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GENERALIZED INTEGRAL EQUATION METHOD FOR AN ELLIPTIC NONLOCAL EQUATION IN MEASURE SPACE*

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Abstract. A solution strategy, called generalized integral equation method, is proposed to solve a class of elliptic nonlocal equations in measure space, within which both the continuous and discrete nonlocal problems can be taken as specific instances. By extracting the main ingredients of integral equation method, we develop a generalized integral equation method in an abstract operator framework. As a matter of fact, the classic integral equation method for continuous local partial differential equations can be categorized into this framework. The key ingredient of the proposed method is to derive the generalized boundary integral equations, which can be coupled appropriately with the interior operator equation to obtain a reduced problem. We prove that the resulting system is well-posed by showing that it admits an equivalent formulation with strong coercivity, and the solution of the reduced problem is the same as that of the original one. The proposed method is applied to a nonlocal equation in two-dimensional space discretized by an asymptotically compatible scheme. Numerical experiments validate the effectiveness.

Keywords. Integral equation method; elliptic nonlocal equations; measure space; asymptotic compatibility.

AMS subject classifications. 31A10; 65N80; 65N22; 65R20.

1. Introduction

Recently, nonlocal models have attracted much attention owing to their wide applications in various research areas, such as nonlocal wave propagation, simulation of nonlocal diffusion processes, peridynamical theory of continuum mechanics and so on, see [3, 6, 32, 41]. In this paper, we propose a space reduction method called *generalized integral equation method* (GIEM) to solve a class of elliptic nonlocal equations in measure space:

$$\sigma u(\mathbf{x}) + \mathcal{L}_\gamma u(\mathbf{x}) = f(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega, \tag{1.1}$$

where σ is a prescribed positive number to ensure ellipticity, $\Omega \subset \mathbb{R}^2$ is an unbounded measure space with measure μ, and $f(\mathbf{x})$ is a prescribed source function in Ω. The nonlocal operator $\mathcal{L}_\gamma$ is defined as

$$\mathcal{L}_\gamma u(\mathbf{x}) = \int_\Omega [u(\mathbf{x}) - u(\mathbf{y})]\gamma\left(\mathbf{x}-\mathbf{y}, \frac{\mathbf{x}+\mathbf{y}}{2}\right)\mu(d\mathbf{y}), \tag{1.2}$$

where the two-variable function γ, called kernel function, is supposed to be nonnegative and satisfies

$$\gamma(\alpha,\beta) = \gamma(-\alpha,\beta), \quad \forall \alpha,\beta \in \Omega, \tag{1.3}$$

$$\gamma(\alpha,\beta) = 0, \quad |\alpha| > \delta > 0. \tag{1.4}$$

As for the kernel function $\gamma(\alpha,\beta)$, the first parameter α measures the distance between two location points, and the second parameter β relates to a reference point. The

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minimum positive constant δ in (1.4) is called the horizon of nonlocal kernel function. Here and hereafter, we use $|\cdot|$ to indicate the maximum norm. If γ genuinely depends on the second variable, the nonlocal medium is called inhomogeneous. Otherwise, the nonlocal medium is called homogeneous. In this paper, we allow the kernel to be inhomogeneous [29] and the following moment condition

$$0<a(\mathbf{x})=\frac{1}{2}\int_{\Omega}(\mathbf{s}\otimes\mathbf{s})\gamma(\mathbf{s},\mathbf{x})\mu(d\mathbf{s})<\infty \tag{1.5}$$

is fulfilled in the whole definition domain. Additionally, we assume that there exists a constant $C_1(\delta)$ such that the kernel function $\gamma(\mathbf{s},\mathbf{x})$ satisfies

$$\int_{\Omega}\gamma(\mathbf{s},\mathbf{x})\mu(d\mathbf{s})<C_1(\delta),\quad \forall\mathbf{x}\in\Omega. \tag{1.6}$$

For continuous problems on unbounded domains, any naive grid-based spatial discretization method, such as finite element method (FEM) and finite difference method (FDM) would result in algebraic systems which involve an infinite number of degrees of freedom. Therefore, additional techniques should be developed. Over the past few decades, many methods have been developed, and among them, artificial boundary method (ABM) and integral equation method (IEM) are very popular, see [13, 25]. Both methods are based on the idea of domain decomposition. After introducing a suitable artificial boundary, the unbounded domain is decomposed into two pieces such that the interior region includes all singularities and inhomogeneities, and the exterior region admits nice local symmetry. By delicately exploring this local symmetry, some kind of relations between the Dirichlet data and the Neumann data can be set up. ABM aims at an explicit expression of the Dirichlet-to-Neumann (DtN) mapping [10, 12], while IEM uses boundary integral equations involving both Dirichlet and Neumann data as implicit artificial boundary conditions [2, 23, 37]. ABM relies heavily both on the geometry and the underlying symmetry of governing equation. Comparatively, IEM is more flexible, and the price to pay is the introduction of additional boundary unknowns. Current research on unbounded domain problems faces two difficulties: fast evaluation of exact boundary conditions [16, 19, 20, 26] and derivation of highly accurate boundary conditions for emerging mathematical models, such as nonlocal model problems [9, 34, 38–40].

The goal of this paper is to develop a general space reduction method for solving nonlocal problems on unbounded domains, by taking (1.1) as an example. The proposed method is inspired by the IEM, and actually takes IEM as a special instance. Right in this sense, we term this method generalized integral equation method (GIEM). It is known that a prerequisite of IEM is the existence of a handy expression of background Green's function, with which two boundary integral equations can be set up by exploring the potential theory. These boundary integral equations can then be applied to reduce the computational domain, either to a manifold of lower dimension, or to a bounded truncated subdomain. The idea of IEM has been partially applied for handling some discrete lattice models and nonlocal models in peridynamics [8, 14, 17, 22, 35, 36]. In this paper, by extracting the main ingredients of IEM, we propose a general space reduction method for an abstract structural operator equation. Both local problems and nonlocal problems can be categorized into this framework after performing suitable domain decomposition. Analogous to IEM, we start with an abstract interface problem and derive generalized integral expressions. From these integral expressions, we can set up two generalized boundary integral equations. We explain how these generalized boundary integral equations can be coupled with interior operator equations. To this end, many coupling techniques can be applied. In this paper, we borrow the idea of symmetric coupling proposed by Han [11] and Costabel [5].

The rest of this paper is organized as follows. In Section 2, domain decomposition technique is applied to the model nonlocal elliptic equation to obtain an equivalent operator equation with tridiagonal structure. In Section 3, we present the derivation of GIEM for the abstract operator equations with tridiagonal structure. In Section 4, we explain how to couple the generalized boundary integral equations with the interior equations to obtain a symmetric coupling problem and prove the well-posedness of the resulting reduced coupling system. In Section 5, we consider a specific two-dimensional elliptic nonlocal equation and employ the quadrature-based finite difference scheme for spatial discretization. Three numerical examples are reported to validate the optimal convergence rate and the asymptotic compatibility property. In the end, the conclusion is drawn in Section 6.

2. Domain decomposition and equivalent operator equation

Let us introduce the following linear space

$$L^2(\Omega,\mu)=\left\{u\in L^1_{loc}(\Omega,\mu):\int_\Omega u^2(\mathbf{x})\mu(d\mathbf{x})<\infty\right\}.$$

This is a Hilbert space with inner product

$$(u,v)_\Omega=\int_\Omega u(\mathbf{x})v(\mathbf{x})\mu(d\mathbf{x}).$$

The induced norm by the above inner product will be denoted by $\|\cdot\|_\Omega$ in the sequel. Considering the kernel function γ is symmetric, it is straightforward to verify

$$(\mathcal{L}_\gamma u,v)_\Omega=\frac{1}{2}\int_\Omega\int_\Omega[u(\mathbf{x})-u(\mathbf{y})][v(\mathbf{x})-v(\mathbf{y})]\gamma\left(\mathbf{x}-\mathbf{y},\frac{\mathbf{y}+\mathbf{x}}{2}\right)\mu(d\mathbf{y})\mu(d\mathbf{x}), \tag{2.1}$$

which implies that the nonlocal operator $\mathcal{L}_\gamma$ is symmetric and nonnegative. Let us introduce

$$\mathbb{H}=\left\{u\in L^2(\Omega,\mu):\int_\Omega\int_\Omega[u(\mathbf{x})-u(\mathbf{y})]^2\gamma\left(\mathbf{x}-\mathbf{y},\frac{\mathbf{y}+\mathbf{x}}{2}\right)\mu(d\mathbf{y})\mu(d\mathbf{x})<\infty\right\}.$$

The inner product in this space is naturally specified as

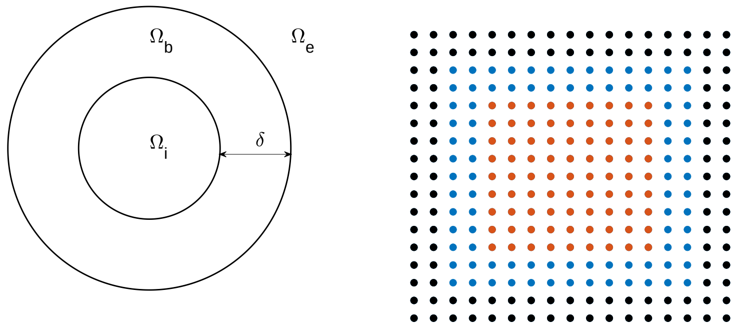
$$\begin{aligned}&(u,v)_\mathbb{H}=(u,v)_\Omega\\&+\frac{1}{2}\int_\Omega\int_\Omega[u(\mathbf{x})-u(\mathbf{y})][v(\mathbf{x})-v(\mathbf{y})]\gamma\left(\mathbf{x}-\mathbf{y},\frac{\mathbf{y}+\mathbf{x}}{2}\right)\mu(d\mathbf{y})\mu(d\mathbf{x}),\quad\forall u,v\in\mathbb{H}.\end{aligned}$$

We denote by $\|\cdot\|_\mathbb{H}$ the induced norm. Obviously, $\mathbb{H}$ is an algebraic linear subspace of $L^2(\Omega,\mu)$. Under the assumption of (1.6), the equivalence of the norms in $\mathbb{H}$ and $L^2(\Omega,\mu)$ can be proved using Hölder inequality [28]. For more general kernel functions, one can consult [29] for more subtle discussions.

As an implication of (1.6), the operator $\sigma I+\mathcal{L}_\gamma$ is both bounded and coercive in $L^2(\Omega,\mu)$. Therefore, the nonlocal problem (1.1) is well-posed, which means that for any $f\in L^2(\Omega,\mu)$, the problem (1.1) admits a unique solution in $L^2(\Omega,\mu)$.

For any measurable subset $D\subset\Omega$, we define

$$\mathrm{Ext}(D)=\left\{\mathbf{x}\in\Omega\backslash D\mid\exists\mathbf{y}\in D\ s.t.\ \gamma\left(\mathbf{x}-\mathbf{y},\frac{\mathbf{x}+\mathbf{y}}{2}\right)\neq0\right\}.$$



(a) (b)

FIG. 2.1. *Schematic diagrams of domain decomposition. Left: continuous measure space. Right: discrete measure space.* Ω_i*: the area where red grids are located;* Ω_b*: the area where blue grids are located;* Ω_e*: the area where black grids are located.*

Let $\Omega=\Omega_i\cup\Omega_b\cup\Omega_e$ be a non-overlapping domain decomposition of total measure space Ω satisfying

$$\mathrm{Ext}(\Omega_i)=\mathrm{Ext}(\Omega_e)=\Omega_b. \tag{2.2}$$

We assume

$$f(\mathbf{x})=0,\quad \forall \mathbf{x}\in\Omega_e, \tag{2.3}$$

$$\gamma\left(\mathbf{x}-\mathbf{y},\frac{\mathbf{x}+\mathbf{y}}{2}\right)=\gamma_0(\mathbf{x}-\mathbf{y}),\quad \forall \mathbf{x},\mathbf{y}\in\Omega_b\cup\Omega_e, \tag{2.4}$$

where $\gamma_0(\mathbf{x})$ is a homogeneous kernel function. The requirements (2.3)-(2.4) can be easily fulfilled if the source function f admits a compact support, and the kernel function γ becomes homogeneous when spatial points are far away from the origin. Domain decomposition diagrams are given in Figure 2.1 to make (2.2) easier to understand for either continuous or discrete case.

Applying the above domain decomposition technique, we can rewrite the nonlocal elliptic problem (1.1) into an equivalent component form: find

$$(u_i,u_b,u_e)\in L^2(\Omega_i,\mu)\times L^2(\Omega_b,\mu)\times L^2(\Omega_e,\mu)$$

such that

$$\begin{aligned}
&\sigma u_i(\mathbf{x})+\int_{\Omega_i}[u_i(\mathbf{x})-u_i(\mathbf{y})]\gamma\left(\mathbf{x}-\mathbf{y},\frac{\mathbf{y}+\mathbf{x}}{2}\right)\mu(d\mathbf{y})\\
&\qquad+\int_{\Omega_b}[u_i(\mathbf{x})-u_b(\mathbf{y})]\gamma\left(\mathbf{x}-\mathbf{y},\frac{\mathbf{y}+\mathbf{x}}{2}\right)\mu(d\mathbf{y})=f(\mathbf{x}),\ \forall \mathbf{x}\in\Omega_i,\\
&\sigma u_b(\mathbf{x})+\int_{\Omega_b}[u_b(\mathbf{x})-u_b(\mathbf{y})]\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y})\\
&\qquad+\int_{\Omega_i}[u_b(\mathbf{x})-u_i(\mathbf{y})]\gamma\left(\mathbf{x}-\mathbf{y},\frac{\mathbf{y}+\mathbf{x}}{2}\right)\mu(d\mathbf{y})
\end{aligned}$$

$$+\int_{\Omega_e}[u_b(\mathbf{x})-u_e(\mathbf{y})]\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y})=f(\mathbf{x}),\ \forall \mathbf{x}\in\Omega_b, \tag{2.5}$$

$$\sigma u_e(\mathbf{x})+\int_{\Omega_e}[u_e(\mathbf{x})-u_e(\mathbf{y})]\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y})$$
$$+\int_{\Omega_b}[u_e(\mathbf{x})-u_b(\mathbf{y})]\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y})=0,\ \forall \mathbf{x}\in\Omega_e.$$

For the sake of brevity of formulations, let us introduce the following operators:

$$\begin{aligned}
L_{ii}u_i(\mathbf{x})&=\sigma u_i(\mathbf{x})+\int_{\Omega_i}[u_i(\mathbf{x})-u_i(\mathbf{y})]\gamma\left(\mathbf{x}-\mathbf{y},\frac{\mathbf{y}+\mathbf{x}}{2}\right)\mu(d\mathbf{y})\\
&\quad+u_i(\mathbf{x})\int_{\Omega_b}\gamma\left(\mathbf{x}-\mathbf{y},\frac{\mathbf{y}+\mathbf{x}}{2}\right)\mu(d\mathbf{y}),\ \forall \mathbf{x}\in\Omega_i,\\
L_{ee}u_e(\mathbf{x})&=\sigma u_e(\mathbf{x})+\int_{\Omega_e}[u_e(\mathbf{x})-u_e(\mathbf{y})]\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y})\\
&\quad+u_e(\mathbf{x})\int_{\Omega_b}\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y}),\ \forall \mathbf{x}\in\Omega_e,\\
L_{bb}^i u_b(\mathbf{x})&=\frac{\sigma}{2}u_b(\mathbf{x})+\frac{1}{2}\int_{\Omega_b}[u_b(\mathbf{x})-u_b(\mathbf{y})]\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y})\\
&\quad+u_b(\mathbf{x})\int_{\Omega_i}\gamma\left(\mathbf{x}-\mathbf{y},\frac{\mathbf{y}+\mathbf{x}}{2}\right)\mu(d\mathbf{y}),\ \forall \mathbf{x}\in\Omega_b,\\
L_{bb}^e u_b(\mathbf{x})&=\frac{\sigma}{2}u_b(\mathbf{x})+\frac{1}{2}\int_{\Omega_b}[u_b(\mathbf{x})-u_b(\mathbf{y})]\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y})\\
&\quad+u_b(\mathbf{x})\int_{\Omega_e}\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y}),\ \forall \mathbf{x}\in\Omega_b,
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
L_{ib}u_b(\mathbf{x})&=-\int_{\Omega_b}u_b(\mathbf{y})\gamma\left(\mathbf{x}-\mathbf{y},\frac{\mathbf{y}+\mathbf{x}}{2}\right)\mu(d\mathbf{y}),\ \forall \mathbf{x}\in\Omega_i,\\
L_{bi}u_i(\mathbf{x})&=-\int_{\Omega_i}u_i(\mathbf{y})\gamma\left(\mathbf{x}-\mathbf{y},\frac{\mathbf{y}+\mathbf{x}}{2}\right)\mu(d\mathbf{y}),\ \forall \mathbf{x}\in\Omega_b,\\
L_{be}u_e(\mathbf{x})&=-\int_{\Omega_e}u_e(\mathbf{y})\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y}),\ \forall \mathbf{x}\in\Omega_b,\\
L_{eb}u_b(\mathbf{x})&=-\int_{\Omega_b}u_b(\mathbf{y})\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y}),\ \forall \mathbf{x}\in\Omega_e.
\end{aligned} \tag{2.7}$$

Furthermore, let us set

$$L_{bb}=L_{bb}^i+L_{bb}^e,\quad X_i=L^2(\Omega_i,\mu),\quad X_b=L^2(\Omega_b,\mu),\quad X_e=L^2(\Omega_e,\mu).$$

Applying the above operators, we can rewrite the problem (2.5) into a more handy algebraic form: find

$$(u_i,u_b,u_e)\in X_i\times X_b\times X_e$$

such that

$$\begin{bmatrix} L_{ii} & L_{ib} & 0\\ L_{bi} & L_{bb} & L_{be}\\ 0 & L_{eb} & L_{ee}\end{bmatrix}\begin{bmatrix}u_i\\ u_b\\ u_e\end{bmatrix}=\begin{bmatrix}f_i\\ f_b\\ 0\end{bmatrix}, \tag{2.8}$$

where we have set $f_i = f|_{\Omega_i}$ and $f_b = f|_{\Omega_b}$. Note that the (ie)-block and the (ei)-block are zero operators due to the Assumption (2.2).

THEOREM 2.1. *Under the assumption of the kernel function (1.3)-(1.6), the two linear operators*

$$L_{ii}^b := \begin{bmatrix} L_{ii} & L_{ib} \\ L_{bi} & L_{bb}^i \end{bmatrix}, \quad L_{ee}^b := \begin{bmatrix} L_{bb}^e & L_{be} \\ L_{eb} & L_{ee} \end{bmatrix}$$

are symmetric bounded coercive operators in $X_i \times X_b$ and $X_b \times X_e$, respectively.

Proof. The symmetry and the boundedness are obvious. It suffices to prove the coercivity. For any

$$u = (u_i, u_b) \in X_i \times X_b,$$

we have

$$(L_{ii}^b u, u)_{X_i \times X_b} = (L_{ii} u_i, u_i)_{X_i} + (L_{bi} u_i, u_b)_{X_b} + (L_{ib} u_b, u_i)_{X_i} + (L_{bb}^i u_b, u_b)_{X_b}.$$

Resorting to (2.1), we have

$$\begin{aligned}
(L_{ii}^b u, u)_{X_i \times X_b} = {} & \sigma \int_{\Omega_i} u_i^2(\mathbf{x}) \mu(d\mathbf{x}) \\
& + \frac{1}{2} \int_{\Omega_i} \int_{\Omega_i} [u_i(\mathbf{x}) - u_i(\mathbf{y})]^2 \gamma\left(\mathbf{x} - \mathbf{y}, \frac{\mathbf{y}+\mathbf{x}}{2}\right) \mu(d\mathbf{y}) \mu(d\mathbf{x}) + \frac{\sigma}{2} \int_{\Omega_b} u_b^2(\mathbf{x}) \mu(d\mathbf{x}) \\
& + \frac{1}{4} \int_{\Omega_b} \int_{\Omega_b} [u_b(\mathbf{x}) - u_b(\mathbf{y})]^2 \gamma\left(\mathbf{x} - \mathbf{y}, \frac{\mathbf{y}+\mathbf{x}}{2}\right) \mu(d\mathbf{y}) \mu(d\mathbf{x}) \\
& + \int_{\Omega_i} \int_{\Omega_b} [u_i(\mathbf{x}) - u_b(\mathbf{y})]^2 \gamma(\mathbf{x} - \mathbf{y}, \frac{\mathbf{y}+\mathbf{x}}{2}) \mu(d\mathbf{y}) \mu(d\mathbf{x}) \\
\geq {} & \frac{\sigma}{2} \left(\int_{\Omega_i} u_i^2(\mathbf{x}) \mu(d\mathbf{x}) + \int_{\Omega_b} u_b^2(\mathbf{x}) \mu(d\mathbf{x}) \right),
\end{aligned}$$

which validates the coercivity of operator L_{ii}^b in $X_i \times X_e$. The proof of coercivity for the operator L_{ee}^b in $X_b \times X_e$ is analogous, and we omit it here. ∎

3. Generalized integral equation method

In this section, we intend to propose a general space reduction method, called *generalized integral equation method*, for operator equations with tridiagonal block structure as in (2.8). In many cases, after performing domain decomposition, people derive a coupling problem which admits tridiagonal block structure if it is formulated appropriately into the form of operator equations. Usually, it is the third component which presents troubles from the computational point of view. Take (2.8) as an example. If Ω is unbounded, so is the subset Ω_e, and the third component u_e might involve a huge number of degrees of freedom. For this kind of problem, people have tried to develop some analytical tools to remove the cumbersome component u_e, and transform their considered problem into a new equivalent form, but easier to solve. This is the basic idea of space reduction, for which Integral equation method (IEM) is an excellent choice. The goal of this section is to extract the essential ingredients of IEM and generalize them into a framework of abstract operator equation. In the sequel, we mimic the derivation of IEM formulated in [21], and employ the symbol system in the classic IEM book [25].

3.1. Generalized integral expressions and generalized boundary integral equations. Let T be a symmetric bounded coercive linear operator, which acts on a product Hilbert space $V_i\times V_b\times V_e$, and admits the following tridiagonal block structure

$$T=\begin{bmatrix} T_{ii} & T_{ib} & 0\\ T_{bi} & T_{bb} & T_{be}\\ 0 & T_{eb} & T_{ee}\end{bmatrix}. \tag{3.1}$$

We assume that T_{bb} admits the following splitting

$$T_{bb}=T^i_{bb}+T^e_{bb},$$

such that the following two operators

$$T^b_{ii}:=\begin{bmatrix} T_{ii} & T_{ib}\\ T_{bi} & T^i_{bb}\end{bmatrix},\quad T^b_{ee}:=\begin{bmatrix} T^e_{bb} & T_{be}\\ T_{eb} & T_{ee}\end{bmatrix}$$

are symmetric bounded coercive operators in $V_i\times V_b$ and $V_b\times V_e$, respectively. More importantly, we assume that the inverse of T, denoted by G, can be determined in some manner and admits the following component form

$$G=\begin{bmatrix} G_{ii} & G_{ib} & G_{ie}\\ G_{bi} & G_{bb} & G_{be}\\ G_{ei} & G_{eb} & G_{ee}\end{bmatrix}. \tag{3.2}$$

In the sequel, let us introduce some definitions related to the operator T.

DEFINITION 3.1. *We call $(u_i,D_i)\in V_i\times V_b$ an interior solution pair if*

$$T_{ii}u_i+T_{ib}D_i=0.$$

Analogously, we call $(u_e,D_e)\in V_e\times V_b$ an exterior solution pair if

$$T_{ee}u_e+T_{eb}D_e=0.$$

The "interior Neumann" and "exterior Neumann" data for solution pairs (u_i,D_i) and (u_e,D_e) are defined as follows:

$$N_i=\mathcal{N}_i(u_i,D_i):=T_{bi}u_i+T^i_{bb}D_i,\quad N_e=\mathcal{N}_e(u_e,D_e):=-T_{be}u_e-T^e_{bb}D_e. \tag{3.3}$$

Given any *jump pair* $(\tilde D,\tilde N)\in V_b\times V_b$, let us consider the following "interface" problem: find $(u_i,D_i)\in V_i\times V_b$ and $(u_e,D_e)\in V_e\times V_b$, such that

$$T_{ii}u_i+T_{ib}D_i=0, \tag{3.4}$$
$$T_{ee}u_e+T_{eb}D_e=0, \tag{3.5}$$
$$D_e-D_i=\tilde D, \tag{3.6}$$
$$N_e-N_i=\mathcal{N}_e(u_e,D_e)-\mathcal{N}_i(u_i,D_i)=\tilde N. \tag{3.7}$$

The following theorem presents the generalized integral expressions in terms of the jump pair $(\tilde D,\tilde N)$.

THEOREM 3.1 (generalized integral expressions). *If the operator T satisfies the conditions specified at the beginning of this section, then the interface problem (3.4)-(3.7) is uniquely solvable, with*

$$D_i=-G_{bb}\tilde N-(G_{bb}T^e_{bb}+G_{be}T_{eb})\tilde D, \tag{3.8}$$

$$D_e = -G_{bb}\tilde{N} + (G_{bb}T^i_{bb} + G_{bi}T_{ib})\tilde{D}, \tag{3.9}$$

$$\begin{aligned} N_i = &-(T_{bi}G_{ib} + T^i_{bb}G_{bb})\tilde{N} \\ &-(T_{bi}G_{ib}T^e_{bb} + T_{bi}G_{ie}T_{eb} + T^i_{bb}G_{bb}T^e_{bb} + T^i_{bb}G_{be}T_{eb})\tilde{D}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} N_e = &\ (T_{be}G_{eb} + T^e_{bb}G_{bb})\tilde{N} \\ &+(T_{be}G_{eb}T^e_{bb} + T_{be}G_{ee}T_{eb} - T^e_{bb}G_{bb}T^i_{bb} - T^e_{bb}G_{bi}T_{ib})\tilde{D}, \end{aligned} \tag{3.11}$$

$$u_i = -G_{ib}\tilde{N} - (G_{ib}T^e_{bb} + G_{ie}T_{eb})\tilde{D}, \tag{3.12}$$

$$u_e = -G_{eb}\tilde{N} - (G_{eb}T^e_{bb} + G_{ee}T_{eb})\tilde{D}. \tag{3.13}$$

Proof. Let $(u_i, D_i) \in V_i \times V_b$ and $(u_e, D_e) \in V_e \times V_b$ be a solution of interface problem (3.4)-(3.7). A direct computation shows that

$$\begin{bmatrix} T_{ii} & T_{ib} & 0 \\ T_{bi} & T_{bb} & T_{be} \\ 0 & T_{eb} & T_{ee} \end{bmatrix} \begin{bmatrix} u_i \\ D_i \\ u_e \end{bmatrix} = \begin{bmatrix} 0 \\ -\tilde{N} - T^e_{bb}\tilde{D} \\ -T_{eb}\tilde{D} \end{bmatrix}. \tag{3.14}$$

Acting G onto the both sides of (3.14), we obtain

$$\begin{bmatrix} u_i \\ D_i \\ u_e \end{bmatrix} = \begin{bmatrix} G_{ii} & G_{ib} & G_{ie} \\ G_{bi} & G_{bb} & G_{be} \\ G_{ei} & G_{eb} & G_{ee} \end{bmatrix} \begin{bmatrix} 0 \\ -\tilde{N} - T^e_{bb}\tilde{D} \\ -T_{eb}\tilde{D} \end{bmatrix}, \tag{3.15}$$

from which, we derive

$$\begin{aligned} u_i &= -G_{ib}\tilde{N} - (G_{ib}T^e_{bb} + G_{ie}T_{eb})\tilde{D}, \\ u_e &= -G_{eb}\tilde{N} - (G_{eb}T^e_{bb} + G_{ee}T_{eb})\tilde{D}, \end{aligned}$$

and

$$D_i = -G_{bb}\tilde{N} - (G_{bb}T^e_{bb} + G_{be}T_{eb})\tilde{D}. \tag{3.16}$$

According to the Dirichlet jump condition (3.6), we derive

$$D_e = -G_{bb}\tilde{N} - (G_{bb}T^e_{bb} + G_{be}T_{eb})\tilde{D} + \tilde{D} = -G_{bb}\tilde{N} + (G_{bb}T^i_{bb} + G_{bi}T_{ib})\tilde{D}. \tag{3.17}$$

Furthermore, we have

$$\begin{aligned} N_i &= T_{bi}u_i + T^i_{bb}D_i \\ &= -(T_{bi}G_{ib} + T^i_{bb}G_{bb})\tilde{N} \\ &\quad -(T_{bi}G_{ib}T^e_{bb} + T_{bi}G_{ie}T_{eb} + T^i_{bb}G_{bb}T^e_{bb} + T^i_{bb}G_{be}T_{eb})\tilde{D}, \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} N_e &= -T_{be}u_e - T^e_{bb}D_e \\ &= (T_{be}G_{eb} + T^e_{bb}G_{bb})\tilde{N} \\ &\quad +(T_{be}G_{eb}T^e_{bb} + T_{be}G_{ee}T_{eb} - T^e_{bb}G_{bb}T^i_{bb} - T^e_{bb}G_{bi}T_{ib})\tilde{D}. \end{aligned} \tag{3.19}$$

The uniqueness follows since the coefficient matrix in (3.14) is invertible. This ends the proof. ∎

For the sake of brevity of notations, let us introduce the following four operators acting on V_b:

$$V = G_{bb}, \tag{3.20}$$

$$K = G_{bi}T_{ib} + G_{bb}T^i_{bb} - \frac{I}{2}, \tag{3.21}$$

$$K' = T_{bi}G_{ib} + T^i_{bb}G_{bb} - \frac{I}{2}, \tag{3.22}$$

$$W = T_{bi}G_{ib}T^e_{bb} + T_{bi}G_{ie}T_{eb} + T^i_{bb}G_{bb}T^e_{bb} + T^i_{bb}G_{be}T_{eb}. \tag{3.23}$$

Note that K' is indeed the adjoint of operator K. With these operators, we can rewrite the expressions (3.8)-(3.11) into the following compact form

$$D_i = K\tilde{D} - V\tilde{N} - \frac{\tilde{D}}{2}, \quad N_i = -W\tilde{D} - K'\tilde{N} - \frac{\tilde{N}}{2}, \tag{3.24}$$

$$D_e = K\tilde{D} - V\tilde{N} + \frac{\tilde{D}}{2}, \quad N_e = -W\tilde{D} - K'\tilde{N} + \frac{\tilde{N}}{2}, \tag{3.25}$$

where N_i and N_e are the Neumann data associated with the solution pairs (u_i, D_i) and (u_e, D_e), with u_i and u_e being determined by (3.12) and (3.13), respectively. The readers might notice that the above four expressions have the same form as the boundary data expressions for the local PDE problems by the integral equation method. Actually, we have tried most to stick to the notations employed in the IEM book [25]. This also justifies why we define the Neumann data as in (3.3), and the jump data as in (3.6)-(3.7).

From (3.24)-(3.25), we can also derive the analogs of the first and second kinds of integral equations for the local PDEs. Actually, if (D_i, N_i) is an interior Cauchy data pair, letting $(0,0)$ be the trivial exterior solution pair, the jumps are

$$\tilde{D} = -D_i, \quad \tilde{N} = -N_i.$$

Substituting the above into (3.24), we derive

$$\frac{D_i}{2} + KD_i - VN_i = 0,$$
$$\frac{N_i}{2} - K'N_i - WD_i = 0.$$

Applying (3.12), we know the interior solution is simply

$$u_i = G_{ib}N_i + (G_{ib}T^e_{bb} + G_{ie}T_{eb})D_i.$$

Correspondingly, if (D_e, N_e) is an exterior Cauchy data pair, by letting $(0,0)$ be the trivial interior solution pair, the jumps are simply

$$\tilde{D} = D_e, \quad \tilde{N} = N_e.$$

Substituting the above into (3.25), we derive

$$\frac{D_e}{2} - KD_e + VN_e = 0, \tag{3.26}$$

$$\frac{N_e}{2} + K'N_e + WD_e = 0. \tag{3.27}$$

Applying (3.13), we know the exterior solution is simply

$$u_e = -G_{eb}N_e - (G_{eb}T^e_{bb} + G_{ee}T_{eb})D_e. \tag{3.28}$$

3.2. Coercivity of the generalized integral operator V and W.

THEOREM 3.2. *Under the assumption for the operator T, the two operators $V:V_b\to V_b$ and $W:V_b\to V_b$ are symmetric, bounded and coercive.*

Proof. The symmetry and boundedness can be derived from the definition directly. It suffices to prove the coercivity, that is, to prove that there exist two positive numbers C_1 and C_2 such that

$$(V\alpha,\alpha)_{V_b}\geq C_1\|\alpha\|^2_{V_b},\quad \forall\alpha\in V_b, \tag{3.29}$$

$$(W\alpha,\alpha)_{V_b}\geq C_2\|\alpha\|^2_{V_b},\quad \forall\alpha\in V_b. \tag{3.30}$$

We only provide a proof for (3.30), since the proof of (3.29) is analogous. Let us consider the interface problem (3.4)-(3.7) with specific jump conditions $\tilde{D}=\alpha$ and $\tilde{N}=0$. We denote the interior and exterior solution pairs by (ϕ_i,ϕ_b) and (ψ_e,ψ_b), respectively. Then according to the jump condition $\tilde{D}=\alpha$, we have

$$\psi_b-\phi_b=\alpha.$$

From (3.24) and (3.25), we know that

$$W\alpha=-N_i=-N_e.$$

Inserting the definition expressions of N_i and N_e (see (3.3)), we derive

$$\begin{aligned}(W\alpha,\alpha)_{V_b}=&(N_i,\phi_b)_{V_b}-(N_e,\psi_b)_{V_b}\\ =&(T^i_{bb}\phi_b+T_{bi}\phi_i,\phi_b)_{V_b}+(T^e_{bb}\psi_b+T_{be}\psi_e,\psi_b)_{V_b}\\ =&(T^i_{bb}\phi_b+T_{bi}\phi_i,\phi_b)_{V_b}+(T_{ii}\phi_i+T_{ib}\phi_b,\phi_i)_{V_i}\\ &+(T^e_{bb}\psi_b+T_{be}\psi_e,\psi_b)_{V_b}+(T_{ee}\psi_e+T_{eb}\psi_b,\psi_e)_{V_e}\\ =&(T^b_{ii}\phi,\phi)_{V_i\times V_b}+(T^b_{ee}\psi,\psi)_{V_b\times V_e},\end{aligned}$$

where the newly introduced variables ϕ and ψ are defined as follows

$$\phi=(\phi_i,\phi_b),\quad \psi=(\psi_e,\psi_b).$$

The last equality is valid since the solution pairs satisfy the homogenous equation. Thanks to the assumption that T^b_{ii} and T^b_{ee} are bounded and coercive, we know that there exists a positive number $C_2>0$ such that

$$(W\alpha,\alpha)_{V_b}\geq C_2\|\alpha\|^2_{V_b},\ \forall\alpha\in\ V_b.$$

This finishes the proof. ∎

REMARK 3.1. The deduction performed in this section was originally stimulated by the nonlocal problem (1.1). However, the key point to derive the generalized integral equations is the tridiagonal block structure of operator equations. The terminology *generalized integral equation method* is justified since the classic IEM can be actually categorized into this operator framework.

4. A symmetric coupling method

Though existent theoretically, the inverse of coefficient matrix of operator equations can be expressed analytically only when the equations admit some global symmetry. In more interesting cases, the best one can expect from (2.8) is the local symmetry, namely, the source function f and the kernel function satisfy (2.3) and (2.4). In these cases, we can embed the sub-blocks L_{be}, L_{eb} and L_{ee} into another operator T with global symmetry (see (3.1)), which acts on a new product space $V_i \times V_b \times V_e$ and satisfies

$$X_b = V_b, \ X_e = V_e, \tag{4.1}$$

and

$$L_{be} = T_{be}, \ L_{eb} = T_{eb}, \ L_{ee} = T_{ee}. \tag{4.2}$$

Note that *the space* V_i *might be completely different from* X_i. We call the operator Equation (2.8) with the above features *locally perturbed operator equation*.

For locally perturbed operator equations, we can apply the GIEM and transform them into new equations without the solution component u_e in X_e. The main idea will be explained in the following subsection, by taking (2.8) as an example.

4.1. Derivation of a coupled operator equation. In this section, we would like to couple the interior equation with the boundary integral equations to obtain a reduced system.

According to our Assumptions (4.1)-(4.2), the third sub-equation of (2.8) reads as

$$T_{eb} u_b + T_{ee} u_e = 0,$$

which implies that (u_e, u_b) is actually an exterior solution pair. Setting

$$N_b = -T^e_{bb} u_b - T_{be} u_e, \tag{4.3}$$

by (3.26)-(3.27), we have

$$\frac{u_b}{2} - K u_b + V N_b = 0, \tag{4.4}$$

$$\frac{N_b}{2} + K' N_b + W u_b = 0. \tag{4.5}$$

Inserting (4.5) into the second sub-equation of (2.8), we obtain

$$\begin{aligned} f_b &= L_{bi} u_i + L_{bb} u_b - T^e_{bb} u_b + T^e_{bb} u_b + T_{be} u_e \\ &\overset{(4.3)}{=} L_{bi} u_i + L^i_{bb} u_b - N_b \\ &\overset{(4.5)}{=} L_{bi} u_i + L^i_{bb} u_b + W u_b + \left(K' - \frac{I}{2} \right) N_b. \end{aligned} \tag{4.6}$$

Now gathering the first sub-equation of (2.8), (4.6) and (4.4), we obtain

$$\begin{bmatrix} L_{ii} & L_{ib} & 0 \\ L_{bi} & L^i_{bb} + W & K' - \frac{I}{2} \\ 0 & K - \frac{I}{2} & -V \end{bmatrix} \begin{bmatrix} u_i \\ u_b \\ N_b \end{bmatrix} = \begin{bmatrix} f_i \\ f_b \\ 0 \end{bmatrix}, \tag{4.7}$$

where V, K, K', W are defined as in (3.20)-(3.23).

The coefficient matrix of (4.7) is symmetric, but generally indefinite. However, by changing the sign of the third sub-equation of (4.7), we derive the following obvious equivalent form

$$\begin{bmatrix} L_{ii} & L_{ib} & 0 \\ L_{bi} & L^i_{bb} + W & K' - \frac{I}{2} \\ 0 & -K + \frac{I}{2} & V \end{bmatrix} \begin{bmatrix} u_i \\ u_b \\ N_b \end{bmatrix} = \begin{bmatrix} f_i \\ f_b \\ 0 \end{bmatrix}. \tag{4.8}$$

Unlike (4.7), though nonsymmetric, the coefficient matrix of (4.8) is positive definite, which will be proved in the next subsection. For the ease of reference, the coefficient matrix of (4.8) will be denoted by L_p. The integral form of the reduced nonlocal equations is given in the Appendix.

4.2. Well-posedness of reduced problems. As stated in Section 4.1, the final coupling system can be expressed out in two equivalent forms. Each has its own merits, as implied in the following theorem.

THEOREM 4.1. *Suppose T_{ii}^b, T_{ee}^b, L_{ii}^b and L_{ee}^b are symmetric, bounded and coercive operators. The Equation (4.8) (or (4.7)) is equivalent to (2.8), in the sense that both are well-posed and the first two components of their solutions are exactly the same. Besides, the third components are related by*

$$N_b = -T_{bb}^e u_b - T_{be} u_e, \tag{4.9}$$

$$u_e = -G_{eb} N_b - (G_{eb} T_{bb}^e + G_{ee} T_{eb}) u_b. \tag{4.10}$$

Proof. By Theorem 2.1 and Theorem 3.2, for any $u = (u_i, u_b, N_b^1) \in X_i \times X_b \times X_b$, it holds that

$$\begin{aligned}
(u, L_p u) &= (u_i, L_{ii} u_i + L_{ib} u_b)_{X_i} + (N_b^1, (-K + \frac{I}{2}) u_b + V N_b^1)_{X_b} \\
&\quad + (u_b, L_{bi} u_i + L_{bb}^i u_b + W u_b + (K' - \frac{I}{2}) N_b^1)_{X_b} \\
&= (u_i, L_{ii} u_i)_{X_i} + (u_i, L_{ib} u_b)_{X_i} + (u_b, L_{bi} u_i)_{X_b} \\
&\quad + (u_b, L_{bb}^i u_b)_{X_b} + (u_b, W u_b)_{X_b} + (N_b^1, V N_b^1)_{X_b} \\
&\geq \frac{\sigma}{2} \left(||u_i||_{X_i}^2 + ||u_b||_{X_b}^2 \right) + C_4 ||u_b||_{X_b}^2 + C_2 ||N_b^1||_{X_b}^2 \\
&\geq \min\left(\frac{\sigma}{2}, C_4, C_2 \right) \left(||u_i||_{X_i}^2 + ||u_b||_{X_b}^2 + ||N_b^1||_{X_b}^2 \right).
\end{aligned}$$

Besides, for any $v = (v_i, v_b, N_b^2) \in X_i \times X_b \times X_b$, we have

$$\begin{aligned}
(u, L_p v) &= (u_i, L_{ii} v_i + L_{ib} v_b)_{X_i} + (N_b^1, (-K + \frac{I}{2}) v_b + V N_b^2)_{X_b} \\
&\quad + (u_b, L_{bi} v_i + L_{bb}^i v_b + W v_b + (K' - \frac{I}{2}) N_b^2)_{X_b} \\
&\leq C_0 \left(||u_i||_{X_i} ||v_i||_{X_i} + ||u_b||_{X_b} ||v_b||_{X_b} + ||N_b^1||_{X_b} ||N_b^2||_{X_b} \right) \\
&\leq C_0 \left(||u_i||_{X_i} + ||u_b||_{X_b} + ||N_b^1||_{X_b} \right) \left(||v_i||_{X_i} + ||v_b||_{X_b} + ||N_b^2||_{X_b} \right).
\end{aligned}$$

These imply that the operator L_p is bounded and coercive. Therefore, the Equation (4.8) is uniquely solvable. Since the Equation (4.8) is derived from (2.8), we know that the first two components of (4.8) are exactly the same as those of (2.8). Therefore, (4.9) is obvious.

In addition, it is straightforward to verify that

$$\begin{bmatrix} T_{ii} & T_{ib} & 0 \\ T_{bi} & T_{bb} & T_{be} \\ 0 & T_{eb} & T_{ee} \end{bmatrix} \begin{bmatrix} 0 \\ u_b \\ u_e \end{bmatrix} = \begin{bmatrix} T_{ib} u_b \\ T_{bb}^i u_b - N_b \\ 0 \end{bmatrix}.$$

By acting G onto the both sides, the third component then reads as

$$\begin{aligned} u_e &= G_{ei}T_{ib}u_b + G_{eb}(T^i_{bb}u_b - N_b) \\ &= -G_{eb}N_b + G_{ei}T_{ib}u_b + G_{eb}T_{bb}u_b - G_{eb}T^e_{bb}u_b \\ &= -G_{eb}N_b - G_{ee}T_{eb}u_b - G_{eb}T^e_{bb}u_b \\ &= -G_{eb}N_b - (G_{eb}T^e_{bb} + G_{ee}T_{eb})u_b. \end{aligned}$$

This finishes the proof. ∎

5. Numerical experiments

We present some numerical tests by using GIEM to solve an instance of nonlocal problem (1.1). More precisely, we are concerned with the following continuous nonlocal model in two dimensions:

$$\sigma q(\mathbf{x}) + \mathcal{L}_{\gamma_\delta} q(\mathbf{x}) = f(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^2, \tag{5.1}$$

$$q(\mathbf{x}) \to 0, \quad \text{as } |\mathbf{x}| \to 0. \tag{5.2}$$

Here $\sigma > 0$ is a prescribed constant and $f(\mathbf{x})$ is a source function in $L^2(\mathbb{R}^2)$. The nonlocal operator $\mathcal{L}_{\gamma_\delta}$ is given by (1.2). Note that to demonstrate the asymptotic performance, we have taken the horizon parameter δ as an explicit asymptotic argument. The scaled kernel function γ_δ is defined by

$$\gamma_\delta(\alpha,\beta) = \frac{1}{\delta^4}\gamma\left(\frac{\alpha}{\delta},\beta\right), \tag{5.3}$$

where γ indicates a father kernel function. With this definition, we have

$$\mathcal{L}_{\gamma_\delta} q(\mathbf{x}) = \frac{1}{\delta^4}\int_{\mathbb{R}^2}[q(\mathbf{x}) - q(\mathbf{y})]\gamma\left(\frac{\mathbf{x}-\mathbf{y}}{\delta},\frac{\mathbf{y}+\mathbf{x}}{2}\right)d\mathbf{y}.$$

It is known that nonlocal models can be taken as a generalization of local models. Actually, since

$$(\mathcal{L}_{\gamma_\delta} q, p) = \frac{1}{2\delta^4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}[q(\mathbf{x}) - q(\mathbf{y})][p(\mathbf{x}) - p(\mathbf{y})]\gamma\left(\frac{\mathbf{x}-\mathbf{y}}{\delta},\frac{\mathbf{x}+\mathbf{y}}{2}\right)d\mathbf{y}d\mathbf{x}, \tag{5.4}$$

we derive

$$\begin{aligned} &\lim_{\delta\to 0^+}(\mathcal{L}_{\gamma_\delta} q, p) \\ =&\lim_{\delta\to 0^+}\frac{1}{2\delta^4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}[\nabla q(\mathbf{x})\cdot(\mathbf{y}-\mathbf{x})\otimes(\mathbf{y}-\mathbf{x})\cdot\nabla p(\mathbf{x})]\gamma\left(\frac{\mathbf{x}-\mathbf{y}}{\delta},\frac{\mathbf{x}+\mathbf{y}}{2}\right)d\mathbf{y}d\mathbf{x} \\ =&\lim_{\delta\to 0^+}\frac{1}{2\delta^4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}[\nabla q(\mathbf{x})\cdot\mathbf{s}\otimes\mathbf{s}\cdot\nabla p(\mathbf{x})]\gamma\left(\frac{\mathbf{s}}{\delta},\mathbf{x}+\frac{\mathbf{s}}{2}\right)d\mathbf{s}d\mathbf{x} \\ =&\frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}[\nabla q(\mathbf{x})\cdot\mathbf{s}\otimes\mathbf{s}\cdot\nabla p(\mathbf{x})]\gamma(\mathbf{s},\mathbf{x})d\mathbf{s}d\mathbf{x} = \int_{\mathbb{R}^2}\nabla q(\mathbf{x})\cdot a(\mathbf{x})\cdot\nabla p(\mathbf{x})d\mathbf{x}, \end{aligned}$$

where we have set

$$a(\mathbf{x}) = \frac{1}{2}\int_{\mathbb{R}^2}(\mathbf{s}\otimes\mathbf{s})\gamma(\mathbf{s},\mathbf{x})d\mathbf{s}.$$

Note that in general the function a is a symmetric positive definite (SPD) tensor of second order. The above deduction reveals that the nonlocal operator $\mathcal{L}_{\gamma_\delta}$ converges to the local differential operator $\mathcal{L}_{loc}$ defined by

$$\mathcal{L}_{loc} q(\mathbf{x}) = -\nabla\cdot(a(\mathbf{x})\nabla q(\mathbf{x})).$$

In the case that the source function f admits a compact support and the father kernel function γ becomes homogeneous when location points are far away from the origin, we can apply GIEM to solve (5.1)-(5.2) numerically. To achieve this, we have two choices:

- Choice A: deduce first a coupling system of continuous nonlocal equation by GIEM, and then employ quadrature schemes to form a discrete algebraic system with finite dofs;
- Choice B: discretize first the continuous nonlocal equation to derive a discrete nonlocal problem, and then apply GIEM to derive a discrete algebraic system with finite dofs.

In our opinion, Choice B is more preferable, since it is much easier to maintain the structural feature of continuous problem—symmetry or coercivity. Besides, starting from a discrete problem with GIEM will refrain us from computing singular integrals, since in this case the Green's function is a lattice function without singularity.

5.1. Spatial discretization. There are many ways to perform spatial discretization for the nonlocal operators. In this paper, we use the quadrature-based finite difference scheme developed in [7,28]. This kind of scheme is known to be promising to maintain the asymptotic feature for the discrete algebraic system.

Let $\mathcal{T}_h$ denote a uniform rectangular grid over $\mathbb{R}^2$ with mesh size being h in both coordinate directions. Let $\Phi_{\mathbf{n}}(\mathbf{x})$ denote the standard continuous piecewise bilinear basis function at the point $\mathbf{x_n}=\mathbf{n}h$. The nonlocal operator $\mathcal{L}_{\gamma_\delta}$ can be approximated at point $\mathbf{x_n}$ by

$$\mathcal{L}_{\gamma_\delta,h}q(\mathbf{x_n})=\frac{1}{\delta^4}\int_{\mathbf{y}\in\mathbb{R}^2}\mathcal{I}_h\left(\frac{q(\mathbf{x_n})-q(\mathbf{y})}{w(\mathbf{x_n}-\mathbf{y})}\right)w(\mathbf{x_n}-\mathbf{y})\gamma\left(\frac{\mathbf{x_n}-\mathbf{y}}{\delta},\frac{\mathbf{x_n}+\mathbf{x_m}}{2}\right)d\mathbf{y},$$

where $\mathcal{I}_h$ denotes the piecewise bilinear interpolation operator associated with grid $\mathcal{T}_h$, and $w(\mathbf{y})$ indicates the following weight function

$$w(\mathbf{y})=\frac{|y_1|^2+|y_2|^2}{|y_1|+|y_2|},\quad \forall\mathbf{y}=(y_1,y_2)^\top\in\mathbb{R}^2.$$

By definition, it is obvious that $w(-\mathbf{y})=w(\mathbf{y})$. The introduction of weight function $w(\mathbf{y})$ is to obtain an asymptotically compatible scheme. It was first proposed and applied to the constant diffusion coefficient in [7] and further discussions about inhomogeneous coefficient (inhomogeneous kernel) case can be found in [28]. After inserting the expression of interpolation operator $\mathcal{I}_h$, we have

$$\begin{aligned}
&\mathcal{L}_{\gamma_\delta,h}q(\mathbf{x_n})\\
&=\sum_{\mathbf{m}\in\mathbb{Z}^2}\frac{q(\mathbf{x_n})-q(\mathbf{x_m})}{\delta^4 w(\mathbf{x_n}-\mathbf{x_m})}\int_{\mathbb{R}^2}\Phi_{\mathbf{m}}(\mathbf{y})\gamma\left(\frac{\mathbf{x_n}-\mathbf{y}}{\delta},\frac{\mathbf{x_n}+\mathbf{x_m}}{2}\right)w(\mathbf{x_n}-\mathbf{y})d\mathbf{y}\\
&=\sum_{\mathbf{m}\in\mathbb{Z}^2}\frac{q(\mathbf{x_n})-q(\mathbf{x_m})}{\delta^4 w(\mathbf{x_n}-\mathbf{x_m})}\int_{\mathbb{R}^2}\Phi(\mathbf{x_n}-\mathbf{x_m}-\mathbf{s})\gamma\left(\frac{\mathbf{s}}{\delta},\frac{\mathbf{x_n}+\mathbf{x_m}}{2}\right)w(\mathbf{s})d\mathbf{s}\\
&=:\sum_{\mathbf{m}\in\mathbb{Z}^2}a_{\mathbf{n},\mathbf{m}}\left[q(\mathbf{x_n})-q(\mathbf{x_m})\right],
\end{aligned}\tag{5.5}$$

where we have put

$$a_{\mathbf{n},\mathbf{m}}=\frac{1}{\delta^4 w(\mathbf{x_n}-\mathbf{x_m})}\int_{\mathbb{R}^2}\Phi(\mathbf{x_n}-\mathbf{x_m}-\mathbf{s})\gamma\left(\frac{\mathbf{s}}{\delta},\frac{\mathbf{x_n}+\mathbf{x_m}}{2}\right)w(\mathbf{s})d\mathbf{s}.$$

It is straightforward to verify that $a_{\mathbf{n},\mathbf{m}}$ is nonnegative and satisfies

$$a_{\mathbf{n},\mathbf{m}} = a_{\mathbf{m},\mathbf{n}}, \quad \forall \mathbf{n},\mathbf{m} \in \mathbb{Z}^2,$$
$$a_{\mathbf{n},\mathbf{m}} = 0, \quad |\mathbf{n}-\mathbf{m}| > L,$$

where $L = \lceil \delta/h \rceil$. Additionally, by the homogeneity Assumption (2.4), we have

$$c_{\mathbf{m}} = a_{\mathbf{n},\mathbf{n}+\mathbf{m}}, \ |\mathbf{n}| > M, |\mathbf{m}| \le L,$$

where $M = \lceil W/h \rceil$ and W represents the size of the interior and boundary domain.

After performing the spatial discretization, the original problem (5.1)-(5.2) is then transformed into the following one:

$$\sigma q(\mathbf{x_n}) + \sum_{\mathbf{m}\in\mathbb{Z}^2} a_{\mathbf{n},\mathbf{m}}[q(\mathbf{x_n}) - q(\mathbf{x_m})] = f(\mathbf{x_n}), \quad \forall \mathbf{n} \in \mathbb{Z}^2, \tag{5.6}$$

$$q(\mathbf{x_n}) \to 0, \ \text{as } |\mathbf{x}| \to 0. \tag{5.7}$$

In the following numerical tests, we assume that the father kernel function has the following form

$$\gamma(\alpha,\beta) = \zeta(\beta) H(\alpha), \tag{5.8}$$

where $H(\alpha)$ is a nonnegative function satisfying

$$H(-\alpha) = H(\alpha), \quad \forall \alpha \in \mathbb{R}^2,$$
$$H(\alpha) = 0, \quad \forall \alpha \in \mathbb{R}^2 \text{ with } |\alpha| > 1,$$

and $\zeta(\beta)$ satisfies

$$\zeta(\beta) = 1, \quad |\beta| > K.$$

In this case, we have

$$a_{\mathbf{n},\mathbf{m}} = \frac{1}{\delta^4 w(\mathbf{x_n} - \mathbf{x}_m)} \zeta\left(\frac{\mathbf{x_n}+\mathbf{x_m}}{2}\right) \int_{\mathbb{R}^2} \Phi(\mathbf{x_n} - \mathbf{x_m} - \mathbf{s}) H\left(\frac{\mathbf{s}}{\delta}\right) w(\mathbf{s}) d\mathbf{s}.$$

The involved integrals are then computed with a high-accuracy quadrature scheme.

We are now ready to apply GIEM for the discrete nonlocal problem (5.6)-(5.7). According to Theorem 4.1, the resulting coupling algebraic system can be efficiently solved using GMRES iterator [24].

5.2. The Green's function. Research on the Green's function for nonlocal models such as peridynamics and elasticity can be found in [27, 30, 31, 33]. The discrete Green's function with asymptotically compatible quadrature-based finite difference scheme was presented in [7], which can be taken as a nonlocal analog for local PDEs [4].

The Green's function corresponding to (5.6) and (5.7) is given by

$$\sigma G_{\mathbf{n}} + \sum_{|\mathbf{m}|\le L} c_{\mathbf{m}}[G_{\mathbf{n}} - G_{\mathbf{n}+\mathbf{m}}] = \delta_{\mathbf{n},0}, \quad \forall \mathbf{n} \in \mathbb{Z}^2, \tag{5.9}$$

$$G_{\mathbf{n}} \to 0, \ \text{as } |\mathbf{n}| \to 0, \tag{5.10}$$

where $\delta_{\mathbf{n},0}$ denotes the Kronecker symbol. The two-dimensional discrete Fourier transform is defined as

$$(\mathcal{F}G)_{\mathbf{k}} = \sum_{\mathbf{n}\in\mathbb{Z}^2} G_{\mathbf{n}} e^{-i\mathbf{n}\cdot\mathbf{k}}, \quad \forall \mathbf{k}\in\mathbb{R}^2.$$

Performing the above onto both sides of (5.9), we derive

$$(\mathcal{F}G)_{\mathbf{k}} = (\sigma + \sum_{|\mathbf{m}|\le L} c_{\mathbf{m}} e^{-i\mathbf{m}\cdot\mathbf{k}})^{-1}.$$

The inverse discrete Fourier transform leads to

$$G_{\mathbf{n}} = \frac{1}{4\pi^2}\int_{(0,2\pi)^2} (\sigma + \sum_{|\mathbf{m}|\le L} c_{\mathbf{m}} e^{-i\mathbf{m}\cdot\mathbf{k}})^{-1} e^{i\mathbf{n}\cdot\mathbf{k}} d\mathbf{k}, \qquad \forall \mathbf{n}\in\mathbb{Z}^2.$$

Since σ is greater than zero, the unique solution to (5.9) and (5.10) decays exponentially when $|\mathbf{n}|\to+\infty$. Therefore, we can apply the composite trapezoidal formula to discretize the above integral and evaluate it by fast Fourier transform (FFT). If σ reaches zero, the evaluation of Green's function will be much more complicated. But for a simple case, i.e., with the Green's function being the following form

$$G_{\mathbf{n}} = \frac{1}{4\pi^2}\int_{(0,2\pi)^2} \frac{e^{i\mathbf{n}\cdot\mathbf{k}} d\mathbf{k}}{\sigma + 4 - 2(\cos k_1 + \cos k_2)},$$

the reference [15] studied the property of Green's function for $\sigma>0$, $\sigma<0$ and $\sigma\sim 0$, which is useful for the understanding of the nature of singularity and for numerical calculations.

5.3. Numerical results. In the sequel, we set $\sigma=0.01$ and $L=\delta/h$, with h being the grid step size. The function $H(\alpha)$ (see (5.8)) is set as

$$H(\alpha) = 10\exp(-20|\alpha|^2), \quad \forall \alpha\in\mathbb{R}^2. \tag{5.11}$$

Though this function merely vanishes at infinity, due to the fast decaying of Gaussian function, we can still take the horizon of H as 1. This is already a good enough approximation since $H(z)\approx 2\times 10^{-8}$ for all $z\in S^2$. In order to investigate the performance of the proposed method and the discretization scheme, we report numerical results from three perspectives:

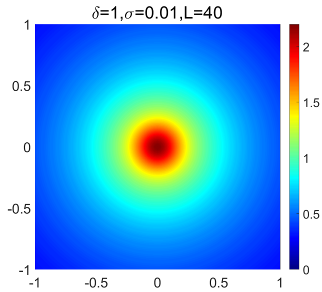
- The convergence order of numerical scheme for the nonlocal equation by refining h with a prescribed δ;
- The asymptotic compatibility of numerical scheme by refining δ and h simultaneously with their ratio fixed;
- The number of iterations with preconditioning as the mesh is refined..

Example 1. As the first example, we consider a homogenous kernel, i.e.

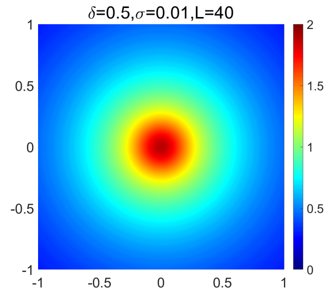
$$\zeta(\beta) = 1, \quad \forall \beta\in\mathbb{R}^2.$$

In this case, the kernel function $\gamma_\delta(\alpha,\beta)$ is simply

$$\gamma_\delta(\alpha,\beta) = \frac{1}{\delta^4} H\left(\frac{\alpha}{\delta}\right), \quad \forall \alpha,\beta\in\mathbb{R}^2.$$



(a)



(b)

FIG. 5.1. *Numerical solutions. Left:* $\delta=1$, $L=40$. *Right:* $\delta=0.5$, $L=40$.

L	$\delta=1$	Order	Iters	$\delta=0.5$	Order	Iters
10	1.649e-2	-	18	3.800e-2	-	23
16	6.626e-2	1.94	19	1.492e-2	1.99	23
22	3.534e-2	1.97	19	7.905e-3	1.99	24
28	2.189e-3	1.99	19	4.884e-3	2.00	24
34	1.487e-3	1.99	19	3.313e-3	2.00	24
40	1.077e-3	1.99	19	2.395e-3	2.00	24

TABLE 5.1. L^2*-errors, convergence orders and number of iterations for Example 1.*

The source function is given by

$$f(x,y)=\exp(-36(x^2+y^2)), \quad \forall (x,y)\in\mathbb{R}^2,$$

which can be taken as a function compactly supported into the unit square $[-1,1]\times[-1,1]$. We illustrate the numerical solutions with the proposed method in Figure 5.1. To investigate the accuracy of discretization scheme, we list in Table 5.1 the L^2-errors, the convergence order and the number of iterations by GMRES for $\delta=1$ and $\delta=0.5$. When $\delta=1$, the exact solution can be expressed out into the following form

$$u(\mathbf{x})=\frac{1}{144\pi}\int_{\mathbb{R}^2}\frac{e^{i\xi\cdot\mathbf{x}-\frac{|\xi|^2}{144}}}{\sigma+\frac{\pi}{2}(1-e^{-\frac{|\xi|^2}{80}})}d\xi, \quad \forall \mathbf{x}\in\mathbb{R}^2.$$

It can be seen from the above formula that the exact solution is expressed in an integral form. But considering that the integrand decays exponentially, we can select a computational domain sufficiently large to be truncated and then apply high-accuracy quadrature scheme to calculate the solution at the given point. The L^2-norm of errors between the exact and numerical solutions is defined as follows

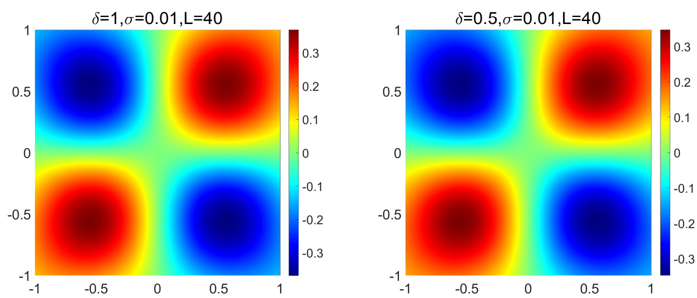
$$\|err\|_2=\sqrt{\sum_{|\mathbf{n}|\le M+L}(u^{\mathbf{n}}_{exa}-u^{\mathbf{n}}_{num})^2},$$

where $u^{\mathbf{n}}_{exa}$ and $u^{\mathbf{n}}_{num}$ denote the exact and numerical solutions at the point $\mathbf{x}_{\mathbf{n}}$, respectively. A second-order spatial convergence rate with fixed δ is clearly observed in

L	$\delta=1$	Order	Iters	$\delta=0.5$	Order	Iters
10	7.669e-3	-	18	1.675e-2	-	27
16	2.923e-3	2.05	20	6.416e-3	2.04	31
22	1.542e-3	2.01	20	3.390e-3	2.00	32
28	9.2773-4	2.11	20	2.056e-3	2.07	33
34	6.102e-4	2.16	20	1.367e-3	2.10	33
40	4.229e-4	2.26	20	9.610e-4	2.17	33

TABLE 5.2. *L^2-errors, convergence orders and number of iterations for Example 2.*

Table 5.1. Besides, the number of iterations is fairly stable, which is in accordance with Theorem 4.1.



(a) (b)

FIG. 5.2. *Numerical solutions. Left: $\delta=1$, $L=40$. Right: $\delta=0.5$, $L=40$.*

Example 2. As the second example, we employ a spatially inhomogeneous kernel function, i.e.,

$$\zeta(\beta)=1+\exp(-25|\beta|^2), \quad \forall\beta\in\mathbb{R}^2,$$

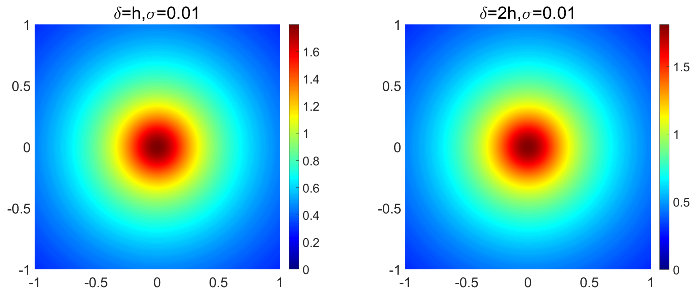
which leads to the kernel function

$$\gamma_\delta(\alpha,\beta)=\frac{1}{\delta^4}\zeta(\beta)H\left(\frac{\alpha}{\delta}\right), \quad \forall\alpha,\beta\in\mathbb{R}^2.$$

The source function is given by

$$f(x,y)=\begin{cases}\frac{\sin(\pi x)\sin(\pi y)}{\pi^2}, & (x,y)\in[-1,1]\times[-1,1],\\ 0, & \text{otherwise.}\end{cases}$$

The computational domain is also set as $[-1,1]\times[-1,1]$. From the expression of $\zeta(\beta)$, we know that the kernel function can be taken homogeneous outside of the computational domain. The numerical solutions are illustrated in Figure 5.2. In Table 5.2, we show the L^2-errors and the number of iterations by GMRES. The reference solution is computed by setting the mesh size as $h=2^{-6}$. For this inhomogeneous numerical test, a second



(a) (b)

FIG. 5.3. *Numerical solutions. Left: $\delta=h$, $L=1$. Right: $\delta=2h$, $L=2$.*

h	$\|\|err\|\|_2$	Order	$\|\|err\|\|_\infty$	Order	Iters
2^{-3}	1.029e-2	-	6.048e-2	-	15
2^{-4}	2.733e-3	1.91	1.405e-2	2.11	19
2^{-5}	7.021e-4	1.96	3.455e-3	2.02	25
2^{-6}	1.769e-4	1.99	8.602e-4	2.01	28
2^{-7}	4.430e-5	2.00	2.148e-4	2.00	28
2^{-8}	1.107e-5	2.00	5.368e-5	2.00	27
2^{-9}	2.767e-6	2.00	1.341e-5	2.00	25

TABLE 5.3. *L^2-errors and L^∞-errors δ-convergence orders for Example 3 with $\delta=h$.*

h	$\|\|err\|\|_2$	order	$\|\|err\|\|_\infty$	Order	Iters
2^{-3}	1.162e-2	-	6.878e-2	-	22
2^{-4}	3.090e-3	1.91	1.594e-2	2.11	29
2^{-5}	7.913e-4	1.97	3.915e-3	2.02	36
2^{-6}	1.992e-4	1.99	9.745e-4	2.01	51
2^{-7}	4.988e-5	2.00	2.432e-4	2.00	89
2^{-8}	1.248e-5	2.00	6.085e-5	2.00	135
2^{-9}	3.125e-6	2.00	1.523e-5	2.00	173

TABLE 5.4. *L^2-errors and L^∞-errors δ-convergence orders for Example 3 with $\delta=2h$.*

order convergence rate and a stable number of iterations after preconditioning are also observed.

Example 3. In the third example, the source function is same as that of the first example, and the kernel function is given by

$$\gamma_\delta(\alpha,\beta)=\frac{1}{\delta^4}H\left(\frac{\alpha}{\delta}\right), \quad \forall \alpha,\beta\in\mathbb{R}^2.$$

We are mainly concerned with the δ-convergence of the discrete scheme (5.5). To achieve this, we maintain the ratio $\delta/h=\mathcal{O}(1)$ and compute both the L^2-errors and L^∞-errors.

Note that for this example, the limiting local equation is given by

$$\sigma u(\mathbf{x}) - \frac{\pi}{160}\Delta u(\mathbf{x}) = f(\mathbf{x}). \tag{5.12}$$

We set the computational domain as $[-1,1]\times[-1,1]$. Using Fourier transform, we obtain the exact solution of integral form

$$u(\mathbf{x}) = \frac{1}{144\pi}\int_{\mathbb{R}^2} \frac{e^{i\xi\cdot\mathbf{x} - \frac{|\xi|^2}{144}}}{\sigma + \frac{\pi}{160}|\xi|^2} d\xi, \quad \forall \mathbf{x}\in\mathbb{R}^2.$$

Since σ is a positive number, the integrand in the above integral decays exponentially. Therefore, similar to the first example, we can choose a sufficiently large computational region and apply high-accuracy quadrature scheme to approximate the integral at the given point. The numerical solutions obtained by the proposed method are illustrated in Figure 5.3.

In Table 5.3 and Table 5.4, we list the L^2-errors and L^∞-errors for $\delta = h$ and $\delta = 2h$, respectively. A second order convergence rate is obviously observed, which validates the asymptotic compatibility property. However, unlike the previous two numerical examples, the number of iterations with GMRES is less stable.

6. Conclusion

We proposed a general space reduction method, called generalized integral equation method, to solve a class of elliptic nonlocal equations in measure space. By extracting the main ingredients and mimicking the derivation of integral equation method (IEM) for the continuous PDEs, we set up a theory for a class of structural operator equations, which actually presents a sufficiently large framework. As a matter of fact, the classical IEM can be categorized into this framework. The idea of symmetric coupling was borrowed to reduce linear systems only with local symmetry. Besides, we proved the well-posedness of the reduced coupling systems.

There are many issues worthy of further study. Even for the nonlocal problems of elliptic type, we have not considered the fast algorithm of generalized integral operators. Though the idea of cluster method or multipole expansion can be conceptually applied to speed up the evaluation process, the details should be worked out. For time-dependent nonlocal problems, the situation might be more challenging. The convolution quadrature method [1, 18] is promising and results in a stable fully discrete numerical scheme. However, both the memory cost and the computation complexity would be too much to bear. We will report any relevant progress in a future work.

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Appendix. Integral form of the reduced nonlocal problem. In the appendix, we will present the integral form of the reduced nonlocal problem (4.7). According to the above deduction, we know that the original problem can be transformed into an equivalent reduced operator equation. In order to make the operator Equation (4.8) easier to understand, let us express it out more explicitly. Considering the last two sub-equations of (4.8) involve generalized boundary integral operators, we will derive the integral representations of these operators in the sequel.

First, let us recall the definitions of operator T. In the component form, it reads as ($\alpha,\gamma\in\{i,b,e\}$, $\alpha\neq\gamma$ and $\beta\in\{i,e\}$)

$$T_{\alpha\gamma}u_\gamma(\mathbf{x}) = -\int_{\Omega_\gamma} u_\gamma(\mathbf{y})\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y}),\ \forall \mathbf{x}\in\Omega_\alpha,$$

$$\begin{aligned}T_{\beta\beta}u_\beta(\mathbf{x}) =& \sigma u_\beta(\mathbf{x}) + \int_{\Omega_\beta}[u_\beta(\mathbf{x})-u_\beta(\mathbf{y})]\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y})\\ &+u_\beta(\mathbf{x})\int_{\Omega_b}\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y}),\ \forall \mathbf{x}\in\Omega_i,\\ T_{bb}^{\beta}u_b(\mathbf{x}) =& \frac{\sigma}{2}u_b(\mathbf{x}) + \frac{1}{2}\int_{\Omega_b}[u_b(\mathbf{x})-u_b(\mathbf{y})]\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y})\\ &+u_b(\mathbf{x})\int_{\Omega_\beta}\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y}),\ \forall \mathbf{x}\in\Omega_b,\end{aligned} \tag{A.1}$$

and

$$T_{bb}^e = L_{bb}^e,\quad T_{ee}=L_{ee},\quad T_{eb}=L_{eb},\quad T_{be}=L_{be}.$$

See (2.6) and (2.7) for the definition of component operators of L.

A direct computation shows that ($\alpha,\beta,\gamma\in\{i,b,e\}$, $\beta\neq\gamma$)

$$\begin{aligned}G_{\alpha\beta}T_{\beta\gamma}\varphi(\mathbf{x}) &= -\int_{\Omega_\beta}\int_{\Omega_\gamma}G(\mathbf{x},\mathbf{z})\varphi(\mathbf{y})\gamma_0(\mathbf{z}-\mathbf{y})\mu(d\mathbf{y})\mu(d\mathbf{z}),\ \forall\mathbf{x}\in\Omega_\alpha,\\ T_{\beta\gamma}G_{\gamma\alpha}\varphi(\mathbf{x}) &= -\int_{\Omega_\gamma}\int_{\Omega_\alpha}G(\mathbf{z},\mathbf{y})\varphi(\mathbf{y})\gamma_0(\mathbf{x}-\mathbf{z})\mu(d\mathbf{y})\mu(d\mathbf{z}),\ \forall\mathbf{x}\in\Omega_\beta,\end{aligned}$$

and ($\alpha\in\{i,b,e\}\,\text{and}\,\beta\in\{i,e\}$)

$$\begin{aligned}&G_{\alpha b}T_{bb}^{\beta}\varphi(\mathbf{x})\\ =&\frac{\sigma}{2}\int_{\Omega_b}G(\mathbf{x},\mathbf{z})\varphi(\mathbf{z})\mu(d\mathbf{z}) + \int_{\Omega_b}\int_{\Omega_\beta}G(\mathbf{x},\mathbf{z})\varphi(\mathbf{z})\gamma_0(\mathbf{z}-\mathbf{y})\mu(d\mathbf{y})\mu(d\mathbf{z})\\ &+\frac{1}{2}\int_{\Omega_b}\int_{\Omega_b}G(\mathbf{x},\mathbf{z})[\varphi(\mathbf{z})-\varphi(\mathbf{y})]\gamma_0(\mathbf{z}-\mathbf{y})\mu(d\mathbf{y})\mu(d\mathbf{z})\\ =&\frac{\sigma}{2}\int_{\Omega_b}G(\mathbf{x},\mathbf{z})\varphi(\mathbf{z})\mu(d\mathbf{z}) + \int_{\Omega_b}\int_{\Omega_\beta}G(\mathbf{x},\mathbf{z})\varphi(\mathbf{z})\gamma_0(\mathbf{z}-\mathbf{y})\mu(d\mathbf{y})\mu(d\mathbf{z})\\ &+\frac{1}{2}\int_{\Omega_b}\int_{\Omega_b}[G(\mathbf{x},\mathbf{z})-G(\mathbf{x},\mathbf{y})]\varphi(\mathbf{z})\gamma_0(\mathbf{z}-\mathbf{y})\mu(d\mathbf{y})\mu(d\mathbf{z}),\quad\forall\mathbf{x}\in\Omega_\alpha,\\ &T_{bb}^{\beta}G_{b\alpha}\varphi(\mathbf{x})\\ =&\frac{\sigma}{2}\int_{\Omega_b}G(\mathbf{x},\mathbf{y})\varphi(\mathbf{y})\mu(d\mathbf{y}) + \int_{\Omega_\beta}\int_{\Omega_\alpha}G(\mathbf{x},\mathbf{y})\varphi(\mathbf{y})\gamma_0(\mathbf{x}-\mathbf{z})\mu(d\mathbf{y})\mu(d\mathbf{z})\\ &+\frac{1}{2}\int_{\Omega_b}\int_{\Omega_b}[G(\mathbf{x},\mathbf{y})-G(\mathbf{z},\mathbf{y})]\varphi(\mathbf{y})\gamma_0(\mathbf{x}-\mathbf{z})\mu(d\mathbf{y})\mu(d\mathbf{z}),\quad\forall\mathbf{x}\in\Omega_b.\end{aligned}$$

Let us introduce the following functions ($\alpha\in\{i,b,e\}$)

$$\begin{aligned}G_\alpha^{(1)}(\mathbf{x},\mathbf{z}) &:= \int_{\Omega_\alpha}[G(\mathbf{x},\mathbf{z})-G(\mathbf{y},\mathbf{z})]\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y}),\\ G_\alpha^{(2)}(\mathbf{x},\mathbf{z}) &:= \int_{\Omega_\alpha}[G(\mathbf{x},\mathbf{z})-G(\mathbf{x},\mathbf{y})]\gamma_0(\mathbf{z}-\mathbf{y})\mu(d\mathbf{y}).\end{aligned}$$

Obviously, it holds that

$$G_\alpha^{(1)}(\mathbf{x},\mathbf{z}) = G_\alpha^{(2)}(\mathbf{z},\mathbf{x}).$$

Therefore, for $\alpha\in\{i,b,e\}$, $\beta\in\{i,e\}$, it holds that

$$\begin{aligned}\left(G_{\alpha\beta}T_{\beta b}+G_{\alpha b}T_{bb}^{\beta}\right)\varphi(\mathbf{x})&=\frac{\sigma}{2}\int_{\Omega_b}G(\mathbf{x},\mathbf{z})\varphi(\mathbf{z})\mu(d\mathbf{z})\\&+\frac{1}{2}\int_{\Omega_b}\int_{\Omega_b}[G(\mathbf{x},\mathbf{z})-G(\mathbf{x},\mathbf{y})]\varphi(\mathbf{z})\gamma_0(\mathbf{z}-\mathbf{y})\mu(d\mathbf{y})\mu(d\mathbf{z})\\&+\int_{\Omega_b}\int_{\Omega_\beta}[G(\mathbf{x},\mathbf{z})-G(\mathbf{x},\mathbf{y})]\varphi(\mathbf{z})\gamma_0(\mathbf{z}-\mathbf{y})\mu(d\mathbf{y})\mu(d\mathbf{z})\\&=\int_{\Omega_b}\left[\frac{\sigma}{2}G(\mathbf{x},\mathbf{z})+\frac{1}{2}G_b^{(2)}(\mathbf{x},\mathbf{z})+G_\beta^{(2)}(\mathbf{x},\mathbf{z})\right]\varphi(\mathbf{z})\mu(d\mathbf{z}),\ \forall\mathbf{x}\in\Omega_\alpha,\end{aligned}$$

and

$$\begin{aligned}\left(T_{b\beta}G_{\beta b}+T_{bb}^{\beta}G_{bb}\right)\varphi(\mathbf{x})&=\frac{\sigma}{2}\int_{\Omega_b}G(\mathbf{x},\mathbf{y})\varphi(\mathbf{y})\mu(d\mathbf{y})\\&+\frac{1}{2}\int_{\Omega_b}\int_{\Omega_b}[G(\mathbf{x},\mathbf{y})-G(\mathbf{z},\mathbf{y})]\varphi(\mathbf{y})\gamma_0(\mathbf{x}-\mathbf{z})\mu(d\mathbf{y})\mu(d\mathbf{z})\\&+\int_{\Omega_b}\int_{\Omega_\beta}[G(\mathbf{x},\mathbf{z})-G(\mathbf{y},\mathbf{z})]\varphi(\mathbf{z})\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y})\mu(d\mathbf{z})\\&=\int_{\Omega_b}\left[\frac{\sigma}{2}G(\mathbf{x},\mathbf{z})+\frac{1}{2}G_b^{(1)}(\mathbf{x},\mathbf{z})+G_\beta^{(1)}(\mathbf{x},\mathbf{z})\right]\varphi(\mathbf{z})\mu(d\mathbf{z}),\ \forall\mathbf{x}\in\Omega_b.\end{aligned}$$

Applying the above formulae, we derive

$$\begin{aligned}K\varphi(\mathbf{x})&=\left(G_{bi}T_{ib}+G_{bb}T_{bb}^{i}-\frac{I}{2}\right)\varphi(\mathbf{x})\\&=\left(G_{bi}T_{ib}+G_{bb}T_{bb}^{i}\right)\varphi(\mathbf{x})-\frac{1}{2}\left(G_{bi}T_{ib}+G_{bb}T_{bb}+G_{be}T_{eb}\right)\varphi(\mathbf{x})\\&=\frac{1}{2}\left(G_{bi}T_{ib}+G_{bb}T_{bb}^{i}\right)\varphi(\mathbf{x})-\frac{1}{2}\left(G_{bb}T_{bb}^{e}+G_{be}T_{eb}\right)\varphi(\mathbf{x})\\&=\frac{1}{2}\int_{\Omega_b}[G_i^{(2)}(\mathbf{x},\mathbf{z})-G_e^{(2)}(\mathbf{x},\mathbf{z})]\varphi(\mathbf{z})\mu(d\mathbf{z}),\quad\forall\mathbf{x}\in\Omega_b,\end{aligned}$$

and

$$\begin{aligned}K'\varphi(\mathbf{x})&=\left(T_{bi}G_{ib}+T_{bb}^{i}G_{bb}-\frac{I}{2}\right)\varphi(\mathbf{x})\\&=\left(T_{bi}G_{ib}+T_{bb}^{i}G_{bb}\right)\varphi(\mathbf{x})-\frac{1}{2}\left(T_{bi}G_{ib}+T_{bb}G_{bb}+T_{be}G_{eb}\right)\varphi(\mathbf{x})\\&=\frac{1}{2}\left(T_{bi}G_{ib}+T_{bb}^{i}G_{bb}\right)\varphi(\mathbf{x})-\frac{1}{2}\left(T_{bb}^{e}G_{bb}+T_{be}G_{eb}\right)\varphi(\mathbf{x})\\&=\frac{1}{2}\int_{\Omega_b}[G_i^{(1)}(\mathbf{x},\mathbf{z})-G_e^{(1)}(\mathbf{x},\mathbf{z})]\varphi(\mathbf{z})\mu(d\mathbf{z}),\quad\forall\mathbf{x}\in\Omega_b.\end{aligned}$$

For ease of exposition of the following deduction, let us introduce the following operator

$$L^{(1)}\varphi(\mathbf{x})=\int_{\Omega_b}\left[\frac{\sigma}{2}G(\mathbf{x},\mathbf{y})+\frac{1}{2}G_b^{(2)}(\mathbf{x},\mathbf{y})+G_e^{(2)}(\mathbf{x},\mathbf{y})\right]\varphi(\mathbf{y})\mu(d\mathbf{y}).$$

With the above preparations, we can simplify the action of operator W as follows

$$W\varphi(\mathbf{x})=\left(T_{bi}G_{ib}T_{bb}^{e}+T_{bi}G_{ie}T_{eb}+T_{bb}^{i}G_{bb}T_{bb}^{e}+T_{bb}^{i}G_{be}T_{eb}\right)\varphi(\mathbf{x})$$

$$=T_{bi}(G_{ib}T^{e}_{bb}+G_{ie}T_{eb})\varphi(\mathbf{x})+T^{i}_{bb}(G_{bb}T^{e}_{bb}+G_{be}T_{eb})\varphi(\mathbf{x})$$
$$=T_{bi}L^{(1)}\varphi(\mathbf{x})+T^{i}_{bb}L^{(1)}\varphi(\mathbf{x}).$$

Applying the definitions of T_{bi} and T^{i}_{bb} leads to

$$W\varphi(\mathbf{x})=\frac{\sigma}{2}L^{(1)}\varphi(\mathbf{x})+\frac{1}{2}\int_{\Omega_b}\Big[L^{(1)}\varphi(\mathbf{x})-L^{(1)}\varphi(\mathbf{y})\Big]\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y})$$
$$+\int_{\Omega_i}\Big[L^{(1)}\varphi(\mathbf{x})-L^{(1)}\varphi(\mathbf{y})\Big]\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y}). \tag{A.2}$$

In addition, let us introduce new functions

$$G^{(1,2)}_{\alpha,\beta}(\mathbf{x},\mathbf{z}):=\int_{\Omega_\beta}[G^{(1)}_{\alpha}(\mathbf{x},\mathbf{z})-G^{(1)}_{\alpha}(\mathbf{x},\mathbf{t})]\gamma_0(\mathbf{z}-\mathbf{t})\mu(d\mathbf{t})$$
$$=\int_{\Omega_\beta}\Big\{\int_{\Omega_\alpha}[G(\mathbf{x},\mathbf{z})-G(\mathbf{y},\mathbf{z})]\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y})$$
$$-\int_{\Omega_\alpha}[G(\mathbf{x},\mathbf{t})-G(\mathbf{y},\mathbf{t})]\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y})\Big\}\gamma_0(\mathbf{z}-\mathbf{t})\mu(d\mathbf{t})$$
$$=\int_{\Omega_\beta}\int_{\Omega_\alpha}[G(\mathbf{x},\mathbf{z})-G(\mathbf{x},\mathbf{t})+G(\mathbf{y},\mathbf{t})-G(\mathbf{y},\mathbf{z})]\gamma_0(\mathbf{x}-\mathbf{y})\gamma_0(\mathbf{z}-\mathbf{t})\mu(d\mathbf{y})\mu(d\mathbf{t}),$$

and

$$G^{(2,1)}_{\alpha,\beta}(\mathbf{x},\mathbf{z}):=\int_{\Omega_\beta}[G^{(2)}_{\alpha}(\mathbf{x},\mathbf{z})-G^{(2)}_{\alpha}(\mathbf{t},\mathbf{z})]\gamma_0(\mathbf{x}-\mathbf{t})\mu(d\mathbf{t})$$
$$=\int_{\Omega_\beta}\Big\{\int_{\Omega_\alpha}[G(\mathbf{x},\mathbf{z})-G(\mathbf{x},\mathbf{y})]\gamma_0(\mathbf{z}-\mathbf{y})\mu(d\mathbf{y})$$
$$-\int_{\Omega_\alpha}[G(\mathbf{t},\mathbf{z})-G(\mathbf{t},\mathbf{y})]\gamma_0(\mathbf{z}-\mathbf{y})\mu(d\mathbf{y})\Big\}\gamma_0(\mathbf{x}-\mathbf{t})\mu(d\mathbf{t})$$
$$=\int_{\Omega_\beta}\int_{\Omega_\alpha}[G(\mathbf{x},\mathbf{z})-G(\mathbf{x},\mathbf{y})+G(\mathbf{t},\mathbf{y})-G(\mathbf{t},\mathbf{z})]\gamma_0(\mathbf{x}-\mathbf{y})\gamma_0(\mathbf{z}-\mathbf{t})\mu(d\mathbf{y})\mu(d\mathbf{z}),$$

where $\alpha,\beta\in\{i,b\}$. A simple calculation reveals that

$$G^{(2,1)}_{\alpha,\beta}(\mathbf{x},\mathbf{z})=G^{(1,2)}_{\beta,\alpha}(\mathbf{x},\mathbf{z}),\qquad G^{(1,2)}_{\alpha,\beta}(\mathbf{z},\mathbf{x})=G^{(2,1)}_{\alpha,\beta}(\mathbf{x},\mathbf{z}).$$

Now since

$$\int_{\Omega_\alpha}\Big[L^{(1)}\varphi(\mathbf{x})-L^{(1)}\varphi(\mathbf{y})\Big]\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y})$$
$$=\int_{\Omega_\alpha}\int_{\Omega_b}\Big[\frac{\sigma}{2}(G(\mathbf{x},\mathbf{z})-G(\mathbf{y},\mathbf{z}))+\frac{1}{2}\Big(G^{(2)}_{b}(\mathbf{x},\mathbf{z})-G^{(2)}_{b}(\mathbf{y},\mathbf{z})\Big)+G^{(2)}_{e}(\mathbf{x},\mathbf{z})-G^{(2)}_{e}(\mathbf{y},\mathbf{z})\Big]$$
$$\varphi(\mathbf{z})\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{z})\mu(d\mathbf{y})$$
$$=\int_{\Omega_b}\int_{\Omega_\alpha}\Big[\frac{\sigma}{2}(G(\mathbf{x},\mathbf{z})-G(\mathbf{y},\mathbf{z}))+\frac{1}{2}\Big(G^{(2)}_{b}(\mathbf{x},\mathbf{z})-G^{(2)}_{b}(\mathbf{y},\mathbf{z})\Big)+G^{(2)}_{e}(\mathbf{x},\mathbf{z})-G^{(2)}_{e}(\mathbf{y},\mathbf{z})\Big]$$
$$\varphi(\mathbf{z})\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y})\mu(d\mathbf{z})$$
$$=\int_{\Omega_b}\Big[\frac{\sigma}{2}G^{(1)}_{\alpha}(\mathbf{x},\mathbf{z})+\frac{1}{2}G^{(2,1)}_{b,\alpha}(\mathbf{x},\mathbf{z})+G^{(2,1)}_{e,\alpha}(\mathbf{x},\mathbf{z})\Big]\varphi(\mathbf{z})\mu(d\mathbf{z}),$$

we can further simplify (A.2) as follows

$$\begin{aligned}
W\varphi(\mathbf{x}) = &\int_{\Omega_b}\Big\{\tfrac{\sigma}{2}\Big[\tfrac{\sigma}{2}G(\mathbf{x},\mathbf{z})+\tfrac{1}{2}G_b^{(2)}(\mathbf{x},\mathbf{z})+G_e^{(2)}(\mathbf{x},\mathbf{z})\Big]+\tfrac{1}{2}\Big[\tfrac{\sigma}{2}G_b^{(1)}(\mathbf{x},\mathbf{z})+\tfrac{1}{2}G_{b,b}^{(2,1)}(\mathbf{x},\mathbf{z})\\
&+G_{e,b}^{(2,1)}(\mathbf{x},\mathbf{z})\Big]+\Big[\frac{\sigma}{2}G_i^{(1)}(\mathbf{x},\mathbf{z})+\frac{1}{2}G_{b,i}^{(2,1)}(\mathbf{x},\mathbf{z})+G_{e,i}^{(2,1)}(\mathbf{x},\mathbf{z})\Big]\Big\}\varphi(\mathbf{z})\mu(d\mathbf{z})\\
=&\int_{\Omega_b}\Big[\frac{\sigma^2}{4}G(\mathbf{x},\mathbf{z})+\frac{\sigma}{4}\Big(G_b^{(1)}(\mathbf{x},\mathbf{z})+G_b^{(2)}(\mathbf{x},\mathbf{z})+2G_i^{(1)}(\mathbf{x},\mathbf{z})+2G_e^{(2)}(\mathbf{x},\mathbf{z})\Big)\\
&+\frac{1}{4}\Big(G_{b,b}^{(2,1)}(\mathbf{x},\mathbf{z})+2G_{b,i}^{(2,1)}(\mathbf{x},\mathbf{z})+2G_{e,b}^{(2,1)}(\mathbf{x},\mathbf{z})+4G_{e,i}^{(2,1)}(\mathbf{x},\mathbf{z})\Big)\Big]\varphi(\mathbf{z})\mu(d\mathbf{z}).
\end{aligned}$$

Therefore, inserting the expressions of generalized boundary integral operators into the resulting system, we can rewrite the final system (4.8) into the following integral form:

- The first sub-equation reads as

$$\begin{aligned}
\sigma u_i(\mathbf{x})+&\int_{\Omega_i}[u_i(\mathbf{x})-u_i(\mathbf{y})]\gamma\Big(\mathbf{x}-\mathbf{y},\frac{\mathbf{y}+\mathbf{x}}{2}\Big)\mu(dy)\\
&+\int_{\Omega_b}[u_i(\mathbf{x})-u_b(\mathbf{y})]\gamma\Big(\mathbf{x}-\mathbf{y},\frac{\mathbf{y}+\mathbf{x}}{2}\Big)\mu(d\mathbf{y})=f_i(\mathbf{x});
\end{aligned}$$

- The second sub-equation reads as

$$\begin{aligned}
&\frac{\sigma}{2}u_b(\mathbf{x})+\frac{1}{2}\int_{\Omega_b}[u_b(\mathbf{x})-u_b(\mathbf{y})]\gamma_0(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y})\\
&\quad+\int_{\Omega_i}[u_b(\mathbf{x})-u_i(\mathbf{y})]\gamma\Big(\mathbf{x}-\mathbf{y},\frac{\mathbf{y}+\mathbf{x}}{2}\Big)\mu(d\mathbf{y})\\
&\quad+\int_{\Omega_b}\Big[\frac{\sigma^2}{4}G(\mathbf{x},\mathbf{z})+\frac{\sigma}{4}\Big(G_b^{(1)}(\mathbf{x},\mathbf{z})+G_b^{(2)}(\mathbf{x},\mathbf{z})+2G_i^{(1)}(\mathbf{x},\mathbf{z})+2G_e^{(2)}(\mathbf{x},\mathbf{z})\Big)\\
&\quad+\frac{1}{4}\Big(G_{b,b}^{(2,1)}(\mathbf{x},\mathbf{z})+2G_{b,i}^{(2,1)}(\mathbf{x},\mathbf{z})+2G_{e,b}^{(2,1)}(\mathbf{x},\mathbf{z})+4G_{e,i}^{(2,1)}(\mathbf{x},\mathbf{z})\Big)\Big]u_b(\mathbf{z})\mu(d\mathbf{z})\\
&\quad+\frac{1}{2}\int_{\Omega_b}\Big[G_i^{(1)}(\mathbf{x},\mathbf{y})-G_e^{(1)}(\mathbf{x},\mathbf{y})\Big]N_b(\mathbf{y})\mu(d\mathbf{y})-\frac{1}{2}N_b(\mathbf{x})=f_b(\mathbf{x});
\end{aligned}$$

- The third sub-equation reads as

$$\begin{aligned}
-\frac{1}{2}\int_{\Omega_b}\Big[G_i^{(2)}(\mathbf{x},\mathbf{y})-G_e^{(2)}(\mathbf{x},\mathbf{y})\Big]u_b(\mathbf{y})\mu(d\mathbf{y})+\frac{1}{2}u_b(\mathbf{x})&\\
+\int_{\Omega_b}G(\mathbf{x},\mathbf{y})N_b(\mathbf{y})\mu(d\mathbf{y})=0.&
\end{aligned}$$

REFERENCES

[1] L. Banjai, C. Lubich, and F.J. Sayas, *Stable numerical coupling of exterior and interior problems for the wave equation*, Numer. Math., 129(4):611–646, 2015. 6

[2] G. Bao, L. Xu, and T. Yin, *Boundary integral equation methods for the elastic and thermoelastic waves in three dimensions*, Comput. Meth. Appl. Mech. Eng., 354:464–486, 2019. 1

[3] F. Bobaru and M. Duangpanya, *The peridynamic formulation for transient heat conduction*, Int. J. Heat Mass Transf., 53(19-20):4047–4059, 2010. 1

[4] F. Chung and S. Yau, *Discrete Green's function*, J. Comb. Theory Ser. A, 91(1-2):191–214, 2000. 5.2

[5] M. Costabel, *Symmetric methods for the coupling of finite elements and boundary elements (invited contribution)*, in C.A. Brebbia, W.L. Wendland, and G. Kuhn (eds.), Mathematical and Computational Aspects. Boundary Elements IX, Springer, 411–420, 1987. 1
[6] Q. Du, M. Gunzburger, R.B. Lehoucq, and K. Zhou, *Analysis and approximation of nonlocal diffusion problems with volume constraints*, SIAM Rev., 54(4):667–696, 2012. 1
[7] Q. Du, Y. Tao, X. Tian, and J. Yang, *Asymtotically compatible discretization of multidimensional nonlocal diffusion models and approximation of nonlocal Green's functions*, IMA J. Numer. Anal., 39(2):607–625, 2019. 5.1, 5.2
[8] Q. Du, H. Han, J. Zhang, and C. Zheng, *Numerical solution of a two-dimensional nonlocal wave equation on unbounded domains*, SIAM J. Sci. Comput., 40(3):A1430–A1445, 2018. 1
[9] Q. Du, J. Zhang, and C. Zheng, *Nonlocal wave propagation in unbounded multi-scale media*, Commun. Comput. Phys., 24(4):1049–1072, 2018. 1
[10] D. Givoli, *Non-reflecting boundary conditions*, J. Comput. Phys., 94(1):1–29, 1991. 1
[11] H. Han, *A new class of variational formulations for the coupling of finite and boundary element methods*, J. Comput. Math., 8(3):223–232, 1990. 1
[12] H. Han and X. Wu, *The approximation of the exact boundary conditions at an artificial boundary for linear elastic equations and its applications*, Math. Comput., 59(199):21–37, 1992. 1
[13] H. Han and X. Wu, *Artificial Boundary Method*, Tsinghua University Press, Springer, 2013. 1
[14] S. Ji, Y. Yang, G. Pang, and X. Antoine, *Accurate artificial boundary conditions for the semi-discretized linear Schrödinger and heat equations on rectangular domains*, Comput. Phys. Commun., 222:84–93, 2018. 1
[15] S. Katsura and S. Inawashiro, *Lattice Green's functions for rectangular and square lattices at arbitrary points*, J. Math. Phys., 12(8):1622, 1971. 5.2
[16] B. Li, J. Zhang, and C. Zheng, *Stability and error analysis for a second-order fast approximation of the one-dimensional Schrödinger equation under absorbing boundary conditions*, SIAM J. Sci. Comput., 40(6):A4083–A4104, 2018. 1
[17] X. Li, *An atomistic-based boundary element method for the reduction of molecular statics models*, Comput. Meth. Appl. Mech. Eng., 225:1–13, 2012. 1
[18] C. Lubich, *Convolution quadrature revisited*, BIT Numer. Math., 44(3):503–514, 2004. 6
[19] C. Lubich and A. Schädle, *Fast convolution for nonreflecting boundary conditions*, SIAM J. Sci. Comput., 24(1):161–182, 2002. 1
[20] X. Ma and C. Zheng, *Fast finite element method for the three-dimensional Poisson equation in infinite domains*, Commun. Comput. Phys., 24:1101–1120, 2018. 1
[21] J.C. Nédélec, *Acoustic and Electromagnetic Equations: Integral Representations for Harmonic Problems*, Springer New York, 2001. 3
[22] G. Pang, Y. Yang, and S. Tang, *Exact boundary condition for semi-discretized Schrödinger equation and heat equation in a rectangular domain*, J. Sci. Comput., 72(1):1–13, 2017. 1
[23] T. Qiu, A. Rieder, F.J. Sayas, and S. Zhang, *Time-domain boundary integral equation modeling of heat transmission problems*, Numer. Math., 143(1):223–259, 2019. 1
[24] Y. Saad, *Iterative Methods for Sparse Linear Systems*, SIAM, 2003. 5.1
[25] S.A. Sauter and C. Schwab, *Boundary Element Methods*, Springer-Verlag, Berlin, 2011. 1, 3, 3.1
[26] T. Sun, J. Wang, and C. Zheng, *Fast evaluation of artificial boundary conditions for advection diffusion equations*, SIAM J. Numer. Anal., 58(6):3530–3579, 2020. 1
[27] S. Silling, M. Zimmermann, and R. Abeyaratne, *Deformation of a peridynamic bar*, J. Elast., 73(1-3):173–190, 2003. 5.2
[28] H. Tian, L. Ju, and Q. Du, *A conservative nonlocal convection–diffusion model and asymptotically compatible finite difference discretization*, Comput. Meth. Appl. Mech. Eng., 320:46–67, 2017. 2, 5.1
[29] X. Tian and Q. Du, *Trace theorem for some nonlocal function spaces with heterogeneous localization*, SIAM J. Math. Anal., 49(2):1621–1644, 2017. 1, 2
[30] L. Wang, J. Xu, and J. Wang, *The Green's functions for peridynamic non-local diffusion*, Proc. R. Soc. A-Math. Phys. Eng. Sci., 472:(2193), 2016. 5.2
[31] L. Wang, J. Xu, and J. Wang, *Static and dynamic Green's functions in peridynamics*, J. Elast., 126(1):95–125, 2017. 5.2
[32] O. Weckner and R. Abeyaratne, *The effect of long-range forces on the dynamics of a bar*, J. Mech. Phys. Solids, 53(3):705–728, 2005. 1
[33] O. Weckner, G. Brunk, and M. Epton, *Green's functions in non-local three-dimensional linear elasticity*, Proc. R. Soc. A-Math. Phys. Eng. Sci., 465(2111):3463–3487, 2009. 5.2
[34] R.A. Wildman and G.A. Gazonas, *A perfectly matched layer for peridynamics in two dimensions*, J. Mech. Mater. Struct., 7(8):765–781, 2013. 1
[35] L. Xue, L. Wang, J. Xu, and J. Wang, *The boundary element method of peridynamics*, Int. J. Numer. Meth. Eng., 122(20):5558–5593, 2021. 1

[36] Y. Yang and Y. Liu, *Modeling of cracks in two-dimensional elastic bodies by coupling the boundary element method with peridynamics*, Int. J. Solids Struct., 217:74–89, 2021. 1

[37] T. Yin, G.C. Hsiao, and L. Xu, *Boundary integral equation methods for the two-dimensional fluid-solid interaction problem*, SIAM J. Numer. Anal., 55(5):2361–2393, 2017. 1

[38] W. Zhang, J. Yang, J. Zhang, and Q. Du, *Artificial boundary conditions for nonlocal heat equations on unbounded domain*, Commun. Comput. Phys., 21(1):16–39, 2017. 1

[39] C. Zheng, Q. Du, X. Ma, and J. Zhang, *Stability and error analysis for a second-order fast approximation of the local and nonlocal diffusion equations on the real line*, SIAM J. Numer. Anal., 58(3):1893–1917, 2020. 1

[40] C. Zheng, J. Hu, Q. Du, and J. Zhang, *Numerical solution of the nonlocal diffusion equation on the real line*, SIAM J. Sci. Comput., 39(5):A1951–A1968, 2017. 1

[41] K. Zhou and Q. Du, *Mathematical and numerical analysis of linear peridynamic models with nonlocal boundary conditions*, SIAM J. Numer. Anal., 48(5):1759–1780, 2010. 1