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EMERGENT BEHAVIORS OF KURAMOTO MODEL WITH FRUSTRATION UNDER SWITCHING TOPOLOGY*

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Abstract. In this paper, we study the emergent behavior of Kuramoto model with switching topology under the effect of uniform frustration. In our frameworks, the switching interaction topology contains a spanning tree in any switching mode. For the initial configuration distributed in an open half circle, we first exploit a similar procedure in [T. Zhu, Netw. Heterog. Media, 17(2):255–291, 2022] to conclude that the Kuramoto oscillators will be pushed into a small region at some instant before the first network switching. Then in a large coupling and small frustration regime, we lift the Kuramoto model to the second-order formulation and apply the matrix theory-based approach in [J.-G. Dong, S.-Y. Ha and D. Kim, Anal. Appl., 19(2):305–342, 2021] to derive the exponential fast frequency synchronization.

Keywords. Synchronization; Kuramoto model; frustration; switching topology; spanning tree.

AMS subject classifications. 34D06; 34C15; 93C15; 34K33.

1. Introduction

Synchronization in complex systems has attracted a lot of attention and been studied in different disciplines such as physics, biology, and engineering [1, 7, 25, 36], etc. This phenomenon can be observed ubiquitously in our natural world, for example, synchronous flashing of fireflies, chorusing of crickets, and rhythmic heart beating [2, 38], etc. Owing to the increasing applications in the control of sensor networks, power networks, and internet networks [11, 12, 31], the efforts have been made towards how a population of oscillators can organize themselves into a synchronized state via weak interactions. Thus, many models have been proposed to address synchronization phenomena among which we focus our interest on the Kuramoto model [23, 24]. There is vast literature on the large-time behaviors of the Kuramoto model such as complete synchronization, partially phase-locked states, nonlinear stability, and discrete dynamics [3, 13, 16, 17, 20, 28, 40].

Recently, some variants of the Kuramoto model were proposed for the needs of modeling real physical and biological systems. Kuramoto and Sakaguchi [34] introduced frustration (or phase shift) into the sinusoidal coupling function in the Kuramoto model so that richer dynamical phenomena would be observed than that without frustration. On the other hand, in previous studies, the emergent dynamics of the Kuramoto model has been studied under fixed time-invariant interaction topology such as the complete graph [3], the graph with hierarchical leadership [19], and the general digraph with spanning tree [8]. Thus, in this paper, we consider the Kuramoto model under the effect of frustration and the switching topology which varies with time. To fix the idea, let $\theta_i = \theta_i(t)$ be the phase of the i-th oscillator, and we denote the network topology at time t via the (0,1)-adjacency matrix $(\chi_{ij}^{\sigma(t)})$:

$$\chi_{ij}^{\sigma(t)} = \begin{cases} 1, & \text{if } j \text{ transmits information to } i \text{ at time } t, \\ 0, & \text{otherwise.} \end{cases}$$

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For the switching law $\sigma(t)$, we assume that there is a finite set $\mathcal{P}=\{1,2,\dots,p\}$ of *modes* among which the network topology can switch. More precisely, the switching law $\sigma(t)$: $[0,\infty)\to\mathcal{P}$ is assumed to be a piecewise constant function and continuous from the right. The set of discontinuous points of $\sigma(t)$ is referred to as the sequence of switching instants, and one topology mode in $\mathcal{P}$ will be activated at the switching instant. Without loss of generality, we denote the switching instants by the sequence $\{t_l\}_{l=0}^{\infty}$ with $t_0=0$ and $\lim_{l\to+\infty}t_l=+\infty$. In the presence of frustration, the dynamics of the Kuramoto model with switching topology is governed by the following ODE system:

$$\begin{cases}\dot{\theta}_i(t)=\Omega_i+K\displaystyle\sum_{j=1}^{N}\chi_{ij}^{\sigma(t)}\sin(\theta_j(t)-\theta_i(t)+\alpha), \quad t>0,\ i=1,2,\dots,N,\\ \theta_i(0)=\theta_i^0,\end{cases} \tag{1.1}$$

where θ_i and Ω_i are the phase and natural frequency of the i-th oscillator respectively, $K>0$ is the coupling strength, and $\alpha\in(0,\frac{\pi}{2})$ is the uniform frustration between oscillators. For the case with nonpositive frustration, we can reformulate such a system into system (1.1) by taking $\hat{\theta}_i=-\theta_i$ for $i=1,2,\dots,N$. For any time $t\geq 0$, we assume that $\chi_{ii}^{\sigma(t)}=1$ for $i=1,\dots,N$, that is to say, i-th oscillator transmits information to itself at any time. Moreover, we introduce the auxiliary variable *frequency* $w_i(t)=\dot{\theta}_i(t)$ and lift the system (1.1) to the second-order formulation, i.e., we directly differentiate (1.1) to obtain the following second-order Kuramoto model

$$\begin{cases}\dot{\theta}_i(t)=w_i(t), \quad t>0,\ i=1,2,\dots,N,\\ \dot{w}_i(t)=K\displaystyle\sum_{j=1}^{N}\chi_{ij}^{\sigma(t)}\cos(\theta_j(t)-\theta_i(t)+\alpha)(w_j(t)-w_i(t)).\end{cases} \tag{1.2}$$

This second-order model is the so-called "augmented Kuramoto model" introduced in [16]. In [20], the authors observed a formal analogue between the Cucker-Smale flocking model and the Kuramoto model for synchronization, and exploited Lyapunov functional approach to derive frequency synchronization for the second-order Kuramoto model with complete graph. Moreover, the authors in [13,16] showed the complete synchronization for the continuous and discrete augmented Kuramoto model with complete graph in terms of initial data and system parameters.

Incorporating the two additional structures, frustration and switching topology into the original Kuramoto model is more realistic and practical in some sense. The author in [6] observed that the frustration is common in disordered interactions, and by varying the value of α in numerical simulations, the author in [41] found that frustration can induce the desynchronization. The effect of frustration has also been a focus of interest due to the relation to networks of oscillators [26,30,32,37]. On the other hand, observations on the behaviors of animals in [29,33] reveal that the switching interaction topology is common in nature, which naturally motivates us to study the Kuramoto model with switching topology. However, these two structures cause a lot of difficulties in mathematical analysis mainly due to the lack of conservation law and the absence of gradient flow structure. As far as the author knows, there are few rigorous theoretical results on the Kuramoto model with switching topology under the effect of frustration.

The work in papers [8,10,14,15,18,27] is closely related to this paper. More precisely, for all-to-all and symmetric case with frustration, when the initial data is confined in half circle, the authors in [14] presented sufficient conditions to ensure complete synchronization. On the other hand, for non-all-to-all case without frustration, in [10], the authors studied the generalized Kuramoto model with switching topology. They provided sufficient frameworks leading to frequency synchronization for the configuration initially distributed over the half circle. However, they demanded that any pair of oscillators have one common neighbor in any switching mode, so that the dissipation mechanism can be captured by the good property of sine function. In [8], the authors considered the general digraph with spanning tree and derived a complete synchronization by lifting the Kuramoto model to second-order formulation. But their analytical tool only works when the initial configuration is distributed in a quarter circle. Finally, for the non-symmetric case with frustration, in [15], the authors considered a complete graph which is a small perturbation of all-to-all network, and showed the complete synchronization for the Kuramoto model with frustrations when the initial phases are restricted in half circle.

In this paper, we consider the synchronization of the Kuramoto model with switching topology under the effect of uniform frustration. We assume the following two structural constraints on the switching network.

Assumption 1.1. *There is a minimal dwelling time $\tau>0$ such that the switching instants $\{t_l\}_{l=0}^{\infty}$ with $t_0=0$ satisfy*

$$t_{l+1}-t_l\geq\tau,\quad \textit{for all } l\in\mathbb{N}. \tag{1.3}$$

Assumption 1.2. *For any interval $[t_l,t_{l+1})$ with $l\in\mathbb{N}$, the corresponding digraph $\mathcal{G}$ in this interval has a spanning tree. In other words, each mode in the set $\mathcal{P}=\{1,2\dots,p\}$ contains a spanning tree.*

First, under the Assumption 1.2 of the switching topology with spanning tree structure in any mode of $\mathcal{P}$, before the first network switching, we apply the idea of digraph decomposition in [21] and similar procedure in [42], to find some finite time at which the oscillators stay in a small arc less than a quarter circle (see Lemma 3.3). Second, based on the estimate in Lemma 3.3, under a priori condition, we apply the matrix and graph theories and the method in [5, 9] for the second-order Kuramoto model (1.2) to derive the frequency diameter a priori tends to zero exponentially fast (see Lemma 4.3). Lastly, we show that a priori condition can be ensured by the sufficient frameworks, which ultimately yields the complete synchronization. The main contribution of this work is presented as below:

Theorem 1.1. *Suppose Assumption 1.1 and Assumption 1.2 hold, and let $(\theta_i(t),\omega_i(t))$ be a solution to system (1.1) and (1.2) with initial configuration satisfying*

$$D(\theta(0))<\zeta<\gamma<\pi, \tag{1.4}$$

where ζ,γ are positive constants. Moreover, for a given sufficiently small positive constant $D^\infty<\min\{\zeta,\frac{\pi}{2}\}$ and a given positive constant $\xi<\frac{\pi}{2}$, we assume the coupling strength K and the frustration α satisfy

$$\begin{aligned}
&\tan\alpha<\frac{1}{\left(1+\frac{(r_0+1)\zeta}{\zeta-D(\theta(0))}\right)2Nc}\frac{\beta^{r_0+1}D^\infty}{[4(2N+1)c]^{r_0}},\quad \max\left\{\frac{\zeta-D(\theta(0))}{D(\Omega)+2NK\sin\alpha},\frac{2}{K}\right\}<\tau,\\
&\left(1+\frac{(r_0+1)\zeta}{\zeta-D(\theta(0))}\right)\frac{c[4(2N+1)c]^{r_0}}{\beta^{r_0+1}D^\infty}\left(\frac{D(\Omega)}{K\cos\alpha}+\frac{2N\sin\alpha}{\cos\alpha}\right)<1,\\
&D^\infty+\alpha+[D(\Omega)+2KN(D^\infty+\alpha)]\frac{2(N-1)}{Ke^{-2N(N-1)}(\cos\xi)^{N-1}}\leq\xi,
\end{aligned} \tag{1.5}$$

where $D(\theta(t)) = \max_{1\le i,j\le N} |\theta_i(t)-\theta_j(t)|$, η,β *and* c *are constants subject to the conditions*

$$\eta > \max\left\{\frac{1}{\sin\gamma}, \frac{2}{1-\frac{\zeta}{\gamma}}\right\}, \quad \beta = 1-\frac{2}{\eta}, \quad c = \frac{\left(\sum_{j=1}^{N-1}\eta^j A(2N,j)+1\right)\gamma}{\sin\gamma}, \tag{1.6}$$

$0 \le r_0 < N$ *is a constant given in* (3.6), *and* $A(2N,j)$ *denotes the number of permutations. Then there exists time* $t_* > 0$ *such that*

$$D(\omega(t)) \le (1-\delta)^{\lfloor \frac{t-t_*}{\hat{h}} \rfloor} D(\omega(t_*)), \quad t \ge t_*,$$

where $\hat{h} = \frac{2(N-1)}{K}$, $\delta = e^{-2N(N-1)}(\cos\xi)^{N-1}$, *and* $D(\omega(t)) = \max_{1\le i,j\le N}|\omega_i(t)-\omega_j(t)|$.

Remark 1.1. We further make some comments on Theorem 1.1. In fact, the complicated conditions (1.5) are for technical reasons. As we consider the case that all oscillators in system (1.1) are initially distributed in an open half circle, i.e., $D(\theta(0)) < \pi$, it is obvious that we can find admissible parameters ζ and γ satisfying $D(\theta(0)) < \zeta < \gamma$ in (1.4). Once ζ and γ are fixed, the values of constants η,β and c in (1.6) can be successively determined. Then sufficiently large K and sufficiently small α can guarantee the validity of constraints in (1.5).

The rest of this paper is organized as follows. In Section 2, we recall basic concepts related to the directed graph and matrix theories, and review some previous results . In Section 3, for the initial configuration confined in an open half circle, we provide a preparatory estimate that there exists a finite time at which the phase diameter can be bounded by a small value less than $\frac{\pi}{2}$. In Section 4, we use the induction argument based on matrix-graph theories, and in a sufficient regime, we present the emergence of frequency synchronization for the Kuramoto model with frustration under switching topology. Section 5 is devoted to a brief summary of our work and the future investigation.

2. Preliminaries and previous results

In this section, we first introduce some basic concepts on the directed graph , and next review the previous results on the dynamical estimates of the Kuramoto model with frustration on a static digraph. Then we recall some definitions and lemmas related to matrix theory, and for simplicity provide some notations which will be frequently used in later sections.

2.1. Directed graph. We consider a digraph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ consisting of a finite set $\mathcal{V} = \{1,\ldots,N\}$ of vertices and a set $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$ of directed arcs. And assume that the oscillators are located at vertices and interact with each other via the underlying network topology. For each vertex i, we denote the set of its neighbors by $\mathcal{N}_i$, which is the set of vertices that directly influence vertex i. Now, let $\theta_i = \theta_i(t)$ be the phase of the oscillator at vertex i, and define the $(0,1)$-adjacency matrix (χ_{ij}) as follows:

$$\chi_{ij} = \begin{cases} 1 & \text{if the } j\text{-th oscillator influences the } i\text{-th oscillator}, \\ 0 & \text{otherwise}. \end{cases} \tag{2.1}$$

Then, the set of neighbors of i-th oscillator is actually $\mathcal{N}_i := \{j : \chi_{ij} > 0\}$.

Definition 2.1 ([21]). *For a given set of* $\{\mathcal{N}_i\}_{i=1}^N$ *in system* (1.1), *we have the following definition.*

(1) *The digraph* $\mathcal{G}=(\mathcal{V},\mathcal{E})$ *associated to* (1.1) *consists of a finite set* $\mathcal{V}=\{1,2,\dots,N\}$ *of vertices, and a set* $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}$ *of arcs with ordered pair* $(j,i)\in\mathcal{E}$ *if* $j\in\mathcal{N}_i$.

(2) *A path in* $\mathcal{G}$ *from* i_1 *to* i_k *is a sequence* $i_1,i_2,\dots,i_k$ *such that*

$$i_s\in\mathcal{N}_{i_{s+1}}\quad \textit{for } 1\le s\le k-1.$$

If there exists a path from j *to* i*, then vertex* i *is said to be reachable from vertex* j.

(3) *The digraph contains a spanning tree if we can find a vertex such that any other vertex of* $\mathcal{G}$ *is reachable from it.*

Based on Definition 2.1, we further denote the interaction graph at time t by $\mathcal{G}^{\sigma(t)}=(\mathcal{V},\mathcal{E}^{\sigma(t)})$, the $(0,1)$-adjacency matrix of the digraph $\mathcal{G}^{\sigma(t)}$ by $\chi_{ij}^{\sigma(t)}$, and the neighbor set of the i-th oscillator in $\mathcal{G}^{\sigma(t)}$ by $\mathcal{N}_i^{\sigma(t)}$.

2.2. Previous results in fixed digraph case. In this part, we provide the previous results of the Kuramoto model with frustration on a general digraph. More precisely, in [42], the author dealt with the synchronization of the following system

$$\dot{\theta}_i(t)=\Omega_i+K\sum_{j=1}^{N}\chi_{ij}\sin(\theta_j(t)-\theta_i(t)+\alpha),\quad t>0,\ i=1,2,\dots,N,\ \alpha\in(0,\frac{\pi}{2}),\tag{2.2}$$

where the digraph is static and contains a spanning tree, which is registered by the $(0,1)$-adjacency matrix (χ_{ij}) in (2.1) independent of time t. In fact, for the system (2.2), we can see it as a special case of system (1.1) without the occurrence of network switching. In the following, we introduce some technical concepts and lemmas introduced in [21] before we review the main result on the dynamics of system (2.2) in [42].

We first recall the idea of directed graph decomposition in [21]. Let $\mathcal{G}=(\mathcal{V},\mathcal{E}),\mathcal{V}_1\subset\mathcal{V}$, and a subgraph $\mathcal{G}_1=(\mathcal{V}_1,\mathcal{E}_1)$ is the digraph with vertex set $\mathcal{V}_1$ and arc set $\mathcal{E}_1$, where $\mathcal{E}_1$ consists of the directed arcs starting from vertices in $\mathcal{V}_1$. For a given digraph $\mathcal{G}=(\mathcal{V},\mathcal{E})$, we will identify a subgraph $\mathcal{G}_1=(\mathcal{V}_1,\mathcal{E}_1)$ with its vertex set $\mathcal{V}_1$ for convenience. The definition of node is given as below.

Definition 2.2. *For any two sets* $\mathcal{V}_1$ *and* $\mathcal{V}_2$ *of vertices, if there exists* $i_1\in\mathcal{V}_1$ *and* $i_2\in\mathcal{V}_2$ *such that* $i_1\in\mathcal{N}_{i_2}$*, then we say* $\mathcal{V}_2$ *is affected by* $\mathcal{V}_1$*. Otherwise, we say* $\mathcal{V}_2$ *is not affected by* $\mathcal{V}_1$*.*

Definition 2.3 ([21](Node)). *Let* $\mathcal{G}=(\mathcal{V},\mathcal{E})$ *be a digraph. A subgraph* $\mathcal{G}_1=(\mathcal{V}_1,\mathcal{E}_1)$ *is called a node if it is strongly connected, i.e., for any subset* $\mathcal{V}_2$ *of* $\mathcal{V}_1$*,* $\mathcal{V}_2$ *is affected by* $\mathcal{V}_1\setminus\mathcal{V}_2$*. Moreover, if* $\mathcal{V}_1$ *is not affected by* $\mathcal{V}\setminus\mathcal{V}_1$*, we say* $\mathcal{V}_1$ *is a maximum node.*

Intuitively, a node is a set of agents which can be viewed as a "large" agent. The concept of node can help us to simplify the structure of the digraph.

Lemma 2.1 ([21]). *Any digraph* $\mathcal{G}$ *contains at least one maximum node. A digraph* $\mathcal{G}$ *contains a unique maximum node if and only if* $\mathcal{G}$ *has a spanning tree.*

Proof. For the detailed proof of this lemma, please see the proofs of Lemma 2.4 and Lemma 2.6 in [21]. □

Lemma 2.2 ([21](Node decomposition)). *Let* $\mathcal{G}$ *be any digraph. Then we can decompose* $\mathcal{G}$ *to be a union as* $\mathcal{G}=\bigcup_{i=0}^{r}(\bigcup_{j=1}^{k_i}\mathcal{G}_i^j)$ *such that*

(1) $\mathcal{G}_0^j$ *are the maximum nodes of* $\mathcal{G}$*, where* $1\le j\le k_0$.

(2) *For any* p,q *where* $1\le p\le r$ *and* $1\le q\le k_p$, $\mathcal{G}_p^q$ *are the maximum nodes of* $\mathcal{G}\setminus(\bigcup_{i=0}^{p-1}(\bigcup_{j=1}^{k_i}\mathcal{G}_i^j))$.

Proof. As $\mathcal{G}$ is assumed to be any digraph, according to Lemma 2.1, we see that $\mathcal{G}$ contains at least one maximum node. Therefore, we can find all maximum nodes of $\mathcal{G}$ and denote them by $\mathcal{G}_0^j$ with $1\le j\le k_0$, where k_0 is the number of maximum nodes. Next, we get rid of $\bigcup_{j=1}^{k_0}\mathcal{G}_0^j$ and collect all maximum nodes in the remaining digraph $\mathcal{G}\setminus(\bigcup_{j=1}^{k_0}\mathcal{G}_0^j)$. Denote all maximum nodes of the remainder $\mathcal{G}\setminus(\bigcup_{j=1}^{k_0}\mathcal{G}_0^j)$ by $\mathcal{G}_1^j$ with $1\le j\le k_1$, provided that there are k_1 maximum nodes for $\mathcal{G}\setminus(\bigcup_{j=1}^{k_0}\mathcal{G}_0^j)$. Then we can repeat the same process and find the maximum nodes $\mathcal{G}_p^q$ of $\mathcal{G}\setminus(\bigcup_{i=0}^{p-1}(\bigcup_{j=1}^{k_i}\mathcal{G}_i^j))$ with $1\le q\le k_p$. As $\mathcal{G}$ consists of finite N oscillators, after r steps, we can construct $\mathcal{G}=(\bigcup_{i=0}^{r}(\bigcup_{j=1}^{k_i}\mathcal{G}_i^j))$ where $0\le r<N$. ∎

REMARK 2.1. Lemma 2.2 shows a clear hierarchical structure on a general digraph. For the Kuramoto model (2.2) with frustration on a fixed digraph $\mathcal{G}$ in [42], the digraph $\mathcal{G}$ is assumed to contain a spanning tree. It follows from Lemma 2.1 that $\mathcal{G}$ contains only one maximum node $\mathcal{G}_0$. Based on the definition of maximum node, for $1\le q\ne q'\le k_p$, we see that $\mathcal{G}_p^q$ and $\mathcal{G}_p^{q'}$ do not affect each other. Actually, $\mathcal{G}_p^q$ will only be affected by $\mathcal{G}_0$ and $\mathcal{G}_i^j$, where $1\le i\le p-1$, $1\le j\le k_i$. Therefore, in the proof of the main result in [42], for system (2.2), without loss of generality assume $k_i=1$ for all $1\le i\le r$. Thus, the decomposition can be expressed by

$$\mathcal{G}=\bigcup_{i=0}^{r}\mathcal{G}_i,\quad 0\le r<N, \tag{2.3}$$

where $\mathcal{G}_p$ is a maximum node of $\mathcal{G}\setminus(\bigcup_{i=0}^{p-1}\mathcal{G}_i)$.

Now, we review the previous results on the emergent dynamics of system (2.2) in [42].

PROPOSITION 2.1 ([42]). *Let* $\theta(t)$ *be a solution to system* (2.2) *and suppose the initial data satisfy* (1.4), *i.e.,*

$$D(\theta(0))<\zeta<\gamma<\pi,$$

where ζ *and* γ *are positive constants. Then the following assertion holds*

$$D(\theta(t))<\zeta,\quad \forall\ t\in[0,\bar{t})\quad \textit{where}\quad \bar{t}=\frac{\zeta-D(\theta(0))}{D(\Omega)+2NK\sin\alpha}.$$

PROPOSITION 2.2 ([42]). *Let* $\theta(t)$ *be a solution to system* (2.2), *and assume the network topology* (χ_{ij}) *contains a spanning tree and the initial data satisfy* (1.4), *i.e.,*

$$D(\theta(0))<\zeta<\gamma<\pi, \tag{2.4}$$

where ζ,γ *are positive constants. Moreover, for a given positive constant* $D^\infty<\min\{\frac{\pi}{2},\zeta\}$, *we assume the coupling strength* K *and the frustration* α *satisfy*

$$\tan\alpha<\frac{1}{\left(1+\frac{(r+1)\zeta}{\zeta-D(\theta(0))}\right)2Nc}\frac{\beta^{r+1}D^\infty}{[4(2N+1)c]^r}, \tag{2.5}$$

$$\left(1+\frac{(r+1)\zeta}{\zeta-D(\theta(0))}\right)\frac{c[4(2N+1)c]^r}{\beta^{r+1}D^\infty}\left(\frac{D(\Omega)}{K\cos\alpha}+\frac{2N\sin\alpha}{\cos\alpha}\right)<1, \tag{2.6}$$

where η,β,c are constants subject to the following conditions

$$\eta>\max\left\{\frac{1}{\sin\gamma},\frac{2}{1-\frac{\zeta}{\gamma}}\right\},\quad \beta=1-\frac{2}{\eta},\quad c=\frac{\left(\sum_{j=1}^{N-1}\eta^j A(2N,j)+1\right)\gamma}{\sin\gamma}, \tag{2.7}$$

$0\le r<N$ *is defined in* (2.3)*, and* $A(2N,j)$ *denotes the number of permutations. Then there exists time* $\hat{t}_*>0$ *such that*

$$D(\theta(t))\le D^\infty,\quad \forall t\in[\hat{t}_*,\infty),$$

where $\hat{t}_*$ *satisfies the following estimate*

$$\hat{t}_*<\frac{(r+1)\zeta}{\frac{K\cos\alpha}{c}\frac{\beta^{r+1}D^\infty}{[4(2N+1)c]^r}-(D(\Omega)+2NK\sin\alpha)}<\bar{t}=\frac{\zeta-D(\theta(0))}{D(\Omega)+2NK\sin\alpha}. \tag{2.8}$$

Remark 2.2. Proposition 2.2 states that for the Kuramoto model (2.2) on a fixed digraph containing a spanning tree, all oscillators confined initially in an open half circle will concentrate into a region of quarter cirlce after some finite time $\hat{t}_*$.

2.3. State-transition matrix and scrambling matrix. In this part, we first introduce a state-transition matrix. Let $t_0\in\mathbb{R}$ and $A:[t_0,\infty)\to\mathbb{R}^{N\times N}$ be an $N\times N$ matrix of piecewise continuous functions. Consider the following time-dependent linear ordinary differential equation:

$$\frac{dx(t)}{dt}=A(t)x(t),\quad t>t_0. \tag{2.9}$$

Then the solution of (2.9) is expressed by

$$x(t)=\Phi(t,t_0)x(t_0),$$

where $\Phi(t,t_0)$ is the state-transition matrix or fundamental matrix. From (C.27) in [35], the state-transition matrix $\Phi(t,t_0)$ can be written via the Peano-Baker formula as below:

$$\begin{aligned}\Phi(t,t_0)=I&+\int_{t_0}^t A(s_1)ds_1+\int_{t_0}^t\int_{t_0}^{s_1}A(s_1)A(s_2)ds_2ds_1+\cdots+\\ &+\int_{t_0}^t\int_{t_0}^{s_1}\cdots\int_{t_0}^{s_{n-1}}A(s_1)A(s_2)\cdots A(s_n)ds_n\cdots ds_2ds_1+\cdots\\ &=I+\sum_{n=1}^\infty\int_{t_0}^t\int_{t_0}^{s_1}\cdots\int_{t_0}^{s_{n-1}}A(s_1)A(s_2)\cdots A(s_n)ds_n\cdots ds_2ds_1,\end{aligned} \tag{2.10}$$

where I is an $N\times N$ identity matrix.

Lemma 2.3 ([4]). *Let $t_0\in\mathbb{R},a\in\mathbb{R}$ and $A:[t_0,\infty)\to\mathbb{R}^{N\times N}$ be an $N\times N$ matrix of continuous functions. And we set $\Phi(t,t_0)$ and $\Psi(t,t_0)$ to be the state-transition matrices corresponding to the following linear ODEs, respectively:*

$$\frac{dx(t)}{dt}=A(t)x(t)\quad \text{and}\quad \frac{dx(t)}{dt}=[A(t)+aI]x(t),\quad t>t_0.$$

Then, the following relation between $\Phi(t,t_0)$ *and* $\Psi(t,t_0)$ *holds,*

$$\Phi(t,t_0)=e^{-a(t-t_0)}\Psi(t,t_0), \quad t\geq t_0. \tag{2.11}$$

Proof. The proof of this lemma can be found in Lemma A.1 in [4]. □

Next we provide fundamental concepts of stochastic matrix, scrambling matrix, and adjacency matrix in the following definition.

DEFINITION 2.4 ([9]). *Let* $A=(a_{ij})$ *be a nonnegative* $N\times N$ *matrix.*

(1) *A is a stochastic matrix, if its row-sum is equal to unity:*

$$\sum_{j=1}^{N} a_{ij}=1, \quad 1\leq i\leq N.$$

(2) *A is a scrambling matrix, if for each pair of indexes* i *and* j*, there exists an index* k *such that* $a_{ik}>0$ *and* $a_{jk}>0$.

(3) *A is an adjacency matrix of a digraph* $\mathcal{G}$ *if the following relation holds:*

$$a_{ij}>0\Leftrightarrow(j,i)\in\mathcal{E}.$$

And we write $\mathcal{G}=\mathcal{G}(A)$.

In addition to Definition 2.4, we define the ergodicity coefficient of a nonnegative $N\times N$ matrix $A=(a_{ij})$ as below,

$$\mu(A):=\min_{i,j}\sum_{k=1}^{N}\min\{a_{ik},a_{jk}\}. \tag{2.12}$$

REMARK 2.3. From the definition in (2.12), it is easy to see that

(1) A is scrambling if and only if $\mu(A)>0$.

(2) For nonnegative matrices A and B, we have

$$A\geq B\Rightarrow\mu(A)\geq\mu(B).$$

The ergodicity coefficient plays a key role in the following lemma which exhibits a contraction property.

LEMMA 2.4 ([4]). *Assume that a nonnegative* $N\times N$ *matrix* $A=(a_{ij})$ *is stochastic, and let* $w=(w_1,\dots,w_N)$ *and* $v=(v_1,\dots,v_N)$ *such that*

$$w=Av.$$

Then we have

$$\max_{i,j}||w_i-w_j||\leq(1-\mu(A))\max_{i,j}||v_i-v_j||,$$

where $||\cdot||$ *denotes the* l_2*-norm in Euclidean space* $\mathbb{E}$.

Proof. The proof of this lemma can be found in Lemma 2.1 in [4]. □

REMARK 2.4. For a nonnegative stochastic $N\times N$ matrix $A=(a_{ij})$, we have

$$\mu(A)\leq 1.$$

In fact, from the definition of stochastic matrix in Definition 2.4 and (2.12), we immediately get

$$\mu(A)=\min_{i,j}\sum_{k=1}^{N}\min\{a_{ik},a_{jk}\}\leq\min_{i,j}\sum_{k=1}^{N}a_{ik}=1,$$

where we use $\sum_{k=1}^{N}a_{ik}=1$ for all $1\leq i\leq N$.

Now we introduce an important property of scrambling matrix which will be crucially used in later analysis.

Lemma 2.5 ([39]). *Assume that $A_i, 1\leq i\leq N-1$ are nonnegative $N\times N$ matrices with positive diagonal elements and $\mathcal{G}(A_i)$ has a spanning tree for all $1\leq i\leq N-1$. Then $A_1A_2\cdots A_{N-1}$ is scrambling.*

Proof. The proof of this lemma can be found in Theorem 5.1 in [39]. □

Finally, we review the concept of frequency synchronization and some notations.

Definition 2.5 ([10]). *Let $\theta(t)=(\theta_1(t),\dots,\theta_N(t))$ be a solution to the system (1.1). Then the configuration $\theta(t)$ exhibits frequency synchronization asymptotically if and only if the relative frequencies tend to zero asymptotically:*

$$\lim_{t\to+\infty}|\dot{\theta}_i(t)-\dot{\theta}_j(t)|=0,\quad \forall\ 1\leq i,j\leq N.$$

Notations: For the sake of convenience, we introduce some notations such as the diameter $D(\Omega)$ of natural frequency, the extreme phases and frequencies, the diameters $D(\theta(t))$ and $D(\omega(t))$ of phases and frequencies:

$$\begin{aligned}
&\theta(t)=(\theta_1(t),\dots,\theta_N(t)),\quad \omega(t)=(\omega_1(t),\dots,\omega_N(t)),\quad D(\Omega)=\max_{1\leq i\leq N}\Omega_i-\min_{1\leq i\leq N}\Omega_i,\\
&\theta_M(t)=\max_{1\leq i\leq N}\theta_i(t),\quad \theta_m(t)=\min_{1\leq i\leq N}\theta_i(t),\quad D(\theta(t))=\theta_M(t)-\theta_m(t),\\
&\omega_M(t)=\max_{1\leq i\leq N}w_i(t),\quad \omega_m(t)=\min_{1\leq i\leq N}w_i(t),\quad D(\omega(t))=\omega_M(t)-\omega_m(t).
\end{aligned}$$

Let $\mathbb{N}$ denote the set of all natural numbers including zero. For matrices $A=(a_{ij})_{m\times n}$ and $B=(b_{ij})_{m\times n}$, $A\geq B$ means that $a_{ij}\geq b_{ij}$ for all $i,j=1,\dots,N$. Let I denote the identity matrix with appropriate dimensions. For $a\in\mathbb{R}$, $\lfloor a\rfloor$ denotes the greatest integer not exceeding a. $A(n,m)=\frac{n!}{(n-m)!}$ denotes the number of permutations.

3. Entrance to a small region

In this section, we provide a preparatory estimate on the phase diameter $D(\theta(t))$. More precisely, under the Assumption 1.2 of the switching topology with spanning tree in any mode of $\mathcal{P}$, we will find a finite time at which the size of phase diameter is bounded by a small value less than $\frac{\pi}{2}$ before the first network switching. We first show the existence of solution to system (1.1).

Lemma 3.1. *Suppose Assumption 1.1 is fulfilled, then there exists a unique solution $\theta(t)$ of system (1.1) on $[0,+\infty)$ with initial data $\theta_i(0)=\theta_i^0, 1\leq i\leq N$. Moreover, this solution $\theta(t)$ is continuous and piecewise smooth.*

Proof. Based on the existence of the dwelling time τ for the switching instants $\{t_l\}_{l=0}^{\infty}$ with $t_0=0$ in Assumption 1.1, it is known that the switching law $\sigma(t):[0,\infty)\to$

$\mathcal{P}=\{1,2,\dots,p\}$ is a piecewise constant function, which is continuous from the right. Therefore, let $\sigma(t)=p_0\in\mathcal{P}$ for $t\in[0,t_1)$, then the system (1.1) can be written as

$$\dot{\theta}_i(t)=\Omega_i+K\sum_{j=1}^{N}\chi_{ij}^{p_0}\sin(\theta_j(t)-\theta_i(t)+\alpha),\quad t\in[0,t_1),$$

subject to the initial data $\theta_i(0)=\theta_i^0$. On the interval $[0,t_1)$, the right-hand vector field of above system is real analytic, thus the existence and uniqueness of a real analytic solution $\theta(t)$ for $t\in[0,t_1)$ are guaranteed by the Cauchy theorem in ordinary differential equations (see Exercise 3.3 in Hale's book [22] for details).

Then let $\sigma(t)=p_1\in\mathcal{P}$ for $t\in[t_1,t_2)$, the system (1.1) can be written as

$$\dot{\theta}_i(t)=\Omega_i+K\sum_{j=1}^{N}\chi_{ij}^{p_1}\sin(\theta_j(t)-\theta_i(t)+\alpha),\quad t\in[t_1,t_2),$$

subject to the initial data $\theta_i(t_1)$. We proceed as before for $t\in[0,t_1)$ to derive the existence and uniqueness of a real analytic solution $\theta(t)$ for $t\in[t_1,t_2)$. The process can be continued to obtain the existence and uniqueness of solution $\theta(t)$ to system (1.1) on $[0,+\infty)$. Moreover, this solution is continuous and piecewise analytic or piecewise smooth since analytic function is smooth. ∎

For the Kuramoto model (1.1) with switching network, we consider the case that the initial configuration is distributed in an open half circle, i.e., $D(\theta(0))<\pi$, then we can find positive constants ζ and γ satisfying

$$D(\theta(0))<\zeta<\gamma<\pi. \tag{3.1}$$

In the subsequence, we present a local estimate which states that the phase diameter remains less than π in short time.

Lemma 3.2. *Let $\theta(t)$ be a solution to system (1.1) and assume that the initial data satisfy (1.4), i.e.,*

$$D(\theta(0))<\zeta<\gamma<\pi,$$

where ζ,γ are positive constants. Then we have the following assertion

$$D(\theta(t))<\zeta,\quad t\in[0,\bar{t})\quad \text{where}\quad \bar{t}=\frac{\zeta-D(\theta(0))}{D(\Omega)+2NK\sin\alpha}. \tag{3.2}$$

Proof. From system (1.1), we see that the dynamics of phase diameter satisfies the following differential inequality

$$\begin{aligned}\dot{D}(\theta(t))&=\frac{d}{dt}(\theta_M(t)-\theta_m(t))\\&=\Omega_M+K\sum_{j=1}^{N}\chi_{Mj}^{\sigma(t)}\sin(\theta_j(t)-\theta_M(t)+\alpha)\\&\quad-\Omega_m-K\sum_{j=1}^{N}\chi_{mj}^{\sigma(t)}\sin(\theta_j(t)-\theta_m(t)+\alpha)\end{aligned}$$

$$
\begin{aligned}
&\le D(\Omega)+K\sum_{j=1}^{N}\chi_{Mj}^{\sigma(t)}[\sin(\theta_j(t)-\theta_M(t))\cos\alpha+\cos(\theta_j(t)-\theta_M(t))\sin\alpha]\\
&\quad -K\sum_{j=1}^{N}\chi_{mj}^{\sigma(t)}[\sin(\theta_j(t)-\theta_m(t))\cos\alpha+\cos(\theta_j(t)-\theta_m(t))\sin\alpha]\\
&= D(\Omega)+K\cos\alpha\left(\sum_{j=1}^{N}\chi_{Mj}^{\sigma(t)}\sin(\theta_j(t)-\theta_M(t))-\sum_{j=1}^{N}\chi_{mj}^{\sigma(t)}\sin(\theta_j(t)-\theta_m(t))\right)\\
&\quad +K\sin\alpha\left(\sum_{j=1}^{N}\chi_{Mj}^{\sigma(t)}\cos(\theta_j(t)-\theta_M(t))-\sum_{j=1}^{N}\chi_{mj}^{\sigma(t)}\cos(\theta_j(t)-\theta_m(t))\right). \qquad (3.3)
\end{aligned}
$$

Here, θ_M and θ_m denote the maximal phase and minimal phase respectively. Thus when the phase diameter $D(\theta(t))$ is less than π, and whichever topology mode is activated, we have

$$
\begin{aligned}
&\chi_{Mj}^{\sigma(t)}\sin(\theta_j(t)-\theta_M(t))\le 0 \quad \text{and} \quad \chi_{mj}^{\sigma(t)}\sin(\theta_j(t)-\theta_m(t))\ge 0,\\
&\left|\sum_{j=1}^{N}\chi_{Mj}^{\sigma(t)}\cos(\theta_j(t)-\theta_M(t))-\sum_{j=1}^{N}\chi_{mj}^{\sigma(t)}\cos(\theta_j(t)-\theta_m(t))\right|\le 2N,
\end{aligned}
$$

which yields from (3.3) that

$$
\dot{D}(\theta(t))\le D(\Omega)+2NK\sin\alpha. \qquad (3.4)
$$

We see from (3.4) that the growth of phase diameter is bounded above by the linear growth with positive slope $D(\Omega)+2NK\sin\alpha$ when $D(\theta(t))\le\zeta<\pi$. Thus, we set

$$
\bar{t}=\frac{\zeta-D(\theta(0))}{D(\Omega)+2NK\sin\alpha},
$$

then it immediately yields from (3.4) that $D(\theta(t))$ is less than ζ before $\bar{t}$, i.e.,

$$
D(\theta(t))<\zeta, \quad \forall t\in[0,\bar{t}),
$$

which yields the desired result. ◻

REMARK 3.1. Note that under the assumption of system (1.1) and system (2.2) with the same initial setting (3.1), for the system (1.1) with switching network, the local estimate in Lemma 3.2 is the same as that of Proposition 2.1 for the Kuramoto system (2.2) with frustration on a static digraph.

We next study the dynamics of system (1.1) before the occurrence of the first network switching. Based on Assumption 1.1, we know that there is no network switching during time interval $[0,\tau)$, thus we assume $\sigma(t)=p_0\in\mathcal{P}$ for $t\in[0,\tau)$, then the system (1.1) can be written as below

$$
\dot{\theta}_i(t)=\Omega_i+K\sum_{j=1}^{N}\chi_{ij}^{p_0}\sin(\theta_j(t)-\theta_i(t)+\alpha), \quad t\in[0,\tau), \qquad (3.5)
$$

where $(\chi_{ij}^{p_0})$ is the $(0,1)$-adjacency matrix of the digraph $\mathcal{G}^{p_0}$ in time interval $[0,\tau)$. It is known from the structural Assumption 1.2 that the digraph $\mathcal{G}^{p_0}$ has a spanning tree.

Then according to Remark 2.1, without loss of generality, we exploit the idea of digraph decomposition in Lemma 2.2 to decompose $\mathcal{G}^{p_0}$ into a union as

$$\mathcal{G}^{p_0} = \bigcup_{i=0}^{r_0} \mathcal{G}_i^{p_0}, \quad 0 \le r_0 < N, \tag{3.6}$$

where $\mathcal{G}_i^{p_0}$ is a maximum node of $\mathcal{G}^{p_0} \setminus (\bigcup_{k=0}^{i-1} \mathcal{G}_k^{p_0})$. For the dynamics of system (3.5) in $[0,\tau)$, in addition to almost the same assumptions (2.5) and (2.6) in Proposition 2.2, we further assume the coupling strength K large enough such that

$$\bar{t} = \frac{\zeta - D(\theta(0))}{D(\Omega) + 2NK\sin\alpha} < \tau, \tag{3.7}$$

where $\bar{t}$ is given in Lemma 3.2 or Proposition 2.1 and τ is the minimal dwelling time defined in (1.3). Then for the system (3.5) with initial data confined in an open half circle, we will provide one analogue of Proposition 2.2, which states that large coupling strength K and small frustration α will push oscillators into a small region of quarter circle at some time before $\bar{t} \in [0,\tau)$.

Lemma 3.3. *Let $\theta(t)$ be a solution to system (3.5), and assume the initial configuration satisfies (1.4), i.e.,*

$$D(\theta(0)) < \zeta < \gamma < \pi,$$

where ζ, γ are positive constants. Moreover, for a given small positive constant $D^\infty < \min\{\zeta, \frac{\pi}{2}\}$, we assume the coupling strength K and the frustration α satisfy

$$\tan\alpha < \frac{1}{\left(1 + \frac{(r_0+1)\zeta}{\zeta - D(\theta(0))}\right) 2Nc} \frac{\beta^{r_0+1} D^\infty}{[4(2N+1)c]^{r_0}}, \tag{3.8}$$

$$\frac{\zeta - D(\theta(0))}{D(\Omega) + 2NK\sin\alpha} < \tau, \quad \left(1 + \frac{(r_0+1)\zeta}{\zeta - D(\theta(0))}\right) \frac{c[4(2N+1)c]^{r_0}}{\beta^{r_0+1} D^\infty} \left(\frac{D(\Omega)}{K\cos\alpha} + \frac{2N\sin\alpha}{\cos\alpha}\right) < 1, \tag{3.9}$$

where η, β and c are constants subject to the following conditions

$$\eta > \max\left\{\frac{1}{\sin\gamma}, \frac{2}{1 - \frac{\zeta}{\gamma}}\right\}, \quad \beta = 1 - \frac{2}{\eta}, \quad c = \frac{\left(\sum_{j=1}^{N-1} \eta^j A(2N,j) + 1\right)\gamma}{\sin\gamma}, \tag{3.10}$$

and r_0 is given in (3.6), $A(2N,j)$ denotes the number of permutations. Then there exists $t_* > 0$ such that

$$D(\theta(t)) \le D^\infty, \quad t \in [t_*, \tau),$$

where t_* is bounded above by $\bar{t}$ given in Lemma 3.2, i.e.,

$$t_* < \frac{(r_0+1)\zeta}{\frac{K\cos\alpha}{c} \frac{\beta^{r_0+1} D^\infty}{[4(2N+1)c]^{r_0}} - (D(\Omega) + 2NK\sin\alpha)} < \bar{t} = \frac{\zeta - D(\theta(0))}{D(\Omega) + 2NK\sin\alpha} < \tau. \tag{3.11}$$

Proof. We see that the network topology mode p_0 in system (3.5) is fixed, thus compared to system (2.2) dealt with in the whole time interval $[0, +\infty)$ in [42], the slight difference is that we address system (3.5) in finite time interval $[0,\tau)$. In fact, for the system (3.5) with initial data subject to the constraint (3.1), based on the decomposition (3.6) and the same local estimate in Lemma 3.2, under the additional assumption in (3.9), i.e.,

$$\bar{t} = \frac{\zeta - D(\theta(0))}{D(\Omega) + 2NK\sin\alpha} < \tau,$$

we can almost apply the same inductive method in [42] to study the dynamics of system (3.5) in time interval $[0,\tau)$ and derive the desired result. As the proof of this lemma is almost the same as that of Proposition 2.2 and rather lengthy, in view of the length of paper, we omit its detailed proof and readers may refer to [42] for more details. □

REMARK 3.2. From Lemma 3.3, it is known that before the first network switching occurs, i.e., for the system (3.5) in $[0,\tau)$, we can find a finite time $t_* > 0$ at which the phase diameter of oscillators is bounded above by a small value, i.e.,

$$D(\theta(t_*)) \le D^\infty, \quad t_* < \bar{t} < \tau. \tag{3.12}$$

4. Frequency synchronization

In this section, we provide a proof of Theorem 1.1 on the frequency synchronization of the Kuramoto model with uniform frustration and switching topology. In the work [5,9] on the flocking estimates for the Cucker-Smale model with switching topology, the linear structure in velocity coupling is the key ingredient for using induction argument based on matrix-graph theories. Therefore, although the sinusoidal coupling in the Kuramoto model is nonlinear, based on the preparatory estimate (3.12) at t_* in Lemma 3.3, we consider system (1.1) starting from t_*, and lift it to the second-order formulation (1.2), which enjoys the linearity in frequency coupling similar to the Cucker-Smale model. Then in a large coupling strength and small frustration regime, we apply the matrix theory-based approach for the second-order Kuramoto system (1.2) to show the complete synchronization.

More precisely, we first show a priori frequency synchronization, i.e., under a priori condition

$$\sup_{t_* \le t \le \infty} D(\theta(t)) + \alpha \le \xi < \frac{\pi}{2}, \tag{4.1}$$

where $\xi < \frac{\pi}{2}$ is a constant, we verify the exponential decay of frequency diameter after t_*. Next, under additional suitable assumptions on the system parameters, we verify that the priori condition (4.1) can be guaranteed for all time after t_*. This procedure finally yields the proof of Theorem 1.1 on the complete synchronization.

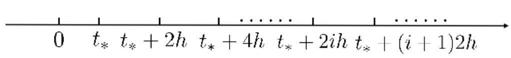

FIG. 4.1. *Interval division*

To this end, we set $h = \frac{1}{K}$ which is the reciprocal of the coupling strength and $\hat{h} = (N-1)2h$. We exploit $2h$ as a unity to divide the positive real axis starting from t_* in Lemma 3.3 and obtain a sequence of intervals

$$[t_* + i2h, t_* + (i+1)2h) \quad \text{with } i \in \mathbb{N}, \tag{4.2}$$

which are presented in Figure 4.1. From Assumption 1.1, we know that there is a minimal positive dwelling time τ for switching law $\sigma(t)$. Thus we can let the coupling strength sufficiently large such that, there is at most one switching instant in each interval $[t_*+i2h, t_*+(i+1)2h)$, in other words, all switching instants $\{t_l\}_{l=0}^{\infty}$ are distributed in different time intervals in (4.2). Actually, we can achieve this under the assumption that the coupling strength $K > \frac{2}{\tau}$, i.e., $2h < \tau$.

4.1. A priori synchronization. In this subsection, we provide that the frequency diameter decays to zero exponentially fast after t_* under a priori condition (4.1).

For this, we consider the second-order Kuramoto model in time interval $[t_*, +\infty)$,

$$\begin{cases} \dot{\theta}_i(t) = w_i(t), \quad t > t_*, \ i = 1, 2, \dots, N, \\ \dot{w}_i(t) = K \displaystyle\sum_{j=1}^{N} \chi_{ij}^{\sigma(t)} \cos(\theta_j(t) - \theta_i(t) + \alpha)(w_j(t) - w_i(t)), \end{cases} \tag{4.3}$$

subject to the initial data $(\theta_i(t_*), \omega_i(t_*)) = (\theta_i(t_*), \dot{\theta}_i(t_*))$. In the following, we rewrite the second-order system (4.3) in a more concise form. Recall that at any instant, we assume the digraph $\mathcal{G}^{\sigma(t)}$ has a self-loop at each vertex i, i.e., $i \in \mathcal{N}_i^{\sigma(t)}$. Let $A^{\sigma(t)}(t) = (a_{ij}^{\sigma(t)}(t))$ be the adjacency matrix of $\mathcal{G}^{\sigma(t)}$ where

$$a_{ij}^{\sigma(t)}(t) := \chi_{ij}^{\sigma(t)} \cos(\theta_j(t) - \theta_i(t) + \alpha).$$

Note that $a_{ii}^{\sigma(t)}(t) = \cos\alpha$ for all $t > 0$. Then define the Laplacian matrix of $A^{\sigma(t)}(t)$ by

$$L^{\sigma(t)}(t) := D^{\sigma(t)}(t) - A^{\sigma(t)}(t), \tag{4.4}$$

where

$$D^{\sigma(t)}(t) := \mathrm{diag}(d_1^{\sigma(t)}(t), \dots, d_N^{\sigma(t)}(t)), \quad d_i^{\sigma(t)}(t) := \sum_{j=1}^{N} a_{ij}^{\sigma(t)}(t), \ 1 \le i \le N.$$

Then the system (4.3) can be rewritten as

$$\begin{cases} \dot{\theta}(t) = \omega(t), \quad t \ge t_*, \\ \dot{\omega}(t) = -K L^{\sigma(t)}(t) \omega(t). \end{cases} \tag{4.5}$$

Let $\Phi(s_2, s_1)$ be the state-transition matrix associated to $(4.5)_2$. Then from (2.9), the solution formula for $\omega(t)$ is given by

$$\omega(t_* + m\hat{h}) = \Phi\left(t_* + m\hat{h}, t_* + (m-1)\hat{h}\right) \omega(t_* + (m-1)\hat{h}), \quad m \ge 1, \ m \in \mathbb{N}. \tag{4.6}$$

We first show that the frequency diameter is nonincreasing under a priori condition (4.1) in the following lemma.

LEMMA 4.1. *Let (θ_i, ω_i) be a solution to system (4.3), and suppose the following priori condition holds*

$$\sup_{t_* \le t < \infty} D(\theta(t)) + \alpha \le \xi < \frac{\pi}{2}.$$

Then the frequency diameter is nonincreasing

$$\frac{d}{dt}D(\omega(t)) \leq 0, \qquad t \geq t_*.$$

Proof. We apply the Equation (4.3) to cope with the dynamics of frequency diameter

$$\begin{aligned}\frac{d}{dt}D(\omega(t)) &= \frac{d}{dt}(\omega_M(t)-\omega_m(t))\\ &= K\sum_{j=1}^{N}\chi_{Mj}^{\sigma(t)}\cos(\theta_j(t)-\theta_M(t)+\alpha)(\omega_j(t)-\omega_M(t))\\ &\quad -K\sum_{j=1}^{N}\chi_{mj}^{\sigma(t)}\cos(\theta_j(t)-\theta_m(t)+\alpha)(\omega_j(t)-\omega_m(t))\\ &\leq 0,\end{aligned}$$

where the indicies M and m denote the extreme frequencies and we use a priori condition (4.1) to get for $t \geq t_*$,

$$\begin{aligned}&|\theta_j(t)-\theta_M(t)+\alpha| \leq |\theta_j(t)-\theta_M(t)|+\alpha \leq D(\theta(t))+\alpha \leq \xi < \frac{\pi}{2},\\ &\text{similarly,}\quad |\theta_j(t)-\theta_m(t)+\alpha| \leq \xi < \frac{\pi}{2}.\end{aligned}$$

This yields the desired result. $\square$

Next, we study the lower bound of the ergodicity coefficient of matrix $\Phi\left(t_*+m\hat{h}, t_*+(m-1)\hat{h}\right)$ in (4.6) and its property.

LEMMA 4.2. *Suppose that Assumption 1.1 and Assumption 1.2 hold, and let $(\theta(t),\omega(t))$ be a solution to system (4.5) satisfying a priori condition*

$$\sup_{t_* \leq t < \infty} D(\theta(t))+\alpha \leq \xi < \frac{\pi}{2}. \tag{4.7}$$

Moreover assume the coupling strength K large enough such that $\frac{2}{K} < \tau$. Then we have
(1) *The ergodicity coefficient $\mu\left(\Phi\left(t_*+m\hat{h}, t_*+(m-1)\hat{h}\right)\right)$ in (4.6) is bounded below*

$$\mu\left(\Phi\left(t_*+m\hat{h}, t_*+(m-1)\hat{h}\right)\right) \geq e^{-KN2h(N-1)}(hK\cos\xi)^{N-1}. \tag{4.8}$$

(2) *The state-transition matrix $\Phi\left(t_*+m\hat{h}, t_*+(m-1)\hat{h}\right)$ in (4.6) is stochastic, where $h=\frac{1}{K}$ and $\hat{h}=(N-1)2h$.*

Proof.
(1) Based on the interval division (4.2) and the condition $\frac{2}{K} < \tau$, we know that there is at most one switching instant during the time interval $[t_*+(m-1)\hat{h}, t_*+(m-1)\hat{h}+2h)$. It's more simpler if the switching instant is exactly located in the left-end point of interval $[t_*+(m-1)\hat{h}, t_*+(m-1)\hat{h}+2h)$ or there is no switching instant at all during the interval, thus for simplicity, we only deal with the situation that there exists one switching instant, say t_{l_2}, in the interval $(t_*+(m-1)\hat{h}, t_*+(m-1)\hat{h}+2h)$. Set $t_{l_1} = t_*+$

$(m-1)\hat{h}$ and $t_{l_3}=t_*+(m-1)\hat{h}+2h$. And let $\sigma(t)=p_r$ for $t\in[t_{l_r},t_{l_{r+1}})$ where $r=1,2$. Then we have

$$\Phi\left(t_*+(m-1)\hat{h}+2h,t_*+(m-1)\hat{h}\right)=\Phi^{p_2}(t_{l_3},t_{l_2})\Phi^{p_1}(t_{l_2},t_{l_1}) \tag{4.9}$$

where for $r=1,2$, $\Phi^{p_r}\left(t_{l_{r+1}},t_{l_r}\right)$ is the state-transition matrix of $-KL^{p_r}(t)$ on $[t_{l_r},t_{l_{r+1}})$.

We next apply (4.4) and a priori condition (4.7) to estimate the coefficient matrix of $(4.5)_2$ on interval $[t_{l_r},t_{l_{r+1}})$,

$$-KL^{p_r}(t)=K(A^{p_r}(t)-D^{p_r}(t))\geq K(\underline{A}^{p_r}-NI)=K\underline{A}^{p_r}-KNI, \tag{4.10}$$

where the matrix $\underline{A}_{p_r}(t)=(\underline{a}_{ij}^{p_r})$ is defined by

$$\underline{a}_{ij}^{p_r}:=\begin{cases}\chi_{ij}^{p_r}\cos\xi, & i\neq j,\\ \cos\alpha, & i=j.\end{cases}$$

Here, we use that for $t\in[t_{l_r},t_{l_{r+1}})$, the elements $a_{ij}^{p_r}(t)$ of the adjacency matrix $A^{p_r}(t)=(a_{ij}^{p_r}(t))$ in (4.10) satisfy

$$|\theta_j(t)-\theta_i(t)+\alpha|\leq|\theta_j(t)-\theta_i(t)|+\alpha\leq D(\theta(t))+\alpha\leq\xi,\quad a_{ii}^{p_r}(t)=\cos\alpha,$$
$$a_{ij}^{p_r}(t)=\chi_{ij}^{p_r}\cos(\theta_j(t)-\theta_i(t)+\alpha)\geq\chi_{ij}^{p_r}\cos\xi,\quad a_{ij}^{p_r}(t)\leq 1,\quad d_i^{p_r}(t)=\sum_{j=1}^{N}a_{ij}^{p_r}(t)\leq N.$$

Thus, it follows from (4.10) that

$$-KL^{p_r}(t)+KNI\geq K\underline{A}^{p_r},\quad\text{for } t\in[t_{l_r},t_{l_{r+1}}). \tag{4.11}$$

On the other hand, let $\Psi^{p_r}(t_{l_{r+1}},t_{l_r})$ be the state transition matrix of $-KL^{p_r}(t)+KNI$ on interval $[t_{l_r},t_{l_{r+1}})$. Then it yields from Lemma 2.3 that

$$\Phi^{p_r}(t_{l_{r+1}},t_{l_r})=e^{-KN(t_{l_{r+1}}-t_{l_r})}\Psi^{p_r}(t_{l_{r+1}},t_{l_r}). \tag{4.12}$$

Combing the Peano-Baker series for $\Psi^{p_r}(t_{l_{r+1}},t_{l_r})$ in (2.10) and (4.11), we immediately derive

$$\begin{aligned}\Psi^{p_r}(t_{l_{r+1}},t_{l_r})=&I+\int_{t_{l_r}}^{t_{l_{r+1}}}(-KL^{p_r}(s_1)+KNI)ds_1\\&+\int_{t_{l_r}}^{t_{l_{r+1}}}\int_{t_{l_r}}^{s_1}(-KL^{p_r}(s_1)+KNI)(-KL^{p_r}(s_2)+KNI)ds_2ds_1+\cdots\\&+\int_{t_{l_r}}^{t_{l_{r+1}}}\int_{t_{l_r}}^{s_1}\cdots\int_{t_{l_r}}^{s_{n-1}}(-KL^{p_r}(s_1)+KNI)(-KL^{p_r}(s_2)+KNI)\cdots\\&\times(-KL^{p_r}(s_n)+KNI)ds_n\cdots ds_2ds_1+\cdots\\\geq&I+\int_{t_{l_r}}^{t_{l_{r+1}}}K\underline{A}^{p_r}ds_1+\int_{t_{l_r}}^{t_{l_{r+1}}}\int_{t_{l_r}}^{s_1}(K\underline{A}^{p_r})^2ds_2ds_1+\cdots\\&+\int_{t_{l_r}}^{t_{l_{r+1}}}\int_{t_{l_r}}^{s_1}\cdots\int_{t_{l_r}}^{s_{n-1}}(K\underline{A}^{p_r})^n ds_n\cdots ds_2ds_1+\cdots\\=&I+\sum_{n=1}^{\infty}\frac{\left((t_{l_{r+1}}-t_{l_r})K\underline{A}^{p_r}\right)^n}{n!}\geq I+(t_{l_{r+1}}-t_{l_r})K\underline{A}^{p_r},\end{aligned} \tag{4.13}$$

Then it follows from (4.12) and (4.13) that

$$\Phi^{p_r}(t_{l_{r+1}},t_{l_r}) \geq e^{-KN(t_{l_{r+1}}-t_{l_r})}\left(I+(t_{l_{r+1}}-t_{l_r})K\underline{A}^{p_r}\right), \qquad r=1,2. \tag{4.14}$$

Thus we apply (4.9) and (4.14) to obtain that

$$\begin{aligned}
&\Phi\left(t_*+(m-1)\hat{h}+2h, t_*+(m-1)\hat{h}\right)\\
&\geq e^{-KN(t_{l_3}-t_{l_2})}(I+(t_{l_3}-t_{l_2})K\underline{A}^{p_2})e^{-KN(t_{l_2}-t_{l_1})}(I+(t_{l_2}-t_{l_1})K\underline{A}^{p_1})\\
&= e^{-KN2h}(I+(t_{l_3}-t_{l_2})K\underline{A}^{p_2})(I+(t_{l_2}-t_{l_1})K\underline{A}^{p_1})\\
&\geq e^{-KN2h}((t_{l_3}-t_{l_2})K\underline{A}^{p_2}+(t_{l_2}-t_{l_1})K\underline{A}^{p_1}).
\end{aligned} \tag{4.15}$$

Since $t_{l_2}\in(t_{l_1},t_{l_3})$ and the length of interval (t_{l_1},t_{l_3}) is $2h$, there are two possibilities that either $t_{l_3}-t_{l_2}\geq h$ or $t_{l_2}-t_{l_1}\geq h$. Thus for the convenience of analysis, without loss of generality, say $t_{l_2}-t_{l_1}\geq h$, it yields from (4.15) that

$$\Phi\left(t_*+(m-1)\hat{h}+2h, t_*+(m-1)\hat{h}\right) \geq e^{-KN2h}hK\underline{A}^{p_1} \geq e^{-KN2h}hK\cos\xi F_1, \tag{4.16}$$

where we use $\cos\alpha \geq \cos\xi$ and the matrix $F_1=(f_{ij}^1)$ is associated to $\underline{A}^{p_1}$ and defined by

$$f_{ij}^1=\begin{cases}\chi_{ij}^{p_1}, & i\neq j,\\ 1, & i=j.\end{cases}$$

Note that F_1 is the (0,1)-adjacency matrix of the digraph $\mathcal{G}$ on $[t_{l_1},t_{l_2})$ and from Assumption 1.2 the digraph $\mathcal{G}(F_1)$ has a spanning tree. Then we can repeat the same analysis in (4.16) to conclude that for all $1\leq i\leq N-1$,

$$\Phi\left(t_*+(m-1)\hat{h}+2ih, t_*+(m-1)\hat{h}+2(i-1)h\right) \geq e^{-KN2h}hK\cos\xi F_i, \tag{4.17}$$

where F_i is the (0,1)-matrix with the property that the digraph $\mathcal{G}(F_i)$ has a spanning tree. Therefore, it follows from (4.17) that

$$\begin{aligned}
\Phi\left(t_*+m\hat{h}, t_*+(m-1)\hat{h}\right) &= \prod_{i=1}^{N-1}\Phi\left(t_*+(m-1)\hat{h}+2ih, t_*+(m-1)\hat{h}+2(i-1)h\right)\\
&\geq \prod_{i=1}^{N-1}\left(e^{-KN2h}hK\cos\xi F_i\right)\\
&= e^{-KN2h(N-1)}(hK\cos\xi)^{N-1}\prod_{i=1}^{N-1}F_i.
\end{aligned} \tag{4.18}$$

Here, it is easy to see from Lemma 2.5 that the matrix $\prod_{i=1}^{N-1}F_i$ is scrambling. Moreover, as each matrix F_i is (0,1)-matrix, it follows from (2.12) that

$$\mu\left(\prod_{i=1}^{N-1}F_i\right)\geq 1. \tag{4.19}$$

Thus we combine (4.18) and (4.19) to derive that

$$\mu\left(\Phi\left(t_*+m\hat{h}, t_*+(m-1)\hat{h}\right)\right) \geq e^{-KN2h(N-1)}(hK\cos\xi)^{N-1}, \quad m\geq 1,\ m\in\mathbb{Z},$$

which yields the first part of this lemma.

(2) For the second part of this lemma, from (4.18), we see that the matrix $\Phi\left(t_*+m\hat{h},t_*+(m-1)\hat{h}\right)$ is nonnegative, thus we only need to show that the sum of each row of $\Phi\left(t_*+m\hat{h},t_*+(m-1)\hat{h}\right)$ is equal to 1. In fact, according to the definition of the Laplacian matrix $L^{\sigma(t)}(t)$, the constant state $\omega(t)=[\omega_1(t),\ldots,\omega_N(t)]^\top=[1,\ldots,1]^\top$ is a solution to $(4.5)_2$ due to the fact that

$$L_{\sigma(t)}(t)[1,\ldots,1]^\top\equiv[0,\ldots,0]^\top,\quad \text{for all } t\geq 0.$$

Thus it satisfies (4.6), i.e.,

$$[1,\ldots,1]^\top=\Phi\left(t_*+m\hat{h},t_*+(m-1)\hat{h}\right)[1,\ldots,1]^\top,\quad m\geq 1,\ m\in\mathbb{Z},$$

which means that $\Phi\left(t_*+m\hat{h},t_*+(m-1)\hat{h}\right)$ is stochastic for all $m\geq 1,\ m\in\mathbb{Z}$. $\square$

Now for the Kuramoto model (4.3) with switching topology under the effect of frustration, we will show the frequency synchronization under a priori boundedness of phase diameter after t_*.

LEMMA 4.3. *Suppose that Assumption 1.1 and Assumption 1.2 hold, and let $(\theta(t),\omega(t))$ be a solution to system (4.5) satisfying a priori condition*

$$\sup_{t_*\leq t<\infty} D(\theta(t))+\alpha\leq\xi<\frac{\pi}{2}. \tag{4.20}$$

Moreover assume the coupling strength K large enough such that $\frac{2}{K}<\tau$. Then we conclude that

$$D(\omega(t))\leq(1-\delta)^{\left\lfloor\frac{t-t_*}{\hat{h}}\right\rfloor}D(\omega(t_*)),\quad t\geq t_*,$$

where $\delta=e^{-2N(N-1)}(\cos\xi)^{N-1}, h=\frac{1}{K}$ and $\hat{h}=(N-1)2h$.

Proof. It is known from Lemma 4.2 that $\Phi\left(t_*+m\hat{h},t_*+(m-1)\hat{h}\right)$ is stochastic. Thus, we apply (4.6) and the contraction property in Lemma 2.4 to have

$$D\left(\omega\left(t_*+m\hat{h}\right)\right)\leq\left(1-\mu\left(\Phi\left(t_*+m\hat{h},t_*+(m-1)\hat{h}\right)\right)\right)D\left(\omega\left(t_*+(m-1)\hat{h}\right)\right).$$

Then we further exploit the above inequality, (4.8) and the process of iteration to derive

$$\begin{aligned}D\left(\omega\left(t_*+m\hat{h}\right)\right)&\leq\left(1-e^{-KN2h(N-1)}(hK\cos\xi)^{N-1}\right)D\left(\omega\left(t_*+(m-1)\hat{h}\right)\right)\\&\leq\left(1-e^{-KN2h(N-1)}(hK\cos\xi)^{N-1}\right)^m D(\omega(t_*)),\qquad m\geq 1,\ m\in\mathbb{Z}.\end{aligned} \tag{4.21}$$

On the other hand, under a priori condition (4.20), we see from Lemma 4.1 that the frequency diameter is nonincreasing for $t\geq t_*$. Therefore, it follows from (4.21) that

$$D(\omega(t))\leq D\left(\omega\left(t_*+\left\lfloor\frac{t-t_*}{\hat{h}}\right\rfloor\hat{h}\right)\right)\leq(1-\delta)^{\left\lfloor\frac{t-t_*}{\hat{h}}\right\rfloor}D(\omega(t_*)),\quad t\geq t_*,$$

where we set

$$\delta=e^{-KN2h(N-1)}(hK\cos\xi)^{N-1}. \tag{4.22}$$

Now we substitute $h=\frac{1}{K}$ into (4.22) to get $\delta=e^{-2N(N-1)}(\cos\xi)^{N-1}$. Thus, we complete the proof of this lemma. $\square$

4.2. Complete synchronization. In this part, we verify that a priori condition (4.1) can be guaranteed by an additional assumption on system parameters, together with Lemma 4.3 which eventually leads to the proof of Theorem 1.1 on the complete synchronization. In addition to the constraints on K and α in Lemma 3.3 and Lemma 4.2, we further assume K and α satisfy the following constraint

$$D^\infty + [D(\Omega) + 2KN(D^\infty + \alpha)] \frac{2(N-1)}{Ke^{-2N(N-1)}(\cos\xi)^{N-1}} \le \xi - \alpha. \tag{4.23}$$

Note that for a given sufficiently small D^∞, sufficiently large K and small enough α can fulfill the condition (4.23). We now show that the additional condition (4.23) can guarantee a priori condition (4.1) after time t_* given in Lemma 3.3.

LEMMA 4.4. *Suppose assumptions in Lemma 3.3 and Lemma 4.2 are fullfilled, and moreover for a given constant $\xi < \frac{\pi}{2}$, we assume*

$$D^\infty + [D(\Omega) + 2KN(D^\infty + \alpha)] \frac{2(N-1)}{Ke^{-2N(N-1)}(\cos\xi)^{N-1}} \le \xi - \alpha. \tag{4.24}$$

Then the following assertion holds

$$\sup_{t_* \le t < +\infty} D(\theta(t)) + \alpha \le \xi < \frac{\pi}{2}. \tag{4.25}$$

Proof. We prove this by continuity argument. In fact, we define a set

$$\mathcal{A} := \{T > t_* : D(\theta(t)) < \xi - \alpha, \ \forall t \in [t_*, T)\}.$$

From (3.12) and the constraint (4.24), we see that $D(\theta(t_*)) \le D^\infty < \xi - \alpha$. Due to the continuity of $D(\theta(t))$, there exists $T > 0$ such that

$$D(\theta(t)) < \xi - \alpha, \quad \text{for } t \in [t_*, T).$$

This yields that the set $\mathcal{A}$ is nonempty. Then we set $T^* = \sup \mathcal{A}$. We claim that $T^* = +\infty$. Suppose not, i.e., $T^* < +\infty$. It is clear to see that

$$D(\theta(t)) < \xi - \alpha, \ t \in [t_*, T^*) \quad \text{and} \quad D(\theta(T^*)) = \xi - \alpha. \tag{4.26}$$

Next, based on the estimate of phase diameter in (3.12) at t_*, we further estimate the frequency diameter at time t_*. According to system (3.5) and $t_* \in [0, \tau)$, it yields that

$$\begin{aligned} D(\omega(t_*)) &= \omega_M(t_*) - \omega_m(t_*) = \dot\theta_M(t_*) - \dot\theta_m(t_*) \\ &= \Omega_M + K \sum_{j=1}^N \chi_{Mj}^{p_0} \sin(\theta_j(t_*) - \theta_M(t_*) + \alpha) \\ &\quad - \Omega_m - K \sum_{j=1}^N \chi_{mj}^{p_0} \sin(\theta_j(t_*) - \theta_m(t_*) + \alpha) \\ &\le D(\Omega) + 2KN(D^\infty + \alpha), \end{aligned} \tag{4.27}$$

where the indexes M and m denote the extreme frequencies and we exploit the following facts

$$\begin{aligned} &|\sin x| \le |x|, \quad x \in (-\infty, +\infty), \\ &|\theta_j(t_*) - \theta_M(t_*) + \alpha| \le |\theta_j(t_*) - \theta_M(t_*)| + \alpha \le D(\theta(t_*)) + \alpha \le D^\infty + \alpha, \\ &\text{similarly, } |\theta_j(t_*) - \theta_m(t_*) + \alpha| \le D^\infty + \alpha. \end{aligned}$$

Then we apply $(4.3)_1$, (3.12), (4.26), (4.27) and Lemma 4.3 before T^* to obtain

$$\begin{aligned}
&|\theta_i(T^*)-\theta_j(T^*)|\\
&\leq |\theta_i(t_*)-\theta_j(t_*)|+\int_{t_*}^{T^*}|\omega_i(s)-\omega_j(s)|ds \leq D(\theta(t_*))+\int_{t_*}^{T^*}D(\omega(s))ds\\
&\leq D(\theta(t_*))+\int_{t_*}^{T^*}(1-\delta)^{\lfloor\frac{s-t_*}{\hat{h}}\rfloor}D(\omega(t_*))ds\\
&< D(\theta(t_*))+\int_{t_*}^{+\infty}(1-\delta)^{\lfloor\frac{s-t_*}{\hat{h}}\rfloor}D(\omega(t_*))ds = D(\theta(t_*))+D(\omega(t_*))\hat{h}\sum_{n=0}^{+\infty}(1-\delta)^n\\
&\leq D^\infty+[D(\Omega)+2KN(D^\infty+\alpha)]\hat{h}\frac{1}{\delta}\\
&= D^\infty+[D(\Omega)+2KN(D^\infty+\alpha)]\frac{2(N-1)}{Ke^{-2N(N-1)}(\cos\xi)^{N-1}}\leq \xi-\alpha.
\end{aligned}$$

Thus it follows from the above inequality that

$$D(\theta(T^*))=\max_{1\leq i,j\leq N}|\theta_i(T^*)-\theta_j(T^*)|<\xi-\alpha,$$

which yields a contradiction to (4.26). Thus $T^*=\infty$, i.e.,

$$D(\theta(t))<\xi-\alpha, \quad t\in[t_*,+\infty).$$

This implies that

$$\sup_{t_*\leq t<+\infty} D(\theta(t))+\alpha\leq\xi<\frac{\pi}{2}.$$

□

Lemma 4.4 states that under an additional assumption (4.23), the phase diameter plus frustration is uniformly bounded by ξ after t_*. Thus we can exploit Lemma 4.3 to derive the complete synchronization . Now, we are ready to prove Theorem 1.1.

Proof. (**Proof of Theorem 1.1.**) Under the sufficient conditions (1.5), we can apply Lemma 3.3, Lemma 4.3 and Lemma 4.4 to complete the proof of Theorem 1.1. □

5. Summary

In this paper, we study the emergent dynamics of the Kuramoto model with frustration under switching topology. When the initial configuration is confined in half circle, under the structural assumption that the network topology in any mode contains a spanning tree, the method in [42] can be directly applied in a sufficient regime before the first network switching occurs, so that we can find a finite time at which the oscillators concentrate into a small arc less than a quarter circle. Then we apply the method based on matrix-graph theories in [9] for the second-order Kuramoto system and present sufficient conditions leading to the exponentially fast emergence of frequency synchronization. In our framework, the size of frustration is sufficiently small and the coupling strength is sufficiently large. However, our analytical approach requires that the switching interaction topology contains a spanning tree in any mode. It is an interesting issue whether we can relax this structural constraint to the union digraph with a spanning tree in a sequence of time-blocks in [9] for the half circle case. This question will be investigated in the future work.

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