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LOCAL SENSITIVITY ANALYSIS FOR THE VICSEK-TYPE SELF-ORGANIZED HYDRODYNAMIC MODEL*

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Abstract. We consider the Vicsek-type self-organized hydrodynamic (SOH) model with random inputs. In the absence of random inputs, the SOH model was derived as a macroscopic limit from the Vicsek model. Considering the uncertainty, which is modeled by a random parameter and enters into the SOH model through the initial data, we perform the local sensitivity analysis to study the regularity with respect to the random parameter and the stability of solutions to the random SOH model.

Keywords. Self-organized hydrodynamic model; Vicsek model; local sensitivity analysis.

AMS subject classifications. 35A01; 49K40; 35Q35.

1. Introduction

1.1. Backgrounds. Self-organized motion is ubiquitous in nature. It corresponds to the formation of large scale coherent structures that emerge from many interactions between individuals without leaders. Well-known examples are bird flocks, fish schools or insect swarms. Furthermore, self-organization also takes place at the microscopic level, for example in bacterial suspensions and sperm dynamics. There are fascinating examples of self-organized models. Among these models, the Vicsek model [24] has received particular attention due to the simplicity and the universality of its qualitative feature. Vicsek model is a time-discrete particle system, each agent moves with the same constant speed and tries to align its velocity orientation with the average velocity orientation of its neighborhood in some sensing region, up to some noise.

Particle simulations tend to be very costly for large number of individuals, and macroscopic simulations are more efficient. Usually, it is hard to derive the macroscopic model directly from the particle model, the kinetic model serves as a bridge. Take Vicsek model as an example. Degond and Motsch [7] provided a time-continuous version of Vicsek model, then through mean field limit they proposed the so-called self-organized kinetic (SOK) model, and finally they derived the macroscopic limit–the self-organized hydrodynamic (SOH) model. For different modelling choices, many modifications of dynamics in [7] have been proposed in the literature, see, for example, [3–6, 9, 10]. We refer the reader to [2, 3, 17, 18] for rigorous derivations of SOK models and SOH models.

For kinetic systems, there are many sources of uncertainty, such as measure errors of initial data or boundary data, and incomplete knowledge of interaction mechanism between particles [20]. Uncertainty in these kinetic equations can be kept and enters into the fluid limit, leading to errors in initial data, boundary data and coefficients of the corresponding hydrodynamic system. The study of uncertainty for kinetic models and hydrodynamic limit models has been an active field. How the uncertainty in the model input affects the model output is related to the content of sensitivity analysis [19]. Local sensitivity analysis focuses on the derivatives of the response with respect to the

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input [23]. We refer the reader to [8, 11–16, 20–22, 26] and the references within for local sensitivity analysis results for Boltzmann equation, Vlasov-Poisson-Fokker-Planck model, Cucker-Smale model and Kuramoto model, but an exhaustive bibliography is out of reach.

1.2. The SOH model. In this paper we consider the SOH model proposed in [6]. Besides the alignment rule considered in [7], Degond-Liu-Motsch-Panferov [6] also considered an additional attraction-repulsion rule. Therefore, the kinetic model set up in [6] contains both an alignment potential and an attraction-repulsion potential. These alignment and attraction-repulsion relations can be kept and result in pressure, viscosity terms and capillary force in the fluid limit. The authors in [6] investigated four different scaling relations and obtained correspondingly four hydrodynamic limit systems. The first three hydrodynamic systems in [6] can be unified in the following form

$$\begin{cases} \partial_t \rho + c_1 \nabla_x \cdot (\rho \Omega) = 0, \\ \rho(\partial_t \Omega + c_2 \Omega \cdot \nabla_x \Omega) + c_3 P_{\Omega^\perp} \nabla_x \rho = \mu P_{\Omega^\perp} \Delta_x (\rho \Omega), \\ |\Omega| = 1, \\ (\rho, \Omega)|_{t=0} = (\rho^{in}(x), \Omega^{in}(x)), \quad |\Omega^{in}| = 1. \end{cases} \tag{1.1}$$

Here coefficients c_1, c_2, c_3 and $\mu \geq 0$ are given constants, defined by integrals related to the alignment kernel and the attraction-repulsion kernel in the kinetic model. Under different scaling relations, they have different values.

SOH system (1.1) governs the dynamics for density $\rho = \rho(t,x)$ and mean velocity orientation $\Omega(t,x)$. The operator $P_{\Omega^\perp} = Id - \Omega \otimes \Omega$ denotes the orthogonal projection onto the plane orthogonal to Ω. (1.1) obeys a geometric constraint $|\Omega| = 1$. Note that (1.1) is a non-conservative system, the moment and energy laws are invalid. As a result, it is rather hard to verify the global well-posedness. The authors in [6] adopted the polar coordinates $\Omega = (\cos\phi, \sin\phi)$ in 2D to write the equations for Ω into coordinate, then they proved the local-in-time existence result with $\mu \geq 0$ by energy method. In 3D, using the spherical coordinates $\Omega = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, (1.1) can be rewritten as

$$\begin{cases} \partial_t \varrho + c_1 \Omega \cdot \nabla_x \varrho + c_1(\Omega_\theta \cdot \nabla_x \theta + \Omega_\varphi \cdot \nabla_x \varphi) = 0, \\ \partial_t \theta + c_2 \Omega \cdot \nabla_x \theta + c_3 \Omega_\theta \cdot \nabla_x \varrho + \mu(-\Delta_x \theta + \cos\theta\sin\theta |\nabla_x \varphi|^2 - 2\nabla_x \varrho \nabla_x \theta) = 0, \\ \sin^2\theta \partial_t \varphi + c_2 \sin^2\theta \Omega \cdot \nabla_x \varphi + D_\varphi(\varrho, \theta, \varphi) = 0, \end{cases}$$

where $\varrho = \ln\rho$ and

$$D_\varphi(\varrho,\theta,\varphi) = c_3 \Omega_\varphi \cdot \nabla_x \varrho + \mu(-\sin^2\theta \Delta_x \varphi - \cos\theta\sin\theta \nabla_x \theta \cdot \nabla_x \varphi - 2\sin^2\theta \nabla_x \varrho \nabla_x \varphi).$$

For the inviscid problem, namely, $\mu = 0$, by using the symmetrizable hyperbolic system theory, the authors in [6] established the local existence result under the initial assumption $\sin\theta_0 > 0$. For the viscous case, namely, $\mu > 0$, in order to bypass the singularity $\frac{1}{\sin^2\theta}$, Zhang-Jiang [25] made an additional non-degenerate a priori assumption $\sin\theta > 0$, then they established the local existence result by using suitable symmetrical structure. Very recently, Jiang-Luo-Zhang [17] adopted the stereographic projection transform

$$\Omega = \left(\frac{2\phi}{W}, \frac{2\psi}{W}, \frac{\phi^2+\psi^2-1}{W}\right) \quad with \quad W = \phi^2 + \psi^2 + 1,$$

to deal with $|\Omega|=1$. More precisely, following the transform $\varrho=\log\rho$ and stereographic projection transform, (1.1) can be rewritten as:

$$\begin{cases} \partial_t\varrho+c_1\Omega\cdot\nabla_x\varrho+c_1\nabla_x\cdot\Omega=0,\\ \partial_t\phi-\mu\Delta_x\phi+c_2\Omega\cdot\nabla_x\phi+H^\phi(\varrho,\phi,\psi)=0,\\ \partial_t\psi-\mu\Delta_x\psi+c_2\Omega\cdot\nabla_x\psi+H^\psi(\varrho,\phi,\psi)=0,\\ (\varrho,\phi,\psi)(0,x)=(\varrho^{in}(x),\phi^{in}(x),\psi^{in}(x)), \end{cases} \tag{1.2}$$

where H^ϕ and H^ψ stand for

$$\begin{aligned} H^\phi(\varrho,\phi,\psi)=&-2\mu\nabla_x\varrho\cdot\nabla_x\phi+\tfrac{c_3}{4}\nabla_x\varrho\cdot W^2\Omega_\phi\\ &+\tfrac{2\mu\phi}{W}|\nabla_x\phi|^2+\tfrac{4\mu\psi}{W}\nabla_x\phi\cdot\nabla_x\psi-\tfrac{2\mu\phi}{W}|\nabla_x\psi|^2,\\ H^\psi(\varrho,\phi,\psi)=&-2\mu\nabla_x\varrho\cdot\nabla_x\psi+\tfrac{c_3}{4}\nabla_x\varrho\cdot W^2\Omega_\psi\\ &-\tfrac{2\mu\psi}{W}|\nabla_x\phi|^2+\tfrac{4\mu\phi}{W}\nabla_x\phi\cdot\nabla_x\psi+\tfrac{2\mu\psi}{W}|\nabla_x\psi|^2, \end{aligned}$$

and the vector Ω_ϕ and Ω_ψ are the partial derivatives of Ω with respect to the variable ϕ and ψ:

$$\Omega_\phi=\big(\tfrac{2(1-\phi^2+\psi^2)}{W^2},-\tfrac{4\phi\psi}{W^2},\tfrac{4\phi}{W^2}\big)\quad \textit{and}\quad \Omega_\psi=\big(-\tfrac{4\phi\psi}{W^2},\tfrac{2(1+\phi^2-\psi^2)}{W^2},\tfrac{4\psi}{W^2}\big).$$

This is a non-singularity system. By energy method, the authors in [17] established a local-in-time existence result for (1.2) with $\mu>0$ in Sobolev spaces $H^s(\mathbb{R}^3)$ $(s\geq 3)$ without additional assumptions.

By using embedding inequalities and interpolation inequalities in noninteger Sobolev spaces, we can reduce the index $s\geq 3$ required in [17] to $s\geq 2$. Our main local well-posedness result for SOH system (1.2) is stated as follows:

THEOREM 1.1. *Let $\mu>0$ and $s\geq 2$. If the initial data $(\varrho^{in}(x),\phi^{in}(x),\psi^{in}(x))\in(H^s(\mathbb{R}^3))^3$, then there exists a finite time $T>0$, depending only on s and the initial data, such that system (1.2) admits a unique solution (ϱ,ϕ,ψ) satisfying*

$$\varrho(t,x)\in C([0,T];H^s),\quad \phi(t,x),\psi(t,x)\in C([0,T];H^s)\cap L^2(0,T;\dot{H}^{s+1}).$$

Moreover, the energy bound

$$\sup_{t\in[0,T]}\big(\|\varrho\|^2_{H^s}+\|\phi\|^2_{H^s}+\|\psi\|^2_{H^s}\big)+\int_0^T(\|\nabla_x\phi\|^2_{H^s}+\|\nabla_x\psi\|^2_{H^s})dt\leq C$$

holds for some positive constant C, which depends only on s and the initial data.

1.3. The SOH model with random inputs. As mentioned before, there are many sources of uncertainty in SOH model (1.2), leading to errors in the initial data and coefficients c_1,c_2,c_3 and μ. To incorporate uncertainty, we consider a random vector z. We are interested in the effect of randomness. For simplicity, we will assume that z is a one-dimensional variable, and lies in I, which is an open interval of $\mathbb{R}$. In fact, allowing z varying in higher dimensions will not effect our results essentially, but it will make the computations more tedious. For the same reason, we only consider uncertainty coming from the initial data, and assume c_1,c_2,c_3 and μ are still given constants. In this setting, the dynamics of $(\rho(z,t,x),\Omega(z,t,x))$ are governed by the random SOH model

$$\begin{cases} \partial_t\varrho+c_1\Omega\cdot\nabla_x\varrho+c_1\nabla_x\cdot\Omega=0,\\ \partial_t\phi-\mu\Delta_x\phi+c_2\Omega\cdot\nabla_x\phi+H^\phi(\varrho,\phi,\psi)=0,\\ \partial_t\psi-\mu\Delta_x\psi+c_2\Omega\cdot\nabla_x\psi+H^\psi(\varrho,\phi,\psi)=0,\\ (\varrho,\phi,\psi)(z,0,x)=(\varrho^{in}(z,x),\phi^{in}(z,x),\psi^{in}(z,x)). \end{cases} \tag{1.3}$$

We perform the local sensitivity analysis for the random SOH model (1.3), which focuses on the dynamic behaviour of derivatives with respect to the random variable z. Our main local sensitivity results are two-fold. First, taking l-nd derivative with respect to z-variable on system (1.3), we formally get

$$\begin{cases} \partial_t\partial_z^l\varrho+c_1\sum\limits_{m=0}^{l}C_l^m\partial_z^m\Omega\cdot\nabla_x\partial_z^{l-m}\varrho+c_1\nabla_x\cdot\partial_z^l\Omega=0,\\ \partial_t\partial_z^l\phi-\mu\Delta_x\partial_z^l\phi+c_2\sum\limits_{m=0}^{l}C_l^m\partial_z^m\Omega\cdot\nabla_x\partial_z^{l-m}\phi+H_{zl}^\phi=0,\\ \partial_t\partial_z^l\psi-\mu\Delta_x\partial_z^l\psi+c_2\sum\limits_{m=0}^{l}C_l^m\partial_z^m\Omega\cdot\nabla_x\partial_z^{l-m}\psi+H_{zl}^\psi=0, \end{cases} \tag{1.4}$$

where

$$\begin{aligned} H_{zl}^\phi=&-2\mu\sum_{m=0}^{l}C_l^m\nabla_x\partial_z^m\varrho\cdot\nabla_x\partial_z^{l-m}\phi+\tfrac{c_3}{4}\sum_{m=0}^{l}C_l^m\nabla_x\partial_z^m\varrho\cdot\partial_z^{l-m}(W_2\Omega_\phi)\\ &+2\mu\sum_{l_1+l_2+l_3=l}\tfrac{l!}{l_1!l_2!l_3!}\partial_z^{l_1}\tfrac{\phi}{W}\nabla_x\partial_z^{l_2}\phi\cdot\nabla_x\partial_z^{l_3}\phi+4\mu\sum_{l_1+l_2+l_3=l}\tfrac{l!}{l_1!l_2!l_3!}\partial_z^{l_1}\tfrac{\psi}{W}\nabla_x\partial_z^{l_2}\phi\cdot\partial_z^{l_3}\psi\\ &-2\mu\sum_{l_1+l_2+l_3=l}\tfrac{l!}{l_1!l_2!l_3!}\partial_z^{l_1}\tfrac{\phi}{W}\nabla_x\partial_z^{l_2}\psi\cdot\nabla_x\partial_z^{l_3}\psi,\\ H_{zl}^\psi=&-2\mu\sum_{m=0}^{l}C_l^m\nabla_x\partial_z^m\varrho\cdot\nabla_x\partial_z^{l-m}\psi+\tfrac{c_3}{4}\sum_{m=0}^{l}C_l^m\nabla_x\partial_z^m\varrho\cdot\partial_z^{l-m}(W_2\Omega_\psi)\\ &-2\mu\sum_{l_1+l_2+l_3=l}\tfrac{l!}{l_1!l_2!l_3!}\partial_z^{l_1}\tfrac{\psi}{W}\nabla_x\partial_z^{l_2}\phi\cdot\nabla_x\partial_z^{l_3}\phi+4\mu\sum_{l_1+l_2+l_3=l}\tfrac{l!}{l_1!l_2!l_3!}\partial_z^{l_1}\tfrac{\phi}{W}\nabla_x\partial_z^{l_2}\phi\cdot\nabla_x\partial_z^{l_3}\psi\\ &+2\mu\sum_{l_1+l_2+l_3=l}\tfrac{l!}{l_1!l_2!l_3!}\partial_z^{l_1}\tfrac{\psi}{W}\nabla_x\partial_z^{l_2}\psi\cdot\nabla_x\partial_z^{l_3}\psi. \end{aligned}$$

If the initial data $(\varrho^{in}(z,x),\phi^{in}(z,x),\psi^{in}(z,x))$ are q-order regular with respect to z, namely $(\partial_z^l\varrho^{in}(x),\partial_z^l\phi^{in}(x),\partial_z^l\psi^{in}(x))|_{l=1}^q$ exist, then we show $(\partial_z^l\varrho(z,t,x)$, $\partial_z^l\phi(z,t,x)$, $\partial_z^l\psi(z,t,x))|_{l=1}^q$ also exist and solve (1.4) rigorously. Our main result is stated as follows.

THEOREM 1.2. *Let $\mu>0$ and q,s be positive integers with $s\geq 2+q$. If for each $z\in I$, the initial data $(\varrho^{in}(z,x),\phi^{in}(z,x),\psi^{in}(z,x))$ satisfy*

$$\begin{aligned} &(\varrho^{in}(z,x),\phi^{in}(z,x),\psi^{in}(z,x))\in(H^s(\mathbb{R}^3))^3,\\ &(\partial_z^l\varrho^{in}(z,x),\partial_z^l\phi^{in}(z,x),\partial_z^l\psi^{in}(z,x))\in(H^{s-l}(\mathbb{R}^3))^3, l=1,2,\dots q, \end{aligned}$$

and

$$\sup_{z\in I}(\|\varrho^{in}(z)\|_{H^s}^2+\|\phi^{in}(z)\|_{H^s}^2+\|\psi^{in}(z)\|_{H^s}^2)\triangleq\epsilon_0^{in}<\infty, \tag{1.5}$$

$$\sup_{z\in I}\sum_{1\leq l\leq q}(\|\partial_z^l\varrho^{in}(z)\|_{H^{s-l}}^2+\|\partial_z^l\phi^{in}(z)\|_{H^{s-l}}^2+\|\partial_z^l\psi^{in}(z)\|_{H^{s-l}})\triangleq\epsilon_1^{in}<\infty, \tag{1.6}$$

then for each $z\in I$, there is a finite time $T>0$, depending only on s, ϵ_0^{in} and ϵ_1^{in}, independent of z, such that (1.3) admits a unique solution

$$(\varrho(z,t,x)\in C([0,T];H^s),\quad \phi(z,t,x),\psi(z,t,x)\in C([0,T];H^s)\cap L^2(0,T;\dot{H}^{s+1}).$$

Furthermore, the solution is regular in z-variable, i.e., for each $z\in I$, for each $1\le l\le q$, $(\partial_z^l\varrho(z,t,x),\partial_z^l\phi(z,t,x),\ \partial_z^l\psi(z,t,x))$ exist and solve (1.4) satisfying

$$\partial_z^l\varrho(z,t,x)\in C([0,T];H^{s-l}),\quad \partial_z^l\phi(z,t,x),\partial_z^l\psi(z,t,x)\in C([0,T];H^{s-l})\cap L^2(0,T;\dot{H}^{s-l+1}).$$

Moreover, the energy bound

$$\sup_{z\in I}\sum_{0\le l\le q}\Big\{\sup_{t\in[0,T]}(\|\partial_z^l\varrho\|_{H^{s-l}}^2+\|\partial_z^l\phi\|_{H^{s-l}}^2+\|\partial_z^l\psi\|_{H^{s-l}}^2)$$
$$+\int_0^T(\|\nabla_x\partial_z^l\phi\|_{H^{s-l}}^2+\|\nabla_x\partial_z^l\psi\|_{H^{s-l}}^2)\mathrm{d}t\Big\}\le C$$

holds for some positive constant C, which depends only on s, ϵ_0^{in}, and ϵ_1^{in}.

We mention that the uniform bound Assumptions (1.5) and (1.6) ensure the existence time interval $T>0$ is independent of the random variable $z\in I$, and guarantee the existence of $(\partial_z^l\varrho(z,t,x),\partial_z^l\phi(z,t,x),\ \partial_z^l\psi(z,t,x))$. This theorem shows that if the initial data has $W^{q,\infty}$ regularity with respect to the parameter z, then the regularity can be preserved in time.

Second, we provide the stability estimates for the z-variations to system (1.3). For each z, define

$$E_T(\varrho(z),\phi(z),\psi(z))=\sum_{0\le l\le q}\Big\{\sup_{t\in[0,T]}(\|\partial_z^l\varrho\|_{H^{s-l}}^2+\|\partial_z^l\phi\|_{H^{s-l}}^2+\|\partial_z^l\psi\|_{H^{s-l}}^2)\Big\}$$
$$+\int_0^T(\|\nabla_x\phi\|_{H^s}^2+\|\nabla_x\psi\|_{H^s}^2)\mathrm{d}t.$$

We state our main result.

THEOREM 1.3. *Under the same assumptions of Theorem 1.2, let (ϱ,ϕ,ψ) and $(\bar\varrho,\bar\phi,\bar\psi)$ be two solutions on $[0,T]$ with initial data $(\varrho^{in},\phi^{in},\psi^{in})$ and $(\bar\varrho^{in},\bar\phi^{in},\bar\psi^{in})$, respectively. Then there exists a positive constant C, depending on time interval T and energy $E_T(\varrho(z),\phi(z),\psi(z))$ and $E_T(\bar\varrho(z),\bar\phi(z),\bar\psi(z))$ such that*

$$\sum_{0\le l\le q}\Big\{\sup_{0\le t\le T}\|\partial_z^l(\varrho-\bar\varrho)(z)\|_{H^{s-l-1}}^2+\|\partial_z^l(\phi-\bar\phi)(z)\|_{H^{s-l-1}}^2+\|\partial_z^l(\psi-\bar\psi)(z)\|_{H^{s-l-1}}^2)$$
$$+\int_0^T(\|\nabla_x\partial_z^l(\phi-\bar\phi)(z)\|_{H^{s-l-1}}^2+\|\nabla_x\partial_z^l(\psi-\bar\psi)(z)\|_{H^{s-l-1}}^2)\mathrm{d}\tau\Big\}$$
$$\le C(\|\partial_z^l(\varrho^{in}-\bar\varrho^{in})(z)\|_{H^{s-l-1}}^2+\|\partial_z^l(\phi^{in}-\bar\phi^{in})(z)\|_{H^{s-l-1}}^2$$
$$+\|\partial_z^l(\psi^{in}-\bar\psi^{in})(z)\|_{H^{s-l-1}}^2).$$

1.4. Organization of this paper. The paper is organized as follows. In Section 2, we establish the a priori estimates and stability estimates for (1.2), then we prove Theorem 1.1. In Section 3, we establish the a priori estimates and stability estimates for (1.4), then we prove Theorem 1.2 and Theorem 1.3. In Appendix, we list some inequalities that are frequently used in this paper.

1.5. Notations. We introduce some notations which will be frequently used in the following text. Denote by $\|\cdot\|_{L^p}$ the norm of the Lebesgue space $L^p(\mathbb{R}^3)$ and $\|\cdot\|_{H^s}$ the norm of Sobolev spaces $H^s(\mathbb{R}^3)$. Symbols $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{H^s}$ present the usual L^2-inner product and H^s-inner product, respectively. Let $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ be a multi-index with its length $|\alpha|=\sum_{i=1}^{3}\alpha_i$, and $\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ be a multi-derivative operator. We also use the notation ∇_x^k to denote ∂_x^α for all $|\alpha|=k$. The notation $A\lesssim B$ indicates that there exists a constant $C>0$ such that $A\le CB$.

2. Local existence result for the deterministic system (1.2)

In this section, we establish the local existence result for system (1.2). As mentioned in the Introduction, we aim to reduce the index to $s\ge 2$. Hence, more elaborate estimates are required. In order to make this article more concise, we list some lengthy but straightforward calculations in the Appendix.

2.1. A priori estimates for (1.2). Define the energy functionals

$$\mathcal{E}_0(t)=\|\varrho(t)\|_{H^s}^2+\|\phi(t)\|_{H^s}^2+\|\psi(t)\|_{H^s}^2,\quad \mathcal{D}_0(t)=\mu(\|\nabla_x\phi(t)\|_{H^s}^2+\|\nabla_x\psi(t)\|_{H^s}^2).$$

We state the following a priori estimate.

LEMMA 2.1. *Let $s\ge 2$. Assume that (ϱ,ϕ,ψ) is a sufficiently smooth solution to system (1.2) on time interval $[0,T]$. Then there is a constant $C>0$, depending only upon s and the coefficients of system (1.2), such that for all $t\in[0,T]$,*

$$\tfrac{d}{dt}\mathcal{E}_0(t)+\mathcal{D}_0(t)\le C(1+\mathcal{E}_0^{s+3}(t)).$$

Proof. The proof is based on the classical energy method. For any integer k, $0\le k\le s$ with $s\ge 2$, applying the derivative operator ∇_x^k on system (1.2), and taking the L^2 product with $(\nabla^k\varrho,\nabla^k\phi,\nabla^k\psi)$, we obtain

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}(\|\nabla_x^k\varrho\|_{L^2}^2+\|\nabla_x^k\phi\|_{L^2}^2+\|\nabla_x^k\psi\|_{L^2}^2)+\mu(\|\nabla_x^{k+1}\phi\|_{L^2}^2+\|\nabla_x^{k+1}\psi\|_{L^2}^2)\\&=-c_1\langle\nabla_x^k(\Omega\cdot\nabla_x\varrho),\nabla_x^k\varrho\rangle-c_2\langle\nabla_x^k(\Omega\cdot\nabla_x\phi),\nabla_x^k\phi\rangle-c_2\langle\nabla_x^k(\Omega\cdot\nabla_x\psi),\nabla_x^k\psi\rangle\\&\quad-c_1\langle\nabla_x^k\nabla_x\cdot\Omega,\nabla_x^k\varrho\rangle-\langle\nabla_x^kH^\phi,\nabla_x^k\phi\rangle-\langle\nabla_x^kH^\psi,\nabla_x^k\psi\rangle.\end{aligned}\tag{2.1}$$

We control the right-hand side terms one by one.

For the convection term, according to Lemma A.1, we have

$$\langle\nabla_x^k(\Omega\cdot\nabla_x\varrho),\nabla_x^k\varrho\rangle\lesssim(\|\nabla_x\Omega\|_{H^2}+\|\nabla_x\Omega\|_{H^k})\|\varrho\|_{H^k}^2\lesssim\|\nabla_x\Omega\|_{H^s}\|\varrho\|_{H^s}^2.$$

Similarly, we have

$$\langle\nabla_x^k(\Omega\cdot\nabla_x\phi),\nabla_x^k\phi\rangle+\langle\nabla_x^k(\Omega\cdot\nabla_x\psi),\nabla_x^k\psi\rangle\lesssim\|\nabla_x\Omega\|_{H^s}(\|\phi\|_{H^s}^2+\|\psi\|_{H^s}^2).$$

It is straightforward to get

$$\langle\nabla_x^k\nabla_x\cdot\Omega,\nabla_x^k\varrho\rangle\lesssim\|\nabla_x\Omega\|_{H^s}\|\varrho\|_{H^s}.$$

We now treat $\langle\nabla_x^kH^\phi,\nabla_x^k\phi\rangle$ and $\langle\nabla_x^kH^\psi,\nabla_x^k\psi\rangle$. We only consider three terms: $\langle\nabla_x^k(\nabla_x\varrho\cdot\nabla_x\phi),\nabla_x^k\phi\rangle$, $\nabla_x^k(\nabla_x\varrho\cdot(W^2\Omega_\phi),\nabla_x^k\phi\rangle$, $\langle\nabla_x^k(\frac{\psi}{W}\nabla_x\phi\cdot\nabla_x\psi),\nabla_x^k\phi\rangle$, since the other terms can be treated in a similar way. In the case $k=0$, using $H^2(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$, direct calculations lead to

$$\langle\nabla_x\varrho\cdot\nabla_x\phi,\phi\rangle\lesssim\|\nabla_x\varrho\|_{L^2}\|\nabla_x\phi\|_{L^2}\|\phi\|_{L^\infty}\lesssim\|\varrho\|_{H^s}\|\phi\|_{H^s}^2,$$

$$\langle \nabla_x \varrho \cdot (W^2 \Omega_\phi), \phi \rangle \lesssim \|\nabla_x \varrho\|_{L^2} \|W^2 \Omega_\phi\|_{L^\infty} \|\phi\|_{L^2} \lesssim \|W^2 \Omega_\phi\|_{L^\infty} \|\varrho\|_{H^s} \|\phi\|_{H^s},$$

and

$$\langle \tfrac{\psi}{W} \nabla_x \phi \cdot \nabla_x \psi, \phi \rangle \lesssim \|\tfrac{\psi}{W}\|_{L^\infty} \|\nabla_x \phi\|_{L^2} \|\nabla_x \psi\|_{L^2} \|\phi\|_{L^\infty} \lesssim \|\tfrac{\psi}{W}\|_{L^\infty} (\|\phi\|_{H^s} + \|\psi\|_{H^s})^3.$$

Estimates for cases $1 \le k \le s$ are much more complex. Note that ϱ satisfies a transport equation without diffusion. We use integration by parts to control $\nabla_x^k \nabla \varrho$. More precisely, applying Lemma A.2-(1), we have

$$\begin{aligned}\langle \nabla_x^k (\nabla_x \varrho \cdot (W^2 \Omega_\phi)), \nabla_x^k \phi \rangle &\lesssim \|\varrho\|_{H^k} (\|W^2 \Omega_\phi\|_{L^\infty} + \|\nabla_x (W^2 \Omega_\phi)\|_{H^{k-1}}) \|\nabla_x \phi\|_{H^k} \\ &\lesssim \|\varrho\|_{H^s} (\|W^2 \Omega_\phi\|_{L^\infty} + \|\nabla_x (W^2 \Omega_\phi)\|_{H^{s-1}}) \|\nabla_x \phi\|_{H^s},\end{aligned}$$

and applying Lemma A.2-(2) leads to

$$\langle \nabla_x^k (\nabla_x \varrho \cdot \nabla_x \phi), \nabla_x^k \phi \rangle \lesssim \|\varrho\|_{H^k} (\|\nabla_x \phi\|_{L^\infty} \| + \|\nabla_x^2 \phi\|_{H^{k-\frac{3}{2}}}) \|\nabla_x \phi\|_{H^k}.$$

Since we aim to show that $s \ge 2$ is enough to close the energy, instead of using $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, we use embedding inequalities in noninteger Sobolev space: $H^\mu(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ with $\mu > \frac{3}{2}$. We take $\mu = \frac{7}{4}$ here. Hence, we have

$$\begin{aligned}\langle \nabla_x^k (\nabla_x \varrho \cdot \nabla_x \phi), \nabla_x^k \phi \rangle &\lesssim \|\varrho\|_{H^k} (\|\nabla_x \phi\|_{H^{\frac{7}{4}}} + \|\nabla_x \phi\|_{H^{k-\frac{1}{2}}}) \|\nabla_x \phi\|_{H^k} \\ &\lesssim \|\varrho\|_{H^s} \|\nabla_x \phi\|_{H^{s-\frac{1}{4}}} \|\nabla_x \phi\|_{H^s}.\end{aligned}$$

Finally, applying Lemma A.3-(1) together with $H^{\frac{7}{4}}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, we have

$$\begin{aligned}\langle \nabla_x^k (\tfrac{\psi}{W} \nabla_x \phi \cdot \nabla_x \psi), \nabla_x^k \phi \rangle &\lesssim \big(\|\tfrac{\psi}{W}\|_{L^\infty} \|\phi\|_{H^k} \|\nabla_x \psi\|_{L^\infty} + \|\tfrac{\psi}{W}\|_{L^\infty} \|\nabla_x \phi\|_{L^\infty} \|\psi\|_{H^k} \\ &\quad + (\|\tfrac{\psi}{W}\|_{L^\infty} + \|\nabla_x \tfrac{\psi}{W}\|_{H^{k-1}}) \|\phi\|_{H^k} \|\psi\|_{H^k} \big) \|\nabla_x \phi\|_{H^k} \\ &\lesssim \big(\|\tfrac{\psi}{W}\|_{L^\infty} \|\phi\|_{H^k} \|\nabla_x \psi\|_{H^{\frac{7}{4}}} + \|\tfrac{\psi}{W}\|_{L^\infty} \|\psi\|_{H^k} \|\nabla_x \phi\|_{H^{\frac{7}{4}}} \\ &\quad + (\|\tfrac{\psi}{W}\|_{L^\infty} + \|\nabla_x \tfrac{\psi}{W}\|_{H^{k-1}}) \|\phi\|_{H^k} \|\psi\|_{H^k} \big) \|\nabla_x \phi\|_{H^k} \\ &\lesssim \big(\|\tfrac{\psi}{W}\|_{L^\infty} \|\phi\|_{H^s} \|\nabla_x \psi\|_{H^{s-\frac{1}{4}}} + \|\tfrac{\psi}{W}\|_{L^\infty} \|\psi\|_{H^s} \|\nabla_x \phi\|_{H^{s-\frac{1}{4}}} \\ &\quad + (\|\tfrac{\psi}{W}\|_{L^\infty} + \|\nabla_x \tfrac{\psi}{W}\|_{H^{s-1}}) \|\phi\|_{H^s} \|\psi\|_{H^s} \big) \|\nabla_x \phi\|_{H^s}.\end{aligned}$$

Plugging all the above inequalities into (2.1), and summing over k from 0 to s, we have

$$\begin{aligned}&\tfrac{1}{2} \tfrac{d}{dt} (\|\varrho\|_{H^s}^2 + \|\phi\|_{H^s}^2 + \|\psi\|_{H^s}^2) + \mu (\|\nabla_x \phi\|_{H^s}^2 + \|\nabla_x \psi\|_{H^s}^2) \\ &\lesssim \|\nabla_x \Omega\|_{H^s} (\|\varrho\|_{H^s}^2 + \|\phi\|_{H^s}^2 + \|\psi\|_{H^s}^2 + \|\varrho\|_{H^s}) + \|\varrho\|_{H^s} (\|\phi\|_{H^s}^2 + \|\psi\|_{H^s}^2) \\ &\quad + \|\varrho\|_{H^s} (\|W^2 \Omega_\phi\|_{L^\infty} \|\phi\|_{H^s} + \|W^2 \Omega_\psi\|_{L^\infty} \|\psi\|_{H^s}) + (\|\tfrac{\phi}{W}\|_{L^\infty} + \|\tfrac{\psi}{W}\|_{L^\infty}) (\|\phi\|_{H^s} + \|\psi\|_{H^s})^3 \\ &\quad + \|\varrho\|_{H^s} (\|\nabla_x \phi\|_{H^{s-\frac{1}{4}}} \|\nabla_x \phi\|_{H^s} + \|\nabla_x \psi\|_{H^{s-\frac{1}{4}}} \|\nabla_x \psi\|_{H^s}) \\ &\quad + \|\varrho\|_{H^s} (\|W^2 \Omega_\phi\|_{L^\infty} + \|\nabla_x (W^2 \Omega_\phi)\|_{H^{s-1}}) \|\nabla_x \phi\|_{H^s} \\ &\quad + \|\varrho\|_{H^s} (\|W^2 \Omega_\psi\|_{L^\infty} + \|\nabla_x (W^2 \Omega_\psi)\|_{H^{s-1}}) \|\nabla_x \psi\|_{H^s} \\ &\quad + (\|\tfrac{\phi}{W}\|_{L^\infty} + \|\tfrac{\psi}{W}\|_{L^\infty}) (\|\phi\|_{H^s} + \|\psi\|_{H^s}) (\|\nabla_x \phi\|_{H^{s-\frac{1}{4}}} + \|\nabla_x \psi\|_{H^{s-\frac{1}{4}}}) \\ &\quad \times (\|\nabla_x \phi\|_{H^s} + \|\nabla_x \psi\|_{H^s}) \\ &\quad + (\|\tfrac{\phi}{W}\|_{L^\infty} + \|\tfrac{\psi}{W}\|_{L^\infty} + \|\nabla_x \tfrac{\phi}{W}\|_{H^{s-1}} + \|\nabla_x \tfrac{\psi}{W}\|_{H^{s-1}}) \\ &\quad \times (\|\phi\|_{H^s} + \|\psi\|_{H^s})^2 (\|\nabla_x \phi\|_{H^s} + \|\nabla_x \psi\|_{H^s}).\end{aligned} \tag{2.2}$$

Applying Lemma A.6 to (2.2) to control the terms related with Ω, and using the Sobolev interpolation formulas

$$\|\nabla_x\phi\|_{H^{s-\frac14}} \lesssim \|\nabla_x\phi\|_{H^{s-1}}^{\frac14}\|\nabla_x\phi\|_{H^s}^{\frac34} \lesssim \|\phi\|_{H^s}^{\frac14}\|\nabla_x\phi\|_{H^s}^{\frac34},$$
$$\|\nabla_x\psi\|_{H^{s-\frac14}} \lesssim \|\nabla_x\psi\|_{H^{s-1}}^{\frac14}\|\nabla_x\psi\|_{H^s}^{\frac34} \lesssim \|\psi\|_{H^s}^{\frac14}\|\nabla_x\psi\|_{H^s}^{\frac34},$$

together with the definition of $\mathcal{E}_0$ and $\mathcal{D}_0$, (2.2) reads

$$\frac12\frac{d}{dt}\mathcal{E}_0+\mathcal{D}_0 \lesssim \mathcal{D}_0^{\frac12}(1+\mathcal{E}_0)^{\frac{s}{2}}(\mathcal{E}_0+\mathcal{E}_0^{\frac12})+(1+\mathcal{E}_0)^{\frac32}\mathcal{E}_0^{\frac12}+\mathcal{E}_0^{\frac58}\mathcal{D}_0^{\frac78}+(1+\mathcal{E}_0)^{\frac{s+2}{2}}\mathcal{D}_0^{\frac12}. \tag{2.3}$$

Finally, applying the Young inequality to control $\mathcal{D}_0^{\frac12}$ and $\mathcal{D}_0^{\frac78}$ by $\mathcal{D}_0$, we have

$$\frac{d}{dt}\mathcal{E}_0+\mathcal{D}_0 \le C(1+\mathcal{E}_0^{s+3}).$$

This completes the proof of the lemma. ∎

2.2. Stability estimates for system (1.2).

LEMMA 2.2. *Let $s\ge 2$. Let (ϱ,ϕ,ψ) and $(\bar\varrho,\bar\phi,\bar\psi)$ are two solutions to system (1.2) in the space*

$$C([0,T];H^s)\times\big(C([0,T];H^s)\cap L^2(0,T;\dot H^{s+1})\big)^2.$$

Let

$$A_0(t)=1+\|\varrho(t)\|_{H^s}^2+\|\phi(t)\|_{H^s}^2+\|\psi(t)\|_{H^s}^2+\|\bar\varrho(t)\|_{H^s}^2+\|\bar\phi(t)\|_{H^s}^2+\|\bar\psi(t)\|_{H^s}^2,$$
$$B_0(t)=\|\nabla_x\phi(t)\|_{H^s}^2+\|\nabla_x\psi(t)\|_{H^s}^2+\|\nabla_x\bar\phi(t)\|_{H^s}^2+\|\nabla_x\bar\psi(t)\|_{H^s}^2,$$

and

$$X_0(t)=\|(\varrho-\bar\varrho)(t)\|_{H^{s-1}}^2+\|(\phi-\bar\phi)(t)\|_{H^{s-1}}^2+\|(\psi-\bar\psi)(t)\|_{H^{s-1}}^2,$$
$$Y_0(t)=\mu(\|\nabla_x(\phi-\bar\phi)(t)\|_{H^{s-1}}^2+\|\nabla_x(\psi-\bar\psi)(t)\|_{H^{s-1}}^2).$$

Then, we have

$$\frac{d}{dt}X_0(t)+Y_0(t)\le CA_0^{2s+4}(t)(1+B_0^{\frac34}(t))X_0(t).$$

Proof. The difference between (ϱ,ϕ,ψ) and $(\bar\varrho,\bar\phi,\bar\psi)$ reads

$$\begin{cases}\partial_t(\varrho-\bar\varrho)+c_1\bar\Omega\cdot\nabla_x(\varrho-\bar\varrho)+c_1(\Omega-\bar\Omega)\cdot\nabla_x\varrho+c_1\nabla_x\cdot(\Omega-\bar\Omega)=0,\\ \partial_t(\phi-\bar\phi)+c_2\bar\Omega\cdot\nabla_x(\phi-\bar\phi)-\mu\Delta_x(\phi-\bar\phi)+c_2(\Omega-\bar\Omega)\cdot\nabla_x\phi=G_2,\\ \partial_t(\psi-\bar\psi)+c_2\bar\Omega\cdot\nabla_x(\psi-\bar\psi)-\mu\Delta_x(\psi-\bar\psi)+c_2(\Omega-\bar\Omega)\cdot\nabla_x\psi=G_3,\end{cases} \tag{2.4}$$

with

$$\bar\Omega=\Omega(\bar\phi,\bar\psi),\quad G_2=-H^\phi(\varrho,\phi,\psi)+H^\phi(\bar\varrho,\bar\phi,\bar\psi),\quad G_3=-H^\psi(\varrho,\phi,\psi)+H^\psi(\bar\varrho,\bar\phi,\bar\psi).$$

For any integer k, $0\le k\le s-1$, applying the derivative operator ∇_x^k on (2.4), and taking the L^2 product with $(\nabla_x^k(\varrho-\bar\varrho),\nabla_x^k(\phi-\bar\phi),\nabla_x^k(\psi-\bar\psi))$, we obtain

$$\frac12\frac{d}{dt}(\nabla_x^k(\varrho-\bar\varrho)\|_{L^2}^2+\|\nabla_x^k(\phi-\bar\phi)\|_{L^2}^2+\|\nabla_x^k(\psi-\bar\psi)\|_{L^2}^2)$$
$$+\mu(\|\nabla_x^{k+1}(\phi-\bar\phi)\|_{L^2}^2+\|\nabla_x^{k+1}(\psi-\bar\psi)\|_{L^2}^2)$$

$$
\begin{aligned}
=&-c_1\langle\nabla_x^k(\bar\Omega\cdot\nabla_x(\varrho-\bar\varrho)),\nabla_x^k(\varrho-\bar\varrho)\rangle-c_2\langle\nabla_x^k(\bar\Omega\cdot\nabla_x(\phi-\bar\phi)),\nabla_x^k(\phi-\bar\phi)\rangle\\
&-c_2\langle\nabla_x^k(\bar\Omega\cdot\nabla_x(\psi-\bar\psi)),\nabla_x^k(\psi-\bar\psi)\rangle-c_1\langle\nabla_x^k((\Omega-\bar\Omega)\cdot\nabla_x\varrho),\nabla_x^k(\varrho-\bar\varrho)\rangle\\
&-c_2\langle\nabla_x^k((\Omega-\bar\Omega)\cdot\nabla_x\phi),\nabla_x^k(\phi-\bar\phi)\rangle-c_2\langle\nabla_x^k((\Omega-\bar\Omega)\cdot\nabla_x\psi),\nabla_x^k(\psi-\bar\psi)\rangle\\
&-c_1\langle\nabla_x^k\nabla_x\cdot(\Omega-\bar\Omega),\nabla_x^k(\varrho-\bar\varrho)\rangle+\langle\nabla_x^kG_2,\nabla_x^k(\phi-\bar\phi)\rangle+\langle\nabla_x^kG_3,\nabla_x^k(\psi-\bar\psi)\rangle. \qquad (2.5)
\end{aligned}
$$

For convection terms, thanks to Lemma A.1 again, we have

$$
\begin{aligned}
\langle\nabla_x^k(\bar\Omega\cdot\nabla_x(\varrho-\bar\varrho)),\nabla_x^k(\varrho-\bar\varrho)\rangle&\lesssim(\|\nabla_x\bar\Omega\|_{H^2}+\|\nabla_x\bar\Omega\|_{H^k})\|\varrho-\bar\varrho\|_{H^k}^2)\\
&\lesssim\|\nabla_x\bar\Omega\|_{H^s}\|\varrho-\bar\varrho\|_{H^{s-1}}^2.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\langle\nabla_x^k(\bar\Omega\cdot\nabla_x(\phi-\bar\phi)),\nabla_x^k(\phi-\bar\phi)\rangle+\langle\nabla_x^k(\bar\Omega\cdot\nabla_x(\psi-\bar\psi)),\nabla_x^k(\psi-\bar\psi)\rangle\\
\lesssim&\|\nabla_x\bar\Omega\|_{H^s}(\|\phi-\bar\phi\|_{H^{s-1}}^2+\|\psi-\bar\psi\|_{H^{s-1}}^2).
\end{aligned}
$$

By virtue of Lemma A.4, we infer that

$$
\begin{aligned}
\langle\nabla_x^k((\Omega-\bar\Omega)\cdot\nabla_x\varrho),\nabla_x^k(\varrho-\bar\varrho)\rangle&\lesssim\|\Omega-\bar\Omega\|_{H^{k+1}}(\|\nabla_x\varrho\|_{H^1}+\|\nabla_x\varrho\|_{H^k})\|\varrho-\bar\varrho\|_{H^k}\\
&\lesssim\|\Omega-\bar\Omega\|_{H^s}\|\varrho\|_{H^s}\|\varrho-\bar\varrho\|_{H^{s-1}}.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
&\langle\nabla_x^k((\Omega-\bar\Omega)\cdot\nabla_x\phi),\nabla_x^k(\phi-\bar\phi)\rangle+\langle\nabla_x^k((\Omega-\bar\Omega)\cdot\nabla_x\psi),\nabla_x^k(\psi-\bar\psi)\rangle\\
\lesssim&\|\Omega-\bar\Omega\|_{H^s}(\|\phi\|_{H^s}+\|\psi\|_{H^s})(\|\phi-\bar\phi\|_{H^{s-1}}+\|\psi-\bar\psi\|_{H^{s-1}}).
\end{aligned}
$$

Direct calculation leads to

$$
\langle\nabla_x^k(\nabla_x\cdot(\Omega-\bar\Omega)),\nabla_x^k(\varrho-\bar\varrho)\rangle\lesssim\|\Omega-\bar\Omega\|_{H^{k+1}}\|\varrho-\bar\varrho\|_{H^k}\lesssim\|\Omega-\bar\Omega\|_{H^s}\|\varrho-\bar\varrho\|_{H^{s-1}}.
$$

Next, we control G_2 and G_3. Since the other terms can be treated in a similar way, we still only consider three terms:

$$
\nabla_x\varrho\cdot\nabla_x\phi-\nabla_x\bar\varrho\cdot\nabla_x\bar\phi=\nabla_x(\varrho-\bar\varrho)\cdot\nabla_x\phi+\nabla_x\bar\varrho\cdot\nabla_x(\phi-\bar\phi)\triangleq R_{11}+R_{12},
$$

$$
\begin{aligned}
\nabla_x\varrho\cdot(W^2\Omega_\phi)-\nabla_x\bar\varrho\cdot(\overline{W}^2\overline{\Omega_\phi})&=\nabla_x(\varrho-\bar\varrho)\cdot(W^2\Omega_\phi)+\nabla_x\bar\varrho\cdot(W^2\Omega_\phi-\overline{W}^2\overline{\Omega_\phi})\\
&\triangleq R_{21}+R_{22},
\end{aligned}
$$

$$
\begin{aligned}
\tfrac{\phi}{W}\nabla_x\phi\cdot\nabla_x\psi-\tfrac{\bar\phi}{\bar W}\nabla_x\bar\phi\cdot\nabla_x\bar\psi=&(\tfrac{\phi}{W}-\tfrac{\bar\phi}{\bar W})\nabla_x\phi\cdot\nabla_x\psi\\
&+\tfrac{\bar\phi}{\bar W}\nabla_x(\phi-\bar\phi)\cdot\nabla_x\psi+\tfrac{\bar\phi}{\bar W}\nabla_x\bar\phi\cdot\nabla_x(\psi-\bar\psi)\\
\triangleq&R_{31}+R_{32}+R_{33}.
\end{aligned}
$$

In the case $k=0$, using $H^1(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3), H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3), H^2(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$, we directly get

$$
\begin{aligned}
&\langle R_{11},\phi-\bar\phi\rangle\lesssim\|\nabla_x(\varrho-\bar\varrho)\|_{L^2}\|\nabla_x\phi\|_{L^\infty}\|\phi-\bar\phi\|_{L^2}\lesssim\|\varrho-\bar\varrho\|_{H^{s-1}}\|\nabla_x\phi\|_{H^s}\|\phi-\bar\phi\|_{H^{s-1}},\\
&\langle R_{12},\phi-\bar\phi\rangle\lesssim\|\nabla_x\bar\varrho\|_{L^6}\|\nabla_x(\phi-\bar\phi)\|_{L^3}\|\phi-\bar\phi\|_{L^2}\lesssim\|\bar\varrho\|_{H^s}\|\nabla_x(\phi-\bar\phi)\|_{H^{s-1}}\|\phi-\bar\phi\|_{H^{s-1}},\\
&\langle R_{21},\phi-\bar\phi\rangle\lesssim\|\nabla_x(\varrho-\bar\varrho)\|_{L^2}\|W^2\Omega_\phi\|_{L^\infty}\|\phi-\bar\phi\|_{L^2}\lesssim\|\varrho-\bar\varrho\|_{H^{s-1}}\|W^2\Omega_\phi\|_{L^\infty}\|\phi-\bar\phi\|_{H^{s-1}},
\end{aligned}
$$

$$
\begin{aligned}
\langle R_{22},\phi-\bar\phi\rangle &\lesssim \|\nabla_x\bar\varrho\|_{L^6}\|W^2\Omega-\overline{W}^2\bar\Omega_\phi\|_{L^3}\|\phi-\bar\phi\|_{L^2}\\
&\lesssim \|\bar\varrho\|_{H^s}\|W^2\Omega_\phi-\overline{W}^2\bar\Omega_\phi\|_{H^{s-1}}\|\phi-\bar\phi\|_{H^{s-1}},\\
\langle R_{31},\phi-\bar\phi\rangle &\lesssim \|\tfrac{\phi}{W}-\tfrac{\bar\phi}{\overline{W}}\|_{L^6}\|\nabla_x\phi\|_{L^6}\|\nabla_x\psi\|_{L^6}\|\phi-\bar\phi\|_{L^2}\\
&\lesssim \|\tfrac{\phi}{W}-\tfrac{\bar\phi}{\overline{W}}\|_{H^{s-1}}\|\phi\|_{H^s}\|\psi\|_{H^s}\|\phi-\bar\phi\|_{H^{s-1}},\\
\langle R_{32},\phi-\bar\phi\rangle &\lesssim \|\tfrac{\bar\phi}{\overline{W}}\|_{L^\infty}\|\nabla_x(\phi-\bar\phi)\|_{L^6}\|\nabla_x\psi\|_{L^3}\|\phi-\bar\phi\|_{L^2}\\
&\lesssim \|\tfrac{\bar\phi}{\overline{W}}\|_{L^\infty}\|\nabla_x(\phi-\bar\phi)\|_{H^{s-1}}\|\psi\|_{H^s}\|\phi-\bar\phi\|_{H^{s-1}},\\
\langle R_{33},\phi-\bar\phi\rangle &\lesssim \|\tfrac{\bar\phi}{\overline{W}}\|_{L^\infty}\|\nabla_x(\psi-\bar\psi)\|_{H^{s-1}}\|\bar\phi\|_{H^s}\|\phi-\bar\phi\|_{H^{s-1}}.
\end{aligned}
$$

In the case $1\le k\le s-1$, we still use integration by parts to control $\nabla_x^k\nabla_x(\varrho-\bar\varrho))$. Hence, by using Lemma A.2-(1) and $H^{\frac74}(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$, we deduce that

$$
\begin{aligned}
\langle\nabla_x^kR_{11},\nabla_x^k(\phi-\bar\phi)\rangle &\lesssim \|\varrho-\bar\varrho\|_{H^k}(\|\nabla_x\phi\|_{L^\infty}+\|\nabla_x^2\phi\|_{H^{k-1}})\|\nabla_x(\phi-\bar\phi)\|_{H^k}\\
&\lesssim \|\varrho-\bar\varrho\|_{H^k}(\|\nabla_x\phi\|_{H^{\frac74}}+\|\nabla_x\phi\|_{H^k})\|\nabla_x(\phi-\bar\phi)\|_{H^k}\\
&\lesssim \|\varrho-\bar\varrho\|_{H^{s-1}}\|\nabla_x\phi\|_{H^{s-\frac14}}\|\nabla_x(\phi-\bar\phi)\|_{H^{s-1}}\\
&\lesssim \|\varrho-\bar\varrho\|_{H^{s-1}}\|\phi\|_{H^s}^{\frac14}\|\nabla_x\phi\|_{H^s}^{\frac34}\|\nabla_x(\phi-\bar\phi)\|_{H^{s-1}},\\
\langle\nabla_x^kR_{21},\nabla_x^k(\phi-\bar\phi)\rangle &\lesssim \|\varrho-\bar\varrho\|_{H^k}(\|W^2\Omega_\phi\|_{L^\infty}+\|\nabla_x(W^2\Omega_\phi)\|_{H^{k-1}})\|\nabla_x(\phi-\bar\phi)\|_{H^k}\\
&\lesssim \|\varrho-\bar\varrho\|_{H^{s-1}}(\|W^2\Omega_\phi\|_{L^\infty}+\|\nabla_x(W^2\Omega_\phi)\|_{H^{s-1}})\|\nabla_x(\phi-\bar\phi)\|_{H^{s-1}}.
\end{aligned}
$$

Applying Lemma A.2-(3), together with the interpolation inequality

$$
\|\nabla_xf\|_{H^{s-\frac32}}\lesssim\|\nabla_xf\|_{H^{s-2}}^{\frac12}\|\nabla_xf\|_{H^{s-1}}^{\frac12}\lesssim\|f\|_{H^{s-1}}^{\frac12}\|\nabla_xf\|_{H^{s-1}}^{\frac12},
$$

we finally infer that

$$
\begin{aligned}
\langle\nabla_x^kR_{12},\nabla_x^k(\phi-\bar\phi)\rangle &\lesssim \|\bar\varrho\|_{H^{k+1}}\|\nabla_x(\phi-\bar\phi)\|_{H^{k-\frac12}}\|\nabla_x(\phi-\bar\phi)\|_{H^k}\\
&\lesssim \|\bar\varrho\|_{H^s}\|\nabla_x(\phi-\bar\phi)\|_{H^{s-\frac32}}\|\nabla_x(\phi-\bar\phi)\|_{H^{s-1}}\\
&\lesssim \|\bar\varrho\|_{H^s}\|\phi-\bar\phi\|_{H^{s-1}}^{\frac12}\|\nabla_x(\phi-\bar\phi)\|_{H^{s-1}}^{\frac32},\\
\langle\nabla_x^kR_{22},\nabla_x^k(\phi-\bar\phi)\rangle &\lesssim \|\bar\varrho\|_{H^{k+1}}\|W^2\Omega_\phi-\overline{W}^2\bar\Omega_\phi\|_{H^{k-\frac12}}\|\nabla_x(\phi-\bar\phi)\|_{H^k}\\
&\lesssim \|\bar\varrho\|_{H^s}\|W^2\Omega_\phi-\overline{W}^2\bar\Omega_\phi\|_{H^{s-1}}\|\nabla_x(\phi-\bar\phi)\|_{H^{s-1}}.
\end{aligned}
$$

By virtue of Lemma A.3-(3), we get

$$
\begin{aligned}
\langle\nabla_x^kR_{31},\nabla_x^k(\phi-\bar\phi)\rangle &\lesssim \|\tfrac{\phi}{W}-\tfrac{\bar\phi}{\overline{W}}\|_{H^k}\|\phi\|_{H^{k+1}}\|\psi\|_{H^{k+1}}\|\nabla_x(\phi-\bar\phi)\|_{H^k}\\
&\lesssim \|\tfrac{\phi}{W}-\tfrac{\bar\phi}{\overline{W}}\|_{H^{s-1}}\|\phi\|_{H^s}\|\psi\|_{H^s}\|\nabla_x(\phi-\bar\phi)\|_{H^{s-1}}.
\end{aligned}
$$

Using Lemma A.3-(2), we have

$$
\begin{aligned}
\langle\nabla_x^kR_{32},\nabla_x^k(\phi-\bar\phi) &\lesssim (\|\tfrac{\bar\phi}{\overline{W}}\|_{L^\infty}+\|\nabla_x\tfrac{\bar\phi}{\overline{W}}\|_{H^{k-1}})\|\nabla_x(\phi-\bar\phi)\|_{H^{k-\frac12}}\|\psi\|_{H^{k+1}}\|\nabla_x(\phi-\bar\phi)\|_{H^k}\\
&\lesssim (\|\tfrac{\bar\phi}{\overline{W}}\|_{L^\infty}+\|\nabla_x\tfrac{\bar\phi}{\overline{W}}\|_{H^{s-1}})\|\nabla_x(\phi-\bar\phi)\|_{H^{s-\frac32}}\|\psi\|_{H^s}\|\nabla_x(\phi-\bar\phi)\|_{H^{s-1}}\\
&\lesssim (\|\tfrac{\bar\phi}{\overline{W}}\|_{L^\infty}+\|\nabla_x\tfrac{\bar\phi}{\overline{W}}\|_{H^{s-1}})\|\phi-\bar\phi\|_{H^{s-1}}^{\frac12}\|\psi\|_{H^s}\|\nabla_x(\phi-\bar\phi)\|_{H^{s-1}}^{\frac32}.
\end{aligned}
$$

Similarly,

$$\langle\nabla_x^k R_{33},\nabla_x^k(\phi-\bar\phi)\rangle\lesssim(\|\tfrac{\bar\phi}{W}\|_{L^\infty}+\|\nabla_x\tfrac{\bar\phi}{W}\|_{H^{s-1}})\|\psi-\bar\psi\|_{H^{s-1}}^{\frac12}\\ \times\|\nabla_x(\psi-\bar\psi)\|_{H^{s-1}}^{\frac12}\|\bar\phi\|_{H^s}\|\nabla_x(\phi-\bar\phi)\|_{H^{s-1}}.$$

Recall the definition of A_0,B_0,X_0 and Y_0. Summing over k from 0 to $s-1$, the left-hand side of (2.5) reads

$$LHS=\tfrac12\tfrac{\mathrm d}{\mathrm dt}X_0+Y_0,$$

and the right-hand side of (2.5) can be controlled by

$$RHS\lesssim B_0^{\frac12}A_0^{\frac{s}{2}}X_0+A_0^{\frac{s+2}{2}}(X_0^{\frac12}+Y_0^{\frac12})X_0^{\frac12}+B_0^{\frac12}X_0\\ +B_0^{\frac38}A_0^{\frac18}X_0^{\frac12}Y_0^{\frac12}+A_0^{\frac{s+2}{2}}(X_0^{\frac12}+X_0^{\frac14}Y_0^{\frac14})Y_0^{\frac12}, \tag{2.6}$$

where we have used Lemma A.6 and Lemma A.7 to control terms related with Ω. Applying the Young inequality, we derive that

$$RHS\le CA_0^{2s+4}(1+B_0^{\frac34})X_0+\tfrac12Y_0.$$

Combining the left-hand side and the right-hand side then completes the proof of this lemma. ∎

2.3. Proof of Theorem 1.1. The structure of the proof is as follows.

(1) We define a family of approximate solutions.

(2) We prove uniform bounds for these approximate solutions on some time interval.

(3) We show this family of approximate solutions converges to some solution to deterministic system (1.2).

(4) We show the solution is unique.

The details are omitted here. On the one hand, based on the a priori estimate in Lemma 2.1 and the stability estimate in Lemma 2.2, the result is expected. On the other hand, the proofs of Theorem 1.1 and Theorem 1.2 have great similarities, and furthermore, the latter one is more complicated.

3. Local sensitivity analysis results for the random system (1.4)

In this section, we consider the random SOH system (1.4). We aim to study the dynamic properties of derivatives $(\partial_z^l\varrho,\partial_z^l\phi,\partial_z^l\psi)$ (l=1,2,...,q). We rewrite (1.4) as follows:

$$\begin{cases}\partial_t\underline{\partial_z^l\varrho}+c_1\Omega\cdot\nabla_x\underline{\partial_z^l\varrho}+c_1\nabla_x\cdot\underline{\partial_z^l\Omega}=J_1,\\ \partial_t\underline{\partial_z^l\phi}-c_3\Delta_x\underline{\partial_z^l\phi}+2\mu\nabla_x\underline{\partial_z^l\varrho}\cdot\nabla_x\phi-\frac{c_3}{4}\nabla_x\underline{\partial_z^l\varrho}\cdot(W_2\Omega_\phi)-4\mu\frac{\psi}{W}\nabla_x\underline{\partial_z^l\phi}\cdot\nabla_x\psi=J_2,\\ \dots\end{cases} \tag{3.1}$$

with

$$J_1=-c_1\partial_z^l\Omega\cdot\nabla_x\varrho-c_1\sum_{m=1}^{l-1}C_l^m\partial_z^m\Omega\cdot\nabla_x\partial_z^{l-m}\varrho=J_{11}+J_{12},$$

$$J_2=\big(-2\mu\nabla_x\varrho\cdot\nabla_x\partial_z^l\phi+\tfrac{c_3}{4}\nabla_x\varrho\cdot\partial_z^l(W_2\Omega_\phi)+4\mu\partial_z^l\tfrac{\psi}{W}\nabla_x\phi\cdot\nabla_x\psi\big)\\ +\big(-2\mu\sum_{m=1}^{l-1}C_l^m\nabla_x\partial_z^m\varrho\cdot\nabla_x\partial_z^{l-m}\phi+\tfrac{c_3}{4}\sum_{m=1}^{l-1}C_l^m\nabla_x\partial_z^m\varrho\cdot\partial_z^{l-m}(W_2\Omega_\phi)$$

$$
\begin{aligned}
&+4\mu \sum_{l_1+l_2+l_3=l,l_1,l_2,l_3\geq 1} \tfrac{l!}{l_1!l_2!l_3!}\partial_z^{l_1}\tfrac{\psi}{W}\nabla_x\partial_z^{l_2}\phi\cdot\nabla_x\partial_z^{l_3}\psi)+\cdots.\\
=&J_{21}+J_{22}+\cdots.
\end{aligned}
$$

We omit the other terms on the right-hand side because they can be treated in the same way as these listed terms.

3.1. A priori estimates for system (1.4). Define the energy functionals

$$
\begin{aligned}
\mathcal{E}_l(z,t)&=\|\partial_z^l\varrho(z,t)\|^2_{H^{s-l}}+\|\partial_z^l\phi(z,t)\|^2_{H^{s-l}}+\|\partial_z^l\psi(z,t)\|^2_{H^{s-l}},\\
\mathcal{D}_l(z,t)&=\mu(\|\nabla_x\partial_z^l\phi(z,t)\|^2_{H^{s-l}}+\|\nabla_x\partial_z^l\psi(z,t)\|^2_{H^{s-l}}),
\end{aligned}
$$

and

$$
A_{l-1}(z,t)=1+\sum_{m=0}^{l-1}\mathcal{E}_m(z,t).
$$

We mention that the space regularity of $(\partial_z^l\varrho,\partial_z^l\phi,\ \partial_z^l\psi)$ is one-order less than that of $(\partial_z^{l-1}\varrho,\ \partial_z^{l-1}\phi,\ \partial_z^{l-1}\psi)$ here. This can be simply explained by the fact that (3.1)-1 is a transport equation, and the term $\nabla_x\partial_z^{l-1}\varrho$ in (3.1)-1 belongs to $H^{s-l}(\mathbb{R}^3)$ under the assumption $\partial_z^{l-1}\varrho\in H^{s-l+1}(\mathbb{R}^3)$.

LEMMA 3.1. *Assume $l\geq 1$ and $s-l\geq 2$. For each $z\in I$, assume that $(\partial_z^m\varrho(z,t,x)$, $\partial_z^m\phi(z,t,x)$, $\partial_z^m\psi(z,t,x))$, $m=0,l,\dots,l-1$, are sufficiently smooth functions, and $(\partial_z^l\varrho(z,t,x)$, $\partial_z^l\phi(z,t,x)$, $\partial_z^l\psi(z,t,x))$ solves system (3.1) on some time interval $[0,T]$. Then there is a constant $C>0$, depending only upon s and the coefficients of system (3.1), such that for all $0\leq t\leq T$,*

$$
\tfrac{d}{dt}\mathcal{E}_l(z,t)+\mathcal{D}_l(z,t)\leq CA_{l-1}^{2s+2}(z,t)(1+\mathcal{E}_l(z,t)).
$$

Proof. It is obvious that replacing $(\partial_z^l\varrho(z,t,x),\ \partial_z^l\phi(z,t,x),\ \partial_z^l\psi(z,t,x),\partial_z^l\Omega(z,t,x))$ in the underlined parts of (3.1) with $(\varrho(t,x),\phi(t,x),\psi(t,x),\Omega(t,x))$ is just the left-hand side of (1.3). Note that $s-l\geq 2$ here is also consistent with $s\geq 2$ in Lemma 2.1. Hence, by taking energy method, following exactly along the same lines as that in the proof of Lemma 2.1 gives

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}(\|\partial_z^l\varrho\|^2_{H^{s-l}}+\|\partial_z^l\phi\|^2_{H^{s-l}}+\|\partial_z^l\psi\|^2_{H^{s-l}})+\mu(\|\nabla_x\partial_z^l\phi\|^2_{H^{s-l}}+\|\nabla_x\partial_z^l\psi\|^2_{H^{s-l}})\\
\lesssim&\|\nabla_x\Omega\|_{H^{s-l}}\|\partial_z^l\varrho\|^2_{H^{s-l}}+\|\nabla_x\partial_z^l\Omega\|_{H^{s-l}}\|\partial_z^l\varrho\|_{H^{s-l}}+\|\partial_z^l\varrho\|_{H^{s-l}}\|\phi\|_{H^{s-l}}\|\partial_z^l\phi\|_{H^{s-l}}\\
&+\|\partial_z^l\varrho\|_{H^{s-l}}\|W^2\Omega_\phi\|_{L^\infty}\|\partial_z^l\phi\|_{H^{s-l}}+\|\partial_z^l\varrho\|_{H^{s-l}}\|\nabla_x\phi\|_{H^{s-l}}\|\nabla_x\partial_z^l\phi\|_{H^{s-l}}\\
&+\|\tfrac{\psi}{W}\|_{L^\infty}\|\psi\|_{H^{s-l}}\|\partial_z^l\phi\|^2_{H^{s-l}}\\
&+\|\partial_z^l\varrho\|_{H^s}(\|W^2\Omega_\phi\|_{L^\infty}+\|\nabla_x(W^2\Omega_\phi)\|_{H^{s-l}})\|\nabla_x\partial_z^l\phi\|_{H^{s-l}}\\
&+\big(\|\tfrac{\psi}{W}\|_{L^\infty}\|\partial_z^l\phi\|_{H^{s-l}}\|\nabla_x\psi\|_{H^{s-l}}+\|\tfrac{\psi}{W}\|_{L^\infty}\|\psi\|_{H^{s-l}}\|\nabla_x\partial_z^l\phi\|_{H^{s-l-\frac{1}{4}}}\\
&+(\|\tfrac{\psi}{W}\|_{L^\infty}+\|\nabla_x\tfrac{\psi}{W}\|_{H^{s-l-1}})\|\partial_z^l\phi\|_{H^{s-l}}\|\psi\|_{H^{s-l}}\big)\|\nabla_x\partial_z^l\phi\|_{H^{s-l}}\\
&+\text{bounds for the right-hand side terms.}
\end{aligned}
$$

Then we treat the right-hand side terms. Applying H^{s-l} is a Banach algebra, we get

$$
\begin{aligned}
\langle J_{11},\partial_z^l\varrho\rangle_{H^{s-l}}&\lesssim\|\partial_z^l\Omega\cdot\nabla_x\varrho\|_{H^{s-l}}\|\partial_z^l\varrho\|_{H^{s-l}}\\
&\lesssim\|\partial_z^l\Omega\|_{H^{s-l}}\|\nabla_x\varrho\|_{H^{s-l}}\|\partial_z^l\varrho\|_{H^{s-l}},
\end{aligned}
$$

$$\begin{aligned}\langle J_{21},\partial_z^l\phi\rangle_{H^{s-l}} \lesssim &(\|\nabla_x\varrho\|_{H^{s-l}}\|\nabla_x\partial_z^l\phi\|_{H^{s-l}}+\|\nabla_x\varrho\|_{H^{s-l}}\|\partial_z^l(W_2\Omega_\phi)\|_{H^{s-l}}\\ &+\|\partial_z^l\tfrac{\psi}{W}\|_{H^{s-l}}\|\nabla_x\phi\|_{H^{s-l}}\|\nabla_x\psi\|_{H^{s-l}})\|\partial_z^l\phi\|_{H^{s-l}},\\ \langle J_{12},\partial_z^l\phi\rangle \lesssim &\sum_{m=1}^{l-1}\|\partial_z^m\Omega\|_{H^{s-l}}\|\nabla_x\partial_z^{l-m}\varrho\|_{H^{s-l}}\|\partial_z^l\phi\|_{H^{s-l}},\\ \langle J_{22},\partial_z^l\phi\rangle_{H^{s-l}} \lesssim &(\sum_{m=1}^{l-1}\|\nabla_x\partial_z^m\varrho\|_{H^{s-l}}\|\nabla_x\partial_z^{l-m}\phi\|_{H^{s-l}}\\ &+\sum_{m=1}^{l-1}\|\nabla_x\partial_z^m\varrho\|_{H^{s-l}}\|\partial_z^{l-m}(W_2\Omega_\phi)\|_{H^{s-l}}\\ &+\sum_{\substack{l_1+l_2+l_3=l,\\ l_1,l_2,l_3\ge 1}}\|\partial_z^{l_1}\tfrac{\psi}{W}\|_{H^{s-l}}\|\nabla_x\partial_z^{l_2}\phi\|_{H^{s-l}}\|\nabla_x\partial_z^{l_3}\psi\|_{H^{s-l}})\|\partial_z^l\phi\|_{H^{s-l}}.\end{aligned}$$

Applying Lemma A.6 and Lemma A.8 to control the terms related with Ω, recalling the definition of $\mathcal{E}_l,\mathcal{D}_l$, and A_{l-1}, we have

$$\tfrac12\tfrac{d}{dt}\mathcal{E}_l(z,t)+\mathcal{D}_l(z,t)\lesssim A_{l-1}^{s+1}(1+\mathcal{E}_l^{\frac12}+\mathcal{D}_l^{\frac12})\mathcal{E}_l^{\frac12}+A_{l-1}^{\frac{s+1}{2}}\mathcal{E}_l^{\frac12}\mathcal{D}_l^{\frac12}+A_{l-1}^{\frac12}\mathcal{E}_1^{\frac18}\mathcal{D}_l^{\frac78}.$$

Finally, applying the Young inequality to control $\mathcal{D}_l^{\frac12}$ and $\mathcal{D}_l^{\frac78}$ by $\mathcal{D}_l$, we have

$$\tfrac{d}{dt}\mathcal{E}_l+\mathcal{D}_l\le CA_{l-1}^{2s+2}(1+\mathcal{E}_l).$$

This completes the proof. ∎

3.2. Stability estimates for system (1.4). The difference between $(\partial_z^l\varrho,\ \partial_z^l\phi,\ \partial_z^l\psi)$ and $(\partial_z^l\bar\varrho,\partial_z^l\bar\phi,\partial_z^l\bar\psi)$ reads

$$\begin{cases}\partial_t\partial_z^l(\varrho-\bar\varrho)+c_1\bar\Omega\cdot\nabla_x\partial_z^l(\varrho-\bar\varrho)+c_1(\Omega-\bar\Omega)\cdot\nabla_x\partial_z^l\varrho+c_1\nabla_x\cdot\partial_z^l(\Omega-\bar\Omega)=H_1,\\ \partial_t\partial_z^l(\phi-\bar\phi)-c_3\Delta_x\partial_z^l(\phi-\bar\phi)+2\mu\nabla_x\partial_z^l(\varrho-\bar\varrho)\cdot\nabla_x\phi+2\mu\nabla_x\partial_z^l\bar\varrho\cdot\nabla_x(\phi-\bar\phi)\\ \qquad-\frac{c_3}{4}\nabla_x\partial_z^l(\varrho-\bar\varrho)\cdot(W^2\Omega_\phi)-\frac{c_3}{4}\nabla_x\partial_z^l\bar\varrho\cdot(W^2\Omega_\phi-\overline{W}^2\overline{\Omega_\phi})\\ \qquad-4\mu(\frac{\phi}{W}-\frac{\bar\phi}{\overline{W}})\nabla_x\partial_z^l\phi\cdot\nabla_x\psi-4\mu\frac{\bar\phi}{\overline{W}}\nabla_x\partial_z^l(\phi-\bar\phi)\cdot\nabla_x\psi\\ \qquad-4\mu\frac{\bar\phi}{\overline{W}}\nabla_x\partial_z^l\bar\phi\cdot\nabla_x(\psi-\bar\psi)=H_2,\\ \dots\end{cases}\tag{3.2}$$

with

$$\begin{aligned}H_1&=(J_{11}(\varrho,\phi,\psi)-J_{11}(\bar\varrho,\bar\phi,\bar\psi))+(J_{12}(\varrho,\phi,\psi)-(J_{12}(\bar\varrho,\bar\phi,\bar\psi))=H_{11}+H_{12},\\ H_2&=(J_{21}(\varrho,\phi,\psi)-J_{21}(\bar\varrho,\bar\phi,\bar\psi))+(J_{22}(\varrho,\phi,\psi)-(J_{22}(\bar\varrho,\bar\phi,\bar\psi))+\cdots=H_{21}+H_{22}+\cdots.\end{aligned}$$

LEMMA 3.2. *Let $l\ge1$ and $s\ge 2+l$. For each $z\in I$, suppose $(\partial_z^m\varrho(z,t,x),\partial_z^m\phi(z,t,x),\partial_z^m\psi(z,t,x))$ and $(\partial_z^m\bar\varrho(z,t,x),\ \partial_z^m\bar\phi(z,t,x),\partial_z^m\bar\psi(z,t,x))$ belong to spaces*

$$C([0,T];H^{s-m})\times\big(C([0,T];H^{s-m})\cap L^2(0,T;\dot H^{s-m+1})\big)^2, m=0,1,\dots,l-1.$$

and $(\partial_z^l\varrho,\partial_z^l\phi,\partial_z^l\psi)$ and $(\partial_z^l\bar\varrho,\partial_z^l\bar\phi,\partial_z\bar\psi)$ are two solutions to system (1.4) in the space

$$C([0,T];H^{s-l})\times\big(C([0,T];H^{s-l})\cap L^2(0,T;\dot H^{s-l+1})\big)^2.$$

Let

$$A_l(z,t)=1+\sum_{m=0}^{l}(\|\partial_z^m\varrho(z,t)\|^2_{H^{s-m}}+\|\partial_z^m\phi(z,t)\|^2_{H^{s-m}}+\|\partial_z^m\psi(z,t)\|^2_{H^{s-m}})$$
$$+\sum_{m=0}^{l}(\|\partial_z^m\bar\varrho(z,t)\|^2_{H^{s-m}}+\|\partial_z^m\bar\phi(z,t)\|^2_{H^{s-m}}+\|\partial_z^m\bar\psi(z,t)\|^2_{H^{s-m}}).$$

Define

$$X_l(z,t)=\|\partial_z^l(\varrho-\bar\varrho)(z,t)\|^2_{H^{s-l-1}}+\|\partial_z^l(\phi-\bar\phi)(z,t)\|^2_{H^{s-l-1}}+\|\partial_z^l(\psi-\bar\psi)(z,t)\|^2_{H^{s-l-1}},$$
$$Y_l(z,t)=\mu(\|\nabla_x\partial_z^l(\phi-\bar\phi)(z,t)\|^2_{H^{s-l-1}}+\|\nabla_x\partial_z^l(\psi-\bar\psi)(z,t)\|^2_{H^{s-l-1}}),$$

and

$$X_{sum,l}(z,t)=\sum_{m=0}^{l}X_m(z,t).$$

Then, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}X_l(z,t)+Y_l(z,t)\le CA_l^{2s+4}X_{sum,l}.$$

Proof. By taking energy method, following exactly along the same lines as that in the proof of Lemma 2.2 gives

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}(\|\partial_z^l(\varrho-\bar\varrho)\|^2_{H^{s-l-1}}+\|\partial_z^l(\phi-\bar\phi)\|^2_{H^{s-l-1}}+\|\partial_z^l(\psi-\bar\psi\|^2_{H^{s-l-1}})\\
&\quad+\mu(\|\nabla_x\partial_z^l(\phi-\bar\phi)\|^2_{H^{s-l-1}}+\|\nabla_x\partial_z^l(\psi-\bar\psi)\|^2_{H^{s-l-1}})\\
&\lesssim\|\nabla_x\bar\Omega\|_{H^{s-l}}\|\partial_z^l(\varrho-\bar\varrho)\|^2_{H^{s-l-1}}+\|\Omega-\bar\Omega\|_{H^{s-l}}\|\partial_z^l\varrho\|_{H^{s-l}}\|\partial_z^l(\varrho-\bar\varrho)\|_{H^{s-l-1}}\\
&\quad+\|\partial_z^l(\Omega-\bar\Omega)\|_{H^{s-l}}\|\partial_z^l(\varrho-\bar\varrho)\|_{H^{s-l-1}}\\
&\quad+\|\partial_z^l(\varrho-\bar\varrho)\|_{H^{s-l-1}}\|\nabla_x\phi\|_{H^{s-l}}(\|\partial_z^l(\phi-\bar\phi)\|_{H^{s-l-1}}+\|\nabla_x\partial_z^l(\phi-\bar\phi)\|_{H^{s-l-1}})\\
&\quad+\|\partial_z^l\bar\varrho\|_{H^{s-l}}\|\nabla_x(\phi-\bar\phi)\|_{H^{s-l-1}}(\|\partial_z^l(\phi-\bar\phi)\|_{H^{s-l-1}}+\|\nabla_x\partial_z^l(\phi-\bar\phi)\|_{H^{s-l-1}})\\
&\quad+\|\partial_z^l(\varrho-\bar\varrho)\|_{H^{s-l-1}}(\|W^2\Omega_\phi\|_{L^\infty}+\|\nabla_x(W^2\Omega_\phi)\|_{H^{s-l-1}})\\
&\quad\times(\|\partial_z^l(\phi-\bar\phi)\|_{H^{s-l-1}}+\|\nabla_x\partial_z^l(\phi-\bar\phi)\|_{H^{s-l-1}})\\
&\quad+\|\partial_z^l\bar\varrho\|_{H^{s-l}}\|W^2\Omega_\phi-\overline{W}^2\bar\Omega_\phi\|_{H^{s-l-1}}(\|\partial_z^l(\phi-\bar\phi)\|_{H^{s-l-1}}+\|\nabla_x\partial_z^l(\phi-\bar\phi)\|_{H^{s-l-1}})\\
&\quad+\|\tfrac{\phi}{W}-\tfrac{\bar\phi}{\overline{W}}\|_{H^{s-l-1}}\|\partial_z^l\phi\|_{H^{s-l}}\|\psi\|_{H^{s-l}}(\|\partial_z^l(\phi-\bar\phi)\|_{H^{s-l-1}}+\|\nabla_x\partial_z^l(\phi-\bar\phi)\|_{H^{s-l-1}})\\
&\quad+\|\tfrac{\bar\phi}{\overline{W}}\|_{L^\infty}\|\nabla_x\partial_z^l(\phi-\bar\phi)\|_{H^{s-l-1}}\|\psi\|_{H^{s-l}}\|\partial_z^l(\phi-\bar\phi)\|_{H^{s-l-1}}\\
&\quad+\|\tfrac{\bar\phi}{\overline{W}}\|_{L^\infty}\|\nabla_x(\psi-\bar\psi)\|_{H^{s-l-1}}\|\partial_z^l\bar\phi\|_{H^{s-l}}\|\partial_z^l(\phi-\bar\phi)\|_{H^{s-l-1}}\\
&\quad+(\|\tfrac{\bar\phi}{\overline{W}}\|_{L^\infty}+\|\nabla_x\tfrac{\bar\phi}{\overline{W}}\|_{H^{s-l-1}})\|\partial_z^l(\phi-\bar\phi)\|^{\frac{1}{2}}_{H^{s-l-1}}\|\psi\|_{H^{s-l}}\|\nabla_x\partial_z^l(\phi-\bar\phi)\|^{\frac{3}{2}}_{H^{s-l-1}}\\
&\quad+(\|\tfrac{\bar\phi}{\overline{W}}\|_{L^\infty}+\|\nabla_x\tfrac{\bar\phi}{\overline{W}}\|_{H^{s-1}})\|\psi-\bar\psi\|^{\frac{1}{2}}_{H^{s-l-1}}\|\nabla_x(\psi-\bar\psi)\|^{\frac{1}{2}}_{H^{s-l-1}}\\
&\quad\times\|\partial_z^l\bar\phi\|_{H^{s-l}}\|\nabla_x\partial_z^l(\phi-\bar\phi)\|_{H^{s-l-1}}\\
&\quad+\text{bounds for the right-hand side terms.}
\end{aligned}$$

Then we treat the right-hand side terms. Applying Lemma A.5-(2), we get

$$\begin{aligned}
&\langle H_{11}+H_{12},\partial_z^l(\varrho-\bar\varrho)\rangle_{H^{s-l-1}}\\
&\lesssim\sum_{m=1}^{l}\|\partial_z^m(\Omega-\bar\Omega)\cdot\nabla_x\partial_z^{l-m}\varrho-\partial_z^m\bar\Omega\cdot\nabla_x\partial_z^{l-m}(\varrho-\bar\varrho)\|_{H^{s-l-1}}\|\partial_z^l(\varrho-\bar\varrho)\|_{H^{s-l-1}}
\end{aligned}$$

$$\lesssim \sum_{m=1}^{l} (\|\partial_z^m(\Omega-\bar{\Omega})\|_{H^{s-l-1}}\|\nabla_x\partial_z^{l-m}\varrho\|_{H^{s-l}} + \|\partial_z^m\bar{\Omega}\|_{H^{s-l}}\|\nabla_x\partial_z^{l-m}(\varrho-\bar{\varrho})\|_{H^{s-l-1}})\|\partial_z^l(\varrho-\bar{\varrho})\|_{H^{s-l-l}},$$

$$\begin{aligned}
&\langle H_{21}+H_{22},\partial_z^l(\phi-\bar{\phi})\rangle_{H^{s-l-1}}\\
\lesssim&\Big(\sum_{m=1}^{l}\|\nabla_x\partial_z^m(\varrho-\bar{\varrho})\nabla_x\partial_z^{l-m}\phi+\nabla_x\partial_z^m\bar{\varrho}\nabla_x\partial_z^{l-m}(\phi-\bar{\phi})\|_{H^{s-l-1}}\\
&+\sum_{m=1}^{l}\|\nabla_x\partial_z^m(\varrho-\bar{\varrho})\partial_z^{l-m}(W_2\Omega_\phi)+\nabla_x\partial_z^m\bar{\varrho}\partial_z^{l-m}(W_2\Omega_\phi-\overline{W}^2\bar{\Omega}_\phi)\|_{H^{s-l-1}}\\
&+\sum_{l_1+l_2+l_3=l,l_1\geq 1}\|\partial_z^{l_1}(\tfrac{\psi}{W}-\tfrac{\bar{\psi}}{\overline{W}})\nabla_x\partial_z^{l_2}\phi\cdot\nabla_x\partial_z^{l_3}\psi+\partial_z^{l_3}\tfrac{\bar{\psi}}{\overline{W}}\nabla_x\partial_z^{l_2}(\phi-\bar{\phi})\cdot\nabla_x\partial_z^{l_3}\psi\\
&+\partial_z^{l_1}\tfrac{\bar{\psi}}{\overline{W}}\nabla_x\partial_z^{l_2}\bar{\phi}\cdot\nabla_x\partial_z^{l_3}(\psi-\bar{\psi})\|_{H^{s-l-1}}\Big)\|\partial_z^l(\phi-\bar{\phi})\|_{H^{s-l-l}}\\
\lesssim&\Big(\sum_{m=1}^{l}\big(\|\nabla_x\partial_z^m(\varrho-\bar{\varrho})\|_{H^{s-l-1}}\|\nabla_x\partial_z^{l-m}\phi\|_{H^{s-l}}\\
&+\|\nabla_x\partial_z^m\bar{\varrho}\|_{H^{s-l}}\|\nabla_x\partial_z^{l-m}(\phi-\bar{\phi})\|_{H^{s-l-1}}\big)\\
&+\sum_{m=1}^{l}\big(\|\nabla_x\partial_z^m(\varrho-\bar{\varrho})\|_{H^{s-l-1}}\|\partial_z^{l-m}(W_2\Omega_\phi)\|_{H^{s-l}}\\
&+\|\nabla_x\partial_z^m\bar{\varrho}\|_{H^{s-l}}\|\partial_z^{l-m}(W_2\Omega_\phi-\overline{W}^2\bar{\Omega}_\phi)\|_{H^{s-l-1}}\big)\\
&+\sum_{l_1+l_2+l_3=l,l_1\geq 1}\big(\|\partial_z^{l_1}(\tfrac{\psi}{W}-\tfrac{\bar{\psi}}{\overline{W}})\|_{H^{s-l-1}}\|\nabla_x\partial_z^{l_2}\phi\|_{H^{s-l}}\|\nabla_x\partial_z^{l_3}\psi\|_{H^{s-l}}\\
&+\|\partial_z^{l_3}\tfrac{\bar{\psi}}{\overline{W}}\|_{H^{s-l}}\|\nabla_x\partial_z^{l_2}(\phi-\bar{\phi})\|_{H^{s-l-1}}\|\nabla_x\partial_z^{l_3}\psi\|_{H^{s-l}}\\
&+\|\partial_z^{l_1}\tfrac{\bar{\psi}}{\overline{W}}\|_{H^{s-l}}\|\nabla_x\partial_z^{l_2}\bar{\phi}\|_{H^{s-l}}\|\nabla_x\partial_z^{l_3}(\psi-\bar{\psi})\|_{H^{s-l-1}}\big)\Big)\|\partial_z^l(\phi-\bar{\phi})\|_{H^{s-l-l}}.
\end{aligned}$$

The main control inequalities are as follows:

$$\|\nabla_x\partial_z^l(\varrho-\bar{\varrho})\|_{H^{s-l-1}}\leq Y_l^{\frac{1}{2}}$$

and if $m<l$,

$$\|\nabla_x\partial_z^m(\varrho-\bar{\varrho})\|_{H^{s-l-1}}\leq\|\partial_z^m(\varrho-\bar{\varrho})\|_{H^{s-l}}\leq\|\partial_z^m(\varrho-\bar{\varrho})\|_{H^{s-m-1}}\leq X_{sum,l}^{\frac{1}{2}}.$$

Applying Lemma A.6-Lemma A.9, recalling the definition of A_l, X_l, Y_l, and $X_{sum,l}$, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}X_l(z,t)+Y_l(z,t)\lesssim&A_l^{s+2}(X_{sum,l}^{\frac{1}{2}}+Y_l^{\frac{1}{2}})X_{sum,l}^{\frac{1}{2}}\\&+A_l^{\frac{s+2}{2}}X_{sum,l}^{\frac{1}{2}}Y_l^{\frac{1}{2}}+A_l^{\frac{s+2}{2}}X_{sum,l}^{\frac{1}{4}}Y_l^{\frac{3}{4}}.\end{aligned}\tag{3.3}$$

This implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}X_l(z,t)+Y_l(z,t)\leq CA_l^{2s+4}X_{sum,l}.$$

This completes the proof. ∎

3.3. Proof of Theorem 1.2. In this subsection, we prove Theorem 1.2. Suppose that the initial data is $W^{q,\infty}$-regular with respect to the parameter z, we show the regularity can be preserved in time. The key is to rigorously justify $(\partial_z^l\varrho,\partial_z^l\phi,\partial_z^l\psi)|_{l=1}^q$ exist and solve (1.4).

Step 1: Approximate solutions.

(a) Existence for $(\varrho^n(z,t,x),\phi^n(z,t,x),\ \psi^n(z,t,x))|_{n\in\mathbb{N}}$.

For each fixed $z\in I$, we start from $(\varrho^0(z,t,x),\phi^0(z,t,x),\psi^0(z,t,x))=0$, then we define by introducing a sequence $(\varrho^n(z,t,x),\phi^n(z,t,x),\psi^n(z,t,x))|_{n\in\mathbb{N}}$ solving

$$\begin{cases}\partial_t\varrho^{n+1}+c_1\Omega^n\cdot\nabla_x\varrho^{n+1}=-c_1\nabla_x\cdot\Omega^n,\\ \partial_t\phi^{n+1}+c_2\Omega^n\cdot\nabla_x\phi^{n+1}-\mu\Delta_x\phi^{n+1}=-H^\phi(\varrho^n,\phi^n,\psi^n),\\ \partial_t\psi^{n+1}+c_2\Omega^n\cdot\nabla_x\psi^{n+1}-\mu\Delta_x\psi^{n+1}=-H^\psi(\varrho^n,\phi^n,\psi^n),\\ (\varrho^{n+1},\phi^{n+1},\psi^{n+1})(z,0,x)=(\mathbb{J}_{2^{n+1}}\varrho^{in}(z,x),\mathbb{J}_{2^{n+1}}\phi^{in}(z,x),\mathbb{J}_{2^{n+1}}\psi^{in}(z,x)),\end{cases}\tag{3.4}$$

where $\Omega^n=\Omega(\phi^n,\psi^n)$ and $\mathbb{J}_{2^{n+1}}$ denotes the mollification operator

$$\mathbb{J}_{2^{n+1}}u(x)=(2^{n+1})^3\int_{\mathbb{R}^3}g(2^{n+1}(x-y))u(y)dy,$$

with $g\in C_0^\infty(\mathbb{R}^3),g\geq 0$ *and* $\int_{\mathbb{R}^3}g(x)dx=1$. According to the transport and transport diffusion equation theories (see, for example, Chapter 3 in [1])), we may argue by induction that for each $z\in I$, there exists a family of smooth global approximate solutions $(\varrho^n(z,t,x),\phi^n(z,t,x),\ \psi^n(z,t,x))|_{n\in\mathbb{N}}$.

(b) Existence for $(\partial_z\varrho^n(z,t,x),\partial_z\phi^n(z,t,x),\ \partial_z\psi^n(z,t,x))|_{n\in\mathbb{N}}$.

For each $z\in I$, suppose Δz is sufficiently small such that $z+\Delta z\in I$. Define

$$\Upsilon^n=\tfrac{\varrho^n(z+\Delta z)-\varrho^n(z)}{\Delta z},\quad \Phi^n=\tfrac{\phi^n(z+\Delta z)-\phi^n(z)}{\Delta z},\quad \Psi^n=\tfrac{\psi^n(z+\Delta z)-\psi^n(z)}{\Delta z},\quad n\in\mathbb{N}.$$

From system (3.4), we find the system for $(\Upsilon^{n+1},\Phi^{n+1},\Psi^{n+1})$ reads

$$\begin{cases}\partial_t\Upsilon^{n+1}+c_1\Omega^n(z+\Delta z)\cdot\nabla_x\Upsilon^{n+1}=-c_1\tfrac{(\Omega^n(z+\Delta z)-\Omega^n(z))}{\Delta z}\cdot\nabla_x\varrho^{n+1}(z)\\ \qquad\qquad -c_1\tfrac{\nabla_x\cdot(\Omega^n(z+\Delta z)-\Omega^n(z))}{\Delta z},\\ \partial_t\Phi^{n+1}+c_2\Omega^n(z+\Delta z)\cdot\nabla_x\Phi^{n+1}-\mu\Delta_x\Phi^{n+1}=-c_2\tfrac{(\Omega^n(z+\Delta z)-\Omega^n(z))}{\Delta z}\cdot\nabla_x\phi^{n+1}(z)+G_2^n,\\ \partial_t\Psi^{n+1}+c_2\Omega^n(z+\Delta z)\cdot\nabla\Psi^{n+1}-\mu\Delta_x\Psi^{n+1}=-c_2\tfrac{(\Omega^n(z+\Delta z)-\Omega^n(z))}{\Delta z}\cdot\nabla_x\psi^{n+1}(z)+G_3^n,\end{cases}\tag{3.5}$$

with

$$G_2^n=\tfrac{-H^\phi(\varrho^n(z+\Delta z),\phi^n(z+\Delta z),\psi^n(z+\Delta z))+H^\phi(\varrho^n(z),\phi^n(z),\psi^n(z))}{\Delta z},$$
$$G_3^n=\tfrac{-H^\psi(\varrho^n(z+\Delta z),\phi^n(z+\Delta z),\psi^n(z+\Delta z))+H^\psi(\varrho^n(z),\phi^n(z),\psi^n(z))}{\Delta z}.$$

We consider (3.5) as a linear system for $(\Upsilon^{n+1},\Phi^{n+1},\Psi^{n+1})$. On the one hand, under the initial assumption

$$(\partial_z\varrho^{in},\partial_z^{in}\phi,\partial_z^{in}\psi)=\lim_{\Delta z\to 0}(\tfrac{\varrho^{in}(z+\Delta z)-\varrho^{in}(z)}{\Delta z},\tfrac{\phi^{in}(z+\Delta z)-\phi^{in}(z)}{\Delta z},\tfrac{\psi^{in}(z+\Delta z)-\psi^{in}(z)}{\Delta z})$$

exists in H^{s-1}, we deduce that the initial data of system (3.5)

$$(\tfrac{\mathbb{J}_{2^{n+1}}\varrho^{in}(z+\Delta z)-\mathbb{J}_{2^{n+1}}\varrho^{in}(z)}{\Delta z},\tfrac{\mathbb{J}_{2^{n+1}}\phi^{in}(z+\Delta z)-\mathbb{J}_{2^{n+1}}\phi^{in}(z)}{\Delta z},\tfrac{\mathbb{J}_{2^{n+1}}\psi^{in}(z+\Delta z)-\mathbb{J}_{2^{n+1}}\psi^{in}(z)}{\Delta z})$$

$$\rightarrow (\mathbb{J}_{2^{n+1}}\partial_z\varrho^{in}(z),\mathbb{J}_{2^{n+1}}\partial_z\phi^{in}(z),\mathbb{J}_{2^{n+1}}\partial_z\psi^{in}(z)),\ as\ \Delta z\rightarrow 0,$$

in $H^m(\mathbb{R}^3)$, for any $m\in\mathbb{N}$. On the other hand, we suppose that

$$(\partial_z\varrho^n,\partial_z\phi^n,\partial_z\psi^n)=\lim_{\Delta z\rightarrow 0}(\Upsilon^n,\Phi^n,\Psi^n)$$

exists in $C([0,T],H^m)$ for any $T>0$ and any $m\in\mathbb{N}$. Then thanks to the transport equation and transport diffusion equation theories again, we can infer that $(\partial_z\varrho^{n+1},\partial_z\phi^{n+1},\partial_z\psi^{n+1})=\lim_{\Delta z\rightarrow 0}(\Upsilon^{n+1},\Phi^{n+1},\Psi^{n+1})$ also makes sense $C([0,T],H^m)$, for any $T>0$ and any $m\in\mathbb{N}$. Furthermore, by taking limits in system (3.5), we deduce that for each $z\in I$, $(\partial_z\varrho^{n+1},\partial_z\phi^{n+1},\partial_z\psi^{n+1})$ satisfies

$$\begin{cases}\partial_t\partial_z\varrho^{n+1}+c_1\Omega^n\cdot\nabla_x\partial_z\varrho^{n+1}=-c_1\partial_z\Omega^n\cdot\nabla_x\varrho^{n+1}-c_1\nabla_x\cdot\partial_z\Omega^n,\\ \partial_t\partial_z\phi^{n+1}+c_2\Omega^n\cdot\nabla_x\partial_z\phi^{n+1}-\mu\Delta\partial_z\phi^{n+1}=c_2\partial_z\Omega^n\cdot\nabla_x\phi^{n+1}-H^{\phi}_{z1}(\varrho^n,\phi^n,\psi^n),\\ \partial_t\partial_z\psi^{n+1}+c_2\Omega^n\cdot\nabla_x\partial_z\psi^{n+1}-\mu\Delta\partial_z\psi^{n+1}=-c_2\partial_z\Omega^n\cdot\nabla_x\psi^{n+1}-H^{\psi}_{z1}(\varrho^n,\phi^n,\psi^n),\\ (\partial_z\varrho^{n+1},\partial_z\phi^{n+1},\partial_z\psi^{n+1})(z,0,x)=(\mathbb{J}_{2^{n+1}}\partial_z\varrho^{in},\mathbb{J}_{2^{n+1}}\partial_z\phi^{in},\mathbb{J}_{2^{n+1}}\partial_z\psi^{in}).\end{cases} \tag{3.6}$$

Expressions of H^{ϕ}_{z1} and H^{ψ}_{z1} are defined in system (1.4) with $l=1$.

(c) Existence for $(\partial^l_z\varrho^n(z,t,x),\partial^l_z\phi^n(z,t,x),\ \partial^l_z\psi^n(z,t,x))|_{n\in\mathbb{N}}$ **with** $0\le l\le q$**.**

Repeating the same arguments as that for case (1.2), and thus will be omitted, we get that for any positive time T, and any nonnegative integer m,

$$(\partial^l_z\varrho^n(z),\partial^l_z\phi^n(z),\partial^l_z\psi^n(z))_{n\in\mathbb{N}}\in C([0,T];H^m),$$

satisfying the following linear system

$$\begin{cases}\partial_t\partial^l_z\varrho^{n+1}+c_1\Omega^n\cdot\nabla_x\partial^l_z\varrho^{n+1}=-c_1\sum\limits_{m=1}^{l}C^m_l\partial^m_z\Omega^n\cdot\nabla_x\partial^{l-m}_z\varrho^{n+1}-c_1\nabla_x\cdot\partial^l_z\Omega^n,\\ \partial_t\partial^l_z\phi^{n+1}+c_2\Omega^n\cdot\nabla_x\partial^l_z\phi^{n+1}-c_3\Delta_x\partial^l_z\phi^{n+1}=-c_2\sum\limits_{m=1}^{l}C^m_l\partial^m_z\Omega^n\cdot\nabla_x\partial^{l-m}_z\phi^{n+1}\\ \qquad\qquad+H^{\phi}_{zl}(\varrho^n,\phi^n,\psi^n),\\ \partial_t\partial^l_z\psi^{n+1}+c_2\Omega^n\cdot\nabla_x\partial^l_z\psi^{n+1}-c_3\Delta_x\partial^l_z\psi^{n+1}=-c_2\sum\limits_{m=1}^{l}C^m_l\partial^m_z\Omega^n\cdot\nabla_x\partial^{l-m}_z\psi^{n+1}\\ \qquad\qquad+H^{\psi}_{zl}(\varrho^n,\phi^n,\psi^n),\\ (\partial^l_z\varrho^{n+1},\partial^l_z\phi^{n+1},\partial^l_z\psi^{n+1})(0,x,z)=(\mathbb{J}_{2^{n+1}}\partial^l_z\varrho^{in},\mathbb{J}_{2^{n+1}}\partial^l_z\phi^{in},\mathbb{J}_{2^{n+1}}\partial^l_z\psi^{in}).\end{cases} \tag{3.7}$$

Step2: Uniform bounds.

(a) Uniform bound for $(\varrho^n(z,t,x),\phi^n(z,t,x),\ \psi^n(z,t,x))|_{n\in\mathbb{N}}$**.**

For any $n\in\mathbb{N}$, define

$$\begin{aligned}\mathcal{E}_{0,n}(z,t)&=\|\varrho^n\|^2_{H^{s-1}}+\|\phi^n\|^2_{H^{s-1}}+\|\psi^n\|^2_{H^{s-1}},\\ D_{0,n}(z,t)&=\mu(\|\nabla_x\phi^n\|^2_{H^{s-1}}+\|\nabla_x\psi^n\|^2_{H^{s-1}}).\end{aligned}$$

Following exactly the proof of (2.3) in Lemma 2.1, we deduce from system (3.4) that

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\mathcal{E}_{0,n+1}+\mathcal{D}_{0,n+1}\lesssim&\mathcal{D}^{\frac12}_{0,n}(1+\mathcal{E}_{0,n})^{\frac{s}{2}}(\mathcal{E}_{0,n+1}+\mathcal{E}^{\frac12}_{0,n+1})+(1+\mathcal{E}_{0,n})^{\frac{s+1}{2}}\mathcal{E}^{\frac12}_{0,n+1}\\ &+\mathcal{E}^{\frac58}_{0,n}\mathcal{D}^{\frac38}_{0,n}\mathcal{D}^{\frac12}_{0,n+1}+(1+\mathcal{E}_{0,n})^{\frac{s+2}{2}}\mathcal{D}^{\frac12}_{0,n+1}.\end{aligned}$$

Applying the Young inequality to control $\mathcal{D}_{0,n+1}^{\frac{1}{2}}$ by $\mathcal{D}_{0,n+1}$, we derive that

$$\begin{aligned}\tfrac{d}{dt}(1+\mathcal{E}_{0,n+1})+\mathcal{D}_{0,n+1}\lesssim&\mathcal{D}_{0,n}^{\frac{1}{2}}(1+\mathcal{E}_{0,n})^{\frac{s}{2}}(1+\mathcal{E}_{0,n+1})+(1+\mathcal{E}_{0,n})^{\frac{s+1}{2}}(1+\mathcal{E}_{0,n+1})\\&+\mathcal{E}_{0,n}^{\frac{5}{4}}\mathcal{D}_{0,n}^{\frac{3}{4}}+(1+\mathcal{E}_{0,n})^{s+2}\\\lesssim&(1+\mathcal{E}_{0,n}^{s+2})(1+\mathcal{D}_{0,n}^{\frac{3}{4}})(1+\mathcal{E}_{0,n+1}).\end{aligned}$$

According to the Gronwall inequality, it turns out that

$$1+\mathcal{E}_{0,n+1}(z,t)+\int_0^t\mathcal{D}_{0,n+1}(z,\tau)d\tau\le\exp^{\{\int_0^tC(1+\mathcal{E}_{0,n}^{s+2}(\tau))(1+\mathcal{D}_{0,n}^{\frac{3}{4}}(\tau))d\tau\}}(1+\mathcal{E}_{0,n+1}(z,0)).$$

Recall the definition of $\mathcal{E}_0^{in}$. It is obvious that

$$\begin{aligned}\mathcal{E}_{0,n+1}(z,0)=&\|\mathbb{J}_{2^{n+1}}\varrho^{in}(z)\|_{H^s}+\|\mathbb{J}_{2^{n+1}}\phi^{in}(z)\|_{H^s}+\|\mathbb{J}_{2^{n+1}}\psi^{in}(z)\|_{H^s}\\\le&\|\varrho^{in}(z)\|_{H^s}+\|\phi^{in}(z)\|_{H^s}+\|\psi^{in}(z)\|_{H^s}\le\epsilon_0^{in}.\end{aligned}$$

This implies

$$\mathcal{E}_{0,n+1}(z,t)+\int_0^t\mathcal{D}_{0,n+1}(z,\tau)d\tau\le\exp^{\{\int_0^tC(1+\mathcal{E}_{0,n}^{s+2}(\tau))(1+\mathcal{D}_{0,n}^{\frac{3}{4}}(\tau))d\tau\}}(1+\epsilon_0^{in}).\tag{3.8}$$

Define $M_{0,n}(z,t)=\sup\limits_{0\le\tau\le t}\mathcal{E}_{0,n}(z,\tau)+\int_0^t\mathcal{D}_{0,n}(z,\tau)\mathrm{d}\tau$. Since

$$\int_0^t(1+\mathcal{E}_{0,n}^{s+2}(\tau))(1+\mathcal{D}_{0,n}^{\frac{3}{4}}(\tau))d\tau\le\sup_{0\le\tau\le t}(1+\mathcal{E}_{0,n}^{s+2}(\tau))\Big(t+\big(\int_0^t\mathcal{D}_{0,n}(\tau)\mathrm{d}\tau\big)^{\frac{3}{4}}t^{\frac{1}{4}}\Big),$$

we deduce from (3.8) that

$$M_{0,n+1}(z,t)\le(1+\epsilon_0^{in})\exp^{\{C(1+M_{0,n}^{s+2}(z,t))(t+M_{0,n}^{\frac{3}{4}}(z,t)t^{\frac{1}{4}})\}}.$$

For any fixed number $M_0>1+\epsilon_0^{in}$, we take T_{M_0} sufficiently small such that

$$(1+\epsilon_0^{in})\exp^{\{C(1+M_0^{s+2})(T_{M_0}+M_0^{\frac{3}{4}}T_{M_0}^{\frac{1}{4}})\}}\le M_0.$$

The induction hypothesis then implies that for all $n\in\mathbb{N}$, $M_{0,n}(z,T_{M_0})\le M_0$. Note that M_0 and T_{M_0} are independent of z. We thus infer that for any $n\in\mathbb{N}$,

$$\sup_{z\in I}\big(\sup_{0\le\tau\le T_{M_0}}\mathcal{E}_{0,n}(z,\tau)+\int_0^{T_{M_0}}\mathcal{D}_{0,n}(z,\tau)\mathrm{d}\tau\big)\le M_0.\tag{3.9}$$

(b) Uniform bound for $(\partial_z\varrho^n(z,t,x),\partial_z\phi^n(z,t,x),\ \partial_z\psi^n(z,t,x))|_{n\in\mathbb{N}}$**.** Define

$$\begin{aligned}\mathcal{E}_{1,n}(z,t)=&\|\partial_z\varrho^n\|_{H^{s-1}}^2+\|\partial_z\phi^n\|_{H^{s-1}}^2+\|\partial_z\psi^n\|_{H^{s-1}}^2,\\\mathcal{D}_{1,n}(z,t)=&\mu(\|\nabla_x\partial_z\phi^n\|_{H^{s-1}}^2+\|\nabla_x\partial_z\psi^n\|_{H^{s-1}}^2),\quad n\in\mathbb{N}.\end{aligned}$$

Let

$$A_{0,n}(z,t)=1+\|\varrho^n\|_{H^s}^2+\|\phi^n\|_{H^s}^2+\|\psi^n\|_{H^s}^2+\|\varrho^{n+1}\|_{H^s}^2+\|\phi^{n+1}\|_{H^s}^2+\|\psi^{n+1}\|_{H^s}^2.$$

Following exactly along the same lines of (2.6) in Lemma 3.1 we deduce from (3.6) that

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{1,n+1}+\mathcal{D}_{1,n+1}\lesssim A_{0,n}^{s+1}(1+\mathcal{E}_{l,n+1}^{\frac{1}{2}}+\mathcal{E}_{l,n}^{\frac{1}{2}}+\mathcal{D}_{l,n}^{\frac{1}{2}})\mathcal{E}_{l,n+1}^{\frac{1}{2}}+A_{0,n}^{\frac{s+1}{2}}\mathcal{E}_{1,n}^{\frac{1}{2}}\mathcal{D}_{l,n+1}^{\frac{1}{2}}+A_{0,n}^{\frac{1}{2}}\mathcal{E}_{1,n}^{\frac{1}{8}}\mathcal{D}_{l,n+1}^{\frac{7}{8}}.$$

Applying the Young inequality, we infer that

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{1,n+1}+\mathcal{D}_{1,n+1}\leq CA_{0,n}^{2s+2}(1+\mathcal{E}_{1,n+1}+\mathcal{E}_{1,n})+\tfrac{1}{8}\mathcal{D}_{1,n}+\tfrac{1}{2}\mathcal{D}_{1,n+1}.$$

This implies

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(1+\mathcal{E}_{1,n+1})+\mathcal{D}_{1,n+1}\leq CA_{0,n}^{2s+2}(1+\mathcal{E}_{1,n+1})+CA_{0,n}^{2s+2}(1+\mathcal{E}_{1,n})+\tfrac{1}{4}\mathcal{D}_{1,n}.$$

The Gronwall inequality then leads to

$$\begin{aligned}1+\mathcal{E}_{1,n+1}(z,t)+\int_0^t\mathcal{D}_{1,n+1}(z,\tau)\mathrm{d}\tau\leq e^{\int_0^t CA_{0,n}^{2s+2}(z,\tau)\mathrm{d}\tau}\Big(1+\mathcal{E}_{1,n+1}(z,0)\\ +\int_0^t\big(CA_{0,n}^{2s+2}(z,\tau)(1+\mathcal{E}_{1,n}(z,\tau))+\tfrac{1}{4}\mathcal{D}_{1,n}(z,\tau)\big)\mathrm{d}\tau\Big).\end{aligned}\tag{3.10}$$

Denote $A=1+M_0$. According to the uniform bound (3.9), we have for all $0\leq t\leq T_{M_0}$,

$$\int_0^t A_{0,n}^{2s+2}(z,\tau)\mathrm{d}\tau\leq A^{2s+2}t.$$

Recall the definition of ϵ_1^{in}. It is obvious that

$$\begin{aligned}\mathcal{E}_{1,n+1}(z,0)=&\|\mathbb{J}_{2^{n+1}}\partial_z\varrho^{in}(z)\|_{H^{s-1}}+\|\mathbb{J}_{2^{n+1}}\partial_z\phi^{in}(z)\|_{H^{s-1}}+\|\mathbb{J}_{2^{n+1}}\partial_z\psi^{in}(z)\|_{H^{s-1}}\\ \leq&\|\partial_z\varrho^{in}(z)\|_{H^{s-1}}+\|\partial_z\phi^{in}(z)\|_{H^{s-1}}+\|\partial_z\psi^{in}(z)\|_{H^{s-1}}\leq\epsilon_1^{in}.\end{aligned}$$

Define $M_{1,n}(z,t)=\sup\limits_{0\leq\tau\leq t}\mathcal{E}_{1,n}(z,\tau)+\int_0^t\mathcal{D}_{1,n}(z,\tau)\mathrm{d}\tau$. Then it is easy to see

$$\begin{aligned}&\int_0^t\big(CA_{0,n}^{2s+2}(z,\tau)(1+\mathcal{E}_{1,n}(z,\tau))+\tfrac{1}{4}\mathcal{D}_{1,n}(z,\tau)\big)\mathrm{d}\tau\\ \leq&CA^{2s+2}t(1+M_{1,n}(z,t))+\tfrac{1}{4}M_{1,n}(z,t).\end{aligned}$$

Hence, we deduce from (3.10) that

$$M_{1,n+1}(z,t)\leq\exp^{CA^{2s+2}t}\big(1+\epsilon_1^{in}+CA^{2s+2}t(1+M_{1,n}(z,t))+\tfrac{1}{4}M_{1,n}(z,t)\big).$$

For any fixed number $M_1>4(1+\epsilon_1^{in})$, we take T_{M_1} sufficiently small ($T_{M_1}\leq T_{M_0}$) such that

$$\exp^{CA^{2s+2}T_{M_1}}\big(\tfrac{1}{4}M_1+CA^{2s+4}T_{M_1}(1+M_1)+\tfrac{1}{4}M_1\big)\leq M_1.$$

The induction hypothesis then implies that for all $n\in\mathbb{N}$, $M_{1,n}(z,T_{M_1})\leq M_1$. Note that M_1 and T_{M_1} are independent of z. This yields for any $n\in\mathbb{N}$,

$$\sup_{z\in I}\Big(\sup_{0\leq\tau\leq T_{M_1}}\mathcal{E}_{1,n}(z,\tau)+\int_0^{T_{M_1}}\mathcal{D}_{1,n}(z,\tau)\mathrm{d}\tau\Big)\leq M_1.\tag{3.11}$$

(c) Uniform bound for $(\partial_z^l\varrho^n(z,t,x),\partial_z^l\phi^n(z,t,x),\,\partial_z^l\psi^n(z,t,x))|_{n\in\mathbb{N}}$ **with** $0\le l\le q$**.**

Define

$$\mathcal{E}_{l,n}(z,t)=\|\partial_z^l\varrho^n\|_{H^{s-l}}^2+\|\partial_z^l\phi^n\|_{H^{s-l}}^2+\|\partial_z^l\psi^n\|_{H^{s-l}}^2,$$
$$D_{l,n}(z,t)=\mu(\|\nabla_x\partial_z^l\phi^n\|_{H^{s-l}}^2+\|\nabla_x\partial_z^l\psi^n\|_{H^{s-l}}^2),\quad n\in\mathbb{N}.$$

Repeating the same arguments as that for case (2.2), and thus will be omitted, we get that there exist a uniform positive time interval T_M and a uniform bound M, such that

$$\sup_{z\in I}\sum_{l=0}^q\Big(\sup_{0\le\tau\le T_M}\mathcal{E}_{l,n}(z,\tau)+\int_0^{T_M}\mathcal{D}_{l,n}(z,\tau)\mathrm{d}\tau\Big)\le M. \tag{3.12}$$

Step 3: Convergence.

For each $z\in I$, for any $n\in\mathbb{N}$, let

$$\begin{aligned}A_{q,n}(z)=&1+\sum_{l=0}^q\big(\|\partial_z^l\varrho^{n-1}(z)\|_{H^{s-l}}^2+\|\partial_z^l\phi^{n-1}(z)\|_{H^{s-l}}^2+\|\partial_z^l\psi^{n-1}(z)\|_{H^{s-l}}^2\\&+\|\partial_z^l\varrho^{n}(z)\|_{H^{s-l}}^2+\|\partial_z^l\phi^{n}(z)\|_{H^{s-l}}^2+\|\partial_z^l\psi^{n}(z)\|_{H^{s-l}}^2\\&+\|\partial_z^l\varrho^{n+1}(z)\|_{H^{s-l}}^2+\|\partial_z^l\phi^{n+1}(z)\|_{H^{s-l}}^2+\|\partial_z^l\psi^{n+1}(z)\|_{H^{s-l}}^2\big),\\B_{0,n}(z)=&\|\nabla_x\phi^{n-1}(z)\|_{H^s}^2+\|\nabla_x\psi^{n-1}(z)\|_{H^s}^2+\|\nabla_x\phi^{n}(z)\|_{H^s}^2+\|\nabla_x\psi^{n}(z)\|_{H^s}^2\big).\end{aligned}$$

$$X_{l,n}(z,t)=\|\partial_z^l(\varrho^n-\varrho^{n-1})\|_{H^{s-l-1}}^2+\|\partial_z^l(\phi^n-\phi^{n-1})\|_{H^{s-l-1}}^2+\|\partial_z^l(\psi^n-\psi^{n-1})\|_{H^{s-l-1}}^2,$$
$$Y_{l,n}(z,t)=\mu(\|\nabla_x\partial_z^l(\phi^n-\phi^{n-1})\|_{H^{s-l-1}}^2+\|\nabla_x\partial_z^l(\psi^n-\psi^{n-1})\|_{H^{s-l-1}}^2),$$

and

$$X_{sum,q,n}(z,t)=\sum_{l=0}^q X_{l,n}(z,t),\qquad Y_{sum,q,n}(z,t)=\sum_{l=0}^q Y_{l,n}(z,t).$$

Following exactly along the same lines of (2.6) in Lemma 2.2, we deduce from system (3.4) that the difference between $(\varrho^{n+1}-\varrho^n,\phi^{n+1}-\phi^n,\psi^{n+1}-\psi^n)$ satisfies

$$\begin{aligned}\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}X_{0,n+1}+Y_{0,n+1}\lesssim&B_{0,n}^{\frac12}A_{q,n}^{\frac{s}{2}}X_{0,n+1}+A_{q,n}^{\frac{s+2}{2}}(X_{0,n}^{\frac12}+Y_{0,n}^{\frac12})X_{0,n+1}^{\frac12}+B_{0,n}^{\frac12}X_{0,n}^{\frac12}X_{0,n+1}^{\frac12}\\&+B_{0,n}^{\frac38}A_{q,n}^{\frac18}X_{0,n}^{\frac12}Y_{0,n+1}^{\frac12}+A_{q,n}^{\frac{s+2}{2}}(X_{0,n}^{\frac12}+X_{0,n}^{\frac14}Y_{0,n}^{\frac14})Y_{0,n+1}^{\frac12}.\end{aligned}$$

Applying the Young inequality, we infer that

$$\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}X_{0,n+1}+Y_{0,n+1}\le CA_{q,n}^{2s+4}(1+B_{0,n}^{\frac34})(X_{0,n+1}+X_{0,n})+\tfrac{1}{2q}Y_{0,n+1}+\tfrac{1}{16q}Y_{0,n}. \tag{3.13}$$

For any $n\in\mathbb{N}$, define

$$X_{l,n}(z,t)=\|\partial_z^l(\varrho^n-\varrho^{n-1})\|_{H^{s-l-1}}^2+\|\partial_z^l(\phi^n-\phi^{n-1})\|_{H^{s-l-1}}^2+\|\partial_z^l(\psi^n-\psi^{n-1})\|_{H^{s-l-1}}^2,$$
$$Y_{l,n}(z,t)=\mu(\|\nabla_x\partial_z^l(\phi^n-\phi^{n-1})\|_{H^{s-l-1}}^2+\|\nabla_x\partial_z^l(\psi^n-\psi^{n-1})\|_{H^{s-l-1}}^2).$$

Following exactly along the same lines as that for (3.3) in Lemma 3.2, we deduce from system (3.6) that the difference between $(\partial_z^l(\varrho^{n+1}-\varrho^n),\partial_z^l(\phi^{n+1}-\phi^n),\partial_z^l(\psi^{n+1}-\psi^n))$ with $1\le l\le q$ satisfies

$$\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}X_{l,n+1}(z,t)+Y_{l,n+1}(z,t)\lesssim A_{q,n}^{s+2}(X_{sum,l,n+1}^{\frac12}+X_{sum,l,n}^{\frac12}+Y_{l,n+1}^{\frac12}+Y_{l,n}^{\frac12})X_{sum,l,n+1}^{\frac12}$$

$$+A_{q,n}^{\frac{s+2}{2}}X_{sum,l,n}^{\frac{1}{2}}Y_{l,n+1}^{\frac{1}{2}}+A_{q,n}^{\frac{s+2}{2}}X_{sum,l,n}^{\frac{1}{4}}Y_{l,n}^{\frac{1}{4}}Y_{l,n+1}^{\frac{1}{2}}.$$

The Young inequality leads to

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}X_{l,n+1}+Y_{l,n+1}\leq& CA_{q,n}^{2s+4}(X_{sum,l,n+1}+X_{sum,l,n})\\&+\frac{1}{2q}Y_{sum,l,n+1}+\frac{1}{16q}Y_{sum,l,n}.\end{aligned}\tag{3.14}$$

Combining the above inequality with (3.13), summing over l from 0 to q, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}X_{sum,q,n+1}+Y_{sum,q,n+1}\\&\leq CA_{q,n}^{2s+4}(1+B_{0,n}^{\frac{3}{4}})(X_{sum,q,n+1}+X_{sum,q,n})+\frac{1}{2}Y_{sum,l,n+1}+\frac{1}{16}Y_{sum,q,n},\end{aligned}$$

this yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}X_{sum,q,n+1}+Y_{sum,q,n+1}\\&\leq CA_{q,n}^{2s+4}(1+B_{0,n}^{\frac{3}{4}})(X_{sum,q,n+1}+X_{sum,q,n})+\frac{1}{8}Y_{sum,q,n}.\end{aligned}$$

Define

$$E_n(z,t)=\sup_{0\leq\tau\leq t}X_{sum,q,n}(z,\tau)+\int_0^t Y_{sum,q,n}(z,\tau)\mathrm{d}\tau.$$

The Gronwall inequality then leads to

$$\begin{aligned}&E_{n+1}(z,t)\leq e^{\int_0^t CA_{q,n}^{2s+4}(z,\tau)(1+B_{0,n}^{\frac{3}{4}}(z,\tau))\mathrm{d}\tau}\Big(E_{n+1}(z,0)\\&+\int_0^t\big(CA_{q,n}^{2s+4}(z,\tau)(1+B_{0,n}^{\frac{3}{4}}(z,\tau))X_{sum,q,n}(z,\tau)+\frac{1}{8}Y_{sum,q,n}(z,\tau)\big)\mathrm{d}\tau\Big).\end{aligned}\tag{3.15}$$

According to the uniform bounds (3.12), we have for all $0\leq t\leq T$,

$$\int_0^t A_{q,n}^{2s+4}(z,\tau)(1+B_{0,n}^{\frac{3}{4}}(z,\tau))\mathrm{d}\tau\leq M^{2s+4}(t+M^{\frac{3}{4}}t^{\frac{1}{4}}).$$

Plugging the above inequality into (3.15) yields

$$\begin{aligned}E_{n+1}(z,t)\leq&\exp^{CM^{2s+4}(t+M^{\frac{3}{4}}t^{\frac{1}{4}})}\big(E_{n+1}(z,0)\\&+CM^{2s+4}(t+M^{\frac{3}{4}}t^{\frac{1}{4}})E_n(z,t)+\frac{1}{8}E_n(z,t)\big).\end{aligned}\tag{3.16}$$

Recall the definition of $\mathbb{J}_{2^n}$. It can be checked that

$$\begin{aligned}E_{n+1}(z,0)=&\sum_{l=0}^{q}(\|\mathbb{J}_{2^{n+1}}\partial_z^l\varrho^{in}-\mathbb{J}_{2^n}\partial_z^l\varrho^{in}\|_{H^{s-l-1}}^2+\|\mathbb{J}_{2^{n+1}}\partial_z^l\phi^{in}-\mathbb{J}_{2^n}\partial_z^l\phi^{in}\|_{H^{s-l-1}}^2\\&+\|\mathbb{J}_{2^{n+1}}\partial_z^l\psi^{in}-\mathbb{J}_{2^n}\partial_z^l\psi^{in}\|_{H^{s-l-1}}^2)\\\leq&\frac{C}{2^{2n}}\sum_{l=0}^{q}(\|\partial_z^l\varrho^{in}\|_{H^{s-l}}^2+\|\partial_z^l\phi^{in}\|_{H^{s-l}}^2+\|\partial_z^l\psi^{in}\|_{H^{s-l}}^2)\\\leq&\frac{C}{4^n}(\epsilon_0^{in}+\epsilon_1^{in}).\end{aligned}$$

Choose T sufficiently small ($T\le T_M$) such that

$$\exp^{CM^{2s+4}(T+M^{\frac{3}{4}}T^{\frac{1}{4}})}\le 2,\quad \exp^{CM^{2s+4}(T+M^{\frac{3}{4}}T^{\frac{1}{4}})}(CM^{2s+4}(T+M^{\frac{3}{4}}T^{\frac{1}{4}})+\tfrac{1}{8})\le\tfrac{1}{4}.$$

Then we deduce from (3.16) that

$$E_{n+1}(z,T)\le\tfrac{2C}{4^n}(\epsilon_0^{in}+\epsilon_1^{in})+\tfrac{1}{4}E_n(z,T).$$

Note T is independent of z. We finally get

$$\sup_{z\in I}E_{n+1}(z,T)\le\tfrac{2C}{4^n}(\epsilon_0^{in}+\epsilon_1^{in})+\tfrac{1}{4}\sup_{z\in I}E_n(z,T). \tag{3.17}$$

This implies $(\partial_z^l\varrho^n(z),\partial_z^l\varrho^n\phi(z),\partial_z^l\psi(z))|_{n\in\mathbb{N}}$ are Cauchy sequences in $C([0,T];H^{s-l-1})$, $l=0,1,\cdots,q$. Moreover, according to (3.17), these sequences are uniformly convergence with respect to z. Hence, the property of uniform continuity guarantees that there exists a limit $(\varrho(z,x,t),\phi(z,x,t),\psi(z,x,t)$ such that $(\partial_z^l\varrho^n(z),\partial_z^l\varrho^n\phi(z),\partial_z^l\psi(z))$ converge to $(\partial_z^l\varrho(z),\partial_z^l\phi(z),\partial_z^l\psi(z))$ in $C([0,T];H^{s-l-1})$, $l=0,1,\cdots,q$. Moreover, from the uniform energy estimate (3.12), we deduce that for each $z\in I$, $(\partial_z^l\varrho,\partial_z^l\phi,\partial_z^l\psi)$ belongs to $L^\infty(0,T;H^{s-l})$, and $(\nabla_x\partial_z^l\phi,\nabla_x\partial_z^l\psi)$ belongs to $L^2(0,T;H^{s-l})$, $l=0,1,\cdots,q$. More precisely, we have

$$\begin{aligned}\sup_{z\in I}\sum_{l=0}^{q}\Big(\sup_{0\le\tau\le T}&(\|\partial_z^l\varrho(z,t)\|_{H^{s-l}}^2+\|\partial_z^l\phi(z,t)\|_{H^{s-l}}^2+\|\partial_z^l\psi(z,t)\|_{H^{s-l}}^2)\\&+\mu\int_0^T(\|\nabla_x\partial_z^l\phi(z,t)\|_{H^{s-l}}^2+\|\nabla_x\partial_z^l\psi(z,t)\|_{H^{s-l}}^2)\mathrm{d}t\Big)\le M.\end{aligned}$$

Passing to the limit in system (3.4) and (3.6), we conclude that (ϱ,ϕ,ψ) is indeed a solution of the random SOH system (1.3), and $(\partial_z^l\varrho,\partial_z^l\phi,\partial_z^l\psi)$ satisfies system (1.4). Thanks to the transport equation and transport diffusion equation theories again, we can also have $(\partial_z^l\varrho,\partial_z^l\phi,\partial_z^l\psi)\in C([0,T];H^{s-l})$.

Step 4: Uniqueness. Uniqueness is an immediate consequence of the stability result in the next subsection. This completes the whole proof of Theorem 1.2.

3.4. Proof of Theorem 1.3. This is just a matter of combining Lemma 2.2 and Lemma 3.2. Let

$$\begin{aligned}A_q(z,t)=1+&\sum_{l=0}^{q}(\|\partial_z^l\varrho(z,t)\|_{H^{s-l}}^2+\|\partial_z^l\phi(z,t)\|_{H^{s-l}}^2+\|\partial_z^l\psi(z,t)\|_{H^{s-l}}^2)\\&+\sum_{l=0}^{q}(\|\partial_z^l\bar\varrho(z,t)\|_{H^{s-l}}^2+\|\partial_z^l\bar\phi(z,t)\|_{H^{s-l}}^2+\|\partial_z^l\bar\psi(z,t)\|_{H^{s-l}}^2),\\B_0(z,t)=\|&\nabla_x\phi(z,t)\|_{H^s}^2+\|\nabla_x\psi(z,t)\|_{H^s}^2+\|\nabla_x\bar\phi(z,t)\|_{H^s}^2+\|\nabla_x\bar\psi(z,t)\|_{H^s}^2.\end{aligned}$$

Define

$$\begin{aligned}X_l(z,t)&=\|\partial_z^l(\varrho-\bar\varrho)(z,t)\|_{H^{s-l-1}}^2+\|\partial_z^l(\phi-\bar\phi)(z,t)\|_{H^{s-l-1}}^2+\|\partial_z^l(\psi-\bar\psi)(z,t)\|_{H^{s-l-1}}^2,\\Y_l(z,t)&=\mu(\|\nabla_x\partial_z^l(\phi-\bar\phi)(z,t)\|_{H^{s-l-1}}^2+\|\nabla_x\partial_z^l(\psi-\bar\psi)(z,t)\|_{H^{s-l-1}}^2),\end{aligned}$$

and

$$X_{sum,q}(z,t)=\sum_{l=0}^{q}X_l(z,t),\qquad Y_{sum,q}(z,t)=\sum_{l=0}^{q}Y_l(z,t).$$

Form Lemma 2.2, we have

$$\tfrac{d}{dt}X_0(t)+Y_0(t)\le CA_q^{2s+4}(t)(1+B_0^{\frac{3}{4}}(t))X_0(t).$$

Form Lemma 3.2, we have

$$\tfrac{\mathrm{d}}{\mathrm{d}t}X_l(z,t)+Y_l(z,t)\le CA_q^{2s+4}X_{sum,q},\ 1\le l\le q.$$

Summing over l from 0 to q, we get

$$\tfrac{\mathrm{d}}{\mathrm{d}t}X_{sum,q}(t)+Y_{sum,q}(t)\le CA_q^{2s+4}(1+B_0^{\frac{3}{4}})X_{sum,q}.$$

The Gronwall inequality then leads to the desired result.

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Appendix. We list some inequalities that are frequently used in Section 2 and Section 3, and give detailed proofs.

For the convection terms, we have the following estimate.

Lemma A.1. *For any integer $k\ge 0$, we have*

$$\langle\nabla_x^k(\vec{f}\cdot\nabla_x g),\nabla_x^k g\rangle\lesssim(\|\nabla_x\vec{f}\|_{H^2}+\|\nabla_x\vec{f}\|_{H^k})\|g\|_{H^k}^2.$$

Proof. Integrating by parts and using the Sobolev inequalities $H^1(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3), H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3), H^2(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$, we have

$$\begin{aligned}\langle\nabla_x^k(\vec{f}\cdot\nabla_x g),\nabla_x^k g\rangle=&\langle\vec{f}\cdot\nabla_x\nabla_x^k g,\nabla_x^k g\rangle+\langle\nabla_x\vec{f}\cdot\nabla_x\nabla_x^{k-1}g,\nabla_x^k g\rangle\\&+\sum_{\substack{k_1\ge 2,\\k_1+k_2=k}}\langle\nabla_x^{k_1}\vec{f}\cdot\nabla_x\nabla_x^{k_2}g,\nabla_x^k g\rangle\\\lesssim&\|\nabla_x\cdot\vec{f}\|_{L^\infty}\|\nabla_x^k g\|_{L^2}^2+\|\nabla_x\vec{f}\|_{L^\infty}\|\nabla_x^k g\|_{L^2}^2\\&+\sum_{\substack{k_1\ge 2,\\k_1+k_2=k}}\|\nabla_x^{k_1}\vec{f}\|_{L^3}\|\nabla_x^{k_2+1}g\|_{L^6}\|\nabla_x^k g\|_{L^2}\\\lesssim&\|\nabla_x\vec{f}\|_{H^2}\|g\|_{H^k}^2+\sum_{\substack{k_1\ge 2,\\k_1+k_2=k}}\|\nabla_x^{k_1}\vec{f}\|_{H^1}\|\nabla_x^{k_2+1}g\|_{H^1}\|\nabla_x^k g\|_{L^2}\\\lesssim&(\|\nabla_x\vec{f}\|_{H^2}+\|\nabla_x\vec{f}\|_{H^k})\|g\|_{H^k}^2.\end{aligned}$$

This completes the proof. ∎

In order to treat the terms like $\nabla_x\varrho\cdot\nabla_x\phi,\nabla_x\varrho\cdot(W^2\Omega_\phi)$, since ϱ is controlled by a transport equation, we use integration by parts to control $\nabla_x\varrho$. We present the following estimates.

Lemma A.2. *For any integer $k\ge 1$, we have*

$$\langle\nabla_x^k(\nabla_x f\cdot\vec{g}),\nabla_x^k h\rangle\lesssim\|f\|_{H^k}(\|\vec{g}\|_{L^\infty}+\|\nabla_x\vec{g}\|_{H^{k-1}})\|\nabla_x h\|_{H^k},$$

$$\langle \nabla_x^k(\nabla_x f\cdot\vec{g}),\nabla_x^k h\rangle \lesssim \|f\|_{H^k}(\|\vec{g}\|_{L^\infty}+\|\nabla_x\vec{g}\|_{H^{k-\frac{3}{2}}})\|\nabla_x h\|_{H^k},$$
$$\langle \nabla_x^k(\nabla_x f\cdot\vec{g}),\nabla_x^k h\rangle \lesssim \|f\|_{H^{k+1}}\|\vec{g}\|_{H^{k-\frac{1}{2}}}\|\nabla_x h\|_{H^k}.$$

Proof. Integration by parts yields

$$\begin{aligned}\langle \nabla_x^k(\nabla_x f\cdot\vec{g}),\nabla_x^k h\rangle &= -\langle \nabla_x^{k-1}(\nabla_x f\cdot\vec{g}),\nabla_x^{k+1}h\rangle\\ &= -\langle \nabla_x\nabla_x^{k-1}f\cdot\vec{g},\nabla_x^{k+1}h\rangle - \langle \sum_{\substack{k_2\geq 1,\\ k_1+k_2=k-1}} \nabla_x\nabla_x^{k_1}f\cdot\nabla_x^{k_2}\vec{g},\nabla_x^{k+1}h\rangle\\ &\triangleq I_1+I_2.\end{aligned}$$

It is obvious that the first part is controlled by

$$I_1 \lesssim (\|\nabla_x\nabla_x^{k-1}f\|_{L^2}\|\vec{g}\|_{L^\infty}\|\nabla_x^{k+1}h\|_{L^2} \lesssim \|f\|_{H^k}\|\vec{g}\|_{L^\infty}\|\nabla_x h\|_{H^k}.$$

Using $H^1(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$, the second part can be controlled by

$$\begin{aligned}I_2 &\lesssim \sum_{\substack{k_2\geq 1,\\ k_1+k_2=k-1}} \|\nabla_x\nabla_x^{k_1}f\|_{L^6}\|\nabla_x^{k_2}\vec{g}\|_{L^3}\|\nabla_x^{k+1}h\|_{L^2}\\ &\lesssim \sum_{\substack{k_2\geq 1,\\ k_1+k_2=k-1}} \|\nabla_x\nabla_x^{k_1}f\|_{H^1}\|\nabla_x^{k_2}\vec{g}\|_{H^1}\|\nabla_x^{k+1}h\|_{L^2}\\ &\lesssim \|f\|_{H^k}\|\nabla_x\vec{g}\|_{H^{k-1}}\|\nabla_x h\|_{H^k}.\end{aligned}$$

Hence, Lemma A.2-(1) is proved.

Alternatively, if we use embedding inequality $H^{\frac{1}{2}}(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3)$ instead of $H^1(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3)$, then for the second part, we have

$$I_2 \lesssim \|f\|_{H^k}\|\nabla_x\vec{g}\|_{H^{k-\frac{3}{2}}}\|\nabla_x h\|_{H^k}.$$

This leads to Lemma A.2-(2).

Similarly, using $H^{\frac{1}{2}}(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3)$ for the first part, we get

$$\begin{aligned}I_1 &\lesssim \|\nabla_x\nabla_x^{k-1}f\|_{L^6}\|\vec{g}\|_{L^3}\|\nabla_x^{k+1}h\|_{L^2}\\ &\lesssim \|\nabla_x\nabla_x^{k-1}f\|_{H^1}\|\vec{g}\|_{H^{\frac{1}{2}}}\|\nabla_x^{k+1}h\|_{L^2}\\ &\lesssim \|f\|_{H^{k+1}}\|\vec{g}\|_{H^{k-\frac{1}{2}}}\|\nabla_x h\|_{H^k}.\end{aligned}$$

Combining the above two inequalities yields Lemma A.2-(3). ∎

Terms like $\frac{\psi}{W}\nabla_x\phi\cdot\nabla_x\psi$ are treated as follows.

LEMMA A.3. *For any integer $k\geq 1$, we have*

$$\begin{aligned}\langle \nabla_x^k(f\nabla_x g\cdot\nabla_x h),\nabla_x^k u\rangle \lesssim \big(&\|f\|_{L^\infty}\|g\|_{H^k}\|\nabla_x h\|_{L^\infty}+\|f\|_{L^\infty}\|\nabla_x g\|_{L^\infty}\|h\|_{H^k}\\ &+(\|f\|_\infty+\|\nabla_x f\|_{H^{k-1}})\|g\|_{H^k}\|h\|_{H^k}\big)\|\nabla_x u\|_{H^k},\end{aligned}$$
$$\langle \nabla_x^k(f\nabla_x g\cdot\nabla_x h),\nabla_x^k u\rangle \lesssim (\|f\|_{L^\infty}+\|\nabla_x f\|_{H^{k-1}})\|\nabla_x g\|_{H^{k-\frac{1}{2}}}\|h\|_{H^{k+1}}\|\nabla_x u\|_{H^k},$$
$$\langle \nabla_x^k(f\nabla_x g\cdot\nabla_x h),\nabla_x^k u\rangle \lesssim \|f\|_{H^k}\|g\|_{H^{k+1}}\|h\|_{H^{k+1}}\|\nabla_x u\|_{H^k}.$$

Proof. Integration by parts yields

$$
\begin{aligned}
&\langle\nabla_x^k(f\nabla_x g\cdot\nabla_x h),\nabla_x^k u\rangle=-\langle\nabla_x^{k-1}(f\nabla_x g\cdot\nabla_x h),\nabla_x^{k+1}u\rangle\\
&=-\langle\sum_{k_1+k_2=k-1} f\nabla_x\nabla_x^{k_1}g\cdot\nabla_x\nabla_x^{k_2}h,\nabla_x^{k+1}u\rangle\\
&\quad+\langle\sum_{\substack{k_3\geq 1,\\ k_1+k_2+k_3=k-1}}\nabla_x^{k_3}f\nabla_x\nabla_x^{k_1}g\cdot\nabla_x\nabla_x^{k_2}h,\nabla_x^{k+1}u\rangle\\
&=-\langle f\nabla_x\nabla_x^{k-1}g\cdot\nabla_x h,\nabla_x^{k+1}u\rangle+\langle f\nabla_x g\cdot\nabla_x\nabla_x^{k-1}h,\nabla_x^{k+1}u\rangle\\
&\quad+\langle\sum_{\substack{k_1,k_2\geq 1,\\ k_1+k_2=k-1}} f\nabla_x\nabla_x^{k_1}g\cdot\nabla_x\nabla_x^{k_2}h,\nabla_x^{k+1}u\rangle\\
&\quad+\langle\sum_{\substack{k_3\geq 1,\\ k_1+k_2+k_3=k-1}}\nabla_x^{k_3}f\nabla_x\nabla_x^{k_1}g\cdot\nabla_x\nabla_x^{k_2}h,\nabla_x^{k+1}u\rangle\\
&\triangleq I_1+I_2+I_3+I_4.
\end{aligned}
$$

Using $H^1(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3), H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$, it can be checked that

$$
\begin{aligned}
I_1&\lesssim\|f\|_{L^\infty}\|\nabla_x\nabla_x^{k-1}g\|_{L^2}\|\nabla_x h\|_{L^\infty}\|\nabla_x^{k+1}u\|_{L^2}\\
&\lesssim\|f\|_{L^\infty}\|g\|_{H^k}\|\nabla_x h\|_{L^\infty}\|\nabla_x u\|_{H^k},\\
I_2&\lesssim\|f\|_{L^\infty}\|\nabla_x g\|_{L^\infty}\|\nabla_x\nabla_x^{k-1}h\|_{L^2}\|\nabla_x^{k+1}u\|_{L^2}\\
&\lesssim\|f\|_{L^\infty}\|\nabla_x g\|_{L^\infty}\|h\|_{H^k}\|\nabla_x u\|_{H^k},\\
I_3&\lesssim\sum_{\substack{k_1,k_2\geq 1,\\ k_1+k_2=k-1}}\|f\|_{L^\infty}\|\nabla_x\nabla_x^{k_1}g\|_{L^3}\|\nabla_x\nabla_x^{k_2}h\|_{L^6}\|\nabla_x^{k+1}u\|_{L^2}\\
&\lesssim\sum_{\substack{k_1,k_2\geq 1,\\ k_1+k_2=k-1}}\|f\|_{L^\infty}\|\nabla_x\nabla_x^{k_1}g\|_{H^1}\|\nabla_x\nabla_x^{k_2}h\|_{H^1}\|\nabla_x^{k+1}u\|_{L^2}\\
&\lesssim\|f\|_{L^\infty}\|\nabla_x g\|_{H^{k-1}}\|\nabla_x h\|_{H^{k-1}}\|\nabla_x u\|_{H^k},\\
I_4&\lesssim\sum_{\substack{k_3\geq 1,\\ k_1+k_2+k_3=k-1}}\|\nabla_x^{k_3}f\|_{L^6}\|\nabla_x\nabla_x^{k_1}g\|_{L^6}\|\nabla_x\nabla_x^{k_2}h\|_{L^6}\|\nabla_x^{k+1}u\|_{L^2}\\
&\lesssim\sum_{\substack{k_3\geq 1,\\ k_1+k_2+k_3=k-1}}\|\nabla_x^{k_3}f\|_{H^1}\|\nabla_x\nabla_x^{k_1}g\|_{H^1}\|\nabla_x\nabla_x^{k_2}h\|_{H^1}\|\nabla_x u\|_{H^k}\\
&\lesssim\|\nabla_x f\|_{H^{k-1}}\|\nabla_x g\|_{H^{k-1}}\|\nabla_x h\|_{H^{k-1}}\|\nabla_x u\|_{H^k}.
\end{aligned}
$$

Combining the above four inequalities leads to Lemma A.3-(1).

Alternatively, applying $H^{\frac{1}{2}}(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3)$ to I_1 and I_2, we have

$$
\begin{aligned}
I_1&\lesssim\|f\|_{L^\infty}\|\nabla_x\nabla_x^{k-1}g\|_{L^3}\|\nabla_x h\|_{L^6}\|\nabla_x^{k+1}u\|_{L^2}\\
&\lesssim\|f\|_{L^\infty}\|\nabla_x\nabla_x^{k-1}g\|_{H^{\frac{1}{2}}}\|\nabla_x h\|_{H^1}\|\nabla_x^{k+1}u\|_{L^2}\\
&\lesssim\|f\|_{L^\infty}\|\nabla_x g\|_{H^{k-\frac{1}{2}}}\|h\|_{H^{k+1}}\|\nabla_x u\|_{H^k},\\
I_2&\lesssim\|f\|_{L^\infty}\|\nabla_x g\|_{L^3}\|\nabla_x\nabla_x^{k-1}h\|_{L^6}\|\nabla_x^{k+1}u\|_{L^2}\\
&\lesssim\|f\|_{L^\infty}\|\nabla_x g\|_{H^{\frac{1}{2}}}\|\nabla_x\nabla_x^{k-1}h\|_{H^1}\|\nabla_x u\|_{H^k}\\
&\lesssim\|f\|_{L^\infty}\|\nabla_x g\|_{H^{k-\frac{1}{2}}}\|h\|_{H^{k+1}}\|\nabla_x u\|_{H^k}.
\end{aligned}
$$

This leads to Lemma A.3-(2).

Finally, if L^∞-norm of f is not required, we can use the following estimates:

$$\begin{aligned}
I_1 \lesssim & \|f\|_{L^6}\|\nabla_x\nabla_x^{k-1}g\|_{L^6}\|\nabla_x h\|_{L^6}\|\nabla_x^{k+1}u\|_{L^2}\\
\lesssim & \|f\|_{H^1}\|\nabla_x\nabla_x^{k-1}g\|_{H^1}\|\nabla_x h\|_{H^1}\|\nabla_x^{k+1}u\|_{L^2}\\
\lesssim & \|f\|_{H^k}\|g\|_{H^{k+1}}\|h\|_{H^{k+1}}\|\nabla_x u\|_{H^k},\\
I_2 \lesssim & \|f\|_{L^6}\|\nabla_x g\|_{L^6}\|\nabla_x\nabla_x^{k-1}h\|_{L^6}\|\nabla_x^{k+1}u\|_{L^2}\\
\lesssim & \|f\|_{H^k}\|g\|_{H^{k+1}}\|h\|_{H^{k+1}}\|\nabla_x u\|_{H^k},\\
I_3 \lesssim & \sum_{\substack{k_1,k_2\geq 1,\\ k_1+k_2=k-1}} \|f\|_{L^6}\|\nabla_x\nabla_x^{k_1}g\|_{L^6}\|\nabla_x\nabla_x^{k_2}h\|_{L^6}\|\nabla_x^{k+1}u\|_{L^2}\\
\lesssim & \sum_{\substack{k_1,k_2\geq 1,\\ k_1+k_2=k-1}} \|f\|_{H^1}\|\nabla_x\nabla_x^{k_1}g\|_{H^1}\|\nabla_x\nabla_x^{k_2}h\|_{H^1}\|\nabla_x^{k+1}u\|_{L^2}\\
\lesssim & \|f\|_{H^k}\|g\|_{H^k}\|h\|_{H^k}\|\nabla_x u\|_{H^k}.
\end{aligned}$$

Combining these above three inequalities with the inequality for I_4 gives Lemma A.3-(3). This completes the proof. ∎

The following lemmas are used in stability estimates.

LEMMA A.4. *For any integer $k\geq 0$, we have*

$$\langle \nabla_x^k(fg), \nabla_x^k h\rangle \lesssim \|f\|_{H^{k+1}}(\|g\|_{H^1}+\|g\|_{H^k})\|h\|_{H^k}.$$

Proof. Using $H^1(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3), H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3), H^2(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$, we have

$$\begin{aligned}
\langle \nabla_x^k(fg), \nabla_x^k h\rangle = & \langle \nabla_x^k f g, \nabla_x^k h\rangle + \sum_{\substack{k_2\geq 1,\\ k_1+k_2=k}} \langle \nabla_x^{k_1} f \nabla_x^{k_2} g, \nabla_x^k h\rangle\\
\lesssim & \|\nabla_x^k f\|_{L^3}\|g\|_{L^6}\|\nabla_x^k h\|_{L^2} + \sum_{\substack{k_2\geq 1,\\ k_1+k_2=k}} \|\nabla_x^{k_1} f\|_{L^\infty}\|\nabla_x^{k_2} g\|_{L^2}\|\nabla_x^k h\|_{L^2}\\
\lesssim & \|f\|_{H^{k+1}}(\|g\|_{H^1}+\|g\|_{H^k})\|h\|_{H^k}.
\end{aligned}$$

This completes the proof. ∎

LEMMA A.5. *For any integer $r\geq 0$, we have*

$$\|fg\|_{H^r} \lesssim (\|f\|_{L^\infty}+\|\nabla_x f\|_{H^r})\|g\|_{H^r}.$$

Specially, if $r\geq 1$, then we have

$$\|fg\|_{H^r} \lesssim \|f\|_{H^{r+1}}\|g\|_{H^r}.$$

Proof. For any integer $0\leq k\leq r$, we have

$$\nabla_x^k(fg) = f\nabla_x^k g + \sum_{\substack{k_1\geq 1,\\ k_1+k_2=k}} \nabla_x^{k_1} f \nabla_x^{k_2} g.$$

Applying the Sobolev inequalities $H^1(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$ yields

$$\|\nabla_x^k(fg)\|_{L^2} \leq \|f\|_{L^\infty}\|\nabla_x^k g\|_{L^2} + \sum_{\substack{k_1\geq 1,\\ k_1+k_2=k}} \|\nabla_x^{k_1} f\|_{L^6}\|\nabla_x^{k_2} g\|_{L^3}$$

$$\lesssim \|f\|_{L^\infty}\|\nabla_x^k g\|_{L^2} + \sum_{\substack{k_1\geq 1,\\ k_1+k_2=k}} \|\nabla_x^{k_1} f\|_{H^1}\|\nabla_x^{k_2} g\|_{H^1}$$
$$\lesssim (\|f\|_{L^\infty} + \|\nabla_x f\|_{H^k})\|g\|_{H^k}.$$

Summing over k from 0 to r then gives the first inequality. Specially, if $r\geq 1$, since $H^2(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$, we have

$$\|fg\|_{H^r} \lesssim (\|f\|_{L^\infty} + \|\nabla_x f\|_{H^r})\|g\|_{H^r} \lesssim \|f\|_{H^{r+1}}\|g\|_{H^r}.$$

This completes the proof. ∎

In the following, we treat terms related with Ω. Recall that

$$\Omega = (\tfrac{2\phi}{W}, \tfrac{2\psi}{W}, \tfrac{\phi^2+\psi^2-1}{W}) \quad \textit{with} \quad W = \phi^2+\psi^2+1.$$

It can be checked that

$$W^2\Omega_\phi = (2(1-\phi^2+\psi^2), -4\phi\psi, 4\phi),$$
$$W^2\Omega_\psi = (-4\phi\psi, 2(1+\phi^2-\psi^2), 4\psi).$$

We mention that due to the structure of Ω (similarly, $W^2\Omega_\phi, W^2\Omega_\psi$), it is not guaranteed that Ω belongs to L^2 if $\phi,\psi\in H^s$, whereas $\|\Omega\|_{L^\infty}$ and $\|\nabla_x\Omega\|_{L^2}$ are available. We then establish the following estimates.

LEMMA A.6. *Let $s\geq 2$. We have*

$$\begin{aligned}
&\|\tfrac{\phi}{W}\|_{L^\infty} + \|\tfrac{\psi}{W}\|_{L^\infty} \lesssim \|\Omega\|_{L^\infty} \lesssim 1,\\
&\|\tfrac{\phi}{W}\|_{H^s} + \|\tfrac{\psi}{W}\|_{H^s} \lesssim (1+\|\phi\|_{H^s}+\|\psi\|_{H^s})^s,\\
&\|\nabla_x\Omega\|_{H^{s-1}} \lesssim (1+\|\phi\|_{H^s}+\|\psi\|_{H^s})^s,\\
&\|\nabla_x\Omega\|_{H^s} \lesssim (\|\nabla_x\phi\|_{H^s}+\|\nabla_x\psi\|_{H^s})(1+\|\phi\|_{H^s}+\|\psi\|_{H^s})^s,\\
&\|W^2\Omega_\phi\|_{L^\infty} + \|W^2\Omega_\psi\|_{L^\infty} \lesssim (1+\|\phi\|_{H^s}+\|\psi\|_{H^s})^2,\\
&\|\nabla_x(W^2\Omega_\phi)\|_{H^{s-1}} + \|\nabla_x(W^2\Omega_\psi)\|_{H^{s-1}} \lesssim (1+\|\phi\|_{H^s}+\|\psi\|_{H^s})^2.
\end{aligned}$$

Proof. Recall $|\Omega|=1$. Since $\frac{2\phi}{W}$ and $\frac{2\psi}{W}$ are the first two components of vector Ω, we obviously have

$$\|\tfrac{\phi}{W}\|_{L^\infty} + \|\tfrac{\psi}{W}\|_{L^\infty} \lesssim \|\Omega\|_{L^\infty} \lesssim 1,$$
$$\|\nabla_x(\tfrac{\phi}{W})\|_{H^{s-1}} + \|\nabla_x(\tfrac{\psi}{W})\|_{H^{s-1}} \lesssim \|\nabla_x\Omega\|_{H^{s-1}}.$$

Next, by direct calculations, we have for $l\geq 1$,

$$\nabla_x^l\Omega = \sum_{m=1}^{l}\sum_{i+j=m}\frac{\partial^m\Omega}{\partial\phi^i\partial\psi^j}(\phi,\psi)T_{ij}(\phi,\psi), \tag{A.1}$$

where $T_{ij}(\phi,\psi) = \sum_{\substack{\sum_{\alpha=1}^{i}\tau_\alpha + \sum_{\beta=1}^{j}\varsigma_\beta = l,\\ \tau_\alpha,\varsigma_\beta\geq 1}} \nabla_x^{\tau_1}\phi\cdots\nabla_x^{\tau_i}\phi\nabla_x^{\varsigma_1}\psi\cdots\nabla_x^{\varsigma_j}\psi$. Each component of the unit vector field Ω is of the form of a rational fraction with lower power polynomial of the factor than that of the denominator. Its derivatives with respect to ϕ and ψ have the same structure. Hence, for any integer i_1, i_2, one has

$$\|\frac{\partial^{i_1+i_2}\Omega}{\partial\phi^{i_1}\partial\psi^{i_2}}\|_{L^\infty}\lesssim 1. \tag{A.2}$$

Thanks to the Sobolev inequalities $H^1(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3), H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3), H^2(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$, it can be checked that

$$\|\nabla_x\Omega\|_{H^{s-1}}\lesssim(1+\|\phi\|_{H^s}+\|\psi\|_{H^s})^s, \tag{A.3}$$

$$\|\nabla_x\Omega\|_{H^s}\lesssim(\|\nabla_x\phi\|_{H^s}+\|\nabla_x\psi\|_{H^s})(1+\|\phi\|_{H^s}+\|\psi\|_{H^s})^s. \tag{A.4}$$

The above two inequalities can be checked by following Lemma 2.3 in [17]. For the convenience of the reader, here we take $s=2$ as an example and give a verification. In fact, according to the expression (A.1), we have

$$\begin{aligned}\|\nabla_x\Omega\|_{H^1}\lesssim&(\|\nabla_x\phi\|_{L^2}+\|\nabla_x\psi\|_{L^2})+\big(\|\nabla_x^2\phi\|_{L^2}+\|\nabla_x^2\psi\|_{L^2}\\&+(\|\nabla_x\phi\|_{L^3}+\|\nabla_x\psi\|_{L^3})(\|\nabla_x\phi\|_{L^6}+\|\nabla_x\psi\|_{L^6})\big)\\\lesssim&(\|\nabla_x\phi\|_{L^2}+\|\nabla_x\psi\|_{L^2})+\big(\|\nabla_x^2\phi\|_{L^2}+\|\nabla_x^2\psi\|_{L^2}\\&+(\|\nabla_x\phi\|_{H^1}+\|\nabla_x\psi\|_{H^1})(\|\nabla_x\phi\|_{H^1}+\|\nabla_x\psi\|_{H^1})\big)\\\lesssim&(1+\|\phi\|_{H^2}+\|\psi\|_{H^2})^2,\end{aligned}$$

and

$$\begin{aligned}\|\nabla_x\Omega\|_{H^2}\lesssim&(\|\nabla_x\phi\|_{L^2}+\|\nabla_x\psi\|_{L^2})+\big(\|\nabla_x^2\phi\|_{L^2}+\|\nabla_x^2\psi\|_{L^2}\\&+(\|\nabla_x\phi\|_{L^3}+\|\nabla_x\psi\|_{L^3})(\|\nabla_x\phi\|_{L^6}+\|\nabla_x\psi\|_{L^6})\big)\\&+\big(\|\nabla_x^3\phi\|_{L^2}+\|\nabla_x^3\psi\|_{L^2}+(\|\nabla_x^2\phi\|_{L^3}+\|\nabla_x^2\psi\|_{L^3})(\|\nabla_x\phi\|_{L^6}+\|\nabla_x\psi\|_{L^6})\\&+(\|\nabla_x\phi\|_{L^6}+\|\nabla_x\psi\|_{L^6})^3\big)\\\lesssim&(\|\nabla_x\phi\|_{L^2}+\|\nabla_x\psi\|_{L^2})+\big(\|\nabla_x^2\phi\|_{L^2}+\|\nabla_x^2\psi\|_{L^2}\\&+(\|\nabla_x\phi\|_{H^1}+\|\nabla_x\psi\|_{H^1})(\|\nabla_x\phi\|_{H^1}+\|\nabla_x\psi\|_{H^1})\big)\\&+\big(\|\nabla_x^3\phi\|_{L^2}+\|\nabla_x^3\psi\|_{L^2}+(\|\nabla_x^2\phi\|_{H^1}+\|\nabla_x^2\psi\|_{H^1})(\|\nabla_x\phi\|_{H^1}+\|\nabla_x\psi\|_{H^1})\\&+(\|\nabla_x\phi\|_{H^1}+\|\nabla_x\psi\|_{H^1})^3\big)\\\lesssim&(\|\nabla_x\phi\|_{H^2}+\|\nabla_x\psi\|_{H^2})(1+\|\phi\|_{H^2}+\|\psi\|_{H^2})^2.\end{aligned}$$

This completes the verification.

Next, since $W\geq 1$, we deduce from (A.3) that

$$\begin{aligned}\|\tfrac{\phi}{W}\|_{H^s}+\|\tfrac{\psi}{W}\|_{H^s}\lesssim&\|\tfrac{\phi}{W}\|_{L^2}+\|\tfrac{\psi}{W}\|_{L^2}+\|\nabla_x\tfrac{\phi}{W}\|_{H^{s-1}}+\|\nabla_x\tfrac{\psi}{W}\|_{H^{s-1}}\\\lesssim&\|\phi\|_{L^2}+\|\psi\|_{L^2}+\|\nabla_x\Omega\|_{H^{s-1}}\\\lesssim&(1+\|\phi\|_{H^s}+\|\psi\|_{H^s})^s.\end{aligned}$$

Finally, applying $H^2(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$, we get that

$$\|W^2\Omega_\phi\|_{L^\infty}+\|W^2\Omega_\psi\|_{L^\infty}\lesssim(1+\|\phi\|_{L^\infty}+\|\psi\|_{L^\infty})^2\lesssim(1+\|\phi\|_{H^s}+\|\psi\|_{H^s})^2.$$

Since H^s with $s\geq 2$ is a Banach algebra, we infer that

$$\|\nabla_x(W^2\Omega_\phi)\|_{H^{s-1}}+\|\nabla_x(W^2\Omega_\psi)\|_{H^{s-1}}$$

$$
\begin{aligned}
&=\|\nabla_x(2(-\phi^2+\psi^2),-4\phi\psi,4\phi)^T\|_{H^{s-1}}+\|\nabla_x(-4\phi\psi,2(\phi^2-\psi^2),4\psi)^T\|_{H^{s-1}}\\
&\lesssim\|(2(-\phi^2+\psi^2),-4\phi\psi,4\phi)^T\|_{H^s}+\|(-4\phi\psi,2(\phi^2-\psi^2),4\psi)^T\|_{H^s}\\
&\lesssim(1+\|\phi\|_{H^s}+\|\psi\|_{H^s})^2.
\end{aligned}
$$

This completes the proof of the lemma. □

The following lemma is used in stability estimates.

Lemma A.7. *Let $s\geq 2$, $\Omega=\Omega(\phi,\psi)=(\frac{2\phi}{W},\frac{2\psi}{W},\frac{\phi^2+\psi^2-1}{W})$ with $W=W(\phi,\psi)=\phi^2+\psi^2+1$, and $\bar\Omega=\Omega(\bar\phi,\bar\psi),\overline{W}=W(\bar\phi,\bar\psi),\overline{\Omega_\phi}=\Omega_\phi(\bar\phi,\bar\psi)$, then we have*

$$
\begin{aligned}
&\|\Omega-\bar\Omega\|_{H^s}\lesssim(1+\|\phi\|_{H^s}+\|\psi\|_{H^s}+\|\bar\phi\|_{H^s}+\|\bar\psi\|_{H^s})^s\\
&\qquad\times(\|\phi-\bar\phi\|_{H^{s-1}}+\|\psi-\bar\psi\|_{H^{s-1}}+\|\nabla_x(\phi-\bar\phi)\|_{H^{s-1}}+\|\nabla_x(\psi-\bar\psi)\|_{H^{s-1}}),\\
&\|\Omega-\bar\Omega\|_{H^{s-1}}\lesssim(1+\|\phi\|_{H^s}+\|\psi\|_{H^s}+\|\bar\phi\|_{H^s}+\|\bar\psi\|_{H^s})^s(\|\phi-\bar\phi\|_{H^{s-1}}+\|\psi-\bar\psi\|_{H^{s-1}}),\\
&\|W^2\Omega_\phi-\overline{W}^2\overline{\Omega_\phi}\|_{H^{s-1}}+\|W^2\Omega_\psi-\overline{W}^2\overline{\Omega_\psi}\|_{H^{s-1}}\\
&\quad\lesssim(1+\|\phi\|_{H^s}+\|\psi\|_{H^s}+\|\bar\phi\|_{H^s}+\|\bar\psi\|_{H^s})(\|\phi-\bar\phi\|_{H^{s-1}}+\|\psi-\bar\psi\|_{H^{s-1}}).
\end{aligned}
$$

Proof. Define $O(r)=\Omega(r\phi+(1-r)\bar\phi,r\psi+(1-r)\bar\psi)$. The Newton-Leibniz formula gives

$$
O(1)-O(0)=\int_0^1 O'(r)dr,
$$

from which it follows

$$
\begin{aligned}
\Omega-\bar\Omega=\Omega(\phi,\psi)-\Omega(\bar\phi,\bar\psi)=\int_0^1&[\Omega_\phi(r\phi+(1-r)\bar\phi,r\psi+(1-r)\bar\psi)(\phi-\bar\phi)\\
&+\Omega_\psi(r\phi+(1-r)\bar\phi,r\psi+(1-r)\bar\psi)(\psi-\bar\psi)]dr. \qquad (A.5)
\end{aligned}
$$

According to (A.2), we have

$$
\|\Omega_\phi(r\phi+(1-r)\bar\phi,r\psi+(1-r)\bar\psi)\|_{L^\infty}+\|\Omega_\psi(r\phi+(1-r)\bar\phi,r\psi+(1-r)\bar\psi)\|_{L^\infty}\lesssim 1. \quad (A.6)
$$

Since Ω_ϕ, Ω_ψ have the same structure as Ω, replacing Ω with Ω_ϕ and Ω_ψ in (A.2), following along the same lines for (A.3), we deduce that

$$
\begin{aligned}
&\|\nabla_x(\Omega_\phi(r\phi+(1-r)\bar\phi,r\psi+(1-r)\bar\psi))\|_{H^{s-1}}\\
&\quad+\|\nabla_x(\Omega_\psi(r\phi+(1-r)\bar\phi,r\psi+(1-r)\bar\psi))\|_{H^{s-1}}\\
&\qquad\lesssim(1+\|r\phi+(1-r)\bar\phi\|_{H^s}+\|r\psi+(1-r)\bar\psi\|_{H^s})^s\\
&\qquad\lesssim(1+\|\phi\|_{H^s}+\|\bar\phi\|_{H^s}+\|\psi\|_{H^s}+\|\bar\psi\|_{H^s})^s. \qquad (A.7)
\end{aligned}
$$

Recall the Moser-type calculus inequality,

$$
\|fg\|_{H^m}\lesssim(\|f\|_{L^\infty}\|\nabla_x^m g\|_{L^2}+\|\nabla_x^m f\|_{L^2}\|g\|_{L^\infty}).
$$

Since $\|g\|_{L^\infty}\lesssim\|g\|_{H^2}$, we infer that if $s\geq 2$, then

$$
\|fg\|_{H^s}\lesssim(\|f\|_{L^\infty}+\|\nabla_x f\|_{H^{s-1}})\|g\|_{H^s}. \qquad (A.8)
$$

Applying the above inequality to (A.5), together with (A.6) and (A.7), we obtain

$$
\|\Omega-\bar\Omega\|_{H^s}\lesssim\int_0^1(\|\Omega_\phi(r\phi+(1-r)\bar\phi,r\psi+(1-r)\bar\psi)\|_\infty
$$

$$
\begin{aligned}
&+\|\nabla_x\Omega_\phi(r\phi+(1-r)\bar\phi,r\psi+(1-r)\bar\psi)\|_{H^{s-1}})\|\phi-\bar\phi\|_{H^s}\\
&+(\|\Omega_\psi(r\phi+(1-r)\bar\phi,r\psi+(1-r)\bar\psi)\|_\infty\\
&+\|\nabla_x\Omega_\psi(r\phi+(1-r)\bar\phi,r\psi+(1-r)\bar\psi)\|_{H^{s-1}})\|\psi-\bar\psi\|_{H^s}\mathrm{d}r\\
\lesssim&(1+\|\phi\|_{H^s}+\|\bar\phi\|_{H^s}+\|\psi\|_{H^s}+\|\bar\psi\|_{H^s})^s(\|\phi-\bar\phi\|_{H^s}+\|\psi-\bar\psi\|_{H^s})\\
\lesssim&(1+\|\phi\|_{H^s}+\|\psi\|_{H^s}+\|\bar\phi\|_{H^s}+\|\bar\psi\|_{H^s})^s\\
&\times(\|\phi-\bar\phi\|_{H^{s-1}}+\|\psi-\bar\psi\|_{H^{s-1}}+\|\nabla_x(\phi-\bar\phi)\|_{H^{s-1}}+\|\nabla_x(\psi-\bar\psi)\|_{H^{s-1}}).
\end{aligned}
$$

One the other hand, applying Lemma A.5-(1) to (A.5) , we have

$$
\begin{aligned}
&\|\Omega-\bar\Omega\|_{H^{s-1}}\\
\lesssim&\int_0^1(\|\Omega_\phi(r\phi+(1-r)\bar\phi,r\psi+(1-r)\bar\psi)\|_\infty\\
&+\|\nabla_x\Omega_\phi(r\phi+(1-r)\bar\phi,r\psi+(1-r)\bar\psi)\|_{H^{s-1}})\|\phi-\bar\phi\|_{H^{s-1}}\\
&+(\|\Omega_\psi(r\phi+(1-r)\bar\phi,r\psi+(1-r)\bar\psi)\|_\infty\\
&+\|\nabla_x\Omega_\psi(r\phi+(1-r)\bar\phi,r\psi+(1-r)\bar\psi)\|_{H^{s-1}})\|\psi-\bar\psi\|_{H^{s-1}}\mathrm{d}r\\
\lesssim&(1+\|\phi\|_{H^s}+\|\bar\phi\|_{H^s}+\|\psi\|_{H^s}+\|\bar\psi\|_{H^s})^s(\|\phi-\bar\phi\|_{H^{s-1}}+\|\psi-\bar\psi\|_{H^{s-1}}).
\end{aligned}
$$

Recall

$$
\begin{aligned}
&W^2\Omega_\phi-\overline{W}^2\overline{\Omega_\phi}\\
&\quad=(-2(\phi+\bar\phi)(\phi-\bar\phi)+2(\psi+\bar\psi)(\psi-\bar\psi),-4\psi(\phi-\bar\phi)-4\bar\phi(\psi-\bar\psi),4(\phi-\bar\phi)).
\end{aligned}
$$

According to Lemma A.5-(2), we have

$$
\begin{aligned}
&\|W^2\Omega_\phi-\overline{W}^2\overline{\Omega_\phi}\|_{H^{s-1}}\\
&\quad\lesssim(1+\|\phi\|_{H^s}+\|\bar\phi\|_{H^s}+\|\psi\|_{H^s}+\|\bar\psi\|_{H^s})(\|\phi-\bar\phi\|_{H^{s-1}}+\|\psi-\bar\psi\|_{H^{s-1}}).
\end{aligned}
$$

$W^2\Omega_\psi-\overline{W}^2\overline{\Omega_\psi}$ can be controlled in the same way. This completes the whole proof. □

Next, we treat the terms related with derivatives of random variable z.

LEMMA A.8. *Let $l\geq 1$ and $s\geq 2+l$. Define*

$$
A_{l-1}(z,t)=1+\sum_{m=0}^{l-1}(\|\partial_z^m\varrho(z,t)\|_{H^{s-m}}^2+\|\partial_z^m\phi(z,t)\|_{H^{s-m}}^2+\|\partial_z^m\psi(z,t)\|_{H^{s-m}}^2).
$$

Then we have

$$
\begin{aligned}
&\|\partial_z^m\tfrac{\phi}{W}\|_{H^{s-l}}+\|\partial_z^m\tfrac{\phi}{W}\|_{H^{s-l}}\lesssim\|\partial_z^m\Omega\|_{H^{s-l}}\lesssim A_{l-1}^s,\ \textit{if}\ 1\leq m\leq l-1,\\
&\|\partial_z^l\tfrac{\phi}{W}\|_{H^{s-l}}+\|\partial_z^l\tfrac{\phi}{W}\|_{H^{s-l}}\lesssim\|\partial_z^l\Omega\|_{H^{s-l}}\lesssim A_{l-1}^s(\|\partial_z^l\phi\|_{H^{s-l}}+\|\partial_z^l\psi\|_{H^{s-l}}+1),\\
&\|\nabla_x\partial_z^l\Omega\|_{H^{s-l}}\lesssim A_{l-1}^s((\|\nabla_x\partial_z^l\phi\|_{H^{s-l}}+\|\nabla_x\partial_z^l\psi\|_{H^{s-l}}+\|\partial_z^l\phi\|_{H^{s-l}}+\|\partial_z^l\psi\|_{H^{s-l}}+1),\\
&\|\partial_z^m(W^2\Omega_\phi)\|_{H^{s-l}}\lesssim A_{l-1},\ \textit{if}\ 1\leq m\leq l-1,\\
&\|\partial_z^l(W^2\Omega_\phi)\|_{H^{s-l}}\lesssim A_{l-1}(\|\partial_z^l\phi\|_{H^{s-l}}+\|\partial_z^l\psi\|_{H^{s-l}}+1).
\end{aligned}
$$

Proof. For any $1\leq m\leq l$, by direct calculations, we have

$$
\partial_z^m\Omega=\sum_{k=1}^m\sum_{i+j=k}\frac{\partial^k\Omega}{\partial\phi^i\partial\psi^j}(\phi,\psi)T_{ij}(\phi,\psi), \tag{A.9}
$$

where $T_{ij}(\phi,\psi)=\sum_{\substack{\sum_{\alpha=1}^{i}\tau_\alpha+\sum_{\beta=1}^{j}\varsigma_\beta=m,\\ \tau_\alpha,\varsigma_\beta\geq 1}}\partial_z^{\tau_1}\phi\cdots\partial_z^{\tau_i}\phi\partial_z^{\varsigma_1}\psi\cdots\partial_z^{\varsigma_j}\psi$. Note that $\frac{\partial^k\Omega}{\partial\phi^i\partial\psi^j}$ has the same structure as that of Ω. Hence, following exactly along the same lines as that for Lemma A.6, we deduce that

$$\begin{aligned}\|\nabla_x\frac{\partial^k\Omega}{\partial\phi^i\partial\psi^j}\|_{H^{s-l}}+\|\frac{\partial^k\Omega}{\partial\phi^i\partial\psi^j}\|_{L^\infty}&\lesssim\|\nabla_x\frac{\partial^k\Omega}{\partial\phi^i\partial\psi^j}\|_{H^{s-1}}+\|\frac{\partial^k\Omega}{\partial\phi^i\partial\psi^j}\|_{L^\infty}\\&\lesssim(1+\|\phi\|_{H^s}+\|\psi\|_{H^s})^s\lesssim A_{l-1}^{\frac{s}{2}}.\end{aligned}\tag{A.10}$$

Recall H^{s-l} is a Banach algebra, it is easy to see that if $1\leq m\leq l-1$, then

$$\|T_{ij}\|_{H^{s-l}}\lesssim A_{l-1}^{\frac{m}{2}},$$

which leads to

$$\|\partial_z^m\Omega\|_{H^{s-l}}\lesssim\sum_{k=1}^{m}\sum_{i+j=k}(\|\nabla_x\frac{\partial^k\Omega}{\partial\phi^i\partial\psi^j}\|_{H^{s-l-1}}+\|\frac{\partial^k\Omega}{\partial\phi^i\partial\psi^j}\|_{L^\infty})\|T_{ij}\|_{H^{s-l}}\lesssim A_{l-1}^s,$$

where we have used (A.10) and (A.8).

On the other hand, if $m=l$, then

$$\|T_{ij}\|_{H^{s-l}}\lesssim(\|\partial_z^l\phi\|_{H^{s-l}}+\|\partial_z^l\psi\|_{H^{s-l}})A_{l-1}^{\frac{m-1}{2}}+A_{l-1}^{\frac{m}{2}},$$

from which it follows

$$\begin{aligned}\|\partial_z^l\Omega\|_{H^{s-l}}&\lesssim\sum_{k=1}^{l}\sum_{i+j=k}(\|\nabla_x\frac{\partial^k\Omega}{\partial\phi^i\partial\psi^j}\|_{H^{s-l-1}}+\|\frac{\partial^k\Omega}{\partial\phi^i\partial\psi^j}\|_{L^\infty})\|T_{ij}\|_{H^{s-l}}\\&\lesssim A_{l-1}^s(\|\partial_z^l\phi\|_{H^{s-l}}+\|\partial_z^l\psi\|_{H^{s-l}}+1).\end{aligned}$$

Recall $\frac{2\phi}{W}$ and $\frac{2\psi}{W}$ are the first two components of vector Ω, then the first three inequalities hold true.

Next, if $m=l$, we derive from (A.9) that

$$\nabla_x\partial_z^l\Omega=\sum_{k=1}^{l}\sum_{i+j=k}(\nabla_x\frac{\partial^k\Omega}{\partial\phi^i\partial\psi^j}(\phi,\psi)T_{ij}(\phi,\psi)+\frac{\partial^k\Omega}{\partial\phi^i\partial\psi^j}(\phi,\psi)\nabla_xT_{ij}(\phi,\psi)).\tag{A.11}$$

It is easy to check that

$$\|\nabla_xT_{ij}\|_{H^{s-l}}\lesssim(\|\nabla_x\partial_z^l\phi\|_{H^{s-l}}+\|\nabla_x\partial_z^l\psi\|_{H^{s-l}}+\|\partial_z^l\phi\|_{H^{s-l}}+\|\partial_z^l\psi\|_{H^{s-l}})A_{l-1}^{\frac{m-1}{2}}+A_{l-1}^{\frac{m}{2}}.$$

As a result, we deduce that

$$\begin{aligned}\|\nabla_x\partial_z^m\Omega\|_{H^{s-l}}\lesssim\sum_{k=1}^{l}\sum_{i+j=k}\Big(&\|\nabla_x\frac{\partial^k\Omega}{\partial\phi^i\partial\psi^j}\|_{H^{s-l}}\|T_{ij}\|_{H^{s-l}}\\&+(\|\nabla_x\frac{\partial^k\Omega}{\partial\phi^i\partial\psi^j}\|_{H^{s-l-1}}+\|\tfrac{\partial^k\Omega}{\partial\phi^i\partial\psi^j}\|_{L^\infty})\|\nabla_xT_{ij}\|_{H^{s-l}}\Big)\end{aligned}$$

$$\lesssim A_{l-1}^{s}((\|\nabla_x\partial_z^l\phi\|_{H^{s-l}}+\|\nabla_x\partial_z^l\psi\|_{H^{s-l}}+\|\partial_z^l\phi\|_{H^{s-l}}+\|\partial_z^l\psi\|_{H^{s-l}}+1).$$

Finally, note that

$$\partial_z^m(W^2\Omega_\phi)=(2\sum_{i+j=m}(-\partial_z^i\phi\partial_z^j\phi+\partial_z^i\psi\partial_z^j\psi),-4\sum_{i+j=m}\partial_z^i\phi\partial_z^j\psi,4\partial_z^m\phi), \tag{A.12}$$

$$\partial_z^m(W^2\Omega_\psi)=(-4\sum_{i+j=m}\partial_z^i\phi\partial_z^j\psi,2\sum_{i+j=m}(\partial_z^i\phi\partial_z^j\phi-\partial_z^i\psi\partial_z^j\psi),4\partial_z^m\psi). \tag{A.13}$$

It is easy to see

$$\begin{aligned}&\|\partial_z^m(W^2\Omega_\phi)\|_{H^{s-l}}+\|\partial_z^m(W^2\Omega_\psi)\|_{H^{s-l}}\\ &\qquad\lesssim\sum_{i+j=m}(\|\partial_z^i\phi\|_{H^{s-l}}+\|\partial_z^i\phi\|_{H^{s-l}}+1)(\|\partial_z^j\phi\|_{H^{s-l}}+\|\partial_z^j\psi\|_{H^{s-l}}+1),\end{aligned}$$

from which it follows that if $1\le m\le l-1$, then

$$\|\partial_z^m(W^2\Omega_\phi)\|_{H^{s-l}}+\|\partial_z^m(W^2\Omega_\psi)\|_{H^{s-l}}\lesssim A_{l-1},$$

and if $m=l$, then

$$\begin{aligned}\|\partial_z^l(W^2\Omega_\phi)\|_{H^{s-l}}+\|\partial_z^l(W^2\Omega_\psi)\|_{H^{s-l}}&\lesssim A_{l-1}+A_{l-1}^{\frac{1}{2}}(\|\partial_z^l\phi\|_{H^{s-l}}+\|\partial_z^l\psi\|_{H^{s-l}})\\ &\lesssim A_{l-1}(\|\partial_z^l\phi\|_{H^{s-l}}+\|\partial_z^l\psi\|_{H^{s-l}}+1).\end{aligned}$$

This completes the whole proof. ∎

LEMMA A.9. *Let $l\ge 1$ and $s\ge 2+l$, $\phi=\phi(z,x),\psi=\psi(z,x),\bar\phi=\bar\phi(z,x),\bar\psi=\bar\psi(z,x)$. Define $\Omega=\Omega(\phi,\psi)=(\frac{2\phi}{W},\frac{2\psi}{W},\frac{\phi^2+\psi^2-1}{W})$, $W=W(\phi,\psi)=\phi^2+\psi^2+1$, and $\bar\Omega=\Omega(\bar\phi,\bar\psi),\overline{W}=W(\bar\phi,\bar\psi)$, $\bar\Omega_\phi=\Omega_\phi(\bar\phi,\bar\psi)$, $\bar\Omega_\psi=\Omega_\psi(\bar\phi,\bar\psi)$. Let*

$$\begin{aligned}A_l(z,t)=&1+\sum_{m=0}^{l}(\|\partial_z^m\varrho(z,t)\|_{H^{s-m}}^2+\|\partial_z^m\phi(z,t)\|_{H^{s-m}}^2+\|\partial_z^m\psi(z,t)\|_{H^{s-m}}^2)\\ &+\sum_{m=0}^{l}(\|\partial_z^m\bar\varrho(z,t)\|_{H^{s-m}}^2+\|\partial_z^m\bar\phi(z,t)\|_{H^{s-m}}^2+\|\partial_z^m\bar\psi(z,t)\|_{H^{s-m}}^2),\\ X_{sum,l}(z,t)=&\sum_{m=0}^{l}(\|\partial_z^m(\varrho-\bar\varrho)(z,t)\|_{H^{s-m-1}}^2+\|\partial_z^m(\phi-\bar\phi)(z,t)\|_{H^{s-m-1}}^2\\ &+\|\partial_z^m(\psi-\bar\psi)(z,t)\|_{H^{s-m-1}}^2),\\ Y_l(z,t)=&\|\nabla_x\partial_z^l(\phi-\bar\phi)(z,t)\|_{H^{s-m-1}}^2+\|\nabla_x\partial_z^l(\psi-\bar\psi)(z,t)\|_{H^{s-m-1}}^2.\end{aligned}$$

Then we have for all $1\le m\le l$,

$$\|\partial_z^m(\Omega-\bar\Omega)\|_{H^{s-l-1}}\lesssim A_l^{s+1}X_{sum,l}^{\frac{1}{2}}$$

$$\|\partial_z^m(\Omega-\bar\Omega)\|_{H^{s-l}}\lesssim A_l^{s+1}(X_{sum,l}^{\frac{1}{2}}+Y_l^{\frac{1}{2}})$$

$$\|\partial_z(W^2\Omega_\phi-\overline{W}^2\bar\Omega_\phi)\|_{H^{s-l-1}}+\|\partial_z(W^2\Omega_\psi-\overline{W}^2\bar\Omega_\psi)\|_{H^{s-l-1}}\lesssim A_l^{\frac{1}{2}}X_{sum,l}^{\frac{1}{2}}.$$

Proof. According to the relation (A.5), we have

$$\partial_z^m(\Omega-\bar\Omega)=\int_0^1\Omega_\phi(r\phi+(1-r)\bar\phi,r\psi+(1-r)\bar\psi)\partial_z^m(\phi-\bar\phi)\mathrm{d}r$$

$$
\begin{aligned}
&+\sum_{k=1}^{m}\int_0^1 \partial_z^k\Omega_\phi(r\phi+(1-r)\bar{\phi},r\psi+(1-r)\bar{\psi})\partial_z^{m-k}(\phi-\bar{\phi})dr\\
&+\int_0^1 \Omega_\psi(r\phi+(1-r)\bar{\phi},r\psi+(1-r)\bar{\psi})\partial_z^{m}(\psi-\bar{\psi})\mathrm{d}r\\
&+\sum_{k=1}^{m}\int_0^1 \partial_z^k\Omega_\psi(r\phi+(1-r)\bar{\phi},r\psi+(1-r)\bar{\psi})\partial_z^{m-k}(\psi-\bar{\psi})dr\\
&\triangleq I_1+I_2+II_1+II_2.
\end{aligned}
$$

For $1\le m\le l$, from inequalities (A.6) and (A.7), we have

$$
\begin{aligned}
&\|\Omega_\phi(r\phi+(1-r)\bar{\phi},r\psi+(1-r)\bar{\psi})\|_{L^\infty}\\
&\qquad\qquad +\|\nabla_x\Omega_\phi(r\phi+(1-r)\bar{\phi},r\psi+(1-r)\bar{\psi})\|_{H^{s-1}})\lesssim A_l^{\frac{s}{2}}.
\end{aligned}
$$

Since Ω and Ω_ϕ have the same structure, similar to Lemma A.8, we deduce that

$$
\sum_{k=1}^{m}\|\partial_z^k\Omega_\phi(r\phi+(1-r)\bar{\phi},r\psi+(1-r)\bar{\psi})\|_{H^{s-l}}\lesssim A_l^{s+1}.
$$

According to Lemma A.5-1 and A.5-2, we have

$$
\begin{aligned}
&\|I_1\|_{H^{s-l-1}}+\|I_2\|_{H^{s-l-1}}\\
&\quad\lesssim\int_0^1(\|\Omega_\phi(r\phi+(1-r)\bar{\phi},r\psi+(1-r)\bar{\psi})\|_{L^\infty}\\
&\quad\quad+\|\nabla_x\Omega_\phi(r\phi+(1-r)\bar{\phi},r\psi+(1-r)\bar{\psi})\|_{H^{s-l-1}})\|\partial_z^m(\phi-\bar{\phi})\|_{H^{s-l-1}})\\
&\quad\quad+\sum_{k=1}^{m}\int_0^1\|\partial_z^k\Omega_\phi(r\phi+(1-r)\bar{\phi},r\psi+(1-r)\bar{\psi})\|_{H^{s-l}}\|\partial_z^{m-k}(\phi-\bar{\phi})\|_{H^{s-l-1}}\mathrm{d}r\\
&\quad\lesssim A_1^{s+1}X_{sum,l}^{\frac{1}{2}}.
\end{aligned}
$$

On the other hand, according to Lemma A.5-1 and by using the fact that H^{s-l} is a Banach algebra, we have

$$
\begin{aligned}
&\|I_1\|_{H^{s-l}}+\|I_2\|_{H^{s-l}}\\
&\lesssim(\int_0^1(\|\Omega_\phi(r\phi+(1-r)\bar{\phi},r\psi+(1-r)\bar{\psi})\|_{L^\infty}\\
&\quad+\|\nabla_x\Omega_\phi(r\phi+(1-r)\bar{\phi},r\psi+(1-r)\bar{\psi})\|_{H^{s-l}})\|\partial_z^m(\phi-\bar{\phi})\|_{H^{s-l}})\\
&\quad+\sum_{k=1}^{m}\int_0^1\|\partial_z^k\Omega_\phi(r\phi+(1-r)\bar{\phi},r\psi+(1-r)\bar{\psi})\|_{H^{s-l}}\|\partial_z^{m-k}(\phi-\bar{\phi})\|_{H^{s-l}}\mathrm{d}r.
\end{aligned}
$$

Note that

$$
\|\partial_z^l(\phi-\bar{\phi})\|_{H^{s-l}}\le\|\partial_z^l(\phi-\bar{\phi})\|_{H^{s-l-1}}+\|\nabla_x\partial_z^l(\phi-\bar{\phi})\|_{H^{s-l}}\le X_{sum,l}^{\frac{1}{2}}+Y_l^{\frac{1}{2}},
$$

and if $k<l$,

$$
\|\partial_z^k(\phi-\bar{\phi})\|_{H^{s-l}}\le\|\partial_z^k(\phi-\bar{\phi})\|_{H^{s-k-1}}\le X_{sum,l}^{\frac{1}{2}}.
$$

This implies

$$\|I_1\|_{H^{s-l}}+\|I_2\|_{H^{s-l}}\lesssim A_l^{s+1}(X_{sum,l}^{\frac{1}{2}}+Y_l^{\frac{1}{2}}).$$

II_1 and II_2 can be controlled in the same way. Hence, the first two inequalities hold true.

Finally, from relation (A.12),

$$\partial_z^m(W^2\Omega_\phi-\overline{W}^2\bar{\Omega}_\phi)=(2\sum_{i+j=m}(-\partial_z^i(\phi-\bar{\phi})\partial_z^j\bar{\phi}-\partial_z^i\bar{\phi}\partial_z^j(\phi-\bar{\phi})+\partial_z^i(\psi-\bar{\psi})\partial_z^j\psi+\partial_z^i\bar{\psi}\partial_z^j(\psi-\bar{\psi})),$$
$$-4\sum_{i+j=m}(\partial_z^i(\phi-\bar{\psi})\partial_z^j\psi+\partial_z^i\bar{\phi}\partial_z^j(\psi-\bar{\psi})),4\partial_z^m(\phi-\bar{\phi})).$$

According to Lemma A.5-(2), it is easy to see

$$\|\partial_z^m(W^2\Omega_\phi-\overline{W}^2\bar{\Omega}_\phi)\|_{H^{s-l-1}}\lesssim A_l^{\frac{1}{2}}X_{sum,l}^{\frac{1}{2}}.$$

This completes the proof of the whole lemma. □

REFERENCES

[1] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der mathematischen Wissenschaften, Springer, Heidelberg, 343, 2011. 3.3

[2] F. Bolley, J.A. Cañizo, and J.A. Carrillo, *Mean-field limit for the stochastic Vicsek model*, Appl. Math. Lett., 25(3):339–343, 2012. 1.1

[3] M. Briant, A. Diez, and S. Merino-Aceituno, *Cauchy theory for general kinetic Vicsek models in collective dynamics and mean-field limit approximations*, SIAM J. Math. Anal., 54(1):1131–1168, 2022. 1.1

[4] P. Degond, A. Frouvelle, and J.-G. Liu, *Macroscopic limits and phase transition in a system of self-propelled particles*, J. Nonlinear Sci., 23(3):427–456, 2013. 1.1

[5] P. Degond, A. Frouvelle, and J.-G. Liu, *Phase transitions, hysteresis, and hyperbolicity for self-organized alignment dynamics*, Arch. Ration. Mech. Anal., 216(1):63–115, 2015. 1.1

[6] P. Degond, J.-G. Liu, S. Motsch, and V. Panferov, *Hydrodynamic models of self-organized dynamics: derivation and existence theory*, Meth. Appl. Anal., 20(2):89–114, 2013. 1.1, 1.2, 1.2

[7] P. Degond and S. Motsch, *Continuum limit of self-driven particles with orientation interaction*, Math. Model. Meth. Appl. Sci., 18:1193–1215, 2008. 1.1, 1.2

[8] Z.-Y. Ding, S.-Y. Ha, and S. Jin, *A local sensitivity analysis in Landau damping for the kinetic Kuramoto equation with random inputs*, Quart. Appl. Math., 79(2):229–264, 2021. 1.1

[9] A. Frouvelle, *A continuum model for alignment of self-propelled particles with anisotropy and density-dependent parameters*, Math. Model. Meth. Appl. Sci., 22(7):1250011, 2012. 1.1

[10] A. Frouvelle and J.-G. Liu, *Dynamics in a kinetic model of oriented particles with phase transition*, SIAM J. Math. Anal., 44(2):791–826, 2012. 1.1

[11] S.-Y. Ha and S. Jin, *Local sensitivity analysis for the Cucker-Smale model with random inputs*, Kinet. Relat. Models, 11(4):859–889, 2018. 1.1

[12] S.-Y. Ha, S. Jin, and J. Jung, *A local sensitivity analysis for the kinetic Cucker-Smale equation with random inputs*, J. Differ. Equ., 265(8):3618–3649, 2018. 1.1

[13] S.-Y. Ha, S. Jin, and J. Jung, *A local sensitivity analysis for the kinetic Kuramoto equation with random inputs*, Netw. Heterog. Media, 14(2):317–340, 2019. 1.1

[14] S.-Y. Ha, S. Jin, and J. Jung, *Local sensitivity analysis for the Kuramoto-Daido model with random inputs in a large coupling regime*, SIAM J. Math. Anal., 52(2):2000–2040, 2020. 1.1

[15] S.-Y. Ha, S. Jin, J. Jung, and W. Shim, *A local sensitivity analysis for the hydrodynamic Cucker-Smale model with random inputs*, J. Differ. Equ., 268(2):636–679, 2020. 1.1

[16] J. Hu and S. Jin, *Uncertainty quantification for kinetic equations*, in S. Jin and L. Pareschi (eds.), Uncertainty Quantification for Kinetic and Hyperbolic Equations, Springer, 14:193–229, 2017. 1.1

[17] N. Jiang, Y.-L. Luo, and T.-F. Zhang, *Coupled self-organized hydrodynamics and Navier-Stokes models: local well-posedness and the limit from the self-organized kinetic-fluid models*, Arch. Ration. Mech. Anal., 236(1):329–387, 2020. 1.1, 1.2, 1.2, 3.4

[18] N. Jiang, L. Xiong, and T.-F. Zhang, *Hydrodynamic limits of the kinetic self-organized models*, SIAM J. Math. Anal., 48(5):3383–3411, 2016. 1.1

[19] A. Saltelli, M. Ratto, T. Andres, T. Campolongo, J. Cariboni, D. Gatelli, M. Saisana, and S. Tarantola, *Global Sensitivity Analysis. The Primer*, John Wiley & Sons, Ltd., Chichester, 2008. 1.1

[20] S. Jin, *Mathematical analysis and numerical methods for multiscale kinetic equations with uncertainties*, in B. Sirakov, P.N. de Souza, and M. Viana (eds.), Proceedings of the International Congress of Mathematicians, World Sci. Publ., Hackensack, NJ, 3611–3639, 2019. 1.1

[21] S. Jin and Y.-D. Li, *Local sensitivity analysis and spectral convergence of the stochastic Galerkin method for discrete-velocity Boltzmann equations with multi-scales and random inputs*, Kinet. Relat. Models, 12(5):969–993, 2019. 1.1

[22] S. Jin and Y. Zhu, *Hypocoercivity and uniform regularity for the Vlasov-Poisson-Fokker-Planck system with uncertainty and multiple scales*, SIAM J. Math. Anal., 50(2):1790–1816, 2018. 1.1

[23] R.C. Smith, *Uncertainty Quantification: Theory, Implementation, and Applications*, Computational Science & Engineering, SIAM, Philadelphia, PA, 12, 2014. 1.1

[24] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, *Novel type of phase transition in a system of self-driven particles*, Phys. Rev. Lett., 75(6):1226–1229, 1995. 1.1

[25] T.-F. Zhang and N. Jiang, *A local existence of viscous self-organized hydrodynamic model*, Nonlinear Anal. Real World Appl., 34:495–506, 2017. 1.2

[26] Y. Zhu, *A local sensitivity and regularity analysis for the Vlasov-Poisson-Fokker-Planck system with multi-dimensional uncertainty and the spectral convergence of the stochastic Galerkin method*, Netw. Heterog. Media, 14(4):677–707, 2019. 1.1