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GLOBAL EXISTENCE FOR QUASILINEAR DEGENERATE TWO-SPECIES CHEMOTAXIS SYSTEM WITH SMALL INITIAL DATA*

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Abstract. This article is devoted to the analysis of quasilinear degenerate chemotaxis system with two-species in dimension $d \geq 3$. The global existence of weak solution to the chemotaxis system with small initial data is proved for the super-critical case in both parabolic-elliptic and fully parabolic types.

Keywords. Degenerate parabolic system; chemotaxis; global existence.

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1. Introduction

It is well-known that a degenerate chemotaxis system is used to describe the interaction between two species over $\mathbb{R}^d$, $d \geq 3$ and takes the following form

$$\begin{cases} u_t = \Delta u^{m_1} - \nabla \cdot (u \nabla v), & x \in \mathbb{R}^d, t > 0, \\ \tau v_t = \Delta v - \tau v + w, & x \in \mathbb{R}^d, t > 0, \\ w_t = \Delta w^{m_2} - \nabla \cdot (w \nabla z), & x \in \mathbb{R}^d, t > 0, \\ \tau z_t = \Delta z - \tau z + u, & x \in \mathbb{R}^d, t > 0, \end{cases} \tag{1.1}$$

where u and w represent the macrophages density and the tumor cells density, respectively. v and z are the concentrations of chemical signals secreted by w and u independently. Here $m_1 > 1$, $m_2 > 1$ denote the diffusion exponents and τ takes value as: $\tau = 0$ or $\tau = 1$. This system is derived by a simplification of the chemotaxis system proposed in [22] to describe the process of macrophage-facilitated breast cancer cells invasion. In this paper, we would like to consider the Cauchy problem of (1.1) with the following initial data:

$$\begin{aligned} u(x,0) = u_0(x),\ w(x,0) = w_0(x), \\ \tau v(x,0) = \tau v_0(x),\ \tau z(x,0) = \tau z_0(x). \end{aligned} \tag{1.2}$$

The following total masses are conserved for $t > 0$

$$\|u(t)\|_{L^1(\mathbb{R}^d)} = \|u_0\|_{L^1(\mathbb{R}^d)} = M_1, \quad \|w(t)\|_{L^1(\mathbb{R}^d)} = \|w_0\|_{L^1(\mathbb{R}^d)} = M_2.$$

For parabolic-elliptic type (i.e. $\tau = 0$), $(1.1)_2$ and $(1.1)_4$ read as

$$-\Delta v = w,\ -\Delta z = u, \tag{1.3}$$

where the foundational solutions of (1.3) are given by

$$v(x,t) = c_d \int_{\mathbb{R}^d} \frac{w(y,t)}{|x-y|^{d-2}} dy, \quad z(x,t) = c_d \int_{\mathbb{R}^d} \frac{u(y,t)}{|x-y|^{d-2}} dy.$$

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Note that c_d is the value of the surface area of the unit sphere in $\mathbb{R}^d$:

$$c_d = \frac{\Gamma(d/2+1)}{d(d-2)\pi^{d/2}}. \tag{1.4}$$

A straightforward computation shows that the following free energy functional

$$\mathcal{F}[u,w](t) = \frac{1}{m_1-1}\int_{\mathbb{R}^d} u^{m_1}(x,t)dx + \frac{1}{m_2-1}\int_{\mathbb{R}^d} w^{m_2}(x,t)dx - c_d \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{u(x,t)w(y,t)}{|x-y|^{d-2}} dxdy$$

exists. For $m_c := 2-2/d$, based on the free energy functional, the authors [7] have proved that the following curves (colored red in Figure 1.1)

$$\text{Line } L_1: \; m_1m_2 + 2m_1/d = m_1+m_2 \quad \text{with} \quad m_1 \in [m_c, d/2), \quad m_2 \in (1, m_c];$$
$$\text{Line } L_2: \; m_1m_2 + 2m_2/d = m_1+m_2 \quad \text{with} \quad m_1 \in (1, m_c], \quad m_2 \in [m_c, d/2)$$

are the exactly sharp conditions, which separate the global existence and blow up of solutions to (1.1). Above the two red curves in the sense that $m_1m_2+2m_1/d > m_1+m_2$ or $m_1m_2+2m_2/d > m_1+m_2$ (we call it the sub-critical case) or on the red curves (critical case) with small initial data, weak solutions globally exist. While blow up will occur for certain large initial data on or below the red curves with $m_1>1$, $m_2>1$ (super-critical case). However, the global well-posedness with small initial data is still unsolved in the super-critical case. In the two-dimensional case, [11, 13] indicate that critical mass phenomenon exists by means of the Moser-Trudinger inequality [29] and the existence of the free energy. The properties of solutions to the Neumann initial boundary value problem are also analyzed by [26, 27, 36, 38, 39] through the free energy, including global existence and blow up.

Since it seems that no free energy functional exists for the fully parabolic case (i.e. $\tau=1$), some additional technical difficulties arise naturally. When $m_1=m_2=1$, by applying the standard energy proof to (1.1), in a bounded domain [27] shows that behaviors of two species are effected by each other, and if the product of two species masses is suitably small, then the solution is global and converges to constant equilibria.

Note that system (1.1) is a general model of classical chemotaxis system with one species ([21], see [28] including volume-filling effects),

$$\begin{cases} u_t = \Delta u^{m_1} - \nabla\cdot(u\nabla v), & x\in\mathbb{R}^d, t>0, \\ \tau v_t = \Delta v - v + u, & x\in\mathbb{R}^d, t>0, \end{cases} \tag{1.5}$$

where $m_1>0$, $\tau=0,1$, u denotes the density of cells and v represents the concentration of the chemical produced by u directly. A large number of research results indicate that the main conclusions and techniques used in the proof of global existence or blow up for (1.1) and (1.5) are quite similar, see the introduction part in [7]. Here we give a short summary of results for (1.5).

The number $m_c = 2-2/d$ divides the global existence into two cases with and without smallness assumption on the size of initial data. In [31–33], the authors have considered (1.5) with $\tau=0$ and proved that solutions exist globally for general initial data provided $m_1>m_c$, whereas blow up occurs for some certain large initial data if $m_1\le m_c$. When $m_1=m_c$ and $(1.5)_2$ is replaced by $-\Delta v = u$, an important observation

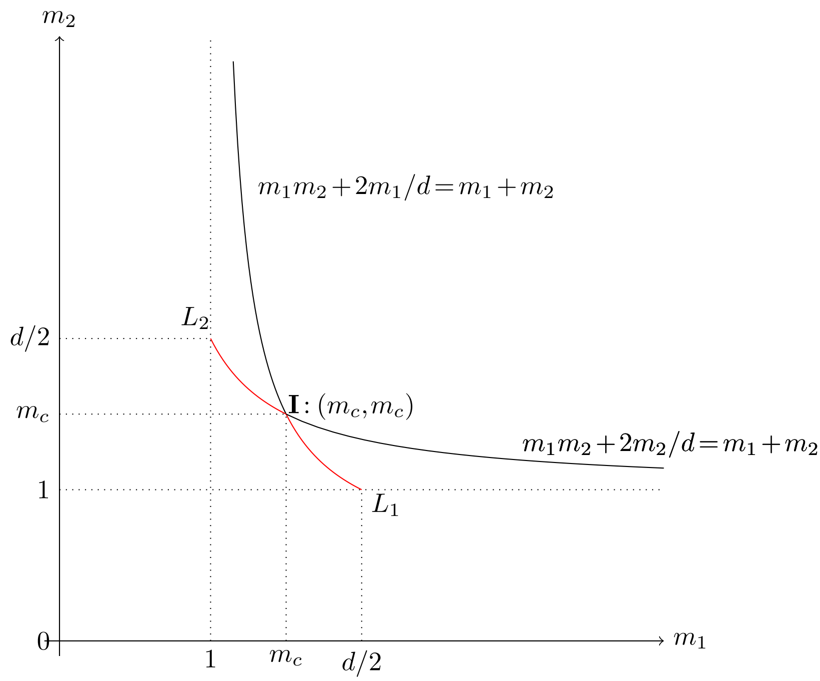

FIG. 1.1. *Parameter lines determining the critical regimes.*

obtained by [4] shows that there exists a critical value $M_c > 0$ such that if total mass $\|u\|_{L^1(\mathbb{R}^d)} = M_1 < M_c$, the solutions exist globally, while the solutions blow up in finite time if $M_1 > M_c$. Another number $m_* := 2d/(d+2) > 0$ was found in [8] to make sure conformal invariance of the corresponding free energy of (1.5). When $m_1 = m_*$, a criterion could be found on L^{m_*}-norm of a family of positive stationary solutions for global existence and blow up [8]. If $m_1 \in (m_*, m_c)$, the condition for the criterion is replaced by L^{m_*}-norm of initial data [9]. For general $m_1 > 0$, small data global solutions, finite-time blow-up behavior, the hyper-contractive estimates and the existence of steady states for (1.5) have been studied in [2].

When $\tau = 1$, [17, 18] proved that for $m_1 > m_c$ or $m_1 < m_c$ with small initial data, global weak solution will exist, and the authors [14, 20] employed Moser's iteration to obtain the uniform bound of solutions. Blow-up solutions were constructed in [16, 19] for large initial data in a ball. We also refer to [15, 35] for the global existence in the non-degenerate case, and [10, 37] for blow-up arguments.

In this paper, when $(m_1, m_2) \in (1, d/2)^2$ fulfills

$$m_1 + m_2 > m_1m_2 + 2m_1/d \quad \text{and} \quad m_1 + m_2 > m_1m_2 + 2m_2/d,$$

the global existence of solutions to (1.1) with small initial data will be considered. As we mentioned above, the area below the red curve in Figure 1.1 is unknown. Hence this project gives an essentially complete characterization of global existence or nonexistence for (1.1).

For reader's convenience, defining

$$p := \frac{d(m_1+m_2-m_1m_2)}{2m_2} \tag{1.6}$$

and

$$q := \frac{d(m_1+m_2-m_1m_2)}{2m_1}, \tag{1.7}$$

we note that $(p,q)\in(1,\infty)^2$ below the red curve in Figure 1.1.

Now, we give the definition of weak solution to (1.1).

DEFINITION 1.1. *Let $m_1>1$, $m_2>1$, $d\geq 3$, $\tau=0,1$, and $T>0$. Suppose that nonnegative initial data satisfies*

$$\begin{aligned}&(u_0,w_0)\in\left(L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)\right)^2, \quad (u_0,w_0)\in(H^1(\mathbb{R}^d))^2,\\ &(\tau v_0,\tau z_0)\in(L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))^2, \quad (\tau\Delta v_0,\tau\Delta z_0)\in(L^{r_0}(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))^2\end{aligned}$$

with $r_0>1$ large enough, then nonnegative functions (u,v,w,z) defined in $\mathbb{R}^d\times(0,T)$ is called a weak solution if

- *(i)* $(u,w)\in(C([0,T];L^1(\mathbb{R}^d))\cap L^\infty(\mathbb{R}^d\times(0,T)))^2$, $(u^{m_1},w^{m_2})\in(L^2(0,T;H^1(\mathbb{R}^d)))^2$;
- *(ii)* $(v,z)\in\left(L^\infty(0,T;H^1(\mathbb{R}^d))\right)^2$;
- *(iii)* *For $\phi\in C_0^\infty(\mathbb{R}^d\times[0,T))$, (u,w) satisfies*
$$\int_0^T\int_{\mathbb{R}^d} u\phi_t\,dxdt+\int_{\mathbb{R}^d}u_0(x)\phi(x,0)dx=\int_0^T\int_{\mathbb{R}^d}(\nabla u^{m_1}-u\nabla v)\cdot\nabla\phi\,dxdt,$$
$$\int_0^T\int_{\mathbb{R}^d} w\phi_t\,dxdt+\int_{\mathbb{R}^d}w_0(x)\phi(x,0)dx=\int_0^T\int_{\mathbb{R}^d}(\nabla w^{m_2}-w\nabla z)\cdot\nabla\phi\,dxdt.$$
- *(iv)* *For $\phi\in C_0^\infty(\mathbb{R}^d\times[0,T))$, if $\tau=1,(v,z)$ satisfies*
$$\int_0^T\int_{\mathbb{R}^d}\nabla v\cdot\nabla\phi\,dxdt+\int_0^T\int_{\mathbb{R}^d}(v\phi-w\phi-v\phi_t)dxdt=\int_{\mathbb{R}^d}v_0(x)\phi(x,0)dx,$$
$$\int_0^T\int_{\mathbb{R}^d}\nabla z\cdot\nabla\phi\,dxdt+\int_0^T\int_{\mathbb{R}^d}(z\phi-u\phi-z\phi_t)dxdt=\int_{\mathbb{R}^d}z_0(x)\phi(x,0)dx.$$
If $\tau=0,(v,z)$ is given by $v=\mathcal{K}*w$ and $z=\mathcal{K}*u$, where $\mathcal{K}=\dfrac{c_d}{|x|^{d-2}}$.

Remark that (u,v,w,z) is called a global weak solution if $T>0$ can be chosen arbitrarily.

Note that the initial condition $(u_0,w_0)\in(H^1(\mathbb{R}^d))^2$ ensures the convergence in the approximate solutions in Section 3, but they can be certainly removed by applying a slightly different proof used in [17]. We first give a description of the global existence for the parabolic-elliptic type.

THEOREM 1.1. *Let $d\geq 3$, $\tau=0$ and $T>0$. Let $(m_1,m_2)\in(1,d/2)^2$ fulfills*

$$m_1+m_2>m_1m_2+2m_1/d \text{ and } m_1+m_2>m_1m_2+2m_2/d. \tag{1.8}$$

Assume that initial data (u_0,w_0) fulfills the assumptions in Definition 1.1 and the following smallness condition:

$$\|u_0\|_{L^p(\mathbb{R}^d)}\le \alpha_1,\quad \|w_0\|_{L^q(\mathbb{R}^d)}\le \alpha_2,$$
$$\|u_0\|_{L^{p+1-p/q}(\mathbb{R}^d)}\le \alpha_3,\quad \|w_0\|_{L^{q+1-q/p}(\mathbb{R}^d)}\le \alpha_4,$$

where $p>1$, $q>1$ are given by (1.6) and (1.7), $\alpha_i=\alpha_i(m_1,m_2,d)>0$, $i=1,\cdots,4$ is a constant given in Lemma 3.3. Then the Cauchy problem (1.1)-(1.2) possesses a global weak solution. Moreover, it is uniformly bounded in the sense that there exists a constant $K_1=K_1(M_1,M_2,\|u_0\|_{L^\infty(\mathbb{R}^d)},\ \|w_0\|_{L^\infty(\mathbb{R}^d)},m_1,m_2,d)>0$ such that

$$\|u(t)\|_{L^r(\mathbb{R}^d)}+\|w(t)\|_{L^r(\mathbb{R}^d)}\le K_1,$$

for $r\in[1,\infty]$ and $t\in(0,T)$.

It is clear that the conclusions in Theorem 1.1 hold for $m_1=m_2\in(1,m_c)$ under some smallness assumption on the initial data (see [33, Proposition 10] or [2, Theorem 2.11]). Here the asserted results below the red curve in Figure 1.1 are obtained by combining the energy estimates for u with w together (see Lemmas 3.3 and 3.4), which covers the case $m_1=m_2\in(1,m_c)$.

The second result is concerned about the global existence for the fully parabolic type.

THEOREM 1.2. *Let $d\ge 3$, $\tau=1$ and $T>0$. Let $(m_1,m_2)\in(1,d/2)^2$ fulfill (1.8), and let $p>1$ and $q>1$ be given by (1.6) and (1.7). Assume that initial data (u_0,v_0,w_0,z_0) fulfills the assumptions in Definition 1.1 and the following smallness conditions:*

$$\|u_0\|_{L^p(\mathbb{R}^d)}\le\alpha_u,\quad \|w_0\|_{L^q(\mathbb{R}^d)}\le \alpha_w,$$
$$\|u_0\|_{L^{p+1-p/q}(\mathbb{R}^d)}\le\alpha_w,\ \ \|w_0\|_{L^{q+1-q/p}(\mathbb{R}^d)}\le \alpha_u,$$
$$\|\Delta v_0\|_{L^{q+1}(\mathbb{R}^d)}\le\alpha_u,\ \ \|\Delta v_0\|_{L^{q(1+1/p)}(\mathbb{R}^d)}\le \alpha_w,$$
$$\|\Delta z_0\|_{L^{p+1}(\mathbb{R}^d)}\le\alpha_w,\ \ \|\Delta z_0\|_{L^{p(1+1/q)}(\mathbb{R}^d)}\le \alpha_u,$$

where α_u,α_w are some positive constants depending on m_1, m_2 and d. Then (1.1)-(1.2) possesses a global weak solution. Moreover, it is uniformly bounded in the sense that

$$\|u(t)\|_{L^r(\mathbb{R}^d)}+\|w(t)\|_{L^r(\mathbb{R}^d)}\le K_2$$

for $r\in[1,\infty]$ and $t\in(0,T)$, where $K_2=K_2(M_1,M_2,\|u_0\|_{L^\infty(\mathbb{R}^d)},\|w_0\|_{L^\infty(\mathbb{R}^d)},\|v_0\|_{L^r(\mathbb{R}^d)},$ $\|\Delta v_0\|_{L^r(\mathbb{R}^d)},\|z_0\|_{L^r(\mathbb{R}^d)},\|\Delta z_0\|_{L^r(\mathbb{R}^d)},m_1,m_2,\ d)>0$.

The proof of Theorem 1.2 mainly follows the well-known maximum Sobolev regularity, which is introduced by Ishida and Yokota [17,18] to deal with one species chemotaxis problem (1.5) with $\tau=1$, see Lemma 2.2. It plays an important role in establishing L^r-estimate for Δv in parabolic evolution equation and finally L^∞-estimate for solutions of (1.5). Then we will combine the above maximum Sobolev regularity with the energy estimates for (u,w) to get the desired conclusions in Theorem 1.2. In addition, one can derive the uniform-in-time boundedness of solutions by the arguments of [20].

At last, we list some interesting questions that have not been completely solved before. For $\tau=0$, we can extend the idea and results in this paper to the case $m_1,m_2\in(0,1]$ below the red curve in Figure 1.1. Colorful and various properties such as extinction phenomenon, decay rate or blow-up behavior for the solutions to (1.5) of fast diffusion type are given in [2, 34]. We will consider these properties for (1.1) in our future work. Compared with the blow-up initial condition in [7, Theorem 5.3], the initial conditions in Theorem 1.1 ensuring the global existence could be weaker or improved. Due to the lack of effective tools, we also leave it for future study. The hyper-contractivity estimates for the multi-dimensional system (1.5) have been investigated extensively, such as [5] for $m_1=1, d=2$, [6] for $m_1=1, d=3$, and [2, 3] for general $m>0$, $d\geq 3$. We plan to make further advances on this aspect for system (1.1). Another interesting question is to find the sharp condition on global existence without any size assumption on the initial data for the fully parabolic case $\tau=1$. While for the parabolic-elliptic case $\tau=0$, it has been actually solved for (1.1) in terms of the free energy functional (see [7]). However, since there is no valid free energy functional for $\tau=1$, it becomes mathematically complex and challenging.

The rest of this paper is organized as follows. In Section 2, we provide some basic inequalities. Section 3 deals with the global existence of weak solutions for the case $\tau=0$. Section 4 is devoted to the global weak solutions for $\tau=1$.

2. Notations and basic inequalities

We denote the norm in the Sobolev space $W^{m,r}(\mathbb{R}^d)$ by $\|\cdot\|_{m,r}$ for $m\geq 0$, $r\geq 1$. We use $\|\cdot\|_r$ to denote the usual norm in $L^r(\mathbb{R}^d)$ space if $m=0$. For a Banach space X and $0<T\leq\infty$, the norms of f in $L^r(0,T;X)$ and $W^{1,r}(0,T;X)$ with $r\geq 1$ are given by

$$\|f\|_{L^r(0,T;X)} := \left(\int_0^T \|f\|_X^r dt\right)^{\frac{1}{r}}$$

and

$$\|f\|_{W^{1,r}(0,T;X)} := \|f\|_{L^r(0,T;X)} + \|f_t\|_{L^r(0,T;X)},$$

respectively. If $r=\infty$,

$$\|f\|_{L^\infty(0,T;X)} := \sup_{t\in(0,T)} \|f(t)\|_X.$$

Denote $Q_T := \mathbb{R}^d\times(0,T)$ and

$$\mathbb{W}_r^{2,1}(Q_T) := \left\{u\in L^r(0,T;W^{2,r}(\mathbb{R}^d))\cap W^{1,r}(0,T;L^r(\mathbb{R}^d))\right\}.$$

For convenience, we collect some useful frequently used inequalities for later. Recall the Hardy-Littlewood-Sobolev inequality

$$\left|\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{f(x)f(y)}{|x-y|^{d-2}} dxdy\right| \leq C_{LHS}\|f\|_{2d/(d+2)}^2,$$

where $C_{HLS}=S_d/c_d$ ([24]), c_d is defined as (1.4) and S_d is the given sharp constant for the following Sobolev inequality

$$\|f\|_{2d/(d-2)}^2 \leq S_d\|\nabla f\|_2^2, \quad S_d = \frac{4}{d(d-2)} 2^{-2/d}\pi^{-1-1/d}\Gamma\left(\frac{d+1}{2}\right)^{2/d}. \tag{2.1}$$

In view of the interpolation inequality and (2.1), it gives that

$$\|g\|_{b/a} \leq \|g\|_1^{1-\theta}\|g\|_{2d/(a(d-2))}^\theta = \|g\|_1^{1-\theta}\|g^{1/a}\|_{2d/(d-2)}^{\theta a} \leq S_d^{\theta a/2}\|g\|_1^{1-\theta}\|\nabla g^{1/a}\|_2^{\theta a},$$

provided that $1<b/a<2d/(a(d-2))$, $a>0$ and $b>0$. Setting $g=f^{\frac{a(r+m-1)}{2}}$ with $m>0$, $r\geq 1$, one has

$$\left\|f^{\frac{a(r+m-1)}{2}}\right\|_{b/a}\leq S_d^{\frac{\theta a}{2}}\left\|f^{\frac{a(r+m-1)}{2}}\right\|_1^{1-\theta}\left\|\nabla f^{\frac{r+m-1}{2}}\right\|_2^{\theta a},$$

i.e.,

$$\|f\|_{\frac{b(r+m-1)}{2}}\leq S_d^{\frac{\theta}{r+m-1}}\|f\|_{\frac{a(r+m-1)}{2}}^{1-\theta}\left\|\nabla f^{\frac{r+m-1}{2}}\right\|_2^{\frac{2\theta}{r+m-1}}, \tag{2.2}$$

where $\theta\in(0,1)$ satisfies

$$\theta=\left(\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a}-\frac{d-2}{2d}\right)^{-1}.$$

Therefore, the Gagliardo-Nirenberg inequality for $d\geq 3$ reads as:

LEMMA 2.1. *Let $d\geq 3$, $m>0$, $r\geq 1$, and let $\kappa_1\geq 1$, $\kappa_2>1$ fulfill*

$$1<\frac{\kappa_2}{\kappa_1}<\frac{(r+m-1)d}{(d-2)\kappa_1}. \tag{2.3}$$

Then for $f\in L^{\kappa_1}(\mathbb{R}^d)$ and $f^{\frac{r+m-1}{2}}\in H^1(\mathbb{R}^d)$,

$$\|f\|_{\kappa_2}\leq S_d^{\frac{\theta}{r+m-1}}\|f\|_{\kappa_1}^{1-\theta}\left\|\nabla f^{\frac{r+m-1}{2}}\right\|_2^{\frac{2\theta}{r+m-1}}, \tag{2.4}$$

where $\theta\in(0,1)$ fulfills

$$\theta=\frac{r+m-1}{2}\left(\frac{1}{\kappa_1}-\frac{1}{\kappa_2}\right)\left(\frac{1}{d}-\frac{1}{2}+\frac{r+m-1}{2\kappa_1}\right)^{-1}.$$

In addition, if $\frac{2\kappa_2\theta}{r+m-1}<2$ in the sense that

$$\frac{\kappa_2}{\kappa_1}<\frac{2}{d}+\frac{r+m-1}{\kappa_1},$$

then for any $\eta>0$,

$$\|f\|_{\kappa_2}^{\kappa_2}\leq\eta S_d\|\nabla f^{\frac{r+m-1}{2}}\|_2^2+\eta^{-\frac{\kappa_2\theta}{r+m-\kappa_2\theta-1}}\|f\|_{\kappa_1}^{\kappa_2(1-\theta)\cdot\frac{r+m-1}{r+m-\kappa_2\theta-1}}. \tag{2.5}$$

Proof. Applying $\kappa_1=\frac{a(r+m-1)}{2}$, $\kappa_2=\frac{b(r+m-1)}{2}$ to (2.2), we note that the condition $1<b/a<2d/(a(d-2))$ is the same with (2.3), then (2.4) is obtained. (2.5) could be obtained by Young's inequality directly. ∎

LEMMA 2.2. *Let $d\geq 3$, $T>0$. Let ψ be a unique mild solution to the Cauchy problem*

$$\begin{cases}\frac{\partial}{\partial t}\psi=\Delta\psi-\psi+h, & (x,t)\in\mathbb{R}^d\times(0,T),\\ \psi(x,0)=\psi_0(x), & x\in\mathbb{R}^d,\end{cases}$$

with $\psi_0\in L^r(\mathbb{R}^d)$ and $h\in L^1(0,T;L^r(\mathbb{R}^d))$, $1\leq r\leq\infty$. Then ψ has a form of expression given by:

$$\psi(t)=e^{t(\Delta-1)}\psi_0+\int_0^t e^{(t-s)(\Delta-1)}h(s)ds,\ t\in[0,T],$$

where

$$\left(e^{t\Delta}h\right)(x,t)=\frac{1}{(4\pi t)^{\frac{d}{2}}}\int_{\mathbb{R}^d}e^{-\frac{|x-y|^2}{4t}}h(y,t)dy.$$

Moreover, ψ satisfies the following properties:

(i) **L^{r_1}-L^{r_2} estimates.** *Let $1\le r_1\le r_2\le\infty$ and $\frac{1}{r_1}-\frac{1}{r_2}<\frac{1}{d}$. Assume that $\psi_0\in W^{1,r_2}(\mathbb{R}^d)$, $h\in L^\infty(0,T;L^{r_1}(\mathbb{R}^d))$. Then for every $t\in[0,T]$, there exists a positive constant $C>0$ depending only on r_1, r_2 and d such that*

$$\|\psi(t)\|_{r_2}\le\|\psi_0\|_{r_2}+C\|h\|_{L^\infty(0,T;L^{r_1}(\mathbb{R}^d))},$$
$$\|\nabla\psi(t)\|_{r_2}\le\|\nabla\psi_0\|_{r_2}+C\|h\|_{L^\infty(0,T;L^{r_1}(\mathbb{R}^d))}.$$

(ii) **Maximal Sobolev regularity.** *Let $1<r<\infty$. Assume that $\psi_0\in W^{2,r}(\mathbb{R}^d)$, $h\in L^r(0,T;L^r(\mathbb{R}^d))$. Then for every $t\in[0,T]$, there exists a positive constant $K_r>0$ depending only on r and d such that*

$$\|\Delta\psi\|_{L^r(0,t;L^r(\mathbb{R}^d))}\le\|\Delta\psi_0\|_r\left(1-e^{-rt}\right)^{\frac{1}{r}}+K_r\|h\|_{L^r(0,t;L^r(\mathbb{R}^d))}.$$

Proof. The first assertion just follows from the semigroup theory with L^p-L^q-estimates for the heat semigroup (see [18, Lemma 2.1]). The second is a particular consequence of well-known results on maximal Sobolev regularity in parabolic evolution equations (see [12, Theorem 3.1], [23, Section 3, Chapter IV], [30, Theorem 3, Chapter IV]). ◻

3. Global existence for $\tau=0$

3.1. Approximated system. We consider the approximated problem

$$\begin{cases}\partial_t u_\epsilon=\Delta(u_\epsilon+\epsilon)^{m_1}-\nabla\cdot(u_\epsilon\nabla v_\epsilon), & x\in\mathbb{R}^d,t>0,\\ \tau\partial_t v_\epsilon=\Delta v_\epsilon-\tau v_\epsilon+w_\epsilon, & x\in\mathbb{R}^d,t>0,\\ \partial_t w_\epsilon=\Delta(w_\epsilon+\epsilon)^{m_2}-\nabla\cdot(w_\epsilon\nabla z_\epsilon), & x\in\mathbb{R}^d,t>0,\\ \tau\partial_t z_\epsilon=\Delta z_\epsilon-\tau z_\epsilon+u_\epsilon, & x\in\mathbb{R}^d,t>0,\\ u_\epsilon(x,0)=u_{0\epsilon}(x),w_\epsilon(x,0)=w_{0\epsilon}(x), & x\in\mathbb{R}^d,\\ \tau v_\epsilon(x,0)=\tau v_{0\epsilon}(x),\ \tau z(x,0)=\tau z_{0\epsilon}(x), & x\in\mathbb{R}^d,\end{cases}\tag{3.1}$$

where $\epsilon\in(0,1)$, $\tau=0,1$. Here $u_{0\epsilon},w_{0\epsilon},\tau v_{0\epsilon},\tau z_{0\epsilon}\in C_0^\infty(\mathbb{R}^d)$ are approximations of u_0, w_0, τv_0 and τz_0, respectively, given by

$$u_{0\epsilon}:=(u_0*\rho_\epsilon)\zeta_\epsilon,\ w_{0\epsilon}:=(w_0*\rho_\epsilon)\zeta_\epsilon,$$
$$\tau v_{0\epsilon}:=(\tau v_0*\rho_\epsilon)\zeta_\epsilon,\ \tau z_{0\epsilon}:=(\tau z_0*\rho_\epsilon)\zeta_\epsilon,$$

with a sequence of mollifiers, where ρ_ϵ is a mollifier such that

$$0\le\rho_\epsilon\in C_0^\infty(\mathbb{R}^d),\ \mathrm{supp}\,\rho_\epsilon\subset\overline{B(0,\epsilon)},\ \int_{\mathbb{R}^d}\rho_\epsilon(x)dx=1,$$

$\zeta_\epsilon(x):=\zeta(\epsilon x)$ is a cut-off function and $\zeta(x)\in C_0^\infty(\mathbb{R}^d)$ is defined by

$$0\le\zeta\le1,\ \ \zeta(x):=\begin{cases}1,\ |x|\le1,\\ 0,\ |x|\ge2,\end{cases}$$

see [18].

By virtue of the analytic semigroup theory [1, Theorem IV.1.5.1] and the fixed point theorem, utilizing the similar arguments of [33, Proposition 8, Lemmas 11 and 12] dealing with only one species chemotaxis system, we can find the local existence of strong solution for (1.1) with $\tau=0,1$:

LEMMA 3.1. *Let $m_1,m_2>1$, $\epsilon\in(0,1)$ and $\tau=0,1$. Then there exists a maximal existence time*

$$T^{\epsilon}_{\max}\in(0,\infty]$$

such that (3.1) has a unique nonnegative strong solution $(u_\epsilon,v_\epsilon,w_\epsilon,z_\epsilon)\in\left(\mathbb{W}^{2,1}_r(Q_T)\right)^4$ with some $r>d$ satisfying

$$\begin{aligned}&\|u_\epsilon(t)\|_1=\|u_{0\epsilon}\|_1\le\|u_0\|_1=M_1,\\&\|w_\epsilon(t)\|_1=\|w_{0\epsilon}\|_1\le\|w_0\|_1=M_2\quad \text{for } t\in[0,T^{\epsilon}_{\max}),\end{aligned}\tag{3.2}$$

where $T^{\epsilon}_{\max}$ depends on ϵ, $\|u_{0\epsilon}\|_{2,d+2}$, $\|w_{0\epsilon}\|_{2,d+2}$, $\|\tau v_{0\epsilon}\|_{1,\infty}$, $\|\tau\Delta v_{0\epsilon}\|_\infty$, $\|\tau z_{0\epsilon}\|_{1,\infty}$, $\|\tau\Delta z_{0\epsilon}\|_\infty$, m_1, m_2 and d. Moreover, if $T^{\epsilon}_{\max}<\infty$, then

$$\limsup_{t\to T^{\epsilon}_{\max}}(\|u_\epsilon(t)\|_\infty+\|w_\epsilon(t)\|_\infty)=\infty.$$

We remark that if $r>d$ is sufficiently large, one can derive $T_0\in(0,T^{\epsilon}_{\max})$ independent of ϵ and $C_0>0$ depending only on $\|u_0\|_r$ and $\|w_0\|_r$ such that the solutions of (3.1) satisfy the following a priori estimate:

$$\|u_\epsilon(t)\|_r+\|w_\epsilon(t)\|_r\le C_0 \text{ for all } t\in[0,T_0] \text{ and } \epsilon\in(0,1).$$

See [31, Proposition 4.1] for $\tau=0$ and [20, Lemma 3.1] for $\tau=1$.

The following choices of some positive numbers will allow one to establish decay inequalities for both parabolic-elliptic and fully parabolic types of (3.1).

LEMMA 3.2. *Let $p>1$ and $q>1$ be given by (1.6) and (1.7). Then there exist $\bar{k}>1$, $\bar{l}>1$, $r_1>1$ and $r_2>1$ such that for any $k>\bar{k}$ and $l>\bar{l}$, one has*

$$k>p\left(1-\frac{1}{q}\right),\tag{3.3}$$

$$l>q\left(1-\frac{1}{p}\right),\tag{3.4}$$

$$\frac{1}{r_1}<1-\frac{d-2}{(l+m_2-1)d},\tag{3.5}$$

$$\frac{1}{r_1}>\max\left\{1-\frac{1}{q},\frac{d-2}{d}\cdot\frac{k}{k+m_1-1}\right\},\tag{3.6}$$

$$\frac{1}{r_2}<1-\frac{d-2}{(k+m_1-1)d},\tag{3.7}$$

$$\frac{1}{r_2}>\max\left\{1-\frac{1}{p},\frac{d-2}{d}\cdot\frac{l}{l+m_2-1}\right\}.\tag{3.8}$$

In particular, r_1 and r_2 can be further chosen to fulfill

$$r_1 \cdot \frac{\frac{k}{p}-\frac{1}{r_1}}{\frac{1}{d}-\frac{1}{2}+\frac{k+m_1-1}{2p}} = r_1' \cdot \frac{\frac{1}{q}-\frac{1}{r_1'}}{\frac{1}{d}-\frac{1}{2}+\frac{l+m_2-1}{2q}} = 2 \tag{3.9}$$

and

$$r_2 \cdot \frac{\frac{l}{q}-\frac{1}{r_2}}{\frac{1}{d}-\frac{1}{2}+\frac{l+m_2-1}{2q}} = r_2' \cdot \frac{\frac{1}{p}-\frac{1}{r_2'}}{\frac{1}{d}-\frac{1}{2}+\frac{k+m_1-1}{2p}} = 2, \tag{3.10}$$

where $r_1' = \frac{r_1}{r_1-1}$ and $r_2' = \frac{r_2}{r_2-1}$.

Proof. First, notice that (3.3) holds out by choosing $\bar{k} = p\left(1-\frac{1}{q}\right)$. Let

$$l := \frac{(k+m_1-1)q}{p} - m_2 + 1, \tag{3.11}$$

then (3.4) is also true by picking $\bar{l} = q\left(1-\frac{1}{p}\right)$ because of

$$\begin{aligned} l >& \frac{\left(p\left(1-\frac{1}{q}\right)+m_1-1\right)q}{p} - m_2 + 1 \\ =& q + \frac{(m_1-1)q}{p} - m_2 \\ =& q\left(1-\frac{1}{p}\right). \end{aligned}$$

A direct computation shows that

$$1 - \frac{d-2}{(l+m_2-1)d} > 1 - \frac{1}{q}$$

and

$$1 - \frac{d-2}{(l+m_2-1)d} > \frac{d-2}{d} \cdot \frac{k}{k+m_1-1}$$

by means of (3.3), (3.4), (3.11) as well as

$$\begin{aligned} \frac{1}{q} &= \frac{2}{d} + \frac{m_1-1}{p}, \\ \frac{1}{p} &= \frac{2}{d} + \frac{m_2-1}{q}. \end{aligned}$$

Therefore it is possible to pick $r_1 \in (1, q/(q-1))$ meeting (3.5) and (3.6). Similarly, the choice of $r_2 \in (1, p/(p-1))$ satisfying (3.7) and (3.8) can be obtained.

In order to show (3.9) and (3.10), we define

$$r_1 = \frac{p}{q} \cdot \frac{1}{k} + 1$$

and

$$r_2 = \frac{q}{p} \cdot \frac{1}{l} + 1. \tag{3.12}$$

Then we have

$$r_1' = \frac{q}{p}k + 1 = l + m_2 + \frac{2}{d}q - 1,$$

$$r_2' = \frac{p}{q}l + 1 = k + m_1 + \frac{2}{d}p - 1.$$

By means of (3.3) and (3.4), a simple calculation yields the following inequalities

$$\frac{d-2}{(l+m_2-1)d} < \frac{1}{r_1'} = \frac{1}{l+m_2+\frac{2}{d}q-1},$$

$$\frac{1}{q} > \frac{1}{r_1'} = \frac{1}{\frac{q}{p}k+1},$$

$$\frac{1}{r_1} = \frac{1}{\frac{p}{q}\cdot\frac{1}{k}+1} > \frac{d-2}{d} \cdot \frac{k}{k+m_1-1},$$

$$r_1 \cdot \frac{\frac{k}{p} - \frac{1}{r_1}}{\frac{1}{d} - \frac{1}{2} + \frac{k+m_1-1}{2p}} = \frac{\frac{kr_1}{p} - 1}{\frac{1}{d} - \frac{1}{2} + \frac{k+m_1-1}{2p}} = \frac{\frac{k}{p}\left(\frac{p}{q}\cdot\frac{1}{k}+1\right) - 1}{\frac{1}{d} - \frac{1}{2} + \frac{k+m_1-1}{2p}} = \frac{\frac{k}{p} + \frac{1}{q} - 1}{\frac{k}{2p} + \frac{1}{2q} - \frac{1}{2}} = 2,$$

$$r_1' \cdot \frac{\frac{1}{q} - \frac{1}{r_1'}}{\frac{1}{d} - \frac{1}{2} + \frac{l+m_2-1}{2q}} = \frac{\frac{r_1'}{q} - 1}{\frac{1}{d} - \frac{1}{2} + \frac{l+m_2-1}{2q}} = \frac{\frac{l+m_2-1}{q} + \frac{2}{d} - 1}{\frac{1}{d} - \frac{1}{2} + \frac{l+m_2-1}{2q}} = 2.$$

Hence r_1 fulfills (3.5), (3.6) and (3.9), where r_2 defined by (3.12) satisfies (3.7), (3.8) and (3.10). □

We next establish L^k-bound for u_ϵ and L^l-bound for w_ϵ with some suitable $k,l>1$ under smallness assumptions on initial data.

LEMMA 3.3. *Let $p,q>1$ be given by (1.6) and (1.7), $T>0$. Assume that the initial data (u_0,w_0) satisfies*

$$\|u_0\|_p < \alpha_1 := \min\{\delta_{u,p,p^*}, \delta_{u,q^*,q}, \delta_{u,s_1,s_1^*}, \delta_{u,s_1^{**},s_1}, \delta_{u,s_2,s_2^*}, \delta_{u,s_2^{**},s_2}\},$$

$$\|u_0\|_{q^*} < \alpha_2 := \min\{\delta_{w,p,p^*}^{q/q^*}, \delta_{w,q^*,q}^{q/q^*}, \delta_{w,s_1,s_1^*}^{q/q^*}, \delta_{w,s_1^{**},s_1}^{q/q^*}, \delta_{w,s_2,s_2^*}^{q/q^*}, \delta_{w,s_2^{**},s_2}^{q/q^*}\},$$

$$\|w_0\|_q < \alpha_3 := \min\{\delta_{w,p,p^*}, \delta_{w,q^*,q}, \delta_{w,s_1,s_1^*}, \delta_{w,s_1^{**},s_1}, \delta_{w,s_2,s_2^*}, \delta_{w,s_2^{**},s_2}\},$$

$$\|w_0\|_{p^*} < \alpha_4 := \min\{\delta_{u,p,p^*}^{p/p^*}, \delta_{u,q^*,q}^{p/p^*}, \delta_{u,s_1,s_1^*}^{p/p^*}, \delta_{u,s_1^{**},s_1}^{p/p^*}, \delta_{u,s_2,s_2^*}^{p/p^*}, \delta_{u,s_2^{**},s_2}^{p/p^*}\},$$

where

$$p^* = q + 1 - q/p, \; q^* = p + 1 - p/q,$$

$$s_1 = m_1 + d + 2, \; s_1^* = s_1 q/p + 1 - q/p, \; s_1^{**} = s_1 p/q + 1 - p/q,$$
$$s_2 = m_2 + d + 2, \; s_2^* = s_2 q/p + 1 - q/p, \; s_2^{**} = s_2 p/q + 1 - p/q,$$

$$\delta_{u,k,l} := \left(\frac{m_1 k(k-1)}{(k+m_1-1)^2(k+l-2)S_d}\right)^{\frac{d}{2p}},$$
$$\delta_{w,k,l} := \left(\frac{m_2 l(l-1)}{(l+m_2-1)^2(k+l-2)S_d}\right)^{\frac{d}{2q}}.$$

Then there exists a positive constant $C>0$ independent of T and ϵ such that

$$\sup_{0<t<T}\|u_\epsilon(t)\|_{p_0} \le C \text{ and } \sup_{0<t<T}\|w_\epsilon(t)\|_{p_0}\le C,$$

where $p_0 := \max\{s_1, s_2\}$.

Proof. Let $k>1$, $l=(k+m_1-1)q/p-m_2+1=kq/p+1-q/p$. Multiplying $(3.1)_1$ and $(3.1)_3$ by ku_ϵ^{k-1} and lw_ϵ^{l-1}, respectively, it gives

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}\left(u_\epsilon^k + w_\epsilon^l\right)dx + \frac{4m_1k(k-1)}{(k+m_1-1)^2}\int_{\mathbb{R}^d}\left|\nabla u_\epsilon^{\frac{k+m_1-1}{2}}\right|^2 dx \\
&\quad + \frac{4m_2l(l-1)}{(l+m_2-1)^2}\int_{\mathbb{R}^d}\left|\nabla w_\epsilon^{\frac{l+m_2-1}{2}}\right|^2 dx \\
&\le -(k-1)\int_{\mathbb{R}^d} u_\epsilon^k \Delta v_\epsilon dx - (l-1)\int_{\mathbb{R}^d} w_\epsilon^l \Delta z_\epsilon dx \\
&= (k-1)\int_{\mathbb{R}^d} u_\epsilon^k w_\epsilon dx + (l-1)\int_{\mathbb{R}^d} w_\epsilon^l u_\epsilon dx,
\end{aligned} \tag{3.13}$$

which together with Hölder's inequality yields that

$$\begin{aligned}
&(k-1)\int_{\mathbb{R}^d} u_\epsilon^k w_\epsilon dx + (l-1)\int_{\mathbb{R}^d} w_\epsilon^l u_\epsilon dx \\
&\le (k-1)\|u_\epsilon\|_{kr_1}^k \|w_\epsilon\|_{r_1'} + (l-1)\|w_\epsilon\|_{lr_2}^l \|u_\epsilon\|_{r_2'}
\end{aligned} \tag{3.14}$$

for some $r_1, r_2 > 1$, $r_1' = r_1/(r_1-1)$ and $r_2' = r_2/(r_2-1)$. Next, based on the choices of k, l, r_1 and r_2 in (3.3)-(3.10), we use the left-hand side terms of (3.13) to control the right-hand side terms of (3.14).

The conditions (3.3), (3.5) and (3.6) ensure that

$$p < kr_1 < \frac{(k+m_1-1)d}{d-2},$$
$$q < r_1' < \frac{(l+m_2-1)d}{d-2},$$

which allow us to employ Young's inequality and the Gagliardo-Nirenberg inequality (see Lemma 2.1) to obtain

$$\begin{aligned}
(k-1)\|u_\epsilon\|_{kr_1}^k \|w_\epsilon\|_{r_1'} &\le (k-1)\|u_\epsilon\|_{kr_1}^{kr_1} + (k-1)\|w_\epsilon\|_{r_1'}^{r_1'} \\
&\le (k-1) S_d^{\frac{kr_1\theta_1}{k+m_1-1}} \|u_\epsilon\|_p^{kr_1(1-\theta_1)} \left\|\nabla u_\epsilon^{\frac{k+m_1-1}{2}}\right\|_2^{\frac{2kr_1\theta_1}{k+m_1-1}} \\
&\quad + (k-1) S_d^{\frac{r_1'\theta_2}{l+m_2-1}} \|w_\epsilon\|_q^{r_1'(1-\theta_2)} \left\|\nabla w_\epsilon^{\frac{l+m_2-1}{2}}\right\|_2^{\frac{2r_1'\theta_2}{l+m_2-1}}
\end{aligned}$$

$$=(k-1)S_d\|u_\epsilon\|_p^{\frac{2}{d}p}\left\|\nabla u_\epsilon^{\frac{k+m_1-1}{2}}\right\|_2^2+(k-1)S_d\|w_\epsilon\|_q^{\frac{2}{d}q}\left\|\nabla w_\epsilon^{\frac{l+m_2-1}{2}}\right\|_2^2 \tag{3.15}$$

with

$$\theta_1=\frac{k+m_1-1}{2}\frac{\frac{1}{p}-\frac{1}{kr_1}}{\frac{1}{d}-\frac{1}{2}+\frac{k+m_1-1}{2p}}\in(0,1),$$

$$\theta_2=\frac{l+m_2-1}{2}\frac{\frac{1}{q}-\frac{1}{r_1'}}{\frac{1}{d}-\frac{1}{2}+\frac{l+m_2-1}{2q}}\in(0,1),$$

where we have used (3.9)-(3.10). Also, the second term at the right-hand side of (3.14) could be dominated by the choices of $r_1>1$, $r_2>1$. In fact, by invoking the existence of $r_2>1$ such that

$$q<lr_2<\frac{(l+m_2-1)d}{d-2}$$

and

$$p<r_2'<\frac{(k+m_1-1)d}{d-2},$$

it follows from Young's inequality and the Gagliardo-Nirenberg inequality that

$$\begin{aligned}&(l-1)\|w_\epsilon\|_{lr_2}^l\|u_\epsilon\|_{r_2'}\le(l-1)\|u_\epsilon\|_{r_2'}^{r_2'}+(l-1)\|w_\epsilon\|_{lr_2}^{lr_2}\\ \le&(l-1)S_d\|u_\epsilon\|_p^{\frac{2}{d}p}\left\|\nabla u_\epsilon^{\frac{k+m_1-1}{2}}\right\|_2^2+(l-1)S_d\|w_\epsilon\|_q^{\frac{2}{d}q}\left\|\nabla w_\epsilon^{\frac{l+m_2-1}{2}}\right\|_2^2\end{aligned} \tag{3.16}$$

by (3.10), which together with (3.13)-(3.16) gives that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^d}\left(u_\epsilon^k+w_\epsilon^l\right)dx\\ &+\left(\frac{4m_1k(k-1)}{(k+m_1-1)^2}-(k+l-2)S_d\|u_\epsilon\|_p^{\frac{2}{d}p}\right)\int_{\mathbb{R}^d}\left|\nabla u_\epsilon^{\frac{k+m_1-1}{2}}\right|^2dx\\ &+\left(\frac{4m_2l(l-1)}{(l+m_2-1)^2}-(k+l-2)S_d\|w_\epsilon\|_q^{\frac{2}{d}q}\right)\int_{\mathbb{R}^d}\left|\nabla w_\epsilon^{\frac{l+m_2-1}{2}}\right|^2dx\\ \le&0.\end{aligned}$$

Integrating the above inequality over $(0,t)$, one has

$$\begin{aligned}&\int_{\mathbb{R}^d}\left(u_\epsilon^k+w_\epsilon^l\right)dx\\ &+\int_0^t\left(\frac{4m_1k(k-1)}{(k+m_1-1)^2}-(k+l-2)S_d\|u_\epsilon(s)\|_p^{\frac{2}{d}p}\right)\cdot\int_{\mathbb{R}^d}\left|\nabla u_\epsilon^{\frac{k+m_1-1}{2}}(s)\right|^2dxds\\ &+\int_0^t\left(\frac{4m_2l(l-1)}{(l+m_2-1)^2}-(k+l-2)S_d\|w_\epsilon(s)\|_q^{\frac{2}{d}q}\right)\cdot\int_{\mathbb{R}^d}\left|\nabla w_\epsilon^{\frac{l+m_2-1}{2}}(s)\right|^2dxds\end{aligned}$$

$$\leq \int_{\mathbb{R}^d} u_{0\epsilon}^k dx + \int_{\mathbb{R}^d} w_{0\epsilon}^l dx. \tag{3.17}$$

With $k=p$, $l=p^*$ in (3.17), if $\|u_0\|_p<\delta_{u,p,p^*}$ and $\|w_0\|_q<\delta_{w,p,p^*}$, then there exists $T_1>0$ such that

$$\int_{\mathbb{R}^d} u_\epsilon^p dx \leq \int_{\mathbb{R}^d} u_{0\epsilon}^p dx + \int_{\mathbb{R}^d} w_{0\epsilon}^{p^*} dx \quad \text{for} \quad t\in(0,T_1]. \tag{3.18}$$

In addition, with $k=q^*$, $l=q$ in (3.17), if $\|u_0\|_p<\delta_{u,q^*,q}$ and $\|w_0\|_q<\delta_{w,q^*,q}$, there exists $T_2>0$ such that

$$\int_{\mathbb{R}^d} w_\epsilon^q dx \leq \int_{\mathbb{R}^d} u_{0\epsilon}^{q^*} dx + \int_{\mathbb{R}^d} w_{0\epsilon}^{q} dx \quad \text{for} \quad t\in(0,T_2]. \tag{3.19}$$

By inserting (3.18) and (3.19) into (3.17), and by choosing $k=p, l=p^*$ and $k=q^*$, $l=q$ it is clear that if

$$\|u_0\|_{q^*} \leq \min\{\delta_{w,p,p^*}^{q/q^*}, \delta_{w,q^*,q}^{q/q^*}\},$$
$$\|w_0\|_{p^*} \leq \min\{\delta_{u,p,p^*}^{p/p^*}, \delta_{u,q^*,q}^{p/p^*}\}$$

is true, then (3.18)-(3.19) hold for $t\in(0,2\min\{T_1,T_2\}]$ by the continuity. Repeating the above procedures, one has (3.18) and (3.19) for $t\in(0,T)$. Consequently, choosing $k=s_1,s_2,s_1^{**},s_2^{**}$, $l=s_1^*,s_2^*,s_1,s_2$ in (3.17), respectively, with the smallness assumption on initial data in the lemma, we obtain from (3.17) that

$$\|u_\epsilon\|_{L^\infty(0,T;L^{p_0}(\mathbb{R}^d))} \leq C \text{ and } \|w_\epsilon\|_{L^\infty(0,T;L^{p_0}(\mathbb{R}^d))} \leq C.$$

Then we have finished our proof. ∎

By means of Lemma 3.3, we have L^∞-bound for the solutions.

LEMMA 3.4. *Let $T>0$. Under the same assumptions on the initial data (u_0,w_0) in Lemma 3.3, there exists a positive constant $C>0$ independent of T and ϵ such that*

$$\sup_{0<t<T} (\|u_\epsilon(t)\|_r + \|w_\epsilon(t)\|_r) \leq C,\ r\in[1,\infty]. \tag{3.20}$$

In addition,

$$\sup_{0<t<T} (\|v_\epsilon(t)\|_r + \|z_\epsilon(t)\|_r) \leq C,\ r\in(d/(d-2),\infty], \tag{3.21}$$

$$\sup_{0<t<T} (\|\nabla v_\epsilon(t)\|_r + \|\nabla z_\epsilon(t)\|_r) \leq C,\ r\in(d/(d-1),\infty]. \tag{3.22}$$

Proof. The proof of the lemma relies on a bootstrap iterative technique, which has been used in [3, 25] to get the L^∞-bound on a degenerate one population Keller-Segel model (1.5). The main core is to establish the following inequality

$$\frac{d}{dt}\int_{\mathbb{R}^d} (u_\epsilon^{p_k}+w_\epsilon^{p_k}) dx \leq -\int_{\mathbb{R}^d} (u_\epsilon^{p_k}+w_\epsilon^{p_k}) dx$$
$$+q_k\left(\left(\int_{\mathbb{R}^d} u_\epsilon^{p_{k-1}} dx\right)^{\gamma_1} + \left(\int_{\mathbb{R}^d} w_\epsilon^{p_{k-1}} dx\right)^{\gamma_2}\right.$$

$$+\left(\int_{\mathbb{R}^d} u_\epsilon^{p_{k-1}}dx\right)^{\gamma_3}+\left(\int_{\mathbb{R}^d} w_\epsilon^{p_{k-1}}dx\right)^{\gamma_4}\Bigg), \tag{3.23}$$

where

$$p_k := \max\{2^k+m_1+d+1, 2^k+m_2+d+1\},$$
$$q_k = C(m_1,m_2,d)p_k^\alpha$$

with $C(m_1,m_2,d)>1$, $\alpha\le d+1$, $k\in\mathbb{N}$, $\gamma_i\le 2$, $i=1,\dots,4$. To begin with, under the assumptions of initial data, it leads to

$$\|u_\epsilon\|_{L^\infty(0,T;L^{p_0}(\mathbb{R}^d))}\le C, \quad \|w_\epsilon\|_{L^\infty(0,T;L^{p_0}(\mathbb{R}^d))}\le C \tag{3.24}$$

with $p_0:=\max\{m_1+d+2, m_2+d+2\}$ from Lemma 3.3.

Now we would like to get the estimate (3.23). Testing $(3.1)_1$ with $p_k u_\epsilon^{p_k-1}$, multiplying $(3.1)_3$ by $p_k w_\epsilon^{p_k-1}$, then summing them up and using Young's inequality we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}(u_\epsilon^{p_k}+w_\epsilon^{p_k})dx\\
\le&-\frac{4m_1p_k(p_k-1)}{(p_k+m_1-1)^2}\int_{\mathbb{R}^d}\left|\nabla u_\epsilon^{\frac{p_k+m_1-1}{2}}\right|^2dx-\frac{4m_2p_k(p_k-1)}{(p_k+m_2-1)^2}\int_{\mathbb{R}^d}\left|\nabla w_\epsilon^{\frac{p_k+m_2-1}{2}}\right|^2dx\\
&+(p_k-1)\int_{\mathbb{R}^d}u_\epsilon^{p_k}w_\epsilon dx+(p_k-1)\int_{\mathbb{R}^d}u_\epsilon w_\epsilon^{p_k}dx\\
\le&-\frac{4m_1p_k(p_k-1)}{(p_k+m_1-1)^2}\int_{\mathbb{R}^d}\left|\nabla u_\epsilon^{\frac{p_k+m_1-1}{2}}\right|^2dx-\frac{4m_2p_k(p_k-1)}{(p_k+m_2-1)^2}\int_{\mathbb{R}^d}\left|\nabla w_\epsilon^{\frac{p_k+m_2-1}{2}}\right|^2dx\\
&+2(p_k-1)\int_{\mathbb{R}^d}u_\epsilon^{p_k+1}dx+2(p_k-1)\int_{\mathbb{R}^d}w_\epsilon^{p_k+1}dx.
\end{aligned} \tag{3.25}$$

Since

$$1<\frac{p_k+1}{p_{k-1}}<\min\left\{\frac{(p_k+m_1-1)d}{(d-2)p_{k-1}}, \frac{p_k+m_1-1}{p_{k-1}}+\frac{2}{d}\right\},$$

it enables us to make use of (2.5) to find a positive constant $\eta_1>0$ such that

$$\begin{aligned}
\|u_\epsilon\|_{p_k+1}^{p_k+1}\le&\eta_1 S_d\left\|\nabla u_\epsilon^{\frac{p_k+m_1-1}{2}}\right\|_2^2\\
&+\eta_1^{-\frac{(p_k+1)\theta_3}{p_k+m_1-(p_k+1)\theta_3-1}}\|u_\epsilon\|_{p_{k-1}}^{(p_k+1)(1-\theta_3)\cdot\frac{p_k+m_1-1}{p_k+m_1-(p_k+1)\theta_3-1}}\\
=&\eta_1 S_d\left\|\nabla u_\epsilon^{\frac{p_k+m_1-1}{2}}\right\|_2^2+\eta_1^{-\frac{p_k-p_{k-1}+1}{\frac{2}{d}p_{k-1}+m_1-2}}\|u_\epsilon\|_{p_{k-1}}^{\gamma_1 p_{k-1}}
\end{aligned} \tag{3.26}$$

with

$$\theta_3=\frac{p_k+m_1-1}{2}\left(\frac{1}{p_{k-1}}-\frac{1}{p_k+1}\right)\left(\frac{1}{d}-\frac{1}{2}+\frac{p_k+m_1-1}{2p_{k-1}}\right)^{-1},$$

where

$$\gamma_1:=(1-\theta_3)\cdot\frac{(p_k+1)(p_k+m_1-1)}{p_{k-1}(p_k+m_1-(p_k+1)\theta_3-1)}$$

$$
\begin{aligned}
&=(1-\theta_3)\cdot\frac{p_k+1}{p_{k-1}}\cdot\frac{\frac{2}{d}-1+\frac{p_k+m_1-1}{p_{k-1}}}{\frac{2}{d}+\frac{m_1-2}{p_{k-1}}}\\
&=(1-\theta_3)\cdot\frac{(p_k+1)\cdot\left(\frac{2}{d}-1+\frac{p_k+m_1-1}{p_{k-1}}\right)}{\frac{2}{d}p_{k-1}+m_1-2}\\
&=\frac{(p_k+1)\cdot\left(\frac{2}{d}-1+\frac{p_k+m_1-1}{p_{k-1}}\right)}{\frac{2}{d}p_{k-1}+m_1-2}-\frac{(p_k+1)\cdot(p_k+m_1-1)\left(\frac{1}{p_{k-1}}-\frac{1}{p_k+1}\right)}{\frac{2}{d}p_{k-1}+m_1-2}\\
&=\frac{(p_k+1)\cdot\left(\frac{2}{d}+\frac{m_1-2}{p_k+1}\right)}{\frac{2}{d}p_{k-1}+m_1-2}\\
&=\frac{\frac{2}{d}(p_k+1)+m_1-2}{\frac{2}{d}p_{k-1}+m_1-2}\leq 2.
\end{aligned}
$$

By a similar computation, $\|w_\epsilon\|_{p_k+1}^{p_k+1}$ can be also bounded by

$$
\|w_\epsilon\|_{p_k+1}^{p_k+1}\leq \eta_1 S_d\left\|\nabla w_\epsilon^{\frac{p_k+m_2-1}{2}}\right\|_2^2+\eta_1^{-\frac{p_k-p_{k-1}+1}{\frac{2}{d}p_{k-1}+m_2-2}}\|w_\epsilon\|_{p_{k-1}}^{\gamma_2 p_{k-1}} \tag{3.27}
$$

with

$$
\gamma_2:=\frac{\frac{2}{d}(p_k+1)+m_2-2}{\frac{2}{d}p_{k-1}+m_2-2}\leq 2.
$$

By inserting (3.26) - (3.27) into (3.25), it is easy to see that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}(u_\epsilon^{p_k}+w_\epsilon^{p_k})dx\leq\left(2\eta_1(p_k-1)S_d-\frac{4m_1p_k(p_k-1)}{(p_k+m_1-1)^2}\right)\int_{\mathbb{R}^d}\left|\nabla u_\epsilon^{\frac{p_k+m_1-1}{2}}\right|^2dx\\
&+\left(2\eta_1(p_k-1)S_d-\frac{4m_2p_k(p_k-1)}{(p_k+m_2-1)^2}\right)\int_{\mathbb{R}^d}\left|\nabla w_\epsilon^{\frac{p_k+m_2-1}{2}}\right|^2dx\\
&+2(p_k-1)\eta_1^{-\frac{p_k-p_{k-1}+1}{\frac{2}{d}p_{k-1}+m_1-2}}\left(\int_{\mathbb{R}^d}u_\epsilon^{p_{k-1}}dx\right)^{\gamma_1}+2(p_k-1)\eta_1^{-\frac{p_k-p_{k-1}+1}{\frac{2}{d}p_{k-1}+m_2-2}}\left(\int_{\mathbb{R}^d}w_\epsilon^{p_{k-1}}dx\right)^{\gamma_2}.
\end{aligned}
$$

Because $p_k>\max\{m_1+1,m_2+1\}$, then it is possible to choose $\eta_1=\min\Big\{\frac{m_1}{4(p_k-1)S_d},$ $\frac{m_2}{4(p_k-1)S_d}\Big\}>0$ such that

$$
\begin{aligned}
2\eta_1(p_k-1)S_d-\frac{4m_1p_k(p_k-1)}{(p_k+m_1-1)^2}<-\frac{m_1}{2},\\
2\eta_1(p_k-1)S_d-\frac{4m_2p_k(p_k-1)}{(p_k+m_2-1)^2}<-\frac{m_2}{2},
\end{aligned}
$$

which induces that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}(u_\epsilon^{p_k}+w_\epsilon^{p_k})dx\\
&\leq-\frac{m_1}{2}\int_{\mathbb{R}^d}\left|\nabla u_\epsilon^{\frac{p_k+m_1-1}{2}}\right|^2dx-\frac{m_2}{2}\int_{\mathbb{R}^d}\left|\nabla w_\epsilon^{\frac{p_k+m_2-1}{2}}\right|^2dx
\end{aligned}
$$

$$+C(m_1,m_2,d)^{-\frac{p_k-p_{k-1}+1}{\frac{2}{d}p_{k-1}+m_1-2}}(p_k-1)^{\frac{p_k-\left(1-\frac{2}{d}\right)p_{k-1}+m_1-1}{\frac{2}{d}p_{k-1}+m_1-2}}\left(\int_{\mathbb{R}^d}u_\epsilon^{p_{k-1}}dx\right)^{\gamma_1}$$
$$+C(m_1,m_2,d)^{-\frac{p_k-p_{k-1}+1}{\frac{2}{d}p_{k-1}+m_2-2}}(p_k-1)^{\frac{p_k-\left(1-\frac{2}{d}\right)p_{k-1}+m_2-1}{\frac{2}{d}p_{k-1}+m_2-2}}\left(\int_{\mathbb{R}^d}w_\epsilon^{p_{k-1}}dx\right)^{\gamma_2}.$$

On the other hand, for $\eta_2>0$, one has

$$\|u_\epsilon\|_{p_k}^{p_k}\leq\eta_2S_d\left\|\nabla u_\epsilon^{\frac{p_k+m_1-1}{2}}\right\|_2^2+\eta_2^{-\frac{p_k-p_{k-1}}{\frac{2}{d}p_{k-1}+m_1-1}}\|u_\epsilon\|_{p_{k-1}}^{\gamma_3p_{k-1}},$$
$$\|w_\epsilon\|_{p_k}^{p_k}\leq\eta_2S_d\left\|\nabla w_\epsilon^{\frac{p_k+m_2-1}{2}}\right\|_2^2+\eta_2^{-\frac{p_k-p_{k-1}}{\frac{2}{d}p_{k-1}+m_2-1}}\|w_\epsilon\|_{p_{k-1}}^{\gamma_4p_{k-1}},$$

where

$$\gamma_3:=\frac{\frac{2}{d}p_k+m_1-1}{\frac{2}{d}p_{k-1}+m_1-1}\leq2,\quad\gamma_4:=\frac{\frac{2}{d}p_k+m_2-1}{\frac{2}{d}p_{k-1}+m_2-1}\leq2.$$

Taking $\eta_2=\min\left\{\frac{m_1}{2S_d},\frac{m_2}{2S_d}\right\}$, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}(u_\epsilon^{p_k}+w_\epsilon^{p_k})dx\leq&-\int_{\mathbb{R}^d}(u_\epsilon^{p_k}+w_\epsilon^{p_k})dx\\&+C(m_1,m_2,d)^{-\frac{p_k-p_{k-1}+1}{\frac{2}{d}p_{k-1}+m_1-2}}(p_k-1)^{\frac{p_k-\left(1-\frac{2}{d}\right)p_{k-1}+m_1-1}{\frac{2}{d}p_{k-1}+m_1-2}}\left(\int_{\mathbb{R}^d}u_\epsilon^{p_{k-1}}dx\right)^{\gamma_1}\\&+C(m_1,m_2,d)^{-\frac{p_k-p_{k-1}+1}{\frac{2}{d}p_{k-1}+m_2-2}}(p_k-1)^{\frac{p_k-\left(1-\frac{2}{d}\right)p_{k-1}+m_2-1}{\frac{2}{d}p_{k-1}+m_2-2}}\left(\int_{\mathbb{R}^d}w_\epsilon^{p_{k-1}}dx\right)^{\gamma_2}\\&+C(m_1,m_2,d)^{-\frac{p_k-p_{k-1}}{\frac{2}{d}p_{k-1}+m_1-1}}\left(\int_{\mathbb{R}^d}u_\epsilon^{p_{k-1}}dx\right)^{\gamma_3}\\&+C(m_1,m_2,d)^{-\frac{p_k-p_{k-1}}{\frac{2}{d}p_{k-1}+m_2-1}}\left(\int_{\mathbb{R}^d}w_\epsilon^{p_{k-1}}dx\right)^{\gamma_4}.\end{aligned}\tag{3.28}$$

By observing that for $k\geq0$, it gives

$$\frac{p_k-\left(1-\frac{2}{d}\right)p_{k-1}+m_1-1}{\frac{2}{d}p_{k-1}+m_1-2}\leq d+1,$$
$$\frac{p_k-\left(1-\frac{2}{d}\right)p_{k-1}+m_2-1}{\frac{2}{d}p_{k-1}+m_2-2}\leq d+1.$$

Moreover, it is obvious that

$$\frac{p_k-p_{k-1}+1}{\frac{2}{d}p_{k-1}+m_1-2}\sim O(1),\quad\frac{p_k-p_{k-1}+1}{\frac{2}{d}p_{k-1}+m_2-2}\sim O(1),$$
$$\frac{p_k-p_{k-1}}{\frac{2}{d}p_{k-1}+m_1-1}\sim O(1),\quad\frac{p_k-p_{k-1}}{\frac{2}{d}p_{k-1}+m_2-1}\sim O(1),$$

as $k\to\infty$, then we infer that (3.23) holds out from (3.28).

Let $y_k(t):=\int_{\mathbb{R}^d}(u_\epsilon^{p_k}+w_\epsilon^{p_k})dx$, then (3.23) implies that

$$
\begin{aligned}
\left(e^t y_k(t)\right)' \leq & q_k\Bigg(\left(\left(\int_{\mathbb{R}^d} u_\epsilon^{p_{k-1}}dx\right)^{\gamma_1}+\left(\int_{\mathbb{R}^d} w_\epsilon^{p_{k-1}}dx\right)^{\gamma_2}\right.\\
&\left.+\left(\int_{\mathbb{R}^d} u_\epsilon^{p_{k-1}}dx\right)^{\gamma_3}+\left(\int_{\mathbb{R}^d} w_\epsilon^{p_{k-1}}dx\right)^{\gamma_4}\right)e^t\\
\leq & 4q_k \max\left\{1,\sup_{t\geq 0} y_{k-1}^{\gamma_{\max}}(t)\right\}e^t
\end{aligned}
$$

with $\gamma_{\max}:=\max\{\gamma_1,\dots,\gamma_4\}\leq 2$, $q_k=C(m_1,m_2,d)p_k^\alpha$, $\alpha\leq d+1$. Integrating the above inequality over $(0,t)$ yields that

$$
\begin{aligned}
y_k(t)\leq & (1-e^{-t})4q_k\max\left\{1,\sup_{t\geq 0} y_{k-1}^{\gamma_{\max}}(t)\right\}+e^{-t}y_k(0)\\
\leq & 2^2 q_k \max\left\{\sup_{t\geq 0} y_{k-1}^2(t),K\right\},
\end{aligned}
$$

where $K:=\max\{y_k(0),1\}$ is a constant. Invoking iterative arguments, we finally find a positive number $C(m_1,m_2,d)>1$ such that

$$
\begin{aligned}
y_k(t)\leq & 2^2 q_k(2^2q_{k-1})^2(2^2q_{k-2})^{2^2}\dots(2^2q_1)^{2^{k-1}}\max\left\{\sup_{t\geq 0} y_0^{2^k}(t),K^{2^{k-1}}\right\}\\
\leq & (C(m_1,m_2,d))^{2^k-1}2^{2^{k+1}-2}(2^{d+1})^{k+2(k-1)+2^2(k-2)+\dots+2^{k-1}(k-(k-1))}\\
& \cdot\max\left\{\sup_{t\geq 0} y_0^{2^k}(t),K^{2^{k-1}}\right\}\\
= & (C(m_1,m_2,d))^{2^k-1}2^{2^{k+1}-2}(2^{d+1})^{2^{k+1}-k-2}\cdot\max\left\{\sup_{t\geq 0} y_0^{2^k}(t),K^{2^{k-1}}\right\}. \qquad (3.29)
\end{aligned}
$$

Taking both sides of (3.29) to the power $1/p_k$, it shows that

$$
\begin{aligned}
\|u_\epsilon(t)\|_{p_k}+\|w_\epsilon(t)\|_{p_k}\leq & 2^{1-\frac{1}{p_k}}\left(\|u\|_{p_k}^{p_k}+\|w\|_{p_k}^{p_k}\right)^{\frac{1}{p_k}}\\
\leq & 2^{2d+5}C(m_1,m_2,d)\max\left\{\sup_{t\geq 0} y_0(t),K\right\},
\end{aligned}
$$

where as $k\to\infty$ we obtain

$$
\|u_\epsilon(t)\|_\infty+\|w_\epsilon(t)\|_\infty\leq 2^{2d+5}C(m_1,m_2,d)\max\left\{\sup_{t\geq 0} y_0(t),K\right\}.
$$

Since the inequality (3.24) implies that there exists a positive constant $C>0$ independent of $k>0$ such that

$$
\sup_{t\geq 0} y_0(t)=\sup_{t\geq 0}\left(\|u_\epsilon(t)\|_{p_0}^{p_0}+\|w_\epsilon(t)\|_{p_0}^{p_0}\right)\leq C,
$$

we obtain L^∞-bound for u_ϵ and w_ϵ. Therefore one combines the L^∞-bound with the L^1-norms (3.2) to get (3.20).

Now we would like to achieve the regularities for v_ϵ and z_ϵ. By means of the following representation,

$$
v_\epsilon=\mathcal{K}*w_\epsilon=c_d\int_{\mathbb{R}^d}\frac{w_\epsilon(y)}{|x-y|^{d-2}}dy,\ z_\epsilon=\mathcal{K}*u_\epsilon=c_d\int_{\mathbb{R}^d}\frac{u_\epsilon(y)}{|x-y|^{d-2}}dy,
$$

an application of the weak Young inequality [34, formula (9), pp. 107] and the boundedness of (u_ϵ, w_ϵ) guarantees that

$$\begin{aligned}
\|v_\epsilon\|_r \le & c_d \left\| w_\epsilon * \frac{1}{|x|^{d-2}} \right\|_r \le C\|w_\epsilon\|_{\frac{dr}{d+2r}} \left\| \frac{1}{|x|^{d-2}} \right\|_{\omega, \frac{d}{d-2}} < \infty, \quad r \in (d/(d-2), \infty), \\
\|z_\epsilon\|_r \le & c_d \|u_\epsilon\|_{\frac{dr}{d+2r}} \left\| \frac{1}{|x|^{d-2}} \right\|_{\omega, \frac{d}{d-2}} < \infty, \quad r \in (d/(d-2), \infty), \\
\|\nabla v_\epsilon\|_r \le & c_d (d-2) \left\| w_\epsilon * \frac{x}{|x|^d} \right\|_r \\
\le & C\|w_\epsilon\|_{\frac{dr}{d+r}} \left\| \frac{1}{|x|^{d-1}} \right\|_{\omega, \frac{d}{d-1}} < \infty, \quad r \in (d/(d-1), \infty), \\
\|\nabla z_\epsilon\|_r \le & C\|w_\epsilon\|_{\frac{dr}{d+r}} \left\| \frac{1}{|x|^{d-1}} \right\|_{\omega, \frac{d}{d-1}} < \infty, \quad r \in (d/(d-1), \infty),
\end{aligned}$$

where $\|\cdot\|_{\omega,r}$ with $r>1$ is defined by

$$\|f\|_{\omega,r} = \sup_A |A|^{-(r-1)/r} \int_A |f(x)| dx$$

for any measurable function f and arbitrary measure finite set A. Next, one makes use of the well-known Calderon-Zygmund inequality to obtain a constant $C = C(r) > 0$ with $r \in (1, \infty)$ such that

$$\|\partial_{x_i} \partial_{x_j} v_\epsilon\|_r \le C\|w_\epsilon\|_r < \infty, \ \|\partial_{x_i} \partial_{x_j} z_\epsilon\|_r \le C\|u_\epsilon\|_r < \infty, \quad 1 \le i, j \le d,$$

which combines with the Morrey's inequality to ensure that

$$\|(v_\epsilon, z_\epsilon)\|_r + \|(\nabla v_\epsilon, \nabla z_\epsilon)\|_r \le C \quad \text{for} \quad r \in (d, \infty].$$

Thus we obtained the claimed estimates for v_ϵ and z_ϵ and the proof is complete. ∎

3.2. Convergence. Some addition regularities of solutions to an approximated system (3.1) will be established at the first step, which ensure the convergence of (3.1) to the claimed weak solution of (1.1)-(1.2).

LEMMA 3.5. *Let $T > 0$. Under the same assumptions on the initial data (u_0, w_0) in Lemma 3.3, then there exists a positive constant $C > 0$ independent of ϵ such that*

$$\left\| \partial_t (u_\epsilon + \epsilon)^{\frac{m_1+1}{2}} \right\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} + \sup_{0<t<T} \|\nabla (u_\epsilon + \epsilon)^{m_1}\|_2^2 \le C, \tag{3.30}$$

$$\left\| \partial_t (w_\epsilon + \epsilon)^{\frac{m_2+1}{2}} \right\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} + \sup_{0<t<T} \|\nabla (w_\epsilon + \epsilon)^{m_2}\|_2^2 \le C, \tag{3.31}$$

$$\left\| \nabla (u_\epsilon + \epsilon)^{\frac{m_1+1}{2}} \right\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} \le C, \tag{3.32}$$

$$\left\| \nabla (w_\epsilon + \epsilon)^{\frac{m_2+1}{2}} \right\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} \le C. \tag{3.33}$$

Proof. Testing (3.1)$_1$ by $\partial_t(u_\epsilon+\epsilon)^{m_1}$, integrating it over $\mathbb{R}^d$, and using Young's inequality, we see that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|\nabla(u_\epsilon+\epsilon)^{m_1}|^2dx+\frac{4m_1}{(m_1+1)^2}\int_{\mathbb{R}^d}\left|\partial_t(u_\epsilon+\epsilon)^{\frac{m_1+1}{2}}\right|^2dx\\
=&-\frac{2m_1}{m_1+1}\int_{\mathbb{R}^d}(u_\epsilon+\epsilon)^{\frac{m_1-1}{2}}\nabla\cdot(u_\epsilon\nabla v_\epsilon)\cdot\partial_t(u_\epsilon+\epsilon)^{\frac{m_1+1}{2}}dx\\
\le&\frac{2m_1}{(m_1+1)^2}\int_{\mathbb{R}^d}\left|\partial_t(u_\epsilon+\epsilon)^{\frac{m_1+1}{2}}\right|^2dx+C(m_1)\|\nabla v_\epsilon\|^2_{L^\infty(Q_T)}\int_{\mathbb{R}^d}\left|\nabla(u_\epsilon+\epsilon)^{\frac{m_1+1}{2}}\right|^2dx\\
&+C(m_1)\left(\|u_\epsilon\|_{L^\infty(Q_T)}+\epsilon\right)^{m_1+1}\int_{\mathbb{R}^d}|\Delta v_\epsilon|^2dx.
\end{aligned}$$

Integrating the above inequality over $(0,t)$, $t\in(0,T)$, it leads to

$$\begin{aligned}
&\frac{2m_1}{(m_1+1)^2}\left\|\partial_t(u_\epsilon+\epsilon)^{\frac{m_1+1}{2}}\right\|^2_{L^2(0,t;L^2(\mathbb{R}^d))}\\
\le&-\frac{1}{2}\left(\|\nabla(u_\epsilon+\epsilon)^{m_1}(t)\|_2^2-\|\nabla(u_{0\epsilon}+\epsilon)^{m_1}\|_2^2\right)\\
&+C(m_1)\|\nabla v_\epsilon\|^2_{L^\infty(Q_T)}\left\|\nabla(u_\epsilon+\epsilon)^{\frac{m_1+1}{2}}\right\|^2_{L^2(0,t;L^2(\mathbb{R}^d))}\\
&+C(m_1)\left(\|u_\epsilon\|_{L^\infty(Q_T)}+\epsilon\right)^{m_1+1}\|\Delta v_\epsilon\|^2_{L^2(0,t;L^2(\mathbb{R}^d))}.
\end{aligned}\tag{3.34}$$

In order to deal with $\nabla(u_\epsilon+\epsilon)^{\frac{m_1+1}{2}}$ in $L^2(0,t;L^2(\mathbb{R}^d))$, we test (3.1)$_1$ by u_ϵ and integrate it over $(0,t)$ and Young's inequality to have

$$\begin{aligned}
&\|u_\epsilon(t)\|_2^2+\frac{8m_1}{(m_1+1)^2}\left\|\nabla(u_\epsilon+\epsilon)^{\frac{m_1+1}{2}}\right\|^2_{L^2(0,t;L^2(\mathbb{R}^d))}\\
\le&\|u_\epsilon\|^3_{L^3(0,t;L^3(\mathbb{R}^d))}+\|w_\epsilon\|^3_{L^3(0,t;L^3(\mathbb{R}^d))}+\|u_{0\epsilon}\|_2^2.
\end{aligned}\tag{3.35}$$

Here, we obtain (3.30) for any $\epsilon\in(0,1)$ by means of (3.20), (3.22), (3.34), (3.35) and the facts that

$$u_0\in H^1(\mathbb{R}^d),\quad u_0\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d),\quad -\Delta v_\epsilon=w_\epsilon.$$

Obviously, (3.32) holds true due to (3.34) - (3.35). Moreover, (3.31) and (3.33) can be obtained in a similar argument. ∎

At the second step, we claim the following convergence through Lemma 3.5.

LEMMA 3.6. *Let $(u_\epsilon,v_\epsilon,w_\epsilon,z_\epsilon)$ be a unique solution of (3.1) and $T>0$. Then with the regularities of solutions obtained in above lemmas at hand, one can pick up a subsequence $(\epsilon_n)_n$ of ϵ satisfying*

$$u_{\epsilon_n}\to u \text{ weakly-}* \text{ in } L^\infty(0,T;L^r(\mathbb{R}^d)),\ \forall r\in[1,\infty],\tag{3.36}$$
$$u_{\epsilon_n}\to u \text{ strongly in } C(0,T;L^r_{loc}(\mathbb{R}^d)),\ \forall r\in[1,\infty),\tag{3.37}$$
$$\nabla(u_{\epsilon_n}+\epsilon_n)^{m_1}\to\nabla u^{m_1} \text{ weakly in } L^2(0,T;L^2(\mathbb{R}^d)),\tag{3.38}$$
$$w_{\epsilon_n}\to w \text{ weakly-}* \text{ in } L^\infty(0,T;L^r(\mathbb{R}^d)),\ \forall r\in[1,\infty],\tag{3.39}$$
$$w_{\epsilon_n}\to w \text{ strongly in } C(0,T;L^r_{loc}(\mathbb{R}^d)),\ \forall r\in[1,\infty),\tag{3.40}$$
$$\nabla(w_{\epsilon_n}+\epsilon_n)^{m_2}\to\nabla w^{m_2} \text{ weakly in } L^2(0,T;L^2(\mathbb{R}^d)),\tag{3.41}$$

$$v_{\epsilon_n}(t) \to v(t) \textit{ weakly-}* \textit{ in } L^\infty(0,T;L^r(\mathbb{R}^d)),\ \forall r \in (d/(d-2),\infty], \tag{3.42}$$
$$\nabla v_{\epsilon_n}(t) \to \nabla v(t) \textit{ weakly-}* \textit{ in } L^\infty(0,T;L^r(\mathbb{R}^d)),\ \forall r \in (d/(d-1),\infty], \tag{3.43}$$
$$z_{\epsilon_n}(t) \to z(t) \textit{ weakly-}* \textit{ in } L^\infty(0,T;L^r(\mathbb{R}^d)),\ \forall r \in (d/(d-2),\infty], \tag{3.44}$$
$$\nabla z_{\epsilon_n}(t) \to \nabla z(t) \textit{ weakly-}* \textit{ in } L^\infty(0,T;L^r(\mathbb{R}^d)),\ \forall r \in (d/(d-1),\infty], \tag{3.45}$$

as $\epsilon_n \to 0$.

Proof. It follows from L^∞-bound (3.20) that there exist subsequences $\{u_{\epsilon_n}\}$ and $\{w_{\epsilon_n}\}$, and their corresponding limit function $(u,w) \in (L^\infty(0,T;L^\infty(\mathbb{R}^d)))^2$ such that (3.36) and (3.39) hold true. Because

$$\begin{aligned}
&\|\partial_t u_\epsilon^{m_1}\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} + \sup_{0<t<T} \|\nabla u_\epsilon^{m_1}\|_2^2 \\
\le& \|\partial_t (u_\epsilon+\epsilon)^{m_1}\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} + \sup_{0<t<T} \|\nabla (u_\epsilon+\epsilon)^{m_1}\|_2^2 \\
\le& \frac{4m_1^2}{(m_1+1)^2}\left(\|u_\epsilon\|_{L^\infty(Q_T)}+\epsilon\right)^{m_1-1} \left\|\partial_t (u_\epsilon+\epsilon)^{\frac{m_1+1}{2}}\right\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} \\
&+ \sup_{0<t<T} \|\nabla (u_\epsilon+\epsilon)^{m_1}\|_2^2
\end{aligned}$$

holds, (3.20), (3.30)-(3.31) imply that $(u_\epsilon^{m_1}, w_\epsilon^{m_2}) \in (L^\infty(0,T;H^1(\mathbb{R}^d)) \cap W^{1,2}(0,T;L^2(\mathbb{R}^d)))^2$, utilizing the Aubin-Lions lemma, we can find a subsequence of $\{u_{\epsilon_n}\}$ and $\{w_{\epsilon_n}\}$, respectively, such that

$$\begin{aligned}
u_{\epsilon_n}^{m_1} &\to \xi_1 \text{ strongly in } C(0,T;L^2_{loc}(\mathbb{R}^d)), \\
w_{\epsilon_n}^{m_2} &\to \xi_2 \text{ strongly in } C(0,T;L^2_{loc}(\mathbb{R}^d)).
\end{aligned}$$

For $x,y \ge 0$, $r \ge 1$, the fact $|x-y|^{2r} \le |x^r - y^r|^2$ tells us that

$$\begin{aligned}
u_{\epsilon_n} &\to \xi_1^{\frac{1}{m_1}} = u \text{ strongly in } C(0,T;L^{2m_1}_{loc}(\mathbb{R}^d)), \\
w_{\epsilon_n} &\to \xi_2^{\frac{1}{m_2}} = w \text{ strongly in } C(0,T;L^{2m_2}_{loc}(\mathbb{R}^d)).
\end{aligned}$$

Then based on the uniform boundedness of (u_ϵ, w_ϵ) in (3.20), it leads to (3.37) and (3.40). In addition, (3.32) - (3.33) ensure that

$$\begin{aligned}
\nabla (u_{\epsilon_n}+\epsilon_n)^{\frac{m_1+1}{2}} &\to \nabla u^{\frac{m_1+1}{2}} \text{ weakly } \textit{in } L^2(0,T;L^2(\mathbb{R}^d)), \\
\nabla (w_{\epsilon_n}+\epsilon_n)^{\frac{m_2+1}{2}} &\to \nabla w^{\frac{m_2+1}{2}} \text{ weakly } \textit{in } L^2(0,T;L^2(\mathbb{R}^d)).
\end{aligned}$$

Thanks to

$$\nabla u^{m_1} = \frac{2m_1}{m_1+1} u^{\frac{m_1-1}{2}} \nabla u^{\frac{m_1+1}{2}}, \nabla w^{m_2} = \frac{2m_2}{m_2+1} w^{\frac{m_2-1}{2}} \nabla w^{\frac{m_2+1}{2}},$$

(3.38) and (3.41) come into existence.

In view of (3.21)-(3.22), we can extract subsequences $\{v_{\epsilon_n}\}$ and $\{z_{\epsilon_n}\}$, and their corresponding limit function $(v,z) \in (L^\infty(0,T;W^{1,r}(\mathbb{R}^d)))^2$ with $r \in (d/(d-2),\infty]$ to let (3.42)-(3.45) be true. ∎

Proof. **(Proof of Theorem 1.1.)** Given $\phi\in C_0^\infty(\mathbb{R}^d\times[0,T))$ with $T>0$, testing equations in $(3.1)_1$ and $(3.1)_3$ by ϕ and integrating over $\mathbb{R}^d\times(0,T)$, we have

$$\int_0^T\int_{\mathbb{R}^d}(\nabla(u_{\epsilon_n}+\epsilon_n)^{m_1}\cdot\nabla\phi-u_{\epsilon_n}\nabla v_{\epsilon_n}\cdot\nabla\phi-u_{\epsilon_n}\phi_t)\,dxdt=\int_{\mathbb{R}^d}u_{0\epsilon_n}(x)\phi(x,0)dx,$$
$$\int_0^T\int_{\mathbb{R}^d}(\nabla(w_{\epsilon_n}+\epsilon_n)^{m_2}\cdot\nabla\phi-w_{\epsilon_n}\nabla z_{\epsilon_n}\cdot\nabla\phi-w_{\epsilon_n}\phi_t)\,dxdt=\int_{\mathbb{R}^d}w_{0\epsilon_n}(x)\phi(x,0)dx$$

with $v_{\epsilon_n}=\mathcal{K}*w_{\epsilon_n}$ and $z_{\epsilon_n}=\mathcal{K}*u_{\epsilon_n}$. From (3.38) and (3.41), it shows that

$$\int_0^T\int_{\mathbb{R}^d}\nabla(u_{\epsilon_n}+\epsilon_n)^{m_1}\cdot\nabla\phi dxdt\to\int_0^T\int_{\mathbb{R}^d}\nabla u^{m_1}\cdot\nabla\phi dxdt,$$
$$\int_0^T\int_{\mathbb{R}^d}\nabla(w_{\epsilon_n}+\epsilon_n)^{m_2}\cdot\nabla\phi dxdt\to\int_0^T\int_{\mathbb{R}^d}\nabla w^{m_2}\cdot\nabla\phi dxdt.$$

The strong convergences (3.37) and (3.40) for $\{u_{\epsilon_n}\}$ and $\{w_{\epsilon_n}\}$ together with (3.43) and (3.45) infer that

$$\int_0^T\int_{\mathbb{R}^d}u_{\epsilon_n}\nabla v_{\epsilon_n}\cdot\nabla\phi dxdt\to\int_0^T\int_{\mathbb{R}^d}u\nabla v\cdot\nabla\phi dxdt,$$
$$\int_0^T\int_{\mathbb{R}^d}w_{\epsilon_n}\nabla z_{\epsilon_n}\cdot\nabla\phi dxdt\to\int_0^T\int_{\mathbb{R}^d}w\nabla z\cdot\nabla\phi dxdt.$$

Moreover, (3.36) and (3.39) ensure that

$$\int_0^T\int_{\mathbb{R}^d}u_{\epsilon_n}\phi_t dxdt\to\int_0^T\int_{\mathbb{R}^d}u\phi_t dxdt,$$
$$\int_0^T\int_{\mathbb{R}^d}w_{\epsilon_n}\phi_t dxdt\to\int_0^T\int_{\mathbb{R}^d}w\phi_t dxdt.$$

Finally, the constructed initial data implies that

$$\int_{\mathbb{R}^d}u_{0\epsilon_n}(x)\phi(x,0)dx\to\int_{\mathbb{R}^d}u_0(x)\phi(x,0)dx,$$
$$\int_{\mathbb{R}^d}w_{0\epsilon_n}(x)\phi(x,0)dx\to\int_{\mathbb{R}^d}w_0(x)\phi(x,0)dx.$$

Therefore, the arbitrary choice of $T>0$ implies that a global weak solution (u,v,w,z) over $\mathbb{R}^d\times(0,T)$ has been formed as Definition 1.1. At the same time, this weak solution is uniformly bounded with respect to time since

$$\|u\|_{L^\infty(0,T;L^r(\mathbb{R}^d))}\le\liminf_{n\to\infty}\|u_{\epsilon_n}\|_{L^\infty(0,T;L^r(\mathbb{R}^d))}\le C,$$
$$\|w\|_{L^\infty(0,T;L^r(\mathbb{R}^d))}\le\liminf_{n\to\infty}\|w_{\epsilon_n}\|_{L^\infty(0,T;L^r(\mathbb{R}^d))}\le C,$$

for $r\in[1,\infty]$ by the facts (3.36), (3.39) and (3.20). ∎

4. Global existence for $\tau=1$

In this section, the global existence of solution for the fully parabolic type of (3.1) in the sense that $\tau=1$ will be considered. We follow the same argument of Lemmas 3.3 and 3.4 to establish a priori estimate for (u_ϵ,w_ϵ).

Lemma 4.1. *Let $T>0$. Let $(m_1,m_2)\in(1,d/2)^2$ satisfy*

$$m_1+m_2>m_1m_2+2m_1/d \text{ and } m_1+m_2>m_1m_2+2m_2/d.$$

Assume that $(u_\epsilon,v_\epsilon,w_\epsilon,z_\epsilon)$ is a unique solution of (3.1) with the initial data $(u_{0\epsilon},v_{0\epsilon},w_{0\epsilon},z_{0\epsilon})$. Then

$$\begin{aligned}&\int_{\mathbb{R}^d}(u_\epsilon^k+w_\epsilon^l)dx+\int_0^t\left(A_1(1)\cdot\left\|\nabla u_\epsilon^{\frac{k+m_1-1}{2}}(s)\right\|_2^2+A_2(1)\cdot\left\|\nabla w_\epsilon^{\frac{l+m_2-1}{2}}(s)\right\|_2^2\right)ds\\&\le\|u_{0\epsilon}\|_k^k+\|w_{0\epsilon}\|_l^l+(k-1)\|\Delta v_{0\epsilon}\|_{r_1'}^{r_1'}+(l-1)\|\Delta z_{0\epsilon}\|_{r_2'}^{r_2'}\end{aligned}\tag{4.1}$$

for $t\in(0,T)$, where k, l, r_1, r_2, $r_1'=\frac{r_1}{r_1-1}$ and $r_2'=\frac{r_2}{r_2-1}$ are given in Lemma 3.2, $K_{r_1'}$ and $K_{r_2'}$ are defined in Lemma 2.2, and

$$A_1(\eta)=\left(\frac{4m_1k(k-1)}{(k+m_1-1)^2}-\left[\frac{(k-1)(K_{r_1'}+1)}{\eta^{r_1-1}}+\frac{(l-1)K_{r_2'}}{\eta^{\frac{1}{r_2-1}}}\right]S_d\|u_\epsilon(s)\|_p^{\frac{2}{d}p}\right),$$

$$A_2(\eta)=\left(\frac{4m_2l(l-1)}{(l+m_2-1)^2}-\left((k-1)K_{r_1'}+(l-1)(K_{r_2'}+1)\right)\eta S_d\|w_\epsilon(s)\|_q^{\frac{2}{d}q}\right).$$

Proof. Multiplying $(3.1)_1$ by ku_ϵ^{k-1}, testing $(3.1)_3$ with lw_ϵ^{l-1} and summing them up, we can see that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^d}\left(u_\epsilon^k+w_\epsilon^l\right)dx+\frac{4m_1k(k-1)}{(k+m_1-1)^2}\int_{\mathbb{R}^d}\left|\nabla u_\epsilon^{\frac{k+m_1-1}{2}}\right|^2dx\\&\quad+\frac{4m_2l(l-1)}{(l+m_2-1)^2}\int_{\mathbb{R}^d}\left|\nabla w_\epsilon^{\frac{l+m_2-1}{2}}\right|^2dx\\&\le-(k-1)\int_{\mathbb{R}^d}u_\epsilon^k\Delta v_\epsilon dx-(l-1)\int_{\mathbb{R}^d}w_\epsilon^l\Delta z_\epsilon dx\\&=:I_1+I_2.\end{aligned}\tag{4.2}$$

Combining the estimates for (u_ϵ,w_ϵ) in Lemma 3.3 with the maximal Sobolev regularity in Lemma 2.2, we will control the terms I_1 and I_2 by the gradient terms at the left-hand side of (4.2). Firstly, we choose positive constants k and l satisfying

$$k>\bar{k}=p\left(1-\frac{1}{q}\right),$$

$$l>\bar{l}=q\left(1-\frac{1}{p}\right),$$

$$l=\frac{(k+m_1-1)q}{p}-m_2+1=\frac{kq}{p}+1-\frac{q}{p}.$$

Let $r_1\in(1,q/(q-1))$ and $r_2\in(1,p/(p-1))$ satisfy

$$r_1=\frac{p}{q}\cdot\frac{1}{k}+1,$$

$$r_2=\frac{q}{p}\cdot\frac{1}{l}+1.$$

Applying the same arguments in Lemma 3.2 shows that r_1 and r_2 satisfy (3.5)-(3.10). By the maximal Sobolev regularity, Hölder's inequality with the exponents $r_1>1$ and $r_1'=\frac{r_1}{r_1-1}>1$, and Young's inequality, integrating I_1 over $(0,t)$, we can find $K_{r_1'}>0$ and $\eta\in(0,1)$ such that

$$
\begin{aligned}
\int_0^t I_1 ds \le& (k-1)\int_0^t\int_{\mathbb{R}^d} u_\epsilon^k|\Delta v_\epsilon| dx\\
\le&(k-1)\left(\int_0^t\int_{\mathbb{R}^d} u_\epsilon^{kr_1}dxds\right)^{\frac{1}{r_1}}\left(\int_0^t\int_{\mathbb{R}^d}|\Delta v_\epsilon|^{r_1'}dxds\right)^{\frac{1}{r_1'}}\\
\le&(k-1)\|\Delta v_{0\epsilon}\|_{r_1'}\left(1-e^{-r_1't}\right)^{\frac{1}{r_1'}}\left(\int_0^t\int_{\mathbb{R}^d} u_\epsilon^{kr_1}dxds\right)^{\frac{1}{r_1}}\\
&+(k-1)K_{r_1'}\left(\int_0^t\int_{\mathbb{R}^d} u_\epsilon^{kr_1}dxds\right)^{\frac{1}{r_1}}\left(\int_0^t\int_{\mathbb{R}^d} w_\epsilon^{r_1'}dxds\right)^{\frac{1}{r_1'}}\\
\le&(k-1)\eta\|\Delta v_{0\epsilon}\|_{r_1'}^{r_1'}+\frac{(k-1)(K_{r_1'}+1)}{\eta^{r_1-1}}\int_0^t\int_{\mathbb{R}^d} u_\epsilon^{kr_1}dxds\\
&+(k-1)K_{r_1'}\eta\int_0^t\int_{\mathbb{R}^d} w_\epsilon^{r_1'}dxds.
\end{aligned}\tag{4.3}
$$

For the term I_2, by utilizing Hölder's inequality with the exponents $r_2>1$ and $r_2'=\frac{r_2}{r_2-1}>1$, the maximal Sobolev regularity and Young's inequality, then there exist $K_{r_2'}>0$ and $\eta\in(0,1)$ such that

$$
\begin{aligned}
\int_0^t I_2 ds \le& (l-1)\int_0^t\int_{\mathbb{R}^d} w_\epsilon^l|\Delta z_\epsilon| dxds\\
\le&(l-1)\left(\int_0^t\int_{\mathbb{R}^d} w_\epsilon^{lr_2}dxds\right)^{\frac{1}{r_2}}\left(\int_0^t\int_{\mathbb{R}^d}|\Delta z_\epsilon|^{r_2'}dxds\right)^{\frac{1}{r_2'}}\\
\le&\frac{l-1}{\eta^{\frac{1}{r_2-1}}}\|\Delta z_{0\epsilon}\|_{r_2'}^{r_2'}+\frac{(l-1)K_{r_2'}}{\eta^{\frac{1}{r_2-1}}}\int_0^t\int_{\mathbb{R}^d} u_\epsilon^{r_2'}dxds+(l-1)(K_{r_2'}+1)\eta\int_0^t\int_{\mathbb{R}^d} w_\epsilon^{lr_2}dxds,
\end{aligned}
$$

which together with (4.3) implies that

$$
\begin{aligned}
\int_0^t (I_1+I_2)ds\le&(k-1)\eta\|\Delta v_{0\epsilon}\|_{r_1'}^{r_1'}+\frac{l-1}{\eta^{\frac{1}{r_2-1}}}\|\Delta z_{0\epsilon}\|_{r_2'}^{r_2'}\\
&+\frac{(k-1)(K_{r_1'}+1)}{\eta^{r_1-1}}\int_0^t\int_{\mathbb{R}^d} u_\epsilon^{kr_1}dxds+\frac{(l-1)K_{r_2'}}{\eta^{\frac{1}{r_2-1}}}\int_0^t\int_{\mathbb{R}^d} u_\epsilon^{r_2'}dxds\\
&+(k-1)K_{r_1'}\eta\int_0^t\int_{\mathbb{R}^d} w_\epsilon^{r_1'}dxds+(l-1)(K_{r_2'}+1)\eta\int_0^t\int_{\mathbb{R}^d} w_\epsilon^{lr_2}dxds.
\end{aligned}\tag{4.4}
$$

The choices of k, l, r_1 and r_2 help us obtain

$$
\|u_\epsilon\|_{kr_1}^{kr_1}\le S_d\|u_\epsilon\|_p^{\frac{2}{d}p}\left\|\nabla u_\epsilon^{\frac{k+m_1-1}{2}}\right\|_2^2,
$$

$$
\|u_\epsilon\|_{r_2'}^{r_2'}\le S_d\|u_\epsilon\|_p^{\frac{2}{d}p}\left\|\nabla u_\epsilon^{\frac{k+m_1-1}{2}}\right\|_2^2
$$

by the Gagliardo-Nirenberg inequality. Therefore, we have

$$\begin{aligned}&\frac{(k-1)(K_{r_1'}+1)}{\eta^{r_1-1}}\int_0^t\int_{\mathbb{R}^d}u_\epsilon^{kr_1}dxds+\frac{(l-1)K_{r_2'}}{\eta^{\frac{1}{r_2-1}}}\int_0^t\int_{\mathbb{R}^d}u_\epsilon^{r_2'}dxds\\&\le\left[\frac{(k-1)(K_{r_1'}+1)}{\eta^{r_1-1}}+\frac{(l-1)K_{r_2'}}{\eta^{\frac{1}{r_2-1}}}\right]S_d\int_0^t\|u_\epsilon(s)\|_p^{\frac{2}{d}p}\|\nabla u_\epsilon^{\frac{k+m_1-1}{2}}(s)\|_2^2ds.\end{aligned}\tag{4.5}$$

Similarly, we get

$$\begin{aligned}&(k-1)K_{r_1'}\eta\int_0^t\int_{\mathbb{R}^d}w_\epsilon^{r_1'}dxds+(l-1)(K_{r_2'}+1)\eta\int_0^t\int_{\mathbb{R}^d}w_\epsilon^{lr_2}dxds\\&\le\big((k-1)K_{r_1'}+(l-1)(K_{r_2'}+1)\big)\eta S_d\int_0^t\|w_\epsilon(s)\|_q^{\frac{2}{d}q}\|\nabla w_\epsilon^{\frac{l+m_2-1}{2}}(s)\|_2^2ds\end{aligned}$$

by the facts that

$$\begin{aligned}\|w_\epsilon\|_{r_1'}^{r_1'}\le S_d\|w_\epsilon\|_q^{\frac{2}{d}q}\left\|\nabla w_\epsilon^{\frac{l+m_2-1}{2}}\right\|_2^2,\\\|w_\epsilon\|_{lr_2}^{lr_2}\le S_d\|w_\epsilon\|_q^{\frac{2}{d}q}\left\|\nabla w_\epsilon^{\frac{l+m_2-1}{2}}\right\|_2^2.\end{aligned}$$

This along with (4.4) and (4.5) yields

$$\begin{aligned}&\int_0^t(I_1+I_2)ds\\&\le(k-1)\eta\|\Delta v_{0\epsilon}\|_{r_1'}^{r_1'}+\frac{l-1}{\eta^{\frac{1}{r_2-1}}}\|\Delta z_{0\epsilon}\|_{r_2'}^{r_2'}\\&\quad+\left[\frac{(k-1)(K_{r_1'}+1)}{\eta^{r_1-1}}+\frac{(l-1)K_{r_2'}}{\eta^{\frac{1}{r_2-1}}}\right]S_d\int_0^t\|u_\epsilon(s)\|_p^{\frac{2}{d}p}\left\|\nabla u_\epsilon^{\frac{k+m_1-1}{2}}(s)\right\|_2^2ds\\&\quad+\big((k-1)K_{r_1'}+(l-1)(K_{r_2'}+1)\big)\eta S_d\int_0^t\|w_\epsilon(s)\|_q^{\frac{2}{d}q}\left\|\nabla w_\epsilon^{\frac{l+m_2-1}{2}}(s)\right\|_2^2ds.\end{aligned}\tag{4.6}$$

By means of (4.2) and (4.6), we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^d}(u_\epsilon^k+w_\epsilon^l)dx+\int_0^tA_1(\eta)\cdot\left\|\nabla u_\epsilon^{\frac{k+m_1-1}{2}}(s)\right\|_2^2ds+\int_0^tA_2(\eta)\cdot\left\|\nabla w_\epsilon^{\frac{l+m_2-1}{2}}(s)\right\|_2^2ds\\&\le\|u_{0\epsilon}\|_k^k+\|w_{0\epsilon}\|_l^l+(k-1)\eta\|\Delta v_{0\epsilon}\|_{r_1'}^{r_1'}+\frac{l-1}{\eta^{\frac{1}{r_2-1}}}\|\Delta z_{0\epsilon}\|_{r_2'}^{r_2'}.\end{aligned}$$

In particular, this lemma is complete if choosing $\eta=1$. □

Based on (4.1), an a priori estimate of solutions will be derived later.

LEMMA 4.2. *Let $T>0$. If there exist some positive constants α_u, α_v, α_w, $\alpha_z>0$ such that (u_0,v_0,w_0,z_0) satisfies*

$$\begin{aligned}&\|u_0\|_p\le\alpha_u,\quad\|w_0\|_q\le\alpha_w,\\&\|u_0\|_{p+1-p/q}\le\alpha_w,\ \|w_0\|_{q+1-q/p}\le\alpha_u,\\&\|\Delta v_0\|_{q+1}\le\alpha_u,\ \|\Delta v_0\|_{q(1+1/p)}\le\alpha_w,\\&\|\Delta z_0\|_{p+1}\le\alpha_w,\ \|\Delta z_0\|_{p(1+1/q)}\le\alpha_u,\end{aligned}\tag{4.7}$$

then there exists a positive constant $C>0$ independent of ϵ and T such that

$$\sup_{0<t<T} \|u_\epsilon\|_{k^*} \le C, \tag{4.8}$$

$$\sup_{0<t<T} \|w_\epsilon\|_{l^*} \le C, \tag{4.9}$$

$$\left\| \nabla u_\epsilon^{\frac{k^*+m_1-1}{2}} \right\|_{L^2(0,T;L^2(\mathbb{R}^d))} \le C, \tag{4.10}$$

$$\left\| \nabla w_\epsilon^{\frac{l^*+m_2-1}{2}} \right\|_{L^2(0,T;L^2(\mathbb{R}^d))} \le C \tag{4.11}$$

for some $k^*>1, l^*=\frac{k^*q}{p}+1-\frac{q}{p}>1$. As a consequence, we have

$$\sup_{0<t<T} (\|u_\epsilon(t)\|_r + \|w_\epsilon(t)\|_r) \le C, \tag{4.12}$$

$$\sup_{0<t<T} (\|v_\epsilon(t)\|_r + \|z_\epsilon(t)\|_r) \le C, \tag{4.13}$$

$$\sup_{0<t<T} (\|\nabla v_\epsilon(t)\|_r + \|\nabla z_\epsilon(t)\|_r) \le C, \tag{4.14}$$

for $r\in[1,\infty]$.

Proof. Define

$$\delta^1_{u,k,l} := \left(\frac{m_1 k(k-1)}{(k+m_1-1)^2 \left((k-1)(K_{r_1'}+1)+(l-1)K_{r_2'}\right) S_d} \right)^{\frac{d}{2p}},$$

$$\delta^1_{w,k,l} := \left(\frac{m_2 l(l-1)}{(l+m_2-1)^2 \left((k-1)K_{r_1'}+(l-1)(K_{r_2'}+1)\right) S_d} \right)^{\frac{d}{2q}}.$$

Choosing $k=p$, $l=p^*=q+1-\frac{q}{p}$, $r_1=1+\frac{1}{q}$, $r_2=1+\frac{1}{p+p/q-1}$ in (4.1), if

$$\|u_0\|_p < \delta^1_{u,p,p^*}, \quad \|w_0\|_q < \delta^1_{w,p,p^*}$$

is true, there exists a positive constant $T_1>0$ such that

$$\|u_\epsilon\|_p^p \le \|u_{0\epsilon}\|_p^p + \|w_{0\epsilon}\|_{p^*}^{p^*} + (p-1)\|\Delta v_{0\epsilon}\|_{q+1}^{q+1} + \frac{q(p-1)}{p}\|\Delta z_{0\epsilon}\|_{p(1+1/q)}^{p(1+1/q)} \tag{4.15}$$

for $t\in(0,T_1]$. With $k=q^*=p+1-\frac{p}{q}$, $l=q$, $r_1=1+\frac{1}{q+q/p-1}$, $r_2=1+\frac{1}{p}$ in (4.1) and $\|u_0\|_p<\delta^1_{u,q^*,q}$, $\|w_0\|_q<\delta^1_{w,q^*,q}$, we can find a positive constant $T_2>0$ such that

$$\|w_\epsilon\|_q^q \le \|u_{0\epsilon}\|_{q^*}^{q^*} + \|w_{0\epsilon}\|_q^q + \frac{p(q-1)}{q}\|\Delta v_{0\epsilon}\|_{q(1+1/p)}^{q(1+1/p)} + (q-1)\|\Delta z_{0\epsilon}\|_{p+1}^{p+1} \tag{4.16}$$

for $t\in(0,T_2]$. Under the following assumptions

$$\|u_0\|_p < \min\{\delta^1_{u,p,p^*}, \delta^1_{u,q^*,q}\}, \qquad \|w_0\|_q < \min\{\delta^1_{w,p,p^*}, \delta^1_{w,q^*,q}\},$$

$$\|u_0\|_{q^*} < \min\{(\delta^1_{w,p,p^*})^{\frac{q}{q^*}}, (\delta^1_{w,q^*,q})^{\frac{q}{q^*}}\}, \quad \|w_0\|_{p^*} < \min\{(\delta^1_{u,p,p^*})^{\frac{p}{p^*}}, (\delta^1_{u,q^*,q})^{\frac{p}{p^*}}\},$$

$$\|\Delta v_0\|_{q+1} < \frac{1}{(p-1)^{\frac{1}{q+1}}} \min\{(\delta^1_{u,p,p^*})^{\frac{p}{q+1}}, (\delta^1_{u,q^*,q})^{\frac{p}{q+1}}\},$$

$$\|\Delta v_0\|_{q(1+1/p)} < \left(\frac{q}{p(q-1)}\right)^{\frac{p}{q(p+1)}} \min\{(\delta^1_{w,p,p^*})^{\frac{p}{p+1}}, (\delta^1_{w,q^*,q})^{\frac{p}{p+1}}\},$$

$$\|\Delta z_0\|_{p+1} < \frac{1}{(q-1)^{\frac{1}{p+1}}} \min\{(\delta^1_{w,p,p^*})^{\frac{q}{p+1}}, (\delta^1_{w,q^*,q})^{\frac{q}{p+1}}\},$$

$$\|\Delta z_0\|_{p(1+1/q)} < \left(\frac{p}{q(p-1)}\right)^{\frac{q}{p(q+1)}} \min\{(\delta^1_{u,p,p^*})^{\frac{q}{q+1}}, (\delta^1_{u,q^*,q})^{\frac{q}{q+1}}\},$$

recalling (4.1), applying (4.15)-(4.16), then one can find that

$$\|u_\epsilon(t)\|_p^{\frac{2}{d}p} < 4^{\frac{2}{d}} \min\{(\delta^1_{u,p,p^*})^{\frac{2}{d}p}, (\delta^1_{u,q^*,q})^{\frac{2}{d}p}\},$$

$$\|w_\epsilon(t)\|_q^{\frac{2}{d}q} < 4^{\frac{2}{d}} \min\{(\delta^1_{w,p,p^*})^{\frac{2}{d}q}, (\delta^1_{w,q^*,q})^{\frac{2}{d}q}\}$$

is true for $t \in (0, \min\{T_1, T_2\}]$, which induces that (4.15)-(4.16) are valid for $t \in (0, 2\min\{T_1, T_2\}]$ due to the property of continuity again. Consequently, one has (4.15)-(4.16) for $t \in (0,T)$. Inserting the above facts into (4.1), we have

$$\|u_\epsilon\|_{L^\infty\left(0,T;L^{k^*}(\mathbb{R}^d)\right)} \le C, \quad \|w_\epsilon\|_{L^\infty\left(0,T;L^{l^*}(\mathbb{R}^d)\right)} \le C$$

for some $k^* > 0$ and $l^* > 0$ if additional smallness assumptions are added:

$$\|u_0\|_p < \delta^1_{u,k^*,l^*}, \quad \|w_0\|_q < \delta^1_{w,k^*,l^*},$$

$$\|u_0\|_{q^*} < (\delta^1_{w,k^*,l^*})^{\frac{q}{q^*}}, \quad \|w_0\|_{p^*} < (\delta^1_{u,k^*,l^*})^{\frac{p}{p^*}},$$

$$\|\Delta v_0\|_{q+1} < \frac{1}{(p-1)^{\frac{1}{q+1}}} (\delta^1_{u,k^*,l^*})^{\frac{p}{q+1}}, \|\Delta v_0\|_{q(1+1/p)} < \left(\frac{q}{p(q-1)}\right)^{\frac{p}{q(p+1)}} (\delta^1_{w,k^*,l^*})^{\frac{p}{p+1}},$$

$$\|\Delta z_0\|_{p+1} < \frac{1}{(q-1)^{\frac{1}{p+1}}} (\delta^1_{w,k^*,l^*})^{\frac{q}{p+1}}, \|\Delta z_0\|_{p(1+1/q)} < \left(\frac{p}{q(p-1)}\right)^{\frac{q}{p(q+1)}} (\delta^1_{u,k^*,l^*})^{\frac{q}{q+1}}.$$

Moreover, we also have (4.10)-(4.11). By virtue of the $L^{r_1} - L^{r_2}$ estimates for the heat semigroup in Lemma 2.2 and the generalized Moser's iteration technique (see [33, Proposition 10], [18, Section 5]), one can obtain the L^∞-estimates (4.12) for (u_ϵ, w_ϵ). Moreover, applying the $L^{r_1} - L^{r_2}$ estimates of Lemma 2.2 for (v_ϵ, z_ϵ) yields the desired results (4.13)-(4.14). ◻

Proof. **(Proof of Theorem 1.2.)** We first multiply $(3.1)_1$ by $\partial_t (u_\epsilon + \epsilon)^{m_1}$ and $(3.1)_3$ by $\partial_t (w_\epsilon + \epsilon)^{m_2}$, and repeat the arguments of Lemmas 3.5 - 3.6, then we find a positive constant $C > 0$ independent of ϵ such that

$$\begin{aligned} &\|\partial_t u_\epsilon^{m_1}\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} + \sup_{0<t<T} \|\nabla u_\epsilon^{m_1}\|_2^2 \le C, \\ &\|\partial_t w_\epsilon^{m_2}\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} + \sup_{0<t<T} \|\nabla w_\epsilon^{m_2}\|_2^2 \le C. \end{aligned} \tag{4.17}$$

The regularity properties (4.17) and (4.8)-(4.14) guarantee that there exist $(u,w)\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ and $(v,z)\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ such that the convergences in Lemma 3.6 will hold out.

Taking $\phi\in C_0^\infty(\mathbb{R}^d\times[0,T))$ with $T>0$, testing both $(3.1)_1$ and $(3.1)_3$ by ϕ and integrating them over $\mathbb{R}^d\times(0,T)$, we can obtain

$$\int_0^T\int_{\mathbb{R}^d} u\phi_t dxdt+\int_{\mathbb{R}^d} u_0(x)\phi(x,0)dx=\int_0^T\int_{\mathbb{R}^d}(\nabla u^{m_1}-u\nabla v)\cdot\nabla\phi dxdt,$$

$$\int_0^T\int_{\mathbb{R}^d} w\phi_t dxdt+\int_{\mathbb{R}^d} w_0(x)\phi(x,0)dx=\int_0^T\int_{\mathbb{R}^d}(\nabla w^{m_2}-w\nabla z)\cdot\nabla\phi dxdt,$$

by the same procedures as those in the proof of Theorem 1.1. Besides, we get

$$\int_0^T\int_{\mathbb{R}^d} u_{\epsilon_n}\phi dxdt\to\int_0^T\int_{\mathbb{R}^d} u\phi dxdt,\ \int_0^T\int_{\mathbb{R}^d} w_{\epsilon_n}\phi dxdt\to\int_0^T\int_{\mathbb{R}^d} w\phi dxdt.$$

By terms of (3.43) and (3.45), it implies that

$$\int_0^T\int_{\mathbb{R}^d}\nabla v_{\epsilon_n}\cdot\nabla\phi dxdt\to\int_0^T\int_{\mathbb{R}^d}\nabla v\cdot\nabla\phi dxdt,$$

$$\int_0^T\int_{\mathbb{R}^d}\nabla z_{\epsilon_n}\cdot\nabla\phi dxdt\to\int_0^T\int_{\mathbb{R}^d}\nabla z\cdot\nabla\phi dxdt.$$

Moreover, (3.42) and (3.44) show that

$$\int_0^T\int_{\mathbb{R}^d} v_{\epsilon_n}\phi dxdt\to\int_0^T\int_{\mathbb{R}^d} v\phi dxdt,\quad \int_0^T\int_{\mathbb{R}^d} v_{\epsilon_n}\phi_t dxdt\to\int_0^T\int_{\mathbb{R}^d} v\phi_t dxdt,$$

$$\int_0^T\int_{\mathbb{R}^d} z_{\epsilon_n}\phi dxdt\to\int_0^T\int_{\mathbb{R}^d} z\phi dxdt,\quad \int_0^T\int_{\mathbb{R}^d} z_{\epsilon_n}\phi_t dxdt\to\int_0^T\int_{\mathbb{R}^d} z\phi_t dxdt,$$

which guarantees that

$$\int_0^T\int_{\mathbb{R}^d}\nabla v\cdot\nabla\phi dxdt+\int_0^T\int_{\mathbb{R}^d}(v\phi-w\phi-v\phi_t)dxdt=\int_{\mathbb{R}^d} v_0(x)\phi(x,0)dx,$$

$$\int_0^T\int_{\mathbb{R}^d}\nabla z\cdot\nabla\phi dxdt+\int_0^T\int_{\mathbb{R}^d}(z\phi-u\phi-z\phi_t)dxdt=\int_{\mathbb{R}^d} z_0(x)\phi(x,0)dx.$$

According to Definition 1.1, the global weak solution (u,v,w,z) is formed over $\mathbb{R}^d\times(0,T)$. We have finished the proof of Theorem 1.2. □

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