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ON THE GLOBAL WELL-POSEDNESS AND OPTIMAL LARGE-TIME BEHAVIOR OF STRONG SOLUTION FOR A MULTI-DIMENSIONAL TWO-FLUID PLASMA MODEL*

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Abstract. This article is concerned with the Cauchy problem to a multi-dimensional two-fluid plasma model in critical functional framework which is not related to the energy space. When the initial data are close to a stable equilibrium state in the sense of suitable L^p-type Besov norms, the global well-posedness for the multi-dimensional system is shown. As a consequence, one may exhibit the unique global solution for a class of large highly oscillating initial velocities in physical dimensions $N=2,3$. Furthermore, based on refined time weighted inequalities in the Fourier spaces, we also establish optimal large-time behavior for the constructed global solutions under a mild additional decay assumption involving only the low frequencies of the initial data.

Keywords. Bipolar compressible Navier-Stokes-Poisson system; Global well-posedness; Optimal large-time behavior; L^p-type critical Besov spaces.

AMS subject classifications. 35Q35; 35D05; 76W05.

1. Introduction and main results

In this paper, we consider the following multi-dimensional two-fluid plasma model, namely the bipolar compressible Navier-Stokes-Poisson system in $\mathbb{R}_+ \times \mathbb{R}^N$:

$$\begin{cases} \partial_t \rho_1 + \operatorname{div}(\rho_1 u_1) = 0, \\ \partial_t(\rho_1 u_1) + \operatorname{div}(\rho_1 u_1 \otimes u_1) + \nabla P_1(\rho_1) = \rho_1 \nabla\phi + \mu_1 \Delta u_1 + \mu_2 \nabla \operatorname{div} u_1, \\ \partial_t \rho_2 + \operatorname{div}(\rho_2 u_2) = 0, \\ \partial_t(\rho_2 u_2) + \operatorname{div}(\rho_2 u_2 \otimes u_2) + \nabla P_2(\rho_2) = -\rho_2 \nabla\phi + \mu_1 \Delta u_2 + \mu_2 \nabla \operatorname{div} u_2, \\ \lambda_D^2 \Delta\phi = \rho_1 - \rho_2, \quad \lim\limits_{|x|\to\infty} \phi(x) = 0 \end{cases} \tag{1.1}$$

subject to the initial data

$$(\rho_1, u_1, \rho_2, u_2, \phi)|_{t=0} = (\rho_{1,0}, u_{1,0}, \rho_{2,0}, u_{2,0}, \phi_0)(x), \quad x \in \mathbb{R}^N. \tag{1.2}$$

Here $\rho_i(t,x)$ denotes the density, $u_i(t,x)$ stands for the velocity field, $P_i(t,x)$ is pressure, and ϕ is the electrostatic potential. The constant N stands for the space dimension with $N\geq 2$. The viscosity coefficients μ_1 and μ_2 satisfy the usual physical conditions $\mu_1>0$, $\mu_1+2\mu_2>0$. $\lambda_D>0$ presents the Debye length, and we can take $\lambda_D=1$ without loss of generality. From a physical point of view, the motion of the ion-dust plasma, the self-gravitational viscous gaseous stars and the charged particles in semiconductor devices can be governed by the macroscopic fluid equation such as the compressible Euler equation and the compressible Navier-Stokes equation under the self-consistent electromagnetic fields. However, in some semiconductor devices and plasmas, the Lorenz force caused by magnetic field is relatively small and can be neglected in some sense, so that it is appropriate for us to consider the effect of the electric field alone. In general,

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the bipolar Navier-Stokes-Poisson system (1.1) can be used to model the transport of charged particles, for instance, holes and electrons in semiconductor devices, and ions and electrons in plasmas under the influence of electrostatic force governed by the self-consistent Poisson equation. Moreover, when we only focus on the dynamics of one particle in semiconductor devices and plasmas, system (1.1) reduces to the unipolar Navier-Stokes-Poisson system. More details on the compressible Navier-Stokes-Poisson system can be found in, e.g. [13, 22] and references therein.

In recent years, there have been a great number of mathematical studies about the compressible Navier-Stokes-Poisson equations. In what follows, we only recall some of them related to our interest. The global existence and L^2-decay rates of the smooth solution of the initial value problem for the unipolar compressible Navier-Stokes-Poisson system in $\mathbb{R}^3$ are proved in [19, 34]. Wang [28] obtained the optimal asymptotic decay rates of solutions just by pure energy estimates. The global existence and optimal L^2-decay rates of solutions for the compressible unipolar Navier-Stokes-Poisson system with external force are discussed in [35]. The point-wise estimates of the smooth solutions for the three-dimensional unipolar isentropic compressible Navier-Stokes-Poisson equation are obtained in [25]. Large-time behavior of the spherically symmetric Navier-Stokes-Poisson system with degenerate viscosity coefficients and the vacuum in the three dimensions is investigated in [33]. Hao and Li [17] proved the global existence and uniqueness of strong solutions for the unipolar isentropic compressible Navier-Stokes-Poisson equation with three and higher dimensions in hybrid Besov spaces. Zheng [32] removed the extra-assumptions on the velocity from [17] and extended the global existence result to the critical L^p framework by using the arguments in [4, 7, 10]. Chikami and Danchin [8] further improved the known result in [32] and established the unique global solvability and time decay rates in any dimension $N \geq 2$ for small perturbations of a linearly stable constant state. Bie et al. [2], Shi and Xu [24] also obtained optimal time decay rates in critical L^p framework, respectively. The first author and Li [30] proved the unique global solvability of multi-dimensional compressible Navier-Stokes-Poisson model with capillarity effect when the initial data are close to a stable equilibrium state in critical Besov spaces.

For the more complicated bipolar Navier-Stokes-Poisson system (1.1), there are very few results due to the interplay of two particles which counteracts the influence of electric field. Li et al. [20], Wang and Xu [27] obtained the global existence and optimal decay rates of classical solution around a constant state for the bipolar compressible Navier-Stokes-Poisson system by a detailed analysis of Green's function to the corresponding linearized equations in $\mathbb{R}^3$. Zhao and Li [36] showed the global existence and optimal L^2-decay rate of solutions for the compressible bipolar Navier-Stokes-Poisson system with external force. Wu and Wang [26] further investigated the pointwise estimates for the isentropic compressible bipolar Navier-Stokes-Poisson system in dimension three and other odd dimensions $N \geq 5$. Lin et al. [21] constructed the global well-posedness of strong solutions to system (1.1) with the initial data close to an equilibrium state in Besov spaces in dimensions $N \geq 3$. Wu, Zhang and Zhang [29] showed the global existence and time decay rates for the 3D bipolar compressible Navier-Stokes-Poisson system with unequal viscosities. Finally, we also mention that there are some studies about the asymptotic stability of nonlinear waves such as the viscous shock wave, the rarefaction wave, the contact discontinuity and stationary wave and their combinations to system (1.1), for example, we refer to [15, 16, 31] and some references therein.

Here, it should be pointed out that the existing results mentioned above including the global existence and large-time behavior of solution to system (1.1) are mainly based on L^2-framework and the dimensions of spaces are only limited to $N\geq 3$. However, to our knowledge, so far there are few results on the global existence and large-time behavior to the model in L^p-framework. Therefore, in this paper, we shall investigate some mathematical properties of system (1.1) in critical L^p-framework. Let us emphasize that this framework allows us to construct global solutions for highly oscillating initial velocities in larger spaces in physical dimensions $N=2,3$. At this stage, let us recall the scaling analysis of system (1.1) to guess which spaces may be critical. One can check that if $(\rho_1,u_1,\rho_2,u_2,\phi)$ solves (1.1), so does $(\rho_{1\lambda},u_{2\lambda},\rho_{1\lambda},u_{2\lambda},\phi_\lambda)$ where:

$$\begin{aligned}
&\rho_{1\lambda}(t,x)=\rho_1(\lambda^2t,\lambda x), \quad u_{1\lambda}(t,x)=\lambda u_1(\lambda^2t,\lambda x),\\
&\rho_{2\lambda}(t,x)=\rho_2(\lambda^2t,\lambda x), \quad u_{2\lambda}(t,x)=\lambda u_2(\lambda^2t,\lambda x),\\
&\qquad\qquad \phi_\lambda(t,x)=\lambda^{-2}\phi(\lambda^2t,\lambda x)
\end{aligned}\tag{1.3}$$

provided that the pressure laws $P_i(i=1,2)$ have been changed into λ^2P_i for all $\lambda>0$. Due to the mixed hyperbolic-parabolic property of system (1.1), the system has to be handled differently in the low and high frequencies respectively. Roughly speaking, the first order terms predominate in low frequencies, so that system (1.1) has to be treated by means of hyperbolic energy methods, which implies that we must treat the low frequencies regime only in spaces constructed on L^2, as it is classical that hyperbolic systems are ill-posed in general L^p spaces. In contrast, in the high frequencies, a L^p approach may be used. The main aim of this article is to address the global existence and optimal large-time behavior of strong solution to the Cauchy problem (1.1)-(1.2) in critical L^p-framework. More importantly, our result allows us to cover any dimension $N\geq 2$.

Our first main result on global well-posedness then reads as follows.

THEOREM 1.1. *Let $N\geq 2$, and the p satisfy*

$$2\leq p\leq \min\{4,\frac{2N}{N-2}\}, \textit{additionally}, p\neq 4, \textit{if} N=2. \tag{1.4}$$

Assume that $P_1'(\bar\rho)>0$, $P_2'(\bar\rho)>0$ for $\bar\rho>0$. There exists a small enough constant α_0 such that if

$$\begin{aligned}
X_0 \overset{def}{=} &\|(c_{1,0},c_{2,0})\|^{\ell}_{\dot B^{\frac N2-2}_{2,1}}+\|(u_{1,0},u_{2,0})\|^{\ell}_{\dot B^{\frac N2-1}_{2,1}}+\|(c_{1,0},c_{2,0})\|^{h}_{\dot B^{\frac Np}_{p,1}}+\|(u_{1,0},u_{2,0})\|^{h}_{\dot B^{\frac Np-1}_{p,1}}\\
&<\alpha_0
\end{aligned}$$

with $c_{1,0}=\rho_{1,0}-\bar\rho$, $c_{2,0}=\rho_{2,0}-\bar\rho$, then the Cauchy problem (1.1)-(1.2) admits a unique global solution (c_1,u_1,c_2,u_2) satisfying that for all $t\geq 0$,

$$X(t)\lesssim X_0, \tag{1.5}$$

where

$$\begin{aligned}
X(t)\overset{def}{=}&\|(c_1,c_2)\|^{\ell}_{\widetilde L^\infty_t(\dot B^{\frac N2-2}_{2,1})}+\|(u_1,u_2)\|^{\ell}_{\widetilde L^\infty_t(\dot B^{\frac N2-1}_{2,1})}+\|(c_1,c_2)\|^{h}_{\widetilde L^\infty_t(\dot B^{\frac Np}_{p,1})}\\
&+\|(u_1,u_2)\|^{h}_{\widetilde L^\infty_t(\dot B^{\frac Np-1}_{p,1})}+\|(c_1,c_2)\|^{\ell}_{L^1_t(\dot B^{\frac N2}_{2,1})}+\|(u_1,u_2)\|^{\ell}_{L^1_t(\dot B^{\frac N2+1}_{2,1})}\\
&+\|(c_1,c_2)\|^{h}_{L^1_t(\dot B^{\frac Np}_{p,1})}+\|(u_1,u_2)\|^{h}_{L^1_t(\dot B^{\frac Np+1}_{p,1})}.
\end{aligned}\tag{1.6}$$

Our second main result on the optimal large-time behavior of strong solution is stated as follows.

THEOREM 1.2. *Let* $(c_{1,0},u_{1,0},c_{2,0},u_{2,0})$ *fulfill the assumptions of Theorem 1.1,* (c_1,u_1,c_2,u_2) *be the corresponding global solution to the Cauchy problem (1.1)-(1.2), and set* $\langle\tau\rangle=:\sqrt{1+\tau^2}$ *and* $\alpha=s_1+\frac{N}{2}+\frac{1}{2}-\varepsilon$ *for sufficiently small* $\varepsilon>0$. *Let the real number* s_1 *satisfy*

$$1-\frac{N}{2}<s_1\le s_0 \tag{1.7}$$

with $s_0=:\frac{2N}{p}-\frac{N}{2}$. *There exists a positive constant* c *such that if*

$$D(0)=:\|(\Lambda^{-1}(\rho_{1,0}+\rho_{2,0}-2\bar\rho),u_{1,0}+u_{2,0},\Lambda^{-1}(\rho_{1,0}-\rho_{2,0}),u_{1,0}-u_{2,0})\|^{\ell}_{\dot B^{-s_1}_{2,\infty}}<c,$$

then we have for all $t\ge 0$

$$D(t)\lesssim\Big(D(0)+\|(\nabla(\rho_{1,0}+\rho_{2,0}-2\bar\rho),u_{1,0}+u_{2,0},\nabla(\rho_{1,0}-\rho_{2,0}),u_{1,0}-u_{2,0})\|^{h}_{\dot B^{\frac{N}{p}-1}_{p,1}}\Big), \tag{1.8}$$

which implies that

$$\begin{aligned}&\sup_{s\in[\varepsilon-s_1,\frac{N}{2}+1]}\Big(\|\langle\tau\rangle^{\frac{s+s_1}{2}}(c_1,c_2)\|^{\ell}_{L^\infty_t(\dot B^{s-1}_{2,1})}+\|\langle\tau\rangle^{\frac{s+s_1}{2}}(u_1,u_2)\|^{\ell}_{L^\infty_t(\dot B^{s}_{2,1})}\Big)\\&\quad+\|\langle\tau\rangle^{\alpha}(c_1,c_2)\|^{h}_{\widetilde L^\infty_t(\dot B^{\frac{N}{p}}_{p,1})}+\|\langle\tau\rangle^{\alpha}(u_1,u_2)\|^{h}_{\widetilde L^\infty_t(\dot B^{\frac{N}{p}-1}_{p,1})}+\|\tau^{\alpha}(\nabla u_1,\nabla u_2)\|^{h}_{\widetilde L^\infty_t(\dot B^{\frac{N}{p}}_{p,1})}\\&\lesssim D(0)+\|(\nabla(\rho_{1,0}+\rho_{2,0}-2\bar\rho),u_{1,0}+u_{2,0},\nabla(\rho_{1,0}-\rho_{2,0}),u_{1,0}-u_{2,0})\|^{h}_{\dot B^{\frac{N}{p}-1}_{p,1}}.\end{aligned} \tag{1.9}$$

Here

$$\begin{aligned}D(t)=:&\sup_{s\in[\varepsilon-s_1,\frac{N}{2}+1]}\Big(\|\langle\tau\rangle^{\frac{s+s_1}{2}}(\rho_1+\rho_2-2\bar\rho,\rho_1-\rho_2)\|^{\ell}_{\dot B^{s-1}_{2,1}}\\&+\|\langle\tau\rangle^{\frac{s+s_1}{2}}(u_1+u_2,u_1-u_2)\|^{\ell}_{\dot B^{s}_{2,1}}\Big)+\|\langle\tau\rangle^{\alpha}(\rho_1+\rho_2-2\bar\rho,\rho_1-\rho_2)\|^{h}_{\widetilde L^\infty_t(\dot B^{\frac{N}{p}}_{p,1})}\\&+\|\langle\tau\rangle^{\alpha}(u_1+u_2,u_1-u_2)\|^{h}_{\widetilde L^\infty_t(\dot B^{\frac{N}{p}-1}_{p,1})}+\|\tau^{\alpha}(\nabla(u_1+u_2),\nabla(u_1-u_2))\|^{h}_{\widetilde L^\infty_t(\dot B^{\frac{N}{p}}_{p,1})}.\end{aligned}$$

We would like to give some comments on our main results.

REMARK 1.1. Compared with [21], in Theorem 1.1, the regularity indices for the high frequency part of u_{10} and u_{20} may be negative. Especially, this allows us to obtain the global well-posedness of system (1.1) for the highly oscillating initial velocities u_{10} and u_{20}. For example, let

$$u_{10}(x)=:\sin\Big(\frac{x_1}{\varepsilon_1}\Big)\phi(x),\quad u_{20}(x)=:\sin\Big(\frac{x_1}{\varepsilon_2}\Big)\phi(x),\quad \phi(x)\in\mathcal S(\mathbb R^N).$$

Thus for any $\varepsilon_i>0(i=1,2)$

$$\|u_{10}\|^{h}_{\dot B^{\frac{N}{p}-1}_{p,1}}\le C\varepsilon_1^{1-\frac{N}{p}},\quad \|u_{20}\|^{h}_{\dot B^{\frac{N}{p}-1}_{p,1}}\le C\varepsilon_2^{1-\frac{N}{p}}\quad\text{for}\quad p>N.$$

Hence such data with small enough $\varepsilon_i(i=1,2)$ generate global unique solutions in dimension $N=2,3$.

Remark 1.2. In Theorems 1.1 and 1.2, we obtain the global well-posedness and optimal time decay rates for multi-dimensional system (1.1) in critical L^p-framework, respectively. Additionally, in Theorem 1.2, the regularity index s can take both negative and nonnegative values, rather than only nonnegative integers, which extends the classical decay results in high Sobolev regularity in [20, 26, 27].

Remark 1.3. It should be emphasized here that, in Theorems 1.1 and 1.2, our conclusions hold in critical regularity framework and the dimensions of spaces are more extensive and are not limited to $N=3$.

As a consequence of Theorem 1.2, we can show the following decay rates of L^p norm of (c_1,u_1,c_2,u_2).

Corollary 1.1. *The solution (c_1,u_1,c_2,u_2) constructed in Theorem 1.2 fulfills*

$$\|\Lambda^s(c_1,c_2)\|_{L^p}\lesssim\Big(D(0)+\|(\nabla n_{1,0},\omega_{1,0},\nabla n_{2,0},\omega_{2,0})\|^h_{\dot B^{\frac{N}{p}-1}_{p,1}}\Big)\langle\tau\rangle^{-\frac{s+s_1+1}{2}}\ \textit{if}-s_1-1<s\le\frac{N}{p},$$

$$\|\Lambda^s(u_1,u_2)\|_{L^p}\lesssim\Big(D(0)+\|(\nabla n_{1,0},\omega_{1,0},\nabla n_{2,0},\omega_{2,0})\|^h_{\dot B^{\frac{N}{p}-1}_{p,1}}\Big)\langle\tau\rangle^{-\frac{s+s_1}{2}}\ \textit{if}-s_1<s\le\frac{N}{p}+1,$$

where the fractional derivative operator Λ^ℓ is defined by $\Lambda^\ell u=:\mathcal{F}^{-1}(|\xi|^\ell\mathcal{F}u)$.

Remark 1.4. In Corollary 1.1, for the three-dimensional case, taking $p=2$, $s=0$ and $s_1=\frac{1}{2}$, we see that the corresponding decay rates are consistent with the optimal time decay rates in [19] for the unipolar Navier-Stokes-Poisson model and the time decay rates in [20] for bipolar compressible Navier-Stokes-Poisson system, respectively.

Moreover, one has more L^q-L^r time decay rates of (c_1,u_1,c_2,u_2).

Corollary 1.2. Let the assumptions of Theorem 1.2 be satisfied with $p=2$. For $2\le r\le\infty$ and $l\in\mathbb{R}$ the corresponding solution (c_1,u_1,c_2,u_2) fulfills

$$\|\Lambda^l(c_1,c_2)\|_{L^r}\lesssim\Big(D(0)+\|(\nabla n_{1,0},\omega_{1,0},\nabla n_{2,0},\omega_{2,0})\|^h_{\dot B^{\frac{N}{p}-1}_{p,1}}\Big)\langle\tau\rangle^{-\frac{s_1}{2}-\frac{N}{2}(\frac{1}{2}-\frac{1}{r})-\frac{l+1}{2}}$$
$$\text{if}\quad -s_1-1<l+N(\frac{1}{2}-\frac{1}{r})\le\frac{N}{2},$$
$$\|\Lambda^k(u_1,u_2)\|_{L^r}\lesssim\Big(D(0)+\|(\nabla n_{1,0},\omega_{1,0},\nabla n_{2,0},\omega_{2,0})\|^h_{\dot B^{\frac{N}{p}-1}_{p,1}}\Big)\langle\tau\rangle^{-\frac{s_1}{2}-\frac{N}{2}(\frac{1}{2}-\frac{1}{r})-\frac{k}{2}}$$
$$\text{if}\quad -s_1<k+N(\frac{1}{2}-\frac{1}{r})\le\frac{N}{2}+1.$$

Before going into the proof of the theorems, let us outline the main ideas and difficulties in proving our main results. Due to the similarity between the compressible Navier-Stokes equations and the bipolar compressible Navier-Stokes-Poisson system (1.1), we can borrow some arguments and ideas in [4, 7, 10, 12, 18] to prove the existence and optimal large-time behavior of strong solution for system (1.1). However, it is nontrivial to apply directly the ideas to system (1.1) because the non-conservative structures of the system and the presence of the nonlocal electric field $\nabla\phi$, coupled with the nonlinear interplay of two carriers through the electric field, ultimately bring us more new difficulties in our analysis. First, system (1.1) has no conservative structures because of the presence of the terms $\rho_1\nabla\phi$ and $\rho_2\nabla\phi$ in the two momentum equations, which leads to the essential difficulty in the bipolar case. To get around this difficulty, we consider a new system (3.2) with the new variables $(\rho_1+\rho_2, u_1+u_2, \rho_1-\rho_2, u_1-u_2)$ which is equivalent to the original system (1.1). Here, the new system has some good mathematical structures, for instance, the partial system including the variables $\rho_1+\rho_2$ and u_1+u_2 is conservative. Hence, it is possible to help us to obtain some desired estimates in our analysis. Second, the nonlocal term $\nabla(-\Delta)^{-1}(\rho_1-\rho_2)$ (that is, $\nabla(-\Delta)^{-1}(c_1-c_2)$ in (3.2)) arising from the lower order electrostatic potential ϕ plays a bad role in the low frequencies. Indeed, we observe that the term $|\xi|^{-1}$ appearing in $\widehat{G}_{21}$ and $\widehat{G}_{23}$ of $\widehat{G}(\xi,t)$ in [27] is singular and causes some difficulty in low frequencies, which leads us to not getting the estimates of $\widehat{G}(\xi,t)$ like the heat kernel and also prevents us from dealing with two-dimensional case (see [32]). Obviously, the bad term $|\xi|^{-1}$ exactly comes from the symbol of the nonlocal term $\nabla(-\Delta)^{-1}(c_1-c_2)$. In order to overcome the difficulty, we notice an important fact that $\nabla(-\Delta)^{-1}(c_1-c_2)$ should have the same regularity as $\Delta(u_1-u_2)$ in (3.2), which induces us to introduce two new unknowns $a_1=\Lambda^{-1}(c_1+c_2)$ and $a_2=\Lambda^{-1}(c_1-c_2)$ and then change system (3.2) into (3.7) without the pseudo-differential operator of order -1 in the linear part of the momentum equations. Then, we can present an explicit derivation of the Fourier transform of Green's matrix $G(x,t)$ corresponding to the linearized system of (3.7) and then carry out its spectral analysis. In particular, we can exhibit that $G(x,t)$ behaves like the heat kernel in the low frequencies, which plays a key role in obtaining the global priori estimates and optimal large-time behavior. In the high frequencies, in order to cover more general values of the integration parameter p, we need to exploit the damping effects of the densities and parabolic properties of the velocities, respectively. For this purpose, as in [18], we introduce two suitable effective velocity fields (named viscous effective flux in Hoff's work [14]) $e_i =: \nabla\big(-\Delta\big)^{-1}(\frac{c^2}{\nu}n_i - \mathrm{div}\omega_i)(i=1,2)$ with the variables $n_1=c_1+c_2, n_2=c_1-c_2, \omega_1=u_1+u_2, \omega_2=u_1-u_2$. Then we easily obtain two standard heat equations for the variables e_1 and e_2 with some low order terms and two transport equations for the variables n_1 and n_2 with damping terms:

$$\begin{aligned}\partial_t n_1+\frac{c^2}{\nu}n_1=&-\mathrm{div}e_1-\Big(\frac{n_1+n_2}{2}\Big)\cdot\mathrm{div}\Big(\frac{\omega_1+\omega_2}{2}\Big)-\Big(\frac{n_1-n_2}{2}\Big)\cdot\mathrm{div}\Big(\frac{\omega_1-\omega_2}{2}\Big)\\&-\Big(\frac{\omega_1+\omega_2}{2}\Big)\cdot\nabla\Big(\frac{n_1+n_2}{2}\Big)-\Big(\frac{\omega_1-\omega_2}{2}\Big)\cdot\nabla\Big(\frac{n_1-n_2}{2}\Big),\end{aligned}$$

and

$$\begin{aligned}\partial_t n_2+\frac{c^2}{\nu}n_2=&-\mathrm{div}e_2-\Big(\frac{n_1+n_2}{2}\Big)\cdot\mathrm{div}\Big(\frac{\omega_1+\omega_2}{2}\Big)+\Big(\frac{n_1-n_2}{2}\Big)\cdot\mathrm{div}\Big(\frac{\omega_1-\omega_2}{2}\Big)\\&-\Big(\frac{\omega_1+\omega_2}{2}\Big)\cdot\nabla\Big(\frac{n_1+n_2}{2}\Big)+\Big(\frac{\omega_1-\omega_2}{2}\Big)\cdot\nabla\Big(\frac{n_1-n_2}{2}\Big).\end{aligned}$$

Noticing that, different from the standard barotropic compressible Navier-Stokes equations, there are two convection terms $\Big(\frac{\omega_1+\omega_2}{2}\Big)\cdot\nabla\Big(\frac{n_1+n_2}{2}\Big)$ and $\Big(\frac{\omega_1-\omega_2}{2}\Big)\cdot\nabla\Big(\frac{n_1-n_2}{2}\Big)$ which easily lose one derivative for functions n_1 and n_2. To deal with them, we need to take full advantage of the mathematical structures which can help us tackle this problem. Finally, we investigate how global strong solution constructed above looks like for large time. Our main ideas are based on refined time-weighted energy inequalities in the Fourier spaces. In the low frequency parts, by making good use of the decay estimates to the Fourier transform of Green's function, it is possible to adapt the standard Duhamel's principle handling those nonlinear terms. With the aid of the low and high frequency decomposition and the nonclassical product estimates in Besov spaces, one can obtain the desired estimates. In the high frequency parts, we can deal with the decay estimates of the solutions using a similar method as that in the proof for global existence in Section 5 together with elaborate nonlinear estimates. Furthermore, in order to close the estimate from time-weighted energy functional, we also exploit some decay estimates with gain of regularity for the high frequencies of velocity fields. The analysis strongly relies on the parabolic maximal regularity for the Lamé semi-group (that is the same as for the heat semi-group, see Remark 2.1).

The rest of the paper unfolds as follows. In the next section, we recall some basic facts about Littlewood-Paley decomposition, Besov spaces and some useful lemmas. In Section 3, we rewrite the original system (1.1)-(1.2) into a new system. Section 4 is devoted to giving the spectral analysis of the semigroup of the linearized system. Then, we shall prove Theorems 1.1 and 1.2 in Sections 5 and 6, respectively.

Notations. We assume C be a positive generic constant throughout this paper that may vary at different places and denote $A\le CB$ by $A\lesssim B$. We shall also need the notations

$$z^\ell =: \sum_{j\le k_0}\dot\Delta_j z \quad \text{and} \quad z^h =: z-z^\ell, \quad \text{for some } k_0\in\mathbb{Z}.$$

$$\|z\|^\ell_{\dot B^s_{2,1}} =: \sum_{j\le k_0} 2^{js}\|\dot\Delta_j z\|_{L^2} \quad \text{and} \quad \|z\|^h_{\dot B^s_{2,1}} =: \sum_{j\ge k_0} 2^{js}\|\dot\Delta_j z\|_{L^2}, \quad \text{for some } k_0\in\mathbb{Z}.$$

Noting the small overlap between low and high frequencies, we have

$$\|z^\ell\|_{\dot B^s_{2,1}} \lesssim \|z\|^\ell_{\dot B^s_{2,1}} \quad \text{and} \quad \|z^h\|_{\dot B^s_{2,1}} \lesssim \|z\|^h_{\dot B^s_{2,1}}.$$

2. Littlewood-Paley theory and some lemmas

In this section, we first introduce some notations and basic theories about the Littlewood-Paley decomposition, Besov spaces, then we also list some inequalities, which will be used in Sections 5–6.

First, let $\mathcal{S}(\mathbb{R}^N)$ be the Schwartz class of rapidly decreasing functions. Given $f\in\mathcal{S}(\mathbb{R}^N)$, its Fourier transform $\mathcal{F}f=\widehat{f}$ is defined by

$$\widehat{f}(\xi)=\int_{\mathbb{R}^N} e^{-ix\cdot\xi}f(x)dx.$$

Let (χ,φ) be a couple of smooth functions valued in $[0,1]$ such that χ is supported in the ball $\{\xi\in\mathbb{R}^N:\ |\xi|\le\frac{4}{3}\}$, φ is supported in the shell $\{\xi\in\mathbb{R}^N:\ \frac{3}{4}\le|\xi|\le\frac{8}{3}\}$, $\varphi(\xi):=\chi(\xi/2)-\chi(\xi)$ and

$$\chi(\xi)+\sum_{j\ge0}\varphi(2^{-j}\xi)=1 \text{ for } \forall\ \xi\in\mathbb{R}^N, \quad \sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1 \text{ for } \forall\ \xi\in\mathbb{R}^N\setminus\{0\}.$$

The homogeneous frequency localization operators $\dot\Delta_j$ and $\dot S_j$ are defined by

$$\dot\Delta_j f\triangleq\mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}f), \quad \dot S_j f\triangleq\sum_{q\le j-1}\dot\Delta_q f \quad \text{for} \quad j\in\mathbb{Z}.$$

We denote the space $\mathcal{S}'_h(\mathbb{R}^N)$ by the dual space of $\mathcal{S}'(\mathbb{R}^N)=\{f\in\mathcal{S}(\mathbb{R}^N): D^\alpha\hat f(0)=0, \text{where } \alpha \text{ is multi-index}\}$, it also can be identified by the quotient space of $\mathcal{S}'(\mathbb{R}^N)/\mathbb{P}$ with the polynomial space $\mathbb{P}$. The formal equality

$$f=\sum_{j\in\mathbb{Z}}\dot\Delta_j f$$

holds true for $f\in\mathcal{S}'_h(\mathbb{R}^N)$ and is called the homogeneous Littlewood-Paley decomposition. One easily verifies that with our choice of φ,

$$\dot\Delta_j\dot\Delta_q f\equiv 0 \quad\text{if}\quad |j-q|\geq 2 \quad\text{and}\quad \dot\Delta_j(\dot S_{q-1}f\dot\Delta_q f)\equiv 0 \quad\text{if}\quad |j-q|\geq 5.$$

Now let us show the definitions of the homogeneous Besov spaces.

DEFINITION 2.1. *Let $\mathcal{S}'$ be the space of all tempered distributions. For $s\in\mathbb{R}$, $1\leq p\leq\infty$, we define the homogeneous Besov space $\dot B^s_{p,1}$ to be*

$$\dot B^s_{p,1}=\Big\{f\in\mathcal{S}'_h:\|f\|_{\dot B^s_{p,1}}<\infty\Big\}$$

with

$$\mathcal{S}'_h=\Big\{f\in\mathcal{S}':\sum_{j\in\mathbb{Z}}\dot\Delta_j f=f\in\mathcal{S}'\Big\} \quad\textit{and}\quad \|f\|_{\dot B^s_{p,1}}=\sum_{j\in\mathbb{Z}}2^{js}\|\dot\Delta_j f\|_{L^p}.$$

DEFINITION 2.2. *Let $s\in\mathbb{R}$,$(r,\rho,p)\in[1,+\infty]^3$ and $T\in(0,+\infty]$. We say then that $f\in L^\rho_T(\dot B^s_{p,r})$, if*

$$\|f\|_{L^\rho_T(\dot B^s_{p,r})}\triangleq\Big\|\big\|2^{qs}\|\dot\Delta_q f\|_{L^p}\big\|_{\ell^r}\Big\|_{L^\rho_T}<+\infty.$$

Here, we introduce the Besov-Chemin-Lerner space $\widetilde L^\rho_T(\dot B^s_{p,r})$ which is initiated in [6].

DEFINITION 2.3. *Let $s\leq\frac{N}{p}$ (respectively $s\in\mathbb{R}$), $(r,\rho,p)\in[1,+\infty]^3$ and $T\in(0,+\infty]$. We define $\widetilde L^\rho_T(\dot B^s_{pr})$ as the completion of $C([0,T];\mathcal{S}'_h)$ by the norm*

$$\|f\|_{\tilde L^\rho_T(\dot B^s_{p,r})}\triangleq\Big\|2^{js}\|\dot\Delta_j f(t)\|_{L^\rho(0,T;L^p)}\Big\|_{\ell^r}<\infty$$

with the usual change if $r=\infty$.

Obviously, $\tilde L^1_T(\dot B^s_{p,1})=L^1_T(\dot B^s_{p,1})$. By a direct application of Minkowski's inequality, we have the following relations between these spaces

$$L^\rho_T(\dot B^s_{p,r})\hookrightarrow\tilde L^\rho_T(\dot B^s_{p,r}) \text{ if } \quad r\geq\rho,\quad \tilde L^\rho_T(\dot B^s_{p,r})\hookrightarrow L^\rho_T(\dot B^s_{p,r}) \text{ if } \quad \rho\geq r.$$

In what follows, some properties of the Besov spaces are listed.

LEMMA 2.1 (See [1]).

(1) *For any $p\in[1,\infty]$ we have the continuous embedding*

$$\dot B^0_{p,1}\hookrightarrow L^p\hookrightarrow\dot B^0_{p,\infty}.$$

(2) *If $s\in\mathbb{R}$, $1\leq p_1\leq p_2\leq\infty$ and $1\leq r_1\leq r_2\leq\infty$, then $\dot B^s_{p_1,r_1}\hookrightarrow\dot B^{s-N(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$.*

(3) *The space $\dot B^{\frac{N}{p}}_{p,1}$ is continuously embedded in the set of bounded continuous functions (going to 0 at infinity if $p<\infty$).*

(4) *For $1\leq p,r_1,r_2,r\leq\infty$, $\sigma_1\neq\sigma_2$ and $\theta\in(0,1)$, then*

$$\|f\|_{\dot B^{\theta\sigma_2+(1-\theta)\sigma_1}_{p,r}}\leq C\|f\|^{1-\theta}_{\dot B^{\sigma_1}_{p,r_1}}\|f\|^\theta_{\dot B^{\sigma_2}_{p,r_2}}.$$

LEMMA 2.2 (See [1]). *For all $1\le r,p,p_1,p_2\le+\infty$, there exists a positive universal constant such that*

$$\begin{aligned}
&\|fg\|_{\dot B^s_{p,r}}\lesssim\|f\|_{L^\infty}\|g\|_{\dot B^s_{p,r}}+\|g\|_{L^\infty}\|f\|_{\dot B^s_{p,r}}, \quad \text{if} \quad s>0;\\
&\|fg\|_{\dot B^{s_1+s_2-\frac{N}{p}}_{p,r}}\lesssim\|f\|_{\dot B^{s_1}_{p,r}}\|g\|_{\dot B^{s_2}_{p,\infty}}, \quad \text{if} \quad s_1,s_2<\frac{N}{p}, \quad \text{and} \quad s_1+s_2>0;\\
&\|fg\|_{\dot B^s_{p,r}}\lesssim\|f\|_{\dot B^s_{p,r}}\|g\|_{\dot B^{\frac{N}{p}}_{p,\infty}\cap L^\infty}, \quad \text{if} \quad |s|<\frac{N}{p};\\
&\|fg\|_{\dot B^s_{2,1}}\lesssim\|f\|_{\dot B^{N/2}_{2,1}}\|g\|_{\dot B^s_{2,1}}, \quad \text{if} \quad s\in(-N/2,N/2].
\end{aligned}$$

The basic tool of the paradifferential calculus is Bony's decomposition [3]. Formally, the product of two tempered distributions u and v may be decomposed into

$$uv=\dot T_uv+\dot T_vu+\dot R(u,v)$$

with

$$\dot T_uv=\sum_{j\in\mathbb Z}\dot S_{j-1}u\dot\Delta_jv, \quad \dot R(u,v)=\sum_{j\in\mathbb Z}\dot\Delta_ju\widetilde{\dot\Delta_j}v, \quad \widetilde{\dot\Delta_j}v=\sum_{|j-j'|\le1}\dot\Delta_{j'}v.$$

As a consequence, the estimates for the paraproduct and remainder operators can be given by

PROPOSITION 2.1 (See [18]). *Let $N\ge2$, $s\in\mathbb R$ and $2\le p\le\min(4,\frac{2N}{N-2})$, we have*

$$\|T_fg\|_{L^1_t(\dot B^{s-1+\frac{N}{2}-\frac{N}{p}}_{2,1})}\le C\|f\|_{L^\infty_t(\dot B^{\frac{N}{p}-1}_{p,1})}\|g\|_{L^1_t(\dot B^s_{p,1})}.$$

In particular, for $s\in\mathbb R$, $m\ge0$, we also have

$$\|(T_fg)^\ell\|_{\dot B^{s-1+\frac{N}{2}-\frac{N}{p}}_{2,1}}\le C\|f\|_{\dot B^{\frac{N}{p}-1}_{p,1}}\|g\|_{\dot B^{s-m}_{p,1}}.$$

PROPOSITION 2.2 (See [18]). *Let $N\ge2$, $s>1-\min(\frac{N}{p},\frac{N}{p'})$ and $1\le p\le4$, we have*

$$\|R(f,g)\|_{\dot B^{s-1+\frac{N}{2}-\frac{N}{p}}_{2,1}}\le C\|f\|_{\dot B^{\frac{N}{p}-1}_{p,1}}\|g\|_{\dot B^s_{p,1}}.$$

LEMMA 2.3 (See [8,9]).
(i) *Let $s>0$, $1\le p,r\le\infty$ and $u\in\dot B^s_{p,r}\cap L^\infty$. If $F\in W^{[s]+2,\infty}_{loc}(\mathbb R^N)$ with $F(0)=0$, then $F(u)\in\dot B^s_{p,r}$. Moreover, there exists a function of one variable C_0 depending only on s and F, and such that*

$$\|F(u)\|_{\dot B^s_{p,r}}\le C_0(\|u\|_{L^\infty})\|u\|_{\dot B^s_{p,r}}.$$

(ii) *Let $s>-N\min(\frac1p,1-\frac1p)$ and $u\in\dot B^s_{p,1}\cap\dot B^{\frac{N}{p}}_{p,1}$, we have $F(u)\in\dot B^s_{p,1}\cap\dot B^{\frac{N}{p}}_{p,1}$ and*

$$\|F(u)\|_{\dot B^s_{p,1}}\le C_0(1+\|u\|_{\dot B^{\frac{N}{p}}_{p,1}})\|u\|_{\dot B^s_{p,1}}.$$

Proposition 2.3 (See [12]). *Let the real numbers σ_1, σ_2, p_1 and p_2 satisfy*

$$\sigma_1+\sigma_2>0, \quad \sigma_1\le\frac{N}{p_1}, \quad \sigma_2\le\frac{N}{p_2}, \quad \sigma_1\ge\sigma_2, \quad \frac{1}{p_1}+\frac{1}{p_2}\le 1,$$

then

$$\|fg\|_{\dot{B}^{\sigma_2}_{q,1}}\lesssim\|f\|_{\dot{B}^{\sigma_1}_{p_1,1}}\|g\|_{\dot{B}^{\sigma_2}_{p_2,1}} \quad \text{with} \quad \frac{1}{q}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\sigma_1}{N}.$$

Let the exponents $\sigma>0$ and $1\le p_1,p_2,q\le\infty$ satisfy

$$\frac{N}{p_1}+\frac{N}{p_2}-N\le\sigma\le\min(\frac{N}{p_1},\frac{N}{p_2}) \quad \text{with} \quad \frac{1}{q}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\sigma}{N},$$

then

$$\|fg\|_{\dot{B}^{-\sigma}_{q,\infty}}\lesssim\|f\|_{\dot{B}^{\sigma}_{p_1,1}}\|g\|_{\dot{B}^{-\sigma}_{p_2,\infty}}.$$

Proposition 2.4 (See [12]). *Let $q_0\in\mathbb{Z}$, and denote $\dot{S}_{q_0}u\triangleq u^\ell$ and for any $s\in\mathbb{R}$, there exists a universal integer N_0 such that for any $2\le p\le 4$ and $\sigma>0$, then*

$$\|uv^h\|^\ell_{\dot{B}^{-s_0}_{2,\infty}}\le C(\|u\|_{\dot{B}^{\sigma}_{p,1}}+\|\dot{S}_{q_0+N_0}u\|_{L^{p^*}})\|v^h\|_{\dot{B}^{-\sigma}_{p,\infty}},$$

and

$$\|u^hv\|^\ell_{\dot{B}^{-s_0}_{2,\infty}}\le C(\|u^h\|_{\dot{B}^{\sigma}_{p,1}}+\|\dot{S}_{q_0+N_0}u^h\|_{L^{p^*}})\|v\|_{\dot{B}^{-\sigma}_{p,\infty}}$$

with $s_0=\frac{2N}{p}-\frac{N}{2}$ and $\frac{1}{p^*}=\frac{1}{2}-\frac{1}{p}$, and C depending only on k_0,N and σ.

The following Bernstein inequality and Bernstein-like inequality will be frequently used.

Lemma 2.4 (See [5]). *Let $1\le p\le q\le+\infty$. Assume that $f\in L^p(\mathbb{R}^N)$, then for any $\gamma\in(\mathbb{N}\cup\{0\})^N$, there exist constants C_1, C_2 independent of f, j such that*

$$\begin{aligned}&\|\partial^\gamma f\|_q\le C_1 2^{j|\gamma|+jN(\frac{1}{p}-\frac{1}{q})}\|f\|_p \ \text{ for } \ \mathrm{supp}\hat{f}\subseteq\{|\xi|\le A_0 2^j\},\\ &\|f\|_p\le C_2 2^{-j|\gamma|}\sup_{|\beta|=|\gamma|}\|\partial^\beta f\|_p \ \text{ for } \ \mathrm{supp}\hat{f}\subseteq\{A_1 2^j\le|\xi|\le A_2 2^j\}.\end{aligned}$$

Lemma 2.5 (See [11]). *If $\mathrm{supp}\mathcal{F}f\subset\xi\in\mathbb{R}^N:r_1\lambda\le|\xi|\le r_2\lambda$, then there exists c depending only on N,r_1 and r_2 so that for all $1<p<\infty$,*

$$c\lambda^2(\frac{p-1}{p})\int_{\mathbb{R}^N}|f|^p dx\le(p-1)\int_{\mathbb{R}^N}|\nabla f|^2|f|^{p-2}dx=-\int_{\mathbb{R}^N}\Delta f|f|^{p-2}f dx.$$

We shall also use the following important commutator estimates in the proof of our main results.

Lemma 2.6 (See [1]). *Let $1\le p,p_1\le\infty$, $1\le r\le\infty$ and $\sigma\in\mathbb{R}$. There exists a constant $C>0$ depending only on σ such that for all $q\in\mathbb{Z}$, we have*

$$\|[v\cdot\nabla,\partial_\ell\dot{\Delta}_q]a\|_{L^p}\le Cc_q 2^{-q(\sigma-1)}\|v\|_{\dot{B}^{\frac{N}{p_1}+1}_{p_1,1}}\|a\|_{\dot{B}^{\sigma}_{p,1}}, if -\min(\frac{N}{p_1},\frac{N}{p'})<\sigma\le 1+\min(\frac{N}{p_1},\frac{N}{p})$$

and

$$\|[v\cdot\nabla,\dot{\Delta}_q]a\|_{L^p}\leq Cc_q2^{-q\sigma}\|\nabla v\|_{\dot{B}^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty}\|a\|_{\dot{B}^{\sigma}_{p,1}},\quad \text{if}\quad -\min(\frac{N}{p_1},\frac{N}{p'})<\sigma<1+\frac{N}{p_1},$$

where the commutator $[\cdot,\cdot]$ *is defined by* $[f,g]=fg-gf$ *and* $(c_j)_{j\in\mathbb{Z}}$ *denotes a sequence such that* $\sum_{q\in\mathbb{Z}}c_q\leq 1$.

We also recall some maximal regularity properties for the heat equation.

PROPOSITION 2.5 (See [1]). *Assume* $\mu>0$, $\sigma\in\mathbb{R},(p,r)\in[1,\infty]^2$ *and* $1\leq\rho_2\leq\rho_1\leq\infty$. *Let* u *satisfy*

$$\begin{cases}\partial_t u-\mu\Delta u=f,\\ u\,|_{t=0}=u_0.\end{cases}$$

Then for all $T>0$ *the following a priori estimate is fulfilled*

$$\mu^{\frac{1}{\rho_1}}\|u\|_{\widetilde{L}^{\rho_1}_T(\dot{B}^{\sigma+\frac{2}{\rho_1}}_{p,r})}\lesssim\|u_0\|_{\dot{B}^{\sigma}_{p,r}}+\mu^{\frac{1}{\rho_2}-1}\|f\|_{\widetilde{L}^{\rho_2}_T(\dot{B}^{\sigma-2+\frac{2}{\rho_2}}_{p,r})}.$$

REMARK 2.1. The solutions to the following Lamé system

$$\begin{cases}\partial_t u-\mu\Delta u-(\lambda+\mu)\nabla\mathrm{div}u=f,\\ u|_{t=0}=u_0,\end{cases}$$

also fulfill the above inequality, whenever $\mu>0$ and $\lambda+2\mu>0$.

Finally, we give several important inequalities to conclude this section.

LEMMA 2.7 (See [1]). *For all* $\sigma>0$, *there exists a positive constant* C_σ *such that*

$$\sup_{t\geq 0}\sum_{q\in\mathbb{Z}}(2^qt^{\frac12})^\sigma e^{-c_02^{2q}t}\leq C_\sigma.$$

LEMMA 2.8 (See [23]). *Let* $r_1,r_2>0$ *satisfy* $\max\{r_1,r_2\}>1$. *Then*

$$\int_0^t(1+t-\tau)^{-r_1}(1+\tau)^{-r_2}d\tau\leq C(r_1,r_2)(1+t)^{-\min\{r_1,r_2\}}.$$

LEMMA 2.9 (See [12]). *Let* $0\leq r_1\leq r_2$ *with* $r_2>1$. *Then*

$$\int_0^t(1+t-\tau)^{-r_1}\tau^{-\theta}(1+\tau)^{\theta-r_2}d\tau\leq C(r_1,r_2)(1+t)^{-r_1}.$$

3. Reformulation of the original system

First, we can assume that $\bar\rho=1$ and $\sqrt{P_i'(\bar\rho)}=c$ without loss of generality. Let $c_1=\rho_1-1,c_2=\rho_2-1$, then the Cauchy problem (1.1)-(1.2) rewrites as

$$\begin{cases}\partial_tc_1+\mathrm{div}u_1=-\mathrm{div}(c_1u_1),\\ \partial_tu_1-\mu_1\Delta u_1-\mu_2\nabla\mathrm{div}u_1+c^2\nabla c_1-\nabla\phi=-\dfrac{c_1}{1+c_1}\mathcal{A}u_1-\Big(\dfrac{\nabla P_1(1+c_1)}{1+c_1}-c^2\nabla c_1\Big)-u_1\nabla u_1,\\ \partial_tc_2+\mathrm{div}u_2=-\mathrm{div}(c_2u_2),\\ \partial_tu_2-\mu_1\Delta u_2-\mu_2\nabla\mathrm{div}u_2+c^2\nabla c_2+\nabla\phi=-\dfrac{c_2}{1+c_2}\mathcal{A}u_2-\Big(\dfrac{\nabla P_2(1+c_2)}{1+c_2}-c^2\nabla c_2\Big)-u_2\nabla u_2,\\ \Delta\phi=c_1-c_2,\\ (c_1,u_1,c_2,u_2)|_{t=0}=(c_{1,0},u_{1,0},c_{2,0},u_{2,0}),\end{cases}\tag{3.1}$$

where

$$\mathcal{A}=:\mu_1\Delta+\mu_2\nabla\mathrm{div}.$$

Denoting $n_1=:c_1+c_2, n_2=:c_1-c_2, \omega_1=:u_1+u_2, \omega_2=:u_1-u_2$, which gives, equivalently

$$c_1=\frac{n_1+n_2}{2}, c_2=\frac{n_1-n_2}{2}, u_1=\frac{\omega_1+\omega_2}{2}, u_2=\frac{\omega_1-\omega_2}{2}.$$

Then it follows that system (3.1) can be reformulated into the following Cauchy problem for the new unknown $(n_1,\omega_1,n_2,\omega_2,\nabla\phi)$

$$\begin{cases}\partial_t n_1+\mathrm{div}\omega_1=E_1(n_1,n_2,\omega_1,\omega_2),\\ \partial_t\omega_1-\mu_1\Delta\omega_1-\mu_2\nabla\mathrm{div}\omega_1+c^2\nabla n_1=F_1(n_1,n_2,\omega_1,\omega_2),\\ \partial_t n_2+\mathrm{div}\omega_2=E_2(n_1,n_2,\omega_1,\omega_2),\\ \partial_t\omega_2-\mu_1\Delta\omega_2-\mu_2\nabla\mathrm{div}\omega_2+c^2\nabla n_2-2\nabla\phi=F_2(n_1,n_2,\omega_1,\omega_2),\\ \Delta\phi=n_2\end{cases}\tag{3.2}$$

with initial data

$$\begin{aligned}(n_1,\omega_1,n_2,\omega_2,\nabla\phi)|_{t=0}&=(n_{1,0},\omega_{1,0},n_{2,0},\omega_{2,0},\nabla\phi_0)\\ &=:(c_{1,0}+c_{2,0},u_{1,0}+u_{2,0},c_{1,0}-c_{2,0},u_{1,0}-u_{2,0}),\end{aligned}$$

where for $i=1,2$,

$$E_i=:-\mathrm{div}\Big(\frac{n_1+n_2}{2}\frac{\omega_1+\omega_2}{2}\Big)+(-1)^i\mathrm{div}\Big(\frac{n_1-n_2}{2}\frac{\omega_1-\omega_2}{2}\Big),$$

and

$$\begin{aligned}F_i=:&-\frac{n_1+n_2}{2+n_1+n_2}\mathcal{A}\Big(\frac{\omega_1+\omega_2}{2}\Big)+(-1)^i\frac{n_1-n_2}{2+n_1-n_2}\mathcal{A}\Big(\frac{\omega_1-\omega_2}{2}\Big)\\ &-\Big(\frac{2P_1'(1+\frac{n_1+n_2}{2})}{2+n_1+n_2}-c^2\Big)\nabla\Big(\frac{n_1+n_2}{2}\Big)+(-1)^i\Big(\frac{2P_2'(1+\frac{n_1-n_2}{2})}{2+n_1-n_2}-c^2\Big)\nabla\Big(\frac{n_1-n_2}{2}\Big)\\ &-\Big(\frac{\omega_1+\omega_2}{2}\Big)\cdot\nabla\Big(\frac{\omega_1+\omega_2}{2}\Big)+(-1)^i\Big(\frac{\omega_1-\omega_2}{2}\Big)\cdot\nabla\Big(\frac{\omega_1-\omega_2}{2}\Big).\end{aligned}$$

Next, let us decompose ω_i into $\omega_i=\mathcal{P}\omega_i+\mathcal{Q}\omega_i (i=1,2)$, where $\mathcal{P}$ and $\mathcal{Q}$ are the projectors onto divergence-free and potential vector-fields, respectively (hence $\mathcal{P}=\mathrm{Id}+(-\Delta)^{-1}\nabla\mathrm{div}$). Setting $\nu=\mu_1+\mu_2$, and applying the orthogonal projectors $\mathcal{P}$ and $\mathcal{Q}$, respectively, to $(3.2)_2$ and $(3.2)_4$ yield that

$$\begin{cases}\partial_t n_1+\mathrm{div}\mathcal{Q}\omega_1=E_1,\\ \partial_t\mathcal{Q}\omega_1-\nu\Delta\mathcal{Q}\omega_1+c^2\nabla n_1=\mathcal{Q}F_1,\\ \partial_t n_2+\mathrm{div}\mathcal{Q}\omega_2=E_2,\\ \partial_t\mathcal{Q}\omega_2-\nu\Delta\mathcal{Q}\omega_2+c^2\nabla n_2-2\nabla\Delta^{-1}n_2=\mathcal{Q}F_2,\end{cases}\tag{3.3}$$

and

$$\begin{cases}\partial_t\mathcal{P}\omega_1-\mu_1\Delta\mathcal{P}\omega_1=\mathcal{P}F_1,\\ \partial_t\mathcal{P}\omega_2-\mu_1\Delta\mathcal{P}\omega_2=\mathcal{P}F_2.\end{cases}\tag{3.4}$$

Setting $v_1 =: \Lambda^{-1}\mathrm{div}\mathcal{Q}\omega_1, v_2 =: \Lambda^{-1}\mathrm{div}\mathcal{Q}\omega_2$, here $\Lambda^s h =: \mathcal{F}^{-1}(|\xi|^s \widehat{h})$ for $s \in \mathbb{R}$, then we see that, from (3.3), (n_1, v_1, n_2, v_2) satisfies

$$\begin{cases} \partial_t n_1 + \Lambda v_1 = E_1, \\ \partial_t v_1 - \nu\Delta v_1 - c^2\Lambda n_1 = \Lambda^{-1}\mathrm{div}F_1, \\ \partial_t n_2 + \Lambda v_2 = E_2, \\ \partial_t v_2 - \nu\Delta v_2 - c^2\Lambda n_2 - 2\Lambda^{-1}n_2 = \Lambda^{-1}\mathrm{div}F_2 \end{cases} \tag{3.5}$$

with initial data

$$(n_1, v_1, n_2, v_2)|_{t=0} = (n_{10}, v_{10}, n_{20}, v_{20}). \tag{3.6}$$

Here we would like to point out that, different from the compressible Navier-Stokes equations, the nonlocal term $\Lambda^{-1}n_2$ in system (3.5) causes some trouble in low frequencies in our analysis. To tackle this problem, we further introduce two new unknowns $a_1 = \Lambda^{-1}n_1$ and $a_2 = \Lambda^{-1}n_2$, and the Cauchy problem (3.5)-(3.6) is equivalent to the following form

$$\begin{cases} \partial_t a_1 + v_1 = \Lambda^{-1}E_1, \\ \partial_t v_1 - \nu\Delta v_1 + c^2\Delta a_1 = \Lambda^{-1}\mathrm{div}F_1, \\ \partial_t a_2 + v_2 = \Lambda^{-1}E_2, \\ \partial_t v_2 - \nu\Delta v_2 + c^2\Delta a_2 - 2a_2 = \Lambda^{-1}\mathrm{div}F_2, \\ (a_1, v_1, a_2, v_2)|_{t=0} = (\Lambda^{-1}n_{10}, v_{10}, \Lambda^{-1}n_{20}, v_{20}). \end{cases} \tag{3.7}$$

4. The spectral analysis of the linearized system

In this section, we shall give a detailed analysis of Green's function and exploit the smoothing effects of Green's matrix in the low frequencies. To begin with, let us consider the Cauchy problem for the corresponding linearized system of (3.7) without outer forces, namely

$$\begin{cases} \partial_t a_1 + v_1 = 0, \\ \partial_t v_1 - \nu\Delta v_1 + c^2\Delta a_1 = 0, \\ \partial_t a_2 + v_2 = 0, \\ \partial_t v_2 - \nu\Delta v_2 + c^2\Delta a_2 - 2a_2 = 0, \\ (a_1, v_1, a_2, v_2)|_{t=0} = (a_{10}, v_{10}, a_{20}, v_{20}). \end{cases} \tag{4.1}$$

Then, the solution of (4.1) can be expressed as

$$\begin{pmatrix} a_1 \\ v_1 \\ a_2 \\ v_2 \end{pmatrix} = G(x,t) * \begin{pmatrix} a_{1,0} \\ v_{1,0} \\ a_{2,0} \\ v_{2,0} \end{pmatrix}, \tag{4.2}$$

where $G(x,t)$ is Green's matrix for system (4.1).

Now we first present an explicit derivation of the Fourier transform $\widehat{G}(\xi,t)$ of Green's matrix $G(x,t)$ in the following lemma.

LEMMA 4.1. *The Fourier transform $\widehat{G}(\xi,t)$ of Green's matrix $G(x,t)$ for the linearized system (4.1) is given by*

$$\widehat{G}(\xi,t)=:\begin{pmatrix} \frac{\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t}}{\lambda_+-\lambda_-} & -\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-} & 0 & 0 \\ -\frac{e^{\lambda_-t}-e^{\lambda_+t}}{\lambda_+-\lambda_-}c^2|\xi|^2 & \frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-} & 0 & 0 \\ 0 & 0 & \frac{\tilde\lambda_+e^{\tilde\lambda_-t}-\tilde\lambda_-e^{\tilde\lambda_+t}}{\tilde\lambda_+-\tilde\lambda_-} & -\frac{e^{\tilde\lambda_+t}-e^{\tilde\lambda_-t}}{\tilde\lambda_+-\tilde\lambda_-} \\ 0 & 0 & -\frac{e^{\tilde\lambda_-t}-e^{\tilde\lambda_+t}}{\tilde\lambda_+-\tilde\lambda_-}(c^2|\xi|^2+2) & \frac{\tilde\lambda_+e^{\tilde\lambda_+t}-\tilde\lambda_-e^{\tilde\lambda_-t}}{\tilde\lambda_+-\tilde\lambda_-} \end{pmatrix} \tag{4.3}$$

with

$$\lambda_\pm=-\frac{\nu}{2}|\xi|^2\pm\frac{1}{2}\sqrt{\nu^2|\xi|^4-4c^2|\xi|^2},\quad \tilde\lambda_\pm=-\frac{\nu}{2}|\xi|^2\pm\frac{1}{2}\sqrt{\nu^2|\xi|^4-4(c^2|\xi|^2+2)}.$$

Proof. Applying the Fourier transforms to $(4.1)_1$ and $(4.1)_2$ leads to

$$\begin{cases}\partial_t\hat a_1+\hat v_1=0,\\ \partial_t\hat v_1+\nu|\xi|^2\hat v_1-c^2|\xi|^2\hat a_1=0.\end{cases} \tag{4.4}$$

From (4.4), it is easy to get

$$\partial_{tt}\hat v_1+\nu|\xi|^2\partial_t\hat v_1-c^2|\xi|^2\partial_t\hat a_1=0.$$

Further, we have

$$\begin{cases}\partial_{tt}\hat v_1+\nu|\xi|^2\partial_t\hat v_1+c^2|\xi|^2\hat v_1=0,\\ \hat v_1(\xi,0)=\hat v_{10}(\xi),\partial_t\hat v_1(\xi,0)=-\nu|\xi|^2\hat v_{10}+c^2|\xi|^2\hat a_{10}.\end{cases} \tag{4.5}$$

It is easy to check that $\lambda_\pm$ are two roots of (4.5). Thus, we assume that the solution of (4.5) has the form

$$\hat v_1(\xi,t)=A(\xi)e^{\lambda_-(\xi)t}+B(\xi)e^{\lambda_+(\xi)t}. \tag{4.6}$$

It follows from the initial conditions, that

$$A(\xi)=\frac{-(\lambda_++\nu|\xi|^2)\hat v_{10}+c^2|\xi|^2\hat a_{10}}{\lambda_--\lambda_+},\ B(\xi)=\frac{(\lambda_-+\nu|\xi|^2)\hat v_{10}-c^2|\xi|^2\hat a_{10}}{\lambda_--\lambda_+},$$

which together with (4.6) implies

$$\hat v_1(\xi,t)=\frac{e^{\lambda_-t}-e^{\lambda_+t}}{\lambda_--\lambda_+}c^2|\xi|^2\hat a_{10}(\xi)+\frac{\lambda_-e^{\lambda_-t}-\lambda_+e^{\lambda_+t}}{\lambda_--\lambda_+}\hat v_{10}(\xi). \tag{4.7}$$

Moreover, from the first equation of (4.4), we see

$$\hat a_1(\xi,t)=\hat a_1(\xi,0)+\int_0^t\hat v_1(\xi,\tau)d\tau,$$

which together with (4.7) and the following relations $\lambda_\pm+\lambda_\mp=-\nu|\xi|^2,\lambda_+\lambda_-=c^2|\xi|^2$ yields

$$\hat a_1(\xi,t)=\frac{\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t}}{\lambda_+-\lambda_-}\hat a_{10}(\xi)-\frac{e^{\lambda_-t}-e^{\lambda_+t}}{\lambda_--\lambda_+}\hat v_{10}(\xi). \tag{4.8}$$

Next, in the completely same way, dealing with the Fourier transforms to $(4.1)_3$ and $(4.1)_4$, we also obtain

$$\hat{v}_2(\xi,t)=\frac{e^{\tilde{\lambda}_-t}-e^{\tilde{\lambda}_+t}}{\tilde{\lambda}_- -\tilde{\lambda}_+}(c^2|\xi|^2+2)\hat{a}_{20}(\xi)+\frac{\tilde{\lambda}_-e^{\tilde{\lambda}_-t}-\tilde{\lambda}_+e^{\tilde{\lambda}_+t}}{\tilde{\lambda}_- -\tilde{\lambda}_+}\hat{v}_{20}(\xi), \tag{4.9}$$

and

$$\hat{a}_2(\xi,t)=\frac{\tilde{\lambda}_+e^{\tilde{\lambda}_-t}-\tilde{\lambda}_-e^{\tilde{\lambda}_+t}}{\tilde{\lambda}_+ -\tilde{\lambda}_-}\hat{a}_{20}(\xi)-\frac{e^{\tilde{\lambda}_-t}-e^{\tilde{\lambda}_+t}}{\tilde{\lambda}_- -\tilde{\lambda}_+}\hat{v}_{20}(\xi). \tag{4.10}$$

Therefore, combining (4.7), (4.8), (4.9) and (4.10), we get (4.3). □

With Lemma 4.1 at hand, we shall derive the pointwise estimates for $\widehat{G}(\xi,t)$ in the low frequencies, which behaves like the heat kernel.

Lemma 4.2. *Let $G(x,t)$ be Green's matrix of Lemma 4.1. Given $R>0$, there is a positive number ϑ such that for $|\xi|<R$, it holds that*

$$|\widehat{G}(\xi,t)|\le Ce^{-\frac{\vartheta}{2}|\xi|^2t}, \tag{4.11}$$

where $C=C(R)$.

Proof. First, recalling

$$\lambda_\pm=-\frac{\nu}{2}|\xi|^2\pm\frac{1}{2}\sqrt{\nu^2|\xi|^4-4c^2|\xi|^2},$$

we readily gather the following simple facts:
If $|\xi|<\frac{2c}{\nu}$, then both $\lambda_\pm$ are two complex conjugated eigenvalues

$$\lambda_\pm=-\frac{\nu}{2}|\xi|^2(1\mp iS_1)\quad\text{with}\quad S_1=\sqrt{\frac{4c^2}{\nu^2|\xi|^2}-1}.$$

If $|\xi|=\frac{2c}{\nu}$, then both $\lambda_\pm$ have real double roots

$$\lambda_\pm=-\frac{\nu}{2}|\xi|^2.$$

If $\frac{2c}{\nu}<|\xi|\le R_1$, then both $\lambda_\pm$ are two real eigenvalues

$$\lambda_\pm=-\frac{\nu}{2}|\xi|^2\Big(1\mp\sqrt{1-\frac{4c^2}{\nu^2|\xi|^2}}\Big)\le-\frac{\nu}{2}|\xi|^2\Big(1\mp\sqrt{1-\frac{4c^2}{\nu^2R_1^2}}\Big),$$

and there is also a positive constant ϑ_1 depending on R_1 such that for $0\le|\xi|\le R_1$, it holds that

$$\mathrm{Re}(\lambda_\pm)\le-\vartheta_1|\xi|^2, \tag{4.12}$$

and

$$|e^{\lambda_\pm(\xi)t}|\le e^{-\vartheta_1|\xi|^2t}. \tag{4.13}$$

Now let $r=\frac{2c}{\nu}$ be the resonant value of $|\xi|$, and choose a small positive number μ. First, for $0\le|\xi|<r-\mu$, we denote $b=\frac{1}{2}\sqrt{4c^2|\xi|^2-\nu^2|\xi|^4}$, then $b>0$ and $\lambda_\pm=-\frac{\nu}{2}|\xi|^2\pm bi$. Further, employing Euler's formula, it is easy to see

$$\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-}=\frac{\sin(bt)}{b}e^{-\frac{\nu}{2}|\xi|^2t},\quad \frac{\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t}}{\lambda_+-\lambda_-}=[\cos(bt)+\frac{\nu}{2}\frac{\sin(bt)}{b}|\xi|^2]e^{-\frac{\nu}{2}|\xi|^2t},$$
$$\frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-}=[\cos(bt)-\frac{\nu}{2}\frac{\sin(bt)}{b}|\xi|^2]e^{-\frac{\nu}{2}|\xi|^2t}.$$

Therefore, for $0\le|\xi|<r-\mu$, we have

$$|\widehat{G}_{11}(\xi,t)|,|\widehat{G}_{12}(\xi,t)|,|\widehat{G}_{21}(\xi,t)|,|\widehat{G}_{22}(\xi,t)|\le Ce^{-\frac{\vartheta}{2}|\xi|^2t}. \tag{4.14}$$

Next, for $r-\mu\le|\xi|<r+\mu$, applying (4.13), we see

$$|\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-}|\le t\sup_{0\le s\le1}e^{t\mathrm{Re}(s\lambda_-+(1-s)\lambda_+)}\le te^{-\vartheta|\xi|^2t}\le Ce^{\frac{-\vartheta}{2}|\xi|^2t},$$
$$|\frac{\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t}}{\lambda_+-\lambda_-}|\le|\lambda_+\frac{e^{\lambda_-t}-e^{\lambda_+t}}{\lambda_+-\lambda_-}|+|e^{\lambda_+t}|\le Ce^{\frac{-\vartheta}{2}|\xi|^2t},$$
$$|\frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-}|\le|\lambda_+\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-}|+|e^{\lambda_-t}|\le Ce^{\frac{-\vartheta}{2}|\xi|^2t}.$$

Then, for $r-\mu\le|\xi|<r+\mu$, we also have

$$|\widehat{G}_{11}(\xi,t)|,|\widehat{G}_{12}(\xi,t)|,|\widehat{G}_{21}(\xi,t)|,|\widehat{G}_{22}(\xi,t)|\le Ce^{-\frac{\vartheta}{2}|\xi|^2t}. \tag{4.15}$$

Finally, for $r+\mu\le|\xi|\le R_1$, we have

$$|\lambda_+-\lambda_-|=\nu|\xi|^2\sqrt{1-r^2|\xi|^{-2}}\ge\nu|\xi|^2\sqrt{1-(\frac{r}{r+\mu})^2}\ge C^{-1},$$

which implies

$$|\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-}|\le t\sup_{0\le s\le1}e^{tRe(s\lambda_-+(1-s)\lambda_+)}\le te^{-\vartheta|\xi|^2t}\le Ce^{\frac{-\vartheta}{2}|\xi|^2t},$$
$$|\frac{\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t}}{\lambda_+-\lambda_-}|\le|\lambda_+\frac{e^{\lambda_-t}-e^{\lambda_+t}}{\lambda_+-\lambda_-}|+|e^{\lambda_+t}|\le Ce^{\frac{-\vartheta}{2}|\xi|^2t},$$
$$|\frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-}|\le|\lambda_+\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-}|+|e^{\lambda_-t}|\le Ce^{\frac{-\vartheta}{2}|\xi|^2t}.$$

Hence, for $r+\mu\le|\xi|\le R_1$, we also have

$$|\widehat{G}_{11}(\xi,t)|,|\widehat{G}_{12}(\xi,t)|,|\widehat{G}_{21}(\xi,t)|,|\widehat{G}_{22}(\xi,t)|\le Ce^{-\frac{\vartheta}{2}|\xi|^2t}. \tag{4.16}$$

On the other hand, considering

$$\tilde{\lambda}_\pm=-\frac{\nu}{2}|\xi|^2\pm\frac{1}{2}\sqrt{\nu^2|\xi|^4-4(c^2|\xi|^2+2)},$$

we have the following facts:

If $|\xi|<\sqrt{\frac{2c^2+2\sqrt{c^4+2\nu^2}}{\upsilon^2}}$, then both $\tilde{\lambda}_\pm$ are two complex conjugated eigenvalues

$$\tilde{\lambda}_\pm=-\frac{\nu}{2}|\xi|^2(1\mp iS_2)\quad\text{with}\quad S_2=\sqrt{\frac{4(c^2|\xi|^2+2)}{\nu^2|\xi|^4}-1}.$$

If $|\xi|=\sqrt{\frac{2c^2+2\sqrt{c^4+2\nu^2}}{\upsilon^2}}$, then both $\tilde{\lambda}_\pm$ have real double roots

$$\tilde{\lambda}_\pm=-\frac{\nu}{2}|\xi|^2.$$

If $\sqrt{\frac{2c^2+2\sqrt{c^4+2\nu^2}}{\upsilon^2}}<|\xi|\le R_2$, then both $\tilde{\lambda}_\pm$ are two real eigenvalues

$$\tilde{\lambda}_\pm=-\frac{\nu}{2}|\xi|^2\Big(1\mp\sqrt{1-\frac{4(c^2|\xi|^2+2)}{\nu^2|\xi|^4}}\Big)\le-\frac{\nu}{2}|\xi|^2\Big(1\mp\sqrt{1-\frac{4c^2}{\nu^2R_2^2}}\Big),$$

and there is also a positive constant ϑ_2 depending on R_2 such that for $0\le|\xi|\le R_2$, it holds that

$$\mathrm{Re}(\tilde{\lambda}_\pm)\le-\vartheta_2|\xi|^2,\quad |e^{\tilde{\lambda}_\pm(\xi)t}|\le e^{-\vartheta_2|\xi|^2t}.$$

Using similar derivations of (4.14)-(4.16), we also deduce that

$$|\widehat{G}_{33}(\xi,t)|,|\widehat{G}_{34}(\xi,t)|,|\widehat{G}_{43}(\xi,t)|,|\widehat{G}_{44}(\xi,t)|\le Ce^{-\frac{\vartheta_2}{2}|\xi|^2t}.$$

Finally, taking $R=\min\{R_1,R_2\}$, $\vartheta=\min\{\vartheta_1,\vartheta_2\}$, we conclude that (4.11) holds. This completes the proof of Lemma 4.2. ∎

Next we shall exploit the following smoothing effects of Green's matrix $G(x,t)$ in the low frequencies which plays an important role in this paper.

Lemma 4.3. *Let $\mathcal{C}$ be a ring centered at 0 in $\mathbb{R}^N$. Then there exist several positive constants R_0,C and c such that, if $\mathrm{supp}\hat{u}\subset\lambda\mathcal{C}$ and $\lambda\le R_0$, then we have*

$$\|G*u\|_{L^2}\le Ce^{-c\lambda^2t}\|u\|_{L^2}. \tag{4.17}$$

Proof. Using the Plancherel theorem and (4.11) yields that

$$\|G*u\|_{L^2}=\|\hat{G}(\xi)\hat{u}(\xi)\|_{L^2}\le C\|e^{-\frac{\vartheta}{2}|\xi|^2t}\hat{u}\|_{L^2}\le Ce^{-c\lambda^2t}\|u\|_{L^2}.$$

This completes the proof of Lemma 4.3. ∎

To conclude this section, we shall also give the following lemma about some optimal *a priori* estimates in the low frequencies for the solution to system (3.7) assuming that E_1, F_1, E_2, and F_2 are given.

Lemma 4.4. *Let (a_1,v_1,a_2,v_2) be a solution of system (3.7), and m_0 be any integer number. There exists a positive constant C depending only on ν,c^2 and m_0, such that the following inequality holds for all $t\ge0$ and $1\le r\le\infty$*

$$\|(a_1,v_1,a_2,v_2)\|^{\ell}_{\tilde{L}^r_t(\dot{B}^{s+\frac{2}{r}}_{2,1})}\le C\Big(\|(a_{1,0},v_{1,0},a_{2,0},v_{2,0})\|^{\ell}_{\dot{B}^s_{2,1}}+\|(\Lambda^{-1}E_1,\Lambda^{-1}E_2,F_1,F_2)\|^{\ell}_{\tilde{L}^1_t(\dot{B}^s_{2,1})}\Big). \tag{4.18}$$

Proof. From (4.2) and Duhamel's principle, the solution of system (3.7) can be expressed as

$$\begin{pmatrix} a_1 \\ v_1 \\ a_2 \\ v_2 \end{pmatrix} = G(t) * \begin{pmatrix} a_{1,0} \\ v_{1,0} \\ a_{2,0} \\ v_{2,0} \end{pmatrix} + \int_0^t G(t-\tau) * \begin{pmatrix} \Lambda^{-1} E_1 \\ F_1 \\ \Lambda^{-1} E_2 \\ F_2 \end{pmatrix} d\tau.$$

Applying homogeneous frequency localization operators $\dot{\Delta}_j$ on both sides of the equation above yields

$$\begin{pmatrix} \dot{\Delta}_j a_1 \\ \dot{\Delta}_j v_1 \\ \dot{\Delta}_j a_2 \\ \dot{\Delta}_j v_2 \end{pmatrix} = G(t) * \begin{pmatrix} \dot{\Delta}_j a_{1,0} \\ \dot{\Delta}_j v_{1,0} \\ \dot{\Delta}_j a_{2,0} \\ \dot{\Delta}_j v_{2,0} \end{pmatrix} + \int_0^t G(t-\tau) * \begin{pmatrix} \Lambda^{-1} \dot{\Delta}_j E_1 \\ \dot{\Delta}_j F_1 \\ \Lambda^{-1} \dot{\Delta}_j E_2 \\ \dot{\Delta}_j F_2 \end{pmatrix} d\tau,$$

which together with Lemma 4.3 and Young's inequality yields

$$\begin{aligned} &\|(\dot{\Delta}_j a_1(t), \dot{\Delta}_j v_1(t), \dot{\Delta}_j a_2(t), \dot{\Delta}_j v_2(t))\|_{L^2} \\ \le & C e^{-c2^{2j}t} \|(\dot{\Delta}_j a_{1,0}, \dot{\Delta}_j v_{1,0}, \dot{\Delta}_j a_{2,0}, \dot{\Delta}_j v_{2,0})\|_{L^2} \\ & + C \int_0^t e^{-c2^{2j}(t-\tau)} \|(\Lambda^{-1}\dot{\Delta}_j E_1, \dot{\Delta}_j F_1, \Lambda^{-1}\dot{\Delta}_j E_2, \dot{\Delta}_j F_2)\|_{L^2} d\tau. \end{aligned}$$

Then, taking L^r norm with respect to t gives

$$\begin{aligned} &\|(\dot{\Delta}_j a_1, \dot{\Delta}_j v_1, \dot{\Delta}_j a_2, \dot{\Delta}_j v_2)\|_{L^r_t L^2} \\ \le & C 2^{-\frac{2j}{r}} \Big(\|(\dot{\Delta}_j a_{1,0}, \dot{\Delta}_j v_{1,0}, \dot{\Delta}_j a_{2,0}, \dot{\Delta}_j v_{2,0})\|_{L^2} + \|(\Lambda^{-1}\dot{\Delta}_j E_1, \dot{\Delta}_j F_1, \Lambda^{-1}\dot{\Delta}_j E_2, \dot{\Delta}_j F_2)\|_{L^1_t L^2} \Big). \end{aligned}$$

Finally, multiplying 2^{js} on both sides of the above equation, and then summing for $j \le m_0$, we get (4.18). This completes the proof of Lemma 4.4. ◻

5. The proof of Theorem 1.1

Our goal in this section is to prove Theorem 1.1. Here, we only exploit an important global *a priori* estimate for system (3.2). The proof is divided into four parts as follows.

Part 1: Low frequencies. Recall that $\mathcal{P}\omega_i (i=1,2)$ are defined by (3.4). According to Proposition 2.5, we easily deduce that

$$\|\mathcal{P}\omega_i\|^{\ell}_{\tilde{L}^\infty_t(\dot{B}^s_{2,1})} + \mu_1 \|\nabla^2 \mathcal{P}\omega_i\|^{\ell}_{\tilde{L}^1_t(\dot{B}^s_{2,1})} \lesssim \|\mathcal{P}\omega_{i,0}\|^{\ell}_{\dot{B}^s_{2,1}} + \|F_i\|^{\ell}_{\tilde{L}^1_t(\dot{B}^s_{2,1})}, \quad i=1,2. \tag{5.1}$$

Combining (4.18) with (5.1) and then taking $s = \frac{N}{2}-1$ yield that

$$\begin{aligned} &\|(a_1,\omega_1,a_2,\omega_2)\|^{\ell}_{\tilde{L}^\infty_t(\dot{B}^{\frac{N}{2}-1}_{2,1})} + \|(a_1,\omega_1,a_2,\omega_2)\|^{\ell}_{L^1_t(\dot{B}^{\frac{N}{2}+1}_{2,1})} \\ \lesssim & \|(a_{1,0},\omega_{1,0},a_{2,0},\omega_{2,0})\|^{\ell}_{\dot{B}^{\frac{N}{2}-1}_{2,1}} + \|(F_1,F_2)\|^{\ell}_{L^1_t(\dot{B}^{\frac{N}{2}-1}_{2,1})} + \|(E_1,E_2)\|^{\ell}_{L^1_t(\dot{B}^{\frac{N}{2}-2}_{2,1})}. \end{aligned} \tag{5.2}$$

As $\Lambda^{-1} n_1 = a_1, \Lambda^{-1} n_2 = a_2$, (5.2) is equivalent to

$$\begin{aligned} &\|(n_1,n_2)\|^{\ell}_{\tilde{L}^\infty_t(\dot{B}^{\frac{N}{2}-2}_{2,1})} + \|(\omega_1,\omega_2)\|^{\ell}_{\tilde{L}^\infty_t(\dot{B}^{\frac{N}{2}-1}_{2,1})} + \|(n_1,n_2)\|^{\ell}_{L^1_t(\dot{B}^{\frac{N}{2}}_{2,1})} + \|(\omega_1,\omega_2)\|^{\ell}_{L^1_t(\dot{B}^{\frac{N}{2}+1}_{2,1})} \\ \lesssim & \|(n_{1,0},n_{2,0})\|^{\ell}_{\dot{B}^{\frac{N}{2}-2}_{2,1}} + \|(\omega_{1,0},\omega_{2,0})\|^{\ell}_{\dot{B}^{\frac{N}{2}-1}_{2,1}} + \|(F_1,F_2)\|^{\ell}_{L^1_t(\dot{B}^{\frac{N}{2}-1}_{2,1})} + \|(E_1,E_2)\|^{\ell}_{L^1_t(\dot{B}^{\frac{N}{2}-2}_{2,1})}. \end{aligned} \tag{5.3}$$

Part 2: High frequencies. First, we bound $\mathcal{P}\omega_i(i=1,2)$ as follows. Thanks to Proposition 2.5 (restricted to the high frequencies), we have

$$\|\mathcal{P}\omega_i\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}+\mu_1\|\mathcal{P}\omega_i\|^h_{L^1_t(\dot{B}^{\frac{N}{p}+1}_{p,1})}\lesssim\|\mathcal{P}\omega_{i,0}\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}}+\|\mathcal{P}F_i\|^h_{L^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})},\quad i=1,2. \tag{5.4}$$

In what follows, we consider system (3.3) coupling the densities and the velocities in the high frequencies. In order to decouple the densities and the velocities so that we can exploit damping effects of $n_i(i=1,2)$, as in [18], we introduce two new effective velocity fields $e_i=:\nabla\big(-\Delta)^{-1}(\frac{c^2}{\nu}n_i-\mathrm{div}\omega_i\big)(i=1,2)$. Then, according to $(3.3)_2$ and $(3.3)_4$, we have

$$\partial_t\Big(\mathcal{Q}\omega_1+\frac{c^2}{\nu}\nabla(-\Delta)^{-1}n_1\Big)-\nu\Delta\Big(\mathcal{Q}\omega_1+\frac{c^2}{\nu}\nabla(-\Delta)^{-1}n_1\Big)=\mathcal{Q}F_1+\frac{c^2}{\nu}\nabla(-\Delta)^{-1}\partial_tn_1, \tag{5.5}$$

and

$$\begin{aligned}&\partial_t\Big(\mathcal{Q}\omega_2+\frac{c^2}{\nu}\nabla(-\Delta)^{-1}n_2\Big)-\nu\Delta\Big(\mathcal{Q}\omega_2+\frac{c^2}{\nu}\nabla(-\Delta)^{-1}n_2\Big)\\=&\mathcal{Q}F_2+\frac{c^2}{\nu}\nabla(-\Delta)^{-1}\partial_tn_2+2\nabla\Delta^{-1}n_2.\end{aligned} \tag{5.6}$$

On the other hand, thanks to the definition of e_i, we may rewrite $(3.3)_1$ and $(3.3)_3$ as the following transport systems with damping terms:

$$\partial_tn_i+\frac{c^2}{\nu}n_i=-\mathrm{div}e_i+E_i,\quad i=1,2, \tag{5.7}$$

which together with (5.5) and (5.6) yields the following two standard heat equations:

$$\partial_te_1-\nu\Delta e_1=\mathcal{Q}F_1+\frac{c^2}{\nu}e_1-\frac{c^4}{\nu^2}\nabla(-\Delta)^{-1}n_1+\frac{c^2}{\nu}\nabla(-\Delta)^{-1}E_1, \tag{5.8}$$

$$\partial_te_2-\nu\Delta e_2=\mathcal{Q}F_2+\frac{c^2}{\nu}e_2-(2+\frac{c^4}{\nu^2})\nabla(-\Delta)^{-1}n_2+\frac{c^2}{\nu}\nabla(-\Delta)^{-1}E_2. \tag{5.9}$$

Hence, owing to Proposition 2.5 and using the fact that $\nabla(-\Delta)^{-1}$ is a homogeneous Fourier multiplier of degree -1, we have

$$\begin{aligned}&\|(e_1,e_2)\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}+\nu\|(\nabla^2e_1,\nabla^2e_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\ \lesssim&\|(e_{1,0},e_{2,0})\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}}+\|(F_1,F_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}+\frac{c^2}{\nu}\|(e_1,e_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\&+\|(\frac{c^4}{\nu^2}\nabla(-\Delta)^{-1}n_1,(2+\frac{c^4}{\nu^2})\nabla(-\Delta)^{-1}n_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\&+\frac{c^2}{\nu}\|(\nabla(-\Delta)^{-1}E_1,\nabla(-\Delta)^{-1}E_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}.\end{aligned} \tag{5.10}$$

Noticin the definition of E_i, from (5.7), we have

$$\partial_tn_1+\frac{c^2}{\nu}n_1=-\mathrm{div}e_1-\Big(\frac{n_1+n_2}{2}\Big)\cdot\mathrm{div}\Big(\frac{\omega_1+\omega_2}{2}\Big)-\Big(\frac{n_1-n_2}{2}\Big)\cdot\mathrm{div}\Big(\frac{\omega_1-\omega_2}{2}\Big)$$

$$-\Big(\frac{\omega_1+\omega_2}{2}\Big)\cdot\nabla\Big(\frac{n_1+n_2}{2}\Big)-\Big(\frac{\omega_1-\omega_2}{2}\Big)\cdot\nabla\Big(\frac{n_1-n_2}{2}\Big), \tag{5.11}$$

and

$$\begin{aligned}\partial_t n_2+\frac{c^2}{\nu}n_2=-\mathrm{div}e_2-\Big(\frac{n_1+n_2}{2}\Big)\cdot\mathrm{div}\Big(\frac{\omega_1+\omega_2}{2}\Big)+\Big(\frac{n_1-n_2}{2}\Big)\cdot\mathrm{div}\Big(\frac{\omega_1-\omega_2}{2}\Big)\\ -\Big(\frac{\omega_1+\omega_2}{2}\Big)\cdot\nabla\Big(\frac{n_1+n_2}{2}\Big)+\Big(\frac{\omega_1-\omega_2}{2}\Big)\cdot\nabla\Big(\frac{n_1-n_2}{2}\Big).\end{aligned} \tag{5.12}$$

Applying the operator $\dot{\Delta}_j$ to (5.11) and (5.12), respectively, and then adding them gives rise to

$$\begin{aligned}&\partial_t\dot{\Delta}_j\Big(\frac{n_1+n_2}{2}\Big)+\frac{c^2}{\nu}\dot{\Delta}_j\Big(\frac{n_1+n_2}{2}\Big)+\Big(\frac{\omega_1+\omega_2}{2}\Big)\cdot\nabla\dot{\Delta}_j\Big(\frac{n_1+n_2}{2}\Big)\\ &=-\mathrm{div}\dot{\Delta}_j\Big(\frac{e_1+e_2}{2}\Big)-\dot{\Delta}_j\Big(\big(\frac{n_1+n_2}{2}\big)\cdot\mathrm{div}\big(\frac{\omega_1+\omega_2}{2}\big)\Big)+\Big[\big(\frac{\omega_1+\omega_2}{2}\big)\cdot\nabla,\dot{\Delta}_j\Big]\Big(\frac{n_1+n_2}{2}\Big).\end{aligned} \tag{5.13}$$

Taking the L^p scalar product with $\dot{\Delta}_j(\frac{n_1+n_2}{2})$ for (5.13) and integrating it with respect to time t, we have

$$\begin{aligned}&\|\dot{\Delta}_j(\frac{n_1+n_2}{2})\|_{L^p}+\frac{c^2}{\nu}\int_0^t\|\dot{\Delta}_j(\frac{n_1+n_2}{2})\|_{L^p}d\tau\\ \lesssim&\|\dot{\Delta}_j(\frac{n_{1,0}+n_{2,0}}{2})\|_{L^p}+\frac{1}{p}\int_0^t\|\mathrm{div}(\frac{\omega_1+\omega_2}{2})\|_{L^\infty}\|\dot{\Delta}_j(\frac{n_1+n_2}{2})\|_{L^p}d\tau\\ &+\int_0^t\Big(\|\dot{\Delta}_j\big((\frac{n_1+n_2}{2})\cdot\mathrm{div}(\frac{\omega_1+\omega_2}{2})\big)\|_{L^p}+\|\mathrm{div}\dot{\Delta}_j(\frac{e_1+e_2}{2})\|_{L^p}\\ &+\|[(\frac{\omega_1+\omega_2}{2})\cdot\nabla,\dot{\Delta}_j](\frac{n_1+n_2}{2})\|_{L^p}\Big)d\tau.\end{aligned} \tag{5.14}$$

By virtue of Lemmas 2.6 and 2.2, from the embedding $\dot{B}^{\frac{N}{p}}_{p,1}\hookrightarrow L^\infty$, multiplying (5.14) by $2^{j\frac{N}{p}}$ and then summing up over $j\geq j_0$ yields

$$\begin{aligned}&\|\frac{n_1+n_2}{2}\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{N}{p}}_{p,1})}+\frac{c^2}{\nu}\|\frac{n_1+n_2}{2}\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}}_{p,1})}\\ \lesssim&\|\frac{n_{1,0}+n_{2,0}}{2}\|^h_{\dot{B}^{\frac{N}{p}}_{p,1}}+\|\mathrm{div}(\frac{e_1+e_2}{2})\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}}_{p,1})}\\ &+\int_0^t\|\frac{n_1+n_2}{2}\|_{\dot{B}^{\frac{N}{p}}_{p,1}}\|\mathrm{div}(\frac{\omega_1+\omega_2}{2})\|_{\dot{B}^{\frac{N}{p}}_{p,1}}d\tau.\end{aligned} \tag{5.15}$$

Similarly, we also have

$$\begin{aligned}&\|\frac{n_1-n_2}{2}\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{N}{p}}_{p,1})}+\frac{c^2}{\nu}\|\frac{n_1-n_2}{2}\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}}_{p,1})}\\ \lesssim&\|\frac{n_{1,0}-n_{2,0}}{2}\|^h_{\dot{B}^{\frac{N}{p}}_{p,1}}\\ &+\|\mathrm{div}(\frac{e_1-e_2}{2})\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}}_{p,1})}+\int_0^t\|\frac{n_1-n_2}{2}\|_{\dot{B}^{\frac{N}{p}}_{p,1}}\|\mathrm{div}(\frac{\omega_1-\omega_2}{2})\|_{\dot{B}^{\frac{N}{p}}_{p,1}}d\tau.\end{aligned} \tag{5.16}$$

Then it follows from (5.15) and (5.16) that

$$
\begin{aligned}
&\|(n_1,n_2)\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{N}{p}}_{p,1})}+\frac{c^2}{\nu}\|(n_1,n_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}}_{p,1})}\\
\lesssim&\|(n_{1,0},n_{2,0})\|^h_{\dot{B}^{\frac{N}{p}}_{p,1}}+\|(\mathrm{div}e_1,\mathrm{div}e_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}}_{p,1})}+\int_0^t\|(n_1,n_2)\|_{\dot{B}^{\frac{N}{p}}_{p,1}}\|(\mathrm{div}\omega_1,\mathrm{div}\omega_2)\|_{\dot{B}^{\frac{N}{p}}_{p,1}}d\tau,
\end{aligned}
\tag{5.17}
$$

which together (5.10) implies that

$$
\begin{aligned}
&\|(n_1,n_2)\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{N}{p}}_{p,1})}+\frac{c^2}{\nu}\|(n_1,n_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}}_{p,1})}+\|(e_1,e_2)\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}+\nu\|(\nabla^2e_1,\nabla^2e_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\
\lesssim&\|(n_{1,0},n_{2,0})\|^h_{\dot{B}^{\frac{N}{p}}_{p,1}}+\|(e_{1,0},e_{2,0})\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}}+\|(F_1,F_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\
&+\frac{c^2}{\nu}\|(\nabla(-\Delta)^{-1}E_1,\nabla(-\Delta)^{-1}E_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}+\frac{c^2}{\nu}\|(e_1,e_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\
&+\|(\frac{c^4}{\nu^2}\nabla(-\Delta)^{-1}n_1,(2+\frac{c^4}{\nu^2})\nabla(-\Delta)^{-1}n_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\
&+\int_0^t\|(n_1,n_2)\|_{\dot{B}^{\frac{N}{p}}_{p,1}}\|(\mathrm{div}\omega_1,\mathrm{div}\omega_2)\|_{\dot{B}^{\frac{N}{p}}_{p,1}}d\tau.
\end{aligned}
\tag{5.18}
$$

Due to the high frequency cut-off, we have

$$
\|(e_1,e_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\lesssim2^{-2j_0}\|(\nabla^2e_1,\nabla^2e_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})},\quad j\geq j_0,
$$

and

$$
\|(\nabla(-\Delta)^{-1}n_1,\nabla(-\Delta)^{-1}n_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\lesssim2^{-2j_0}\|(n_1,n_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}}_{p,1})},\quad j\geq j_0.
$$

Then, taking j_0 suitably large, we deduce from (5.18) that

$$
\begin{aligned}
&\|(n_1,n_2)\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{N}{p}}_{p,1})}+\frac{c^2}{\nu}\|(n_1,n_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}}_{p,1})}+\|(e_1,e_2)\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\
&\quad+\nu\|(\nabla^2e_1,\nabla^2e_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\
\lesssim&\|(n_{1,0},n_{2,0})\|^h_{\dot{B}^{\frac{N}{p}}_{p,1}}+\|(e_{1,0},e_{2,0})\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}}+\|(F_1,F_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\
&\quad+\|(E_1,E_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-2}_{p,1})}+\int_0^t\|(n_1,n_2)\|_{\dot{B}^{\frac{N}{p}}_{p,1}}\|(\mathrm{div}\omega_1,\mathrm{div}\omega_2)\|_{\dot{B}^{\frac{N}{p}}_{p,1}}d\tau.
\end{aligned}
\tag{5.19}
$$

Noticing that $\omega_i=\mathcal{P}\omega_i+\mathcal{Q}\omega_i=\mathcal{P}\omega_i+e_i-\frac{c^2}{\nu}\nabla(-\Delta)^{-1}n_i(i=1,2)$, we deduce that

$$
\begin{aligned}
&\|(\omega_1,\omega_2)\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}+\nu\|(\nabla^2\omega_1,\nabla^2\omega_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\
\lesssim&\|(\mathcal{P}\omega_1,\mathcal{P}\omega_2)\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}+\nu\|(\nabla^2\mathcal{P}\omega_1,\nabla^2\mathcal{P}\omega_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}+\|(e_1,e_2)\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\
&+\nu\|(\nabla^2e_1,\nabla^2e_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}+\frac{c^2}{\nu}2^{-2j_0}\|(n_1,n_2)\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{N}{p}}_{p,1})}+c^2\|(n_1,n_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}}_{p,1})},
\end{aligned}
\tag{5.20}
$$

which together with (5.4) and (5.19) implies that

$$\begin{aligned}&\|(n_1,n_2)\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{N}{p}}_{p,1})}+\frac{c^2}{\nu}\|(n_1,n_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}}_{p,1})}+\|(\omega_1,\omega_2)\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}+\nu\|(\nabla^2\omega_1,\nabla^2\omega_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\&\le C\Big(\|(n_{1,0},n_{2,0})\|^h_{\dot{B}^{\frac{N}{p}}_{p,1}}+\|(\omega_{1,0},\omega_{2,0})\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}}+\|(F_1,F_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\&\quad+\|(E_1,E_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-2}_{p,1})}+\int_0^t\|(n_1,n_2)\|_{\dot{B}^{\frac{N}{p}}_{p,1}}\|(\mathrm{div}\omega_1,\mathrm{div}\omega_2)\|_{\dot{B}^{\frac{N}{p}}_{p,1}}d\tau\Big).\end{aligned}\tag{5.21}$$

Putting together (5.3) and (5.21), we finally get

$$\begin{aligned}X(t)\lesssim&X(0)+\|(F_1,F_2)\|^\ell_{L^1_t(\dot{B}^{\frac{N}{2}-1}_{2,1})}+\|(F_1,F_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}+\|(E_1,E_2)\|^\ell_{L^1_t(\dot{B}^{\frac{N}{2}-2}_{2,1})}\\&+\|(E_1,E_2)\|^h_{\tilde{L}^1_t(\dot{B}^{\frac{N}{p}-2}_{p,1})}+\int_0^t\|(n_1,n_2)\|_{\dot{B}^{\frac{N}{p}}_{p,1}}\|(\mathrm{div}\omega_1,\mathrm{div}\omega_2)\|_{\dot{B}^{\frac{N}{p}}_{p,1}}d\tau.\end{aligned}\tag{5.22}$$

Part 3: Nonlinear estimates. Now, let us bound nonlinear terms in (5.22). For the last term, thanks to Hölder's inequality, Lemmas 2.4 and 2.2, we get

$$\begin{aligned}&\int_0^t\|n_i\|_{\dot{B}^{\frac{N}{p}}_{p,1}}\|\mathrm{div}\omega_i\|_{\dot{B}^{\frac{N}{p}}_{p,1}}d\tau\\&\lesssim\|n_i\|_{L^\infty_t(\dot{B}^{\frac{N}{p}}_{p,1})}\|\nabla\omega_i\|_{L^1_t(\dot{B}^{\frac{N}{p}}_{p,1})}\\&\lesssim\Big(\|n_i\|^\ell_{L^\infty_t(\dot{B}^{\frac{N}{2}-2}_{2,1})}+\|n_i\|^h_{L^\infty_t(\dot{B}^{\frac{N}{p}}_{p,1})}\Big)\Big(\|\omega_i\|^\ell_{L^1_t(\dot{B}^{\frac{N}{2}+1}_{2,1})}+\|\omega_i\|^h_{L^1_t(\dot{B}^{\frac{N}{p}+1}_{p,1})}\Big)\\&\lesssim X^2(t),\ i=1,2.\end{aligned}\tag{5.23}$$

According to Lemmas 2.2 and 2.4, embedding $\dot{B}^{\frac{N}{2}+s}_{2,1}\hookrightarrow\dot{B}^{\frac{N}{p}+s}_{p,1}$ for all $p\ge2, s\in\mathbb{R}$ and interpolation inequality, we have

$$\|E_i\|^h_{L^1_t(\dot{B}^{\frac{N}{p}-2}_{p,1})}\lesssim\|E_i\|_{L^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\lesssim\|\frac{n_1+n_2}{2}\|_{L^2_t(\dot{B}^{\frac{N}{p}}_{p,1})}\|\frac{\omega_1+\omega_2}{2}\|_{L^2_t(\dot{B}^{\frac{N}{p}}_{p,1})}\lesssim X^2(t),\ i=1,2.\tag{5.24}$$

In what follows, we will bound term by term from $F_i (i=1,2)$. For simplicity, denote

$$L_1(n_1+n_2)=\frac{n_1+n_2}{2+n_1+n_2},\quad L_2(n_1-n_2)=\frac{n_1-n_2}{2+n_1-n_2},$$

$$L_3(n_1+n_2)=\frac{2P_1'(1+\frac{n_1+n_2}{2})}{2+n_1+n_2}-c^2=\frac{P_1'(1+\frac{n_1+n_2}{2})}{1+\frac{n_1+n_2}{2}}-P_1'(1),$$

$$L_4(n_1-n_2)=\frac{2P_1'(1+\frac{n_1-n_2}{2})}{2+n_1-n_2}-c^2=\frac{P_1'(1+\frac{n_1-n_2}{2})}{1+\frac{n_1-n_2}{2}}-P_1'(1).$$

Obviously, L_1,L_2,L_3,L_4 are smooth functions vanishing at 0. Thanks to Lemmas 2.4, 2.3 and 2.2, we infer that

$$\begin{aligned}\|L_1(n_1+n_2)\mathcal{A}(\frac{\omega_1+\omega_2}{2})\|^h_{L^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}&\lesssim\|L_1(n_1+n_2)\|_{L^\infty_t(\dot{B}^{\frac{N}{p}}_{p,1})}\|\nabla^2(\omega_1+\omega_2)\|_{L^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\&\lesssim\|n_1+n_2\|_{L^\infty_t(\dot{B}^{\frac{N}{p}}_{p,1})}\|\omega_1+\omega_2\|_{L^1_t(\dot{B}^{\frac{N}{p}+1}_{p,1})}\\&\lesssim X^2(t).\end{aligned}$$

Similarly, we also obtain the corresponding estimates of other terms such as $L_2(n_1 - n_2)\mathcal{A}(\frac{\omega_1-\omega_2}{2})$, $L_3(n_1+n_2)\nabla(\frac{n_1+n_2}{2})$ and $L_4(n_1-n_2)\nabla(\frac{n_1-n_2}{2})$. Here, we omit the details.

By virtue of Lemmas 2.4 and 2.2, we get

$$
\begin{aligned}
&\|(\frac{\omega_1+\omega_2}{2})\cdot\nabla(\frac{\omega_1+\omega_2}{2})\|^h_{L^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\
&\lesssim \|\omega_1+\omega_2\|_{L^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\|\nabla(\omega_1+\omega_2)\|_{L^1_t(\dot{B}^{\frac{N}{p}}_{p,1})}\\
&\lesssim \|\omega_1+\omega_2\|_{L^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\|\omega_1+\omega_2\|_{L^1_t(\dot{B}^{\frac{N}{p}+1}_{p,1})}\\
&\lesssim \Big(\|\omega_1+\omega_2\|^\ell_{L^\infty_t(\dot{B}^{\frac{N}{2}-1}_{2,1})}+\|\omega_1+\omega_2\|^h_{L^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\Big)\\
&\quad\times\Big(\|\omega_1+\omega_2\|^\ell_{L^1_t(\dot{B}^{\frac{N}{2}+1}_{2,1})}+\|\omega_1+\omega_2\|^h_{L^1_t(\dot{B}^{\frac{N}{p}+1}_{p,1})}\Big)\\
&\lesssim X^2(t).
\end{aligned}
$$

Similarly,

$$
\|(\frac{\omega_1-\omega_2}{2})\cdot\nabla(\frac{\omega_1-\omega_2}{2})\|^h_{L^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\lesssim X^2(t).
$$

Hence

$$
\|F_i\|^h_{L^1_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\lesssim X^2(t), i=1,2. \tag{5.25}
$$

Next, we will bound the low frequencies of $E_i(i=1,2)$ and $F_i(i=1,2)$ in $L^1_t(\dot{B}^{\frac{N}{2}-1}_{2,1})$.

For E_i, we have

$$
\|E_i\|^\ell_{L^1_t(\dot{B}^{\frac{N}{2}-2}_{2,1})}\lesssim \|(n_1+n_2)(\omega_1+\omega_2)\|^\ell_{L^1_t(\dot{B}^{\frac{N}{2}-1}_{2,1})}.
$$

Using Bony's decomposition gives

$$
(n_1+n_2)(\omega_1+\omega_2)=T_{(n_1+n_2)}(\omega_1+\omega_2)+R\big((n_1+n_2),(\omega_1+\omega_2)\big)+T_{(\omega_1+\omega_2)}(n_1+n_2).
$$

According to Propositions 2.1 and 2.2, we get

$$
\|T_{(n_1+n_2)}(\omega_1+\omega_2)\|_{L^1_t(\dot{B}^{\frac{N}{2}-1}_{2,1})}\lesssim\|(n_1+n_2)\|_{L^2_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\|(\omega_1+\omega_2)\|_{L^2_t(\dot{B}^{\frac{N}{p}}_{p,1})},
$$

$$
\|T_{(\omega_1+\omega_2)}(n_1+n_2)\|_{L^1_t(\dot{B}^{\frac{N}{2}-1}_{2,1})}\lesssim\|(\omega_1+\omega_2)\|_{L^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\|(n_1+n_2)\|_{L^1_t(\dot{B}^{\frac{N}{p}}_{p,1})},
$$

$$
\|R\big((n_1+n_2),(\omega_1+\omega_2)\big)\|_{L^1_t(\dot{B}^{\frac{N}{2}-1}_{2,1})}\lesssim\|(n_1+n_2)\|_{L^2_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\|(\omega_1+\omega_2)\|_{L^2_t(\dot{B}^{\frac{N}{p}}_{p,1})},
$$

which together with the interpolation inequality yields that

$$
\|E_i\|^\ell_{L^1_t(\dot{B}^{\frac{N}{2}-2}_{2,1})}\lesssim X^2(t), i=1,2. \tag{5.26}
$$

To handle the first term of F_i, employing Bony's decomposition and splitting $(\omega_1+\omega_2)$ into $(\omega_1+\omega_2)^\ell+(\omega_1+\omega_2)^h$, we have

$$\begin{aligned}L_1(n_1+n_2)\mathcal{A}(\omega_1+\omega_2)=&T_{\mathcal{A}(\omega_1+\omega_2)}L_1(n_1+n_2)+R\Big(\mathcal{A}(\omega_1+\omega_2),L_1(n_1+n_2)\Big)\\&+T_{L_1(n_1+n_2)}\mathcal{A}(\omega_1+\omega_2)^\ell+T_{L_1(n_1+n_2)}\mathcal{A}(\omega_1+\omega_2)^h.\end{aligned}$$

It follows from Lemma 2.3(i), Propositions 2.1 and 2.2, that

$$\begin{aligned}&\|T_{\mathcal{A}(\omega_1+\omega_2)}L_1(n_1+n_2)+R\big(\mathcal{A}(\omega_1+\omega_2),L_1(n_1+n_2)\big)\|^\ell_{L^1_t(\dot B^{\frac N2-1}_{2,1})}\\
\lesssim&\|\mathcal{A}(\omega_1+\omega_2)\|_{L^1_t(\dot B^{\frac Np-1}_{p,1})}\|L_1(n_1+n_2)\|_{L^\infty_t(\dot B^{\frac Np}_{p,1})}\\
\lesssim&\|\omega_1+\omega_2\|_{L^1_t(\dot B^{\frac Np+1}_{p,1})}\|n_1+n_2\|_{L^\infty_t(\dot B^{\frac Np}_{p,1})}\\
\lesssim&X^2(t).\end{aligned}$$

According to Lemma 2.2, we infer that

$$\begin{aligned}\|T_{L_1(n_1+n_2)}\mathcal{A}(\omega_1+\omega_2)^\ell\|^\ell_{L^1_t(\dot B^{\frac N2-1}_{2,1})}\lesssim&\|L_1(n_1+n_2)\|_{L^\infty_t(L^\infty)}\|\mathcal{A}(\omega_1+\omega_2)^\ell\|_{L^1_t(\dot B^{\frac N2-1}_{2,1})}\\
\lesssim&\|n_1+n_2\|_{L^\infty_t(\dot B^{\frac Np}_{p,1})}\|\omega_1+\omega_2\|^\ell_{L^1_t(\dot B^{\frac N2+1}_{2,1})}\\
\lesssim&X^2(t).\end{aligned}$$

By virtue of Proposition 2.1 and Lemma 2.3(ii), we have

$$\begin{aligned}&\|T_{L_1(n_1+n_2)}\mathcal{A}(\omega_1+\omega_2)^h\|^\ell_{L^1_t(\dot B^{\frac N2-1}_{2,1})}\\
\lesssim&\|L_1(n_1+n_2)\|_{L^\infty_t(\dot B^{\frac Np-1}_{p,1})}\|\mathcal{A}(\omega_1+\omega_2)^h\|_{L^1_t(\dot B^{\frac Np-1}_{p,1})}\\
\lesssim&(1+\|n_1+n_2\|_{L^\infty_t(\dot B^{\frac Np}_{p,1})})\|n_1+n_2\|_{L^\infty_t(\dot B^{\frac Np-1}_{p,1})}\|\omega_1+\omega_2\|^h_{L^1_t(\dot B^{\frac Np+1}_{p,1})}\\
\lesssim&X^2(t)+X^3(t).\end{aligned}$$

Thus,

$$\|L_1(n_1+n_2)\mathcal{A}(\frac{\omega_1+\omega_2}{2})\|^\ell_{L^1_t(\dot B^{\frac N2-1}_{2,1})}\lesssim X^2(t)+X^3(t).$$

Similarly,

$$\|L_2(n_1-n_2)\mathcal{A}(\frac{\omega_1-\omega_2}{2})\|^\ell_{L^1_t(\dot B^{\frac N2-1}_{2,1})}\lesssim X^2(t)+X^3(t),$$

$$\|L_3(n_1+n_2)\nabla(\frac{n_1+n_2}{2})\|^\ell_{L^1_t(\dot B^{\frac N2-1}_{2,1})}\lesssim X^2(t)+X^3(t),$$

and

$$\|L_4(n_1-n_2)\nabla(\frac{n_1-n_2}{2})\|^\ell_{L^1_t(\dot B^{\frac N2-1}_{2,1})}\lesssim X^2(t)+X^3(t).$$

To handle the term $(\frac{\omega_1+\omega_2}{2})\cdot\nabla(\frac{\omega_1+\omega_2}{2})$, employing Bony's decomposition yields that

$$\begin{aligned}(\omega_1+\omega_2)\cdot\nabla(\omega_1+\omega_2)=&T_{(\omega_1+\omega_2)}\nabla(\omega_1+\omega_2)+R\big((\omega_1+\omega_2),\nabla(\omega_1+\omega_2)\big)\\&+T_{\nabla(\omega_1+\omega_2)}(\omega_1+\omega_2).\end{aligned}$$

Thanks to Proposition 2.1 and Proposition 2.2, we get

$$\begin{aligned}&\|T_{(\omega_1+\omega_2)}\nabla(\omega_1+\omega_2)+R\big((\omega_1+\omega_2),\nabla(\omega_1+\omega_2)\big)\|_{L^1_t(\dot{B}^{\frac{N}{2}-1}_{2,1})}\\
\lesssim&\|\omega_1+\omega_2\|_{L^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\|\nabla(\omega_1+\omega_2)\|_{L^1_t(\dot{B}^{\frac{N}{p}}_{p,1})}\\
\lesssim&X^2(t),\\
&\|T_{\nabla(\omega_1+\omega_2)}(\omega_1+\omega_2)\|_{L^1_t(\dot{B}^{\frac{N}{2}-1}_{2,1})}\\
\lesssim&\|\omega_1+\omega_2\|_{L^2_t(\dot{B}^{\frac{N}{p}}_{p,1})}\|\nabla(\omega_1+\omega_2)\|_{L^2_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\
\lesssim&X^2(t).\end{aligned}$$

Thus,

$$\|(\frac{\omega_1+\omega_2}{2})\cdot\nabla(\frac{\omega_1+\omega_2}{2})\|^\ell_{L^1_t(\dot{B}^{\frac{N}{2}-1}_{2,1})}\lesssim X^2(t).$$

Similarly,

$$\|(\frac{\omega_1-\omega_2}{2})\cdot\nabla(\frac{\omega_1-\omega_2}{2})\|^\ell_{L^1_t(\dot{B}^{\frac{N}{2}-1}_{2,1})}\lesssim X^2(t).$$

Hence

$$\|F_i\|^\ell_{L^1_t(\dot{B}^{\frac{N}{2}-1}_{2,1})}\lesssim X^2(t)+X^3(t), i=1,2. \tag{5.27}$$

Therefore, putting all the above estimates (5.23), (5.24), (5.25), (5.26) and (5.27) together, we conclude that

$$X(t)\lesssim X(0)+X^2(t)+X^3(t). \tag{5.28}$$

Part 4: Global *a priori* estimates. According to (5.28), for some sufficiently small α_0, and applying the standard bootstrap argument, we finally obtain the following global uniform estimates for all $t\geq 0$,

$$X(t)\lesssim X(0).$$

Furthermore, employing the same argument in [7,32], we can also obtain the uniqueness of strong solution to the system. Here, we omit it. This completes the proof of Theorem 1.1.

6. The proof of Theorem 1.2

Our goal in this section is to prove Theorem 1.2. We divide it into the following three steps, according to the three terms of the time-weighted functional $D(t)$ (see (1.9)). In what follows, we shall use frequently the fact that the global solution (c_1,u_1,c_2,u_2) provided by Theorem 1.1 fulfills

$$\|(c_1,c_2)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}}_{p,1})}\leq c\ll 1 \quad \text{for all} \quad t\geq 0. \tag{6.1}$$

Step 1: Bounds for the low frequency regimes. From Lemma 4.2, we have

$$|\mathcal{F}G| \le Ce^{-c_0 t|\xi|^2}, \quad i=1,2, \quad |\xi| \le 2^{j_0},$$

where positive constants c_0 and C depend only on j_0, and $\mathcal{F}$ presents the Fourier transform.
Denoting $U = (\Lambda^{-1}n_1, \omega_1, \Lambda^{-1}n_2, \omega_2)$, we have

$$|\mathcal{F}(G(t)U)(\xi)| \le Ce^{-c_0 t|\xi|^2} |\mathcal{F}U(\xi)| \quad \text{for all} \quad |\xi| \le 2^{j_0}. \tag{6.2}$$

Therefore, using Parseval's inequality and the definition of $\dot{\Delta}_j$, we get for all $j \le j_0$,

$$\|G\dot{\Delta}_j U(t)\|_{L^2} \lesssim e^{-c_0 2^{2j}t} \|\dot{\Delta}_j U(t)\|_{L^2}.$$

Hence, multiplying by $t^{\frac{s_1+s}{2}} 2^{js}$ and then summing up on $j \le j_0$, yield that

$$\begin{aligned}
t^{\frac{s_1+s}{2}} \sum_{j \le j_0} 2^{js} \|G\dot{\Delta}_j U(t)\|_{L^2} &\lesssim t^{\frac{s_1+s}{2}} \sum_{j \le j_0} 2^{js} e^{-c_0 2^{2j}t} \|\dot{\Delta}_j U(t)\|_{L^2} \\
&\lesssim t^{\frac{s_1+s}{2}} \sum_{j \le j_0} 2^{j(s+s_1)} e^{-c_0 2^{2j}t} \|\dot{\Delta}_j U(t)\|_{L^2} 2^{-js_1} \\
&\lesssim \|U(t)\|^{\ell}_{\dot{B}^{-s_1}_{2,\infty}} \sum_{j \le j_0} t^{\frac{s_1+s}{2}} 2^{j(s+s_1)} e^{-c_0 2^{2j}t}.
\end{aligned} \tag{6.3}$$

By Lemma 2.7, we thus have that for $s + s_1 > 0$,

$$\sup_{t \ge 0} t^{\frac{s_1+s}{2}} \|GU(t)\|^{\ell}_{\dot{B}^{s}_{2,1}} \le C_\sigma \|U(t)\|^{\ell}_{\dot{B}^{-s_1}_{2,\infty}}.$$

In addition, it is obvious for $s + s_1 > 0$,

$$\|GU(t)\|^{\ell}_{\dot{B}^{s}_{2,1}} \lesssim \|U(t)\|^{\ell}_{\dot{B}^{-s_1}_{2,\infty}} \sum_{j \le j_0} 2^{j(s+s_1)} \lesssim \|U(t)\|^{\ell}_{\dot{B}^{-s_1}_{2,\infty}}.$$

Then, setting $\langle t \rangle =: \sqrt{1+t^2}$, we get

$$\sup_{t \ge 0} \langle t \rangle^{\frac{s_1+s}{2}} \|GU(t)\|^{\ell}_{\dot{B}^{s}_{2,1}} \lesssim \|U(t)\|^{\ell}_{\dot{B}^{-s_1}_{2,\infty}}. \tag{6.4}$$

Therefore, taking advantage of Duhamel's formula, we have

$$\begin{aligned}
\|(\Lambda^{-1}n_1, \omega_1, \Lambda^{-1}n_2, \omega_2)\|^{\ell}_{\dot{B}^{s}_{2,1}} &\lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \|(\Lambda^{-1}n_{1,0}, \omega_{1,0}, \Lambda^{-1}n_{2,0}, \omega_{2,0})\|^{\ell}_{\dot{B}^{-s_1}_{2,\infty}} \\
&+ \int_0^t \langle t-\tau \rangle^{-\frac{s_1+s}{2}} \|(\Lambda^{-1}E_1, \Lambda^{-1}E_2, F_1, F_2)(\tau)\|^{\ell}_{\dot{B}^{-s_1}_{2,\infty}} d\tau.
\end{aligned} \tag{6.5}$$

PROPOSITION 6.1. *If p and s_1 fulfill (1.4) and (1.7), respectively, we have*

$$\int_0^t \langle t-\tau \rangle^{-\frac{s_1+s}{2}} \|(\Lambda^{-1}E_1, \Lambda^{-1}E_2, F_1, F_2)(\tau)\|^{\ell}_{\dot{B}^{-s_1}_{2,\infty}} d\tau \lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \Big(D^2(t) + X^2(t) \Big), \tag{6.6}$$

where $X(t)$ and $D(t)$ have been defined in (1.6) and (1.9), respectively.

Proof. Using similar derivations of Lemmas of 3.1-3.3 in [24], we can complete the proof of Proposition 6.1. Here, we omit it. ∎

Therefore, combining (6.6) with (6.5) and the fact that $-s_1 < s \leq \frac{N}{2}+1$, we deduce

$$\langle t\rangle^{\frac{s_1+s}{2}} \|(\Lambda^{-1}n_1,\Lambda^{-1}n_2,\omega_1,\omega_2)\|^{\ell}_{\dot{B}^s_{2,1}} \lesssim D(0)+D^2(t)+X^2(t). \tag{6.7}$$

Step 2: Decay estimates for the high frequencies of $(\nabla n_1,\nabla n_2,\omega_1,\omega_2)$**.** This step is devoted to bounding the second and third terms of $D(t)$. Applying the operator $\dot{\Delta}_j$ to (3.4), and multiplying them with $\dot{\Delta}_j\mathcal{P}\omega_i|\dot{\Delta}_j\mathcal{P}\omega_i|^{p-2}$ and then integrating over $\mathbb{R}^N$, we deduce that

$$\frac{1}{p}\frac{d}{dt}\|\dot{\Delta}_j\mathcal{P}\omega_i\|^p_{L^p} - \mu_1\int \Delta\dot{\Delta}_j\mathcal{P}\omega_i|\dot{\Delta}_j\mathcal{P}\omega_i|^{p-2}\dot{\Delta}_j\mathcal{P}\omega_i dx = \int |\dot{\Delta}_j\mathcal{P}\omega_i|^{p-2}\dot{\Delta}_j\mathcal{P}\omega_i\dot{\Delta}_jF_i dx.$$

According to Lemma 2.5, there exists a positive constant c_s such that

$$\frac{d}{dt}\|\dot{\Delta}_j\mathcal{P}\omega_i\|_{L^p} + c_s\mu_1 2^{2j}\|\dot{\Delta}_j\mathcal{P}\omega_i\|_{L^p} \lesssim \|\dot{\Delta}_jF_i\|_{L^p}. \tag{6.8}$$

Now, recall that

$$\begin{cases} \partial_t n_1 + \dfrac{c^2}{\nu}n_1 = -\mathrm{div}e_1 + E_1, \\ \partial_t e_1 - \nu\Delta e_1 = -\nabla(-\Delta)^{-1}\mathrm{div}F_1 + \dfrac{c^2}{\nu}e_1 - \dfrac{c^4}{\nu^2}\nabla(-\Delta)^{-1}n_1 + \dfrac{c^2}{\nu}\nabla(-\Delta)^{-1}E_1, \\ \partial_t n_2 + \dfrac{c^2}{\nu}n_2 = -\mathrm{div}e_2 + E_2, \\ \partial_t e_2 - \nu\Delta e_2 = -\nabla(-\Delta)^{-1}\mathrm{div}F_2 + \dfrac{c^2}{\nu}e_2 - (2+\dfrac{c^4}{\nu^2})\nabla(-\Delta)^{-1}n_2 + \dfrac{c^2}{\nu}\nabla(-\Delta)^{-1}E_2. \end{cases}$$

By a similar derivation in (6.8), for $e_i (i=1,2)$, we get

$$\begin{aligned} &\frac{d}{dt}\|(\dot{\Delta}_je_1,\dot{\Delta}_je_2)\|_{L^p} + c_s\nu 2^{2j}\|(\dot{\Delta}_je_1,\dot{\Delta}_je_2)\|_{L^p} \\ \lesssim& \|\frac{c^2}{\nu}(\dot{\Delta}_je_1,\dot{\Delta}_je_2) + \frac{c^4}{\nu^2}(-\Delta)^{-1}\nabla\dot{\Delta}_jn_1 + (2+\frac{c^4}{\nu^2})(-\Delta)^{-1}\nabla\dot{\Delta}_jn_2\|_{L^p} \\ &+ \|(-\nabla(-\Delta)^{-1}\mathrm{div}\dot{\Delta}_jF_1, -\nabla(-\Delta)^{-1}\mathrm{div}\dot{\Delta}_jF_2)\|_{L^p} \\ &+ \frac{c^2}{\nu}\|(\nabla(-\Delta)^{-1}\dot{\Delta}_jE_1, \nabla(-\Delta)^{-1}\dot{\Delta}_jE_2)\|_{L^p}. \end{aligned} \tag{6.9}$$

Following similar derivations of (5.14) and (5.16), and denoting $R_1 =: [(\frac{\omega_1+\omega_2}{2})\cdot\nabla,\nabla\dot{\Delta}_j](\frac{n_1+n_2}{2}), R_2 =: [(\frac{\omega_1-\omega_2}{2})\cdot\nabla,\nabla\dot{\Delta}_j](\frac{n_1-n_2}{2})$, we deduce that

$$\begin{aligned} &\frac{d}{dt}\|\nabla\dot{\Delta}_j(\frac{n_1+n_2}{2})\|_{L^p} + \frac{c^2}{\nu}\|\nabla\dot{\Delta}_j(\frac{n_1+n_2}{2})\|_{L^p} \\ \lesssim& \frac{1}{p}\|\mathrm{div}(\frac{\omega_1+\omega_2}{2})\|_{L^\infty}\|\nabla\dot{\Delta}_j(\frac{n_1+n_2}{2})\|_{L^p} + \|\nabla\dot{\Delta}_j\big((\frac{n_1+n_2}{2})\cdot\mathrm{div}(\frac{\omega_1+\omega_2}{2})\big)\|_{L^p} \\ &+ C2^{2j}\|\dot{\Delta}_j(\frac{e_1+e_2}{2})\|_{L^p} + \|R_1\|_{L^p}, \end{aligned} \tag{6.10}$$

and

$$\begin{aligned}&\frac{d}{dt}\|\nabla\dot{\Delta}_j(\frac{n_1-n_2}{2})\|_{L^p}+\frac{c^2}{\nu}\|\nabla\dot{\Delta}_j(\frac{n_1-n_2}{2})\|_{L^p}\\ \lesssim&\frac{1}{p}\|\mathrm{div}(\frac{\omega_1-\omega_2}{2})\|_{L^\infty}\|\nabla\dot{\Delta}_j(\frac{n_1-n_2}{2})\|_{L^p}+\|\nabla\dot{\Delta}_j\big((\frac{n_1-n_2}{2})\cdot\mathrm{div}(\frac{\omega_1-\omega_2}{2})\big)\|_{L^p}\\ &\quad+C2^{2j}\|\dot{\Delta}_j(\frac{e_1-e_2}{2})\|_{L^p}+\|R_2\|_{L^p}.\end{aligned}\tag{6.11}$$

Summing up (6.10) to (6.11) yields that

$$\begin{aligned}&\frac{d}{dt}\|(\nabla\dot{\Delta}_jn_1,\nabla\dot{\Delta}_jn_2)\|_{L^p}+\frac{c^2}{\nu}\|(\nabla\dot{\Delta}_jn_1,\nabla\dot{\Delta}_jn_2)\|_{L^p}\\ \lesssim&2^{2j}\|(\dot{\Delta}_je_1,\dot{\Delta}_je_2)\|_{L^p}+\|(R_1,R_2)\|_{L^p}+\frac{1}{p}\|(\mathrm{div}\omega_1,\mathrm{div}\omega_2)\|_{L^\infty}\|(\nabla\dot{\Delta}_jn_1,\nabla\dot{\Delta}_jn_2)\|_{L^p}\\ &\quad+\|\big(\nabla\dot{\Delta}_j(n_1\mathrm{div}\omega_1),\nabla\dot{\Delta}_j(n_2\mathrm{div}\omega_2)\big)\|_{L^p}.\end{aligned}\tag{6.12}$$

Adding up that inequality (multiplied by ηc_s for $\eta>0$) to (6.8) and (6.9), we conclude that

$$\begin{aligned}&\frac{d}{dt}\Big(\|(\dot{\Delta}_j\mathcal{P}\omega_1,\dot{\Delta}_j\mathcal{P}\omega_2)\|_{L^p}+\|(\dot{\Delta}_je_1,\dot{\Delta}_je_2)\|_{L^p}+\eta c_s\|(\nabla\dot{\Delta}_jn_1,\nabla\dot{\Delta}_jn_2)\|_{L^p}\Big)\\ &\quad+c_s\mu_12^{2j}\|(\dot{\Delta}_j\mathcal{P}\omega_1,\dot{\Delta}_j\mathcal{P}\omega_2)\|_{L^p}+c_s\nu2^{2j}\|(\dot{\Delta}_je_1,\dot{\Delta}_je_2)\|_{L^p}+\eta c_s\tfrac{c^2}{\nu}\|(\nabla\dot{\Delta}_jn_1,\nabla\dot{\Delta}_jn_2)\|_{L^p}\\ \lesssim&\frac{c^2}{\nu}\|(\nabla(-\Delta)^{-1}\dot{\Delta}_jE_1,\nabla(-\Delta)^{-1}\dot{\Delta}_jE_2)\|_{L^p}+\|(\dot{\Delta}_jF_1,\dot{\Delta}_jF_2)\|_{L^p}+\|\frac{c^2}{\nu}(\dot{\Delta}_je_1,\dot{\Delta}_je_2)\\ &\quad+\frac{c^4}{\nu^2}(-\Delta)^{-1}\nabla\dot{\Delta}_jn_1+(2+\frac{c^4}{\nu^2})(-\Delta)^{-1}\nabla\dot{\Delta}_jn_2\|_{L^p}+\eta c_s2^{2j}\|(\dot{\Delta}_je_1,\dot{\Delta}_je_2)\|_{L^p}\\ &\quad+\eta c_s\Big(\|(R_1,R_2)\|_{L^p}+\frac{1}{p}\|(\mathrm{div}\omega_1,\mathrm{div}\omega_2)\|_{L^\infty}\|(\nabla\dot{\Delta}_jn_1,\nabla\dot{\Delta}_jn_2)\|_{L^p}\\ &\quad+\|\big(\nabla\dot{\Delta}_j(n_1\mathrm{div}\omega_1),\nabla\dot{\Delta}_j(n_2\mathrm{div}\omega_2)\big)\|_{L^p}\Big).\end{aligned}$$

As $(-\Delta)^{-1}$ is a homogeneous Fourier multiplier of degree -2, for $j\geq j_0-1$, we get

$$\begin{aligned}\|((-\Delta)^{-1}\nabla\dot{\Delta}_jn_1,(-\Delta)^{-1}\nabla\dot{\Delta}_jn_2)\|_{L^p}&\lesssim2^{-2j}\|(\nabla\dot{\Delta}_jn_1,\nabla\dot{\Delta}_jn_2)\|_{L^p}\\ &\lesssim2^{-2j_0}\|(\nabla\dot{\Delta}_jn_1,\nabla\dot{\Delta}_jn_2)\|_{L^p}.\end{aligned}$$

Choosing η small enough, and j_0 suitably large, we conclude that there exists a constant $c_0>0$ such that for all $j\geq j_0-1$,

$$\begin{aligned}&\frac{d}{dt}\Big(\|(\dot{\Delta}_j\mathcal{P}\omega_1,\dot{\Delta}_j\mathcal{P}\omega_2)\|_{L^p}+\|(\dot{\Delta}_je_1,\dot{\Delta}_je_2)\|_{L^p}+\eta c_s\|(\nabla\dot{\Delta}_jn_1,\nabla\dot{\Delta}_jn_2)\|_{L^p}\Big)\\ &\quad+c_0\Big(\|(\dot{\Delta}_j\mathcal{P}\omega_1,\dot{\Delta}_j\mathcal{P}\omega_2)\|_{L^p}+\|(\dot{\Delta}_je_1,\dot{\Delta}_je_2)\|_{L^p}+\eta c_s\|(\nabla\dot{\Delta}_jn_1,\nabla\dot{\Delta}_jn_2)\|_{L^p}\Big)\\ \lesssim&\|(\Lambda^{-1}\dot{\Delta}_jE_1,\Lambda^{-1}\dot{\Delta}_jE_2)\|_{L^p}+\|(\dot{\Delta}_jF_1,\dot{\Delta}_jF_2)\|_{L^p}+\eta c_s\Big(\|(R_1,R_2)\|_{L^p}\\ &\quad+\tfrac{1}{p}\|(\mathrm{div}\omega_1,\mathrm{div}\omega_2)\|_{L^\infty}\|(\nabla\dot{\Delta}_jn_1,\nabla\dot{\Delta}_jn_2)\|_{L^p}+\|\big(\nabla\dot{\Delta}_j(n_1\mathrm{div}\omega_1),\nabla\dot{\Delta}_j(n_2\mathrm{div}\omega_2)\big)\|_{L^p}\Big).\end{aligned}$$

Then integrating in time t yields that

$$\begin{aligned}&e^{c_0t}\|\big((\dot{\Delta}_j\mathcal{P}\omega_1,\dot{\Delta}_j\mathcal{P}\omega_2),(\dot{\Delta}_je_1,\dot{\Delta}_je_2),(\nabla\dot{\Delta}_jn_1,\nabla\dot{\Delta}_jn_2)\big)(t)\|_{L^p}\\ \lesssim&\|\big((\dot{\Delta}_j\mathcal{P}\omega_1,\dot{\Delta}_j\mathcal{P}\omega_2),(\dot{\Delta}_je_1,\dot{\Delta}_je_2),(\nabla\dot{\Delta}_jn_1,\nabla\dot{\Delta}_jn_2)\big)(0)\|_{L^p}+\int_0^te^{c_0\tau}S_j(\tau)d\tau\end{aligned}$$

with $S_j =: \sum_{i=1}^5 S_j^i$ where

$$S_j^1 =: \|(\Lambda^{-1}\dot{\Delta}_j E_1, \Lambda^{-1}\dot{\Delta}_j E_2)\|_{L^p}, \quad S_j^2 =: \|(\dot{\Delta}_j F_1, \dot{\Delta}_j F_2)\|_{L^p}, \quad S_j^3 =: \|(R_1, R_2)\|_{L^p},$$

$$S_j^4 =: \|(\mathrm{div}\omega_1, \mathrm{div}\omega_2)\|_{L^\infty} \|(\nabla\dot{\Delta}_j n_1, \nabla\dot{\Delta}_j n_2)\|_{L^p},$$

and

$$S_j^5 =: \|\big(\nabla\dot{\Delta}_j(n_1 \mathrm{div}\omega_1), \nabla\dot{\Delta}_j(n_2 \mathrm{div}\omega_2)\big)\|_{L^p}.$$

On the other hand, due to

$$\omega_i = e_i - \frac{c^2}{\nu}\nabla(-\Delta)^{-1} n_i + \mathcal{P}\omega_i, \quad i = 1, 2,$$

we have $j \geq j_0 - 1$

$$\begin{aligned} &\|(\dot{\Delta}_j\omega_1, \dot{\Delta}_j\omega_2)\|_{L^p} \\ \lesssim &\|(\dot{\Delta}_j e_1, \dot{\Delta}_j e_2)\|_{L^p} + \|(\dot{\Delta}_j\mathcal{P}\omega_1, \dot{\Delta}_j\mathcal{P}\omega_2)\|_{L^p} + 2^{-2j_0}\|(\dot{\Delta}_j\nabla n_1, \dot{\Delta}_j\nabla n_2)\|_{L^p}. \end{aligned}$$

Therefore, we arrive for all $j \geq j_0 - 1$ and $t \geq 0$ at

$$\begin{aligned} &\|(\dot{\Delta}_j\omega_1, \dot{\Delta}_j\omega_2, \nabla\dot{\Delta}_j n_1, \nabla\dot{\Delta}_j n_2)(t)\|_{L^p} \\ \lesssim & e^{-c_0 t}\|(\dot{\Delta}_j\omega_1, \dot{\Delta}_j\omega_2, \nabla\dot{\Delta}_j n_1, \nabla\dot{\Delta}_j n_2)(0)\|_{L^p} + \int_0^t e^{-c_0(t-\tau)} S_j(\tau)\, d\tau. \end{aligned}$$

Multiplying both sides by $\langle t\rangle^\alpha 2^{(\frac{N}{p}-1)j}$, taking the supremum on $[0,T]$, and summing up over $j \geq j_0$,

$$\begin{aligned} &\|\langle t\rangle^\alpha (n_1, n_2)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}}_{p,1})} + \|\langle t\rangle^\alpha (\omega_1, \omega_2)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{N}{p}-1}_{p,1})} \\ \lesssim &\|(n_{1,0}, n_{2,0})\|^h_{\dot{B}^{\frac{N}{p}}_{p,1}} + \|(\omega_{1,0}, \omega_{2,0})\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}} + \sum_{j \geq j_0} \sup_{0 \leq t \leq T} \Big(\langle t\rangle^\alpha \int_0^t e^{c_0(\tau - t)} 2^{(\frac{N}{p}-1)j} S_j(\tau)\, d\tau\Big). \end{aligned} \tag{6.13}$$

In order to bound the sum, for $0 \leq t \leq 2$, and taking advantage of Lemmas 2.6 and 2.2, we get

$$\begin{aligned} &\sum_{j \geq j_0} \sup_{0 \leq t \leq 2} \Big(\langle t\rangle^\alpha \int_0^t e^{c_0(\tau - t)} 2^{(\frac{N}{p}-1)j} S_j(\tau)\, d\tau\Big) \\ \lesssim &\int_0^2 \sum_{j \geq j_0} 2^{(\frac{N}{p}-1)j} S_j(\tau)\, d\tau \\ \lesssim &\int_0^2 \Big(\|(\Lambda^{-1}E_1, \Lambda^{-1}E_2)\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}} + \|(F_1, F_2)\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}} + \|(n_1, n_2)\|_{\dot{B}^{\frac{N}{p}}_{p,1}} \|(\nabla\omega_1, \nabla\omega_2)\|_{\dot{B}^{\frac{N}{p}}_{p,1}}\Big) d\tau. \end{aligned}$$

Bounding $\Lambda^{-1}E_1, \Lambda^{-1}E_2, F_1, F_2$ as in the proof of Theorem 1.1 leads to

$$\sum_{j \geq j_0} \sup_{0 \leq t \leq 2} \Big(\langle t\rangle^\alpha \int_0^t e^{c_0(\tau - t)} 2^{(\frac{N}{p}-1)j} S_j(\tau)\, d\tau\Big) \lesssim X^2(2). \tag{6.14}$$

To bound the supremum on $[2,T]$, we split the integral on $[0,t]$ into integrals on $[0,1]$ and $[1,t]$, respectively. The $[0,1]$ part of the integral is easy to handle, we have

$$\begin{aligned}
&\sum_{j\ge j_0}\sup_{2\le t\le T}\Big(\langle t\rangle^{\alpha}\int_0^1 e^{c_0(\tau-t)}2^{(\frac{N}{p}-1)j}S_j(\tau)\,d\tau\Big)\\
\lesssim&\sum_{j\ge j_0}\sup_{2\le t\le T}\Big(\langle t\rangle^{\alpha}e^{-\frac{c_0}{2}t}\int_0^1 2^{(\frac{N}{p}-1)j}S_j(\tau)\,d\tau\Big)\\
\lesssim&\int_0^1\sum_{j\ge j_0}2^{(\frac{N}{p}-1)j}S_j(\tau)\,d\tau\\
\lesssim&X^2(1). \qquad (6.15)
\end{aligned}$$

Finally, let us deal with the $[1,t]$ part of the integral for $2\le t\le T$. Using the fact that $\langle\tau\rangle\approx\tau$ when $\tau\ge1$ and Lemma 2.8 yields that

$$\sum_{j\ge j_0}\sup_{2\le t\le T}\Big(\langle t\rangle^{\alpha}\int_1^t e^{c_0(\tau-t)}2^{(\frac{N}{p}-1)j}S_j(\tau)\,d\tau\Big)\lesssim\sum_{j\ge j_0}2^{(\frac{N}{p}-1)j}\sup_{1\le t\le T}t^{\alpha}S_j(t). \qquad (6.16)$$

To bound S_j^1, from the following decomposition:

$$\begin{aligned}
(n_1+n_2)(\omega_1+\omega_2)=&(n_1+n_2)(\omega_1+\omega_2)^h+(n_1+n_2)^h(\omega_1+\omega_2)^{\ell}\\
&+(n_1+n_2)^{\ell}(\omega_1+\omega_2)^{\ell},
\end{aligned}$$

we get

$$\begin{aligned}
\|t^{\alpha}\Lambda^{-1}E_1\|^h_{\widetilde L^{\infty}_T(\dot B^{\frac{N}{p}-1}_{p,1})}\lesssim&\|t^{\alpha}(n_1+n_2)(\omega_1+\omega_2)^h\|^h_{\widetilde L^{\infty}_T(\dot B^{\frac{N}{p}-1}_{p,1})}\\
&+\|t^{\alpha}(n_1+n_2)^h(\omega_1+\omega_2)^{\ell}\|^h_{\widetilde L^{\infty}_T(\dot B^{\frac{N}{p}-1}_{p,1})}\\
&+\|t^{\alpha}(n_1+n_2)^{\ell}(\omega_1+\omega_2)^{\ell}\|^h_{\widetilde L^{\infty}_T(\dot B^{\frac{N}{p}-1}_{p,1})}.
\end{aligned}$$

Thanks to Lemma 2.2, we have

$$\begin{aligned}
\|t^{\alpha}(n_1+n_2)(\omega_1+\omega_2)^h\|^h_{\widetilde L^{\infty}_T(\dot B^{\frac{N}{p}-1}_{p,1})}\lesssim&\|n_1+n_2\|_{\widetilde L^{\infty}_T(\dot B^{\frac{N}{p}}_{p,1})}\|t^{\alpha}(\omega_1+\omega_2)\|^h_{\widetilde L^{\infty}_T(\dot B^{\frac{N}{p}-1}_{p,1})}\\
\lesssim&X(t)D(t),
\end{aligned}$$

$$\begin{aligned}
\|t^{\alpha}(n_1+n_2)^h(\omega_1+\omega_2)^{\ell}\|^h_{\widetilde L^{\infty}_T(\dot B^{\frac{N}{p}-1}_{p,1})}\lesssim&\|t^{\alpha}(n_1+n_2)\|^h_{\widetilde L^{\infty}_T(\dot B^{\frac{N}{p}}_{p,1})}\|\omega_1+\omega_2\|^{\ell}_{\widetilde L^{\infty}_T(\dot B^{\frac{N}{2}-1}_{2,1})}\\
\lesssim&X(t)D(t),
\end{aligned}$$

$$\begin{aligned}
&\|t^{\alpha}(n_1+n_2)^{\ell}(\omega_1+\omega_2)^{\ell}\|^h_{\widetilde L^{\infty}_T(\dot B^{\frac{N}{p}-1}_{p,1})}\\
\lesssim&\|t^{\alpha}(n_1+n_2)^{\ell}(\omega_1+\omega_2)^{\ell}\|^h_{\widetilde L^{\infty}_T(\dot B^{\frac{N}{p}}_{p,1})}\\
\lesssim&\|t^{\frac12(s_1+\frac{N}{2}+1-\varepsilon)}(n_1+n_2)\|^{\ell}_{\widetilde L^{\infty}_T(\dot B^{\frac{N}{2}}_{2,1})}\|t^{\frac12(s_1+\frac{N}{2}-\varepsilon)}(\omega_1+\omega_2)\|^{\ell}_{\widetilde L^{\infty}_T(\dot B^{\frac{N}{2}}_{2,1})}\\
\lesssim&\|t^{\frac12(s_1+\frac{N}{2}+1-\varepsilon)}(n_1+n_2)\|^{\ell}_{L^{\infty}_T(\dot B^{\frac{N}{2}-\varepsilon}_{2,1})}\|t^{\frac12(s_1+\frac{N}{2}-\varepsilon)}(\omega_1+\omega_2)\|^{\ell}_{L^{\infty}_T(\dot B^{\frac{N}{2}-\varepsilon}_{2,1})}\\
\lesssim&D^2(t),
\end{aligned}$$

where we have used the embedding $\dot{B}^{\frac{N}{2}+s}_{2,1} \hookrightarrow \dot{B}^{\frac{N}{p}+s}_{p,1}$ for all $p \geq 2, s \in \mathbb{R}$.
Then

$$\sum_{j \geq j_0} 2^{(\frac{n}{p}-1)j} \sup_{1 \leq t \leq T} t^\alpha S_j^1(t) \lesssim X^2(t) + D^2(t).$$

In what follows, we bound the term $\sum_{j \geq j_0} 2^{(\frac{N}{p}-1)j} \sup_{1 \leq t \leq T} t^\alpha S_j^2(t)$. First, we deal with the term $L_1(n_1+n_2)\mathcal{A}(\omega_1+\omega_2)$ in F_1. Employing the following decomposition

$$L_1(n_1+n_2)\mathcal{A}(\omega_1+\omega_2) = L_1(n_1+n_2)\mathcal{A}(\omega_1+\omega_2)^h + L_1(n_1+n_2)\mathcal{A}(\omega_1+\omega_2)^\ell,$$

and Lemma 2.3, we get

$$\begin{aligned}
\|t^\alpha L_1(n_1+n_2)\mathcal{A}(\omega_1+\omega_2)^h\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} &\lesssim \|n_1+n_2\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})} \|t^\alpha \nabla^2(\omega_1+\omega_2)\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \\
&\lesssim \|n_1+n_2\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})} \|t^\alpha \nabla(\omega_1+\omega_2)\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})} \\
&\lesssim X(t)D(t),
\end{aligned}$$

and

$$\begin{aligned}
&\|t^\alpha L_1(n_1+n_2)\mathcal{A}(\omega_1+\omega_2)^\ell\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \\
&\lesssim \|t^{\frac{1}{2}(s_1+\frac{N}{2}+1-\varepsilon)} L_1(n_1+n_2)\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})} \|t^{\frac{1}{2}(s_1+\frac{N}{2}+1-\varepsilon)} \nabla^2(\omega_1+\omega_2)^\ell\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{2}-1}_{2,1})} \\
&\lesssim \|t^{\frac{1}{2}(s_1+\frac{N}{2}+1-\varepsilon)}(n_1+n_2)\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})} \|t^{\frac{1}{2}(s_1+\frac{N}{2}+1-\varepsilon)}(\omega_1+\omega_2)\|^\ell_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{2}+1-\varepsilon}_{2,1})} \\
&\lesssim \Big(\|t^{\frac{1}{2}(s_1+\frac{N}{2}+1-\varepsilon)}(n_1+n_2)\|^\ell_{L^\infty_T(\dot{B}^{\frac{N}{2}-\varepsilon}_{2,1})} + \|t^\alpha(n_1+n_2)\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})} \Big) \\
&\quad \times \|t^{\frac{1}{2}(s_1+\frac{N}{2}+1-\varepsilon)}(\omega_1+\omega_2)\|^\ell_{L^\infty_T(\dot{B}^{\frac{N}{2}+1-\varepsilon}_{2,1})} \\
&\lesssim D^2(t),
\end{aligned}$$

which implies that

$$\|t^\alpha L_1(n_1+n_2)\mathcal{A}(\frac{\omega_1+\omega_2}{2})\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim D^2(t) + X^2(t).$$

Similarly,

$$\|t^\alpha L_2(n_1-n_2)\mathcal{A}(\frac{\omega_1-\omega_2}{2})\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})} \lesssim D^2(t) + X^2(t).$$

To handle the term $\|t^\alpha(\frac{\omega_1+\omega_2}{2}) \cdot \nabla(\frac{\omega_1+\omega_2}{2})\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})}$, use the following decomposition:

$$\begin{aligned}
(\frac{\omega_1+\omega_2}{2}) \cdot \nabla(\frac{\omega_1+\omega_2}{2}) = (\frac{\omega_1+\omega_2}{2}) \cdot \nabla(\frac{\omega_1+\omega_2}{2})^h + (\frac{\omega_1+\omega_2}{2})^\ell \cdot \nabla(\frac{\omega_1+\omega_2}{2})^\ell \\
+ (\frac{\omega_1+\omega_2}{2})^h \cdot \nabla(\frac{\omega_1+\omega_2}{2})^\ell.
\end{aligned}$$

Using Lemma 2.2, and the embedding $\dot{B}_{2,1}^{\frac{N}{2}+s} \hookrightarrow \dot{B}_{p,1}^{\frac{N}{p}+s}$ for all $p \geq 2, s \in \mathbb{R}$, we have

$$
\begin{aligned}
&\|t^\alpha(\frac{\omega_1+\omega_2}{2}) \cdot \nabla(\frac{\omega_1+\omega_2}{2})^h\|^h_{\widetilde{L}^\infty_T(\dot{B}_{p,1}^{\frac{N}{p}-1})} \\
\lesssim &\|\frac{\omega_1+\omega_2}{2}\|_{\widetilde{L}^\infty_T(\dot{B}_{p,1}^{\frac{N}{p}-1})} \|t^\alpha \nabla(\frac{\omega_1+\omega_2}{2})\|^h_{\widetilde{L}^\infty_T(\dot{B}_{p,1}^{\frac{N}{p}})} \\
\lesssim &X(t)D(t),
\end{aligned}
$$

$$
\begin{aligned}
&\|t^\alpha(\frac{\omega_1+\omega_2}{2})^\ell \cdot \nabla(\frac{\omega_1+\omega_2}{2})^\ell\|^h_{\widetilde{L}^\infty_T(\dot{B}_{p,1}^{\frac{N}{p}-1})} \\
\lesssim &\|t^\alpha(\frac{\omega_1+\omega_2}{2})^\ell \cdot \nabla(\frac{\omega_1+\omega_2}{2})^\ell\|_{\widetilde{L}^\infty_T(\dot{B}_{p,1}^{\frac{N}{p}})} \\
\lesssim &\|t^{\frac{1}{2}(s_1+\frac{N}{2}+1-\varepsilon)} \nabla(\frac{\omega_1+\omega_2}{2})\|^\ell_{\widetilde{L}^\infty_T(\dot{B}_{2,1}^{\frac{N}{2}})} \|t^{\frac{1}{2}(s_1+\frac{N}{2}-\varepsilon)}(\frac{\omega_1+\omega_2}{2})\|^\ell_{\widetilde{L}^\infty_T(\dot{B}_{2,1}^{\frac{N}{2}-\varepsilon})} \\
\lesssim &\|t^{\frac{1}{2}(s_1+\frac{N}{2}+1-\varepsilon)} (\frac{\omega_1+\omega_2}{2})\|^\ell_{L^\infty_T(\dot{B}_{2,1}^{\frac{N}{2}+1-\varepsilon})} \|t^{\frac{1}{2}(s_1+\frac{N}{2}-\varepsilon)}(\frac{\omega_1+\omega_2}{2})\|^\ell_{L^\infty_T(\dot{B}_{2,1}^{\frac{N}{2}-\varepsilon})} \\
\lesssim &D^2(t),
\end{aligned}
$$

and

$$
\begin{aligned}
&\|t^\alpha(\frac{\omega_1+\omega_2}{2})^h \cdot \nabla(\frac{\omega_1+\omega_2}{2})^\ell\|^h_{\widetilde{L}^\infty_T(\dot{B}_{p,1}^{\frac{N}{p}-1})} \\
\lesssim &\|\nabla(\frac{\omega_1+\omega_2}{2})\|^\ell_{\widetilde{L}^\infty_T(\dot{B}_{p,1}^{\frac{N}{p}})} \|t^\alpha(\frac{\omega_1+\omega_2}{2})\|^h_{\widetilde{L}^\infty_T(\dot{B}_{p,1}^{\frac{N}{p}-1})} \\
\lesssim &X(t)D(t).
\end{aligned}
$$

Hence

$$
\sum_{j \geq j_0} 2^{(\frac{n}{p}-1)j} \sup_{1 \leq t \leq T} \|t^\alpha(\frac{\omega_1+\omega_2}{2}) \cdot \nabla(\frac{\omega_1+\omega_2}{2})\|^h_{\widetilde{L}^\infty_T(\dot{B}_{p,1}^{\frac{N}{p}-1})}(t) \lesssim X^2(t)+D^2(t).
$$

Similarly,

$$
\sum_{j \geq j_0} 2^{(\frac{n}{p}-1)j} \sup_{1 \leq t \leq T} \|t^\alpha(\frac{\omega_1-\omega_2}{2}) \cdot \nabla(\frac{\omega_1-\omega_2}{2})\|^h_{\widetilde{L}^\infty_T(\dot{B}_{p,1}^{\frac{N}{p}-1})}(t) \lesssim X^2(t)+D^2(t).
$$

By virtue of Lemma 2.2 and Lemma 2.3(i), we obtain

$$
\begin{aligned}
&\|t^\alpha L_3(n_1+n_2)\nabla(n_1+n_2)\|^h_{\widetilde{L}^\infty_T(\dot{B}_{p,1}^{\frac{N}{p}-1})} \\
\lesssim &\|t^{\frac{1}{2}(s_1+\frac{N}{2}+1-\varepsilon)} L_3(n_1+n_2)\|_{\widetilde{L}^\infty_T(\dot{B}_{p,1}^{\frac{N}{p}})} \|t^{\frac{1}{2}(s_1+\frac{N}{2}+1-\varepsilon)} \nabla(n_1+n_2)\|_{\widetilde{L}^\infty_T(\dot{B}_{p,1}^{\frac{N}{p}-1})} \\
\lesssim &\|t^{\frac{1}{2}(s_1+\frac{N}{2}+1-\varepsilon)}(n_1+n_2)\|^2_{\widetilde{L}^\infty_T(\dot{B}_{p,1}^{\frac{N}{p}})} \\
\lesssim &\Big(\|t^{\frac{1}{2}(s_1+\frac{N}{2}+1-\varepsilon)}(n_1+n_2)\|^\ell_{L^\infty_T(\dot{B}_{2,1}^{\frac{N}{2}-\varepsilon})} + \|t^\alpha(n_1+n_2)\|^h_{\widetilde{L}^\infty_T(\dot{B}_{p,1}^{\frac{N}{p}})}\Big)^2 \\
\lesssim &D^2(t).
\end{aligned}
$$

Similarly,

$$\|t^{\alpha}L_4(n_1-n_2)\nabla(\frac{n_1-n_2}{2})\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})}\lesssim D^2(t).$$

Hence, we end up with

$$\sum_{j\geq j_0}2^{(\frac{n}{p}-1)j}\sup_{1\leq t\leq T}t^{\alpha}S^2_j(t)\lesssim X^2(t)+D^2(t).$$

For the term S^3_j and S^4_j, according to Lemma 2.6, we get

$$\begin{aligned}
&\sum_{j\geq j_0}2^{(\frac{N}{p}-1)j}\sup_{1\leq t\leq T}t^{\alpha}\|R_1\|_{L^p}\\
&\lesssim\|t^{\frac12(s_1+\frac{N}{2}+1-\varepsilon)}\nabla(\omega_1+\omega_2)\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\|t^{\frac12(s_1+\frac{N}{2}+1-\varepsilon)}\nabla(n_1+n_2)\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\
&\lesssim\Big(\|t^{\frac12(s_1+\frac{N}{2}+1-\varepsilon)}(n_1+n_2)\|^\ell_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{2}-\varepsilon}_{2,1})}+\|t^{\alpha}(n_1+n_2)\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\Big)\\
&\quad\times\Big(\|t^{\frac12(s_1+\frac{N}{2}+1-\varepsilon)}(\omega_1+\omega_2)\|^\ell_{L^\infty_T(\dot{B}^{\frac{N}{2}+1-\varepsilon}_{2,1})}+\|t^{\alpha}\nabla(\omega_1+\omega_2)\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\Big)\\
&\lesssim D^2(t),
\end{aligned}$$

and we have the embedding $\dot{B}^{\frac{N}{p}}_{p,1}\hookrightarrow L^\infty$,

$$\begin{aligned}
&\sum_{j\geq j_0}2^{(\frac{N}{p}-1)j}\sup_{1\leq t\leq T}t^{\alpha}\|\nabla\omega_1\|_{L^\infty}\|\nabla\dot{\Delta}_jn_1\|_{L^p}\\
&\lesssim\|t^{\frac12(s_1+\frac{N}{2}+1-\varepsilon)}\nabla\omega_1\|_{\widetilde{L}^\infty_T(L^\infty)}\|t^{\frac12(s_1+\frac{N}{2}+1-\varepsilon)}\nabla n_1\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\
&\lesssim\|t^{\frac12(s_1+\frac{N}{2}+1-\varepsilon)}\nabla\omega_1\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\|t^{\frac12(s_1+\frac{N}{2}+1-\varepsilon)}\nabla n_1\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\
&\lesssim D^2(t).
\end{aligned}$$

To bound the term with S^5_j, by Lemma 2.2, we have

$$\begin{aligned}
\|t^{\alpha}\nabla(n_1\mathrm{div}\omega^h_1)\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})}&\lesssim\|t^{\alpha}n_1\mathrm{div}\omega^h_1\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\\
&\lesssim\|n_1\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\|t^{\alpha}\nabla\omega_1\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\\
&\lesssim X(t)D(t),
\end{aligned}$$

and

$$\begin{aligned}
\|t^{\alpha}\nabla(n_1\mathrm{div}\omega^\ell_1)\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}-1}_{p,1})}&\lesssim\|t^{\alpha}n_1\mathrm{div}\omega^\ell_1\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\\
&\lesssim\|t^{\frac12(s_1+\frac{N}{2}+1-\varepsilon)}n_1\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{p}}_{p,1})}\|t^{\frac12(s_1+\frac{N}{2}+1-\varepsilon)}\mathrm{div}\omega_1\|^\ell_{\widetilde{L}^\infty_T(\dot{B}^{\frac{N}{2}}_{2,1})}\\
&\lesssim D^2(t).
\end{aligned}$$

Therefore, we conclude that

$$\sum_{j\geq j_0}\sup_{2\leq t\leq T}\Big(\langle t\rangle^{\alpha}\int_1^t e^{c_0(\tau-t)}2^{(\frac{N}{p}-1)j}S_j(\tau)\,d\tau\Big)\lesssim X^2(t)+D^2(t).\tag{6.17}$$

Plugging (6.14), (6.17) into (6.13) yields that

$$\begin{aligned}&\|\langle t\rangle^{\alpha}(n_1,n_2)\|^h_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{N}{p}}_{p,1})}+\|\langle t\rangle^{\alpha}(\omega_1,\omega_2)\|^h_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\&\lesssim\|(n_{1,0},n_{2,0})\|^h_{\dot{B}^{\frac{N}{p}}_{p,1}}+\|(\omega_{1,0},\omega_{2,0})\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}}+X^2(t)+D^2(t).\end{aligned}\tag{6.18}$$

Step 3: Decay estimates with gain of regularity for the high frequencies of ω_1 and ω_2. In order to bound the last terms of $D(t)$, we shall use the fact that $\omega_i(i=1,2)$ satisfy the following two equations

$$\partial_t\omega_1-\mathcal{A}\omega_1=F_1-c^2\nabla n_1,\tag{6.19}$$

$$\partial_t\omega_2-\mathcal{A}\omega_2=F_2-c^2\nabla n_2+2\nabla\Delta^{-1}n_2.\tag{6.20}$$

To obtain the desired estimates, we reformulate (6.19)-(6.20) in terms of the weighted unknowns $t^{\alpha}\mathcal{A}\omega_1,t^{\alpha}\mathcal{A}\omega_2$

$$\partial_t(t^{\alpha}\mathcal{A}\omega_1)-\mathcal{A}(t^{\alpha}\mathcal{A}\omega_1)=\alpha t^{\alpha-1}\mathcal{A}\omega_1+t^{\alpha}\mathcal{A}(F_1-c^2\nabla n_1),$$

and

$$\partial_t(t^{\alpha}\mathcal{A}\omega_2)-\mathcal{A}(t^{\alpha}\mathcal{A}\omega_2)=\alpha t^{\alpha-1}\mathcal{A}\omega_2+t^{\alpha}\mathcal{A}(F_2-c^2\nabla n_2+2\nabla\Delta^{-1}n_2).$$

Obviously, $t^{\alpha}\mathcal{A}\omega_1|_{t=0}=t^{\alpha}\mathcal{A}\omega_2|_{t=0}=0$. Then, we deduce, from Remark 2.1, that

$$\begin{aligned}&\|\tau^{\alpha}(\nabla^2\omega_1,\nabla^2\omega_2)\|^h_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\&\lesssim\|\tau^{\alpha-1}(\nabla^2\omega_1,\nabla^2\omega_2)\|^h_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{N}{p}-3}_{p,1})}+\|\tau^{\alpha}(F_1,F_2)\|^h_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\\&\quad+\|\tau^{\alpha}(\nabla n_1,\nabla n_2)\|^h_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}+\|\tau^{\alpha}\nabla\Delta^{-1}n_2\|^h_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}.\end{aligned}$$

Since $\alpha\geq 1$, we get

$$\begin{aligned}&\|\tau^{\alpha-1}(\nabla^2\omega_1,\nabla^2\omega_2)\|^h_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{N}{p}-3}_{p,1})}\lesssim\|\tau^{\alpha-1}(\omega_1,\omega_2)\|^h_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\lesssim\|\langle\tau\rangle^{\alpha}(\omega_1,\omega_2)\|^h_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{N}{p}-1}_{p,1})},\\&\|\tau^{\alpha}(\nabla n_1,\nabla n_2)\|^h_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\lesssim\|\langle\tau\rangle^{\alpha}(\nabla n_1,\nabla n_2)\|^h_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\lesssim\|\langle\tau\rangle^{\alpha}(n_1,n_2)\|^h_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{N}{p}}_{p,1})},\\&\|\tau^{\alpha}\nabla\Delta^{-1}n_2\|^h_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\lesssim 2^{-2j_0}\|\tau^{\alpha}\nabla n_2\|^h_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}\lesssim\|\langle\tau\rangle^{\alpha}n_2\|^h_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{N}{p}}_{p,1})}.\end{aligned}$$

It is clear that $\|\tau^{\alpha}(F_1,F_2)\|^h_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{N}{p}-1}_{p,1})}$ is exactly same as Step 2, we conclude that

$$\|\tau^{\alpha}(\nabla\omega_1,\nabla\omega_2)\|^h_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{N}{p}}_{p,1})}\lesssim X^2(t)+D^2(t).\tag{6.21}$$

Finally, adding up (6.7), (6.18) and (6.21) yields for all $t\geq 0$,

$$D(t)\lesssim D(0)+\|(\nabla n_{1,0},\omega_{1,0},\nabla n_{2,0},\omega_{2,0})\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}}+X^2(t)+D^2(t).\tag{6.22}$$

As Theorem 1.1 ensures that $X(t)$ is small, one can conclude that (1.8) is fulfilled for all time if $D(0),\|(\nabla n_{1,0},\omega_{1,0},\nabla n_{2,0},\omega_{2,0})\|^h_{\dot{B}^{\frac{N}{p}-1}_{p,1}}$ are small enough. This completes the proof of Theorem 1.2.

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