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# SOME MODELS FOR THE INTERACTION OF LONG AND SHORT WAVES IN DISPERSIVE MEDIA. PART II: WELL-POSEDNESS*

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**Abstract.** The (in)validity of a system coupling the cubic, nonlinear Schrödinger equation (NLS) and the Korteweg-de Vries equation (KdV) commonly known as the NLS-KdV system for studying the interaction of long and short waves in dispersive media was discussed in part I of this work [N.V. Nguyen and C. Liu, Water Waves, 2:327–359, 2020]. It was shown that the NLS-KdV system can never be obtained from the full Euler equations formulated in the study of water waves, nor even the *linear* Schrödinger-Korteweg de Vries system where the two equations in the system appear at the same scale in the asymptotic expansion for the temporal and spatial variables. A few alternative models were then proposed for describing the interaction of long and short waves.

In this second installment, the Cauchy problems associated with the alternative models introduced in part I are analyzed. It is shown that all of these models are locally well-posed in some Sobolev spaces. Moreover, they are also globally well-posed in those spaces for a range of suitable parameters.

**Keywords.** Euler equations; linear Schrödinger equation; NLS-equation; KdV-equation; BBM-equation; NLS-KdV system; abcd-system.

**Subject classifications.** 35A35; 35M30; 35Q31; 35Q35; 76B15.

## 1. Introduction

This manuscript is a continuation of our previous work [15] in which we discussed the (in)validity of the system coupling the nonlinear Schrödinger equation and the Korteweg-de Vries equation

$$\begin{cases} iu_t + u_{xx} + a|u|^2 u = -buv, \\ v_t + cvv_x + v_{xxx} = -\dfrac{b}{2}(|u|^2)_x, \end{cases} \tag{1.1}$$

where $x,t \in \mathbb{R}$, $v(x,t)$ is a real-valued function, $u(x,t)$ is complex-valued and $a,b,c$ are real constants. Many articles have mentioned that system (1.1) arises generically as a model for interactions between long gravity waves and capillary waves on the surface of shallow water, or a model for the interaction of Langmuir waves and ion-acoustic waves in plasma physics [1, 11, 16]. Even though countless mathematical papers have studied system (1.1), they each appear to have quoted one another regarding the system's derivation and applications. The only derivation we could find is the paper by Kawahara *et.al.* [14], where the following system

$$\begin{cases} i\left(\dfrac{\partial u}{\partial t_2} + k\dfrac{\partial u}{\partial x_2}\right) + p\dfrac{\partial^2 u}{\partial x_1^2} = quv, \\ \dfrac{\partial v}{\partial t_3} + \dfrac{\partial v}{\partial x_3} + \dfrac{3}{2}v\dfrac{\partial v}{\partial x_1} + r\dfrac{\partial^3 v}{\partial x_1^3} = -s\dfrac{\partial |u|^2}{\partial x_1}, \end{cases} \tag{1.2}$$

was originally introduced with $k,p,q,r$ and $s$ being real constants, $x_n = \epsilon^n x$, $t_n = \epsilon^n t$. Here $\epsilon$ is the small parameter in terms of which the asymptotic expansions were performed. Notice that with regard to the left-hand side, the first equation in (1.2) is *linear*

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whilst that in (1.1) is *nonlinear*. Further, the time scales appearing in (1.2) are inconsistent, with the dynamics of the second equation of (1.2) appearing on a different time scale than that of the first equation. The same is true for the derivation in the context of plasma physics, see [1,11,16], where references lead back to [17] and the system (1.1) is not found in any form.

The analysis in part I, (see also [9]), established unequivocally the fact that, contrary to what has been assumed theretofore, one *cannot* derive the NLS-KdV system (1.1) from the Euler equations used in the study of water waves, *nor* even the coupled linear Schrödinger-KdV system (1.2) consistently where the two equations in the system appear at the same temporal and spatial scales. It is important to point out that a system coupling a linear Schrödinger equation with a KdV equation similar to (1.2) was indeed derived recently for the resonant interaction between short surface waves and long internal waves in a two-layer ocean by Craig, Guyenne, and Sulem [6,7]. However, the underlined physical problem there is completely different from this one. In their work, Craig *et. al.* considered a fluid system in which two immiscible layers are separated by a sharp free interface. They then studied the regime where long waves propagate in the interfacial mode, which are coupled to a modulational regime for the free-surface mode.

After evaluating the (in)validity of systems (1.1) and (1.2) as physical models, the following four systems were put forward for the study of interaction of long and short waves in dispersive media instead

$$\begin{cases} \dfrac{\partial u}{\partial t}+\mu\dfrac{\partial u}{\partial x}+a_0\dfrac{\partial^3 u}{\partial x^3}+ib\dfrac{\partial^2 u}{\partial x^2}=-\dfrac{\partial(uv)}{\partial x}-i\lambda uv,\\ \dfrac{\partial v}{\partial t}+\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial x}+c\dfrac{\partial^3 v}{\partial x^3}=-\dfrac{1}{2}\dfrac{\partial|u|^2}{\partial x}; \end{cases} \tag{1.3}$$

$$\begin{cases} \dfrac{\partial u}{\partial t}+\mu\dfrac{\partial u}{\partial x}-a_1\dfrac{\partial^3 u}{\partial x^2\partial t}+ib\dfrac{\partial^2 u}{\partial x^2}=-\dfrac{\partial(uv)}{\partial x}-i\lambda uv,\\ \dfrac{\partial v}{\partial t}+\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial x}-c\dfrac{\partial^3 v}{\partial x^2\partial t}=-\dfrac{1}{2}\dfrac{\partial|u|^2}{\partial x}; \end{cases} \tag{1.4}$$

$$\begin{cases} \dfrac{\partial u}{\partial t}+\mu\dfrac{\partial u}{\partial x}+a_0\dfrac{\partial^3 u}{\partial x^3}+ib\dfrac{\partial^2 u}{\partial x^2}=-\dfrac{\partial(uv)}{\partial x}-i\lambda uv,\\ \dfrac{\partial v}{\partial t}+\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial x}-c\dfrac{\partial^3 v}{\partial x^2\partial t}=-\dfrac{1}{2}\dfrac{\partial|u|^2}{\partial x}; \end{cases} \tag{1.5}$$

and

$$\begin{cases} \dfrac{\partial u}{\partial t}+\mu\dfrac{\partial u}{\partial x}-a_1\dfrac{\partial^3 u}{\partial x^2\partial t}+ib\dfrac{\partial^2 u}{\partial x^2}=-\dfrac{\partial(uv)}{\partial x}-i\lambda uv,\\ \dfrac{\partial v}{\partial t}+\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial x}+c\dfrac{\partial^3 v}{\partial x^3}=-\dfrac{1}{2}\dfrac{\partial|u|^2}{\partial x}; \end{cases} \tag{1.6}$$

where $\mu,\lambda,a_0,a_1,b,c$ are real constants with $\mu,\lambda,c>0$.

It is universally agreed that for $x,t\in\mathbb{R}$, $v(x,t)$ a real-valued function and $u(x,t)$ a complex-valued function, the dispersive equation

$$v_t+v_x+vv_x+Lv_x=0$$

is referred to as the Korteweg-de Vries (KdV) equation when $L=\partial_{xx}$, and Benjamin-Bona-Mahoney (BBM) equation when $L=-\partial_{xt}$, while the equation

$$iu_t+u_{xx}\pm|u|^2u=0$$

is referred to as the (focusing for the + sign and de-focusing for the − sign) cubic nonlinear Schrödinger (NLS) equation. Following this conventional nomenclature, we will refer to (1.3), (1.4), (1.5) and (1.6) as the Schrödinger KdV-KdV, Schrödinger BBM-BBM, Schrödinger KdV-BBM, and Schrödinger BBM-KdV systems respectively.

Despite bearing some resemblance of the *abcd*-systems derived in [3, 4], the above systems possess some significant differences. In [3, 4], both of the functions $u$ and $v$ are real. Here, $u$ is complex and $v$ is real. Moreover, the couplings between the two equations in any system here are only through the nonlinear terms, in addition to the extra term $iu_{xx}$. It is a well-known fact that the interaction between the nonlinear term $uu_x$ and the dispersive term $u_{xxx}$, $(u_{xxt})$ is what supports the solitary waves in the KdV-equation (BBM-equation), while the terms $u_{xx}$ and $|u|^2u$ play the same role in the cubic NLS-equation. The above systems feature the presence of both the $u_{xx}$ and either the $u_{xxx}$ or $u_{xxt}$ terms, and thus exhibit some very interesting dynamics. In particular, the signs of $a_0,a_1$ and $b$ as well as the interaction among them will play significant roles in determining global well-posedness of the Cauchy problems associated with the above systems and also in determining the existence and stability of solitary-wave solutions of these systems. Readers are referred to the first part for detailed discussions/comparisons between the above four models with other models such as the Davey-Stewartson system [8], the Benney system [2] or the ones put forward by Djordjevic and Redekopp [10], (see also [5]).

All of the above four systems possess at least three conserved quantities when $a_1>0$. The three conserved quantities for (1.3) are

$$H_1(v)=\int_{-\infty}^{\infty} v\ dx, \qquad H_2(u,v)=\int_{-\infty}^{\infty}(|u|^2+v^2)dx,$$
$$H_3(u,v)=\int_{-\infty}^{\infty}\left(\frac{a_0}{2}|u_x|^2+\frac{c}{2}v_x^2-\frac{\mu}{2}|u|^2-\frac{1}{2}|u|^2v-\frac{1}{2}v^2-\frac{1}{6}v^3\right)dx$$
$$+\left(\frac{\lambda a_0}{2}-\frac{b}{2}\right)\mathcal{I}m\int_{-\infty}^{\infty}u\bar{u}_x dx+\left(\frac{b\lambda}{2}-\frac{\lambda^2a_0}{2}\right)\int_{-\infty}^{\infty}v^2dx.$$

The conserved quantities for (1.4) are as follows

$$\mathcal{H}_1(v)=\int_{-\infty}^{\infty} v\ dx, \qquad \mathcal{H}_2(u,v)=\int_{-\infty}^{\infty}(|u|^2+v^2+a_1|u_x|^2+cv_x^2)dx,$$
$$\mathcal{H}_3(u,v)=\int_{-\infty}^{\infty}\left(\frac{\mu}{2}|u|^2+\frac{1}{2}v^2+\frac{1}{6}v^3+\frac{1}{2}|u|^2v\right)dx$$
$$+\frac{(b-a_1\lambda\mu)}{2+2a_1\lambda^2}\ \mathcal{I}m\int_{-\infty}^{\infty}u\bar{u}_x\ dx+\frac{\lambda(b-a_1\lambda\mu)}{2+2a_1\lambda^2}\int_{-\infty}^{\infty}|u|^2\ dx.$$

The conserved quantities for (1.5) are given by

$$I_1(v)=\int_{-\infty}^{\infty} v\ dx, \qquad I_2(u,v)=\int_{-\infty}^{\infty}(|u|^2+v^2+cv_x^2)dx,$$
$$I_3(u,v)=\int_{-\infty}^{\infty}\left(\frac{a_0}{2}|u_x|^2-\frac{\mu}{2}|u|^2-\frac{1}{2}|u|^2v-\frac{1}{2}v^2-\frac{1}{6}v^3\right)dx$$

$$+\Big(\frac{b}{2}-\frac{\lambda a_0}{2}\Big)\ \mathcal{I}m\int_{-\infty}^{\infty} u\overline{u}_x\ dx+\Big(\frac{b\lambda}{2}-\frac{\lambda^2 a_0}{2}\Big)\int_{-\infty}^{\infty}|u|^2\ dx.$$

The conserved quantities for system (1.6) are

$$\mathcal{I}_1(v)=\int_{-\infty}^{\infty} v\ dx, \qquad \mathcal{I}_2(u,v)=\int_{-\infty}^{\infty}(|u|^2+a_1|u_x|^2+v^2)dx,$$

$$\mathcal{I}_3(u,v)=\int_{-\infty}^{\infty}\Big(\frac{c}{2}v_x^2-\frac{\mu}{2}|u|^2-\frac{1}{2}|u|^2v-\frac{1}{2}v^2-\frac{1}{6}v^3\Big)dx$$
$$+\frac{(a_1\lambda\mu-b)}{2+2a_1\lambda^2}\ \mathcal{I}m\int_{-\infty}^{\infty}u\overline{u}_x\ dx+\frac{\lambda(a_1\lambda\mu-b)}{2+2a_1\lambda^2}\int_{-\infty}^{\infty}|u|^2\ dx.$$

In this paper, we will analyze the Cauchy problems associated with the above four models. It is shown that all of these models are locally well-posed in some Sobolev spaces. Moreover, they are also globally well-posed in those spaces for a range of suitable parameters.

## 2. Preliminaries

Commonly used notations will be employed. For $1\le p\le\infty$, we denote by $L^p=L^p(\mathbb{R})$ the space of all measurable functions $f$ on $\mathbb{R}$ for which the norm $\|f\|_{L^p}^p=\int_{\mathbb{R}}|f(x)|^p dx$ is finite for $1\le p<\infty$, and $\|f\|_{L^\infty}$ is the essential supremum of $|f|$ on $\mathbb{R}$. Throughout the paper, whenever $p=2$ we denote the $L^2$-norm of $f$ simply as $\|f\|$. The space $H^s_{\mathbb{R}}=H^s(\mathbb{R})$ and $H^s_{\mathbb{C}}=H^s_{\mathbb{C}}(\mathbb{R})$ are the usual real-valued and complex-valued Sobolev spaces, respectively, consisting of all measurable functions such that $f$ and their derivatives $f^{(k)}$ are in $L^2$, $1\le k\le s$, equipped with the norm $\|f\|_s^2=\int_{\mathbb{R}}(1+|\xi|^2)^s|\hat{f}(\xi)|^2d\xi$ where $\hat{f}$ is the Fourier transform of $f$. For any Banach space $\mathcal{X}$, let $\mathcal{X}^k:\equiv\mathcal{X}\times\mathcal{X}\times\ldots\times\mathcal{X}$ be the $k-$times Cartesian product of $\mathcal{X}$. For $\mathcal{X},\mathcal{Y}$ any two Banach spaces, let $\mathcal{B}(\mathcal{X},\mathcal{Y})$ denote the space of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. For $I$ an arbitrary interval of $\mathbb{R}$ and $\mathcal{Y}$ any Banach space, let $C(I,\mathcal{Y})$ denote the Banach space of continuous maps $f$ which takes $I$ into $\mathcal{Y}$, with the norms given by $\|f\|_{C(I,\mathcal{Y})}=\sup\{\|f(t)\|_{\mathcal{Y}}|t\in I\}$. As usual, the subspace $C_b=C_b(\mathbb{R})$ of $L^\infty(\mathbb{R})$ consists of all bounded, continuous functions and $C_b^n=C_b^n(\mathbb{R})$ denotes the subspace of $C_b$ such that $f$ along with their derivatives $f^{(k)}$ are in $C_b$, $1\le k\le n$. The subspace of $C_b$, such that all of its members along with their derivatives of all orders vanish at infinity, will be denoted as $C_0^\infty$. In the case when $\mathcal{Y}$ is the Sobolev space $H^s$ and $T>0$ fixed, let $C_T:\equiv C([0,T],H^s)$ and $\mathcal{C}_T^k:\equiv C([0,T],H^s\times H^s\times\ldots\times H^s)$, equipped with the norm $\|\vec{f}\|_{\mathcal{C}_T^k}=\max_{1\le i\le k}\{\|f_i\|_{C_T}\}$. Similarly, when $\mathcal{Y}$ is $C_0^\infty$ and $T>0$ fixed, we set $C_{0,T}:\equiv C([0,T],C_0^\infty)$ and $\mathcal{C}_{0,T}^k:\equiv C([0,T],C_0^\infty\times C_0^\infty\times\ldots\times C_0^\infty)$, equipped with the norm $\|\vec{f}\|_{\mathcal{C}_{0,T}^k}=\max_{1\le i\le k}\{\|f_i\|_{C_{0,T}}\}$. Throughout the manuscript, unless otherwise stated explicitly, we will denote the various constants whose precise values are not of importance to us as $C$.

In Section 3, we study first the question of local well-posedness of the above four systems. That is, one imagines being provided with an initial wave profile, say at $t=0$

$$u(x,0)=f(x), \qquad v(x,0)=g(x), \tag{2.1}$$

for $x\in\mathbb{R}$ and then inquiring into the subsequent evolutions using the four systems. A problem is said to be locally well-posed if there exists a time $T>0$ such that a unique solution $\big(u(x,t),v(x,t)\big)$, depending continuously on the initial data $(f,g)$, departs from $(f,g)$ under the influence of the evolution in question for $t\in[0,T]$.

Our four systems under consideration here can be handled in a fairly straightforward manner using the method pioneered by Kato [12, 13] with some appropriate modifications. This method assures the local well-posedness for all the four systems in the Sobolev space $H^s$ for $s>3/2$ with small enough $H^1$-normed data. In the case of the Schrödinger BBM-BBM system (1.4), however, one can also take advantage of the presence of the smoothing operator of the type $(1-k\partial_{xx})^{-1}$, $k>0$ with the method of Contraction Mapping Principle to establish the local well-posedness for (1.4) in $H^s$ for $s\geq 0$, as well as derive several other regularity properties for its solutions that cannot be obtained (or at least it is not trivial to do so) with Kato's method. Thus, we will employ Kato's method for each of the systems (1.3), (1.5) and (1.6), and will call upon the method of Contraction Mapping Principle for system (1.4).

To utilize Kato's method as detailed in [12, 13], it is better to rewrite the Cauchy problems for the systems (1.3), (1.5) and (1.6) compactly as

$$\frac{d\vec{u}}{dt}+\mathcal{A}(t,\vec{u})\vec{u}=0, \quad \vec{u}(0)=(f,g). \tag{2.2}$$

Define

$$L^2_{\mathbb{C}}(\mathbb{R})=\Big\{\phi(x)|\phi:\mathbb{R}\to\mathbb{C}, \int_{\mathbb{R}}|\phi(x)|^2dx<\infty\Big\},$$

$$L^2_{\mathbb{R}}(\mathbb{R})=\Big\{\phi\in L^2_{\mathbb{C}}(\mathbb{R})|\phi:\mathbb{R}\to\mathbb{R}\Big\}.$$

Similarly we can define $H^s_{\mathbb{C}}(\mathbb{R})$ and $H^s_{\mathbb{R}}(\mathbb{R})$. The space $L^2_{\mathbb{C}}(\mathbb{R})$ becomes a real Hilbert space under the inner product

$$(\phi,\psi)_{L^2_{\mathbb{C}}(\mathbb{R})}=\mathrm{Re}\int_{\mathbb{R}}\phi(x)\overline{\psi(x)}dx$$

for any $\phi,\psi\in L^2_{\mathbb{C}}(\mathbb{R})$. Through out the manuscript, we denote $X:\equiv L^2_{\mathbb{C}}(\mathbb{R})\times L^2_{\mathbb{R}}(\mathbb{R})$, $Y:\equiv H^s_{\mathbb{C}}(\mathbb{R})\times H^s_{\mathbb{R}}(\mathbb{R})$ for $s>3/2$, while $Z:\equiv H^s_{\mathbb{C}}(\mathbb{R})\times H^s_{\mathbb{R}}(\mathbb{R})$ for $s\geq 0$. For any $\vec{f}=(f_1,f_2)\in X$ and $\vec{g}=(g_1,g_2)\in Y$, equip them with the respective norms $\|\vec{f}\|^2_X=\sum_{i=1}^2\|f_i\|^2_{L^2}$ and $\|\vec{g}\|^2_Y=\sum_{i=1}^2\|g_i\|^2_{H^s}$. The inner product on $X$ is realized as

$$\left(\vec{f},\vec{h}\right)_X=\mathrm{Re}\int_{\mathbb{R}}f_1\overline{h_1}dx+\mathrm{Re}\int_{\mathbb{R}}f_2\overline{h_2}dx$$

for $\vec{h}=(h_1,h_2)\in X$ and similarly for $Y$. To establish the local well-posedness for the system (2.2), the crux of the matter is to verify that:

(i) the operator $\mathcal{A}(t,\vec{u})$ is quasi-accretive in $X$, and for bounded $\vec{u}\in Y$ the operator $\mathcal{A}(t,\vec{u})+\sigma I$ is surjective for some $\sigma>\beta$, where $I$ denotes the identity operator;

(ii) there exists a time $T$ such that for each $t\in[0,T]$, the mapping $\vec{u}\to\mathcal{A}(t,\vec{u})$ is Lipschitz continuous in the sense that

$$\|\mathcal{A}(t,\vec{u})-\mathcal{A}(t,\vec{v})\|_{\mathcal{B}(Y,X)}\leq C\|\vec{u}-\vec{v}\|_X,$$

for some constant $C$ independent of $t$ and $\vec{u},\vec{v}$; and

(iii) the mapping $t\to\mathcal{A}(t,\vec{u})\in\mathcal{B}(Y,X)$ is strongly continuous for any fixed $\vec{u}\in Y$.

Precisely, we need to show that the following properties hold true.

*Property A*: Suppose that $s>\frac{3}{2}$. Then the operator $\mathcal{A}(t,\vec{u})$ is quasi-accretive in $X$, namely, there exists a positive constant $\beta$ independent of $t,\vec{u}$ and $\vec{v}$ such that

$$\big(\mathcal{A}(t,\vec{u})\vec{v},\vec{v}\big)_X \geq -\beta\|\vec{u}\|_Y\|\vec{v}\|_X^2$$

holds for all $\vec{u}\in Y$ and $\vec{v}\in X$, and for bounded $\vec{u}\in Y$ the operator $\mathcal{A}(t,\vec{u})+\sigma I$ is surjective for some $\sigma>\beta$, where $I$ denotes the identity operator.

*Property B*: Suppose that $s>\frac{3}{2}$ and let $\mathbf{\Lambda}=(1-D^2)^{\frac{1}{2}}I$, where $I$ denotes the $3\times 3$ identity matrix. Then $\mathbf{\Lambda}^{\mathbf{s}}$: $Y\to X$ is an isomorphism of $Y$ onto $X$, and $\forall\vec{u},\vec{v}\in Y$,

(1) $\mathcal{B}(t,\vec{u}):=\mathbf{\Lambda}^s\mathcal{A}(t,\vec{u})\mathbf{\Lambda}^{-s}-\mathcal{A}(t,\vec{u})$ is a bounded operator from $X$ to $X$,

(2) $\|\mathcal{B}(t,\vec{u})\|_{\mathcal{B}(X,X)}\leq C\|\vec{u}\|_Y$,

(3) $\|\mathcal{B}(t,\vec{u})-\mathcal{B}(t,\vec{v})\|_{\mathcal{B}(X,X)}\leq C\|\vec{u}-\vec{v}\|_Y$,

where $C$ is a constant independent of $t$ and $\vec{u},\vec{v}\in Y$.

*Property C*: Suppose that $s>\frac{3}{2}$. Then

$$\|\mathcal{A}(t,\vec{u})-\mathcal{A}(t,\vec{v})\|_{\mathcal{B}(Y,X)}\leq C\|\vec{u}-\vec{v}\|_X$$

holds for all $\vec{u},\vec{v}\in X$ with C a constant independent of $t$ and $\vec{u},\vec{v}$.

*Property D*: The mapping $t\to\mathcal{A}(t,\vec{u})\in\mathcal{B}(Y,X)$ is strongly continuous for any fixed $\vec{u}\in X$.

Once these properties have been verified, Kato's theory [12,13] immediately assures that the system (2.2) is locally well-posed.

THEOREM 2.1.

(i) *Let $s>\frac{3}{2}$. For any $(f,g)\in Y$, there exists a unique solution $(u,v)\in C([0,T],H^s)^2 \bigcap C^1([0,T],H^{s-3})^2$ to the system (2.2) with $T$ having a lower bound depending only on the norm $\|(f,g)\|_Y$.*

(ii) *The mapping $(f,g)\to\big(u(t),v(t)\big)$ is continuous in the $Y$-norm. More precisely, if $(f_n,g_n)\in Y$, $n=1,2,\cdots$, with $\|(f_n,g_n)-(f,g)\|_Y\to 0$ and $T'<T$, the solution $(u_n,v_n)$ for $\big(u_n(0),v_n(0)\big)=(f_n,g_n)$ exists on $[0,T']$ for sufficiently large $n$ and $\|(u_n,v_n)-(u,v)\|_Y\to 0$ uniformly in $t\in[0,T']$.*

As regard to the Contraction Mapping Principle, the technique is self-explanatory and will be described when the system (1.4) is dealt with below. We then extend the local well-posedness theories to global results in Section 4. Since all the local results for the four systems rest upon the fact that the time intervals of existence $T$ depend only on the norms of the initial data $\|(f,g)\|_{H^s_{\mathbb{C}}\times H^s_{\mathbb{R}}}$, ($s\geq 0$ for the Schrödinger BBM-BBM system (1.4) while $s>3/2$ for the other three), the crux of the matter here is to show that the $H^s_{\mathbb{C}}\times H^s_{\mathbb{R}}$-norms of their solutions are bounded on any bounded time interval. The conserved quantities for the four systems will play an important part in establishing these bounds.

## 3. Local well-posedness

In this section, the local well-posedness theories for the above four systems will be established. Recall that in the derivations of the four systems [15], the parameters $\mu,\lambda,a_0,a_1,b$ and $c$ are real numbers with $\mu,\lambda,c>0$. It was also demonstrated that $a_0$ and $a_1$ are negative for sufficiently small wave numbers for any Weber number $W>3$, and are strictly positive for at least a range of wave numbers and Weber number $W$.

We would like to point out that for the local well-posedness theories, we only require that $a_1>0$ due to the invertibility of the operator $(1-a_1\partial_{xx})$. That is, the local results here are established independently of the definitiveness of the signs of $a_0$ and $b$.

**3.1. The Schrödinger KdV-KdV system.** We now consider the Cauchy problem for the Schrödinger KdV-KdV system; that is, we study the problem (1.3)-(2.1). We recall that $X :\equiv L^2_{\mathbb{C}}(\mathbb{R}) \times L^2_{\mathbb{R}}(\mathbb{R})$ and $Y :\equiv H^s_{\mathbb{C}}(\mathbb{R}) \times H^s_{\mathbb{R}}(\mathbb{R})$ and that the Schrödinger KdV-KdV system (1.3) is

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mu \dfrac{\partial u}{\partial x} + a_0 \dfrac{\partial^3 u}{\partial x^3} + ib \dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial (uv)}{\partial x} - i\lambda uv, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial x} + c\dfrac{\partial^3 v}{\partial x^3} = -\dfrac{1}{2}\dfrac{\partial |u|^2}{\partial x}. \end{cases}$$

The local theory for the Schrödinger KdV-KdV system is as follows.

THEOREM 3.1.

(i) *Let $s > \frac{3}{2}$. There exists a $\delta > 0$ such that for any initial data $(f,g) \in Y$ with $\|(f,g)\|_{H^1_{\mathbb{C}}(\mathbb{R})\times H^1_{\mathbb{R}}(\mathbb{R})} \le \delta$, there is a unique solution $(u,v) \in C([0,T],Y) \bigcap C^1([0,T], H^{s-3}_{\mathbb{C}} \times H^{s-3}_{\mathbb{R}})$ to (1.3)-(2.1) with $T$ having a lower bound depending only on the norm $\|(f,g)\|_Y$.*

(ii) *The map $(f,g) \to \big(u(t),v(t)\big)$ is continuous in the $Y$-norm. More precisely, if $(f_n,g_n) \in Y$, $n = 1,2,\cdots$, with $\|(f_n,g_n) - (f,g)\|_Y \to 0$ and $T' < T$, the solution $(u_n,v_n)$ for $\big(u_n(0),v_n(0)\big) = (f_n,g_n)$ exists on $[0,T']$ for sufficiently large $n$ and $\|(u_n,v_n) - (u,v)\|_Y \to 0$ uniformly in $t \in [0,T']$.*

Let

$$Q(t) = e^{-a_0 t D^3 - ibtD^2},\ P(t) = e^{-ctD^3}, \quad D = \frac{d}{dx}.$$

Then

$$Q(t): L^2_{\mathbb{C}}(\mathbb{R}) \to L^2_{\mathbb{C}}(\mathbb{R}), \quad Q(t): H^s_{\mathbb{C}}(\mathbb{R}) \to H^s_{\mathbb{C}}(\mathbb{R}),$$

$$P(t): L^2_{\mathbb{R}}(\mathbb{R}) \to L^2_{\mathbb{R}}(\mathbb{R}), \quad P(t): H^s_{\mathbb{R}}(\mathbb{R}) \to H^s_{\mathbb{R}}(\mathbb{R})$$

are all unitary maps. Set $\vec{u} = \begin{bmatrix} u \\ v \end{bmatrix}$ and consider the transformations $u(x,t) = Q(t)\widetilde{u}(x,t)$, $v(x,t) = P(t)\widetilde{v}(x,t)$. In terms of the new variables $\widetilde{u}, \widetilde{v}$, the system (1.3) reads

$$\begin{cases} Q(t)\widetilde{u}_t + \mu Q(t)\widetilde{u}_x + Q(t)\widetilde{u}_x P(t)\widetilde{v} + Q(t)\widetilde{u} P(t)\widetilde{v}_x + i\lambda Q(t)\widetilde{u} P(t)\widetilde{v} = 0, \\ P(t)\widetilde{v}_t + P(t)\widetilde{v}_x + P(t)\widetilde{v} P(t)\widetilde{v}_x + \mathrm{Re}\Big(\overline{Q(t)\widetilde{u}} Q(t)\widetilde{u}_x\Big) = 0. \end{cases} \tag{3.1}$$

For any $\vec{u} \in Y$ and $t \ge 0$, define the operator

$$\mathcal{A}(t,\vec{u}) = \mathcal{U}(-t) \begin{pmatrix} M_{\mu + P(t)v} D + M_{i\lambda P(t)v} & M_{Q(t)u} D \\ \mathrm{Re}\big(M_{\overline{Q(t)u}} D\big) & M_{1+P(t)v} D \end{pmatrix} \mathcal{U}(t)$$

where $M_f$ is the operator of multiplication by $f$ and

$$\mathcal{U}(t) = \begin{pmatrix} Q(t) & 0 \\ 0 & P(t) \end{pmatrix}.$$

The system (3.1) can be written compactly as

$$\frac{d\vec{u}}{dt} + \mathcal{A}(t,\vec{u})\vec{u} = 0, \quad \vec{u} \in Y \tag{3.2}$$

where the tildes have been dropped for ease of reading. We now proceed to verify properties A through D for the system (3.2). Once these properties are shown to hold true, the local well-posedness for (3.2) is guaranteed by Theorem 2.1, and hence Theorem 3.1 is an immediate consequence. The following lemma will come in handy, which is just a multi-dimensional version of Lemma A.2 in [12].

LEMMA 3.1. *Let $\Lambda=(1-D^2)^{\frac{1}{2}}$, $f\in H^r(\mathbb{R})$ for some $r>\frac{3}{2}$, and let $M_f$ be the operator of multiplication by $f$. Then*

$$\|[\Lambda^r,M_f]\Lambda^{1-r}\|\le c_0\|Df\|_{H^{r-1}},$$

*where the constant $c_0$ depends only on $r$. Here, $[\Lambda^r,M_f]$ denotes the commutator between $\Lambda^r$ and $M_f$, and the norm $\|\cdot\|$ on the left is the operator norm in $L^2(\mathbb{R})$.*

Define

$$\beta_r=\sup\{\|f\|_{L^\infty}:\|f\|_{H^r}=1\} \tag{3.3}$$

for any $r>\frac{1}{2}$. Property A is a direct result of the following.

LEMMA 3.2. *Suppose that $s>\frac{3}{2}$. Then the operator $\mathcal{A}(t,\vec{u})$ is quasi-accretive in $X$, namely,*

$$\big(\mathcal{A}(t,\vec{u})\vec{v},\vec{v}\big)_X\ge -2\beta_{s-1}\|\vec{u}\|_Y\|\vec{v}\|_X^2 \tag{3.4}$$

*holds for all $\vec{u}\in Y$ and $\vec{v}\in H^1_{\mathbb{C}}\times H^1_{\mathbb{R}}$. Moreover $\mathcal{A}(t,\vec{u})+\sigma I$ is surjective from $H^1\times H^1$ to $X$ for any $\sigma>2\beta_{s-1}\|\vec{u}\|_Y$ and $\|\vec{u}\|_{H^1_{\mathbb{C}}\times H^1_{\mathbb{R}}}<\frac{\min\{1,\mu\}}{2\beta_{s-1}}$, where $I$ denotes the identity operator.*

*Proof.* We break the proof into three steps.

Step 1: To prove (3.4), let $\vec{u}=(u_1,u_2)\in Y$, $\vec{w}=(w_1,w_2)\in H^1_{\mathbb{C}}\times H^1_{\mathbb{R}}$ and $\vec{v}=U(-t)\vec{w}$. Then

$$\begin{aligned}
&\big(\mathcal{A}(t,\vec{u})\vec{v},\vec{v}\big)_X=\Big(\mathcal{A}(t,\vec{u})\mathcal{U}(-t)\vec{w},\mathcal{U}(-t)\vec{w}\Big)\\
&=\mathrm{Re}\Big(M_{\mu+P(t)u_2}Dw_1+M_{i\lambda P(t)u_2}w_1+M_{Q(t)u_1}Dw_2,w_1\Big)\\
&\qquad+\Big(\mathrm{Re}\big(M_{\overline{Q(t)u_1}}Dw_1\big)+M_{1+P(t)u_2}Dw_2,w_2\Big)\\
&=\mathrm{Re}\Big((\mu+P(t)u_2)Dw_1,w_1\Big)+\mathrm{Re}\Big(i\lambda(P(t)u_2)w_1,w_1\Big)+\mathrm{Re}\Big(Q(t)u_1Dw_2,w_1\Big)\\
&\qquad+\Big(\mathrm{Re}\big(\overline{Q(t)u_1}Dw_1\big),w_2\Big)+\Big((1+P(t)u_2)Dw_2,w_2\Big).
\end{aligned}$$

As regard to the first term in the above equation, note that $P(t)u_2$ is a real value function, then an integration by parts yields

$$\begin{aligned}
\mathrm{Re}\Big((\mu+P(t)u_2)Dw_1,w_1\Big)&=-\frac{1}{2}\int|w_1|^2P(t)Du_2\ge-\frac{1}{2}\|P(t)Du_2\|_{L^\infty}\|w_1\|_{L^2}^2\\
&\ge-\frac{1}{2}\beta_{s-1}\|P(t)Du_2\|_{s-1}\|w_1\|_{L^2}^2\\
&\ge-\frac{1}{2}\beta_{s-1}\|\vec{u}\|_Y\|\vec{w}\|_X^2.
\end{aligned} \tag{3.5}$$

For the second term we have

$$\mathrm{Re}\Big(i\lambda(P(t)u_2)w_1,w_1\Big)=\mathrm{Re}\int_{\mathbb{R}}i\lambda|w_1|^2P(t)u_2dx=0. \tag{3.6}$$

Similarly, we have for the last term the following estimate

$$\Big((1+P(t)u_2)Dw_2,w_2\Big) \geq -\frac{1}{2}\beta_{s-1}\|\vec{u}\|_Y\|\vec{w}\|_X^2. \tag{3.7}$$

For the third and fourth terms, integrations by parts again reveal

$$\begin{aligned}
&\mathrm{Re}\Big(M_{Q(t)u_1}Dw_2,w_1\Big)+\Big(\mathrm{Re}\big(M_{\overline{Q(t)u_1}}Dw_1\big),w_2\Big)\\
&=\mathrm{Re}\int_{\mathbb{R}}(Q(t)u_1)D(\overline{w_1}w_2)\\
&=-\mathrm{Re}\int_{\mathbb{R}}(Q(t)Du_1)\overline{w_1}w_2\\
&\geq -\beta_{s-1}\|\vec{u}\|_Y\|\vec{w}\|_X^2.
\end{aligned}\tag{3.8}$$

Putting (3.5), (3.6), (3.7) and (3.8) together, we deduce that

$$\Big(\mathcal{A}(t,\vec{u})\vec{v},\vec{v}\Big)_X \geq -2\beta_{s-1}\|\vec{u}\|_Y\|\vec{v}\|_X^2.$$

Thus, (3.4) follows from the above inequality.

Step 2: To show that the range of $\big(\mathcal{A}(t,\vec{u})+\sigma I\big)$, denoted as $\mathrm{R}\big(\mathcal{A}(t,\vec{u})+\sigma I\big)$, is closed in $X$. Let $\vec{v}\in H^1_{\mathbb{C}}\times H^1_{\mathbb{R}}$

$$(\sigma-2\beta_{s-1}\|\vec{u}\|_Y)\|\vec{v}\|_X^2 \leq \big((\mathcal{A}(t,\vec{u})+\sigma I)\vec{v},\vec{v}\big) \leq \|\big(\mathcal{A}(t,\vec{u})+\sigma I\big)\vec{v}\|_X\|\vec{v}\|_X \tag{3.9}$$

due to the Cauchy-Schwarz inequality. This shows that for any $\vec{u}\in Y$,

$$\|\vec{v}\|_X \leq \frac{1}{\sigma-2\beta_{s-1}\|\vec{u}\|_Y}\|\big(\mathcal{A}(t,\vec{u})+\sigma I\big)\vec{v}\|_X. \tag{3.10}$$

Let $\vec{f}=(f_1,f_2)=U(t)(\mathcal{A}(t,\vec{u})+\sigma I)U(-t)\vec{w}$, $\vec{w}=(w_1,w_2)\in H^1_{\mathbb{C}}\times H^1_{\mathbb{R}}$, then

$$\begin{cases} f_1=\sigma w_1+\mu Dw_1+(P(t)u_2)Dw_1+i\lambda(P(t)u_2)w_1+(Q(t)u_1)Dw_2\\ f_2=\sigma w_2+\mathrm{Re}\big(\overline{Q(t)u_1}Dw_1\big)+(1+P(t)u_2)Dw_2.\end{cases}$$

From the first equation, we have

$$\begin{aligned}
\mu\|Dw_1\| &\leq \|f_1\|+\sigma\|w_1\|+\|P(t)u_2\|_{L^\infty}\|Dw_1\|+\lambda\|P(t)u_2\|_{L^\infty}\|w_1\|+\|Q(t)u_1\|_{L^\infty}\|Dw_2\|\\
&\leq \|f_1\|+\frac{\sigma+\lambda\beta_{s-1}\|\vec{u}\|_Y}{\sigma-2\beta_{s-1}\|\vec{u}\|_Y}\|\vec{f}\|+\beta_{s-1}\|u_2\|_{H^1}\|Dw_1\|+\beta_{s-1}\|u_1\|_{H^1}\|Dw_2\|\\
&\leq \frac{2\sigma+(\lambda-2)\beta_{s-1}R}{\sigma-2\beta_{s-1}R}\|\vec{f}\|+\|\vec{u}\|_{H^1_{\mathbb{C}}\times H^1_{\mathbb{R}}}\beta_{s-1}\|Dw_1\|+\|\vec{u}\|_{H^1_{\mathbb{C}}\times H^1_{\mathbb{R}}}\beta_{s-1}\|Dw_2\|
\end{aligned}$$

where (3.10), (3.3) and $R=\|\vec{u}\|_Y$ have been used. It indicates that

$$(\mu-\|\vec{u}\|_{H^1_{\mathbb{C}}\times H^1_{\mathbb{R}}}\beta_{s-1})\|Dw_1\| \leq \frac{2\sigma+(\lambda-2)\beta_{s-1}R}{\sigma-2\beta_{s-1}R}\|\vec{f}\|+\|\vec{u}\|_{H^1_{\mathbb{C}}\times H^1_{\mathbb{R}}}\beta_{s-1}\|Dw_2\|. \tag{3.11}$$

Similarly, we obtain from the second equation that

$$(1-\|\vec{u}\|_{H^1_{\mathbb{C}}\times H^1_{\mathbb{R}}}\beta_{s-1})\|Dw_2\| \leq \frac{2\sigma-2\beta_{s-1}R}{\sigma-2\beta_{s-1}R}\|\vec{f}\|+\|\vec{u}\|_{H^1_{\mathbb{C}}\times H^1_{\mathbb{R}}}\beta_{s-1}\|Dw_1\|. \tag{3.12}$$

Summing up (3.11) and (3.12) leads to

$$(\mu-2\|\vec{u}\|_{H^1_{\mathbb{C}}\times H^1_{\mathbb{R}}}\beta_{s-1})\|Dw_1\|+(1-2\|\vec{u}\|_{H^1_{\mathbb{C}}\times H^1_{\mathbb{R}}}\beta_{s-1})\|Dw_2\|\le\frac{4\sigma+(\lambda-4)\beta_{s-1}R}{\sigma-2\beta_{s-1}R}\|\vec{f}\|. \tag{3.13}$$

Combining (3.13) and (3.10) gives us

$$\|\big(\mathcal{A}(t,\vec{u})+\sigma I\big)\vec{v}\|_X\ge C\|\vec{v}\|_{H^1_{\mathbb{C}}\times H^1_{\mathbb{R}}} \tag{3.14}$$

holding true for $\|\vec{u}\|_{H^1_{\mathbb{C}}\times H^1_{\mathbb{R}}}<\frac{\min\{1,\mu\}}{2\beta_{s-1}}$ for some constant $C>0$ independent of $\vec{v}$. It follows then from (3.14) that $\mathrm{R}\big(\mathcal{A}(t,\vec{u})+\sigma I\big)$ is closed in $X$.

Step 3: To show that $\mathrm{R}\big(\mathcal{A}(t,\vec{u})+\sigma I\big)$ is dense in $X$, let $\vec{y}\in X$ and $\vec{y}\perp\mathrm{R}\big(\mathcal{A}(t,\vec{u})+\sigma I\big)$, namely,

$$\big((\mathcal{A}(t,\vec{u})+\sigma I)\vec{v},\vec{y}\big)=0,\ \forall\vec{v}\in H^1_{\mathbb{C}}\times H^1_{\mathbb{R}}. \tag{3.15}$$

Set $\vec{w}=U(t)\vec{v}$ and $\vec{g}=U(t)\vec{y}$, then $\vec{w}=(w_1,w_2)\in H^1_{\mathbb{C}}\times H^1_{\mathbb{R}}$ and $\vec{g}=(g_1,g_2)\in L^2_{\mathbb{C}}\times L^2_{\mathbb{R}}$. Thus (3.15) can be rewritten as

$$\big((U(t)(\mathcal{A}(t,\vec{u})+\sigma I)U(-t)\vec{w},\vec{g}\big)=0,\ \forall\vec{w}\in H^1_{\mathbb{C}}\times H^1_{\mathbb{R}},$$

which amounts to the same as

$$\begin{aligned}\mathrm{Re}\int_{\mathbb{R}}\big(\sigma w_1+\mu Dw_1+(P(t)u_2)Dw_1+i\lambda(P(t)u_2)w_1+(Q(t)u_1)Dw_2\big)\overline{g_1}dx\\+\int_{\mathbb{R}}\big(\sigma w_2+\mathrm{Re}\big(\overline{Q(t)u_1}Dw_1\big)+(1+P(t)u_2)Dw_2\big)g_2dx=0.\end{aligned} \tag{3.16}$$

Taking $w_1=0$ in (3.16) gives us

$$\mathrm{Re}\int_{\mathbb{R}}\big(\overline{g_1}Q(t)u_1+g_2(1+P(t)u_2)\big)Dw_2dx+\int_{\mathbb{R}}\sigma g_2w_2dx=0, \tag{3.17}$$

while taking $w_2=0$ and replacing $w_1$ by $iw_1$ in (3.16) gives us

$$\int_{\mathbb{R}}\big(\sigma\overline{g_1}+i\lambda\overline{g_1}P(t)u_2\big)w_1dx+\int_{\mathbb{R}}\big(\mu\overline{g_1}+\overline{g_1}P(t)u_2+g_2\overline{Q(t)u_1}\big)Dw_1dx=0. \tag{3.18}$$

Define $h_1=\mathrm{Re}\big(g_1\overline{Q(t)u_1}\big)+g_2(1+P(t)u_2)$, $h_2=\sigma g_1-i\lambda g_1P(t)u_2$ and $h_3=\mu g_1+g_1P(t)u_2+g_2Q(t)u_1$. Notice that $w_2,g_2,P(t)u_2$ are real-valued functions; thus, (3.17) and (3.18) imply that

$$\int_{\mathbb{R}}h_1Dw_2dx+\int_{\mathbb{R}}\sigma g_2w_2dx=0,\quad\int_{\mathbb{R}}\overline{h_2}w_1dx+\int_{\mathbb{R}}\overline{h_3}Dw_1dx=0. \tag{3.19}$$

By Plancherel's identity, we deduce from (3.19) that

$$\int_{\mathbb{R}}\Big(-i\xi\overline{\widehat{h_1}}+\sigma\overline{\widehat{g_2}}\Big)\widehat{w_2}d\xi=0,\quad\int_{\mathbb{R}}\Big(-i\xi\overline{\widehat{h_3}}+\overline{\widehat{h_2}}\Big)\widehat{w_1}d\xi=0,$$

from which we conclude that $-i\xi\overline{\widehat{h_1}}+\sigma\overline{\widehat{g_2}}=0$ for a.e. $\xi$. Thus $\sigma g_2=Dh_1$ and $h_1\in H^1$. Similarly, $-i\xi\overline{\widehat{h_3}}+\overline{\widehat{h_2}}=0$ for a.e. $\xi$, and hence $h_3\in H^1_{\mathbb{C}}$ with $h_2=Dh_3$. In terms of $h_1$ and $h_3$, we have

$$g_2=\frac{(\mu+P(t)u_2)h_1-\mathrm{Re}(\overline{h_3}Q(t)u_1)}{(\mu+P(t)u_2)(1+P(t)u_2-|Q(t)u_1|^2)}.$$

Note that

$$\max\{\|P(t)u_2\|_{L^\infty}, \|Q(t)u_1\|_{L^\infty}\} \le \beta_{s-1}\|\vec{u}\|_{H^1\times H^1},$$

hence the denominator of $g_2$ is nonzero and its absolute value has a positive lower bound when $\|\vec{u}\|_{H^1_{\mathbb{C}}\times H^1_{\mathbb{R}}} < \frac{\min\{1,\mu\}}{2\beta_{s-1}}$ . Thus, $g_2 \in H^1$ and $g_1 \in H^1$. By (3.15) and (3.9) we obtain that $g_1 = g_2 = 0$ which implies that $\vec{y} = 0$. Consequently, the set $\mathrm{R}\big(\mathcal{A}(t,\vec{u}) + \sigma I\big)$ is dense in $X$. Therefore, $\mathrm{R}\big(\mathcal{A}(t,\vec{u}) + \sigma I\big) = X$. This concludes the proof of the lemma. ∎

We will verify property B next.

Lemma 3.3. *Suppose that $s > \frac{3}{2}$ and let $\boldsymbol{\Lambda} = (1-D^2)^{\frac{1}{2}} I$. Then $\boldsymbol{\Lambda}^s : Y \to X$ is an isomorphism of $Y$ onto $X$, and for all $\vec{u},\vec{v} \in Y$*

(1) *$\mathcal{B}(t,\vec{u}) := \boldsymbol{\Lambda}^s \mathcal{A}(t,\vec{u}) \boldsymbol{\Lambda}^{-s} - \mathcal{A}(t,\vec{u})$ is a bounded operator from $X$ to $X$;*
(2) *$\|\mathcal{B}(t,\vec{u})\|_{\mathcal{B}(X,X)} \le \lambda_0 \|\vec{u}\|_Y$;*
(3) *$\|\mathcal{B}(t,\vec{u}) - B(t,\vec{v})\|_{\mathcal{B}(X,X)} \le \lambda_0 \|\vec{u} - \vec{v}\|_Y$,*

*where $\lambda_0$ is independent of $t$ and $\vec{u},\vec{v} \in Y$.*

*Proof.* First, we observe that item (1) is implied by item (2), and item (3) is equivalent to item (2) because $\mathcal{B}(t,\vec{u})$ is linear with respect to $\vec{u}$. Therefore, it suffices to prove item (2). Indeed, we have $\mathcal{U}(t)\mathcal{B}(t,\vec{u})\mathcal{U}(-t) =$

$$\begin{pmatrix} [\Lambda^s, M_{P(t)u_2}]\Lambda^{-s}D + [\Lambda^s, M_{i\lambda P(t)u_2}]\Lambda^{-s} & [\Lambda^s, M_{Q(t)u_1}]\Lambda^{-s}D \\ [\Lambda^s, \mathrm{Re} M_{\overline{Q(t)u_1}}]\Lambda^{-s}D & [\Lambda^s, M_{P(t)u_2}]\Lambda^{-s}D \end{pmatrix}.$$

By Lemma 3.1, we have

$$\|[\Lambda^s, M_{P(t)u_2}]\Lambda^{-s}D\| \le \|[\Lambda^s, M_{P(t)u_2}]\Lambda^{1-s}\|\|\Lambda^{-1}D\| \le c_0\|P(t)Du_2\|_{s-1} \le c_0\|\vec{u}\|_Y;$$

$$\|[\Lambda^s, M_{i\lambda P(t)u_2}]\Lambda^{-s}\| \le \|[\Lambda^s, M_{i\lambda P(t)u_2}]\Lambda^{1-s}\|\|\Lambda^{-1}\| \le c_0\|\lambda P(t)Du_2\|_{s-1} \le |\lambda| c_0\|\vec{u}\|_Y,$$

where the norms in the above inequalities are the operator norms in $L^2(\mathbb{R})$, and where we have used the facts that $\|\Lambda^{-1}D\| \le 1$ and $\|\Lambda^{-1}\| \le 1$. The other terms can be dealt with similarly. Consequently,

$$\|\mathcal{B}(t,\vec{u})\|_{\mathcal{B}(X,X)} \le \lambda_0\|\vec{u}\|_Y. \tag{3.20}$$

Hence the lemma is proved. ∎

Property C for the system (1.3) is the following.

Lemma 3.4. *Suppose that $s > \frac{3}{2}$. Then*

$$\|\mathcal{A}(t,\vec{u}) - \mathcal{A}(t,\vec{v})\|_{\mathcal{B}(Y,X)} \le \lambda_1\|\vec{u} - \vec{v}\|_X$$

*holds for all $\vec{u},\vec{v} \in X$ with $\lambda_1$ a constant independent of $t$ and $\vec{u},\vec{v}$.*

*Proof.* Let $\vec{u} = (u_1,u_2)$ and $\vec{v} = (v_1,v_2)$, then $\mathcal{U}(t)\Big(\mathcal{A}(t,\vec{u}) - \mathcal{A}(t,\vec{v})\Big)\mathcal{U}(-t) =$

$$\begin{pmatrix} M_{P(t)(u_2-v_2)}D + M_{i\lambda P(t)(u_2-v_2)} & M_{Q(t)(u_1-v_1)}D \\ \mathrm{Re}\big(M_{\overline{Q(t)(u_1-v_1)}}D\big) & M_{P(t)(u_2-v_2)}D \end{pmatrix}$$

which reveals that the operator $\mathcal{A}(t,\vec{u})-\mathcal{A}(t,\vec{v})$ is linear with respect to the difference $\vec{u}-\vec{v}$. Thus, it suffices to prove that the following inequality

$$\left\| \begin{pmatrix} M_{P(t)u_2}D+M_{i\lambda P(t)u_2} & M_{Q(t)u_1}D \\ \mathrm{Re}\big(M_{\overline{Q(t)u_1}}D\big) & M_{P(t)u_2}D \end{pmatrix} \right\|_{\mathcal{B}(Y,X)} \le C\beta_{s-1}\|\vec{u}\|_X \tag{3.21}$$

holds for any $\vec{u}\in Y$. From the definition of $\beta$ in (3.3) and an application of the Hölder inequality, we have the following estimate for the first entry of the matrix in (3.21)

$$\|P(t)u_2Df\|_{L^2}\le \|P(t)u_2\|_{L^2}\|Df\|_{L^\infty}\le \beta_{s-1}\|\vec{u}\|_X\|f\|_s,$$

for any $f\in H^s$. All the other terms can be dealt with in the same way. Consequently, (3.21) follows from similar argument used to prove (3.20). □

Property D for the system (1.3) is established next.

LEMMA 3.5. *The mapping $t\to\mathcal{A}(t,\vec{u})\in\mathcal{B}(Y,X)$ is strongly continuous for any fixed $\vec{u}\in Y$.*

*Proof.* First we observe that

$$U(t)\mathcal{A}(t,\vec{u})U(-t)=\begin{pmatrix} M_{\mu+P(t)v}D+M_{i\lambda P(t)v} & M_{Q(t)u}D \\ \mathrm{Re}\big(M_{\overline{Q(t)u}}D\big) & M_{1+P(t)v}D \end{pmatrix}$$

and $\mathcal{U}(t)$ is a unitary operator. Then $\|\mathcal{A}(t,\vec{u})\|_{\mathcal{B}(Y,X)}\le C(1+\|\vec{u}\|_Y)$ for any $\vec{u}\in Y$ and for all $t\ge 0$ by an argument similar to (3.21). Moreover, it is straightforward to see that $P(t)f$, $Q(t)f\in L^2$ are continuous in $t$ for any fixed $f\in L^2$ by the Lebesgue's dominated convergence theorem. Thus, the lemma is proved. □

Theorem 3.1 can now be seen as a direct consequence of Kato's theory outlined above.

**3.2. The Schrödinger KdV-BBM system.** Recall that the Schrödinger KdV-BBM system (1.5) is

$$\begin{cases} u_t+\mu u_x+a_0u_{xxx}+ibu_{xx}=-(uv)_x-i\lambda uv, \\ v_t+v_x+vv_x-cv_{xxt}=-\dfrac{1}{2}(|u|^2)_x. \end{cases}$$

The local theory for the Schrödinger KdV-BBM system is as follows.

THEOREM 3.2.

(i) *Let $s>\frac{3}{2}$. There exists a $\delta>0$ such that for any $(f,g)\in Y$ with $\|(f,g)\|_{H^1_{\mathbb{C}}(\mathbb{R})\times H^1_{\mathbb{R}}(\mathbb{R})}\le\delta$, there is a unique solution $(u,v)\in C([0,T],Y)\bigcap C^1([0,T],H^{s-3}_{\mathbb{C}}\times H^{s-3}_{\mathbb{R}})$ to (1.5)-(2.1) with $T$ having a lower bound depending only on the norm $\|(f,g)\|_Y$.*

(ii) *The map $(f,g)\to\big(u(t),v(t)\big)$ is continuous in the $Y$-norm. More precisely, if $(f_n,g_n)\in Y$, $n=1,2,\cdots$, with $\|(f_n,g_n)-(f,g)\|_Y\to 0$ and $T'<T$, the solution $(u_n,v_n)$ for $\big(u_n(0),v_n(0)\big)=(f_n,g_n)$ exists on $[0,T']$ for sufficiently large $n$ and $\|(u_n,v_n)-(u,v)\|_Y\to 0$ uniformly in $t\in[0,T']$.*

**3.3. The Schrödinger BBM-KdV system.** Recall that the Schrödinger BBM-KdV system (1.6) is

$$\begin{cases} u_t+\mu u_x-a_1u_{xxt}+ibu_{xx}=-(uv)_x-i\lambda uv, \\ v_t+v_x+vv_x+cv_{xxx}=-\dfrac{1}{2}(|u|^2)_x. \end{cases}$$

The local theory for the Schrödinger BBM-KdV system is as follows.

THEOREM 3.3.

(i) *Let* $a_1>0$ *and* $s>\frac{3}{2}$. *There exists a* $\delta>0$ *such that for any initial data* $(f,g)\in Y$ *with* $\|(f,g)\|_{H^1_{\mathbb{C}}(\mathbb{R})\times H^1_{\mathbb{R}}(\mathbb{R})}\le\delta$, *there is a unique solution* $(u,v)\in C([0,T],Y)\bigcap C^1([0,T],H^{s-3}_{\mathbb{C}}\times H^{s-3}_{\mathbb{R}})$ *to* (1.6)-(2.1) *with* $T$ *having a lower bound depending only on the norm* $\|(f,g)\|_Y$.

(ii) *The map* $(f,g)\to\big(u(t),v(t)\big)$ *is continuous in the* $Y$*-norm. More precisely, if* $(f_n,g_n)\in Y$, $n=1,2,\cdots$, *with* $\|(f_n,g_n)-(f,g)\|_Y\to 0$ *and* $T'<T$, *the solution* $(u_n,v_n)$ *for* $\big(u_n(0),v_n(0)\big)=(f_n,g_n)$ *exists on* $[0,T']$ *for sufficiently large* $n$ *and* $\|(u_n,v_n)-(u,v)\|_Y\to 0$ *uniformly in* $t\in[0,T']$.

**3.4. The Schrödinger BBM-BBM system.** Recall that the Schrödinger BBM-BBM system (1.4) is

$$\begin{cases} u_t+\mu u_x-a_1u_{xxt}+ibu_{xx}=-(uv)_x-i\lambda uv,\\ v_t+v_x+vv_x-cv_{xxt}=-\dfrac{1}{2}(|u|^2)_x. \end{cases}$$

Notice that the first equation in (1.4) is the same as the first equation in (1.6), whilst the second equation is the same as the second equation in (1.5). Thus, the local well-posedness for (1.4) in the space $Y:\equiv H^s_{\mathbb{C}}\times H^s_{\mathbb{R}}$, $s>3/2$, follows immediately by the same Kato's method as before. However, due to the presence of the operator $(1-k\partial_{xx})^{-1}$, for $k>0$, this result can also be obtained and improved through an argument via the Contraction Mapping Principle. This approach actually establishes the local well-posedness theory for (1.4) in $Z:\equiv H^s_{\mathbb{C}}\times H^s_{\mathbb{R}}$ for $s\ge 0$, as well as several other regularity properties for its solutions that cannot be obtained (or at least it is not trivial to do so) with Kato's method. Thus, we present our result for system (1.4) through the method of Contraction Mapping Principle.

The local existence result for the system (1.4)-(2.1) is as follows.

THEOREM 3.4. *Let* $a_1>0$ *and* $s\ge 0$ *be given. For any* $(f,g)\in Z$, *there exist a time* $T>0$ *and a unique solution* $(u,v)$ *of* (1.4)-(2.1) *that satisfies* $(u,v)\in C([0,T],Z)$. *Moreover, the correspondence* $(f,g)\to(u,v)$ *is locally Lipschitz continuous.*

*Proof.* Let $u\to u+iw$ and after equating the real and imaginary parts, the system (1.4) is rewritten as

$$\begin{cases} (1-a_1\partial_{xx})u_t=-(\mu u+uv-bw_x)_x+\lambda wv,\\ (1-a_1\partial_{xx})w_t=-(\mu w+wv+bu_x)_x-\lambda uv,\\ (1-c\partial_{xx})v_t=-(v+\frac{1}{2}u^2+\frac{1}{2}w^2+\frac{1}{2}v^2)_x. \end{cases} \tag{3.22}$$

Therefore, we can instead consider the system (3.22)-(2.1) with initial data $\big(u(x,0),w(x,0),v(x,0)\big)=(f,g,h)\in\tilde{Z}:\equiv H^s_{\mathbb{R}}\times H^s_{\mathbb{R}}\times H^s_{\mathbb{R}}$, $s\ge 0$. For any $k>0$, set

$$M_k(z)=\frac{1}{2\sqrt{k}}e^{-\frac{|z|}{\sqrt{k}}},\quad z\in\mathbb{R}.$$

The operator $(1-k\partial_{xx})$ is invertible and

$$(1-k\partial_{xx})^{-1}f=M_k*f:=\int_{-\infty}^{\infty}M_k(x-y)f(y)dy,$$

for all $f \in L^p(\mathbb{R})$ with $p \in [1,\infty)$. Moreover, for all rapidly decreasing functions $f$ on $\mathbb{R}$

$$M_k * f' = (M_k)' * f.$$

Denote $K_k = -(M_k)'$, then we have

$$K_k(z) = \frac{1}{2k}\operatorname{sgn}(z) e^{-\frac{|z|}{\sqrt{k}}}.$$

Therefore, for $a_1 > 0$, (recall that $c > 0$), (3.22) can be formulated as

$$\begin{cases} u(x,t) = f(x) - \displaystyle\int_0^t (1 - a_1\partial_{xx})^{-1}\partial_x\big(\mu u(x,\tau) + u(x,\tau)v(x,\tau) - bw_x(x,\tau)\big)d\tau \\ \qquad\qquad + \lambda \displaystyle\int_0^t (1 - a_1\partial_{xx}^{-1}\big(w(x,\tau)v(x,\tau)\big)d\tau, \\ w(x,t) = g(x) - \displaystyle\int_0^t (1 - a_1\partial_{xx})^{-1}\partial_x\big(\mu w(x,\tau) + w(x,\tau)v(x,\tau) + bu_x(x,\tau)\big)d\tau \\ \qquad\qquad - \lambda \displaystyle\int_0^t (1 - a_1\partial_{xx}^{-1}\big(u(x,\tau)v(x,\tau)\big)d\tau, \\ v(x,t) = h(x) - \displaystyle\int_0^t (1 - c\partial_{xx})^{-1}\partial_x\Big(v(x,\tau) + \frac{1}{2}u^2(x,\tau) + \frac{1}{2}w^2(x,\tau) + \frac{1}{2}v^2(x,\tau)\Big)d\tau. \end{cases} \tag{3.23}$$

The above system can be rewritten compactly as

$$\vec{u} = \mathcal{A}\vec{u},$$

where for

$$\vec{u} = \begin{bmatrix} u(x,t) \\ w(x,t) \\ v(x,t) \end{bmatrix} \in \begin{bmatrix} H^s \\ H^s \\ H^s \end{bmatrix},$$

the operator $\mathcal{A}$ is defined as

$$\begin{aligned} \mathcal{A}\vec{u} = \begin{bmatrix} f(x) \\ g(x) \\ h(x) \end{bmatrix} - \begin{bmatrix} \int_0^t (1 - a_1\partial_{xx})^{-1}\partial_x\big(\mu u(x,\tau) + u(x,\tau)v(x,\tau) - bw_x(x,\tau)\big)d\tau \\ \int_0^t (1 - a_1\partial_{xx})^{-1}\partial_x\big(\mu w(x,\tau) + w(x,\tau)v(x,\tau) + bu_x(x,\tau)\big)d\tau \\ \int_0^t (1 - c\partial_{xx})^{-1}\partial_x\big(v(x,\tau) + \frac{1}{2}u^2(x,\tau) + \frac{1}{2}w^2(x,\tau) + \frac{1}{2}v^2(x,\tau)\big)d\tau \end{bmatrix} \\ + \begin{bmatrix} \lambda\int_0^t (1 - a_1\partial_{xx})^{-1}\big(w(x,\tau)v(x,\tau)\big)d\tau \\ -\lambda\int_0^t (1 - a_1\partial_{xx})^{-1}\big(u(x,\tau)v(x,\tau)\big)d\tau \\ 0 \end{bmatrix}. \end{aligned}$$

We will show that there exists a time $T > 0$ such that for some radius $R > 0$, the operator $\mathcal{A}$ is a Lipschitz continuous *contraction mapping* of $B_R$ into itself. Recall that for any fixed $T > 0$, we denote $C_T :\equiv C([0,T], H^s)$ and $\mathcal{C}_T^3 :\equiv C([0,T], H^s \times H^s \times \times H^s)$. Fix a time $T > 0$ whose value will be determined presently and set

$$B_R = \{\vec{u} : \|\vec{u}\|_{\mathcal{C}_T^3} \leq R\},$$

with the norm being defined as (where $\|f\|_{C_T} = \|f\|_{C([0,T],H^s)}$)

$$\|\vec{u}\|_{\mathcal{C}_T^3} = \max\Big\{\|u\|_{C_T},\ \|w\|_{C_T},\ \|v\|_{C_T}\Big\}.$$

Consider first the case when $s=0$. We recall the following facts:

(a) $\|(1-k\partial_{xx})^{-1}f_{xx}\|_{L^2}\le C\|f\|_{L^2}$;
(b) $\|(1-k\partial_{xx})^{-1}f_x\|_{L^2}\le C\|f\|_{H^{-1}}\le C\|f\|_{L^2}$;
(c) $\|(1-k\partial_{xx})^{-1}(fg)_x\|_{L^2}\le C\|fg\|_{H^{-1}}\le C\|fg\|_{L^1}\le C\|f\|_{L^2}\|g\|_{L^2}$;
(d) $\|(1-k\partial_{xx})^{-1}fg\|_{L^2}\le C\|f\|_{L^2}\|g\|_{L^2}$,

where $C$ denotes various universal constants independent of $f$ and $g$ whose precise values are not of importance to us. For any two pairs $\vec{u}_1=(u_1,w_1,v_1)$ and $\vec{u}_2=(u_2,w_2,v_2)\in\mathcal{C}_T^3$, it must hold true that

$$\mathcal{A}\vec{u}_1-\mathcal{A}\vec{u}_2=\begin{bmatrix}\lambda\int_0^t(1-a_1\partial_{xx})^{-1}(w_1v_1-w_2v_2)d\tau\\ -\lambda\int_0^t(1-a_1\partial_{xx})^{-1}(u_1v_1-u_2v_2)\big)d\tau\\ 0\end{bmatrix}$$
$$+\begin{bmatrix}-\int_0^t(1-a_1\partial_{xx})^{-1}\big(\mu(u_1-u_2)_x+(u_1v_1-u_2v_2)_x-b(w_1-w_2)_{xx}\big)d\tau\\ -\int_0^t(1-a_1\partial_{xx})^{-1}\big(\mu(w_1-w_2)_x+(w_1v_1-w_2v_2)_x+b(u_1-u_2)_{xx}\big)d\tau\\ -\int_0^t(1-c\partial_{xx})^{-1}\big((v_1-v_2)_x+\frac{1}{2}(u_1^2-u_2^2)_x+\frac{1}{2}(w_1^2-w_2^2)_x+\frac{1}{2}(v_1^2-v_2^2)_x\big)d\tau\end{bmatrix}.$$

Using the above stated facts, we obtain for the first component, $\forall t\in[0,T]$

$$\begin{aligned}&-(1-a_1\partial_{xx})^{-1}\big(\mu(u_1-u_2)_x+(u_1v_1-u_2v_2)_x-b(w_1-w_2)_{xx}\big)\\&+\lambda(1-a_1\partial_{xx})^{-1}(w_1v_1-w_2v_2)\le C(\mu,\lambda,a_1,b)\big(\|u_1-u_2\|_{L^2}+\|w_1-w_2\|_{L^2}\\&+\|u_1\|_{L^2}\|v_1-v_2\|_{L^2}+\|v_2\|_{L^2}\|u_1-u_2\|_{L^2}+\|w_1\|_{L^2}\|v_1-v_2\|_{L^2}+\|v_2\|_{L^2}\|w_1-w_2\|_{L^2}\big).\end{aligned}$$

The other two components can be handled similarly. Thus,

$$\begin{aligned}&\|\mathcal{A}\vec{u}_1-\mathcal{A}\vec{u}_2\|_{\mathcal{C}_T^3}\\&\le CT\Big(\|u_1\|_{C_T}+\|u_2\|_{C_T}+\|w_1\|_{C_T}+\|w_2\|_{C_T}+\|v_1\|_{C_T}+\|v_2\|_{C_T}+6\Big)\|\vec{u}_1-\vec{u}_2\|_{\mathcal{C}_T^3}\end{aligned}\tag{3.24}$$

where $C=C(\mu,\lambda,a_1,b,c)$. If $\vec{u}_1$ and $\vec{u}_2\in B_R$, then (3.24) implies that

$$\|\mathcal{A}\vec{u}_1-\mathcal{A}\vec{u}_2\|_{\mathcal{C}_T^3}\le 6CT(1+R)\|\vec{u}_1-\vec{u}_2\|_{\mathcal{C}_T^3}.$$

It is now time to pick $T$ and we take

$$T=\frac{1}{12C(1+R)}.$$

Choose $R$ such that $R=2\max\{\|f\|_{L^2},\|g\|_{L^2},\|h\|_{L^2}\}$. Then

$$\|\mathcal{A}\vec{u}\|_{\mathcal{C}_T^3}\le\|\mathcal{A}\vec{u}-\mathcal{A}\vec{0}\|_{\mathcal{C}_T^3}+\|\mathcal{A}\vec{0}\|_{\mathcal{C}_T^3}\le R,$$

whenever $\vec{u}\in B_R$. Thus, $\mathcal{A}$ is a contraction mapping that takes $B_R$ continuously into itself and the first part of the theorem follows immediately from the Contraction Mapping Principle for $s=0$. Notice that since $T=\frac{1}{12C(1+R)}$, the interval of existence gets smaller as the initial data gets larger and vice versa. However $T$ does not approach infinity as the data approach zero, at least as these estimates indicate.

For the case when $s>0$, the exact same argument can be employed using the space $H^s$ instead of $L^2$, and the fact that

$$\|(1-k\partial_{xx}^2)^{-1}(fg)_x\|_{H^s}\le C\|fg\|_{H^{s-1}}\le C\|f\|_{H^s}\|g\|_{H^s},$$

whose proof is straightforward and can be found, for example, in [4]. The existence of a unique solution for (3.22)-(2.1) is thus established for $s \geq 0$.

To see that the correspondence $(f,g,h) \to (u,w,v)$ is locally Lipschitz continuous, let $(f,g,h)$ and $(\tilde{f},\tilde{g},\tilde{h})$ be two initial data given in $\tilde{Z}$ and let the two unique solutions emanating from them through the evolution (3.22) be, respectively,

$$(u,w,v) = A_{(f,g,h)}(u,w,v) \qquad \text{and} \qquad (\tilde{u},\tilde{w},\tilde{v}) = A_{(\tilde{f},\tilde{g},\tilde{h})}(\tilde{u},\tilde{w},\tilde{v}).$$

Suppose both solutions stay inside the ball $\overline{B_R}$ about the origin in $\tilde{Z}$. It follows from above that both mappings $A_{(f,g,h)}$ and $A_{(\tilde{f},\tilde{g},\tilde{h})}$ are contractions, say with Lipschitz constants $\theta$ and $\tilde{\theta}$ in $(0,1)$. An application of the triangle inequality shows that

$$\begin{aligned}
&\|(u,w,v) - (\tilde{u},\tilde{w},\tilde{v})\|_{\mathcal{C}_T^3} \\
&= \|A_{(f,g,h)}(u,w,v) - A_{(\tilde{f},\tilde{g},\tilde{h})}(\tilde{u},\tilde{w},\tilde{v})\|_{\mathcal{C}_T^3} \\
&\leq \|A_{(f,g,h)}(u,w,v) - A_{(f,g,h)}(\tilde{u},\tilde{w},\tilde{v})\|_{\mathcal{C}_T^3} \\
&\qquad + \|A_{(f,g,h)}(\tilde{u},\tilde{w},\tilde{v}) - A_{(\tilde{f},\tilde{g},\tilde{h})}(\tilde{u},\tilde{w},\tilde{v})\|_{\mathcal{C}_T^3} \\
&\leq \theta \|(u,w,v) - (\tilde{u},\tilde{w},\tilde{v})\|_{\mathcal{C}_T} + \|(f,g,h) - (\tilde{f},\tilde{g},\tilde{h})\|_{\tilde{Z}},
\end{aligned}$$

from which Lipschitz continuity follows with a Lipschitz constant at most $1/(1-\theta)$. This completes the proof of the theorem. ∎

Indeed, due to the presence of the operators $(1 - a_1\partial_{xx})^{-1}$ and $(1 - c\partial_{xx})^{-1}$, more can be acquired as regard the regularity of solution $(u,w,v)$ of (3.22)-(2.1).

Proposition 3.1. *If $f,g,h \in C_b^2(\mathbb{R})$ and $\vec{u}$ is a solution in $\mathcal{C}_T^3$ of the system (3.22)-(2.1), then $\vec{u}, \vec{u}_x$ and $\vec{u}_{xx}$ are infinitely smooth functions of $t$, and $\vec{u}$ solves (3.22) pointwise. More precisely, $\partial_t^m \vec{u}$, $\partial_t^m \vec{u}_x$, and $\partial_t^m \vec{u}_{xx} \in \mathcal{C}_T^3$ for any $m \geq 0$, and the quantities*

$$\begin{cases}
u_t + \mu u_x - a_1 u_{xxt} - b w_{xx} + (uv)_x - \lambda wv = 0, \\
w_t + \mu w_x - a_1 w_{xxt} + b u_{xx} + (wv)_x + \lambda uv = 0, \\
v_t + v_x + v v_x - c v_{xxt} + \frac{1}{2}(u^2 + w^2)_x = 0,
\end{cases}$$

*hold true for all $(x,t) \in \mathbb{R} \times [0,T]$. Furthermore,*

$$\lim_{t \to 0} \vec{u}(x,t) = \begin{bmatrix} f(x) \\ g(x) \\ h(x) \end{bmatrix}$$

*in $C_b^2(\mathbb{R}) \times C_b^2(\mathbb{R}) \times C_b^2(\mathbb{R})$.*

*Proof.* Recall that

$$\begin{cases}
u(x,t) = f(x) + \displaystyle\int_0^t \int_{-\infty}^{\infty} K_{a_1}(x-y)[\mu u(y,\tau) + u(y,\tau)v(y,\tau) + b w_y(y,\tau)]\,dyd\tau \\
\qquad\qquad + \lambda \displaystyle\int_0^t \int_{-\infty}^{\infty} M_{a_1}(x-y) w(y,\tau) v(y,\tau)\,dyd\tau, \\
w(x,t) = g(x) + \displaystyle\int_0^t \int_{-\infty}^{\infty} K_{a_1}(x-y)[\mu w(y,\tau) + w(y,\tau)v(y,\tau) - b u_y(y,\tau)]\,dyd\tau \\
\qquad\qquad - \lambda \displaystyle\int_0^t \int_{-\infty}^{\infty} M_{a_1}(x-y) u(y,\tau) v(y,\tau)\,dyd\tau, \\
v(x,t) = h(x) + \displaystyle\int_0^t \int_{-\infty}^{\infty} K_c(x-y)[v(y,\tau) + \frac{1}{2}u^2(y,\tau) + \frac{1}{2}w^2(y,\tau) + \frac{1}{2}v^2(y,\tau)]\,dyd\tau.
\end{cases}$$

Hence clearly $u,w,v$ are differentiable with respect to $t$ and

$$\begin{cases} u_t(x,t)=\displaystyle\int_{-\infty}^{\infty}\Big(K_{a_1}(x-y)[\mu u(y,t)+u(y,t)v(y,t)-bw_y(y,t)] \\ \qquad +\lambda M_{a_1}(x-y)w(y,t)v(y,t)\Big)dy, \\ w_t(x,t)=\displaystyle\int_{-\infty}^{\infty}\Big(K_{a_1}(x-y)[\mu w(y,t)+w(y,t)v(y,t)+bu_y(y,t)] \\ \qquad -\lambda M_{a_1}(x-y)u(y,t)v(y,t)\Big)dy, \\ v_t(x,t)=\displaystyle\int_{-\infty}^{\infty}K_c(x-y)[v(y,t)+\frac{1}{2}u^2(y,t)+\frac{1}{2}w^2(y,t)+\frac{1}{2}v^2(y,t)]dy, \end{cases}$$

which implies that $\vec{u}_t\in\mathcal{C}_T^3$ as $K_{a_1},K_c\in L^1(\mathbb{R})$. Since $\vec{u}_t\in\mathcal{C}_T^3$, we have

$$\begin{cases} u_{tt}=\displaystyle\int_{-\infty}^{\infty}\Big(K_{a_1}(x-y)[\mu u_t+(uv)_t-bw_{yt}]+\lambda M_{a_1}(x-y)(wv)_t\Big)dy, \\ w_{tt}=\displaystyle\int_{-\infty}^{\infty}\Big(K_{a_1}(x-y)[\mu w_t+(wv)_t+bu_{yt}]-\lambda M_{a_1}(x-y)(uv)_t\Big)dy, \\ v_{tt}=\displaystyle\int_{-\infty}^{\infty}K_c(x-y)[v_t+\frac{1}{2}(u^2+w^2+v^2)_t]dy. \end{cases}$$

An inductive argument now gives $\partial_t^m\vec{u}\in\mathcal{C}_T^3$, for any $m\geq 0$. Notice that indeed we do not need to impose here that $f,g,h\in C_b^2(\mathbb{R})$; that is, $\partial_t^m\vec{u}\in\mathcal{C}_T^3$, for any $m\geq 0$ independent of the smoothness of the initial data $f,g$ and $h$. (See also the following remark at the end of the proof.)

Next, using the convolution property we have

$$\begin{cases} u_x=f'+\int_0^t K_{a_1}'*[\mu u(\cdot,\tau)+u(\cdot,\tau)v(\cdot,\tau)-bw_x(\cdot,\tau)]d\tau-\lambda\int_0^t K_{a_1}*[w(\cdot,\tau)v(\cdot,\tau)]d\tau, \\ w_x=g'+\int_0^t K_{a_1}'*[\mu w(\cdot,\tau)+w(\cdot,\tau)v(\cdot,\tau)+bu_x(\cdot,\tau)]d\tau+\lambda\int_0^t K_{a_1}*[u(\cdot,\tau)v(\cdot,\tau)]d\tau, \\ v_x=h'+\int_0^t K_c'*[v(\cdot,\tau)+\frac{1}{2}\big(u^2(\cdot,\tau)+w^2(\cdot,\tau)+v^2(\cdot,\tau)\big)]d\tau, \end{cases} \tag{3.25}$$

where the distributional derivative $K_k'$ is given by

$$K_k'=-\frac{1}{k}M_k+\frac{\delta}{k}.$$

Therefore, (3.25) can be arranged as

$$\begin{cases} u_x=f'-\dfrac{1}{a_1}\displaystyle\int_0^t\Big(M_{a_1}*[\mu u+uv-bw_x]-[\mu u+uv-bw_x]\Big)d\tau \\ \qquad -\lambda\displaystyle\int_0^t K_{a_1}*[w(\cdot,\tau)v(\cdot,\tau)]d\tau, \\ w_x=g'-\dfrac{1}{a_1}\displaystyle\int_0^t\Big(M_{a_1}*[\mu w+wv+bu_x]-[\mu w+wv+bu_x]\Big)d\tau \\ \qquad +\lambda\displaystyle\int_0^t K_{a_1}*[u(\cdot,\tau)v(\cdot,\tau)]d\tau, \\ v_x=h'-\dfrac{1}{c}\displaystyle\int_0^t\Big(M_c*[v+\frac{1}{2}(u^2+w^2+v^2)]d\tau-[v+\frac{1}{2}(u^2+w^2+v^2)]\Big)d\tau. \end{cases}$$

As $f,g,h\in C_b^2$, it is clear that $u_x,w_x,v_x\in C_T$ due to the facts that $M_{a_1},M_c\in L^1(\mathbb{R})$. Since $u_x$, $w_x$ and $v_x$ are written in terms of $u,w$ and $v$, an inductive argument shows that $\partial_t^m\vec{u}_x\in\mathcal{C}_T^3$, for any $m\geq 0$.

We can likewise proceed to obtain

$$\begin{cases} u_{xx}=f''+\dfrac{1}{a_1}\displaystyle\int_0^t\Big(K_{a_1}*[\mu u+uv-bw_x]+[\mu u_x+(uv)_x-bw_{xx}]\Big)d\tau\\ \qquad -\lambda\displaystyle\int_0^t K_{a_1}'*[w(\cdot,\tau)v(\cdot,\tau)]d\tau,\\ w_{xx}=g''+\dfrac{1}{a_1}\displaystyle\int_0^t\Big(K_{a_1}*[\mu w+wv+bu_x]+[\mu w_x+(wv)_x+bu_{xx}]\Big)d\tau\\ \qquad +\lambda\displaystyle\int_0^t K_{a_1}'*[u(\cdot,\tau)v(\cdot,\tau)]d\tau,\\ v_{xx}=h''+\dfrac{1}{c}\displaystyle\int_0^t K_c*[v+\frac{1}{2}(u^2+w^2+v^2)]d\tau+\frac{1}{c}\int_0^t[v_x+\frac{1}{2}(u^2+w^2+v^2)_x]d\tau. \end{cases}\tag{3.26}$$

A similar argument like above allows us to deduce that $u_{xx},w_{xx},v_{xx}\in C_T$ and inductively, $\partial_t^m\vec{u}_{xx}\in\mathcal{C}_T^3$, for any $m\geq 0$. It is now a straightforward verification using the expressions obtained above that $\vec{u}$ solves (3.22) pointwise. Hence, the Proposition 3.1 is proved. □

Indeed, more can be asserted as regard the regularity of solution for the system (3.22)-(2.1).

Corollary 3.1. *If $f,g,h\in C_b^k(\mathbb{R})$, $k\geq 2$ and $\vec{u}$ is a solution in $\mathcal{C}_T^3$ of (3.22)-(2.1), then $\vec{u}$ solves the system pointwise and $\partial_t^m\partial_x^n\vec{u}\in\mathcal{C}_T^3$, $\forall m\geq 0$ and $0\leq n\leq k$.*

A closer look at the expressions for $u_t,w_t$ and $v_t$ reveals that actually for $s>1/2$, the terms $u_t,w_t$ and $v_t$ all lie in $C\big([0,T],H^{s+1}(\mathbb{R})\big)$. That is, taking one derivative in time yields one more degree of smoothness in space. An inductive argument shows that for $s>1/2$, the quantities $\partial_t^m u,\partial_t^m w,\partial_t^m v\in C\big([0,T],H^{s+1}(\mathbb{R})\big)$ for $m=1,2,\ldots$. However, a solution of (3.22) cannot acquire more spatial regularity than that of the initial data. That is, $\forall k\geq 0$, if $f,g,h\in C_b^k(\mathbb{R})$ but $f,g,h\notin C_b^{k+1}(\mathbb{R})$, then $\vec{u}(\cdot,t)\notin C_b^{k+1}(\mathbb{R})^3$, $\forall t>0$. Because otherwise, suppose that for some $t>0$, $\vec{u}(\cdot,t)\in C_b^{k+1}(\mathbb{R})^3$, then at this value of $t$,

$$\begin{cases} f(x)=u(x,t)-\int_0^t K_{a_1}*[\mu u+uv-bw_x]-\lambda\int_0^t K_{a_1}*(wv)d\tau,\\ g(x)=w(x,t)-\int_0^t K_{a_1}*[\mu w+wv+bu_x]+\lambda\int_0^t K_{a_1}*(uv)d\tau,\\ h(x)=v(x,t)-\int_0^t K_c*[v+\frac{1}{2}(u^2+w^2+v^2)]d\tau. \end{cases}$$

As $\vec{u}\in C_b^{k+1}(\mathbb{R})^3$, so are $(\mu u+uv-bw_x)$, $wv$, $(\mu w+wv+bu_x)$, $uv$, and $\big(v+\frac{1}{2}(u^2+w^2+v^2)\big)$. After the convolutions with $K_{a_1}$ and $K_c$, we conclude that the right-hand sides of the above expressions belong to $C_b^{k+1}(\mathbb{R})^3$, a contradiction.

## 4. Global well-posedness

In this section, we will extend the local existence theories established in the previous section to global results. Since all the local results for the four systems rest upon the fact that the time intervals of existence $T$ depend only on the norms of the initial data $\|(f,g)\|_{H^s\times H^s}$, ($s\geq 0$ for the Schrödinger BBM-BBM system (1.4), while $s>3/2$ for the other three with some $H^1$-norm bounds on the initial data), the crux of the matter here is to show that the $H^s\times H^s$-norms of their solutions are bounded on any bounded time interval. In the case of the Schrödinger BBM-BBM system, this settles immediately as the conserved quantity $\mathcal{H}_2$ is just the weighted $H^1\times H^1$-norm which yields the needed bound on the initial data $\|(f,g)\|_{H^1\times H^1}$. Consequently, the Schrödinger BBM-BBM system (1.4) is globally well-posed in the Sobolev space $H^1\times H^1$. The situation is more challenging for the three systems (1.3), (1.5) and (1.6). As the local theories for those three systems require that the initial data be taken from $H^s\times H^s$ for $s>3/2$, and all the known conserved quantities associated with each of the three systems possess only at most the first derivatives of the functions involved, we need to obtain a-priori $H^s\times H^s$-norm bounds in terms of the initial data through other means. We employ the standard technique of multiplying each given equation in the systems by appropriate corresponding derivatives and then performing integration by parts. The lowest value of $s$ that we can achieve using this approach is $s=2$. Because of this, our global results pertaining to the other three systems are set in $H^2\times H^2$. It is conceivable that one might be able to obtain the a-priori $H^s\times H^s$ bounds, $s>3/2$, on the solutions of the three systems through other techniques, in which case the global well-posedness for these systems would be set in $H^s\times H^s$, $s>3/2$. For ease of notation, through out this section we will denote the norm $\|(f,g)\|_{H^1\times H^1}$ as $\|(f,g)\|_1$.

**4.1. The Schrödinger KdV-KdV system.**

LEMMA 4.1. *Let $s>3/2$ and $a_0>0$. There exist a constant $C>0$ depending only on $a_0,b,c,\lambda,\mu$ and a $\delta>0$ such that*

$$\|(u,v)\|_1\le C(\|(\phi,\psi)\|_1+\|(\phi,\psi)\|_1^{\frac{5}{3}}),\quad t\in[0,T],$$

*for any solution $(u,v)\in C([0,T],Y)$ of the system (1.3) with initial data $(\phi,\psi)\in Y$ with $\|(\phi,\psi)\|_1\le\delta$.*

*Proof.* We recall the following two conserved quantities

$$H_2(u,v)=\int_{-\infty}^{\infty}(|u|^2+v^2)dx, \tag{4.1}$$

and

$$\begin{aligned}H_3(u,v)=&\int_{-\infty}^{\infty}\Big(\frac{a_0}{2}|u_x|^2+\frac{c}{2}v_x^2-\frac{\mu}{2}|u|^2-\frac{1}{2}|u|^2v-\frac{1}{2}v^2-\frac{1}{6}v^3\Big)dx\\&+(\frac{\lambda a_0}{2}-\frac{b}{2})\mathcal{I}m\int_{-\infty}^{\infty}u\bar{u}_xdx+(\frac{b\lambda}{2}-\frac{\lambda^2a_0}{2})\int_{-\infty}^{\infty}v^2dx\end{aligned} \tag{4.2}$$

associated with the Schödinger KdV-KdV system (1.3) when $(u,v)$ is a smooth solution. However, by a standard approximating argument due to the local existence Theorem 3.1, they also hold for any solution in $C([0,T],Y)$. Thus, we can deduce from (4.2) that

$$\begin{aligned}\frac{a_0}{2}\|u_x\|^2+\frac{c}{2}\|v_x\|^2=&H_3(\phi,\psi)+\int_{-\infty}^{\infty}\Big(\frac{\mu}{2}|u|^2+\frac{1}{2}|u|^2v+\frac{1}{2}v^2+\frac{1}{6}v^3\Big)dx\\&-(\frac{\lambda a_0}{2}-\frac{b}{2})\mathcal{I}m\int_{-\infty}^{\infty}u\bar{u}_xdx-(\frac{b\lambda}{2}-\frac{\lambda^2a_0}{2})\int_{-\infty}^{\infty}v^2dx.\end{aligned} \tag{4.3}$$

Combining (4.3) and (4.1), we arrive at

$$\begin{aligned}a_0\|u_x\|^2+c\|v_x\|^2\le&2H_3(\phi,\psi)+\max\Big\{\mu,|1-b\lambda+a_0\lambda^2|\Big\}H_2(\phi,\psi)\\&+\int_{-\infty}^{\infty}\Big(|u|^2v+\frac{1}{3}v^3\Big)dx-(\lambda a_0-b)\mathcal{I}m\int_{-\infty}^{\infty}u\bar{u}_xdx.\end{aligned}$$

Using the inequalities

$$|\int_{-\infty}^{\infty} v|u|^2 dx| \le \|v\|_{L^\infty}\|u\|^2, \qquad |\int_{-\infty}^{\infty} v^3 dx| \le \|v\|_{L^\infty}\|v\|^2,$$

and

$$|\int_{-\infty}^{\infty} u\bar{u}_x dx| \le \|u\|\|u_x\|, \qquad \|v\|_{L^\infty} \le \|v_x\|^{\frac{1}{2}}\|v\|^{\frac{1}{2}},$$

we arrive at

$$\begin{aligned} a_0\|u_x\|^2 + c\|v_x\|^2 \le 2H_3(\phi,\psi) + \max\Big\{\mu, |1-b\lambda+a_0\lambda^2|\Big\}H_2(\phi,\psi) + \|v_x\|^{\frac{1}{2}}[H_2(\phi,\psi)]^{\frac{5}{4}} \\ + |a_0\lambda - b|\|u_x\|[H_2(\phi,\psi)]^{\frac{1}{2}}. \end{aligned} \tag{4.4}$$

On the other hand, it is easy to see that

$$|H_3(\phi,\psi)| \le C_1\Big(\|(\phi,\psi)\|_1^2 + \|(\phi,\psi)\|_1^3\Big), \qquad H_2(\phi,\psi) \le \|(\phi,\psi)\|_1^2, \tag{4.5}$$

for some constant $C_1$ depending only on $a_0, b, c, \lambda$ and $\mu$. Let $\xi(t) = \|(u,v)\|_1$ and $r = \|(\phi,\psi)\|_1$, then it follows from (4.4) and (4.5) that

$$\xi(t)^2 \le C_2[r^2 + r^3 + r^{\frac{10}{3}}],$$

for some constant $C_2$ depending only on $a_0, b, c, \lambda$ and $\mu$ because of Young's inequality. Therefore,

$$\xi(t) \le \sqrt{C_2}(r + r^{\frac{3}{2}} + r^{\frac{5}{3}}) \le 2\sqrt{C_2}(r + r^{\frac{5}{3}})$$

and the lemma follows. ∎

As explained above, the global well-posedness rests upon the argument that the $Y$-norm ($s > 3/2$) of the solutions of (1.3) are bounded on any bounded time interval. However, due to the lack of conservation laws involving higher derivatives of solutions of the Schrödinger KdV-KdV system, we opt for evaluating $\frac{d}{dt}\|u_{xx}\|^2$ and $\frac{d}{dt}\|v_{xx}\|^2$. The following identities will come in handy whose proofs will appear in the Appendix.

Lemma 4.2. *Let $C$ denote various constants depending only on $a_0, b, c, \lambda$ and $\mu$. The following statements hold true*
(1)

$$\frac{d}{dt}(u_{xx}, u_{xx}) = \mathcal{R}e\Big\{4(u, u_{xx}v_{xxx}) + 6(uv_{xx}, u_{xxx}) + 10(vu_{xx}, u_{xxx})\Big\} - \Pi_0, \tag{4.6}$$

*where $\Pi_0$ is given by*

$$\Pi_0 = 2\lambda\mathcal{I}m\{(vu_x, u_{xxx}) + (uv_x, u_{xxx})\}$$

*and satisfies that*

$$|\Pi_0| \le C(\|(\phi,\psi)\|_1^3 + \|(\phi,\psi)\|_1^5)\{\|u_{xx}\|^2 + \|u_{xx}\|\|v_{xx}\|\}.$$

(2)

$$\frac{d}{dt}(v_{xx},v_{xx}) = 10(vv_{xx},v_{xxx}) + 6\mathcal{R}e(u,u_{xx}v_{xxx}) + 4\mathcal{R}e(uv_{xx},u_{xxx}). \tag{4.7}$$

(3)

$$\frac{d}{dt}(v_x^2,v) = 6c(vv_{xx},v_{xxx}) + \Pi_1 \tag{4.8}$$

*where* $\Pi_1$ *is given by*

$$\Pi_1 = \frac{1}{2}(2vv_{xx}+v_x^2,(v^2)_x + (|u|^2)_x),$$

*and satisfies that*

$$|\Pi_1| \le C\big(\|(\phi,\psi)\|_1^3 + \|(\phi,\psi)\|_1^5\big)\|v_{xx}\|. \tag{4.9}$$

(4)

$$\frac{d}{dt}((|u|^2)_x,v_x) = \mathcal{R}e\Big\{6c(u,u_{xx}v_{xxx}) + (4c+2a_0)(uv_{xx},u_{xxx})\Big\} + \Pi_2 \tag{4.10}$$

*where* $\Pi_2$ *is given by*

$$\Pi_2 = \big((|u|^2)_{xx}, v_x + vv_x\big) + 2\mathcal{R}e\big(uv_{xx}, \mu u_x + ibu_{xx} + (uv)_x + i\lambda uv\big)$$

*and satisfies*

$$|\Pi_2| \le C\Big[\|(\phi,\psi)\|_1 + \|(\phi,\psi)\|_1^5\Big]\Big\{\|v_{xx}\| + \|u_{xx}\|^2 + \|v_{xx}\|^2\Big\}. \tag{4.11}$$

(5)

$$\begin{aligned}\frac{d}{dt}(v,|u_x|^2) = \mathcal{R}e\Big\{&(2a_0-2c)(u,u_{xx}v_{xxx}) + (2a_0-2c)(uv_{xx},u_{xxx})\\ &+ 6a_0(vu_{xx},u_{xxx})\Big\} + \Pi_3,\end{aligned} \tag{4.12}$$

*where* $\Pi_3$ *is given by*

$$\Pi_3 = -(|u_x|^2, v_x + vv_x + \frac{1}{2}(|u|^2)_x) + 2\mathcal{R}e\Big\{\big((vu_x)_x, \mu u_x + ibu_{xx} + (uv)_x + i\lambda uv\big)\Big\}$$

*and satisfies*

$$|\Pi_3| \le C\Big[\|(\phi,\psi)\|_1^{\frac{3}{2}} + \|(\phi,\psi)\|_1^5\Big]\Big[\|u_{xx}\| + \|u_{xx}\|^{\frac{3}{2}} + \|v_{xx}\|\Big]. \tag{4.13}$$

With the above lemma in hand, we now proceed to show that the $H^2-$norm of a solution of the Schrödinger KdV-KdV system given in Theorem 3.1 is bounded in any finite time under certain conditions.

LEMMA 4.3. *Let* $a_0 > 0$ *satisfying* $5c^2 + 5a_0c - a_0^2 > 0$ *and let* $(u,v) \in C([0,T], H^2 \times H^2)$ *be any solution to* (1.3)*. Then the following estimate*

$$\begin{aligned}\|u_{xx}\|^2 + \|v_{xx}\|^2 \le \Big\{C_1\Big[&\|(\phi,\psi)\|_1 + \|(\phi,\psi)\|_1^5 + \|\phi_{xx}\|^2\\ &+ \|\psi_{xx}\|^2\Big] + 1\Big\}e^{C_2[\|(\phi,\psi)\|_1 + \|(\phi,\psi)\|_1^5]t} - 1\end{aligned}$$

*holds as long as* $(u,v)$ *exists, where* $C_1,C_2$ *depend only on* $a_0,b,c,\lambda$ *and* $\mu$.

*Proof.* First we observe that the identities (4.6), (4.7), (4.8), (4.10) and (4.12) can be rearranged in the following form

$$\begin{pmatrix} \frac{d}{dt}(u_{xx},u_{xx})+\Pi_0 \\ \frac{d}{dt}(v_{xx},v_{xx}) \\ \frac{d}{dt}((v_x)^2,v)-\Pi_1 \\ \frac{d}{dt}(uu_x,v_x)-\Pi_2 \\ \frac{d}{dt}(v,(u_x)^2)-\Pi_3 \end{pmatrix} = A \begin{pmatrix} (vv_{xx},v_{xxx}) \\ \mathcal{R}e(u,u_{xx}v_{xxx}) \\ \mathcal{R}e(uv_{xx},u_{xxx}) \\ \mathcal{R}e(vu_{xx},u_{xxx}) \end{pmatrix}, \tag{4.14}$$

where the matrix $A$ is given by

$$A=\begin{pmatrix} 0 & 4 & 6 & 10 \\ 10 & 6 & 4 & 0 \\ 6c & 0 & 0 & 0 \\ 0 & 6c & 4c+2a_0 & 0 \\ 0 & 2a_0-2c & 2a_0-2c & 6a_0 \end{pmatrix}.$$

Let $\vec{C}$ denote the row vector $(c_1,c_2,c_3,c_4,c_5)$. We want to find a linear combination of the vector on the left-hand side of (4.14) that equates to zero, or equivalently, we consider the following homogeneous linear system

$$\vec{C}A=\vec{0}.$$

A direct calculation shows that a basis for the solution set is given by

$$c_1=\frac{3a_0}{5}, c_2=\frac{5c^2+5ca_0-a_0^2}{15a_0}, c_3=\frac{5c^2+5ca_0-a_0^2}{-9ca_0}, c_4=-\frac{2a_0+c}{3a_0}, c_5=-1.$$

Thus, we arrive at

$$c_1\left[\frac{d}{dt}(u_{xx},u_{xx})+\Pi_0\right]+c_2\frac{d}{dt}(v_{xx},v_{xx})+c_3\left[\frac{d}{dt}((v_x)^2,v)-\Pi_1\right] \\ +c_4\left[\frac{d}{dt}(uu_x,v_x)-\Pi_2\right]+c_5\left[\frac{d}{dt}(v,(u_x)^2)-\Pi_3\right]=0$$

which can be rewritten compactly as

$$c_1\frac{d}{dt}(u_{xx},u_{xx})+c_2\frac{d}{dt}(v_{xx},v_{xx})=\Pi_4 \tag{4.15}$$

where $\Pi_4$ is defined by

$$\Pi_4=-c_3\frac{d}{dt}((v_x)^2,v)-c_4\frac{d}{dt}(uu_x,v_x)-c_5\frac{d}{dt}(v,(u_x)^2)-c_1\Pi_0+c_3\Pi_1+c_4\Pi_2+c_5\Pi_3.$$

Consequently, an $H^2\times H^2-$bound on the solution of (1.3) can be obtained if $c_1$ and $c_2$ are of the same sign. As $c_1>0$, we need $c_2>0$ which holds true if and only if $5c^2+5a_0c-a_0^2>0$. Now,

$$|\int_0^t \Pi_4 ds|\le\eta(\|(\phi,\psi)\|_1)+\eta(\|(\phi,\psi)\|_1)\int_0^t\left[(\|u_{xx}\|^2+\|v_{xx}\|^2)^{\frac{1}{2}}+\|u_{xx}\|^2+\|v_{xx}\|^2\right]ds \tag{4.16}$$

by (4.9), (4.11), (4.13) and where the function $\eta(r)$ is from Lemma 4.1 given by $\eta(r)=C(r+r^5)$ with the constant $C$ depending only on $a_0,b,c,\lambda,\mu$. Setting $\xi(t)=\|u_{xx}\|^2+\|v_{xx}\|^2$ and combining (4.15) with (4.16), we arrive at

$$\begin{aligned}\min\{c_1,c_2\}\xi(t)&\le\max\{c_1,c_2\}\xi(0)+\eta(\|(\phi,\psi)\|_1)+\eta(\|(\phi,\psi)\|_1)\int_0^t[\sqrt{\xi(s)}+\xi(s)]ds\\&\le\max\{c_1,c_2\}\xi(0)+\eta(\|(\phi,\psi)\|_1)+\eta(\|(\phi,\psi)\|_1)\int_0^t(2\xi(s)+1)ds.\end{aligned}$$

It then follows that

$$2\xi(t)+1\le A_0+A_1\int_0^t(2\xi(s)+1)ds,$$

where

$$A_0=\frac{2}{\min\{c_1,c_2\}}[\max\{c_1,c_2\}\xi(0)+\eta(\|(\phi,\psi)\|_1)]+1,$$

and

$$A_1=\frac{2\eta(\|(\phi,\psi)\|_1)}{\min\{c_1,c_2\}}.$$

Consequently, an application of Gronwall's inequality deduces that

$$2\xi(t)+1\le A_0e^{A_1t}$$

as long as the solution $(u,v)$ exists. Therefore

$$\|u_{xx}\|+\|v_{xx}\|\le\sqrt{2\xi(t)}\le\sqrt{A_0e^{A_1t}-1}.$$

This completes the proof of the lemma. □

Lemmas 4.1 and 4.3 reveal the important fact that any local solution to the Schrödinger KdV-KdV system (1.3) in $H^2\times H^2$ is bounded on finite interval $[0,T]$. Thus, we can repeatedly use the local existence Theorem 3.1 to extend the local solution to the global one.

THEOREM 4.1.

(i) *Let $a_0,c>0$ satisfying $5c^2+5a_0c-a_0^2>0$. There exists a $\delta>0$ such that for any initial data $(f,g)\in H^2\times H^2$ with $\|(f,g)\|_1\le\delta$, there is a unique global solution $(u,v)\in C([0,\infty),H^2\times H^2)\bigcap C^1([0,\infty),H^{-1}\times H^{-1})$ to (1.3)-(2.1).*

(ii) *The map $(f,g)\to\big(u(t),v(t)\big)$ is continuous in the $H^2\times H^2$-norm. More precisely, if $(f_n,g_n)\in H^2\times H^2$, $n=1,2,\cdots$, with $\|(f_n,g_n)-(f,g)\|_{H^2\times H^2}\to0$ and $T<\infty$, the solution $(u_n,v_n)$ for $\big(u_n(0),v_n(0)\big)=(f_n,g_n)$ exists on $[0,T]$ for sufficiently large $n$ and $\|(u_n,v_n)-(u,v)\|_{H^2\times H^2}\to0$ uniformly in $t\in[0,T]$.*

### 4.2. The Schrödinger KdV-BBM system.

THEOREM 4.2. *Let $a_0>0$. There exists a $\delta>0$ such that for any initial data $(f,g)\in H^2\times H^2$ with $\|(f,g)\|_1\le\delta$, there is a unique solution $(u,v)\in C([0,T],H^2\times H^2)\bigcap C^1([0,T],H^{-1}\times H^{-1})$ to (1.5)-(2.1). Moreover, the map $(f,g)\to\big(u(t),v(t)\big)$ is continuous in the $H^2\times H^2$-norm. That is, if $(f_n,g_n)\in H^2\times H^2$, $n=1,2,\cdots$, with $\|(f_n,g_n)-(f,g)\|_{H^2\times H^2}\to0$ and $T<\infty$, the solution $(u_n,v_n)$ for $\big(u_n(0),v_n(0)\big)=(f_n,g_n)$ exists on $[0,T]$ for sufficiently large $n$ and $\|(u_n,v_n)-(u,v)\|_{H^2\times H^2}\to0$ uniformly in $t\in[0,T]$.*

### 4.3. The Schrödinger BBM-KdV system.

THEOREM 4.3. *Let $a_1>0$. There exists a $\delta>0$ such that for any initial data $(f,g)\in H^2\times H^2$ with $\|(f,g)\|_1\le\delta$, there is a unique solution $(u,v)\in C([0,T],H^2\times H^2)\bigcap C^1([0,T],H^{-1}\times H^{-1})$ to (1.6)-(2.1). Moreover, the map $(f,g)\to\big(u(t),v(t)\big)$ is continuous in the $H^2\times H^2$-norm. That is, if $(f_n,g_n)\in H^2\times H^2$, $n=1,2,\cdots$, with $\|(f_n,g_n)-(f,g)\|_{H^2\times H^2}\to 0$ and $T<\infty$, the solution $(u_n,v_n)$ for $\big(u_n(0),v_n(0)\big)=(f_n,g_n)$ exists on $[0,T]$ for sufficiently large $n$ and $\|(u_n,v_n)-(u,v)\|_{H^2\times H^2}\to 0$ uniformly in $t\in[0,T]$.*

### 4.4. The Schrödinger BBM-BBM system.

Recall that after equating the real and imaginary parts, the Schrödinger BBM-BBM system (1.4) becomes the system (3.22). We also recall from the introduction that the subspace of $C_b$, such that all of its members vanish at infinity, will be denoted as $C_0^\infty$, that $C_{0,T}=C([0,T],C_0^\infty)$ and $\mathcal{C}^3_{0,T}=C([0,T],C_0^\infty\times C_0^\infty\times C_0^\infty)$. The precise statement for the global well-posedness of the Schrödinger BBM-BBM system (1.4) is below.

THEOREM 4.4. *Let $f,g\in H^1(\mathbb{R})$. There exists a unique global solution $\vec{u}(x,t)\in H^1(\mathbb{R})\times H^1(\mathbb{R})$ to the Cauchy problem (1.4) with $\vec{u}(0)=(f,g)$ that depends continuously on the initial data. Moreover, if $f,g\in H^1(\mathbb{R})\cap C_b^2(\mathbb{R})$ then $\partial_t^m\partial_x^k\vec{u}\in C([0,T],H^1(\mathbb{R}))$, and $\partial_t^m\partial_x^k\vec{u}\to 0$ as $x\to\pm\infty$, $\forall T>0$, $0\le k\le 2$, $m\ge 0$.*

The following proposition is useful in extending the local existence obtained in Section 3 to a global result.

PROPOSITION 4.1. *Let $\vec{u}$ be the solution in $\mathcal{C}^3_T$ of (3.22)-(2.1) with $f,g,h\in C_b^k(\mathbb{R})$ for some $k\ge 0$ such that*

$$f,g,h,f',g',h'\cdots,f^{(p)},g^{(p)},h^{(p)}\to 0 \quad as \quad x\to\pm\infty,$$

*for some $p\le k$. Then, for $0\le t\le T$,*

$$\partial_x^l\partial_t^m\vec{u}\to 0 \quad as \quad x\to\pm\infty,$$

*for any $0\le l\le p$ and $\forall m\ge 0$.*

*Proof.* Set $\vec{u}=(u,w,v)$. For $n\ge 1$, let $\vec{u}_n=\mathcal{A}\vec{u}_{n-1}$ and $\vec{u}_0=\vec{u}(0)$. Notice that $\vec{u}_0=0$ as $x\to\pm\infty$ by the assumption.

*Claim*: If $\vec{u}\in\mathcal{C}^3_{0,T}$, then $K_{a_1}*u, K_{a_1}*w$ and $K_c*v\in C_{0,T}$. To see this, fix $\epsilon>0$; then, there exists an $\eta_1>0$ such that for all $x\ge\eta_1$

$$\begin{aligned}&\Big|\int_{-\infty}^{\infty}e^{-|x-y|}u(y,t)dy\Big|\le e^{-x}\int_{-\infty}^{\eta_1}e^{y}|u(y,t)|dy+\sup_{y\ge\eta_1}|u(y,t)|\int_{\eta_1}^{\infty}e^{-|x-y|}dy\\ &\le e^{\eta-x}\sup_{y\in\mathbb{R},0\le t\le T}|u(y,t)|+2\sup_{y\ge\eta_1}|u(y,t)|.\end{aligned}$$

Moreover, as $u\to 0$ when $y\to\pm\infty$, for any fixed $t$ there exists an $\eta_2>0$ such that $|u(y,t)|\le\frac{\epsilon}{4}$ for all $y\ge\eta_2$. Set $\eta=\max\{\eta_1,\eta_2\}$. Then for all $x\ge\eta$, the first term is less than $\frac{\epsilon}{2}$. The same can be said when $x\to-\infty$. Thus, $K_{a_1}*u\in C^3_{0,T}$.

Exact same technique applies for all the convolutions appearing in Equation (3.23), using the facts that for $k>0$

$$K_k'=-\frac{1}{k}M_k+\frac{\delta}{k},$$

$$\Big|K_k*(wz)\Big|(x)\le\|w\|_\infty\Big(|K_k|*|z|\Big)(x),$$

for all $x\in\mathbb{R}$. Thus, from the claim, it follows immediately that $\vec{u}_1=\mathcal{A}\vec{u}_0$ vanishes at $\pm\infty$ and by induction, $\vec{u}_n=\mathcal{A}\vec{u}_{n-1}$ also vanishes at $\pm\infty$. As $\mathcal{A}$ is a contraction mapping and $\mathcal{C}^3_{0,T}$ is a Banach space, it follows that $\vec{u}_n\to\vec{u}$ in $\mathcal{C}^3_{0,T}$. Now,

$$\begin{cases} u_t(x,t)=\displaystyle\int_{-\infty}^{\infty}\Big(K_{a_1}(x-y)[\mu u(y,t)+u(y,t)v(y,t)-bw_y(y,t)] \\ \qquad\qquad +\lambda M_{a_1}(x-y)w(y,t)v(y,t)\Big)dy, \\ w_t(x,t)=\displaystyle\int_{-\infty}^{\infty}\Big(K_{a_1}(x-y)[\mu w(y,t)+w(y,t)v(y,t)+bu_y(y,t)] \\ \qquad\qquad -\lambda M_{a_1}(x-y)u(y,t)v(y,t)\Big)dy, \\ v_t(x,t)=\displaystyle\int_{-\infty}^{\infty}K_c(x-y)[v(y,t)+\frac{1}{2}u^2(y,t)+\frac{1}{2}w^2(y,t)+\frac{1}{2}v^2(y,t)]dy, \end{cases}$$

Thus, $u_t,w_t,v_t\in C_{0,T}$, and inductively, $\partial_t^m\vec{u}\in\mathcal{C}^3_{0,T},\forall m\ge 0$. Moreover, if $p\ge 1$, then

$$\begin{cases} u_x=f'-\dfrac{1}{a_1}\displaystyle\int_0^t\Big(M_{a_1}*[\mu u+uv-bw_x]-[\mu u+uv-bw_x]\Big)d\tau \\ \qquad -\lambda\displaystyle\int_0^t K_{a_1}*[w(\cdot,\tau)v(\cdot,\tau)]d\tau, \\ w_x=g'-\dfrac{1}{a_1}\displaystyle\int_0^t\Big(M_{a_1}*[\mu w+wv+bu_x]-[\mu w+wv+bu_x]\Big)d\tau \\ \qquad +\lambda\displaystyle\int_0^t K_{a_1}*[u(\cdot,\tau)v(\cdot,\tau)]d\tau, \\ v_x=h'-\dfrac{1}{c}\displaystyle\int_0^t\Big(M_c*[v+\frac{1}{2}(u^2+w^2+v^2)]d\tau-[v+\frac{1}{2}(u^2+w^2+v^2)]\Big)d\tau. \end{cases}$$

As $(\mu u+uv+bw_x)\in C_{0,T}$, we have $\int_0^t(\mu u+uv+bw_x)d\tau\in C_{0,T}$ and the same thing holds true for each and every term on the right-hand side of the above system. Hence $u_x,w_x,v_x\in C_{0,T}$. The assertion that $u_{xx},w_{xx},v_{xx}\in C_{0,T}$ follows the exact same lines using (3.26) instead. An inductive argument now completes the proof of the proposition.

□

The following proposition follows immediately from the conserved quantity $\mathcal{H}_2(u,v)$ associated with the system (1.4) found in [15] and mentioned in Section 1 above. As this quantity is just the weighted $H^1$-norm of $(u,v)$, we denote it as $H^1_{a_1,c}$-norm.

Proposition 4.2. *Let $f,g\in C_b^k(\mathbb{R})\cap H^1(\mathbb{R})$, $k\ge 2$. Then the solution $\vec{u}(x,t)$ of (1.4)-(2.1) lies in $H^1(\mathbb{R})\times H^1(\mathbb{R})$, for all $0\le t<+\infty$, and*

$$\|\vec{u}(\cdot,t)\|_{H^1_{a_1,c}}=\|\vec{u}(\cdot,0)\|_{H^1_{a_1,c}}=\int_{-\infty}^{\infty}\big(|u|^2+a_1|u_x|^2+v^2+cv_x^2\big)dx.$$

Remark 4.1. The above two propositions say that indeed $\vec{u}$ is a classical solution of the system (1.4) whenever $f,g\in C_b^k(\mathbb{R})\cap H^1(\mathbb{R})$, $k\ge 2$.

*Proof.* **(Proof of Theorem 4.4.)** To extend the local solution obtained in Section 3 to a global one, we can repeat the same argument using $\vec{u}(x,T)$ as the new initial data. Thus we are done if we can show that $\sup_x |\vec{u}(x,t)|$ is bounded on *bounded time intervals.* But this is true since for all functions $h \in H^1$,

$$h^2(x) = 2\int_{-\infty}^{x} h(y)h'(y)dy \le \int_{-\infty}^{\infty} [|h(y)|^2 + |h'(y)|^2]dy = \|h\|_{H^1}^2,$$

which implies that

$$|\vec{u}(x,t)|^2 \le \|\vec{u}\|_{H^1}^2 \le \max\{1,\frac{1}{a_1},\frac{1}{c}\}\|\vec{u}(\cdot,t)\|_{H^1_{a_1,c}}^2 = \max\{1,\frac{1}{a_1},\frac{1}{c}\}\|\vec{u}(\cdot,0)\|_{H^1_{a_1,c}}^2.$$

Thus, there is one universal bound for $\vec{u}(x,t)$ which is independent of $t$. The fact that solution of (1.4) depends continuously on initial data follows from Theorem 3.4 while uniqueness of the solution is straightforward to see. Moreover, if $f,g \in H^1 \cap C_b^2$, then $f,g,f',g' \to 0$ as $x \to \pm\infty$. Thus, it follows from Proposition 4.1 that $\partial_t^m \partial_x^k \vec{u} \in \mathcal{C}_{0,T}^3$, $m \ge 0$. ∎

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**Appendix.** *Proof.* **(Proof of Lemma 4.2.)** To see (1), multiplying the first equation in (1.3) by $D^4\overline{u}$ and the conjugate of the first equation by $D^4u$, adding the results together, followed by an integration with respect to the variable $x$, we obtain

$$\frac{d}{dt}(u_{xx},u_{xx}) = 2\mathcal{R}e\big((uv)_{xx},u_{xxx}\big) - 2\lambda\mathcal{I}m\{(vu_x,u_{xxx}) + (uv_x,u_{xxx})\}. \tag{A.1}$$

On the other hand, noting that

$$\big((uv)_{xx},u_{xxx}\big) = (uv_{xx},u_{xxx}) + 2(u_xv_x,u_{xxx}) + (vu_{xx},u_{xxx}), \tag{A.2}$$

and

$$\begin{aligned}(u_xv_x,u_{xxx}) &= -(u_{xx}v_x,u_{xx}) - (u_xv_{xx},u_{xx})\\ &= 2\mathcal{R}e(vu_{xx},u_{xxx}) + (u,u_{xx}v_{xxx}) + (uv_{xx},u_{xxx}).\end{aligned} \tag{A.3}$$

Statement (1) then follows from (A.1), (A.2) and (A.3).

To see (2), multiplying the second equation in (1.3) by $D^4v$ then integrating with respect to variable $x$, we arrive at

$$\frac{d}{dt}(v_{xx},v_{xx}) = \big((v^2+|u|^2)_{xx},v_{xxx}\big). \tag{A.4}$$

Moreover,

$$\begin{aligned}(|u_x|^2,v_{xxx}) &= -2\mathcal{R}e(u_x,u_{xx}v_{xx})\\ &= 2\mathcal{R}e(u,u_{xx}v_{xxx}) + 2\mathcal{R}e(v_{xx}u,u_{xxx}).\end{aligned} \tag{A.5}$$

This implies that

$$\begin{aligned}\big((|u|^2)_{xx},v_{xxx}\big) &= 2\mathcal{R}e(u,u_{xx}v_{xxx}) + 2(|u_x|^2,v_{xxx})\\ &= 6\mathcal{R}e(u,u_{xx}v_{xxx}) + 4\mathcal{R}e(uv_{xx},u_{xxx}).\end{aligned} \tag{A.6}$$

We also have

$$\big((v^2)_{xx}, v_{xxx}\big) = 10(vv_{xx}, v_{xxx}). \tag{A.7}$$

Statement (2) then follows from (A.4),(A.6) and (A.7).

To see (3), notice first that

$$\begin{aligned}\frac{d}{dt}\big((v_x)^2, v\big) &= (2vv_x, v_{xt}) + \big((v_x)^2, v_t\big)\\ &= -2\big(vv_{xx} + (v_x)^2, v_t\big) + \big((v_x)^2, v_t\big)\\ &= -\big(2vv_{xx} + (v_x)^2, v_t\big).\end{aligned} \tag{A.8}$$

Using the second equation in (1.3), we deduce that

$$\begin{aligned}-\big(2vv_{xx} + (v_x)^2, v_t\big) &= \big(2vv_{xx} + (v_x)^2, v_x + vv_x + cv_{xxx} + \frac{1}{2}(|u|^2)_x\big)\\ &= c(2vv_{xx}, v_{xxx}) + c\big((v_x)^2, v_{xxx}\big)\\ &\qquad + \big(2vv_{xx} + (v_x)^2, v_x + vv_x + \frac{1}{2}(|u|^2)_x\big).\end{aligned} \tag{A.9}$$

Notice that

$$\big((v_x)^2, v_{xxx}\big) = 4(vv_{xx}, v_{xxx}).$$

Then, (A.9) can be rewritten as

$$-\big(2vv_{xx} + (v_x)^2, v_t\big) = 6c(vv_{xx}, v_{xxx}) + \big(2vv_{xx} + (v_x)^2, v_x + vv_x + \frac{1}{2}(|u|^2)_x\big). \tag{A.10}$$

Since

$$\big(2vv_{xx} + (v_x)^2, v_x\big) = 0,$$

the Equation (4.8) follows directly from (A.8) and (A.10). We also have

$$|\big(2vv_{xx}, (v^2)_x + (|u|^2)_x\big)| \le 4\{\|v\|_1^3 + \|u\|_1^2\|v\|_1\}\|v_{xx}\|,$$

by the Hölder inequality, and then an integration by parts yields

$$|\big((v_x)^2, (v^2)_x + (|u|^2)_x\big)| = |(2v_xv_{xx}, v^2 + |u|^2)| \le 2\{\|v\|_1^3 + \|u\|_1^2\|v\|_1\}\|v_{xx}\|.$$

Thus, summing over the above inequalities implies

$$|\Pi_1| \le 3\{\|v\|_1^3 + \|u\|_1^2\|v\|_1\}\|v_{xx}\|.$$

Item (4.9) now follows from Lemma 4.1.

To see (4), notice first that

$$\begin{aligned}\frac{d}{dt}\big((|u|^2)_x, v_x\big) &= 2\mathcal{R}e\Big\{(\overline{u}u_x, v_{xt}) + (u_xv_x, u_t) + (uv_x, u_{xt})\Big\}\\ &= -\big((|u|^2)_{xx}, v_t\big) - 2\mathcal{R}e(uv_{xx}, u_t)\}.\end{aligned} \tag{A.11}$$

Again, using the equations in (1.3), we obtain

$$-\big((|u|^2)_{xx}, v_t\big) = \big((|u|^2)_{xx}, v_x + vv_x + cv_{xxx} + \frac{1}{2}(|u|^2)_x\big)$$

$$
\begin{aligned}
&= c\big((|u|^2)_{xx}, v_{xxx}\big) + \big((|u|^2)_{xx}, v_x + vv_x\big)\\
&= \mathcal{R}e\Big\{6c(u, u_{xx}v_{xxx}) + 4c(uv_{xx}, u_{xxx})\Big\} + \big((|u|^2)_{xx}, v_x + vv_x\big)
\end{aligned}
\tag{A.12}
$$

where an application of (A.6) was used, and

$$
\begin{aligned}
-(uv_{xx}, u_t) &= \big(uv_{xx}, \mu u_x + a_0 u_{xxx} + ibu_{xx} + (uv)_x + i\lambda uv\big)\\
&= a_0(uv_{xx}, u_{xxx}) + \big(uv_{xx}, \mu u_x + ibu_{xx} + (uv)_x + i\lambda uv\big).
\end{aligned}
\tag{A.13}
$$

Combining (A.11), (A.12) and (A.13) we arrive at (4.10). Next, one can follow the same technique used in proving (4.9) above

$$
|\Pi_2| \le C\Big\{\Big(\|u\|_1^2 + \|u\|_1^2\|v\|_1\Big)\|v_{xx}\| + \|u\|_1\|u_{xx}\|\|v_{xx}\|\Big\}
$$

holds. Consequently, (4.11) follows from Lemma 4.1 and Hölder's inequality.

To see (5), notice that

$$
\begin{aligned}
\frac{d}{dt}(v, |u_x|^2) &= (v_t, |u_x|^2) + 2\mathcal{R}e(vu_x, u_{xt})\\
&= \big(|u_x|^2, v_t\big) - 2\mathcal{R}e\big((vu_x)_x, u_t\big)\\
&= -\big(|u_x|^2, v_x + vv_x + cv_{xxx} + \frac{1}{2}(|u|^2)_x\big)\\
&\quad + \mathcal{R}e\Big\{2\big((vu_x)_x, \mu u_x + a_0u_{xxx} + ibu_{xx} + (uv)_x + i\lambda uv\big)\Big\}\\
&= -c(|u_x|^2, v_{xxx}) + 2a_0\mathcal{R}e\big((vu_x)_x, u_{xxx}\big) - \big(|u_x|^2, v_x + vv_x + \frac{1}{2}(|u|^2)_x\big)\\
&\quad + \mathcal{R}e\Big\{2\big((vu_x)_x, \mu u_x + ibu_{xx} + (uv)_x + i\lambda uv\big)\Big\},
\end{aligned}
\tag{A.14}
$$

where we have utilized the two equations in (1.3). By (A.3)

$$
\begin{aligned}
2a_0\mathcal{R}e\big((vu_x)_x, u_{xxx}\big) &= 2a_0\mathcal{R}e\Big\{(vu_{xx}, u_{xxx}) + (v_xu_x, u_{xxx})\Big\}\\
&= \mathcal{R}e\Big\{6a_0(vu_{xx}, u_{xxx}) + 2a_0(u, u_{xx}v_{xxx}) + 2a_0(uv_{xx}, u_{xxx})\Big\},
\end{aligned}
\tag{A.15}
$$

and by (A.5)

$$
-c(|u_x|^2, v_{xxx}) = \mathcal{R}e\Big\{-2c(uv_{xx}, u_{xxx}) - 2c(u, u_{xx}v_{xxx})\Big\}.
\tag{A.16}
$$

Consequently, (4.12) follows immediately from (A.14), (A.15) and (A.16). The estimate

$$
|\Pi_3| \le C\Big\{[\|u\|_1^3 + \|v\|_1^2\|u\|_1 + \|u\|_1\|v\|_1]\|u_{xx}\| + \|u_{xx}\|^{\frac{3}{2}}\|u\|_1^{\frac{1}{2}}\|v\|_1 + \|u\|_1^2\|v\|_1\|v_{xx}\|\Big\}.
$$

can be proved using the same approach as in statements (3) and (4) above. Thus, (4.13) follows from Lemma 4.1 again. □

## REFERENCES

[1] K. Appert and J. Vaclavik, *Dynamics of coupled solitons*, Phys. Fluids, 20:1845–1849, 1977. 1, 1
[2] D.J. Benney, *A general theory for interactions between short and long waves*, Stud. Appl. Math., 56(1):81–94, 1977. 1
[3] J.L. Bona, M. Chen, and S.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I: Derivation and linear theory*, J. Nonlinear Sci., 12:283–318, 2002. 1
[4] J.L. Bona, M. Chen, and S.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II: The nonlinear theory*, Nonlinearity, 17:925–952, 2004. 1, 3.4
[5] U. Brinch-Nielsen and I.G. Jonsson, *Fourth order evolution equations and stability analysis for Stokes waves on arbitrary water depth*, Wave Motion, 8:455–472, 1986. 1
[6] W. Craig, P. Guyenne, and C. Sulem, *The surface signature of internal waves*, J. Fluid Mech., 710:277–303, 2012. 1
[7] W. Craig, P. Guyenne, and C. Sulem, *Coupling between internal and surface waves*, Nat. Hazards, 57:617–642, 2011. 1
[8] A. Davey and K. Stewartson, *On three-dimensional packets of surface waves*, Proc. Roy. Soc. A, 388(1613):101–110, 1974. 1
[9] B. Deconinck, N.V. Nguyen, and B.L. Segal, *The interaction of long and short waves in dispersive media*, J. Phys. A: Math. Theor., 49:415501, 2016. 1
[10] V.D. Djordjevic and L.G. Redekopp, *On two-dimensional packets of capillary-gravity waves*, J. Fluid Mech., 79(4):703–714, 1977. 1
[11] H. Ikezi, K. Nishikawa, H. Hojo, and K. Mima, *Coupled electron-plasma and ion-acoustic solitons excited by parametric instability*, Proc. Int. Conf. Plasma Phys. Contr. Nucl. Fusion Res., Tokyo, 5:239–248, 1974. 1, 1
[12] T. Kato, *Quasi-linear equations of evolution, with applications to partial differential equations*, in W.N. Everitt (eds), Spectral Theory and Differential Equations, Lecture Notes in Mathematics, Springer, 448:25–70, 1975. 2, 2, 3.1
[13] T. Kato, *Linear and quasi-linear equations of evolution of hyperbolic type*, in G. Da Prato and G. Geymonat (eds), Hyperbolicity, C.I.M.E. Summer Schools, Vol 72. Springer, 125–191, 1976. 2, 2
[14] T. Kawahara, N. Sugimoto, and T. Kakutani, *Nonlinear interaction between short and long capillary-gravity waves*, J. Phys. Soc. Jpn., 39(5):1379–1386, 1975. 1
[15] N.V. Nguyen and C. Liu, *Some models for the interaction of long and short waves in dispersive media. Part I: Derivation*, Water Waves, 2:327–359, 2020. 1, 3, 4.4
[16] K. Nishikawa, H. Hojo, K. Mima, and H. Ikezi, *Coupled nonlinear electron-plasma and ion-acoustic waves*, Phys. Rev. Lett., 33:148–151, 1974. 1, 1
[17] V.E. Zakharov, *Collapse of Langmuir waves*, J. Exp. Theor. Phys., 35:908–914, 1972. 1