varepsilon|^2\Big\}^\kappa \\ &\le 2^{\kappa-1}c_5\cdot\Big\{\Big\{\int_\Omega \Big|A^{-\frac{1}{2}}(u_\varepsilon-\overline{u}_0)\Big|^2\Big\}^{\frac{1}{\theta}}+1\Big\} + 2^{\kappa-1}a^\kappa c_5\cdot\Big\{\Big\{\int_\Omega|\nabla v_\varepsilon|^2\Big\}^2+1\Big\} \\ &\le 2^{\kappa-1}c_5\cdot\Big\{c_4\int_\Omega (u_\varepsilon-\overline{u}_0)^2+1\Big\} + 2^{\kappa-1}a^\kappa c_5\cdot\Big\{|\Omega|\int_\Omega|\nabla v_\varepsilon|^4+1\Big\} \\ &\le \frac{1}{4\Gamma_2}\int_\Omega (u_\varepsilon-\overline{u}_0)^2 + \frac{a}{2\Gamma_1}\int_\Omega|\nabla v_\varepsilon|^4 + 2^{\kappa-1}(1+a^\kappa)c_5 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1), \end{aligned}$$

because $2^{\kappa-1}c_5c_4\le\frac{1}{4\Gamma_2}$ and $2^{\kappa-1}a^\kappa c_5|\Omega|\le\frac{a}{2\Gamma_1}$ due to (3.16). Consequently, (3.15) implies that with $c_6:=c_2+2^{\kappa-1}(1+a^\kappa)c_5$ we have

$$y_\varepsilon'(t)+c_5y_\varepsilon^\kappa(t)+g_\varepsilon(t)\le c_6 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1), \tag{3.17}$$

so that since for fixed $\tau>0$,

$$\overline{y}(t) := c_7\cdot\Big(t-\frac{\tau}{2}\Big)^{-\frac{1}{\kappa-1}}+c_7, \qquad t>\frac{\tau}{2},$$

with

$$c_7 := \max\left\{ \big((\kappa-1)c_5\big)^{-\frac{1}{\kappa-1}}, \left(\frac{c_6}{c_5}\right)^{\frac{1}{\kappa}} \right\},$$

satisfies $\overline{y}(t) \nearrow +\infty$ as $t \searrow \frac{\tau}{2}$ and

$$\begin{aligned}
\overline{y}'(t)+c_5\overline{y}^\kappa(t)+g_\varepsilon(t)-c_6 &\ge \overline{y}'(t)+c_5\overline{y}^\kappa(t)-c_6 \\
&= -\frac{c_7}{\kappa-1}\left(t-\frac{\tau}{2}\right)^{-\frac{1}{\kappa-1}-1} + c_5 c_7^\kappa \cdot \left\{ \left(t-\frac{\tau}{2}\right)^{-\frac{1}{\kappa-1}} + 1 \right\}^\kappa - c_6 \\
&\ge -\frac{c_7}{\kappa-1}\left(t-\frac{\tau}{2}\right)^{-\frac{1}{\kappa-1}-1} + c_5 c_7^\kappa \left(t-\frac{\tau}{2}\right)^{-\frac{\kappa}{\kappa-1}} + c_5 c_7^\kappa - c_6 \\
&= c_5 c_7 \cdot \left\{ c_7^{\kappa-1} - \frac{1}{(\kappa-1)c_5} \right\} \cdot \left(t-\frac{\tau}{2}\right)^{-\frac{\kappa}{\kappa-1}} + c_5 \cdot \left\{ c_7^\kappa - \frac{c_6}{c_5} \right\} \\
&\ge 0 \qquad \text{for all } t > \frac{\tau}{2},
\end{aligned}$$

an ODE comparison argument applied to (3.17) shows that $y_\varepsilon(t) \le \overline{y}(t)$ for all $t > \frac{\tau}{2}$ and $\varepsilon \in (0,1)$, and that hence, in particular,

$$y_\varepsilon(t) \le c_8(\tau) := c_7 \cdot \left(\frac{\tau}{2}\right)^{-\frac{1}{\kappa-1}} + c_7 \qquad \text{for all } t > \tau \text{ and } \varepsilon \in (0,1). \tag{3.18}$$

Thereupon, by direct integration in (3.17) we obtain that

$$\int_t^{t+1} g_\varepsilon(s)ds \le y_\varepsilon(t) + \int_t^{t+1} c_6 ds \le c_8(\tau) + c_6 \qquad \text{for all } t > \tau \text{ and } \varepsilon \in (0,1), \tag{3.19}$$

so that in view of (3.13) and (3.14) we infer (3.7) and (3.8) from (3.18) and (3.9)-(3.11) from (3.19) if we choose $C(\tau)$ appropriately large. □

A straightforward estimation of corresponding time derivatives does not only pave the way toward an Aubin-Lions type compactness argument, but beyond this also prepares our analysis of the solution behavior near the initial instant.

LEMMA 3.5. *Suppose that $n \ge 1$, and that (2.5) and (2.4) hold. Then one can find $C > 0$ such that*

$$\|u_{\varepsilon t}(\cdot,t)\|_{(W_N^{2,\infty}(\Omega))^\star} \le C \qquad \textit{for all } t > 0 \textit{ and } \varepsilon \in (0,1) \tag{3.20}$$

and

$$\|v_{\varepsilon t}(\cdot,t)\|_{(W_N^{2,\infty}(\Omega))^\star} \le C \qquad \textit{for all } t > 0 \textit{ and } \varepsilon \in (0,1). \tag{3.21}$$

Proof. We fix $\varphi \in W_N^{2,\infty}(\Omega)$ and use (2.6) to see that

$$\begin{aligned}
\left| \int_\Omega u_{\varepsilon t}\varphi \right| &= \left| \int_\Omega \Delta\big(u_\varepsilon \phi_\varepsilon(v_\varepsilon)\big)\varphi \right| \\
&= \left| \int_\Omega u_\varepsilon \phi_\varepsilon(v_\varepsilon)\Delta\varphi \right| \\
&\le \|u_\varepsilon\|_{L^1(\Omega)} \big\|\phi_\varepsilon(v_\varepsilon)\big\|_{L^\infty(\Omega)} \|\Delta\varphi\|_{L^\infty(\Omega)} \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1)
\end{aligned} \tag{3.22}$$

as well as

$$\begin{aligned}\left|\int_\Omega v_{\varepsilon t}\varphi\right| &= \left|\int_\Omega \Delta v_\varepsilon \varphi - \int_\Omega \frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon}\varphi\right| \\ &= \left|\int_\Omega v_\varepsilon \Delta\varphi - \int_\Omega \frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon}\varphi\right| \\ &\le \|v_\varepsilon\|_{L^1(\Omega)}\|\Delta\varphi\|_{L^\infty(\Omega)} + \|u_\varepsilon\|_{L^1(\Omega)}\|v_\varepsilon\|_{L^\infty(\Omega)}\|\varphi\|_{L^\infty(\Omega)},\end{aligned} \tag{3.23}$$

because $0 \le \frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} \le u_\varepsilon$ in $\Omega\times(0,\infty)$ for all $\varepsilon\in(0,1)$. Since (2.8), (2.9), (2.5), (2.4) and (1.6) guarantee boundedness of $(u_\varepsilon)_{\varepsilon\in(0,1)}$ and $(v_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^\infty((0,\infty);L^1(\Omega))$, and of $(v_\varepsilon)_{\varepsilon\in(0,1)}$ and $(\phi_\varepsilon(v_\varepsilon))_{\varepsilon\in(0,1)}$ in $L^\infty(\Omega\times(0,\infty))$, from (3.22) and (3.23) we immediately obtain (3.20) and (3.21) with some suitably large $C>0$. □

As a preparation for our argument related to the continuity features claimed in Theorem 1.1, let us briefly record the following density property of the space appearing in the boundedness statements from Lemma 3.5.

LEMMA 3.6. *The set $W^{2,\infty}_N(\Omega)$ is dense in $C^0(\overline{\Omega})$.*

Proof. This immediately follows from standard parabolic theory, which namely asserts that if we let $(e^{t\Delta})_{t\ge0}$ denote the Neumann heat semigroup on Ω, then given any $\varphi\in C^0(\overline{\Omega})$ we have $e^{t\Delta}\varphi\in W^{2,\infty}_N(\Omega)$ for all $t>0$ and $e^{t\Delta}\varphi\to\varphi$ in $C^0(\overline{\Omega})$ as $t\searrow0$. □

As a consequence of the above a priori estimates, based on a standard extraction procedure we can now construct a global solution in the sense of (2.1).

LEMMA 3.7. *Let $n\ge1$, and let (2.5) and (2.4) hold. Then there exist $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ as well as nonnegative functions u and v on $\Omega\times(0,\infty)$ such that $\varepsilon_j\searrow0$ as $j\to\infty$, that (1.8) holds, and that as $\varepsilon=\varepsilon_j\searrow0$ we have*

$$u_\varepsilon \rightharpoonup u \qquad \text{in } L^2_{loc}(\overline{\Omega}\times(0,\infty)), \tag{3.24}$$

$$v_\varepsilon\to v \text{ and } \nabla v_\varepsilon\to\nabla v \qquad a.e. \text{ in } \Omega\times(0,\infty), \tag{3.25}$$

$$v_\varepsilon\to v \qquad \text{in } L^2_{loc}((0,\infty);W^{1,q}(\Omega)) \quad \text{for all } q\in[1,\tfrac{2n}{(n-2)_+}), \tag{3.26}$$

$$v_\varepsilon(\cdot,t)\to v(\cdot,t) \qquad \text{in } W^{1,q}(\Omega) \text{ for a.e. } t>0 \quad \text{for each } q\in[1,\tfrac{2n}{(n-2)_+}), \tag{3.27}$$

$$\nabla v_\varepsilon \overset{\star}{\rightharpoonup} \nabla v \qquad \text{in } L^\infty_{loc}((0,\infty);L^2(\Omega)) \qquad \text{and} \tag{3.28}$$

$$\frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon} \rightharpoonup uv \qquad \text{in } L^1_{loc}(\overline{\Omega}\times(0,\infty)). \tag{3.29}$$

In the sense of Definition 2.1, (u,v) forms a global very weak solution of (1.5) which satisfies

$$\int_\Omega u(\cdot,t) = \overline{u}_0|\Omega| \qquad \text{for a.e. } t>0. \tag{3.30}$$

Moreover, for all $\tau>0$ there exists $C(\tau)>0$ such that

$$\int_\Omega \left|A^{-\frac12}\big(u(\cdot,t)-\overline{u}_0\big)\right|^2 \le C(\tau) \qquad \text{for a.e. } t>\tau \tag{3.31}$$

and

$$\int_t^{t+1}\int_\Omega u^2 + \int_t^{t+1}\int_\Omega |\Delta v|^2 + \int_t^{t+1}\int_\Omega |\nabla v|^4 \le C(\tau) \qquad \text{for all } t>\tau, \tag{3.32}$$

and we have

$$A^{-\frac{1}{2}}(u_\varepsilon - \overline{u}_0) \overset{\star}{\rightharpoonup} A^{-\frac{1}{2}}(u-\overline{u}_0) \qquad \text{in } L^\infty_{loc}((0,\infty);L^2(\Omega)) \tag{3.33}$$

as $\varepsilon=\varepsilon_j \searrow 0$.

Proof. From (3.9) it follows that

$$(u_\varepsilon)_{\varepsilon\in(0,1)} \text{ is bounded in } L^2(\Omega\times(\tau,T)) \text{ for all } \tau>0 \text{ and } T>\tau,$$

while (3.10), (3.11), (3.8) and (3.21) guarantee that

$$(v_\varepsilon)_{\varepsilon\in(0,1)} \text{ is bounded in } L^2((\tau,T);W^{2,2}(\Omega)), \text{ in } L^4((\tau,T);W^{1,4}(\Omega)) \\ \text{and in } L^\infty((\tau,T);W^{1,2}(\Omega)) \text{ for all } \tau>0 \text{ and } T>\tau,$$

and that

$$(v_{\varepsilon t})_{\varepsilon\in(0,1)} \text{ is bounded in } L^\infty\big((0,\infty);(W^{2,\infty}_N(\Omega))^\star\big). \tag{3.34}$$

A standard extraction argument based on an Aubin-Lions lemma, and relying on the compactness of the embedding $W^{2,2}(\Omega)\hookrightarrow W^{1,q}(\Omega)$ for all $q\in[1,\frac{2n}{(n-2)_+})$, thus provides $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ as well as nonnegative functions $u\in L^2_{loc}(\overline{\Omega}\times(0,\infty))$ and $v\in L^2_{loc}((0,\infty);W^{2,2}(\Omega))\cap L^4_{loc}((0,\infty));W^{1,4}(\Omega))\cap L^\infty_{loc}((0,\infty);W^{1,2}(\Omega))$ such that $\varepsilon_j\searrow 0$ as $j\to\infty$, and that (3.24), (3.25), (3.26), (3.27) and (3.28) hold as $\varepsilon=\varepsilon_j\searrow 0$, where recalling (2.8), (2.9), (2.5), (3.7) and (3.8), and again using (3.9)-(3.11), we readily obtain that also

$$u\in L^\infty((0,\infty);L^1(\Omega)) \qquad \text{and} \qquad v\in L^\infty(\Omega\times(0,\infty)), \tag{3.35}$$

and that (3.32), (3.31) and (3.33) as well as (3.30) hold. Furthermore, since $0\le\frac{\xi}{1+\varepsilon\xi}\nearrow\xi$ as $\varepsilon\searrow 0$ for all $\xi\ge 0$, and since thus the L^1 convergence feature trivially contained in (3.9) ensures that also $\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\rightharpoonup u$ in $L^1_{loc}(\overline{\Omega}\times(0,\infty))$ as $\varepsilon=\varepsilon_j\searrow 0$ thanks to Lemma A.1, it also follows that (3.29) holds, because $(\frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon})_{\varepsilon\in(0,1)}$ is bounded in $L^2(\Omega\times(\tau,T))$ and hence relatively compact in with respect to the weak topology in $L^1(\Omega\times(\tau,T))$ for all $\tau>0$ and $T>\tau$ due to (3.9) and (2.9), and because whenever $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ is a subsequence of $(\varepsilon_j)_{j\in\mathbb{N}}$ such that $\frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon}\rightharpoonup z$ in $L^1_{loc}(\overline{\Omega}\times(0,\infty))$ with some $z\in L^1_{loc}(\overline{\Omega}\times(0,\infty))$ as $\varepsilon=\varepsilon_{j_k}\searrow 0$, due to the pointwise approximation property in (3.26) a well-known result ([46, Lemma A.1]) becomes applicable so as to identify $z=uv$.

To derive the identities in (2.2) and (2.3) from this, we only need to observe that for each $\varphi\in C_0^\infty(\overline{\Omega}\times(0,\infty))$ fulfilling $\frac{\partial\varphi}{\partial\nu}=0$ on $\partial\Omega\times(0,\infty)$, according to (2.6) we have

$$-\int_0^\infty\int_\Omega u_\varepsilon\varphi_t = \int_0^\infty\int_\Omega u_\varepsilon\phi_\varepsilon(v_\varepsilon)\Delta\varphi \qquad \text{for all } \varepsilon\in(0,1)$$

and

$$-\int_0^\infty\int_\Omega v_\varepsilon\varphi_t = \int_0^\infty\int_\Omega v_\varepsilon\Delta\varphi - \int_0^\infty\int_\Omega \frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon}\varphi \qquad \text{for all } \varepsilon\in(0,1),$$

that (3.24) and (3.26) clearly entail that

$$\int_0^\infty\int_\Omega u_\varepsilon\varphi_t\to\int_0^\infty\int_\Omega u\varphi_t, \quad \int_0^\infty\int_\Omega v_\varepsilon\varphi_t\to\int_0^\infty\int_\Omega v\varphi_t \quad \text{and} \quad \int_0^\infty\int_\Omega v_\varepsilon\Delta\varphi\to\int_0^\infty\int_\Omega v\Delta\varphi$$

as $\varepsilon=\varepsilon_j\searrow 0$, that due to (3.29) we have

$$\int_0^\infty\int_\Omega \frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon}\varphi \to \int_0^\infty\int_\Omega uv\varphi$$

as $\varepsilon=\varepsilon_j\searrow 0$, and that (3.26) together with (2.9), (2.5), (2.4), (1.6) and the dominated convergence theorem ensure that $\phi_\varepsilon(v_\varepsilon)\to\phi(v)$ in $L^2_{loc}(\overline{\Omega}\times(0,\infty))$ and hence, again by (3.24),

$$\int_0^\infty\int_\Omega u_\varepsilon\phi_\varepsilon(v_\varepsilon)\Delta\varphi \to \int_0^\infty\int_\Omega u\phi(v)\Delta\varphi$$

as $\varepsilon=\varepsilon_j\searrow 0$.

It remains to note that in addition to (3.34) we know from Lemma 3.5 that also

$$(u_{\varepsilon t})_{\varepsilon\in(0,1)} \text{ is bounded in } L^\infty\big((0,\infty);(W^{2,\infty}_N(\Omega))^\star\big),$$

so that since both $L^1(\Omega)$ and $L^\infty(\Omega)$ are compactly embedded into $(W^{2,\infty}_N(\Omega))^\star$, in view of (2.8), (2.9) and (2.5) we may twice employ the Arzelà-Ascoli theorem to infer that

$$u_\varepsilon\to u \quad\text{and}\quad v_\varepsilon\to v \qquad \text{in } C^0_{loc}\big([0,\infty);(W^{2,\infty}_N(\Omega))^\star\big) \tag{3.36}$$

as $\varepsilon=\varepsilon_j\searrow 0$, and that

$$u(\cdot,t)\to u_0 \quad\text{and}\quad v(\cdot,t)\to v_0 \quad \text{in } (W^{2,\infty}_N(\Omega))^\star \qquad \text{as } t\searrow 0. \tag{3.37}$$

Indeed, since $W^{2,\infty}_N(\Omega)$ is dense in $C^0(\overline{\Omega})$ by Lemma 3.6, and since the inclusion $C^\infty_0(\Omega)\subset W^{2,\infty}_N(\Omega)$ entails density of $W^{2,\infty}_N(\Omega)$ also in $L^1(\Omega)$, from (3.36), (3.35) and (3.37) it follows by means of a standard argument that actually $u\in C^0_{w-\star}([0,\infty);(C^0(\overline{\Omega}))^\star)$ and $v\in C^0_{w-\star}([0,\infty);L^\infty(\Omega))$ with $u(\cdot,t)\overset{\star}{\rightharpoonup} u_0$ in $(C^0(\overline{\Omega}))^\star$ and $v(\cdot,t)\overset{\star}{\rightharpoonup} v_0$ in $L^\infty(\Omega)$ as $t\searrow 0$. □

Our main result on global solvability in the considered general framework thus becomes an evident consequence:

Proof. (**Proof of Theorem 1.1.**) Since due to (1.6) and (1.7) we can clearly choose $(\phi_\varepsilon)_{\varepsilon\in(0,1)}$ as well as $(u_{0\varepsilon})_{\varepsilon\in(0,1)}$ and $(v_{0\varepsilon})_{\varepsilon\in(0,1)}$ such that (2.4) and (2.5) hold, the statement actually is part of what has been asserted by Lemma 3.7. □

4. Large time relaxation Proof of Theorem 1.2

This section is devoted to the investigation of the large time relaxation feature of (1.5) described in Theorem 1.2. For this purpose, throughout this section we shall assume that ϕ satisfies (1.10), so that in addition to (2.4), $(\phi_\varepsilon)_{\varepsilon\in(0,1)}$ can be chosen in such a way that

$$\sup_{\varepsilon\in(0,1)}\|\phi'_\varepsilon\|_{L^\infty((0,M))}<\infty \qquad \text{for all } M>0. \tag{4.1}$$

In fact, we shall see that under this assumption, unlike in the general setting from Section 3 we can control the integral on the right-hand side of (3.6) in terms of suitably decaying quantities, based on the following observation.

LEMMA 4.1. *Let $n\geq 1$, and assume that (2.5), (2.4) and (4.1) hold. Then given any $q>n$, one can find $\Gamma_3(q)>0$ such that*

$$\frac{d}{dt}\int_\Omega\Big|A^{-\frac12}(u_\varepsilon-\overline{u}_0)\Big|^2+\frac{1}{\Gamma_3(q)}\int_\Omega(u_\varepsilon-\overline{u}_0)^2\leq\Gamma_3(q)\|\nabla v_\varepsilon\|^2_{L^q(\Omega)} \tag{4.2}$$

for all $t>0$ *and* $\varepsilon\in(0,1)$.

Proof. We abbreviate $c_1:=\|v_0\|_{L^\infty(\Omega)}+1$ and may then rely on (4.1) to fix $c_1>0$ such that

$$\left|\phi_\varepsilon'(\xi)\right|\le c_2 \qquad \text{for all } \xi\in[0,c_1] \text{ and any } \varepsilon\in(0,1). \tag{4.3}$$

We moreover employ a Morrey-type estimate to see that thanks to our assumption $q>n$ we can find $c_3=c_3(q)>0$ fulfilling

$$\left|\varphi(x)-\varphi(y)\right|\le c_3\|\nabla\varphi\|_{L^q(\Omega)} \qquad \text{for all } \varphi\in C^1(\overline{\Omega}) \text{ and each } x,y\in\Omega. \tag{4.4}$$

On the right-hand side of (3.6), recalling (2.8), (2.9) and (2.5) we can therefore estimate the integrand according to

$$\begin{aligned}
&\left|\overline{u}_0\phi_\varepsilon\big(v_\varepsilon(x,t)\big)-\overline{u_\varepsilon(\cdot,t)\phi_\varepsilon\big(v_\varepsilon(\cdot,t)\big)}\right|\\
&=\left|\frac{1}{|\Omega|}\cdot\left\{\int_\Omega u_\varepsilon(y,t)dy\right\}\cdot\phi_\varepsilon\big(v_\varepsilon(x,t)\big)-\frac{1}{|\Omega|}\int_\Omega u_\varepsilon(y,t)\phi_\varepsilon\big(v_\varepsilon(y,t)\big)dy\right|\\
&=\frac{1}{|\Omega|}\int_\Omega u_\varepsilon(y,t)\cdot\left|\phi_\varepsilon\big(v_\varepsilon(x,t)\big)-\phi_\varepsilon\big(v_\varepsilon(y,t)\big)\right|dy\\
&\le\frac{1}{|\Omega|}\cdot\left\{\int_\Omega u_\varepsilon(y,t)dy\right\}\cdot\sup_{y\in\Omega}\left|\phi_\varepsilon\big(v_\varepsilon(x,t)\big)-\phi_\varepsilon\big(v_\varepsilon(y,t)\big)\right|\\
&\le\overline{u}_0c_2\cdot\sup_{y\in\Omega}\left|v_\varepsilon(x,t)-v_\varepsilon(y,t)\right|\\
&\le\overline{u}_0c_2c_3\|\nabla v_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \qquad \text{for all } x\in\Omega, t>0 \text{ and } \varepsilon\in(0,1).
\end{aligned}$$

From (3.6) we hence obtain (4.2) if we let $\Gamma_3(q):=\max\{\Gamma_2,\Gamma_2\overline{u}_0^2c_2^2c_3^2|\Omega|\}$. □

Indeed, in low-dimensional cases the right-hand side of (4.2) is, up to an expression already known to decay integrably fast in time, essentially dominated by the dissipation rate encountered in Lemma 3.2. More precisely, taking suitable linear combinations leads to the following.

LEMMA 4.2. *Let* $n\le3$, *and assume (2.5), (2.4) and (4.1). Then there exist* $b>0$ *and* $\Gamma_4>0$ *such that*

$$\mathcal{F}_\varepsilon(t):=\int_\Omega\left|A^{-\frac12}\big(u_\varepsilon(\cdot,t)-\overline{u}_0\big)\right|^2+b\int_\Omega\left|\nabla v_\varepsilon(\cdot,t)\right|^2, \qquad t\ge0,\ \varepsilon\in(0,1), \tag{4.5}$$

satisfies

$$\mathcal{F}_\varepsilon'(t)+\frac{1}{\Gamma_4}\int_\Omega(u_\varepsilon-\overline{u}_0)^2+\frac{1}{\Gamma_4}\int_\Omega|\nabla v_\varepsilon|^4\le\Gamma_4\int_\Omega|\nabla v_\varepsilon|^2 \qquad \textit{for all } t>0 \textit{ and } \varepsilon\in(0,1). \tag{4.6}$$

Proof. Using that $\max\{n,2\}<\frac{2n}{(n-2)_+}$ due to our assumption that $n\le3$, we can pick $q>\max\{n,2\}$ such that $q<\frac{2n}{(n-2)_+}$. We then let $\Gamma_3=\Gamma_3(q)$ be as accordingly provided by Lemma 4.1, and taking Γ_1 from Lemma 3.2 and letting

$$b:=\frac{1}{2\Gamma_1\Gamma_3}, \tag{4.7}$$

we can draw on the compactness of the first among the two continuous embeddings $W^{2,2}(\Omega)\hookrightarrow W^{1,q}(\Omega)\hookrightarrow W^{1,2}(\Omega)$ to infer from an associated Ehrling inequality and standard elliptic regularity theory that there exists $c_1>0$ fulfilling

$$\Gamma_3\|\nabla\varphi\|_{L^q(\Omega)}^2\le\frac{b}{2}\|\Delta\varphi\|_{L^2(\Omega)}^2+c_1\|\nabla\varphi\|_{L^2(\Omega)}^2 \tag{4.8}$$

for all $\varphi\in C^2(\overline{\Omega})$ such that $\frac{\partial\varphi}{\partial\nu}=0$ on $\partial\Omega$. Then from (4.2) we obtain that for all $t>0$ and $\varepsilon\in(0,1)$,

$$\begin{aligned}\frac{d}{dt}\int_\Omega\left|A^{-\frac12}(u_\varepsilon-\overline{u}_0)\right|^2+\frac{1}{\Gamma_3}\int_\Omega(u_\varepsilon-\overline{u}_0)^2 &\le\Gamma_3\|\nabla v_\varepsilon\|_{L^q(\Omega)}^2\\ &\le\frac{b}{2}\int_\Omega|\Delta v_\varepsilon|^2+c_1\int_\Omega|\nabla v_\varepsilon|^2,\end{aligned}$$

which when added to (3.4) shows that with $(\mathcal{F}_\varepsilon)_{\varepsilon\in(0,1)}$ as in (4.5) we have

$$\begin{aligned}\mathcal{F}_\varepsilon'(t)+\frac{1}{\Gamma_3}\int_\Omega(u_\varepsilon-\overline{u}_0)^2+\frac{b}{2}\int_\Omega|\Delta v_\varepsilon|^2+\frac{b}{\Gamma_1}\int_\Omega|\nabla v_\varepsilon|^4&\\ \le\frac{b}{2}\int_\Omega|\Delta v_\varepsilon|^2+c_1\int_\Omega|\nabla v_\varepsilon|^2&\\ +b\Gamma_1\int_\Omega(u_\varepsilon-\overline{u}_0)^2\qquad\text{for all } t>0 \text{ and } \varepsilon\in(0,1).&\end{aligned}$$

In view of (4.7), this yields (4.6) if we choose $\Gamma_4:=\max\{2\Gamma_3,\frac{\Gamma_1}{b},c_1\}$. ∎

According to the decay feature of $t\mapsto\int_\Omega|\nabla v_\varepsilon|^2$ included in (3.2), an analysis of the damped linear ODI in (4.6) already entails the stabilization property of u claimed in Theorem 1.2, and beyond this also provides some further information on decay of the signal gradient.

Lemma 4.3. *Let $n\le3$, and suppose that (2.5), (2.4) and (4.1) hold. Then there exists a null set $N_\star\subset(0,\infty)$ such that $A^{-\frac12}\big(u(\cdot,t)-\overline{u}_0\big)\in L^2(\Omega)$ for all $t\in(0,\infty)\setminus N_\star$ with*

$$A^{-\frac12}\big(u(\cdot,t)-\overline{u}_0\big)\to0\quad\text{in } L^2(\Omega)\qquad\text{as } (1,\infty)\setminus N_\star\ni t\to\infty, \tag{4.9}$$

and one can find $C>0$ such that

$$\int_1^\infty\int_\Omega|\nabla v_\varepsilon|^4\le C\qquad\text{for all } \varepsilon\in(0,1). \tag{4.10}$$

Proof. Since $A^{-\frac12}$ is continuous on $L^2_\perp(\Omega)$, we can fix $c_1>0$ in such a way that

$$\|A^{-\frac12}\varphi\|_{L^2(\Omega)}^2\le c_1\|\varphi\|_{L^2(\Omega)}^2\qquad\text{for all } \varphi\in L^2_\perp(\Omega),$$

whence if we take Γ_4 from Lemma 4.2 and let $c_2:=\frac{1}{c_1\Gamma_4}$, then from (4.6) we infer that for the functions in (4.5) we have

$$\mathcal{F}_\varepsilon'(t)+c_2\mathcal{F}_\varepsilon(t)+\frac{1}{\Gamma_4}\int_\Omega|\nabla v_\varepsilon|^4\le h_\varepsilon(t):=(\Gamma_4+bc_2)\int_\Omega|\nabla v_\varepsilon|^2 \tag{4.11}$$

for all $t>0$ and $\varepsilon\in(0,1)$. If here we first neglect the nonnegative third summand on the left, then by means of a comparison argument we obtain that

$$\mathcal{F}_\varepsilon(t)\le\mathcal{F}_\varepsilon(1)e^{-c_2(t-1)}+\int_1^t e^{-c_2(t-s)}h_\varepsilon(s)ds$$

$$\leq c_3 e^{-c_2(t-1)} + \int_1^t e^{-c_2(t-s)} h_\varepsilon(s)ds \qquad \text{for all } t>1 \text{ and } \varepsilon\in(0,1), \quad (4.12)$$

with $c_3 := \sup_{\varepsilon\in(0,1)} \mathcal{F}_\varepsilon(1)$ being finite due to (3.7) and (3.8).

In order to make appropriate use of this in the framework of the sparse topological information on the approximation properties of $(v_\varepsilon)_{\varepsilon\in(0,1)}$, and especially of $(u_\varepsilon)_{\varepsilon\in(0,1)}$, provided by Lemma 3.7, we note that (3.8) and (3.7) particularly ensure that for all $t_0>1$, with $(\varepsilon_j)_{j\in\mathbb{N}}$ as found there we have $\nabla v_\varepsilon \overset{\star}{\rightharpoonup} \nabla v$ and $A^{-\frac12}(u_\varepsilon - \overline{u}_0) \overset{\star}{\rightharpoonup} A^{-\frac12}(u-\overline{u}_0)$ in $L^\infty((t_0,t_0+1);L^2(\Omega))$ as $\varepsilon=\varepsilon_j\searrow 0$, and that thus, according to lower semicontinuity of the norms in these spaces with respect to the considered convergence type, for

$$\mathcal{F}(t) := \int_\Omega \left|A^{-\frac12}\big(u(\cdot,t)-\overline{u}_0\big)\right|^2 + b\int_\Omega \left|\nabla v(\cdot,t)\right|^2, \qquad t>0, \qquad (4.13)$$

we have

$$\|\mathcal{F}\|_{L^\infty((t_0,t_0+1))} \leq \liminf_{\varepsilon=\varepsilon_j\searrow 0} \|\mathcal{F}_\varepsilon\|_{L^\infty((t_0,t_0+1))} \qquad \text{for all } t_0>1. \qquad (4.14)$$

Here the right-hand side can be controlled by combining (4.12) with (3.26) and (3.2): Indeed, for $t_0>1$ and $t\in(t_0,t_0+1)$, in (4.12) we can estimate

$$\int_1^t e^{-c_2(t-s)} h_\varepsilon(s)ds = e^{-c_2 t}\int_1^t e^{c_2 s} h_\varepsilon(s)ds \leq e^{-c_2 t_0}\int_1^{t_0+1} e^{c_2 s} h_\varepsilon(s)ds \quad \text{for all } \varepsilon\in(0,1),$$

where we may use that (3.26) warrants that

$$h_\varepsilon \to h \quad \text{in } L^1_{loc}((0,\infty)) \quad \text{as } \varepsilon=\varepsilon_j\searrow 0 \qquad (4.15)$$

with $h(t) := (\Gamma_4 + bc_2)\int_\Omega |\nabla v(\cdot,t)|^2$, $t>0$. Therefore, (4.14) along with (4.12) shows that

$$\begin{aligned}\|\mathcal{F}\|_{L^\infty((t_0,t_0+1))} &\leq c_3 e^{-c_2(t_0-1)} + e^{-c_2 t_0}\int_1^{t_0+1} e^{c_2 s} h(s)ds \\ &= c_3 e^{-c_2(t_0-1)} + e^{c_2}\int_1^{t_0+1} e^{-c_2(t_0+1-s)} h(s)ds \qquad \text{for all } t_0>1,\end{aligned}$$

so that since $\int_1^\infty h(s)ds$ is finite by (3.2) and (4.15), and since thus

$$\int_1^{t_0+1} e^{-c_2(t_0+1-s)} h(s)ds = \int_1^\infty \mathbf{1}_{(1,t_0+1)}(s) e^{-c_2(t_0+1-s)} h(s)ds \to 0 \qquad \text{as } t_0\to\infty$$

according to the dominated convergence theorem, it follows that

$$\|\mathcal{F}\|_{L^\infty((t_0,t_0+1))} \to 0 \qquad \text{as } t_0\to\infty.$$

With some suitably chosen null set $N_\star\subset(0,\infty)$, this readily establishes the claimed conclusion together with (4.9), whereas (4.10) can be seen by going back to (4.11) and integrating over $t\in(1,T)$ for $T>1$, which namely reveals that in view of our definition of c_3,

$$\frac{1}{\Gamma_4}\int_1^T\int_\Omega |\nabla v_\varepsilon|^4 \leq \mathcal{F}_\varepsilon(1) + (\Gamma_4 + bc_2)\int_1^T\int_\Omega |\nabla v_\varepsilon|^2$$

$$\leq c_3 + (\Gamma_4 + bc_2) \cdot \frac{1}{2} |\Omega| \cdot \left(\|v_0\|_{L^\infty(\Omega)} + 1 \right)^2 \quad \text{for all } T > 1 \text{ and } \varepsilon \in (0,1),$$

again thanks to (3.2). ∎

Finally, we only need to observe that thanks to the fact that the exponent 4 appearing in (4.10) exceeds the currently considered spatial dimension, through a Morrey-type inequality the latter can be combined with (3.1) and (2.8) so as to yield decay of v in the intended flavor.

LEMMA 4.4. *If $n \leq 3$ and (2.5), (2.4) as well as (4.1) hold, then there exists a null set $N_{\star\star} \subset (0,\infty)$ such that*

$$v(\cdot,t) \to 0 \quad \text{in } L^\infty(\Omega) \qquad \text{as } (1,\infty) \setminus N_{\star\star} \ni t \to \infty. \tag{4.16}$$

Proof. Once more making explicit use of our restriction on n, we again employ a Morrey estimate to find $c_1 > 0$ such that

$$\left\| \varphi - \|\varphi\|_{L^\infty(\Omega)} \right\|_{L^\infty(\Omega)} \leq c_1 \|\nabla \varphi\|_{L^4(\Omega)} \qquad \text{for all } \varphi \in W^{1,4}(\Omega). \tag{4.17}$$

Moreover, combining (3.29) with (3.1) and (3.25) with (4.10) we obtain that

$$\int_1^\infty \int_\Omega uv < \infty \qquad \text{and} \qquad \int_1^\infty \int_\Omega |\nabla v|^4 < \infty,$$

whence given $\eta > 0$ we can choose $t_\eta > 2$ suitably large fulfilling

$$\int_{t_\eta - 1}^{t_\eta} \int_\Omega uv \leq \frac{m\eta}{2} \qquad \text{and} \qquad \int_{t_\eta - 1}^{t_\eta} \left\| \nabla v(\cdot,t) \right\|_{L^4(\Omega)} dt \leq \frac{\eta}{2c_1}, \tag{4.18}$$

where $m := |\Omega| \overline{u}_0$ is positive according to (1.7). To see that, in fact, with some null set $N_{\star\star} \subset (0,\infty)$ we have

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \leq \eta \qquad \text{for all } t \in (t_\eta, \infty) \setminus N_{\star\star}, \tag{4.19}$$

we note that in view of the continuity of the embedding $W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$ we know from (3.27) that $v_\varepsilon(\cdot,t) \to v(\cdot,t)$ in $L^\infty(\Omega)$ for a.e. $t > 0$ as $\varepsilon = \varepsilon_j \searrow 0$, with $(\varepsilon_j)_{j \in \mathbb{N}}$ as provided there. As a consequence of this, namely, we may rely on (2.9) to see that with some null set $N_{\star\star} \subset (0,\infty)$,

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \leq \|v(\cdot,t_0)\|_{L^\infty(\Omega)} \qquad \text{for all } t_0 \in (0,\infty) \setminus N_{\star\star} \text{ and each } t \in (t_0,\infty) \setminus N_{\star\star}, \tag{4.20}$$

while thanks to (3.30), upon enlarging $N_{\star\star}$ if necessary we may assume that moreover

$$\int_\Omega u(\cdot,t) = m \qquad \text{for all } t \in (0,\infty) \setminus N_{\star\star}. \tag{4.21}$$

In particular, using (4.17) and (4.18) we can estimate

$$\int_{t_\eta - 1}^{t_\eta} \|v(\cdot,t)\|_{L^\infty(\Omega)} \cdot m dt$$

$$= \int_{t_\eta - 1}^{t_\eta} \int_\Omega u(x,t) v(x,t) dx dt - \int_{t_\eta - 1}^{t_\eta} \int_\Omega u(x,t) \cdot \left\{ v(x,t) - \|v(\cdot,t)\|_{L^\infty(\Omega)} \right\} dx dt$$

$$\leq \frac{m\eta}{2} + \int_{t_\eta-1}^{t_\eta} \left\{ \int_\Omega u(x,t)dx \right\} \cdot \left\| v(\cdot,t) - \|v(\cdot,t)\|_{L^\infty(\Omega)} \right\|_{L^\infty(\Omega)} dt$$
$$\leq \frac{m\eta}{2} + c_1 m \int_{t_\eta-1}^{t_\eta} \left\| \nabla v(\cdot,t) \right\|_{L^4(\Omega)} dt$$
$$\leq \frac{m\eta}{2} + c_1 m \cdot \frac{\eta}{2c_1},$$

so that since, on the other hand,

$$\int_{t_\eta-1}^{t_\eta} \|v(\cdot,t)\|_{L^\infty(\Omega)} \cdot m dt \geq m \cdot \operatorname*{essinf}_{t\in(t_\eta-1,t_\eta)} \|v(\cdot,t)\|_{L^\infty(\Omega)}$$

by (4.20) and (4.21), it follows that

$$\operatorname*{essinf}_{t\in(t_\eta-1,t_\eta)} \|v(\cdot,t)\|_{L^\infty(\Omega)} \leq \eta.$$

Again thanks to (4.20), this entails (4.19) and thereby implies (4.16). ∎

Our main result on large time stabilization has thereby been achieved already.

Proof. (**Proof of Theorem 1.2.**) Noting that our assumptions in (1.10) enable us to choose $(\phi_\varepsilon)_{\varepsilon\in(0,1)}$ in such a way that both (2.4) and (4.1) hold, we may take (u,v) as accordingly provided by Lemma 3.7, and then conclude as intended by combining Lemma 4.3 with Lemma 4.4 and letting $N := N_\star \cup N_{\star\star}$, with the null sets $N_\star$ and $N_{\star\star}$ introduced there. ∎

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Appendix. As we could not find an appropriate reference for this in the literature, let us finally include a derivation of the following general statement on weak L^1 convergence that has been used in Lemma 3.7.

LEMMA A.1. *Let $N \geq 1$ and $G \subset \mathbb{R}^N$ be measurable with $|G| < \infty$, and suppose that $(\rho_j)_{j\in\mathbb{N}} \subset C^0([0,\infty))$ and $(w_j)_{j\in\mathbb{N}} \subset L^1(G;[0,\infty))$ are such that as $j \to \infty$ we have*

$$\sup_{\xi\in[0,M]} \left|\rho_j(\xi) - \xi\right| \to 0 \qquad \text{for all } M > 0 \tag{A.1}$$

and

$$w_j \rightharpoonup w \qquad \text{in } L^1(G), \tag{A.2}$$

and that there exists $K > 0$ such that

$$\left|\rho_j(\xi)\right| \leq K\xi \qquad \text{for all } \xi \geq 1 \text{ and } j \in \mathbb{N}. \tag{A.3}$$

Then

$$\rho_j(w_j) \rightharpoonup w \quad \text{in } L^1(G) \qquad \text{as } j \to \infty. \tag{A.4}$$

In particular, this conlcusion holds whenever $(\rho_j)_{j\in\mathbb{N}} \subset C^0([0,\infty))$ is such that

$$0 \leq \rho_j(\xi) \nearrow \xi \quad \text{as } j \to \infty \qquad \text{for all } \xi \geq 0. \tag{A.5}$$

Proof. Since $(w_j)_{j\in\mathbb{N}}$ is relatively compact with respect to the weak topology in $L^1(G)$ by (A.2), from the De la Vallée-Poussin theorem we obtain $c_1>0$ and a function $\psi:[0,\infty)\to(0,\infty)$ such that $\frac{\psi(\xi)}{\xi}\to+\infty$ as $\xi\to\infty$, and that

$$\int_G \psi(w_j)\le c_1 \quad \text{for all } j\in\mathbb{N}. \tag{A.6}$$

Now given $0\not\equiv\varphi\in L^\infty(G)$ and $\eta>0$, we abbreviate $c_2:=\|\varphi\|_{L^\infty(G)}$ and first pick $\delta>0$ small enough fulfilling

$$\delta\le\frac{\eta}{4c_1c_2\cdot(K+1)}, \tag{A.7}$$

then choose $M\ge1$ such that

$$\frac{\psi(\xi)}{\xi}\ge\frac{1}{\delta} \quad \text{for all } \xi>M, \tag{A.8}$$

and taking $c_3>0$ large enough such that in accordance with (A.2) we have

$$\int_G w_j\le c_3 \qquad \text{for all } j\in\mathbb{N}, \tag{A.9}$$

we use (A.1) and again (A.2) to fix $j_\eta\in\mathbb{N}$ in such a way that

$$\big|\rho_j(\xi)-\xi\big|\le\frac{\eta}{4c_2|G|} \quad \text{for all } j\ge j_\eta \text{ and } \xi\in[0,M] \tag{A.10}$$

as well as

$$\left|\int_G w_j\varphi-\int_G w\varphi\right|\le\frac{\eta}{2} \quad \text{for all } j\ge j_\eta. \tag{A.11}$$

Then for any $j\ge j_\eta$, in the identity

$$\int_G\rho_j(w_j)\varphi-\int_G w\varphi=\int_G\{\rho_j(w_j)-w_j\}\cdot\varphi+\left\{\int_G w_j\varphi-\int_G w\varphi\right\}, \qquad j\in\mathbb{N}, \tag{A.12}$$

we can use our definition of c_2 to estimate

$$\begin{aligned}\left|\int_G\{\rho_j(w_j)-w_j\}\cdot\varphi\right| &\le c_2\int_G|\rho_j(w_j)-w_j|\\ &= c_2\int_{\{w_j\le M\}}|\rho_j(w_j)-w_j|+c_2\int_{\{w_j>M\}}|\rho_j(w_j)-w_j|\end{aligned} \tag{A.13}$$

for all $j\in\mathbb{N}$, where according to (A.10),

$$c_2\int_{\{w_j\le M\}}|\rho_j(w_j)-w_j|\le c_2\int_{\{w_j\le M\}}\frac{\eta}{4c_2|G|}=\frac{\eta}{4}\cdot\frac{|\{w_j\le M\}|}{|G|}\le\frac{\eta}{4} \qquad \text{for all } j\ge j_\eta. \tag{A.14}$$

Moreover, recalling that $M\ge1$ we may rely on (A.3) to see that thanks to (A.8), (A.6) and (A.7),

$$c_2\int_{\{w_j>M\}}|\rho_j(w_j)-w_j| \le c_2\int_{\{w_j>M\}}|\rho_j(w_j)|+c_2\int_{\{w_j>M\}}w_j$$

$$
\begin{aligned}
&\leq c_2 \cdot (K+1) \int_{\{w_j > M\}} w_j \\
&= c_2 \cdot (K+1) \int_{\{w_j > M\}} \frac{w_j}{\psi(w_j)} \cdot \psi(w_j) \\
&\leq c_2 \cdot (K+1)\delta \int_{\{w_j > M\}} \psi(w_j) \\
&\leq c_2 \cdot (K+1)\delta \cdot c_1 \\
&\leq \frac{\eta}{4} \qquad \text{for all } j \in \mathbb{N}.
\end{aligned}
$$

Together with (A.14) inserted into (A.13), this shows that (A.12) along with (A.11) implies the inequality

$$
\left| \int_G \rho_j(w_j)\varphi - \int_G w\varphi \right| \leq \frac{\eta}{4} + \frac{\eta}{4} + \left| \int_G w_j\varphi - \int_G w\varphi \right| \leq \eta \qquad \text{for all } j \geq j_\eta,
$$

from which (A.4) follows for such $(\rho_j)_{j\in\mathbb{N}}$ due to the fact that $\eta > 0$ and $\varphi \in L^\infty(G) \cong (L^1(G))^\star$ were arbitrary.

The additional claim concerning sequences $(\rho_j)_{j\in\mathbb{N}}$ fulfilling (A.5) results from this upon observing that in this case (A.3) is trivially satisfied, whereas (A.1) is a consequence of Dini's theorem. ∎

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