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GLOBAL-IN-TIME CLASSICAL SOLUTIONS TO TWO-DIMENSIONAL AXISYMMETRIC EULER SYSTEM WITH SWIRL*

GENG LAI† AND MI ZHU‡

Abstract. We study global-in-time classical solutions to the two-dimensional (2D) compressible Euler system with axial symmetry. We derive several groups of suitable characteristic decompositions for the 2D axisymmetric compressible Euler system. Using these characteristic decompositions, we find several classes of expanding initial data to ensure the existence of global-in-time classical solutions. These solutions have an expanding vacuum region centered at the origin.

Keywords. Axisymmetric Euler system; classical solution; characteristic decomposition; vacuum.

AMS subject classifications. 35L65; 35L60; 35L67.

1. Introduction

We consider the two-dimensional (2D) compressible Euler equations

$$\begin{cases} \rho_t+(\rho u_1)_{x_1}+(\rho u_2)_{x_2}=0, \\ (\rho u_1)_t+(\rho u_1^2+p)_{x_1}+(\rho u_1u_2)_{x_2}=0, \\ (\rho u_2)_t+(\rho u_1u_2)_{x_1}+(\rho u_2^2+p)_{x_2}=0, \\ (\rho E)_t+(\rho u_1E+u_1p)_{x_1}+(\rho u_2E+u_2p)_{x_2}=0, \end{cases} \tag{1.1}$$

where (u_1,u_2) is the velocity, ρ is the density, p is the pressure, $E=\frac{1}{2}(u_1^2+u_2^2)+\epsilon$ is the total energy, and ϵ is the internal energy. We choose the equations of state

$$p=e^s\rho^\gamma \quad \text{and} \quad \epsilon=\frac{p}{(\gamma-1)\rho}, \tag{1.2}$$

where s is the entropy and γ is an adiabatic exponent between 1 and 3.

It is well known that the solutions of the Cauchy problem for the compressible Euler equations may blow up in finite time, no matter how smooth and small the initial data are; see, e.g., [1–3, 5, 16, 21, 30, 35, 37]. It is natural to consider what type of initial data are possible to guarantee the existence of global-in-time classical solutions for the compressible Euler equations. For the results on the existence of global-in-time classical solutions for the one-dimensional compressible Euler equations, we refer the reader to [4,6,19,28,41]. Serre [36] first obtained the existence of global-in-time classical solutions for the compressible Euler equations for ideal gases in multi-dimensions, provided that the initial velocity is close to a linear field and the initial density is sufficiently small. Subsequently, Grassin [10] obtained the existence of global-in-time classical solutions for the multi-dimensional compressible Euler equations, provided the initial velocity forces particles to spread out and the initial density is sufficiently small in some norm. Magali [31] extended the result of [10] to a van der Waals gas. Godin [11] studied the lifespan of the classical solutions to the spherically Euler equations for ideal gases with initial data that are a small perturbation, with compact support, to a constant

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†Department of Mathematics, Shanghai University, Shanghai, 200444, P.R. China (laigeng@shu.edu.cn).

‡Department of Mathematics, Shanghai University, Shanghai, 200444, P.R. China (zhumi@shu.edu.cn).

state. Subsequently, Godin [12] obtained the global existence of classical solutions to the spherically symmetric Euler equations for Chaplygin gases with initial data that are a small perturbation, with compact support, to a constant state. Recently, Hou and Yin [13, 14] obtained the existence of global-in-time classical solutions to the axisymmetric non-isentropic Euler equations of Chaplygin gas with swirl.

In this paper we study axisymmetric flows of the system (1.1). That is, we assume the flows have the property

$$\begin{gathered}\rho(x,\theta,t)=\rho(x,t), \quad s(x,\theta,t)=s(x,t),\\ \begin{pmatrix} u_1(x,\theta,t)\\ u_2(x,\theta,t)\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix} u(x,t)\\ v(x,t)\end{pmatrix}\end{gathered} \tag{1.3}$$

for all $t>0$, $\theta\in[0,2\pi)$, and $x>0$, where x and θ are the polar coordinates of the (x_1,x_2)-plane. With this symmetry, system (1.1) can be reduced to

$$\begin{cases}\rho_t+(\rho u)_x+\dfrac{\rho u}{x}=0,\\ (\rho u)_t+(\rho u^2+p)_x+\dfrac{\rho(u^2-v^2)}{x}=0,\\ (\rho v)_t+(\rho uv)_x+\dfrac{2\rho uv}{x}=0,\\ (\rho E)_t+(\rho uE+up)_x+\dfrac{\rho uE}{x}+\dfrac{up}{x}=0.\end{cases} \tag{1.4}$$

Notice now that u and v in (1.4) represent the radial and pure swirl velocities in the flow, respectively.

For 2D isentropic axisymmetric flows without swirl, system (1.4) can be reduced to

$$\begin{cases}\rho_t+(\rho u)_x+\dfrac{\rho u}{x}=0,\\ (\rho u)_t+(\rho u^2+p)_x+\dfrac{\rho u^2}{x}=0.\end{cases} \tag{1.5}$$

There are a lot of important works on the global existence of weak solutions to the system (1.5); we refer the reader to [7–9, 22, 32–34] and the references cited therein. We also refer the reader to the survey paper [15].

For 2D isentropic axisymmetric flows with swirl, system (1.4) can be reduced to

$$\begin{cases}\rho_t+(\rho u)_x+\dfrac{\rho u}{x}=0,\\ (\rho u)_t+(\rho u^2+p)_x+\dfrac{\rho(u^2-v^2)}{x}=0,\\ (\rho v)_t+(\rho uv)_x+\dfrac{2\rho uv}{x}=0.\end{cases} \tag{1.6}$$

As far as we know, the global existence of weak solutions to the Cauchy problem for the system (1.6) is still a difficult problem. However, the Riemann problem for (1.6) for polytropic gases has been solved completely by Zhang and Zheng [38]. We also refer the reader to [39, 40] for some other related works.

We are concerned with global-in-time classical solutions to the system (1.4). It is well-known that the 2D axisymmetric Euler system usually does not have global classical solutions, even with an initial data that is a small perturbation of a constant state; see

Alinhac [1,2]. So, a natural question is what type of initial data are possible to ensure the existence of global-in-time classical solutions. We consider (1.4) with data

$$\begin{cases} (u,v,\rho,s)(x,0)=(\bar{u},\bar{v},\bar{\rho},\bar{s})(x), & x>\varepsilon; \\ (u,v,\rho,s)(x,0)=(u_0,0,\rho_0,s_0), & 0<x<\varepsilon; \\ (\rho u)(0,t)=0, & t>0, \end{cases} \tag{1.7}$$

where $(\bar{u},\bar{v},\bar{\rho},\bar{s})(x)\in C^1[\varepsilon,+\infty)$, $(\bar{u},\bar{v},\bar{\rho},\bar{s})(\varepsilon)=(u_0,0,\rho_0,s_0)$, $(u_0,0,\rho_0,s_0)$ is a constant state, and $u_0>0$. We aim at finding some sufficient conditions on $(\bar{u},\bar{v},\bar{\rho},\bar{s})(x)$ to ensure the existence of global-in-time classical solutions.

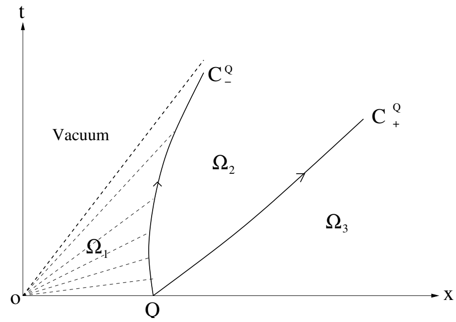

FIG. 1.1. *Wave structure of the solution to the problem (1.4, 1.7).*

Let us briefly describe the main approach to solving the problem (1.4, 1.7). See Figure 1.1. We first consider (1.4) with data

$$\begin{cases} (u,v,\rho,s)(x,0)=(u_0,0,\rho_0,s_0), & x>0; \\ (\rho u)(0,t)=0, & t>0. \end{cases} \tag{1.8}$$

By the result of Zheng [40], we know that the problem (1.4, 1.8) admits a global self-similar solution. Moreover, there exists a $u_*>0$ which depends only on ρ_0, s_0, and γ such that when $u_0>u_*$ the self-similar solution expands to a vacuum, i.e., there exists a circular vacuum region expanding with a constant speed $u_v>0$. Thus, when $u_0>u_*$ we obtain the solution of the problem in a domain Ω_1 bounded by C_-^Q, $x=u_vt$, and the x-axis, where C_-^Q is a C_- characteristic curve issuing from the point $Q=(\varepsilon,0)$. We next solve a Cauchy problem in a triangle domain Ω_3 bounded by the x-axis and a C_+ characteristic curve C_+^Q issuing from Q. Finally, we solve a Goursat problem in a triangle domain Ω_2 bounded by C_+^Q and C_-^Q.

The main difficulty for the global existence is that the a priori C^1 estimates for the solutions to the Cauchy problem and the Goursat problem are hard to obtain. In the paper we use the method of characteristic decompositions. This method was first proposed by Li, Zhang, and Zheng [24] in investigating simple waves of the 2D compressible Euler equations. Recently, this method was used to study the interactions of rarefaction waves; see [17, 18, 23, 25–27]. Motivated by a recent work of Lai and Sheng [20], we derive a group of suitable characteristic decompositions for the system (1.4). Using these characteristic decompositions, we find a sufficient condition on the initial data to ensure the existence of global-in-time classical solutions.

The rest of the paper is organized as follows. In Section 2 we construct the self-similar solution to the Riemann problem (1.4, 1.8). In Section 3 we consider (1.5) with data

$$\begin{cases}(u,\rho)(x,0)=(\bar{u},\bar{\rho})(x), \ x>\varepsilon;\\ (u,\rho)(x,0)=(u_0,\rho_0), \quad 0<x\le\varepsilon;\\ (\rho u)(0,t)=0, \qquad\qquad t>0.\end{cases} \tag{1.9}$$

We find a sufficient condition on $(\bar{u},\bar{\rho})(x)$ to obtain the existence of a global-in-time classical solution. The result is stated as Theorem 3.1. In Section 4 we consider (1.6) with data

$$\begin{cases}(u,v,\rho)(x,0)=(\bar{u},\bar{v},\bar{\rho})(x), \ x>\varepsilon;\\ (u,v,\rho)(x,0)=(u_0,0,\rho_0), \quad 0<x\le\varepsilon;\\ (\rho u)(0,t)=0, \qquad\qquad t>0.\end{cases} \tag{1.10}$$

We find a sufficient condition on $(\bar{u},\bar{v},\bar{\rho})(x)$ to obtain the existence of a global-in-time classical solution. The result is stated as Theorem 4.1. In Section 5 we study the problem (1.4, 1.7). We give a sufficient condition on $(\bar{u},\bar{c},\bar{v},\bar{s})(x)$ to obtain the existence of a global-in-time classical solution to the problem. The result is stated as Theorem 5.1.

2. Self-similar solution to the axisymmetric Euler system

In this section we consider (1.5) with data

$$\begin{cases}(u,\rho)(x,0)=(u_0,\rho_0), \ x>0;\\ (\rho u)(0,t)=0, \qquad t>0.\end{cases} \tag{2.1}$$

Problem (1.5, 2.1) admits a self-similar solution which depends only on the self-similar variable $\xi=x/t$. By the self-similarity, system (1.5) can be written as

$$\begin{cases}-\xi\dfrac{\mathrm{d}\rho}{\mathrm{d}\xi}+\dfrac{\mathrm{d}(\rho u)}{\mathrm{d}\xi}+\dfrac{\rho u}{\xi}=0,\\ -\xi\dfrac{\mathrm{d}u}{\mathrm{d}\xi}+u\dfrac{\mathrm{d}u}{\mathrm{d}\xi}+\dfrac{1}{\rho}\dfrac{\mathrm{d}p}{\mathrm{d}\xi}=0,\end{cases}$$

where we use the equation of state $p=e^{s_0}\rho^\gamma$.

A direct computation yields

$$\begin{cases}\dfrac{\mathrm{d}u}{\mathrm{d}\xi}=-\dfrac{p'(\rho)u}{\xi\left[p'(\rho)-(u-\xi)^2\right]},\\ \dfrac{\mathrm{d}\rho}{\mathrm{d}\xi}=\dfrac{\rho u(u-\xi)}{\xi\left[p'(\rho)-(u-\xi)^2\right]}.\end{cases} \tag{2.2}$$

Let $\eta=1/\xi$. Then the system (2.2) can be converted into

$$\begin{cases}\dfrac{\mathrm{d}u}{\mathrm{d}\eta}=\dfrac{p'(\rho)u\eta}{(c\eta)^2-(1-u\eta)^2},\\ \dfrac{\mathrm{d}\rho}{\mathrm{d}\eta}=\dfrac{\rho u(1-u\eta)}{(c\eta)^2-(1-u\eta)^2}.\end{cases} \tag{2.3}$$

The initial condition $(u,\rho)(x,0)=(u_0,\rho_0)$ can be converted into

$$(u,\rho)\,|_{\eta=0}=(u_0,\rho_0). \tag{2.4}$$

For the initial value problem (2.3, 2.4), we have the following result.

Lemma 2.1. *There exists a $u_*>0$ such that when $u_0>u_*$ the problem (2.3, 2.4) admits a solution $(\hat{u},\hat{\rho})(\eta)$ in $(0,\eta_v)$ for some $\eta_v>\frac{1}{u_0}$. Moreover, the solution satisfies*

- $\lim_{\eta\to\eta_v} \hat{u}(\eta)=u_v$ *and* $\lim_{\eta\to\eta_v} \hat{c}(\eta)=0$, *where* $\hat{c}(\eta)=\sqrt{\gamma e^{s_0}\hat{\rho}^{\gamma-1}(\eta)}$ *and* $u_v=\frac{1}{\eta_v}$;
- $\eta\hat{c}(\eta)<\frac{1}{\sqrt{2}}\big(1-\eta\hat{u}(\eta)\big)$ *for* $0<\eta<\eta_v$;
- $\hat{\rho}'(\eta)<0$ *and* $\hat{u}'(\eta)<0$ *for* $0<\eta<\eta_v$.

Proof. This lemma is due to Zhang and Zheng ([38], Section 5) and Zheng [40]. We also refer the reader to the work [39]. ∎

REMARK 2.1. Actually, the u_* in Lemma 2.1 depends only on ρ_0, s_0, and γ; u_v depends only on u_0, ρ_0, s_0, and γ. Moreover, for any fixed γ and u_0 and s_0, $u_*\to 0$ and $u_v\to u_0$ as $\rho_0\to 0$. We also know $\hat{u}'(\eta)<0$ for $\eta\in(0,\eta_v)$.

3. Axisymmetric isentropic Euler system without swirl

3.1. Main result. In this section we consider the problem (1.5, 1.9). In (1.5) we take the equation of state $p=e^{s_0}\rho^\gamma$, where s_0 is given in (1.7). The main result of this section can be stated as the following theorem.

THEOREM 3.1. *Assume* $u_0>u_*$ *and*

$$c_M=\sup_{x\in[\varepsilon,+\infty)}\bar{c}(x) \;<\; \Big(\frac{\gamma-1}{2}\Big)u_v, \tag{3.1}$$

where the constants u_v *and* u_* *are determined in Lemma 2.1. Assume as well that there exists a constant* $\mathcal{B}\in\big(\frac{1}{2},\min\{\frac{3}{\gamma+1},\frac{1}{3-\gamma}\}\big)$ *such that*

(A1) $(\frac{\gamma+1}{2})\mathcal{B}^2-(\frac{\gamma+7}{4})\mathcal{B}+\frac{3}{4}+\big(\frac{\mathcal{B}}{2}+\frac{1}{4}\big)\frac{c_M}{u_v}<0$;

(A2) $|\bar{c}'(x)|-\big(\frac{\gamma-1}{2}\big)\bar{u}'(x)+\big(\mathcal{B}-\frac{1}{2}\big)\frac{(\gamma-1)\bar{u}(x)}{x}<0$ *for* $x\in[\varepsilon,+\infty)$.

Then the problem (1.5, 1.9) admits a global-in-time classical solution.

REMARK 3.1. It is easy to check that the roots of $\big(\frac{\gamma+1}{2}\big)r^2-\big(\frac{\gamma+7}{4}\big)r+\frac{3}{4}=0$ are $r=\frac{1}{2}$ and $r=\frac{3}{\gamma+1}$. Thus, one can find an expanding initial data such that the assumptions in Theorem 3.1 can be satisfied.

3.2. Characteristic equations and decompositions for (1.5). For smooth flows, system (1.5) can be reduced to

$$\begin{cases} \rho_t+\rho u_x+u\rho_x+\dfrac{\rho u}{x}=0,\\ u_t+uu_x+\dfrac{p_x}{\rho}=0. \end{cases} \tag{3.2}$$

The eigenvalues of system (3.2) are

$$\lambda_+=u+c \quad\text{and}\quad \lambda_-=u-c.$$

The left eigenvectors corresponding to $\lambda_\pm$ are $l_\pm=(c,\pm\rho)$. Multiplying (3.2) on the left by $l_\pm$, we get the characteristic equations

$$c\partial_\pm\rho\pm\rho\partial_\pm u=-\frac{c\rho u}{x}, \tag{3.3}$$

where

$$\partial_\pm=\partial_t+(u\pm c)\partial_x.$$

From $c^2=\gamma e^{s_0}\rho^{\gamma-1}$ we have

$$\partial_\pm\rho=\frac{2c\partial_\pm c}{\gamma(\gamma-1)e^{s_0}\rho^{\gamma-2}}. \tag{3.4}$$

Inserting this into (3.3), we get

$$\begin{cases}\partial_+u=-\dfrac{2}{\gamma-1}\partial_+c-\dfrac{uc}{x},\\ \partial_-u=\dfrac{2}{\gamma-1}\partial_-c+\dfrac{uc}{x}.\end{cases} \tag{3.5}$$

LEMMA 3.1. *For the system (1.5), we have the commutator relation*

$$\partial_+\partial_- -\partial_-\partial_+=\Big(-\frac{1}{2c\mu^2}(\partial_+c+\partial_-c)-\frac{u}{x}\Big)(\partial_+-\partial_-), \tag{3.6}$$

where

$$\mu^2=\frac{\gamma-1}{\gamma+1}.$$

Proof. Using (3.5) and $\partial_x=\frac{\partial_+-\partial_-}{2c}$, we have

$$\begin{aligned}\partial_+\partial_- -\partial_-\partial_+&=(\partial_t+\lambda_+\partial_x)(\partial_t+\lambda_-\partial_x)-(\partial_t+\lambda_-\partial_x)(\partial_t+\lambda_+\partial_x)\\ &=(\partial_+\lambda_- -\partial_-\lambda_+)\partial_x,\\ &=(\partial_+u-\partial_+c-\partial_-u-\partial_-c)\partial_x\\ &=\Big(-\frac{1}{2c\mu^2}(\partial_+c+\partial_-c)-\frac{u}{x}\Big)(\partial_+-\partial_-).\end{aligned} \tag{3.7}$$

This completes the proof. □

PROPOSITION 3.1. *We have the characteristic decompositions*

$$\begin{cases}c\partial_-\partial_+c=\dfrac{1}{2\mu^2}(\partial_+c+\partial_-c)\partial_+c+(\dfrac{3}{2}\partial_+c+\dfrac{1}{2}\partial_-c)\dfrac{uc}{x}+\dfrac{c^2}{2x}(\partial_+c-\partial_-c)+\dfrac{(\gamma-1)u^2c^2}{x^2},\\ c\partial_+\partial_-c=\dfrac{1}{2\mu^2}(\partial_+c+\partial_-c)\partial_-c+(\dfrac{3}{2}\partial_-c+\dfrac{1}{2}\partial_+c)\dfrac{uc}{x}+\dfrac{c^2}{2x}(\partial_+c-\partial_-c)+\dfrac{(\gamma-1)u^2c^2}{x^2}.\end{cases} \tag{3.8}$$

Proof. Using the commutator relation (3.6) for the variable c, we have

$$c\partial_+\partial_-c-c\partial_-\partial_+c=-\frac{\gamma+1}{2(\gamma-1)}(\partial_+c+\partial_-c)(\partial_+c-\partial_-c)-\frac{cu}{x}(\partial_+c-\partial_-c). \tag{3.9}$$

Using the commutator relation (3.6) for the variable u, we have

$$\partial_+\partial_-u-\partial_-\partial_+u=\left(-\frac{1}{2c\mu^2}(\partial_+c+\partial_-c)-\frac{u}{x}\right)(\partial_+u-\partial_-u). \tag{3.10}$$

Inserting (3.5) into this, we get

$$c\partial_+\partial_-c+c\partial_-\partial_+c=\frac{\gamma+1}{2(\gamma-1)}(\partial_+c+\partial_-c)^2+\frac{2cu}{x}(\partial_+c+\partial_-c)+\frac{c^2}{x}(\partial_+c-\partial_-c)+\frac{2(\gamma-1)c^2u^2}{x^2}. \tag{3.11}$$

Combining (3.9) and (3.11), we get the first equation of (3.8). The second equation of (3.8) can be proved similarly. This completes the proof. □

It is convenient to write (3.8) in the form

$$\begin{cases} \partial_-\Big(\dfrac{\partial_+ c}{c}\Big) = \dfrac{1}{2\mu^2}\Big(\dfrac{\partial_+ c}{c}+\dfrac{\partial_- c}{c}\Big)\dfrac{\partial_+ c}{c}+\Big(\dfrac{3}{2}\dfrac{\partial_+ c}{c}+\dfrac{1}{2}\dfrac{\partial_- c}{c}\Big)\dfrac{u}{x} \\ \qquad +\dfrac{c}{2x}\Big(\dfrac{\partial_+ c}{c}-\dfrac{\partial_- c}{c}\Big)+\dfrac{(\gamma-1)u^2}{x^2}-\dfrac{\partial_+ c}{c}\dfrac{\partial_- c}{c}, \\ \partial_+\Big(\dfrac{\partial_- c}{c}\Big) = \dfrac{1}{2\mu^2}\Big(\dfrac{\partial_+ c}{c}+\dfrac{\partial_- c}{c}\Big)\dfrac{\partial_- c}{c}+\Big(\dfrac{3}{2}\dfrac{\partial_- c}{c}+\dfrac{1}{2}\dfrac{\partial_+ c}{c}\Big)\dfrac{u}{x} \\ \qquad +\dfrac{c}{2x}\Big(\dfrac{\partial_+ c}{c}-\dfrac{\partial_- c}{c}\Big)+\dfrac{(\gamma-1)u^2}{x^2}-\dfrac{\partial_+ c}{c}\dfrac{\partial_- c}{c}. \end{cases} \tag{3.12}$$

Let $\mathcal{A}>\mathcal{B}$ be a fixed constant to be determined. We define

$$R_+=\frac{\partial_+ c}{c}+\frac{\mathcal{A}(\gamma-1)u}{x} \quad \text{and} \quad R_-=\frac{\partial_- c}{c}+\frac{\mathcal{A}(\gamma-1)u}{x}. \tag{3.13}$$

Then by (3.12) we have

$$\begin{cases} \partial_+ R_- = a_{11}R_-^2+a_{12}R_+R_-+a_{13}R_-+a_{14}R_++a_{15}, \\ \partial_- R_+ = a_{21}R_+^2+a_{22}R_+R_-+a_{23}R_++a_{24}R_-+a_{25}, \end{cases} \tag{3.14}$$

where

$$a_{14}=\Big\{\frac{\gamma-3}{2}\mathcal{A}+\frac{1}{2}-(2\mathcal{A}-\frac{1}{2})\frac{c}{u}\Big\}\frac{u}{x}, \tag{3.15}$$

$$a_{24}=\Big\{\frac{\gamma-3}{2}\mathcal{A}+\frac{1}{2}+(2\mathcal{A}-\frac{1}{2})\frac{c}{u}\Big\}\frac{u}{x}, \tag{3.16}$$

$$a_{15}=\Big\{2\mathcal{A}^2-3\mathcal{A}+1-2\mathcal{A}(1-\mathcal{A})\frac{c}{u}\Big\}\frac{(\gamma-1)u^2}{x^2}, \tag{3.17}$$

and

$$a_{25}=\Big\{2\mathcal{A}^2-3\mathcal{A}+1+2\mathcal{A}(1-\mathcal{A})\frac{c}{u}\Big\}\frac{(\gamma-1)u^2}{x^2}. \tag{3.18}$$

We define

$$\widehat{R}_+=\frac{\partial_+ c}{c}+\frac{\mathcal{B}(\gamma-1)u}{x} \quad \text{and} \quad \widehat{R}_-=\frac{\partial_- c}{c}+\frac{(\gamma-1)u}{2x}. \tag{3.19}$$

Then by (3.12) we have

$$\begin{cases} \partial_+ \widehat{R}_- = \hat{a}_{11}\widehat{R}_-^2+\hat{a}_{12}\widehat{R}_+\widehat{R}_-+\hat{a}_{13}\widehat{R}_-+\hat{a}_{14}\widehat{R}_++\hat{a}_{15}, \\ \partial_- \widehat{R}_+ = \hat{a}_{21}\widehat{R}_+^2+\hat{a}_{22}\widehat{R}_+\widehat{R}_-+\hat{a}_{23}\widehat{R}_++\hat{a}_{24}\widehat{R}_-+\hat{a}_{25}, \end{cases} \tag{3.20}$$

where

$$\hat{a}_{14}=\Big\{\frac{\gamma-1}{4}-\frac{c}{2u}\Big\}\frac{u}{x}, \tag{3.21}$$

$$\hat{a}_{24}=\Big\{\frac{\gamma-3}{2}\mathcal{B}+\frac{1}{2}+(2\mathcal{B}-\frac{1}{2})\frac{c}{u}\Big\}\frac{u}{x}, \tag{3.22}$$

$$\hat{a}_{15}=\Big\{\frac{1-\gamma}{4}\mathcal{B}+\frac{\gamma+1}{8}-\frac{1}{4}+(\frac{\mathcal{B}}{2}-\frac{3}{4})\frac{c}{u}\Big\}\frac{(\gamma-1)u^2}{x^2}, \tag{3.23}$$

and

$$\hat{a}_{25}=\left\{\frac{\gamma+1}{2}\mathcal{B}^2-\frac{\gamma+7}{4}\mathcal{B}+\frac{3}{4}+(\frac{\mathcal{B}}{2}+\frac{1}{4})\frac{c}{u}\right\}\frac{(\gamma-1)u^2}{x^2}. \tag{3.24}$$

REMARK 3.2. From (3.1) and (A1) we immediately have that if $0<c<c_{_M}$ and $u>u_v$ then $\hat{a}_{14}>0$, $\hat{a}_{24}>0$, $\hat{a}_{15}<0$, and $\hat{a}_{25}<0$.

3.3. Solution in domain Ω_1. Let $C_-^Q: x=x_-(t;\varepsilon)$ be a C_- characteristic curve issuing from the point $Q=(\varepsilon,0)$, i.e.

$$\begin{cases}\dfrac{\mathrm{d}x_-(t;\varepsilon)}{\mathrm{d}t}=\hat{u}(t/x_-)-\hat{c}(t/x_-),\ t>0;\\ x_-(t;0)=\varepsilon.\end{cases} \tag{3.25}$$

We are going to show that C_-^Q does not meet the vacuum boundary $x=u_vt$, $t>0$.

LEMMA 3.1. *For the self-similar solution $(u,\rho)=(\hat{u},\hat{\rho})(t/x)$, we have*

$$-\frac{(2+\sqrt{2})(\gamma-1)u}{2x}<\frac{\partial_-c}{c}<-\frac{(\gamma-1)u}{2x} \quad on \quad C_-^Q. \tag{3.26}$$

Proof. A direct computation yields

$$\partial_-=\left(\frac{1}{x}-\frac{(u-c)\eta}{x}\right)\frac{\mathrm{d}}{\mathrm{d}\eta}. \tag{3.27}$$

By the second equation of (2.3) we have

$$\frac{\mathrm{d}c}{\mathrm{d}\eta}=\frac{\gamma-1}{2}\cdot\frac{cu(1-u\eta)}{(c\eta)^2-(1-u\eta)^2}. \tag{3.28}$$

Thus, by Lemma 2.1 we have

$$\begin{aligned}\frac{\partial_-c}{c}+\frac{(\gamma-1)u}{2x}&=\frac{(\gamma-1)u}{2x}\cdot\left\{\big[1-(u-c)\eta\big]\frac{(1-u\eta)}{(c\eta)^2-(1-u\eta)^2}+1\right\}\\&=\frac{(\gamma-1)u}{2x}\cdot\frac{c\eta}{c\eta+u\eta-1}<0\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}\frac{\partial_-c}{c}+\frac{(2+\sqrt{2})(\gamma-1)u}{2x}&=\frac{(\gamma-1)u}{2x}\cdot\left\{\big[1-(u-c)\eta\big]\frac{(1-u\eta)}{(c\eta)^2-(1-u\eta)^2}+2+\sqrt{2}\right\}\\&=\frac{(\gamma-1)u}{2x}\cdot\frac{(2+\sqrt{2})(c\eta-\frac{1}{\sqrt{2}}(1-u\eta))}{c\eta+u\eta-1}>0\end{aligned} \tag{3.30}$$

along C_-^Q. This completes the proof of the lemma. □

From Lemma 3.1, we have

$$\frac{\partial_- c}{c} > -\frac{(2+\sqrt{2})(\gamma-1)u}{2x} = -\frac{(2+\sqrt{2})(\gamma-1)u}{2(u-c)}\partial_- \ln x \quad \text{on} \quad C_-^Q. \tag{3.31}$$

This immediately implies that C_-^Q does not meet the vacuum boundary $x = u_v t$, $t > 0$. Let Ω_1 be a domain encircled by C_-^Q, the vacuum boundary, and the x-axis. Then the solution in the domain Ω_1 is the self-similar solution $(u,c) = (\hat{u},\hat{c})(t/x)$.

3.4. Solution in domain Ω_3. We first consider (1.5) with data

$$(u,c)(x,0) = (\bar{u},\bar{c})(x), \quad \varepsilon < x < r, \tag{3.32}$$

where $r > \varepsilon$ is an arbitrary constant. The local existence of classical solution to the Cauchy problem (1.5, 3.32) can be obtained by the method of characteristics (cf. [29]). In order to extend the local solution to a global solution, one needs to establish an a priori C^1 norm estimate of the solution.

LEMMA 3.2. *Assume that the Cauchy problem (1.5, 3.32) admits a classical solution in some region. Then there exists a sufficiently large constant $\mathcal{A}$ such that the solution satisfies*

$$\widehat{R}_\pm < 0, \quad \frac{\partial_\pm c}{c} > -\frac{\mathcal{A}(\gamma-1)u}{x}, \quad 0 < c < c_M, \quad \textit{and} \quad u_v < u < \bar{u}(r)+1. \tag{3.33}$$

Proof. We shall prove this lemma by the method of continuity. The proof proceeds in two steps.

Step 1. From $c^2 = \gamma e^{s_0}\rho^{\gamma-1}$ we have

$$\rho_t = \frac{2cc_t}{\gamma(\gamma-1)e^{s_0}\rho^{\gamma-2}} \quad \text{and} \quad \rho_x = \frac{2cc_x}{\gamma(\gamma-1)e^{s_0}\rho^{\gamma-2}}.$$

Inserting this into the first equation of (1.5) we get

$$c_t = -uc_x - \frac{\gamma-1}{2}cu_x - \frac{\gamma-1}{2}\frac{cu}{x}. \tag{3.34}$$

Hence,

$$\partial_\pm c = c_t + (u\pm c)c_x = \pm cc_x - \frac{\gamma-1}{2}cu_x - \frac{\gamma-1}{2}\cdot\frac{cu}{x}. \tag{3.35}$$

Consequently, we have

$$\widehat{R}_+(x,0) = \bar{c}'(x) - \frac{\gamma-1}{2}\bar{u}'(x) + \big(\mathcal{B} - \frac{1}{2}\big)\frac{(\gamma-1)\bar{u}(x)}{x} \tag{3.36}$$

and

$$\widehat{R}_-(x,0) = -\bar{c}'(x) - \frac{\gamma-1}{2}\bar{u}'(x). \tag{3.37}$$

By assumption (A2) we have

$$\widehat{R}_-(x,0) < 0 \quad \text{and} \quad \widehat{R}_+(x,0) < 0 \quad \text{for} \ \ \varepsilon < x < r. \tag{3.38}$$

From (3.35) we have

$$R_{\pm}(x,0)=\pm\bar{c}'(x)-\frac{\gamma-1}{2}\bar{u}'(x)+\big(\mathcal{A}-\frac{1}{2}\big)\frac{(\gamma-1)\bar{u}(x)}{x}. \tag{3.39}$$

By assumption (A2) we have $\bar{u}'(x)>0$ for $x\geq\varepsilon$. Hence, $\bar{u}(x)>u_0$. Consequently, when $\mathcal{A}$ is sufficiently large,

$$R_-(x,0)>0 \quad \text{and} \quad R_+(x,0)>0 \quad \text{for} \quad \varepsilon<x<r. \tag{3.40}$$

Then the inequalities in (3.33) hold on $\{(x,t)\,|\,t=0,\ \varepsilon<x<r\}$.

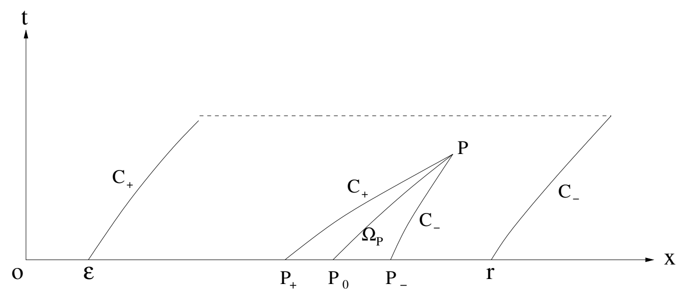

FIG. 3.1. *Domain* Ω_P.

Step 2. Let P be an arbitrary point in the domain. The backward C_+ and C_- characteristic curves issuing from P intersect the x-axis at some points P_+ and P_-, respectively; see Figure 3.1. The backward trajectory line C_0 issuing from P intersects the x-axis at a point P_0. We denote by Ω_P a closed triangle domain closed by $\widehat{P_+P}$, $\widehat{P_-P}$, and $\overline{P_+P_-}$. We are going to prove that if the inequalities in (3.33) hold for all points in $\Omega_P\setminus\{P\}$, then they also hold at P.

From $\widehat{R}_{\pm}<0$ in $\Omega_P\setminus\{P\}$, we immediately have $\partial_{\pm}c<0$ on $\Omega_P\setminus\{P\}$. Hence, we have $c<c_M$ at P. Since $R_{\pm}>0$ for all points in $\Omega_P\setminus\{P\}$, we have $\frac{\partial_0 c}{c}>-\frac{\mathcal{A}(\gamma-1)u}{x}$ in $\Omega_P\setminus\{P\}$. Integrating this along $\widehat{P_0P}$ from P_0 to P, we have

$$c(P)>c(P_0)\Big(\frac{x(P)}{x(P_0)}\Big)^{-\mathcal{A}(\gamma-1)}>0. \tag{3.41}$$

In view of (3.5), we have

$$\partial_-u=\frac{2c\widehat{R}_-}{\gamma-1}<0 \quad \text{and} \quad \partial_+u=-\frac{2c\widehat{R}_+}{\gamma-1}+(\mathcal{B}-\frac{1}{2})\frac{2uc}{x}>0 \quad \text{in} \quad \Omega_P\setminus\{P\}.$$

Thus, we have

$$u_v<u(P_+)<u(P)<u(P_-)<\bar{u}(r)+1.$$

Suppose $\widehat{R}_-(P)=0$. Then by the assumption that the inequalities in (3.33) hold for every point in $\Omega_P\setminus\{P\}$, we have $\partial_+\widehat{R}_-\geq0$ at P. While, by the first equation of (3.20) and Remark 3.2, we have

$$\partial_+\widehat{R}_-=\hat{a}_{14}\widehat{R}_++\hat{a}_{15}<0 \tag{3.42}$$

at P. This leads to a contradiction. Then we get $\widehat{R}_-(P)<0$. Similarly, we have $\widehat{R}_+(P)<0$. From $\widehat{R}_\pm(P)<0$ we also have

$$R_\pm < \frac{\mathcal{A}(\gamma-1)u}{x} \quad \text{at} \quad P. \tag{3.43}$$

Suppose $R_+=0$ at P. Then by the assumption that the inequalities in (3.33) hold for any point in $\Omega_P\setminus\{P\}$ we have $\partial_- R_+\le 0$ at P. While, by (3.43) and the second equation of (3.14), we have

$$\begin{cases} \partial_- R_+ > a_{24}\dfrac{\mathcal{A}(\gamma-1)u}{x}+a_{25} > \Big\{\dfrac{\gamma+1}{2}\mathcal{A}^2-\dfrac{5}{2}\mathcal{A}+1\Big\}\dfrac{(\gamma-1)u^2}{x^2}>0 \text{ if } a_{24}<0; \\ \partial_- R_+ > a_{25} > \Big\{2\big(1-\dfrac{c_M}{u_v}\big)\mathcal{A}^2-\big(3-2\dfrac{c_M}{u_v}\big)\mathcal{A}+1\Big\}\dfrac{(\gamma-1)u^2}{x^2}>0 \quad \text{if } a_{24}\ge 0. \end{cases} \tag{3.44}$$

This leads to a contradiction. We then have $R_+(P)>0$. Similarly, we get $R_-(P)>0$.

We then prove that if the inequalities in (3.33) hold for every point in $\Omega_P\setminus\{P\}$ then these inequalities also hold at P. Therefore, by an argument of continuity we have that the solution satisfies (3.33). This completes the proof of the lemma. ∎

Lemma 3.2 gives a C^0 norm estimate for (c,u) and a gradient estimate for c. The gradient estimate for u can be obtained by (3.5). We then establish an a priori C^1 estimate for the solution. Thus, the existence of a global classical solution can be obtained by the classical extension method (cf. Li [28]). We then get the following conclusion.

LEMMA 3.3. *The Cauchy problem (1.5, 3.32) admits a global classical solution. Moreover, the solution satisfies (3.33).*

Since $r>\varepsilon$ can be arbitrary, we obtain the solution in Ω_3 bounded by C_+^Q and the x-axis, where C_+^Q is a C_+ characteristic curve issuing from Q. Moreover, the solution satisfies

$$\widehat{R}_\pm<0, \quad 0<c<c_M, \quad \text{and} \quad u>u_v. \tag{3.45}$$

3.5. Solution in domain Ω_2. Take any points Q_+ and Q_- on C_+^Q and C_-^Q, respectively. We now consider (1.5) with data

$$(u,c)=\begin{cases} (\hat{u},\hat{c})(t/x) \text{ on } \widehat{QQ_-}; \\ (\tilde{u},\tilde{c})(x,t) \text{ on } \widehat{QQ_+}; \end{cases} \tag{3.46}$$

where $(\tilde{u},\tilde{c})(x,t)$ denotes the solution in Ω_3.

Problem (1.5, 3.46) is a Goursat-type boundary value problem, and the existence of a local C^1 solution is known by the method of characteristics (see [29]). In order to extend the local solution to a global solution, we need to establish an a priori C^1 estimate of the solution.

LEMMA 3.4. *Assume that the Goursat problem (1.5, 3.46) admits a classical solution in some region. Then there exists a sufficiently large $\mathcal{A}>0$ such that*

$$\widehat{R}_\pm<0, \quad \frac{\partial_\pm c}{c}>-\frac{\mathcal{A}(\gamma-1)u}{x}, \quad 0<c<c_M, \quad \text{and} \quad u_v<u<2u(Q_+). \tag{3.47}$$

Proof. The proof of this lemma proceeds in two steps.

Step 1. We first prove that the inequalities in (3.47) hold on $\widehat{QQ_+} \cup \widehat{QQ_-}$.

Firstly, from Lemmas 3.1 and 3.2 we have $0 < c < c_M$ and $u_v < u < 2u(Q_+)$ on $\widehat{QQ_+} \cup \widehat{QQ_-}$, $\widehat{R}_- < 0$ on $\widehat{QQ_-}$, and $\widehat{R}_+ < 0$ on $\widehat{QQ_+}$.

We next prove $\widehat{R}_+ |_{\widehat{QQ_-}} < 0$ and $\widehat{R}_- |_{\widehat{QQ_+}} < 0$. Suppose that there exists a "first" point Q_1 on $\widehat{QQ_+}$ such that $\widehat{R}_-(Q_1) = 0$ and $\widehat{R}_- < 0$ on $\widehat{QQ_1}$. Then we have $\partial_+ \widehat{R}_- \geq 0$ at Q_1. While, as in (3.42), we have $\partial_+ \widehat{R}_- < 0$ at Q_1, which leads to a contradiction. Thus $\widehat{R}_- |_{\widehat{QQ_+}} < 0$. Similarly, we have $\widehat{R}_+ |_{\widehat{QQ_-}} < 0$.

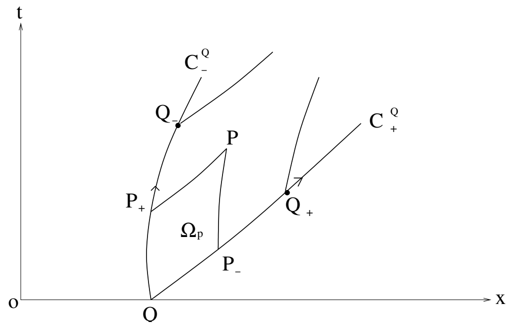

FIG. 3.2. *Global classical solution to the Goursat problem.*

When $\mathcal{A}$ is sufficiently large, we have $R_+ |_{\widehat{QQ_+}} > 0$ and $R_- |_{\widehat{QQ_-}} > 0$. We next prove $R_+ |_{\widehat{QQ_-}} > 0$ and $R_- |_{\widehat{QQ_+}} > 0$. Suppose that there exists a "first" point Q_1 on $\widehat{QQ_+}$ such that $R_-(Q_1) = 0$ and $R_- > 0$ on $\widehat{QQ_1}$. Then we have $\partial_+ R_- \leq 0$ at Q_1. While, as in (3.44) we have $\partial_- R_+ > 0$ at Q_1, which leads to a contradiction. Thus we have $R_+ |_{\widehat{QQ_+}} > 0$. Similarly, we have $R_- |_{\widehat{QQ_-}} > 0$. We then prove (3.47) on $\widehat{QQ_+} \cup \widehat{QQ_-}$.

Step 2. Let P be an arbitrary point in the domain. The backward C_+ and C_- characteristic curves issuing from P intersect $\widehat{QQ_-}$ and $\widehat{QQ_+}$ at some points P_+ and P_-, respectively. The backward trajectory line C_0 issuing from P intersects $\widehat{QQ_+} \cup \widehat{QQ_-}$ at some point P_0. We denote by Ω_P a closed quadrilateral domain closed by $\widehat{P_+P}$, $\widehat{P_-P}$, $\overline{QP_-}$, and $\overline{QP_+}$; see Figure 3.2. As in the proof of Lemma 3.2, we know that if the inequalities in (3.47) hold for every point in $\Omega_P \setminus \{P\}$, then they also hold at P.

Therefore, by an argument of continuity we complete the proof of the lemma. ∎

By Lemma 3.4 we get an a priori C^1 estimate of the solution to the Goursat problem (1.5, 3.46). Thus, the existence of a global classical solution can be obtained by the classical extension method (cf. Li [28]). We then have the following global existence.

LEMMA 3.5. *The Goursat problem (1.5, 3.46) admits a global classical solution.*

Since $Q_\pm$ can be arbitrary, we obtain the solution in a domain Ω_2 bounded by characteristic curves C_+^Q and C_-^Q. We then complete the proof of Theorem 3.1.

4. Axisymmetric isentropic Euler equations with swirl

4.1. Main result. In this section we consider the problem (1.6, 1.10). The main result can be stated as the following theorem.

THEOREM 4.1. *Let*

$$\delta=\sup_{x\in[\varepsilon,+\infty)}\frac{\bar{v}^2(x)}{\bar{c}(x)} \quad \text{and} \quad c_{_M}=\sup_{x\in[\varepsilon,+\infty)}\bar{c}(x).$$

Assume $1<\gamma<5-2\sqrt{2}$ *and there exist constants* $\mathcal{D}\in\left(\frac{1}{2},\min\{\frac{3}{\gamma+1},\frac{1}{3-\gamma}\}\right)$ *and* $\mathcal{D}'\in(\mathcal{D},\frac{2}{\gamma-1})$ *such that*

$$|\bar{c}'(x)|+(\mathcal{D}-\frac{1}{2})\frac{(\gamma-1)\bar{u}(x)}{x}<\frac{\gamma-1}{2}\bar{u}'(x)<-|\bar{c}'(x)|+\left(\mathcal{D}'-\frac{1}{2}\right)\frac{(\gamma-1)\bar{u}(x)}{x} \tag{4.1}$$

for $x\in[\varepsilon,+\infty)$. *Then when* δ *and* $c_{_M}$ *are sufficiently small the problem (1.6, 1.10) admits a global-in-time classical solution.*

REMARK 4.1. It is easy to check that when δ and $c_{_M}$ are sufficiently small, the following properties hold:

(S0) $\frac{c_M}{u_v}<\frac{\gamma-1}{2}-\frac{(\gamma+1)\delta}{2u_v}$;

(S1) $\frac{\gamma+1}{2}\mathcal{D}^2-\frac{\gamma+7}{4}\mathcal{D}+\frac{3}{4}+\left(\frac{\mathcal{D}}{2}+\frac{1}{4}\right)\frac{c_M}{u_v}+\left(\frac{1-3\gamma}{4}\mathcal{D}-\frac{\gamma-15}{8}\right)\frac{\delta}{u_v}<0$;

(S2) $\frac{\gamma+1}{2}\mathcal{C}^2-\left(\frac{5}{2}+\frac{3}{2}\frac{c_M}{u_v}+\frac{\gamma\delta}{u_v}\right)\mathcal{C}+1-\frac{2\delta}{u_v} > 0$;

(S3) $2\left(1-\frac{c_M}{u_v}\right)\mathcal{C}^2-\left(3+2\frac{c_M}{u_v}\right)\mathcal{C}+1-\frac{|\gamma\mathcal{C}-2|\delta}{u_v}>0$;

(S4) $|\bar{c}'(x)|-\frac{\gamma-1}{2}\bar{u}'(x)+(\mathcal{D}-\frac{1}{2})\frac{(\gamma-1)\bar{u}(x)}{x}-\frac{(\gamma-1)\bar{v}^2(x)}{2\bar{c}(x)x}<0$ for $x\in[\varepsilon,+\infty)$;

(S5) $|\bar{c}'(x)|-\frac{\gamma-1}{2}\bar{u}'(x)+\frac{(\gamma-1)\bar{v}^2(x)}{2\bar{c}(x)x}<0$ for $x\in[\varepsilon,+\infty)$;

(S6) $-|\bar{c}'(x)|-\frac{\gamma-1}{2}\bar{u}'(x)-\frac{(\gamma-1)\bar{v}^2(x)}{2\bar{c}(x)x}+(\mathcal{C}-\frac{1}{2})\frac{(\gamma-1)\bar{u}(x)}{x}>0$ for $x\in[\varepsilon,+\infty)$.

Here, the constant

$$\mathcal{C}=\frac{2}{\gamma-1}. \tag{4.2}$$

REMARK 4.2. From $1<\gamma<5-2\sqrt{2}$ we have $\mathcal{C}>1+\frac{\sqrt{2}}{2}$.

4.2. Characteristic equations and decompositions for (1.6). For smooth flows, system (1.6) can be simplified to

$$\begin{cases} \rho_t+\rho u_x+u\rho_x+\dfrac{\rho u}{x}=0,\\ u_t+uu_x+\dfrac{p_x}{\rho}-\dfrac{v^2}{x}=0,\\ v_t+uv_x+\dfrac{uv}{x}=0. \end{cases} \tag{4.3}$$

The eigenvalues of the system (4.3) are

$$\lambda_+=u+c,\quad \lambda_0=u,\quad \lambda_-=u-c.$$

The left eigenvectors corresponding to $\lambda_\pm$ are $l_\pm=(c,\pm\rho,0)$. Multiplying (4.3) on the left by $l_\pm$ we get the characteristic equations

$$c\partial_\pm\rho\pm\rho\partial_\pm u=-\frac{c\rho u}{x}\pm\frac{\rho v^2}{x}, \tag{4.4}$$

where

$$\partial_\pm = \partial_t + (u \pm c)\partial_x.$$

Inserting (3.4) into (4.4), we get

$$\begin{cases} \partial_+ u = -\dfrac{2}{\gamma-1}\partial_+ c - \dfrac{uc}{x} + \dfrac{v^2}{x}, \\ \partial_- u = \dfrac{2}{\gamma-1}\partial_- c + \dfrac{uc}{x} + \dfrac{v^2}{x}. \end{cases} \tag{4.5}$$

REMARK 4.3. From (4.5) we immediately have that the bound of $|\nabla u|$ can be controlled by the bounds of ∇c, u, v, c, and $\frac{1}{x}$.

LEMMA 4.1. *For the system (1.6), we have the commutator relation*

$$\partial_+\partial_- - \partial_-\partial_+ = \left(-\frac{1}{2c\mu^2}(\partial_+ c + \partial_- c) - \frac{u}{x}\right)(\partial_+ - \partial_-). \tag{4.6}$$

Proof. The proof is similar to that for Lemma 3.1, we omit the details. □

PROPOSITION 4.1. *We have the characteristic decompositions*

$$\begin{cases} c\partial_-\left(\partial_+ c - \dfrac{(\gamma-1)v^2}{2x}\right) = \dfrac{1}{2\mu^2}(\partial_+ c + \partial_- c)\partial_+ c + (\dfrac{3}{2}\partial_+ c + \dfrac{1}{2}\partial_- c)\dfrac{uc}{x} + \dfrac{c^2}{2x}(\partial_+ c - \partial_- c) \\ \qquad + \dfrac{(\gamma-1)u^2c^2}{x^2} - \dfrac{(\gamma-1)c^2v^2}{2x^2} + \dfrac{3(\gamma-1)cuv^2}{2x^2}, \\ c\partial_+\left(\partial_- c + \dfrac{(\gamma-1)v^2}{2x}\right) = \dfrac{1}{2\mu^2}(\partial_+ c + \partial_- c)\partial_- c + (\dfrac{3}{2}\partial_- c + \dfrac{1}{2}\partial_+ c)\dfrac{uc}{x} + \dfrac{c^2}{2x}(\partial_+ c - \partial_- c) \\ \qquad + \dfrac{(\gamma-1)u^2c^2}{x^2} - \dfrac{(\gamma-1)c^2v^2}{2x^2} - \dfrac{3(\gamma-1)cuv^2}{2x^2}. \end{cases} \tag{4.7}$$

Proof. Using the commutator relation (4.6) for the variable c, we have

$$c\partial_+\partial_- c - c\partial_-\partial_+ c = -\frac{\gamma+1}{2(\gamma-1)}(\partial_+ c + \partial_- c)(\partial_+ c - \partial_- c) - \frac{cu}{x}(\partial_+ c - \partial_- c). \tag{4.8}$$

Using the commutator relation (4.6) for the variable u, we have

$$\partial_+\partial_- u - \partial_-\partial_+ u = \left(-\frac{1}{2c\mu^2}(\partial_+ c + \partial_- c) - \frac{u}{x}\right)(\partial_+ u - \partial_- u). \tag{4.9}$$

Inserting (4.5) into (4.9), we get

$$\begin{aligned} c\partial_+\partial_- c + c\partial_-\partial_+ c = &\frac{\gamma+1}{2(\gamma-1)}(\partial_+ c + \partial_- c)^2 + \frac{\gamma+3}{2}\frac{cu}{x}(\partial_+ c + \partial_- c) \\ &+ \frac{c^2u^2(\gamma-1)}{x^2} - \frac{c(\gamma-1)}{2}\left[\partial_+\left(\frac{cu}{x}\right) + \partial_-\left(\frac{cu}{x}\right)\right] \\ &- \frac{c(\gamma-1)}{2}\left[\partial_+\left(\frac{v^2}{x}\right) - \partial_-\left(\frac{v^2}{x}\right)\right]. \end{aligned} \tag{4.10}$$

Combining (4.8) and (4.10), we obtain

$$c\partial_-\left(\partial_+ c - \frac{(\gamma-1)v^2}{2x}\right) = \frac{\gamma+1}{2(\gamma-1)}(\partial_+ c + \partial_- c)\partial_+ c + \frac{(\gamma+3)cu}{4x}(\partial_+ c + \partial_- c)$$

$$+\frac{cu}{2x}(\partial_+ c-\partial_- c)-\frac{c(\gamma-1)}{4}\Big[\partial_+\Big(\frac{cu}{x}\Big)+\partial_-\Big(\frac{cu}{x}\Big)\Big]$$
$$+\frac{c^2u^2(\gamma-1)}{2x^2}-\frac{c(\gamma-1)}{4}\Big[\partial_+\Big(\frac{v^2}{x}\Big)+\partial_-\Big(\frac{v^2}{x}\Big)\Big]. \tag{4.11}$$

By a direct computation, we have

$$\partial_+\left(\frac{v^2}{x}\right)+\partial_-\left(\frac{v^2}{x}\right)=\frac{2v}{x}(\partial_+ v+\partial_- v)-\frac{2uv^2}{x^2}=-\frac{6uv^2}{x^2} \tag{4.12}$$

and

$$\partial_+\left(\frac{cu}{x}\right)+\partial_-\left(\frac{cu}{x}\right)=\frac{u}{x}(\partial_+ c+\partial_- c)-\frac{2c}{(\gamma-1)x}(\partial_+ c-\partial_- c)-\frac{2cu^2}{x^2}+\frac{2cv^2}{x^2}. \tag{4.13}$$

Inserting (4.12) and (4.13) into (4.11), we get the first equation of (4.7). The second equation of (4.7) can be proved similarly. This completes the proof. ∎

It is convenient to write (4.7) in the form

$$\begin{cases}\partial_-\Big(\frac{\partial_+ c}{c}-\frac{(\gamma-1)v^2}{2cx}\Big) = \frac{1}{2\mu^2}\Big(\frac{\partial_+ c}{c}+\frac{\partial_- c}{c}\Big)\frac{\partial_+ c}{c}+\Big(\frac{3}{2}\frac{\partial_+ c}{c}+\frac{1}{2}\frac{\partial_- c}{c}\Big)\frac{u}{x} \\ \qquad +\frac{c}{2x}\Big(\frac{\partial_+ c}{c}-\frac{\partial_- c}{c}\Big)+\frac{(\gamma-1)u^2}{x^2}-\frac{(\gamma-1)v^2}{2x^2} \\ \qquad +\frac{3(\gamma-1)uv^2}{2cx^2}-\frac{\partial_+ c\partial_- c}{c^2}+\frac{(\gamma-1)v^2}{2cx}\frac{\partial_- c}{c}, \\ \partial_+\Big(\frac{\partial_- c}{c}+\frac{(\gamma-1)v^2}{2cx}\Big) = \frac{1}{2\mu^2}\Big(\frac{\partial_+ c}{c}+\frac{\partial_- c}{c}\Big)\frac{\partial_- c}{c}+\Big(\frac{3}{2}\frac{\partial_- c}{c}+\frac{1}{2}\frac{\partial_+ c}{c}\Big)\frac{u}{x} \\ \qquad +\frac{c}{2x}\Big(\frac{\partial_+ c}{c}-\frac{\partial_- c}{c}\Big)+\frac{(\gamma-1)u^2}{x^2}-\frac{(\gamma-1)v^2}{2x^2} \\ \qquad -\frac{3(\gamma-1)uv^2}{2cx^2}-\frac{\partial_+ c\partial_- c}{c^2}-\frac{(\gamma-1)v^2}{2cx}\frac{\partial_+ c}{c}.\end{cases} \tag{4.14}$$

We define

$$W_+=\frac{\partial_+ c}{c}-\frac{(\gamma-1)v^2}{2cx}+\frac{\mathcal{C}(\gamma-1)u}{x} \quad \text{and} \quad W_-=\frac{\partial_- c}{c}+\frac{(\gamma-1)v^2}{2cx}+\frac{\mathcal{C}(\gamma-1)u}{x}. \tag{4.15}$$

Then by (4.14) we have

$$\begin{cases}\partial_+ W_-=b_{11}W_-^2+b_{12}W_+W_-+b_{13}W_-+b_{14}W_++b_{15}, \\ \partial_- W_+=b_{21}W_+^2+b_{22}W_+W_-+b_{23}W_++b_{24}W_-+b_{25},\end{cases} \tag{4.16}$$

where

$$b_{14}=\left\{\frac{\gamma-3}{2}\mathcal{C}+\frac{1}{2}-(2\mathcal{C}-\frac{1}{2})\frac{c}{u}-\frac{(\gamma+1)v^2}{4cu}\right\}\frac{u}{x}, \tag{4.17}$$

$$b_{24}=\left\{\frac{\gamma-3}{2}\mathcal{C}+\frac{1}{2}+(2\mathcal{C}-\frac{1}{2})\frac{c}{u}+\frac{(\gamma+1)v^2}{4cu}\right\}\frac{u}{x}, \tag{4.18}$$

$$b_{15}=\left\{2\mathcal{C}^2-3\mathcal{C}+1-2\mathcal{C}(1-\mathcal{C})\frac{c}{u}+(\gamma\mathcal{C}-2)\frac{v^2}{cu}\right\}\frac{(\gamma-1)u^2}{x^2}, \tag{4.19}$$

and

$$b_{25}=\left\{2\mathcal{C}^2-3\mathcal{C}+1+2\mathcal{C}(1-\mathcal{C})\frac{c}{u}-(\gamma\mathcal{C}-2)\frac{v^2}{cu}\right\}\frac{(\gamma-1)u^2}{x^2}. \tag{4.20}$$

We define

$$\widehat{W}_+=\frac{\partial_+c}{c}-\frac{(\gamma-1)v^2}{2cx}+\frac{\mathcal{D}(\gamma-1)u}{x}\quad\text{and}\quad\widehat{W}_-=\frac{\partial_-c}{c}+\frac{(\gamma-1)v^2}{2cx}+\frac{(\gamma-1)u}{2x}. \tag{4.21}$$

Then by (4.14) we have

$$\begin{cases}\partial_+\widehat{W}_-=\hat{b}_{11}\widehat{W}_-^2+\hat{b}_{12}\widehat{W}_+\widehat{W}_-+\hat{b}_{13}\widehat{W}_-+\hat{b}_{14}\widehat{W}_++\hat{b}_{15},\\ \partial_-\widehat{W}_+=\hat{b}_{21}\widehat{W}_+^2+\hat{b}_{22}\widehat{W}_+\widehat{W}_-+\hat{b}_{23}\widehat{W}_++\hat{b}_{24}\widehat{W}_-+\hat{b}_{25},\end{cases} \tag{4.22}$$

where

$$\hat{b}_{14}=\left\{\frac{\gamma-1}{4}-\frac{c}{2u}-\frac{(\gamma+1)v^2}{4cu}\right\}\frac{u}{x}, \tag{4.23}$$

$$\hat{b}_{24}=\left\{\frac{\gamma-3}{2}\mathcal{D}+\frac{1}{2}+(2\mathcal{D}-\frac{1}{2})\frac{c}{u}+\frac{(\gamma+1)v^2}{4cu}\right\}\frac{u}{x}, \tag{4.24}$$

$$\hat{b}_{15}=\left\{\frac{1-\gamma}{4}\mathcal{D}+\frac{\gamma+1}{8}-\frac{1}{4}+(\frac{\mathcal{D}}{2}-\frac{3}{4})\frac{c}{u}+\Big(\frac{\gamma+1}{4}\mathcal{D}+\frac{3\gamma-17}{8}\Big)\frac{v^2}{cu}\right\}\frac{(\gamma-1)u^2}{x^2}, \tag{4.25}$$

and

$$\hat{b}_{25}=\left\{\frac{\gamma+1}{2}\mathcal{D}^2-\frac{\gamma+7}{4}\mathcal{D}+\frac{3}{4}+(\frac{\mathcal{D}}{2}+\frac{1}{4})\frac{c}{u}+\Big(\frac{1-3\gamma}{4}\mathcal{D}-\frac{\gamma-15}{8}\Big)\frac{v^2}{cu}\right\}\frac{(\gamma-1)u^2}{x^2}. \tag{4.26}$$

Remark 4.4. From $\frac{1}{2}<\mathcal{D}<\min\{\frac{3}{\gamma+1},\frac{1}{3-\gamma}\}$, (S0), and (S1), we have that if $0<c\le c_{_M}$, $u\ge u_v$, and $\frac{v^2}{c}\le\delta$ then $\hat{b}_{14}>0$, $\hat{b}_{24}>0$, and $\hat{b}_{25}<0$. Meanwhile, if $0<c\le c_{_M}$, $u\ge u_v$, and $\frac{v^2}{c}\le\delta$ then $\frac{1-\gamma}{4}+\frac{c}{2u}+\frac{(\gamma+1)v^2}{4cu}<0$. Thus $\hat{b}_{15}$ is decreasing with respect to $\mathcal{D}$. Inserting $\mathcal{D}=\frac{1}{2}$ into $\hat{b}_{15}$, we also have that if $0<c\le c_{_M}$, $u\ge u_v$, and $\frac{v^2}{c}\le\delta$ then $\hat{b}_{15}<0$.

4.3. Solution in domain Ω_3. Since $(u,v,\rho)(x,0)=(u_0,0,\rho_0)$ for $0<x\le\varepsilon$, the solution in the region Ω_1 is the self-similar solution $(u,v,\rho)=(\hat{u}(t/x),0,\hat{\rho}(t/x))$, where the function $(\hat{u},\hat{\rho})$ is the solution to the Riemann problem (2.3, 2.4). Moreover, from Lemma 3.1 we also have that the solution satisfies (3.26).

We first consider (1.6) with data

$$(u,v,c)(x,0)=(\bar{u},\bar{v},\bar{c})(x),\quad \varepsilon<x<r, \tag{4.27}$$

where $r>\varepsilon$ can be arbitrary.

Lemma 4.2. *Assume that the initial value problem (1.6, 4.27) admits a classical solution in some domain. Then the solution satisfies*

$$\widehat{W}_\pm<0,\quad W_\pm>0,\quad 0<c<c_{_M},\quad \textit{and}\quad u_v<u<\bar{u}(r)+1. \tag{4.28}$$

Proof. We shall prove this lemma by the method of continuity. The proof proceeds in two steps.

Step 1. From $c^2=\gamma e^{s_0}\rho^{\gamma-1}$ we have

$$\rho_t=\frac{2cc_t}{\gamma(\gamma-1)e^{s_0}\rho^{\gamma-2}}\quad\text{and}\quad\rho_x=\frac{2cc_x}{\gamma(\gamma-1)e^{s_0}\rho^{\gamma-2}}.$$

Inserting this into the first equation of (1.6) we get

$$c_t=-uc_x-\frac{\gamma-1}{2}cu_x-\frac{\gamma-1}{2}\frac{cu}{x}. \tag{4.29}$$

Hence,

$$\partial_\pm c=c_t+(u\pm c)c_x=\pm cc_x-\frac{\gamma-1}{2}cu_x-\frac{\gamma-1}{2}\cdot\frac{cu}{x}, \tag{4.30}$$

and consequently

$$\widehat{W}_+(x,0)=\bar{c}'(x)-\frac{\gamma-1}{2}\bar{u}'(x)+\big(\mathcal{D}-\frac{1}{2}\big)\frac{(\gamma-1)\bar{u}(x)}{x}-\frac{(\gamma-1)\bar{v}^2(x)}{2\bar{c}(x)x} \tag{4.31}$$

and

$$\widehat{W}_-(x,0)=-\bar{c}'(x)-\frac{\gamma-1}{2}\bar{u}'(x)+\frac{(\gamma-1)\bar{v}^2(x)}{2\bar{c}(x)x}. \tag{4.32}$$

Thus, by (S4) and (S5) we have

$$\widehat{W}_\pm(x,0)<0\quad\text{for}\quad\varepsilon<x<r. \tag{4.33}$$

Similarly, by (4.30) we have

$$W_\pm(x,0)=\pm\bar{c}'(x)-\frac{\gamma-1}{2}\bar{u}'(x)+\big(\mathcal{C}-\frac{1}{2}\big)\frac{(\gamma-1)\bar{u}(x)}{x}\mp\frac{(\gamma-1)\bar{v}^2(x)}{2\bar{c}(x)x}. \tag{4.34}$$

Thus, by (S6) we have

$$W_\pm(x,0)>0\quad\text{for}\quad\varepsilon<x<r. \tag{4.35}$$

Step 2. Let P be an arbitrary point in the domain. The backward C_+ and C_- characteristic curves issuing from P intersect the x-axis at some points P_+ and P_-, respectively. The backward trajectory line C_0 issuing from P intersects the x-axis at a point P_0. We denote by Ω_P a closed triangle domain closed by $\widehat{P_+P}$, $\widehat{P_-P}$, and $\overline{P_+P_-}$. We are going to prove that if the inequalities in (4.28) hold for all points in $\Omega_P\setminus\{P\}$, then they also hold at P. We shall prove this by the method of contradiction.

From $\widehat{W}_-<0$ in $\Omega_P\setminus\{P\}$, we immediately have $\partial_-c<0$ in $\Omega_P\setminus\{P\}$. Hence, we have $c<c_{_M}$ at P. From $W_\pm>0$ in $\Omega_P\setminus\{P\}$, we have $\frac{\partial_0 c}{c}>-\frac{\mathcal{C}(\gamma-1)u}{x}$ on $\Omega_P\setminus\{P\}$. Integrating this along $\widehat{P_0P}$ from P_0 to P, we have

$$c(P)>c(P_0)\Big(\frac{x(P)}{x(P_0)}\Big)^{-\mathcal{C}(\gamma-1)}>0. \tag{4.36}$$

From the third equation of (1.6) we have

$$\partial_0\ln v=-\partial_0\ln x. \tag{4.37}$$

Integrating this along $\widehat{P_0P}$ from P_0 to P, we get

$$v(P)=\frac{x(P_0)}{x(P)}v(P_0). \tag{4.38}$$

Combining (4.36) and (4.38) and recalling $\mathcal{C}=\frac{2}{\gamma-1}$ and $x(P)>x(P_0)$, we get

$$(\frac{v^2}{c})(P)<\Big(\frac{x(P_0)}{x(P)}\Big)^{2-\mathcal{C}(\gamma-1)}(\frac{v^2}{c})(P_0)<\delta. \tag{4.39}$$

In view of (4.5), we have

$$\partial_- u=\frac{2c\widehat{W}_-}{\gamma-1}<0 \quad \text{and} \quad \partial_+ u=-\frac{2c\widehat{W}_+}{\gamma-1}+(\mathcal{D}-\frac{1}{2})\frac{2uc}{x}>0 \tag{4.40}$$

in $\Omega_P\setminus\{P\}$. Thus we have

$$u_v<u(P_+)<u(P)<u(P_-)<\bar{u}(r)+1.$$

Suppose $\widehat{W}_-=0$ at P. Then according to that the inequalities in (4.28) hold in $\Omega_P\setminus\{P\}$, we have $\partial_+\widehat{W}_-\geq 0$ at P. While, by the first equation of (4.22) and Remark 4.4, we have

$$\partial_+\widehat{W}_-=\hat{b}_{14}\widehat{W}_++\hat{b}_{15}\ <\ 0 \tag{4.41}$$

at P. This leads to a contradiction. We then get $\widehat{W}_-(P)<0$. Similarly, we have $\widehat{W}_+(P)<0$.

Suppose $W_-=0$ at P. Then according to that the inequalities in (4.28) hold in $\Omega_P\setminus\{P\}$, we have $\partial_+W_-\leq 0$ at P. While, using (S2), (S3), $\frac{\partial_- c}{c}<0$, and the first equation of (4.16), we have

$$\begin{cases} \partial_+W_->b_{14}\dfrac{\mathcal{C}(\gamma-1)u}{x}+b_{15} & \\ \qquad >\Big\{\dfrac{\gamma+1}{2}\mathcal{C}^2-\Big(\dfrac{5}{2}+\dfrac{3}{2}\dfrac{c_{_M}}{u_v}\Big)\mathcal{C}+1-\dfrac{2\delta}{u_v}\Big\}\dfrac{(\gamma-1)u^2}{x^2}>0 & \text{if } b_{14}<0; \\ \partial_+W_->b_{15}>\Big\{2\mathcal{C}^2-\big(3+2\dfrac{c_{_M}}{u_v}\big)\mathcal{C}+1-\dfrac{|\gamma\mathcal{C}-2|\delta}{u_v}\Big\}\dfrac{(\gamma-1)u^2}{x^2}>0 & \text{if } b_{14}\geq 0. \end{cases} \tag{4.42}$$

This leads to a contradiction. We then have $W_-(P)>0$. Similarly, we have $W_+(P)>0$.

We then prove that if the inequalities in (4.28) hold in $\Omega_P\setminus\{P\}$ then they also hold at P. Therefore, by an argument of continuity we complete the proof of the lemma. □

From Lemma 4.2 we actually have that the solution satisfies

$$\widehat{W}_\pm<0, \quad W_\pm>0, \quad 0<c<c_{_M}, \quad \frac{v^2}{c}\leq\delta, \quad \text{and} \quad u>u_v. \tag{4.43}$$

This gives a C^0 norm estimate for (u,v,c) and a gradient estimate for c. The gradient estimate for u can be obtained by (4.5). In order to estimate $|\nabla v|$, we use the commutator relation

$$\partial_0\partial_x v-\partial_x\partial_0 v=-\partial_x u\partial_x v.$$

Inserting the third equation of (1.6) into this, we get

$$\partial_0(\partial_x v)+\Big(\frac{u}{x}+\partial_x u\Big)\partial_x v=\frac{uv}{x^2}-\frac{v\partial_x u}{x}. \tag{4.44}$$

Thus, the value of $\partial_x v$ can be obtained by integrating (4.44) along C_0 characteristic lines. The gradient estimate for v can be obtained by $\partial_x v$ and the third equation of (1.6). We then establish an a priori C^1 estimate for the solution. Thus, the existence of a global classical solution can be obtained by the classical extension method (cf. Li [28]). We then have the following global existence.

LEMMA 4.3. *The Cauchy problem (1.6, 4.27) admits a global classical solution. Moreover, the solution satisfies (4.43).*

Since $r>\varepsilon$ can be arbitrary, we obtain the solution in a domain Ω_3 bounded by C_+^Q and the x-axis. Moreover the solution satisfies (4.43).

4.4. Solution in domain Ω_2. Take any points Q_+ and Q_- on C_+^Q and C_-^Q, respectively. We now consider (1.6) with data

$$(u,v,c)=\begin{cases}(\hat u,\hat v,\hat c)(t/x) \text{ on } \widehat{QQ_-};\\ (\tilde u,\tilde v,\tilde c)(x,t) \text{ on } \widehat{QQ_+};\end{cases} \tag{4.45}$$

where $\hat v=0$ and $(\tilde u,\tilde v,\tilde c)(x,t)$ denotes the solution in Ω_3.

Problem (1.6, 4.45) is a Goursat problem. In order to obtain a global solution, we need to establish an a priori C^1 estimate of the solution.

LEMMA 4.4. *Assume that the Goursat problem (1.6, 4.45) admits a classical solution in some domain. Then the solution satisfies*

$$\widehat{W}_\pm<0,\quad W_\pm>0,\quad 0<c<c_{_M},\quad \textit{and}\quad u_v<u<u(Q_+)+1. \tag{4.46}$$

Proof. The proof of this lemma proceeds in two steps.

Step 1. We first prove that the inequalities in (4.46) hold on $\widehat{QQ_+}\cup\widehat{QQ_-}$.

By (4.28) we know that $W_+>0$, $\widehat{W}_+<0$, $\frac{v^2}{c}\le\delta$, and $0<c<c_{_M}$ on $\widehat{QQ_+}$. By (4.40) we know that $u_v<u<u(Q_+)+1$ on $\widehat{QQ_+}$. By (3.26) we have $\widehat{W}_-<0$ and $W_->0$ on $\widehat{QQ_-}$.

We now prove $\widehat{W}_+|_{\widehat{QQ_-}}<0$ and $\widehat{W}_-|_{\widehat{QQ_+}}<0$. Suppose that there exists a "first" point Q_1 on $\widehat{QQ_+}$ such that $\widehat{W}_-(Q_1)=0$ and $\widehat{W}_-<0$ on $\widehat{QQ_1}$. Then we have $\partial_+\widehat{W}_-\ge 0$ at Q_1. While, as in (4.41), we have $\partial_+\widehat{W}_-<0$ at Q_1, which leads to a contradiction. Thus we have $\widehat{W}_-|_{\widehat{QQ_+}}<0$. Similarly, we have $\widehat{W}_+|_{\widehat{QQ_-}}<0$.

We next prove $W_+|_{\widehat{QQ_-}}>0$ and $W_-|_{\widehat{QQ_+}}>0$. Suppose that there exists a "first" point Q_1 on $\widehat{QQ_+}$ such that $W_-(Q_1)=0$ and $W_->0$ on $\widehat{QQ_1}$. Then we have $\partial_+W_-\le 0$ at Q_1. While, as in (4.42), we have $\partial_+W_->0$ at Q_1, which leads to a contradiction. Thus we have $W_-|_{\widehat{QQ_+}}>0$. Similarly, we have $W_+|_{\widehat{QQ_-}}>0$.

Step 2. Let P be an arbitrary point in the domain. The backward C_+ and C_- characteristic curves issuing from P intersect $\widehat{QQ_-}$ and $\widehat{QQ_+}$ at some points P_+ and P_-, respectively. The backward trajectory line issuing from P intersects $\widehat{QQ_+}\cup\widehat{QQ_-}$ at some point P_0. We denote by Ω_P a closed domain closed by $\widehat{P_+P}$, $\widehat{P_-P}$, $\overline{QP_-}$, and

$\overline{QP_+}$. Then, as in the proof of Lemma 4.2, one has that if the inequalities in (4.46) hold for every point in $\Omega_P\setminus\{P\}$ then they also hold at P.

Therefore, by an argument of continuity we complete the proof of the lemma. □

LEMMA 4.5. *The Goursat problem (1.6, 4.45) admits a global classical solution.*

Proof. This lemma can be proved by Lemma 4.4 and the classical extension method. We omit the details. □

Since $Q_\pm$ can be arbitrary, we obtain the solution in a triangle domain Ω_2 bounded by C_+^Q and C_-^Q. We then finish the proof of Theorem 4.1.

5. Axisymmetric non-isentropic Euler equations with swirl

5.1. Problem and main result. We now consider the problem (1.4, 1.7). Let $\gamma_*\in(1,2)$ be a constant such that $\frac{7-\gamma_*}{2(\gamma_*+1)}=\frac{1}{3-\gamma_*}$. Then we have

$$\frac{1}{2}<\frac{7-\gamma}{2(\gamma+1)}<\min\left\{\frac{3}{\gamma+1},\frac{1}{3-\gamma}\right\}\quad\text{for}\quad\gamma_*<\gamma<3. \tag{5.1}$$

The main result is stated as the following theorem.

THEOREM 5.1. *Let*

$$\delta_1=\sup_{x\in[\varepsilon,+\infty)}\frac{\bar{v}^2(x)}{\bar{c}(x)},\quad \delta_2=\sup_{x\in[\varepsilon,+\infty)}|x\bar{c}\bar{s}'|,\quad \textit{and}\quad c_{M}=\sup_{x\in[\varepsilon,+\infty)}\bar{c}(x).$$

Assume $\gamma_*<\gamma<5-2\sqrt{2}$ *and there exists a constant* $\mathcal{K}\in\left(1,\frac{2}{\gamma-1}\right)$ *such that*

$$|\bar{c}'(x)|+\frac{(\gamma-1)\bar{u}(x)}{2x}<\frac{\gamma-1}{2}\bar{u}'(x)<-|\bar{c}'(x)|+\left(\mathcal{K}-\frac{1}{2}\right)\frac{(\gamma-1)\bar{u}(x)}{x} \tag{5.2}$$

for $x\in[\varepsilon,+\infty)$. *Then when* δ_1, δ_2, *and* c_M *are sufficiently small the problem (1.4, 1.7) admits a global-in-time classical solution.*

REMARK 5.1. Let $\mathcal{F}\in\left(\frac{7-\gamma}{2(\gamma+1)},\min\{\frac{3}{\gamma+1},\frac{1}{3-\gamma}\}\right)$ be a constant. It is easy to check that when δ_1 and δ_2 and c_M are sufficiently small the following properties hold:

(H0) $\frac{c_M}{u_v}<\frac{\gamma-1}{2}-\frac{(\gamma+1)\delta_1}{2u_v}-\frac{\delta_2}{2\gamma u_v}$;

(H1) $\frac{1-\gamma}{4}\mathcal{F}+\frac{\gamma+1}{8}-\frac{1}{4}+\frac{(2\mathcal{F}-1)\delta_2}{8\gamma u_v}<0$;

(H2) $\frac{\gamma-3}{2}\mathcal{F}+\frac{1}{2}-\frac{\delta_2}{4\gamma u_v}>0$;

(H3) $\frac{\gamma+1}{2}\mathcal{F}^2-\frac{\gamma+7}{4}\mathcal{F}+\frac{3}{4}+\left(\frac{\mathcal{F}}{2}+\frac{1}{4}\right)\frac{c_M}{u_v}+\left|\frac{1-3\gamma}{4}\mathcal{F}-\frac{\gamma-15}{8}\right|\frac{\delta_1}{u_v}+\frac{(2\mathcal{F}-1)\delta_2}{8\gamma u_v}<0$;

(H4) $\frac{\gamma+1}{2}\mathcal{C}^2-\left(\frac{5}{2}+\frac{3}{2}\frac{c_M}{u_v}+\frac{\gamma\delta_1}{u_v}+\frac{\delta_2}{4\gamma u_v}\right)\mathcal{C}+1-\frac{2\delta_1}{u_v}>0$;

(H5) $2\left(1-\frac{c_M}{u_v}\right)\mathcal{C}^2-\left(3+2\frac{c_M}{u_v}\right)\mathcal{C}+1-\frac{|\gamma\mathcal{C}-2|\delta_1}{u_v}>0$;

(H6) $|\bar{c}'(x)|-\frac{\gamma-1}{2}\bar{u}'(x)+(\mathcal{F}-\frac{1}{2})\frac{(\gamma-1)\bar{u}(x)}{x}-\frac{(\gamma-1)\bar{v}^2(x)}{2\bar{c}(x)x}+\frac{\bar{c}(x)|\bar{s}'(x)|}{2\gamma}<0$ for $x\in[\varepsilon,+\infty)$;

(H7) $|\bar{c}'(x)|-\frac{\gamma-1}{2}\bar{u}'(x)+\frac{(\gamma-1)\bar{v}^2(x)}{2\bar{c}(x)x}+\frac{\bar{c}(x)|\bar{s}'(x)|}{2\gamma}<0$ for $x\in[\varepsilon,+\infty)$;

(H8) $-|\bar{c}'(x)|-\frac{\gamma-1}{2}\bar{u}'(x)-\frac{(\gamma-1)\bar{v}^2(x)}{2\bar{c}(x)x}-\frac{\bar{c}(x)|\bar{s}'(x)|}{2\gamma}+(\mathcal{C}-\frac{1}{2})\frac{(\gamma-1)\bar{u}(x)}{x}>0$ for $x\in[\varepsilon,+\infty)$,

where $\mathcal{C}=\frac{2}{\gamma-1}$.

5.2. Characteristic equations and decompositions for (1.4). For smooth flows, system (1.4) can be simplified to

$$\begin{cases} \rho_t + (\rho u)_x + \dfrac{\rho u}{x} = 0, \\ u_t + u u_x + \dfrac{p_x}{\rho} - \dfrac{v^2}{x} = 0, \\ v_t + u v_x + \dfrac{uv}{x} = 0, \\ s_t + u s_x = 0. \end{cases} \tag{5.3}$$

The eigenvalues of the system (5.3) are

$$\lambda_+ = u + c, \quad \lambda_0 = u, \quad \lambda_- = u - c.$$

The left eigenvectors corresponding to $\lambda_\pm$ are $l_\pm = (c, \pm\rho, 0, \sqrt{\frac{e^s}{\gamma}}\rho^{\frac{\gamma+1}{2}})$. Multiplying (5.3) on the left by $l_\pm$ we get the characteristic equations

$$c\partial_\pm \rho \pm \rho \partial_\pm u \pm \rho^\gamma e^s s_x = -\frac{\rho c u}{x} \pm \frac{\rho v^2}{x}, \tag{5.4}$$

where

$$\partial_\pm = \partial_t + (u \pm c)\partial_x.$$

From $c^2 = \gamma e^s \rho^{\gamma-1}$ we have

$$\partial_\pm \rho = \frac{2c\partial_\pm c - \gamma \rho^{\gamma-1}\partial_\pm e^s}{\gamma(\gamma-1)e^s \rho^{\gamma-2}}.$$

Combining this with $\partial_\pm s = \pm c s_x$, we have

$$c\partial_\pm \rho = \frac{2\rho \partial_\pm c}{\gamma-1} \mp \frac{\gamma \rho^\gamma e^s}{\gamma-1} s_x.$$

Inserting this into (5.4), we get

$$\begin{cases} \partial_+ u = -\dfrac{2}{\gamma-1}\partial_+ c + \dfrac{c^2}{\gamma(\gamma-1)} s_x - \dfrac{uc}{x} + \dfrac{v^2}{x}, \\ \partial_- u = \dfrac{2}{\gamma-1}\partial_- c + \dfrac{c^2}{\gamma(\gamma-1)} s_x + \dfrac{uc}{x} + \dfrac{v^2}{x}. \end{cases} \tag{5.5}$$

Lemma 5.1. *For smooth flows, we have commutator relation*

$$\partial_+\partial_- - \partial_-\partial_+ = \left(-\frac{1}{2c\mu^2}(\partial_+ c + \partial_- c) - \frac{u}{x}\right)(\partial_+ - \partial_-). \tag{5.6}$$

Proof. The proof is similar to that for Lemma 3.1, we omit the details. □

Lemma 5.2. *For the system (1.4), we have*

$$\partial_0 \left(\frac{s_x}{c^{\frac{2}{\gamma-1}}}\right) = \frac{u}{x} \cdot \frac{s_x}{c^{\frac{2}{\gamma-1}}} \tag{5.7}$$

where $\partial_0 = \partial_t + u\partial_x$.

Proof. By (5.5), we have

$$\begin{aligned}\partial_0\partial_x s &= (\partial_0\partial_x - \partial_x\partial_0)s = (\partial_t + u\partial_x)\partial_x s - \partial_x(\partial_t + u\partial_x)s \\ &= -\partial_x u\partial_x s = \frac{(\partial_+ c + \partial_- c)s_x}{c(\gamma-1)} + \frac{u}{x}s_x. \end{aligned} \tag{5.8}$$

Thus, by $\partial_0 = \frac{\partial_+ + \partial_-}{2}$ we have

$$\partial_0\left(\frac{s_x}{c^{\frac{2}{\gamma-1}}}\right) = \frac{\partial_0\partial_x s}{c^{\frac{2}{\gamma-1}}} - \frac{2}{\gamma-1}\frac{s_x}{c^{\frac{\gamma+1}{\gamma-1}}}\partial_0 c \quad = \frac{u}{x}\cdot\frac{s_x}{c^{\frac{2}{\gamma-1}}}. \tag{5.9}$$

This completes the proof. ∎

PROPOSITION 5.1. *We have the characteristic decompositions*

$$\left\{\begin{aligned} c\partial_-\Big(\partial_+ c - \frac{(\gamma-1)v^2}{2x} - \frac{c^2}{2\gamma}s_x\Big) &= \frac{1}{2\mu^2}(\partial_+ c + \partial_- c)\partial_+ c + (\frac{3}{2}\partial_+ c + \frac{1}{2}\partial_- c)\frac{uc}{x} \\ &\quad + \frac{c^2}{2x}(\partial_+ c - \partial_- c) + \frac{(\gamma-1)u^2c^2}{x^2} - \frac{(\gamma-1)c^2v^2}{2x^2} \\ &\quad + \frac{3(\gamma-1)cuv^2}{2x^2} - \frac{c^4 s_x}{2\gamma x} - \frac{c^3 u s_x}{2\gamma x} \\ &\quad - \frac{c^2(\partial_+ c + \partial_- c)s_x}{2(\gamma-1)}, \\ c\partial_+\Big(\partial_- c + \frac{(\gamma-1)v^2}{2x} + \frac{c^2}{2\gamma}s_x\Big) &= \frac{1}{2\mu^2}(\partial_+ c + \partial_- c)\partial_- c + (\frac{3}{2}\partial_- c + \frac{1}{2}\partial_+ c)\frac{uc}{x} \\ &\quad + \frac{c^2}{2x}(\partial_+ c - \partial_- c) + \frac{(\gamma-1)u^2c^2}{x^2} - \frac{(\gamma-1)c^2v^2}{2x^2} \\ &\quad - \frac{3(\gamma-1)cuv^2}{2x^2} - \frac{c^4 s_x}{2\gamma x} + \frac{c^3 u s_x}{2\gamma x} + \frac{c^2(\partial_+ c + \partial_- c)s_x}{2(\gamma-1)}. \end{aligned}\right. \tag{5.10}$$

Proof. Using the commutator relation (5.6) for the variable c, we have

$$\partial_+\partial_- c - \partial_-\partial_+ c = \left(-\frac{1}{2c\mu^2}(\partial_+ c + \partial_- c) - \frac{u}{x}\right)(\partial_+ c - \partial_- c). \tag{5.11}$$

Using the commutator relation (5.6) for the variable u, we have

$$\partial_+\partial_- u - \partial_-\partial_+ u = \left(-\frac{1}{2c\mu^2}(\partial_+ c + \partial_- c) - \frac{u}{x}\right)(\partial_+ u - \partial_- u). \tag{5.12}$$

Inserting (5.5) into this, we get

$$\begin{aligned} c\partial_+\partial_- c + c\partial_-\partial_+ c &= \frac{\gamma+1}{2(\gamma-1)}(\partial_+ c + \partial_- c)^2 + \frac{2cu}{x}(\partial_+ c + \partial_- c) + \frac{c^2}{x}(\partial_+ c - \partial_- c) \\ &\quad - \frac{(\gamma-1)c}{2}\partial_+\left(\frac{c^2 s_x}{\gamma(\gamma-1)}\right) + \frac{(\gamma-1)c}{2}\partial_-\left(\frac{c^2 s_x}{\gamma(\gamma-1)}\right) \\ &\quad - \frac{c^4}{\gamma x}s_x + \frac{2(\gamma-1)c^2u^2}{x^2} - \frac{(\gamma-1)c^2v^2}{x^2} \\ &\quad - \frac{(\gamma-1)c}{2}\partial_+\left(\frac{v^2}{x}\right) + \frac{(\gamma-1)c}{2}\partial_-\left(\frac{v^2}{x}\right). \end{aligned} \tag{5.13}$$

Combining (5.11) and (5.13), we get

$$
\begin{aligned}
&c\partial_-\Big(\partial_+c-\frac{(\gamma-1)v^2}{2x}-\frac{c^2}{2\gamma}s_x\Big)\\
=&\frac{\gamma+1}{2(\gamma-1)}(\partial_+c+\partial_-c)\partial_+c+(\frac{3}{2}\partial_+c+\frac{1}{2}\partial_-c)\frac{uc}{x}+\frac{c^2}{2x}(\partial_+c-\partial_-c)+\frac{(\gamma-1)u^2c^2}{x^2}\\
&-\frac{(\gamma-1)c^2v^2}{2x^2}-\frac{(\gamma-1)c}{4}\left[\partial_+\left(\frac{c^2s_x}{\gamma(\gamma-1)}\right)+\partial_-\left(\frac{c^2s_x}{\gamma(\gamma-1)}\right)\right]\\
&-\frac{(\gamma-1)c}{4}\left[\partial_+\left(\frac{v^2}{x}\right)+\partial_-\left(\frac{v^2}{x}\right)\right]-\frac{c^4}{2\gamma x}s_x,
\end{aligned}\tag{5.14}
$$

By (5.8), we have

$$
\partial_+(\frac{v^2}{x})+\partial_-(\frac{v^2}{x})=\frac{2v}{x}(\partial_+v+\partial_-v)-\frac{2uv^2}{x^2}=-\frac{6uv^2}{x^2}\tag{5.15}
$$

and

$$
\partial_+\left(\frac{c^2s_x}{\gamma(\gamma-1)}\right)+\partial_-\left(\frac{c^2s_x}{\gamma(\gamma-1)}\right)=\frac{2c}{(\gamma-1)^2}(\partial_+c+\partial_-c)s_x+\frac{2c^2}{\gamma(\gamma-1)}\frac{u}{x}s_x.\tag{5.16}
$$

Inserting (5.15) and (5.16) into (5.14) we get the first equation of (5.10). The second equation of (5.10) can be proved similarly. This completes the proof. ∎

It is convenient to write (5.10) in the form

$$
\begin{cases}
\partial_-\Big(\frac{\partial_+c}{c}-\frac{(\gamma-1)v^2}{2cx}-\frac{c}{2\gamma}s_x\Big)=\frac{1}{2\mu^2}\Big(\frac{\partial_+c}{c}+\frac{\partial_-c}{c}\Big)\frac{\partial_+c}{c}+\Big(\frac{3}{2}\frac{\partial_+c}{c}+\frac{1}{2}\frac{\partial_-c}{c}\Big)\frac{u}{x}\\
\qquad+\frac{c}{2x}\Big(\frac{\partial_+c}{c}-\frac{\partial_-c}{c}\Big)+\frac{(\gamma-1)u^2}{x^2}-\frac{(\gamma-1)v^2}{2x^2}\\
\qquad+\frac{3(\gamma-1)uv^2}{2cx^2}-\frac{\partial_+c\partial_-c}{c^2}+\frac{(\gamma-1)v^2}{2cx}\frac{\partial_-c}{c}-\frac{c^2}{2\gamma x}s_x\\
\qquad-\frac{c}{2\gamma}\frac{u}{x}s_x+\frac{cs_x}{2\gamma}\frac{\partial_-c}{c}-\frac{c}{2(\gamma-1)}\Big(\frac{\partial_+c}{c}+\frac{\partial_-c}{c}\Big)s_x,\\
\partial_+\Big(\frac{\partial_-c}{c}+\frac{(\gamma-1)v^2}{2cx}+\frac{c}{2\gamma}s_x\Big)=\frac{1}{2\mu^2}\Big(\frac{\partial_+c}{c}+\frac{\partial_-c}{c}\Big)\frac{\partial_-c}{c}+\Big(\frac{3}{2}\frac{\partial_-c}{c}+\frac{1}{2}\frac{\partial_+c}{c}\Big)\frac{u}{x}\\
\qquad+\frac{c}{2x}\Big(\frac{\partial_+c}{c}-\frac{\partial_-c}{c}\Big)+\frac{(\gamma-1)u^2}{x^2}-\frac{(\gamma-1)v^2}{2x^2}\\
\qquad-\frac{3(\gamma-1)uv^2}{2cx^2}-\frac{\partial_+c\partial_-c}{c^2}-\frac{(\gamma-1)v^2}{2cx}\frac{\partial_+c}{c}-\frac{c^2}{2\gamma x}s_x\\
\qquad+\frac{c}{2\gamma}\frac{u}{x}s_x-\frac{cs_x}{2\gamma}\frac{\partial_+c}{c}+\frac{c}{2(\gamma-1)}\Big(\frac{\partial_+c}{c}+\frac{\partial_-c}{c}\Big)s_x.
\end{cases}\tag{5.17}
$$

We define

$$
Z_+=\frac{\partial_+c}{c}-\frac{(\gamma-1)v^2}{2cx}-\frac{cs_x}{2\gamma}+\frac{\mathcal{C}(\gamma-1)u}{x},\quad Z_-=\frac{\partial_-c}{c}+\frac{(\gamma-1)v^2}{2cx}+\frac{cs_x}{2\gamma}+\frac{\mathcal{C}(\gamma-1)u}{x}.\tag{5.18}
$$

Then by (5.17) we have

$$
\begin{cases}
\partial_+Z_-=d_{11}Z_-^2+d_{12}Z_+Z_-+d_{13}Z_-+d_{14}Z_++d_{15},\\
\partial_-Z_+=d_{21}Z_+^2+d_{22}Z_+Z_-+d_{23}Z_++d_{24}Z_-+d_{25},
\end{cases}\tag{5.19}
$$

where

$$d_{14}=\Big\{\frac{\gamma-3}{2}\mathcal{C}+\frac{1}{2}-(2\mathcal{C}-\frac{1}{2})\frac{c}{u}-\frac{(\gamma+1)v^2}{4cu}+\frac{xs_x}{4\gamma}\cdot\frac{c}{u}\Big\}\frac{u}{x}, \tag{5.20}$$

$$d_{24}=\Big\{\frac{\gamma-3}{2}\mathcal{C}+\frac{1}{2}+(2\mathcal{C}-\frac{1}{2})\frac{c}{u}+\frac{(\gamma+1)v^2}{4cu}-\frac{xs_x}{4\gamma}\cdot\frac{c}{u}\Big\}\frac{u}{x}, \tag{5.21}$$

$$d_{15}=\Big\{2\mathcal{C}^2-3\mathcal{C}+1-2\mathcal{C}(1-\mathcal{C})\frac{c}{u}+(\gamma\mathcal{C}-2)\frac{v^2}{cu}\Big\}\frac{(\gamma-1)u^2}{x^2}, \tag{5.22}$$

and

$$d_{25}=\Big\{2\mathcal{C}^2-3\mathcal{C}+1+2\mathcal{C}(1-\mathcal{C})\frac{c}{u}-(\gamma\mathcal{C}-2)\frac{v^2}{cu}\Big\}\frac{(\gamma-1)u^2}{x^2}. \tag{5.23}$$

We define

$$\widehat{Z}_+=\frac{\partial_+c}{c}-\frac{(\gamma-1)v^2}{2cx}-\frac{c}{2\gamma}s_x+\frac{\mathcal{F}(\gamma-1)u}{x},\quad \widehat{Z}_-=\frac{\partial_-c}{c}+\frac{(\gamma-1)v^2}{2cx}+\frac{c}{2\gamma}s_x+\frac{(\gamma-1)u}{2x}. \tag{5.24}$$

Then by (5.17) we have

$$\begin{cases}\partial_+\widehat{Z}_-=\hat{d}_{11}\widehat{Z}_-^2+\hat{d}_{12}\widehat{Z}_+\widehat{Z}_-+\hat{d}_{13}\widehat{Z}_-+\hat{d}_{14}\widehat{Z}_++\hat{d}_{15},\\ \partial_-\widehat{Z}_+=\hat{d}_{21}\widehat{Z}_+^2+\hat{d}_{22}\widehat{Z}_+\widehat{Z}_-+\hat{d}_{23}\widehat{Z}_++\hat{d}_{24}\widehat{Z}_-+\hat{d}_{25},\end{cases} \tag{5.25}$$

where

$$\hat{d}_{14}=\Big\{\frac{\gamma-1}{4}-\frac{c}{2u}-\frac{(\gamma+1)v^2}{4cu}+\frac{xs_x}{4\gamma}\cdot\frac{c}{u}\Big\}\frac{u}{x}, \tag{5.26}$$

$$\hat{d}_{24}=\Big\{\frac{\gamma-3}{2}\mathcal{F}+\frac{1}{2}+(2\mathcal{F}-\frac{1}{2})\frac{c}{u}+\frac{(\gamma+1)v^2}{4cu}-\frac{xs_x}{4\gamma}\cdot\frac{c}{u}\Big\}\frac{u}{x}, \tag{5.27}$$

$$\begin{aligned}\hat{d}_{15}=&\Big\{\frac{1-\gamma}{4}\mathcal{F}+\frac{\gamma+1}{8}-\frac{1}{4}+(\frac{\mathcal{F}}{2}-\frac{3}{4})\frac{c}{u}+\Big(\frac{\gamma+1}{4}\mathcal{F}+\frac{3\gamma-17}{8}\Big)\frac{v^2}{cu}\Big\}\frac{(\gamma-1)u^2}{x^2}\\ &+\Big[\frac{(1-2\mathcal{F})xs_x}{8\gamma}\cdot\frac{c}{u}\Big]\frac{(\gamma-1)u^2}{x^2},\end{aligned} \tag{5.28}$$

and

$$\begin{aligned}\hat{d}_{25}=&\Big\{\frac{\gamma+1}{2}\mathcal{F}^2-\frac{\gamma+7}{4}\mathcal{F}+\frac{3}{4}+(\frac{\mathcal{F}}{2}+\frac{1}{4})\frac{c}{u}+\Big(\frac{1-3\gamma}{4}\mathcal{F}-\frac{\gamma-15}{8}\Big)\frac{v^2}{cu}\Big\}\frac{(\gamma-1)u^2}{x^2}\\ &+\Big[\frac{(1-2\mathcal{F})xs_x}{8\gamma}\cdot\frac{c}{u}\Big]\frac{(\gamma-1)u^2}{x^2}.\end{aligned} \tag{5.29}$$

REMARK 5.2. From $\frac{7-\gamma}{2(\gamma+1)}<\mathcal{F}<\min\{\frac{3}{\gamma+1},\frac{1}{3-\gamma}\}$ and (H0)–(H3), we know that if $0<c<c_M$, $u>u_v$, $\frac{v^2}{c}\le\delta_1$, and $|xcs_x|\le\delta_2$ then $\hat{d}_{14}>0$, $\hat{d}_{24}>0$, $\hat{d}_{15}<0$, and $\hat{d}_{25}<0$.

5.3. Solution in domain Ω_3. Since $(u,v,\rho,s)(x,0)=(u_0,0,\rho_0,s_0)$ for $0<x\le\varepsilon$, the solution in the region Ω_1 is the self-similar solution $(u,v,\rho,s)=(\hat{u}(t/x),0,\hat{\rho}(t/x),s_0)$. Moreover, the solution satisfies (3.26).

We first consider (1.4) with data

$$(u,v,c,s)(x,0)=(\bar{u},\bar{v},\bar{c},\bar{s})(x),\quad \varepsilon<x<r, \tag{5.30}$$

where $r>\varepsilon$ can be arbitrary.

LEMMA 5.3. *Assume that the initial value problem (1.4), (5.30) admits a classical solution in some domain. Then the solution satisfies*

$$\widehat{Z}_\pm<0,\quad Z_\pm>0,\quad 0<c<c_{_M},\quad \textit{and}\quad u_v<u<\bar{u}(r)+1. \tag{5.31}$$

Proof. We shall prove this lemma by the method of continuity. The proof proceeds in two steps.

Step 1. From $c^2=\gamma e^s\rho^{\gamma-1}$ we have

$$\rho_t=\frac{2cc_t-\gamma e^s\rho^{\gamma-1}s_t}{\gamma(\gamma-1)e^s\rho^{\gamma-2}}\quad\text{and}\quad \rho_x=\frac{2cc_x-\gamma e^s\rho^{\gamma-1}s_x}{\gamma(\gamma-1)e^s\rho^{\gamma-2}}.$$

Inserting this into the first equation of (1.4) and using the fourth equation of (1.4), we get

$$c_t=-uc_x-\frac{\gamma-1}{2}cu_x-\frac{\gamma-1}{2}\frac{cu}{x}. \tag{5.32}$$

Hence, we have

$$\partial_\pm c=c_t+(u\pm c)c_x=\pm cc_x-\frac{\gamma-1}{2}cu_x-\Big(\frac{\gamma-1}{2}\Big)\frac{cu}{x}. \tag{5.33}$$

Consequently, we get

$$\widehat{Z}_+(x,0)=\bar{c}'(x)-\frac{\gamma-1}{2}\bar{u}'(x)+\big(\mathcal{F}-\frac{1}{2}\big)\frac{(\gamma-1)\bar{u}(x)}{x}-\frac{(\gamma-1)\bar{v}^2(x)}{2\bar{c}(x)x}-\frac{\bar{c}(x)\bar{s}'(x)}{2\gamma} \tag{5.34}$$

and

$$\widehat{Z}_-(x,0)=-\bar{c}'(x)-\frac{\gamma-1}{2}\bar{u}'(x)+\frac{(\gamma-1)\bar{v}^2(x)}{2\bar{c}(x)x}+\frac{\bar{c}(x)\bar{s}'(x)}{2\gamma}. \tag{5.35}$$

Thus, by (H6) and (H7) we have

$$\widehat{Z}_\pm(x,0)<0\quad\text{for}\ \ \varepsilon<x<r. \tag{5.36}$$

Similarly, by (5.33) we have

$$Z_\pm(x,0)=\pm\bar{c}'(x)-\frac{\gamma-1}{2}\bar{u}'(x)+\big(\mathcal{C}-\frac{1}{2}\big)\frac{(\gamma-1)\bar{u}(x)}{x}\mp\frac{(\gamma-1)\bar{v}^2(x)}{2\bar{c}(x)x}\mp\frac{\bar{c}(x)\bar{s}'(x)}{2\gamma}. \tag{5.37}$$

Thus, by (H8) we have

$$Z_\pm(x,0)>0\quad\text{for}\ \ \varepsilon<x<r. \tag{5.38}$$

Step 2. Let P be an arbitrary point in the domain. The backward C_+ and C_- characteristic curves issuing from P intersect the x-axis at some points P_+ and P_-, respectively. The backward trajectory line issuing from P intersects the x-axis at some point P_0. We denote by Ω_P a closed triangle domain closed by $\widehat{P_+P}$, $\widehat{P_-P}$, and $\overline{P_+P_-}$. We are going to prove that if the inequalities in (5.31) hold for all points in $\Omega_P\setminus\{P\}$, then they also hold at P. We shall prove this by the method of contradiction.

From (5.7) we have

$$\partial_0 \ln\left(\frac{s_x}{c^{\frac{2}{\gamma-1}}}\right)=\partial_0 \ln x. \tag{5.39}$$

Integrating this along $\widehat{P_0P}$ from P_0 to P, we get

$$\frac{s_x(P)}{c^{\frac{2}{\gamma-1}}(P)}=\frac{s_x(P_0)}{c^{\frac{2}{\gamma-1}}(P_0)}\cdot\frac{x(P)}{x(P_0)}. \tag{5.40}$$

Since $\widehat{Z}_\pm<0$ in $\Omega_P\setminus\{P\}$, we have $\frac{\partial_0 c}{c}<-\big(\frac{\mathcal{F}}{2}+\frac{1}{4}\big)\frac{(\gamma-1)u}{x}$ in $\Omega_P\setminus\{P\}$. Integrating this along $\widehat{P_0P}$ from P_0 to P, we have

$$c(P)<c(P_0)\Big(\frac{x(P)}{x(P_0)}\Big)^{-(\frac{\mathcal{F}}{2}+\frac{1}{4})(\gamma-1)}>0. \tag{5.41}$$

From $\mathcal{F}>\frac{7-\gamma}{2(\gamma+1)}$ we have $2-(\frac{\mathcal{F}}{2}+\frac{1}{4})(\gamma+1)<0$ and $x(P)>x(P_0)$. Then we obtain

$$(xcs_x)(P)<(xcs_x)(P_0)\Big(\frac{x(P)}{x(P_0)}\Big)^{2-(\frac{\mathcal{F}}{2}+\frac{1}{4})(\gamma+1)}<\delta_2. \tag{5.42}$$

As in (4.39), we get

$$\big(\frac{v^2}{c}\big)(P)<\Big(\frac{x(P_0)}{x(P)}\Big)^{2-\mathcal{C}(\gamma-1)}\big(\frac{v^2}{c}\big)(P_0)<\delta_1. \tag{5.43}$$

In view of (5.5), we have

$$\partial_- u=\frac{2c\widehat{Z}_-}{\gamma-1}<0 \quad\text{and}\quad \partial_+ u=-\frac{2c\widehat{Z}_+}{\gamma-1}+(\mathcal{F}-\frac{1}{2})\frac{2uc}{x}>0 \tag{5.44}$$

in $\Omega_P\setminus\{P\}$. Then we have

$$u_v<u(P_+)<u(P)<u(P_-)<\bar{u}(r)+1.$$

Suppose that $\widehat{Z}_-=0$ at P. Then by the assumption that the inequalities in (5.31) hold in $\Omega_P\setminus\{P\}$, we have $\partial_+\widehat{Z}_-\geq 0$ at P. While, by the first equation of (5.25) and Remark 5.2, we have

$$\partial_+\widehat{Z}_-=\hat{d}_{14}\widehat{Z}_++\hat{d}_{15}\ <\ 0 \tag{5.45}$$

at P. This leads to a contradiction. We then get $\widehat{Z}_-(P)<0$. Similarly, we have $\widehat{Z}_+(P)<0$.

Suppose that $Z_-=0$ at P. Then by the assumption that (5.31) holds in $\Omega_P\setminus\{P\}$ we have $\partial_+Z_-\le 0$ at P. While, by (H4), (H5), and the first equation of (5.19), we have

$$\begin{aligned}\partial_+Z_- &> \frac{d_{14}\mathcal{C}(\gamma-1)u}{x}+d_{15}\\ &>\Big\{\frac{\gamma+1}{2}\mathcal{C}^2-\Big(\frac{5}{2}+\frac{3}{2}\frac{c_{_M}}{u_v}+\frac{\delta_2}{4\gamma u_v}\Big)\mathcal{C}+1-\frac{2\delta_1}{u_v}\Big\}\frac{(\gamma-1)u^2}{x^2}>0 \quad \text{if} \quad d_{14}<0\end{aligned}$$

and

$$\partial_+Z_->d_{15}>\Big\{2\mathcal{C}^2-\big(3+2\frac{c_{_M}}{u_v}\big)\mathcal{C}+1-\frac{|\gamma\mathcal{C}-2|\delta_1}{u_v}\Big\}\frac{(\gamma-1)u^2}{x^2}>0 \quad \text{if} \quad d_{14}\ge 0.$$

This leads to a contradiction. We then have $Z_-(P)>0$. Similarly, we have $Z_+(P)>0$.

We then prove that if the inequalities in (5.31) hold for every point in $\Omega_P\setminus\{P\}$, then they also hold at P. Therefore, by an argument of continuity we complete the proof of the lemma. ∎

From Lemma 5.3 we actually know that the solution satisfies

$$\widehat{Z}_\pm<0, \quad Z_\pm>0, \quad 0<c<c_{_M}, \quad \frac{v^2}{c}\le\delta_1, \quad |xcs_x|<\delta_2 \quad \text{and} \quad u>u_v. \tag{5.46}$$

This gives an a priori C^0 norm estimate for (u,v,c) and gradient estimates for c and s. The gradient estimate for u can be obtained by (5.5). Like (4.44), one has

$$\partial_0(\partial_x v)+\Big(\frac{u}{x}+\partial_x u\Big)\partial_x v=\frac{uv}{x^2}-\frac{v\partial_x u}{x}. \tag{5.47}$$

The gradient estimate for v can be obtained by $\partial_x v$ and the third equation of (1.4). We then establish an a priori C^1 estimate for the solution. Thus, the existence of global classical solution can be obtained by the classical extension method (cf. Li [28]). We then have the following global existence.

LEMMA 5.1. *The Cauchy problem (1.4, 5.30) admits a global classical solution. Moreover, the solution satisfies (5.31).*

Since $r>\varepsilon$ can be arbitrary, we construct the solution in a triangle domain Ω_3 encircled by C_+^Q and the x-axis. Moreover the solution satisfies (5.46).

5.4. Solution in domain Ω_2. Take any points Q_+ and Q_- on C_+^Q and C_-^Q, respectively. We now consider (1.4) with data

$$(u,v,c,s)=\begin{cases}(\hat u,\hat v,\hat c,\hat s)(t/x) \text{ on } \widehat{QQ_-};\\ (\tilde u,\tilde v,\tilde c,\tilde s)(x,t) \text{ on } \widehat{QQ_+};\end{cases} \tag{5.48}$$

where $\hat v\equiv 0$, $\hat s\equiv s_0$, and $(\tilde u,\tilde v,\tilde c,\tilde s)(x,t)$ denotes the solution in Ω_3.

Problem (1.4, 5.48) is a Goursat problem, and the existence of a local C^1 solution is known by the method of characteristics. In order to extend the local solution to a global solution, one needs to establish an a priori C^1 norm estimate of the solution.

LEMMA 5.2. *Assume that the Goursat problem (1.4, 5.48) admits a classical solution. Then the solution satisfies*

$$\widehat{Z}_\pm<0, \quad Z_\pm>0, \quad 0<c<c_{_M}, \quad \textit{and} \quad u_v<u<u(Q_+)+1. \tag{5.49}$$

Proof. The approach to proving this lemma is similar to that of Lemma 4.4, we omit the details. ∎

From Lemma 5.2, (5.5), and (5.47), one can get an a priori C^1 norm estimate of the classical solution to the Goursat problem (1.4), (5.48). Then by the classical extension one gets the following global existence.

LEMMA 5.4. *The Goursat problem (1.4, 5.48) admits a global classical solution.*

Since $Q_\pm$ can be arbitrary, we obtain the solution in a triangle domain Ω_2 bounded by C_+^Q and C_-^Q. Then we complete the proof of Theorem 5.1.

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