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G-MEAN RANDOM ATTRACTORS FOR COMPLEX GINZBURG-LANDAU EQUATIONS WITH PROBABILITY-UNCERTAIN INITIAL DATA*

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Abstract. In this paper, a class of complex Ginzburg-Landau equations with random initial data is investigated, where the randomness may be of probability uncertainty. The existence and uniqueness of global solution for such system are proved under the framework of nonlinear expectation. Then, the existence of pullback G-mean random attractors for the G-mean random dynamical system generated by the solution operators of (1.1) is investigated not only in $L^2_G(\Omega, L^2(\mathbb{R}))$, but also in a weighted space $L^2_G(\Omega, L^2_\sigma(\mathbb{R}))$. Moreover, such attractor is periodic if the nonautonomous deterministic forcing is time periodic.

Keywords. Complex Ginzburg-Landau equation; random initial data; nonlinear expectation; G-mean random dynamical system; G-mean random attractor.

AMS subject classifications. 37H05; 35B40; 35Q56.

1. Introduction

This paper is concerned with the complex Ginzburg-Landau equation with random initial condition in nonlinear expectation space defined on the entire $\mathbb{R}$:

$$\begin{cases} \frac{\partial u}{\partial t} = (\lambda + i\alpha)\Delta u - \rho u - (\kappa + i\beta)|u|^2 u + f(x,t), \ x \in \mathbb{R}, \ t > \tau, \\ u(x,\tau) = u_\tau(x), \ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\tau \in \mathbb{R}$, $u_\tau \in L^2_G(\Omega, L^2(\mathbb{R}))$, $u(x,t)$ is a complex value function, i is the imaginary unit, $\lambda, \alpha, \rho, \kappa, \beta$ are real constants satisfying $\lambda, \rho, \kappa > 0$, and f is given in $L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}))$.

It is well known that the complex Ginzburg-Landau equations have wide range of applications in mechanics, physics and other fields. For decades, the existence and uniqueness of global solutions of Ginzburg-Landau equations have been studied in [13, 14, 17, 25] for deterministic Ginzburg-Landau equations, and in [2, 30, 37] for stochastic Ginzburg-Landau equations.

Observe that in the above mentioned papers, the initial condition is deterministic. However, random disturbance or environmental noise may cause the randomness of measurement error of initial condition, thus taking it into consideration is necessary. Many interesting discoveries about systems with random initial condition have been reported, see, e.g., [4, 29, 40] and the references therein.

One of the basic tasks of the theory of differential equations and dynamical systems is to study the asymptotic behavior of solutions. The long-term behaviors of deterministic systems have been investigated in [6, 31, 32]. In order to investigate the stochastic systems, [1, 11] introduced the concept of pathwise random attractor. For stochastic systems with additive or linear multiplicative noise, pathwise random attractors have been examined in [3, 5, 7, 8, 16, 18, 20, 27, 33]. Moreover, the existence of random attractors for Ginzburg-Landau equations with linear noise has been investigated by many authors, e.g., see [21, 23, 28].

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It is well known that the pathwise theory of random dynamical systems has not been applied successfully to deal with stochastic system with nonlinear white noise or nonlocal stochastic system. In order to overcome this, mean-square random dynamical system was first introduced in [19], which can be applied to study nonlocal stochastic differential equations, stochastic differential equations with random delay [38] and others [36]. However, the main difficulty in applying this theory is the lack of useful characterizations of compact sets of such spaces of mean-square random variables. Therefore, until now, the existence of mean-square random attractors has been established only for some strictly contractive systems. Especially, when the drift term of stochastic partial differential equation is nonlinear, such theory in [19] may not be applicable since the weak continuity of solutions for stochastic systems may be difficult to obtain. Recently, a new type of weak pullback mean random attractor was introduced in Bochner spaces and a theorem on existence and uniqueness of such attractors was proved in [34], which can be used to investigate asymptotic behaviors for some stochastic systems with nonlinear drift term and nonlinear diffusion term [35, 40].

However, probability uncertainty or Knightian uncertainty cannot be characterized well in the previous literature, which often appears in practical problems such as finance and economic problems. In order to characterize such uncertainty, the theory of time-consistent nonlinear expectation, as well as its related stochastic calculus and G-Brownian motion, was developed by [12, 24]. Under Lipschitz condition, the existence and uniqueness of solutions of stochastic system driven by G-Brownian motion have been investigated in [15, 24, 39]. Stability has also been extensively studied in the literature, see, e.g., [10, 22, 26]. In addition, in [9], the mean random dynamical system was introduced in nonlinear expectation framework, which can be used for nonlocal stochastic systems driven by G-Brownian motion.

Inspired by the aforementioned works, in this paper, we will consider the asymptotic dynamics of complex Ginzburg-Landau equations with random initial data in nonlinear expectation space. It is worth mentioning that [40] discussed the weak pullback mean random attractors for Ginzburg-Landau equations defined in Bochner spaces, where the system is defined in a bounded domain O and the initial data are in a classical probability space. The main contributions and the highlights of this paper are listed as follows:

(i) The pathwise random dynamical system theory [1, 8, 11] and mean random dynamic system theory [19, 34] under the classical probability framework have been widely used to study the dynamic behaviors of solutions. But these theories cannot be applied to system (1.1) with random initial condition in nonlinear expectation. Therefore, we generalize the theories of mean random dynamical system in the sense of classical probability in [19, 34] to nonlinear probability case, which can be used to study the long-term behaviors of the solutions for system (1.1).

(ii) Compared with [13, 14, 40], the random initial condition with probabilistic uncertainty is taken into account in complex Ginzburg-Landau equation defined on the entire $\mathbb{R}$. This system can characterize the statistic uncertainty of environmental noise, thus it can be applied to more cases.

(iii) Different from [9], the concept of G-mean random dynamical system is introduced in $L_G^2(\Omega, L^2(\mathbb{R}))$ over $(\Omega, \mathcal{H}, \hat{\mathbb{E}}, \mathcal{F})$, instead of over a filtered space $(\Omega, \mathcal{H}, \hat{\mathbb{E}}, \mathcal{F}, \mathcal{F}_t)$. Meanwhile, the definition of pullback mean random attractor for G-mean random dynamical system is given. And then we prove the existence of such attractors as well as their periodicity when the nonautonomous deterministic forcing f is periodic in time (see Theorem 4.1).

(iv) In addition, the existence and periodicity of such attractors for system (1.1) are investigated in a weighted space $L_G^2(\Omega, L_\sigma^2(\mathbb{R}))$ (see Section 5 for more details), which generalizes the results in Section 4. It is worth noticing that the space $L_G^2(\Omega, L^2(\mathbb{R}))$ is contained in the weighted space $L_G^2(\Omega, L_\sigma^2(\mathbb{R}))$. Especially, when $\sigma > \frac{1}{2}$, $L_G^2(\Omega, L_\sigma^2(\mathbb{R}))$ contains all bounded measurable functions. In this case, the requirement condition of $f(x,t)$ in the weighted space $L_G^2(\Omega, L_\sigma^2(\mathbb{R}))$ is weaker than that in $L_G^2(\Omega, L^2(\mathbb{R}))$.

The outline of this paper is as follows. Some preliminaries are introduced in Section 2. In Section 3, the existence and uniqueness of solutions for system (1.1) in $L_G^2(\Omega, L^2(\mathbb{R}))$ are proved. The existence and periodicity of mean random attractors for system (1.1) in $L_G^2(\Omega, L^2(\mathbb{R}))$ and $L_G^2(\Omega, L_\sigma^2(\mathbb{R}))$ are investigated in Sections 4 and 5, respectively.

Throughout this paper, let $\|\cdot\|_{L^p(\mathbb{R})}$ and $\langle\cdot,\cdot\rangle_{L^p(\mathbb{R})}$ denote the norm and the inner product of $L^p(\mathbb{R})$, respectively. When $p=2$, we will omit the subscript $L^p(\mathbb{R})$ in the above notations for simplicity.

2. Preliminaries

We use the framework and notations of Peng in [24]. Let $(\Omega, \mathcal{F})$ be a given measurable space and $\mathcal{H}$ be a linear space of real-valued functions defined on $(\Omega, \mathcal{F})$ satisfying that $\varphi(\xi_1, \xi_2, \cdots, \xi_n) \in \mathcal{H}$ if $\xi_1, \xi_2, \cdots, \xi_n \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, where $C_{l,Lip}(\mathbb{R}^n)$ denotes the linear space of functions φ satisfying the local Lipschitz condition:

$$|\varphi(x) - \varphi(y)| \le C(1 + |x|^m + |y|^m)|x-y| \text{ for } x, y \in \mathbb{R}^n,$$

where $C > 0$ and $m \in \mathbb{N}$ depend on φ.

Definition 2.1. *A sublinear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\hat{\mathbb{E}}: \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for all $\xi, \zeta \in \mathcal{H}$,*

(i) ***Monotonicity:*** *if $\xi \ge \zeta$, then $\hat{\mathbb{E}}[\xi] \ge \hat{\mathbb{E}}[\zeta]$;*

(ii) ***Constant preserving:*** *$\hat{\mathbb{E}}[c] = c$;*

(iii) ***Sub-additivity:*** *$\hat{\mathbb{E}}[\xi + \zeta] \le \hat{\mathbb{E}}[\xi] + \hat{\mathbb{E}}[\zeta]$;*

(iv) ***Positive homogeneity:*** *$\hat{\mathbb{E}}[\lambda\xi] = \lambda\hat{\mathbb{E}}[\xi]$ for $\lambda \ge 0$.*

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space.

Definition 2.2. *A set function $\mathbb{C}$ on $\mathcal{F}$ is called a capacity if it satisfies*

(i) *$\mathbb{C}(\emptyset) = 0$, $\mathbb{C}(\Omega) = 1$;*

(ii) *$\mathbb{C}(A) \le \mathbb{C}(B)$, $A \subset B$, $A, B \in \mathcal{F}$.*

A capacity $\mathbb{C}$ is said to be sub-additive if it satisfies $\mathbb{C}(A \cup B) \le \mathbb{C}(A) + \mathbb{C}(B)$.

Definition 2.3. *Given a capacity $\mathbb{C}$, a set $A \in \mathcal{F}$ is said to be polar if $\mathbb{C}(A) = 0$. A property is said to hold quasi-surely (q.s.) if it holds outside a polar set.*

Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a sublinear expectation space, then define a capacity: $\mathbb{C}(A) := \hat{\mathbb{E}}(I_A)$, $\forall A \in \mathcal{F}$. Then, $\mathbb{C}$ is a sub-additive capacity. And it was proved in [24] that there exists a family of linear expectations $\mathbb{E}_P : \mathcal{H} \to \mathbb{R}$, indexed by $P \in \mathcal{P}$ such that

$$\hat{\mathbb{E}}[u] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[u], \quad u \in \mathcal{H}, \tag{2.1}$$

where $\mathcal{P}$ is a family of probability measures. Let $L_G^0(\Omega, X)$ be the space of all $(\mathcal{F}, \mathcal{B}(X))$-measurable functions, where X is a Banach space with norm $\|\cdot\|_X$. We observe that $\mathcal{L}^2 := \{u \in L_G^0(\Omega, X) : \hat{\mathbb{E}}[\|u\|_X^2] < \infty\}$ and $\mathcal{N}^2 := \{u \in L_G^0(\Omega, X) : \hat{\mathbb{E}}[\|u\|_X^2] = 0\}$ are linear spaces. Denote $L_G^2(\Omega,X):=\mathcal{L}^2/\mathcal{N}^2$. Similar to the classical result, it is not difficult to prove that $L_G^2(\Omega,X)$ is a Banach space with the norm $\|u\|_{L_G^2}:=(\hat{\mathbb{E}}[\|u\|_X^2])^{\frac{1}{2}}$.

In order to investigate the asymptotic behavior of system (1.1) with random initial conditions, following the idea of [9, 19, 34], we will introduce the concepts of G-mean random dynamical system and pullback G-mean random attractors over $(\Omega,\mathcal{H},\hat{\mathbb{E}},\mathcal{F})$ (not over $(\Omega,\mathcal{H},\hat{\mathbb{E}},\mathcal{F},\mathcal{F}_t)$). For convenience, let $\mathfrak{X}:=L_G^2((\Omega,\mathcal{H},\hat{\mathbb{E}},\mathcal{F});X)=\{\phi|\ \phi$ is $\mathcal{F}$-measurable and $\hat{\mathbb{E}}[\|\phi\|_X^2]<\infty\}$, and $\mathbb{R}^2_{\geq}:=\{(t,t_0)\in\mathbb{R}^2:t\geq t_0\}$.

Definition 2.4. *A family of mappings $\Phi=\{\Phi(t,t_0,\xi_0):(t,t_0)\in\mathbb{R}^2_{\geq}\}$ on nonlinear expectation space is called a G-mean square random dynamical system over $(\Omega,\mathcal{H},\hat{\mathbb{E}},\mathcal{F})$, if the mapping $\Phi(t,t_0,\cdot):\mathfrak{X}\to\mathfrak{X}$, $(t,t_0)\in\mathbb{R}^2_{\geq}$ satisfies:*

(i) ***Initial value property:*** *$\Phi(t_0,t_0,\xi_0)=\xi_0$ for any $\xi_0\in\mathfrak{X}$ and $t_0\in\mathbb{R}$;*

(ii) ***Two-parameter semigroup property:*** *$\Phi(t_2,t_0,\xi_0)=\Phi(t_2,t_1,\Phi(t_1,t_0,\xi_0))$ for every $\xi_0\in\mathfrak{X}$ and $(t_2,t_1),(t_1,t_0)\in\mathbb{R}^2_{\geq}$;*

(iii) ***Continuity property:*** *$(t,t_0,\xi_0)\mapsto\Phi(t,t_0,\xi_0)$ is continuous in the space $\mathbb{R}^2_{\geq}\times\mathfrak{X}$.*

Definition 2.5. *A family $K=\{K(t)\}_{t\in\mathbb{R}}$ of nonempty subsets of $\mathfrak{X}$ for each $t\in\mathbb{R}$ is said to be Φ-invariant if*

$$\Phi(t,t_0,K(t_0))=K(t),\quad \textit{for all } (t,t_0)\in\mathbb{R}^2_{\geq},$$

and Φ-positively invariant if

$$\Phi(t,t_0,K(t_0))\subseteq K(t),\quad \textit{for all } (t,t_0)\in\mathbb{R}^2_{\geq}.$$

Let $D=\{D(t)\subseteq\mathfrak{X}:t\in\mathbb{R}\}$ be a family of bounded nonempty sets and there exists a constant $\lambda>0$ such that

$$\lim_{t\to-\infty}e^{\lambda t}\|D(t)\|_{\mathfrak{X}}^2=0, \tag{2.2}$$

where $\|D(t)\|_{\mathfrak{X}}^2=\sup\limits_{u\in D(t)}\hat{\mathbb{E}}[\|u\|_X^2]$. And D is said to be uniformly bounded if there exists a positive constant r such that, $\hat{\mathbb{E}}[\|u(t)\|_X^2]\leq r$ holds for any $t\in\mathbb{R}$ and $u(t)\in D(t)\in D$. In what follows, we set

$$\mathcal{D}=\{D=\{D(t)\subseteq\mathfrak{X}:D(t)\neq\emptyset \text{ bounded},\ t\in\mathbb{R}\}:D \text{ satisfies (2.2)}\}.$$

Definition 2.6. *A family $K=\{K(t)\}_{t\in\mathbb{R}}\in\mathcal{D}$ is called a $\mathcal{D}$-pullback absorbing family for Φ if for each $t\in\mathbb{R}$ and $D\in\mathcal{D}$, there exists $T'=T'(t,D)>0$ such that*

$$\Phi(t,t-s,D(t-s))\subseteq K(t),\quad s\geq T'.$$

Remark 2.1. Compared with [19,38], in this paper, the absorbing family for G-mean random dynamical system Φ is not required to be uniformly bounded. Obviously, D is uniformly bounded, which implies that it satisfies (2.2). This shows that a uniformly bounded family of nonempty closed subsets $D=\{D(t)\}_{t\in\mathbb{R}}$ belongs to $\mathcal{D}$. Therefore, the requirement conditions of absorbing set are weaker than those in the literature [19,38].

Definition 2.7. *A family $\mathcal{A}=\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is called a $\mathcal{D}$-pullback G-mean random attractor for Φ in $\mathfrak{X}$ if the following conditions are fulfilled:*

(i) $\mathcal{A}(t)$ *is a nonempty compact subset of* $\mathfrak{X}$ *for each* $t\in\mathbb{R}$*;*

(ii) $\mathcal{A}$ *is* Φ*-invariant, i.e.,* $\Phi(t,t_0,\mathcal{A}(t_0))=\mathcal{A}(t)$*, for all* $(t,t_0)\in\mathbb{R}^2_{\geq}$*;*

(iii) $\mathcal{A}$ *pullback attracts every* $D\in\mathcal{D}$*, that is, for every* $t\in\mathbb{R}$*,*

$$\lim_{s\to+\infty} d(\Phi(t,t-s,D(t-s)),\mathcal{A}(t))=0,$$

where $d(A,B):=\sup\limits_{x\in A}\inf\limits_{y\in B}\|x-y\|_{\mathfrak{X}}$ *is the Hausdorff semi-distance, for any* $A,B\subseteq\mathfrak{X}$*.*

In addition, in this paper, the following two lemmas are also frequently used to prove the existence of $\mathcal{D}$-pullback G-mean random attractors for complex Ginzburg-Landau system with random initial condition.

Lemma 2.1 ([23] Gagliardo-Nirenberg's inequality). *Let* $u\in L^q(\mathbb{R})$ *and its derivatives of order* m*,* $D^m u\in L^r(\mathbb{R})$*,* $1\leq q,r\leq\infty$*. For the derivatives* $D^j u$*,* $1\leq j<m$*, there exists* $c=c(m,j,q,r,\theta)$ *such that*

$$\|D^j u\|_{L^p(\mathbb{R})}\leq c\|D^m u\|^{\theta}_{L^r(\mathbb{R})}\|u\|^{1-\theta}_{L^q(\mathbb{R})},$$

where $\frac{1}{p}=j+\theta(\frac{1}{r}-m)+(1-\theta)\frac{1}{q}$*, for all* θ *in the interval* $\frac{j}{m}\leq\theta\leq 1$*.*

Lemma 2.2 ([21]). *For any* $-1<\mu<+\infty$ *and* $x,y\in\mathbb{C}$*, the following inequality holds*

$$|\mathrm{Im}(\bar{x}-\bar{y})(|x|^{\mu}x-|y|^{\mu}y)|\leq\frac{\mu}{2\sqrt{\mu+1}}\mathrm{Re}(\bar{x}-\bar{y})(|x|^{\mu}x-|y|^{\mu}y).$$

3. Existence and uniqueness of solutions for system (1.1)

The objectives of this section are to study the existence and uniqueness of solution for system (1.1) in nonlinear expectation space $(\Omega,\mathcal{H},\hat{\mathbb{E}})$, which are necessary for establishing the G-mean square dynamical system associated with system (1.1). First, the definition of solution for system (1.1) is given below.

Definition 3.1. *Let* $\tau\in\mathbb{R}$ *and* $u_\tau\in L^2_G(\Omega,L^2(\mathbb{R}))$*. A continuous mapping* $u(\cdot)\doteq u(\cdot,\tau,u_\tau):[\tau,\infty)\to L^2_G(\Omega,L^2(\mathbb{R}))$ *is called a solution of system* (1.1) *if*

$$\begin{aligned} u(\cdot,\tau,u_\tau)\in &C\big([\tau,\infty),L^2_G\big(\Omega,L^2(\mathbb{R})\big)\big)\bigcap L^2_G\big(\Omega,L^2_{loc}\big((\tau,\infty),H^1(\mathbb{R})\big)\big)\\ &\bigcap L^4_G\big(\Omega,L^4_{loc}\big((\tau,\infty),L^4(\mathbb{R})\big)\big)\end{aligned}$$

and u satisfies, for every $t>\tau$ and $\xi\in H^1(\mathbb{R})\cap L^4(\mathbb{R})$

$$\begin{aligned}\langle u(t),\xi\rangle=&\langle u_\tau,\xi\rangle-(\lambda+i\alpha)\int_\tau^t\langle\nabla u,\nabla\xi\rangle ds-\rho\int_\tau^t\langle u,\xi\rangle ds\\ &-(\kappa+i\beta)\int_\tau^t\langle|u|^2u,\xi\rangle ds+\int_\tau^t\langle f(s),\xi\rangle ds,\ \text{q.s.},\end{aligned}$$

where $f(\cdot):\mathbb{R}\to L^2(\mathbb{R})$ and $\langle f(s),\xi\rangle=\int_{\mathbb{R}}f(x,s)\xi(x)dx$.

We now prove the existence and uniqueness of solutions to problem (1.1) in the sense of Definition 3.1.

Theorem 3.1. *For every* $\tau\in\mathbb{R}$ *and* $u_\tau\in L^2_G(\Omega,L^2(\mathbb{R}))$*, system* (1.1) *has a unique solution* $u(\cdot,\tau,u_\tau)$ *in the sense of Definition 3.1. Furthermore, this solution is* $\mathcal{F}$*-measurable with respect to* $\omega\in\Omega$*.*

Proof. Let $O_k=\{x\in\mathbb{R},|x|<k\}$ for each $k\in\mathbb{N}$, and consider the following equation defined in O_k

$$\frac{\partial u_k}{\partial t}=(\lambda+i\alpha)\Delta u_k-\rho u_k-(\kappa+i\beta)|u_k|^2u_k+f(x,t),\ t>\tau,\ x\in O_k, \tag{3.1}$$

with boundary condition

$$u_k(x,t)=0,\ t>\tau,\ |x|=k, \tag{3.2}$$

and initial condition

$$u_k(x,\tau)=u_\tau(x),\ x\in O_k. \tag{3.3}$$

For any fixed $k>0$, we can deduce that system (3.1)-(3.3) has one solution u_k (the definition of solution is the same as in Definition 3.1 but replacing $\mathbb{R}$ by O_k). Moreover, for any $t\geq\tau$, $u_k(t,\omega)$ is $\mathcal{F}$-measurable with respect to $\omega\in\Omega$.

Next, we will derive uniform estimates on the solution u_k of (3.1)-(3.3). Taking the inner product of (3.1) and considering the real part, we have

$$\begin{aligned}\frac{d}{dt}\|u_k\|^2=&-2\lambda\|\nabla u_k\|^2-2\rho\|u_k\|^2-2\kappa\|u_k\|^4_{L^4(O_k)}+2\mathrm{Re}\int_{O_k}f(t,x)\overline{u_k}dx\\ \leq&-2\lambda\|\nabla u_k\|^2-\rho\|u_k\|^2-2\kappa\|u_k\|^4_{L^4(O_k)}+\frac{1}{\rho}\|f(t)\|^2.\end{aligned} \tag{3.4}$$

Therefore, for any $t\geq\tau$ and $\omega\in\Omega$, we find

$$\begin{aligned}&\|u_k(t,\omega)\|^2+2\lambda\int_\tau^t\|\nabla u_k(s,\omega)\|^2ds+2\kappa\int_\tau^t\|u_k(s,\omega)\|^4_{L^4(O_k)}ds\\ \leq&\|u_\tau(\omega)\|^2+\frac{1}{\rho}\int_\tau^t\|f(s)\|^2ds.\end{aligned}$$

Then, for every fixed $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $T>0$, we have

$$\|u_k(t,\omega)\|^2\leq\|u_\tau(\omega)\|^2+\frac{1}{\rho}\int_\tau^{\tau+T}\|f(s)\|^2ds,\ t\in[\tau,\tau+T], \tag{3.5}$$

$$\int_\tau^{\tau+T}\|\nabla u_k(s,\omega)\|^2ds\leq\frac{1}{2\lambda}\Big(\|u_\tau(\omega)\|^2+\frac{1}{\rho}\int_\tau^{\tau+T}\|f(s)\|^2ds\Big), \tag{3.6}$$

and

$$\int_\tau^{\tau+T}\|u_k(s,\omega)\|^4_{L^4(O_k)}ds\leq\frac{1}{2\kappa}\Big(\|u_\tau(\omega)\|^2+\frac{1}{\rho}\int_\tau^{\tau+T}\|f(s)\|^2ds\Big). \tag{3.7}$$

It follows from (3.5)-(3.7) that for every fixed $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $T>0$,

$$\begin{aligned}\{u_k(\cdot,\omega)\}_{k=1}^\infty \text{ is bounded in } L^\infty\big((\tau,\tau+T),L^2(O_k)\big)\bigcap L^2\big((\tau,\tau+T),H_0^1(O_k)\big)\\ \bigcap L^4\big((\tau,\tau+T),L^4(O_k)\big)\end{aligned} \tag{3.8}$$

and

$$\{|u_k|^2u_k\}_{k=1}^\infty \text{ is bounded in } L^{\frac{4}{3}}\big((\tau,\tau+T),L^{\frac{4}{3}}(O_k)\big). \tag{3.9}$$

Therefore,

$$\left\{\frac{du_k}{dt}\right\}_{k=1}^{\infty} \text{ is bounded in } L^{\frac{4}{3}}\big((\tau,\tau+T),L^{\frac{4}{3}}(O_k)\big)+L^2\big((\tau,\tau+T),H^{-1}(O_k)\big). \quad (3.10)$$

Consider u_k as a function defined on the entire space $\mathbb{R}$ by setting $u_k(x,t)=0$ for all $|x|>k$ and $t\in[\tau,\tau+T]$. Let $t'\in(\tau,\tau+T]$ be fixed. Then by (3.8)-(3.10), we know that there exists a subsequence $\{u_{k_l}\}_{l=1}^{\infty}$ of $\{u_k\}_{k=1}^{\infty}$ such that

$$\begin{aligned}
u_{k_l}(\cdot,\omega)&\to u(\cdot,\omega) \text{ weak star in } L^{\infty}\big((\tau,\tau+T),L^2(\mathbb{R})\big),\\
u_{k_l}(\cdot,\omega)&\to u(\cdot,\omega) \text{ weakly in } L^{2}\big((\tau,\tau+T),H^1(\mathbb{R})\big),\\
u_{k_l}(\cdot,\omega)&\to u(\cdot,\omega) \text{ weakly in } L^{4}\big((\tau,\tau+T),L^4(\mathbb{R})\big),\\
\frac{d}{dt}u_{k_l}(\cdot,\omega)&\to \frac{d}{dt}u(\cdot,\omega) \text{ weakly in } L^{\frac{4}{3}}\big((\tau,\tau+T),L^{\frac{4}{3}}(\mathbb{R})\big)+L^2\big((\tau,\tau+T),H^{-1}(\mathbb{R})\big),\\
u_{k_l}(t',\omega)&\to v \text{ weakly in } L^2(\mathbb{R}),
\end{aligned}$$

where $u\in L^{\infty}\big((\tau,\tau+T),L^2(\mathbb{R})\big)\cap L^2\big((\tau,\tau+T),H^1(\mathbb{R})\big)\cap L^4\big((\tau,\tau+T),L^4(\mathbb{R})\big)$ and $v\in L^2(\mathbb{R})$. Then by a standard procedure (see [20]), we can deduce that

$$|u_{k_l}(\cdot,\omega)|^2u_{k_l}(\cdot,\omega)\to|u(\cdot,\omega)|^2u(\cdot,\omega) \text{ weakly in } L^{\frac{4}{3}}\big((\tau,\tau+T),L^{\frac{4}{3}}(\mathbb{R})\big).$$

Therefore, letting $l\to\infty$, we have that for any $\xi\in H^1(\mathbb{R})\cap L^4(\mathbb{R})$,

$$\frac{d}{dt}\langle u,\xi\rangle=-(\lambda+i\alpha)\langle\nabla u,\nabla\xi\rangle-\rho\langle u,\xi\rangle-(\kappa+i\beta)\langle|u|^2u,\xi\rangle+\langle f(t),\xi\rangle, \quad (3.11)$$

on $(\tau,\tau+T)$. In addition, $u(\cdot,\omega)\in C([\tau,\tau+T],L^2(\mathbb{R}))$, $u(\tau,\omega)=u_\tau(\omega)$, $u(t',\omega)=v$ and

$$\frac{1}{2}\frac{d}{dt}\|u(t,\omega)\|^2=-\lambda\|\nabla u(t,\omega)\|^2-\rho\|u(t,\omega)\|^2-\kappa\|u(t,\omega)\|^2_{L^4(\mathbb{R})}+\langle f(t),u(t,\omega)\rangle. \quad (3.12)$$

Therefore, we can deduce that for any $t'\in(\tau,\tau+T]$,

$$u_{k_l}(t',\omega)\to u(t',\omega) \text{ weakly in } L^2(\mathbb{R}), \quad (3.13)$$

and $u(\cdot,\omega)$ is a solution of the deterministic system (1.1) with initial condition $u_\tau(\omega)$ for a fixed $\omega\in\Omega$.

On the other hand, for every fixed $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $T>0$, we have

$$\|u(t,\omega)\|^2\le\|u_\tau(\omega)\|^2+\frac{1}{\rho}\int_\tau^{\tau+T}\|f(s)\|^2ds,\ t\in[\tau,\tau+T], \quad (3.14)$$

$$\int_\tau^{\tau+T}\|\nabla u(s,\omega)\|^2ds\le\frac{1}{2\lambda}\Big(\|u_\tau(\omega)\|^2+\frac{1}{\rho}\int_\tau^{\tau+T}\|f(s)\|^2ds\Big), \quad (3.15)$$

and

$$\int_\tau^{\tau+T}\|u(s,\omega)\|^4_{L^4(\mathbb{R})}ds\le\frac{1}{2\kappa}\Big(\|u_\tau(\omega)\|^2+\frac{1}{\rho}\int_\tau^{\tau+T}\|f(s)\|^2ds\Big). \quad (3.16)$$

Since $u_\tau\in L^2_G(\Omega,L^2(\mathbb{R}))$, we see from (3.14) that $u\in L^{\infty}_{loc}\big((\tau,\infty),L^2_G(\Omega,L^2(\mathbb{R}))\big)$. It then follows from (3.14)-(3.16) that

$$u\in L^2_G\big(\Omega,L^2_{loc}\big((\tau,\infty),H^1(\mathbb{R})\big)\big)\bigcap L^4_G\big(\Omega,L^4_{loc}\big((\tau,\infty),L^4(\mathbb{R})\big)\big).$$

Next, we are going to prove that $u\in C\big([\tau,\infty),L_G^2\big(\Omega,L^2(\mathbb{R})\big)\big)$. It follows from (3.12) and the Young inequality that

$$\hat{\mathbb{E}}[\|u(t)\|^2-\|u(s)\|^2]\leq\frac{1}{\rho}\int_s^t\|f(r)\|^2dr. \tag{3.17}$$

For any $t,s\geq\tau$,

$$\hat{\mathbb{E}}[\|u(t)-u(s)\|^2]=\hat{\mathbb{E}}[\|u(t)\|^2+\|u(s)\|^2-2\langle u(t),u(s)\rangle]. \tag{3.18}$$

It is not hard to prove that $L_G^2(\Omega,H^1(\mathbb{R}))\cap L_G^2(\Omega,L^4(\mathbb{R}))$ is dense in $L_G^2(\Omega,L^2(\mathbb{R}))$, thus there exists $\{u_n(s)\}_{n=1}^{\infty}\in L_G^2(\Omega,H^1(\mathbb{R}))\cap L_G^2(\Omega,L^4(\mathbb{R}))$ such that

$$\hat{\mathbb{E}}[\|u(s)-u_n(s)\|^2]\to 0,\ \text{as}\ n\to\infty. \tag{3.19}$$

Combining this with (3.17), we obtain that for any fixed $s\geq\tau$,

$$\begin{aligned}&\hat{\mathbb{E}}[\|u(t)-u(s)\|^2]=\hat{\mathbb{E}}[\|u(t)\|^2+\|u(s)\|^2-2\langle u(t),u_n(s)+u(s)-u_n(s)\rangle]\\&\leq\hat{\mathbb{E}}[\|u(t)\|^2+\|u(s)\|^2-2\langle u(t),u_n(s)\rangle]+2\hat{\mathbb{E}}[\langle u(t),u_n(s)-u(s)\rangle]\\&\leq\hat{\mathbb{E}}\left[2\|u(s)\|^2-2\langle u(t),u_n(s)\rangle+\frac{1}{\rho}\int_s^t\|f(r)\|^2dr\right]+2\hat{\mathbb{E}}[\langle u(t),u_n(s)-u(s)\rangle].\end{aligned}$$

It follows from (3.11) that

$$\begin{aligned}&\langle u(t,\omega),u_n(s,\omega)\rangle\\&=\langle u(s,\omega),u_n(s,\omega)\rangle-\int_s^t(\lambda+i\alpha)\langle\nabla u(r,\omega),\nabla u_n(r,\omega)\rangle dr-\int_s^t\rho\langle u(r,\omega),u_n(s,\omega)\rangle dr\\&\quad-\int_s^t(\kappa+i\beta)\langle|u(r,\omega)|^2u(r,\omega),u_n(s,\omega)\rangle dr+\int_s^t\langle f(r),u_n(s,\omega)\rangle dr.\end{aligned}$$

Without loss of generality, we suppose $t\geq s$, therefore,

$$\begin{aligned}&\hat{\mathbb{E}}[\|u(t)-u(s)\|^2]\leq\hat{\mathbb{E}}\Big[2\|u(s)\|^2-2\langle u(s),u_n(s)\rangle-\int_s^t(\lambda+i\alpha)\langle\nabla u(r),\nabla u_n(s)\rangle dr\\&\quad-\int_s^t\rho\langle u(r),u_n(s)\rangle dr-\int_s^t(\kappa+i\beta)\langle|u(r)|^2u(r),u_n(s)\rangle dr\\&\quad+\int_s^t\langle f(r),u_n(s)\rangle dr+\frac{1}{\rho}\int_s^t\|f(r)\|^2dr\Big]+2\hat{\mathbb{E}}[\langle u(t),u_n(s)-u(s)\rangle]\\&\leq\hat{\mathbb{E}}[2\langle u(s),u(s)-u_n(s)\rangle]+C\hat{\mathbb{E}}\left[\int_s^t\|\nabla u(r)\|\|\nabla u_n(s)\|dr\right]\\&\quad+C\hat{\mathbb{E}}\left[\int_s^t\|u(r)\|\|u_n(s)\|ds\right]+C\hat{\mathbb{E}}\left[\int_s^t\|u(r)\|_{L^4(\mathbb{R})}^3\|u_n(s)\|_{L^4(\mathbb{R})}dr\right]\\&\quad+\hat{\mathbb{E}}\left[\int_s^t\|f(r)\|\|u_n(s)\|dr\right]+\frac{1}{\rho}\int_s^t\|f(r)\|^2dr+2\hat{\mathbb{E}}[\langle u(t),u_n(s)-u(s)\rangle].\end{aligned} \tag{3.20}$$

For the first and last terms on the right-hand side of (3.20), we have

$$\begin{aligned}&2\hat{\mathbb{E}}[\langle u(s),u(s)-u_n(s)\rangle]+2\hat{\mathbb{E}}[\langle u(t),u_n(s)-u(s)\rangle]\\&\leq 2(\hat{\mathbb{E}}[\|u(s)\|^2])^{\frac{1}{2}}(\hat{\mathbb{E}}[\|u(s)-u_n(s)\|^2])^{\frac{1}{2}}+2(\hat{\mathbb{E}}[\|u(t)\|^2])^{\frac{1}{2}}(\hat{\mathbb{E}}[\|u_n(s)-u(s)\|^2])^{\frac{1}{2}}.\end{aligned} \tag{3.21}$$

It then follows from (3.14) and (3.19) that

$$\hat{\mathbb{E}}[\langle u(s),u(s)-u_n(s)\rangle]+2\hat{\mathbb{E}}[\langle u(t),u_n(s)-u(s)\rangle]\to 0,\ \text{as } n\to\infty. \tag{3.22}$$

For the second term on the right-hand side of (3.20), by the Hölder inequality we have

$$\begin{aligned}&C\hat{\mathbb{E}}\left[\int_s^t \|\nabla u(r)\|\|\nabla u_n(s)\|dr\right]\\ \le& C\hat{\mathbb{E}}\left[\left(\int_s^t \|\nabla u(r)\|^2dr\right)^{\frac{1}{2}}\left(\int_s^t \|\nabla u_n(s)\|^2dr\right)^{\frac{1}{2}}\right]\\ \le& C\left(\hat{\mathbb{E}}\Big[\int_s^t \|\nabla u(r)\|^2dr\Big]\right)^{\frac{1}{2}}(\hat{\mathbb{E}}[\|\nabla u_n(s)\|^2])^{\frac{1}{2}}|t-s|^{\frac{1}{2}}.\end{aligned} \tag{3.23}$$

Similarly, we find

$$C\hat{\mathbb{E}}\left[\int_s^t \|u(r)\|\|u_n(s)\|ds\right]\le C\left(\hat{\mathbb{E}}\Big[\int_s^t \|u(r)\|^2dr\Big]\right)^{\frac{1}{2}}(\hat{\mathbb{E}}[\|u_n(s)\|^2])^{\frac{1}{2}}|t-s|^{\frac{1}{2}}, \tag{3.24}$$

$$\begin{aligned}&C\hat{\mathbb{E}}\left[\int_s^t \|u(r)\|^3_{L^4(\mathbb{R})}\|u_n(s)\|_{L^4(\mathbb{R})}dr\right]\\ \le& C\left(\hat{\mathbb{E}}\Big[\int_s^t \|u(r)\|^4_{L^4(\mathbb{R})}dr\Big]\right)^{\frac{4}{3}}(\hat{\mathbb{E}}[\|u_n(s)\|^4_{L^4(\mathbb{R})}])^{\frac{1}{4}}|t-s|^{\frac{1}{4}},\end{aligned} \tag{3.25}$$

$$\hat{\mathbb{E}}\left[\int_s^t \|f(r)\|\|u_n(s)\|dr\right]\le C\left(\hat{\mathbb{E}}\Big[\int_s^t \|f(r)\|^2dr\Big]\right)^{\frac{1}{2}}(\hat{\mathbb{E}}[\|u_n(s)\|^2])^{\frac{1}{2}}|t-s|^{\frac{1}{2}}. \tag{3.26}$$

By (3.22)-(3.26), one can deduce that

$$\hat{\mathbb{E}}[\|u(t)-u(s)\|^2]\to 0,\ \text{as } t\to s,$$

which shows that

$$u\in C([\tau,\infty),L^2_G(\Omega,L^2(\mathbb{R}))). \tag{3.27}$$

Therefore, u is a solution to problem (1.1) in the sense of Definition 3.1.

Next, we will show that the solution is unique. Let u_1, u_2 be any two solutions of system (1.1) with the same initial condition and $v=u_1-u_2$. Note

$$\begin{aligned}\frac{d}{dt}\|v\|^2=&-2\lambda\|\nabla v\|^2-2\rho\|v\|^2-2\mathrm{Re}(\kappa+i\beta)\int_{\mathbb{R}}(|u_1|^2u_1-|u_2|^2u_2)\bar{v}dx\\ \le&-2\lambda\|\nabla v\|^2+2\left|\mathrm{Re}(\kappa+i\beta)\int_{\mathbb{R}}(|u_1|^2u_1-|u_2|^2u_2)\bar{v}dx\right|.\end{aligned}$$

It follows from the Young inequality and Lemma 2.1 that

$$\begin{aligned}&\left|\mathrm{Re}(\kappa+i\beta)\int_{\mathbb{R}}(|u_1|^2u_1-|u_2|^2u_2)\bar{v}dx\right|\\ =&\left|\mathrm{Re}(\kappa+i\beta)\int_{\mathbb{R}}\left[|u_1|^2(u_1-u_2)+(|u_1|^2-|u_2|^2)u_2\right]\bar{v}dx\right|\end{aligned}$$

$$
\begin{aligned}
&\le \kappa\int_{\mathbb{R}}|u_1|^2|v|^2dx+\left|\mathrm{Re}(\kappa+i\beta)\int_{\mathbb{R}}(|u_1|^2-|u_2|^2)u_2\bar{v}dx\right|\\
&\le 3\sqrt{\kappa^2+\beta^2}\int_{\mathbb{R}}(|u_1|^2+|u_2|^2)|v|^2dx\\
&\le 3\sqrt{2(\kappa^2+\beta^2)}\Big(\int_{\mathbb{R}}(|u_1|^4+|u_2|^4)dx\Big)^{\frac{1}{2}}\|v\|^2_{L^4(\mathbb{R})}\\
&\le 3c\sqrt{2(\kappa^2+\beta^2)}\big(\|u_1\|^2_{L^4(\mathbb{R})}+\|u_2\|^2_{L^4(\mathbb{R})}\big)\|\nabla v\|^{\frac{1}{2}}\|v\|^{\frac{3}{2}}\\
&\le 3c\sqrt{2(\kappa^2+\beta^2)}\left[\varepsilon\|\nabla v\|^2+\frac{3}{4}(2\varepsilon)^{-\frac{1}{3}}\big(\|u_1\|^{\frac{8}{3}}_{L^4(\mathbb{R})}+\|u_2\|^{\frac{8}{3}}_{L^4(\mathbb{R})}\big)\|v\|^2\right],
\end{aligned}
$$

where c is defined in Lemma 2.1. Then letting $\varepsilon=\frac{\lambda}{3c\sqrt{2(\kappa^2+\beta^2)}}$, we have

$$
\frac{d}{dt}\|v\|^2\le\frac{9\sqrt{2}c}{4}\sqrt{\kappa^2+\beta^2}(2\varepsilon)^{-\frac{1}{3}}\big(\|u_1\|^{\frac{8}{3}}_{L^4(\mathbb{R})}+\|u_2\|^{\frac{8}{3}}_{L^4(\mathbb{R})}\big)\|v\|^2.
$$

It follows from the Gronwall inequality that

$$
\begin{aligned}
\|u_1(t,\omega)-u_2(t,\omega)\|^2 &\le\|u_{\tau,1}(\omega)-u_{\tau,2}(\omega)\|^2e^{\frac{9\sqrt{2}c}{4}\sqrt{\kappa^2+\beta^2}(2\varepsilon)^{-\frac{1}{3}}\int_\tau^t(\|u_1\|^{\frac{8}{3}}_{L^4(\mathbb{R})}+\|u_2\|^{\frac{8}{3}}_{L^4(\mathbb{R})})ds}\\
&\le\|u_{\tau,1}(\omega)-u_{\tau,2}(\omega)\|^2e^{C[(\int_\tau^t\|u_1\|^4_{L^4(\mathbb{R})}ds)^{\frac{2}{3}}+(\int_\tau^t\|u_2\|^4_{L^4(\mathbb{R})}ds)^{\frac{2}{3}}]}, \quad (3.28)
\end{aligned}
$$

where $C=\frac{9\sqrt{2}c}{4}\sqrt{\kappa^2+\beta^2}(\frac{t-\tau}{2\varepsilon})^{\frac{1}{3}}$. This implies the uniqueness of solution.

Note that (3.13) and the uniqueness of solutions imply that the entire sequence $u_k(t,\omega)\to u(t,\omega)$ weakly in $L^2(\mathbb{R})$. By the measurability of $u_k(t,\omega)$ in ω, the measurability of $u(t,\omega)$ can be obtained directly. The proof is complete. □

REMARK 3.1. In classical probability space, the continuity of the solution with respect to time in the mean sense can be proved by the dominated convergence theorem, see, e.g., [34, 35, 40]. Nevertheless, different from the classical probability space, the dominated convergence theorem usually does not hold in the framework of nonlinear expectation. This gives rise to some difficulties in proving $u\in C([\tau,\infty),L^2_G(\Omega,L^2(\mathbb{R})))$, which is necessary to establish the G-mean square dynamical system associated with (1.1) in the next section.

REMARK 3.2. Inequality (3.28) shows the uniqueness of the solution. However, it does not indicate the continuity of solutions with respect to initial conditions, which will be proved in Lemma 4.1.

4. G-mean random attractors for (1.1) in $L^2_G(\Omega,L^2(\mathbb{R}))$

In this section, we will prove the existence of mean random attractors for system (1.1) in $L^2_G(\Omega,L^2(\mathbb{R}))$. For this purpose, we further assume:

$$
|\beta|\le\sqrt{3}\kappa, \quad (4.1)
$$

$$
\int_{-\infty}^{\tau}e^{\rho s}\|f(s)\|^2ds<\infty,\ \forall\ \tau\in\mathbb{R}. \quad (4.2)
$$

4.1. G-mean random dynamical systems.

To investigate the long-term dynamics of the solutions of problem (1.1), we need to define a random dynamical system based on the solution operators.

Let Φ be a mapping on $\mathbb{R}^2_{\geq} \times L^2_G(\Omega, L^2(\mathbb{R}))$ given by

$$\Phi(t,\tau,u_\tau) = u(t,\tau,u_\tau),\ t \geq \tau,$$

where $u_\tau \in L^2_G(\Omega, L^2(\mathbb{R}))$, and u is the solution of problem (1.1) with initial datum u_τ. By the definition of Φ, we have

$$\hat{\mathbb{E}}[\|\Phi(t,\tau,u_\tau)\|^2] = \hat{\mathbb{E}}[\|u(t,\tau,u_\tau)\|^2] < \infty,$$

and $\Phi(t,\tau,u_\tau)$ is $\mathcal{F}$-measurable since the solution $u(t)$ is $\mathcal{F}$-measurable, which implies that Φ maps $\mathbb{R}^2_{\geq} \times L^2_G(\Omega, L^2(\mathbb{R}))$ into $L^2_G(\Omega, L^2(\mathbb{R}))$.

In addition, by the uniqueness of solutions, we have

$$\Phi(t,\tau,u_\tau) = \Phi(t,s,\Phi(s,\tau,u_\tau)), \tag{4.3}$$

for any (t,s), $(s,\tau) \in \mathbb{R}^2_{\geq}$ and $u_\tau \in L^2_G(\Omega, L^2(\mathbb{R}))$. Furthermore, it is easy to check that

$$\Phi(\tau,\tau,u_\tau) = u_\tau.$$

Thus, according to Definition 2.4, in order to show that the solution of problem (1.1) with random initial condition generates a G-mean random dynamical system, it remains to prove that Φ is continuous in the space $\mathbb{R}^2_{\geq} \times L^2_G(\Omega, L^2(\mathbb{R}))$.

Lemma 4.1. *Assume (4.1) holds. Then, the mapping Φ is uniformly strictly contracting, i.e., for the different initial values $u_{\tau,1}$, $u_{\tau,2} \in L^2_G(\Omega, L^2(\mathbb{R}))$, we have*

$$\hat{\mathbb{E}}[\|\Phi(t,\tau,u_{\tau,1}) - \Phi(t,\tau,u_{\tau,2})\|^2] \leq \hat{\mathbb{E}}[\|u_{\tau,1} - u_{\tau,2}\|^2] e^{-2\rho(t-\tau)}, \tag{4.4}$$

for all $t \geq \tau$.

Proof. Let $u_1(t,\tau,u_{\tau,1})$ and $u_2(t,\tau,u_{\tau,2})$ be two different solutions of (1.1) for the initial values $u_{\tau,1}$, $u_{\tau,2} \in L^2_G(\Omega, L^2(\mathbb{R}))$ and the same initial time τ. By the definition of Φ, we have

$$\hat{\mathbb{E}}[\|\Phi(t,\tau,u_{\tau,1}) - \Phi(t,\tau,u_{\tau,2})\|^2] = \hat{\mathbb{E}}[\|u_1(t,\tau,u_{\tau,1}) - u_2(t,\tau,u_{\tau,2})\|^2].$$

We can deduce that

$$\begin{aligned} &\frac{d}{dt}\|u_1 - u_2\|^2 \\ &= -2\lambda\|\nabla(u_1 - u_2)\|^2 - 2\rho\|u_1 - u_2\|^2 - 2\mathrm{Re}(\kappa + i\beta)\langle |u_1|^2 u_1 - |u_2|^2 u_2, u_1 - u_2\rangle \\ &\leq -2\rho\|u_1 - u_2\|^2 - 2\mathrm{Re}(\kappa + i\beta)\langle |u_1|^2 u_1 - |u_2|^2 u_2, u_1 - u_2\rangle. \end{aligned} \tag{4.5}$$

By Lemma 2.2 and (4.1), the second term on the right-hand side of (4.5) can be bounded by

$$\begin{aligned} &-2\mathrm{Re}(\kappa + i\beta)\langle |u_1|^2 u_1 - |u_2|^2 u_2, u_1 - u_2\rangle \\ &= -2\kappa \int_{\mathbb{R}} \mathrm{Re}\big(\bar{u}_1(x) - \bar{u}_2(x)\big)\big(|u_1(x)|^2 u_1(x) - |u_2(x)|^2 u_2(x)\big)dx \\ &\quad + 2\beta \int_{\mathbb{R}} \mathrm{Im}\big(\bar{u}_1(x) - \bar{u}_2(x)\big)\big(|u_1(x)|^2 u_1(x) - |u_2(x)|^2 u_2(x)\big)dx \\ &\leq 2\kappa\Big(-1 + \frac{|\beta|}{\kappa\sqrt{3}}\Big) \int_{\mathbb{R}} \mathrm{Re}\big(\bar{u}_1(x) - \bar{u}_2(x)\big)\big(|u_1(x)|^2 u_1(x) - |u_2(x)|^2 u_2(x)\big)dx \leq 0. \end{aligned} \tag{4.6}$$

Therefore, we can conclude

$$\frac{d}{dt}\|u_1-u_2\|^2 \le -2\rho\|u_1-u_2\|^2.$$

The Gronwall inequality gives that

$$\hat{\mathbb{E}}[\|u_1(t,\tau,u_{\tau,1})-u_2(t,\tau,u_{\tau,2})\|^2] \le \hat{\mathbb{E}}[\|u_{\tau,1}-u_{\tau,2}\|^2]e^{-2\rho(t-\tau)}.$$

The proof is complete. ∎

REMARK 4.1. It follows from Lemma 4.1 that $\Phi(t,\tau,\cdot)$ maps $L_G^2(\Omega,L^2(\mathbb{R}))$ to $L_G^2(\Omega,L^2(\mathbb{R}))$ continuously. Then combining $u\in C([\tau,\infty),L_G^2(\Omega,L^2(\mathbb{R})))$ with (4.3), we can deduce that the mapping Φ is continuous in the space $\mathbb{R}_{\geq}^2\times L_G^2(\Omega,L^2(\mathbb{R}))$. Therefore, Φ is a G-mean random dynamical system associated with problem (1.1).

4.2. Existence of G-mean random attractors. In this subsection, we prove the existence of $\mathcal{D}$-pullback G-mean random attractors for problem (1.1) with random initial condition. We first construct a pullback absorbing set in $L_G^2(\Omega,L^2(\mathbb{R}))$.

LEMMA 4.2. *Let (4.1) and (4.2) hold. Then for every $\tau\in\mathbb{R}$ and $D\in\mathcal{D}$, there exist $T=T(\tau,D)>0$ and $R(\tau)>0$ such that for all $t\geq T$,*

$$\hat{\mathbb{E}}[\|\Phi(\tau,\tau-t,u_{\tau-t})\|^2]\le R(\tau),$$

where $u_{\tau-t}\in D(\tau-t)$.

Proof. It follows from the Young inequality that

$$\begin{aligned}\frac{d}{dt}\|u\|^2 &= -2\lambda\|\nabla u\|^2-2\rho\|u\|^2-2\kappa\|u\|_{L^4(\mathbb{R})}^4+2\mathrm{Re}\langle f(t),u\rangle\\ &\le -2\lambda\|\nabla u\|^2-2\rho\|u\|^2-2\kappa\|u\|_{L^4(\mathbb{R})}^4+\rho\|u\|^2+\frac{1}{\rho}\|f(t)\|^2\\ &\le -\rho\|u\|^2+\frac{1}{\rho}\|f(t)\|^2.\end{aligned}$$

Therefore,

$$\frac{d}{dt}e^{\rho t}\|u\|^2=\rho e^{\rho t}\|u\|^2+e^{\rho t}\frac{d}{dt}\|u\|^2\le\frac{1}{\rho}e^{\rho t}\|f(t)\|^2.$$

Then integrating on $(\tau-t,\tau)$ with $t\geq 0$, we have

$$\|u(\tau,\tau-t,u_{\tau-t})\|^2\le e^{-\rho t}\|u_{\tau-t}\|^2+\frac{1}{\rho}e^{-\rho\tau}\int_{\tau-t}^{\tau}e^{\rho s}\|f(s)\|^2ds.$$

Therefore,

$$\hat{\mathbb{E}}[\|u(\tau,\tau-t,u_{\tau-t})\|^2]\le e^{-\rho t}\hat{\mathbb{E}}[\|u_{\tau-t}\|^2]+\frac{1}{\rho}e^{-\rho\tau}\int_{-\infty}^{\tau}e^{\rho s}\|f(s)\|^2ds, \tag{4.7}$$

which together with $u_{\tau-t}\in D(\tau-t)$ and $D\in\mathcal{D}$, implies that there exists $T=T(\tau,D)>0$ such that for all $t\geq T$,

$$\hat{\mathbb{E}}[\|\Phi(\tau,\tau-t,u_{\tau-t})\|^2]\le R(\tau),$$

where

$$R(\tau)=1+\frac{1}{\rho}e^{-\rho\tau}\int_{-\infty}^{\tau}e^{\rho s}\|f(s)\|^2ds.$$

The proof is complete. ∎

REMARK 4.2. Define the family of sets $B(\tau)=B_{r(\tau)}$, where $B_{r(\tau)}$ is the ball in $L_G^2(\Omega,L^2(\mathbb{R}))$ centered on the origin with radius $r(\tau)$ specified by

$$r(\tau):=\sqrt{1+\frac{1}{\rho}e^{-\rho\tau}\int_{-\infty}^{\tau}e^{\rho s}\|f(s)\|^2ds}.$$

It is not difficult to prove that the family $B=\{B(\tau):\tau\in\mathbb{R}\}$ belongs to $\mathcal{D}$. Indeed, choosing $\lambda=\rho$ in (2.2), we have

$$\lim_{\tau\to-\infty}e^{\rho\tau}\|B(\tau)\|_{\mathfrak{X}}^2=\lim_{\tau\to-\infty}e^{\rho\tau}+\lim_{\tau\to-\infty}\frac{e^{\rho\tau}e^{-\rho\tau}}{\rho}\int_{-\infty}^{\tau}e^{\rho s}\|f(s)\|^2ds,$$

which together with (4.2) implies

$$\lim_{\tau\to-\infty}e^{\rho\tau}\|B(\tau)\|_{\mathfrak{X}}^2=0.$$

Therefore, $B\in\mathcal{D}$. Further, by Lemma 4.2, we find that for every $\tau\in\mathbb{R}$ and $D=\{D(t)\}_{t\in\mathbb{R}}\in\mathcal{D}$, there exists $T=T(\tau,D)>0$ such that for all $t\geq T$, $\Phi(\tau,\tau-t,u_{\tau-t})\subseteq B(\tau)$. Therefore, B is a $\mathcal{D}$-pullback absorbing family for Φ.

THEOREM 4.1. *Assume (4.1) and (4.2) hold. Then, problem (1.1) has a unique $\mathcal{D}$-pullback G-mean random attractor $\mathcal{A}=\{\mathcal{A}(\tau):\tau\in\mathbb{R}\}$ in $L_G^2(\Omega,L^2(\mathbb{R}))$. Furthermore, if there exists a positive number ϖ such that $f:\mathbb{R}\to L^2(\mathbb{R})$ is ϖ-periodic, then such attractor $\mathcal{A}$ is also ϖ-periodic; that is, $\mathcal{A}(\tau+\varpi)=\mathcal{A}(\tau)$ for all $\tau\in\mathbb{R}$.*

Proof. It follows from Remark 4.2 that $\{B(\tau):\tau\in\mathbb{R}\}$ is a $\mathcal{D}$-pullback absorbing set for Φ.

Next, we will show that $\{\Phi(0,t_n,x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_G^2(\Omega,L^2(\mathbb{R}))$ with values in $B(0)$, where $x_n\in D(t_n)$ for each $n\in\mathbb{N}$, $D\in\mathcal{D}$, and $\{t_n\}_{n\in\mathbb{N}}$ is a monotone decreasing sequence tending to $-\infty$ with $t_1=0$ and $t_n-t_{n+1}\geq T^*(t_n)$, where $T^*(t_n)$ is the absorbing time. Indeed, for any $m>n$,

$$\begin{aligned}\hat{\mathbb{E}}\big[\|\Phi(0,t_n,x_n)-\Phi(0,t_m,x_m)\|^2\big]&=\hat{\mathbb{E}}\big[\|\Phi(0,t_n,x_n)-\Phi(0,t_n,\Phi(t_n,t_m,x_m))\|^2\big]\\&\leq e^{\rho t_n}\hat{\mathbb{E}}\big[\|x_n-\Phi(t_n,t_m,x_m)\|^2\big].\end{aligned}\tag{4.8}$$

It then follows from the construction of the time sequence and the absorbing property of this set that $\Phi(t_n,t_m,x_m)\in B(t_n)$. Therefore,

$$\begin{aligned}\hat{\mathbb{E}}\big[\|\Phi(0,t_n,x_n)-\Phi(0,t_m,x_m)\|^2\big]&\leq 4e^{\rho t_n}\left(1+\frac{1}{\rho}e^{-\rho t_n}\int_{-\infty}^{t_n}e^{\rho s}\|f(s)\|^2ds\right)\\&=4e^{\rho t_n}+\frac{4}{\rho}\int_{-\infty}^{t_n}e^{\rho s}\|f(s)\|^2ds,\end{aligned}\tag{4.9}$$

which implies that $\{\Phi(0,t_n,x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence due to $t_n\to-\infty$ as $n\to\infty$.

Then by the completeness of $L_G^2(\Omega, L^2(\mathbb{R}))$, there exists a unique limit $x^*(0)\in B(0)$ such that

$$\hat{\mathbb{E}}\big[\|\Phi(0,t_n,x_n)-x^*(0)\|^2\big]\to 0 \quad \text{as} \quad n\to\infty.$$

Now define $x^*(t)\doteq\Phi(t,0,x^*(0))$ for $t\geq 0$. We can repeat the above argument with 0 replaced by -1 to obtain a limit $x^*(-1)\in B(-1)$ such that

$$\hat{\mathbb{E}}\big[\|\Phi(-1,t_n,x_n)-x^*(-1)\|^2\big]\to 0 \quad \text{as} \quad n\to\infty.$$

It is clear that $x^*(0)=\Phi(0,-1,x^*(-1))$ due to

$$\begin{aligned}
&\hat{\mathbb{E}}\big[\|x^*(0)-\Phi(0,-1,x^*(-1))\|^2\big]\\
=&\hat{\mathbb{E}}\big[\|x^*(0)-\Phi(0,t_n,x_n)+\Phi(0,t_n,x_n)-\Phi(0,-1,x^*(-1))\\
&\quad+\Phi(0,-1,\Phi(-1,t_n,x_n))-\Phi(0,-1,\Phi(-1,t_n,x_n))\|^2\big]\\
=&\hat{\mathbb{E}}\big[\|x^*(0)-\Phi(0,t_n,x_n)-\Phi(0,-1,x^*(-1))+\Phi(0,-1,\Phi(-1,t_n,x_n))\|^2\big]\\
\leq&\hat{\mathbb{E}}\big[\|x^*(0)-\Phi(0,t_n,x_n)\|^2\big]+\hat{\mathbb{E}}\big[\|\Phi(0,-1,\Phi(-1,t_n,x_n))-\Phi(0,-1,x^*(-1))\|^2\big]\\
\leq&\hat{\mathbb{E}}\big[\|x^*(0)-\Phi(0,t_n,x_n)\|^2\big]+e^{-\rho}\hat{\mathbb{E}}\big[\|\Phi(-1,t_n,x_n)-x^*(-1)\|^2\big].
\end{aligned}$$

Then proceed with this construction via induction for each $-n$ and $-n-1$ to obtain a limit $x^*(-n-1)\in B(-n-1)$ such that $x^*(-n)=\Phi(-n,-n-1,x^*(-n-1))$. And define $x^*(t)=\Phi(t,-n-1,x^*(-n-1))$ for $-n-1<t<-n$. In this way an entire trajectory $x^*(t)$ of Φ is constructed, i.e., with $x^*(t)=\Phi(t,s,x^*(s))$ for all $(t,s)\in\mathbb{R}^2_{\geq}$.

Moreover, by the strictly contracting property all other paths of Φ converge to $x^*(t)$ in the G-mean sense. In fact, $x^*(t)$ is unique and forms a $\mathcal{D}$-pullback G-mean random attractor for Φ consisting of singleton sets $\mathcal{A}=\{x^*(t)\}$. To see this suppose that $\bar{x}^*(t)$ is another entire trajectory with $\bar{x}^*(t)\in\mathcal{A}(t)$ for all $t\in\mathbb{R}$ and $\hat{\mathbb{E}}\big[\|x^*(0)-\bar{x}^*(0)\|^2\big]\geq\varepsilon>0$.

Similarly to (4.8) and (4.9), the strictly contracting condition and uniform boundedness of the entire paths give

$$\begin{aligned}
&\hat{\mathbb{E}}\big[\|\Phi(0,-t,x^*(-t))-\Phi(0,-t,\bar{x}^*(-t))\|^2\big]\\
\leq& e^{\rho t}\hat{\mathbb{E}}\big[\|x^*(-t)-\bar{x}^*(-t)\|^2\big]\\
\leq& 4\left(1+\frac{1}{\rho}e^{\rho t}\int_{-\infty}^{-t}e^{\rho s}\|f(s)\|^2ds\right)e^{-\rho t}\\
\leq& 4e^{-\rho t}+\frac{4}{\rho}\int_{-\infty}^{-t}e^{\rho s}\|f(s)\|^2ds
\end{aligned}$$

for all $t\geq 0$, which implies that there exists $T>0$ such that for all $t\geq T$

$$\hat{\mathbb{E}}\big[\|\Phi(0,-t,x^*(-t))-\Phi(0,-t,\bar{x}^*(-t))\|^2\big]\leq\frac{1}{2}\varepsilon.$$

However, $x^*(0)=\Phi(0,-t,x^*(-t))$ and $\bar{x}^*(0)=\Phi(0,-t,\bar{x}^*(-t))$, so

$$\varepsilon\leq\hat{\mathbb{E}}\big[\|x^*(0)-\bar{x}^*(0)\|^2\big]=\hat{\mathbb{E}}\big[\|\Phi(0,-t,x^*(-t))-\Phi(0,-t,\bar{x}^*(-t))\|^2\big]\leq\frac{1}{2}\varepsilon,$$

for all $t\geq T$, which is a contradiction.

Thus the G-mean random dynamical system Φ has a $\mathcal{D}$-pullback G-mean random attractor $\mathcal{A}=\{x^*(t)\}_{t\in\mathbb{R}}$.

Finally, the periodicity of random attractors will be shown. According to the construction process of $x^*(t)$, for any $t\in\mathbb{R}$ we have $x^*(t+\varpi)=\Phi(t+\varpi,t,x^*(t))$. It then follows from the fact that $f(x,t)$ is ϖ-periodic that Φ is periodic with period ϖ. Noting that Φ satisfies two-parameter semigroup property, therefore, $x^*(t)=x^*(t+\varpi)$, which shows that the attractor $\mathcal{A}$ is also ϖ-periodic. □

Remark 4.3. If $f(x,t)\equiv 0$ for $x\in\mathbb{R}$, we can obtain that 0 is the solution of system (1.1). Then combining with Theorem 4.1, we have $x^*(t)\equiv 0$.

Remark 4.4. In [19, 34, 35, 38, 40], the existence of mean random attractors has been investigated for mean random dynamical system in classical probability space, which cannot be applied to the system with probability-uncertain initial data. However, Theorem 4.1 may be applied to prove the existence of pullback G-mean random attractors for G-mean random dynamical system, so it can be applied to more cases.

5. G-mean random attractors for (1.1) in $L^2_G(\Omega,L^2_\sigma(\mathbb{R}))$

In this section, we will investigate the asymptotic behaviors of the solutions to problem (1.1) in the weighted space $L^2_G(\Omega,L^2_\sigma(\mathbb{R}))$ with a weight function $\phi(x)=(1+|x|^2)^{-\sigma}$ for $x\in\mathbb{R}$, where $\sigma>\frac{1}{2}$ is a fixed number and $L^2_\sigma(\mathbb{R})$ is defined by

$$L^2_\sigma(\mathbb{R})=\left\{u:\mathbb{R}\to\mathbb{R} \text{ is measurable}, \int_{\mathbb{R}}(1+|x|^2)^{-\sigma}|u(x)|^2dx<\infty\right\}$$

with the norm

$$\|u\|_\sigma=\Big(\int_{\mathbb{R}}(1+|x|^2)^{-\sigma}|u(x)|^2dx\Big)^{1/2}.$$

In order to study the dynamics of problem (1.1) in the weighted space $L^2_G(\Omega,L^2_\sigma(\mathbb{R}))$, we have to extend Φ from $L^2_G(\Omega,L^2(\mathbb{R}))$ to $L^2_G(\Omega,L^2_\sigma(\mathbb{R}))$. This extension is possible based on the Lipschitz continuity of solutions in $L^2_G(\Omega,L^2_\sigma(\mathbb{R}))$.

The following lemma is the result about the Lipschitz continuity of solutions in $L^2_G(\Omega,L^2_\sigma(\mathbb{R}))$.

Lemma 5.1. *Suppose $f_1,f_2\in L^2_{loc}(\mathbb{R},L^2(\mathbb{R}))$, $u_{\tau,1},u_{\tau,2}\in L^2_G(\Omega,L^2(\mathbb{R}))$ and (4.1) hold. Let u_1,u_2 be solutions of problem (1.1) with f replaced by f_1 and f_2, respectively. Then for every $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $T>0$, there exists a positive constant $C=C(\tau,T)$ such that for all $t\in[\tau,\tau+T]$,*

$$\hat{\mathbb{E}}[\|u_1(t,\tau,u_{\tau,1})-u_2(t,\tau,u_{\tau,2})\|^2_\sigma]\le C\hat{\mathbb{E}}[\|u_{\tau,1}-u_{\tau,2}\|^2_\sigma]+C\int_\tau^t\|f_1(s)-f_2(s)\|^2_\sigma ds. \quad (5.1)$$

Proof. It follows from (1.1) that

$$\begin{aligned}\frac{\partial(u_1-u_2)}{\partial t}=&(\lambda+i\alpha)\triangle(u_1-u_2)-\rho(u_1-u_2)\\&-(\kappa+i\beta)(|u_1|^2u_1-|u_2|^2u_2)+f_1(t,x)-f_2(t,x).\end{aligned}$$

Taking the inner product with u_1-u_2 in $L^2_\sigma(\mathbb{R})$, we have

$$\begin{aligned}\frac{d}{dt}\|u_1-u_2\|^2_\sigma=&2\mathrm{Re}(\lambda+i\alpha)\big\langle\triangle(u_1-u_2),(u_1-u_2)\big\rangle_\sigma-2\rho\|u_1-u_2\|^2_\sigma\\&-2\mathrm{Re}(\kappa+i\beta)\big\langle|u_1|^2u_1-|u_2|^2u_2,u_1-u_2\big\rangle_\sigma+2\mathrm{Re}\langle f_1(t)-f_2(t),u_1-u_2\rangle_\sigma.\end{aligned} \quad (5.2)$$

Now, we compute the terms on the right-hand of (5.2).

$$\begin{aligned}&2\mathrm{Re}(\lambda+i\alpha)\big\langle\triangle(u_1-u_2),(u_1-u_2)\big\rangle_\sigma\\=&-2\lambda\big\langle\nabla(u_1-u_2),\nabla(u_1-u_2)\cdot\phi(x)\big\rangle-2\mathrm{Re}(\lambda+i\alpha)\big\langle\nabla(u_1-u_2),\nabla\phi(x)\cdot(u_1-u_2)\big\rangle.\end{aligned}$$

By simple calculations, we find that

$$|\nabla\phi(x)|\le\sigma\phi(x),\quad\forall\ x\in\mathbb{R},$$

which, together with the Young inequality, shows that

$$\begin{aligned}&2\mathrm{Re}(\lambda+i\alpha)\big\langle\triangle(u_1-u_2),(u_1-u_2)\big\rangle_\sigma\\\le&-2\lambda\|\nabla(u_1-u_2)\|_\sigma^2+2\sigma\sqrt{\lambda^2+\alpha^2}\big\langle\nabla(u_1-u_2),\phi(x)(u_1-u_2)\big\rangle\\\le&-2\lambda\|\nabla(u_1-u_2)\|_\sigma^2+\lambda\|\nabla(u_1-u_2)\|_\sigma^2+\frac{\sigma^2(\lambda^2+\alpha^2)}{\lambda}\|u_1-u_2\|_\sigma^2\\\le&-\lambda\|\nabla(u_1-u_2)\|_\sigma^2+\frac{\sigma^2(\lambda^2+\alpha^2)}{\lambda}\|u_1-u_2\|_\sigma^2.\end{aligned}\tag{5.3}$$

For the third term on the right-hand side of (5.2), it follows from Lemma 2.2 that

$$\begin{aligned}&-2\mathrm{Re}(\kappa+i\beta)\big\langle|u_1|^2u_1-|u_2|^2u_2,u_1-u_2\big\rangle_\sigma\\=&-2\mathrm{Re}(\kappa+i\beta)\int_{\mathbb{R}}\phi(x)(|u_1|^2u_1-|u_2|^2u_2)(\bar u_1-\bar u_2)dx\\=&-2\kappa\int_{\mathbb{R}}\phi(x)\mathrm{Re}\ L dx+2\beta\int_{\mathbb{R}}\phi(x)\mathrm{Im}\ L dx\\\le&2\kappa\Big(-1+\frac{|\beta|}{\kappa\sqrt3}\Big)\int_{\mathbb{R}}\phi(x)\mathrm{Re}\ L dx,\end{aligned}\tag{5.4}$$

where $L=\big(|u_1(x)|^2u_1(x)-|u_2(x)|^2u_2(x)\big)\big(\bar u_1(x)-\bar u_2(x)\big)$. It then follows from (4.1), (5.2)-(5.4) and the Young inequality that

$$\frac{d}{dt}\|u_1-u_2\|_\sigma^2\le c\|u_1-u_2\|_\sigma^2+\frac1\rho\|f_1(t)-f_2(t)\|_\sigma^2,\tag{5.5}$$

where $c=\frac{\sigma^2(\lambda^2+\alpha^2)}{\lambda}-\rho$. For every $\tau\in\mathbb{R}$, $T>0$ and $t\in[\tau,\tau+T]$, the Gronwall inequality gives that

$$\begin{aligned}&\hat{\mathbb{E}}[\|u_1(t,\tau,u_{\tau,1})-u_2(t,\tau,u_{\tau,2})\|_\sigma^2]\\\le&\hat{\mathbb{E}}[\|u_{\tau,1}-u_{\tau,2}\|_\sigma^2]e^{c(t-\tau)}+\frac1\rho\int_\tau^t\|f_1(s)-f_2(s)\|_\sigma^2e^{c(t-s)}ds.\end{aligned}$$

This completes the proof. □

Next, we extend the mapping Φ from $L^2_G(\Omega,L^2(\mathbb{R}))$ to $L^2_G(\Omega,L^2_\sigma(\mathbb{R}))$. This will enable us to study the dynamics of the system (1.1) in $L^2_G(\Omega,L^2_\sigma(\mathbb{R}))$.

THEOREM 5.1. *Suppose $f\in L^2_{loc}(\mathbb{R},L^2_\sigma(\mathbb{R}))$ and (4.1) hold. Then one can associate problem (1.1) with a continuous system $\Phi:\mathbb{R}^2_\ge\times L^2_G(\Omega,L^2_\sigma(\mathbb{R}))\to L^2_G(\Omega,L^2_\sigma(\mathbb{R}))$ such that for every $(\tau,t)\in\mathbb{R}^2_\ge$ and $u_\tau\in L^2_G(\Omega,L^2(\mathbb{R}))$, $\Phi(t,\tau,u_\tau)=u(t,\tau,u_\tau)$, where $\Phi(t,\tau,u_\tau)$ is the solution of problem (1.1) with initial time τ and initial condition u_τ.*

Proof. Notice that $L^2_G(\Omega, L^2(\mathbb{R}))$ is dense in $L^2_G(\Omega, L^2_\sigma(\mathbb{R}))$. Indeed, $\forall u \in L^2_G(\Omega, L^2_\sigma(\mathbb{R}))$, let $u_n(x) = u(x)I_{\{|x|<n\}}$, where $n \in \mathbb{N}^+$. Then, for any positive integer n, we have

$$\begin{aligned}\hat{\mathbb{E}}\Big[\int_{\mathbb{R}} |u(x)I_{\{|x|<n\}}|^2 dx\Big] =& \hat{\mathbb{E}}\Big[\int_{|x|<n} |u(x)|^2 dx\Big] \\ =& (1+n^2)^\sigma \hat{\mathbb{E}}\Big[\int_{|x|<n} (1+n^2)^{-\sigma} |u(x)|^2 dx\Big] \\ \le& (1+n^2)^\sigma \hat{\mathbb{E}}\Big[\int_{|x|<n} (1+x^2)^{-\sigma} |u(x)|^2 dx\Big] \\ \le& (1+n^2)^\sigma \hat{\mathbb{E}}\Big[\int_{\mathbb{R}} (1+x^2)^{-\sigma} |u(x)|^2 dx\Big] \\ <& \infty,\end{aligned}$$

which implies $u_n \in L^2_G(\Omega, L^2(\mathbb{R}))$. In addition,

$$\hat{\mathbb{E}}\Big[\|u(x) - u_n(x)\|^2_\sigma\Big] = \hat{\mathbb{E}}\Big[\int_{|x|\ge n} (1+x^2)^{-\sigma} |u(x)|^2 dx\Big],$$

which, together with $u \in L^2_G(\Omega, L^2_\sigma(\mathbb{R}))$, implies

$$\hat{\mathbb{E}}\Big[\|u(x) - u_n(x)\|^2_\sigma\Big] \to 0, \text{ as } n \to \infty.$$

Therefore, $L^2_G(\Omega, L^2(\mathbb{R}))$ is dense in $L^2_G(\Omega, L^2_\sigma(\mathbb{R}))$. Then $L^2_G(\Omega, L^2(\mathbb{R})) \times L^2((\tau, \tau+T), L^2(\mathbb{R}))$ is dense in $L^2_G(\Omega, L^2_\sigma(\mathbb{R})) \times L^2((\tau, \tau+T), L^2_\sigma(\mathbb{R}))$.

Given $(u_\tau, f) \in L^2_G(\Omega, L^2_\sigma(\mathbb{R})) \times L^2((\tau, \tau+T), L^2_\sigma(\mathbb{R}))$, there exists a sequence $(u_n, f_n) \in L^2_G(\Omega, L^2(\mathbb{R})) \times L^2((\tau, \tau+T), L^2(\mathbb{R}))$ such that $(u_n, f_n) \to (u_\tau, f)$ in $L^2_G(\Omega, L^2_\sigma(\mathbb{R})) \times L^2((\tau, \tau+T), L^2_\sigma(\mathbb{R}))$. Let $\tau \in \mathbb{R}$ and $T > 0$, by Lemma 5.1, we find that $\{u(\cdot, \tau, (u_n, f_n))\}_{n=1}^\infty$ is a Cauchy sequence in $C([\tau, \tau+T], L^2_G(\Omega, L^2_\sigma(\mathbb{R})))$, so $\lim_{n\to\infty} u(\cdot, \tau, (u_n, f_n))$ exists in $C([\tau, \tau+T], L^2_G(\Omega, L^2_\sigma(\mathbb{R})))$. It is evident that this limit does not depend on the choice of (u_n, f_n). Let $\tilde{\Phi}$ be a mapping from $L^2_G(\Omega, L^2_\sigma(\mathbb{R})) \times L^2((\tau, \tau+T), L^2_\sigma(\mathbb{R}))$ to $C([\tau, \tau+T], L^2_G(\Omega, L^2_\sigma(\mathbb{R})))$ such that for every $(u_\tau, f) \in L^2_G(\Omega, L^2_\sigma(\mathbb{R})) \times L^2((\tau, \tau+T), L^2_\sigma(\mathbb{R}))$ and $(u_n, f_n) \in L^2_G(\Omega, L^2(\mathbb{R})) \times L^2((\tau, \tau+T), L^2(\mathbb{R}))$

$$\tilde{\Phi}(\cdot, \tau, (u_\tau, f)) = \lim_{n\to\infty} u(\cdot, \tau, (u_n, f_n)), \tag{5.6}$$

where $(u_n, f_n) \to (u_\tau, f)$ in $L^2_G(\Omega, L^2_\sigma(\mathbb{R})) \times L^2((\tau, \tau+T), L^2_\sigma(\mathbb{R}))$. It follows from Lemma 5.1 that $\tilde{\Phi}(\cdot, \tau, (u_\tau, f))$ is Lipschitz continuous in (u_τ, f) in $L^2_G(\Omega, L^2_\sigma(\mathbb{R})) \times L^2((\tau, \tau+T), L^2_\sigma(\mathbb{R}))$.

Note that for every $t \ge \tau$, $u(t, \tau, \omega, (u_n, f_n))$ is $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R})))$-measurable and the embedding $L^2(\mathbb{R}) \hookrightarrow L^2_\sigma(\mathbb{R})$ is continuous. Therefore, $u(t, \tau, \omega, (u_n, f_n))$ is $(\mathcal{F}, \mathcal{B}(L^2_\sigma(\mathbb{R})))$-measurable, which along with (5.6) implies that $\tilde{\Phi}(\cdot, \tau, (u_\tau, f))$ is $(\mathcal{F}, \mathcal{B}(L^2_\sigma(\mathbb{R})))$-measurable for all $t \ge \tau$. One can check that $\tilde{\Phi}$ is continuous in $\mathbb{R}^2_\ge \times L^2_G(\Omega, L^2_\sigma(\mathbb{R}))$. Actually, $\tilde{\Phi}$ is an extension of Φ to the weighted space $L^2_G(\Omega, L^2_\sigma(\mathbb{R}))$, and we will not distinguish Φ and $\tilde{\Phi}$ in the sequel. ∎

In order to study the existence of pullback mean attractors of Φ in $L^2_G(\Omega, L^2_\sigma(\mathbb{R}))$, we need to prove that Φ is uniformly strictly contracting and has a pullback absorbing family in $L_G^2(\Omega, L_\sigma^2(\mathbb{R}))$. When deriving the uniform estimates on solutions, we need the following condition on f:

$$\int_{-\infty}^{\tau} e^{\rho s}\|f(s)\|_\sigma^2 ds < \infty, \ \forall \tau \in \mathbb{R}, \tag{5.7}$$

which weakens condition (4.2).

REMARK 5.1. As we will see later, when studying the existence of pullback mean attractors of Φ in $L_G^2(\Omega, L_\sigma^2(\mathbb{R}))$, it is convenient to use an equivalent norm for $L_G^2(\Omega, L_\sigma^2(\mathbb{R}))$ which is defined by the weight function $\phi_\delta(x) = (1+|\delta x|^2)^{-\sigma}$, $x \in \mathbb{R}$, where

$$\delta = \min\Big\{1, \frac{\sqrt{\lambda\rho}}{\sqrt{(\lambda^2+\alpha^2)\sigma}}\Big\}. \tag{5.8}$$

By simple calculations, we can obtain that

$$|\nabla\phi_\delta(x)| \le \sigma\delta\phi_\delta(x), \ \forall\ x \in \mathbb{R}, \tag{5.9}$$

and

$$\hat{\mathbb{E}}\left[\|u\|_\sigma^2\right] \le \hat{\mathbb{E}}\Big[\int_{\mathbb{R}} \phi_\delta(x)|u(x)|^2 dx\Big] \le \delta^{-2\sigma}\hat{\mathbb{E}}\left[\|u\|_\sigma^2\right], \ \forall\ u \in L_G^2(\Omega, L_\sigma^2(\mathbb{R})), \tag{5.10}$$

which shows that the weighted space $L_G^2(\Omega, L_\sigma^2(\mathbb{R}))$ has an equivalent norm which is given by $\left(\hat{\mathbb{E}}\Big[\int_{\mathbb{R}} \phi_\delta(x)|u(x)|^2 dx\Big]\right)^{\frac{1}{2}}$ for $u \in L_G^2(\Omega, L_\sigma^2(\mathbb{R}))$.

LEMMA 5.2. *Assume (4.1) holds. Then the G-mean random dynamical system Φ is uniformly strictly contracting in $L_G^2(\Omega, L_\sigma^2(\mathbb{R}))$, i.e., for the different initial values $u_{\tau,1}$, $u_{\tau,2} \in L_G^2(\Omega, L_\sigma^2(\mathbb{R}))$, there exists a constant M (which is independent of τ) such that*

$$\hat{\mathbb{E}}[\|\Phi(t,\tau,u_{\tau,1}) - \Phi(t,\tau,u_{\tau,2})\|_\sigma^2] \le M\hat{\mathbb{E}}[\|u_{\tau,1} - u_{\tau,2}\|_\sigma^2] e^{-\rho(t-\tau)}, \tag{5.11}$$

for all $t \ge \tau$.

Proof. Let $u_1(t,\tau,u_{\tau,1})$ and $u_2(t,\tau,u_{\tau,2})$ be two different solutions of (1.1) from the different initial values $u_{\tau,1}$, $u_{\tau,2} \in L_G^2(\Omega, L_\sigma^2(\mathbb{R}))$ for the same initial time τ. Taking the inner product of $u_1 - u_2$ with $\phi_\delta(u_1 - u_2)$ in $L^2(\mathbb{R})$ and then taking the real part,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} \phi_\delta(x)|u_1-u_2|^2 dx = &-2\lambda\int_{\mathbb{R}} \phi_\delta(x)|\nabla(u_1-u_2)|^2 dx - 2\rho\int_{\mathbb{R}} |u_1-u_2|^2\phi_\delta(x) dx \\ &- 2\mathrm{Re}(\lambda+i\alpha)\int_{\mathbb{R}} (\bar{u}_1-\bar{u}_2)\nabla(u_1-u_2)\cdot\nabla\phi_\delta(x) dx \\ &- 2\mathrm{Re}(\kappa+i\beta)\big\langle |u_1|^2u_1 - |u_2|^2u_2, \phi_\delta(x)(u_1-u_2)\big\rangle. \end{aligned} \tag{5.12}$$

For the third term on the right-hand side of (5.12), by (5.9) and the Young inequality,

$$\begin{aligned}&-2\mathrm{Re}(\lambda+i\alpha)\int_{\mathbb{R}} (\bar{u}_1-\bar{u}_2)\nabla(u_1-u_2)\cdot\nabla\phi_\delta(x) dx \\ \le& 2\sigma\delta\sqrt{\lambda^2+\alpha^2}\int_{\mathbb{R}} \big|(\bar{u}_1-\bar{u}_2)\nabla(u_1-u_2)\phi_\delta(x)\big| dx \\ \le& \frac{\sigma^2\delta^2(\lambda^2+\alpha^2)}{\lambda}\int_{\mathbb{R}} |u_1-u_2|^2\phi_\delta(x) dx + \lambda\int_{\mathbb{R}} |\nabla(u_1-u_2)|^2\phi_\delta(x) dx, \end{aligned} \tag{5.13}$$

and together with (4.1), (5.8) and Lemma 2.2, we deduce from (5.12) that

$$\frac{d}{dt}\int_{\mathbb{R}}\phi_\delta(x)|u_1-u_2|^2dx \le -\rho\int_{\mathbb{R}}|u_1-u_2|^2\phi_\delta(x)dx. \tag{5.14}$$

The Gronwall inequality gives that

$$\begin{aligned}&\hat{\mathbb{E}}\left[\int_{\mathbb{R}}\phi_\delta(x)|u_1(t,\tau,u_{\tau,1})-u_2(t,\tau,u_{\tau,2})|^2dx\right]\\ \le&\hat{\mathbb{E}}\left[\int_{\mathbb{R}}\phi_\delta(x)|u_{\tau,1}-u_{\tau,2}|^2dx\right]e^{-\rho(t-\tau)},\end{aligned} \tag{5.15}$$

which, together with (5.10) and the definition of Φ, implies that

$$\hat{\mathbb{E}}[\|\Phi(t,\tau,u_{\tau,1})-\Phi(t,\tau,u_{\tau,2})\|_\sigma^2]\le\delta^{-2\sigma}\hat{\mathbb{E}}\left[\|u_{\tau,1}-u_{\tau,2}\|_\sigma^2\right]e^{-\rho(t-\tau)}. \tag{5.16}$$

The proof is complete. □

LEMMA 5.3. *Let (4.1) and (5.7) hold. Then for every $\tau\in\mathbb{R}$ and $D\in\mathcal{D}$, there exist $T=T(\tau,D)>0$ and $\tilde{R}(\tau)>0$ such that for all $t\ge T$,*

$$\hat{\mathbb{E}}[\|\Phi(\tau,\tau-t,u_{\tau-t})\|_\sigma^2]\le\tilde{R}(\tau), \tag{5.17}$$

where $u_{\tau-t}\in D(\tau-t)$.

Proof. Taking the inner product of u with $\phi_\delta u$ in $L^2(\mathbb{R})$, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}\phi_\delta(x)|u|^2dx=&-2\lambda\int_{\mathbb{R}}\phi_\delta(x)|\nabla u|^2dx-2\mathrm{Re}(\lambda+i\alpha)\int_{\mathbb{R}}\bar{u}\nabla u\cdot\nabla\phi_\delta(x)dx\\ &-2\rho\int_{\mathbb{R}}|u|^2\phi_\delta(x)dx-2\kappa\int_{\mathbb{R}}|u|^4\phi_\delta(x)dx+2\mathrm{Re}\int_{\mathbb{R}}f(x,t)\phi_\delta(x)\bar{u}dx.\end{aligned}$$

Similar to (5.13), we have

$$\begin{aligned}-2\mathrm{Re}(\lambda+i\alpha)\int_{\mathbb{R}}\bar{u}\nabla u\cdot\nabla\phi_\delta(x)dx\le&2\sigma\delta\sqrt{\lambda^2+\alpha^2}\int_{\mathbb{R}}|\bar{u}\nabla u\cdot\nabla\phi_\delta(x)|dx\\ \le&\frac{\sigma^2\delta^2(\lambda^2+\alpha^2)}{\lambda}\int_{\mathbb{R}}|u|^2\phi_\delta(x)dx+\lambda\int_{\mathbb{R}}|\nabla u|^2\phi_\delta(x)dx.\end{aligned}$$

Therefore, we can deduce that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}\phi_\delta(x)|u|^2dx\le&\frac{\sigma^2\delta^2(\lambda^2+\alpha^2)}{\lambda}\int_{\mathbb{R}}|u|^2\phi_\delta(x)dx-2\rho\int_{\mathbb{R}}|u|^2\phi_\delta(x)dx\\ &+\frac{2}{\rho}\int_{\mathbb{R}}|f(x,t)|^2\phi_\delta(x)dx+\frac{\rho}{2}\int_{\mathbb{R}}|u|^2\phi_\delta(x)dx\\ \le&-\frac{\rho}{2}\int_{\mathbb{R}}|u|^2\phi_\delta(x)dx+\frac{2}{\rho}\int_{\mathbb{R}}|f(x,t)|^2\phi_\delta(x)dx.\end{aligned} \tag{5.18}$$

It follows from that

$$\frac{d}{dt}e^{\frac{\rho t}{2}}\int_{\mathbb{R}}\phi_\delta(x)|u|^2dx=\frac{\rho}{2}e^{\frac{\rho t}{2}}\|u\|^2+e^{\frac{\rho t}{2}}\frac{d}{dt}\|u\|^2\le\frac{2}{\rho}e^{\frac{\rho t}{2}}\int_{\mathbb{R}}|f(x,t)|^2\phi_\delta(x)dx.$$

Then integrating on $(\tau-t,\tau)$ with $t\geq 0$, we have

$$\begin{aligned}&\hat{\mathbb{E}}\left[\int_{\mathbb{R}}\phi_\delta(x)|u(\tau,\tau-t,u_{\tau-t})|^2dx\right]\\ \leq & e^{-\frac{\rho t}{2}}\hat{\mathbb{E}}\left[\int_{\mathbb{R}}\phi_\delta(x)|u_{\tau-t}|^2dx\right]+\frac{2}{\rho}e^{-\frac{\rho\tau}{2}}\int_{\tau-t}^{\tau}\int_{\mathbb{R}}e^{\rho s}|f(x,s)|^2\phi_\delta(x)dxds. \qquad (5.19)\end{aligned}$$

Using (5.10) and the definition of Φ again, we have

$$\hat{\mathbb{E}}[\|\Phi(\tau,\tau-t,u_{\tau-t})\|_\sigma^2]\leq e^{-\frac{\rho t}{2}}\delta^{-2\sigma}\hat{\mathbb{E}}[\|u_{\tau-t}\|_\sigma^2]+\frac{1}{\rho\delta^{2\sigma}}e^{-\rho\tau}\int_{-\infty}^{\tau}e^{\rho s}\|f(s)\|^2ds,$$

which, together with $u_{\tau-t}\in D(\tau-t)$ and $D\in\mathcal{D}$, implies that there exists $T=T(\tau,D)>0$ such that for all $t\geq T$,

$$\hat{\mathbb{E}}[\|\Phi(\tau,\tau-t,u_{\tau-t})\|^2]\leq\tilde{R}(\tau),$$

where

$$\tilde{R}(\tau)=1+\frac{1}{\rho\delta^{2\sigma}}e^{-\rho\tau}\int_{-\infty}^{\tau}e^{\rho s}\|f(s)\|^2ds.$$

The proof is complete. □

THEOREM 5.2. *Under assumptions (4.1) and (5.7), then the G-mean random dynamical system Φ associated with problem (1.1) has a unique $\mathcal{D}$-pullback G-mean random attractor $\mathcal{A}=\{\mathcal{A}(\tau):\tau\in\mathbb{R}\}$ in $L_G^2(\Omega,L_\sigma^2(\mathbb{R}))$. Furthermore, if there exists a positive number ϖ such that $f:\mathbb{R}\to L^2(\mathbb{R})$ is ϖ-periodic, then such an attractor $\mathcal{A}$ is also ϖ-periodic; that is, $\mathcal{A}(\tau+\varpi)=\mathcal{A}(\tau)$ for all $\tau\in\mathbb{R}$.*

Proof. By Lemmas 5.2 and 5.3, and using the method similar to Theorem 4.1, the proof can be completed. □

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