

GLOBAL DYNAMICS IN A CHEMOTAXIS MODEL DESCRIBING TUMOR ANGIOGENESIS WITH/WITHOUT MITOSIS IN ANY DIMENSION*

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Abstract. In this work, we study the following Neumann-initial boundary value problem for a three-component chemotaxis model describing tumor angiogenesis:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi_1 \nabla \cdot (u \nabla w) + u(a - \mu u^\theta), & x \in \Omega, t > 0, \\ v_t = d \Delta v + \xi_2 \nabla \cdot (v \nabla w) + u - v, & x \in \Omega, t > 0, \\ 0 = \Delta w + u - \bar{u}, \quad \int_{\Omega} w = 0, \quad \bar{u} := \frac{1}{|\Omega|} \int_{\Omega} u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \Omega, \end{cases}$$

in a bounded smooth but not necessarily convex domain $\Omega \subset \mathbb{R}^n (n \geq 2)$ with model parameters $\xi_1, \xi_2, d, \theta > 0, a, \chi, \mu \geq 0$. Based on subtle energy estimates, we first identify two positive constants ξ_0 and μ_0 such that the above problem allows only global classical solutions with qualitative bounds provided one of the following conditions holds:

$$(1) \xi_1 \geq \xi_0 \chi^2; \quad (2) \theta = 1, \mu \geq \max \left\{ 1, \chi^{\frac{8+2n}{5+n}} \right\} \mu_0 \chi^{\frac{2}{5+n}}; \quad (3) \theta > 1, \mu > 0.$$

Then, due to the obtained qualitative bounds, upon deriving higher order gradient estimates, we show exponential convergence of bounded solutions to the spatially homogeneous equilibrium (i) for μ large if $\mu > 0$, (ii) for d large if $a = \mu = 0$ and (iii) for merely $d > 0$ if $\chi = a = \mu = 0$. As a direct consequence of our findings, all solutions to the above system with $\chi = a = \mu = 0$ are globally bounded and they converge to constant equilibrium, and therefore, no patterns can arise.

Keywords. Chemotaxis; tumor angiogenesis; convection; qualitative boundedness; global stability.

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1. Introduction and statement of main results

To describe the branching of capillary sprouts during angiogenesis, Orme & Chaplain [32] proposed the following reaction-advection-diffusion system

$$\begin{cases} u_t = d_1 \Delta u - \chi \nabla \cdot (u \nabla v) + \xi_1 \nabla \cdot (u \nabla w), \\ v_t = d_2 \Delta v + \xi_2 \nabla \cdot (v \nabla w) + \alpha u - \beta v, \\ w_t = d_3 \Delta w + \gamma u - \delta w, \end{cases} \quad (1.1)$$

with positive parameters $d_1, d_2, d_3, \chi, \xi_1, \xi_2, \beta, \delta, \alpha, \gamma$, where u, v and w denote the density of endothelial cells (ECs), adhesive sites, and the matrix (including fibronectin, laminin, and collagen IV), respectively. Different from the classical mathematical models of tumor angiogenesis with chemotaxis as the principle mechanism of cell motion ([8, 36]), the model (1.1) was proposed based on the experimental observations that during angiogenesis process ECs secrete a matrix consisting of fibronectin, laminin and collagen

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IV [34] and the movement of ECs is affected by the distribution of adhesive sites on this matrix. More specifically, the following two processes are essentially incorporated in (1.1) (see [32, 34]):

- ECs secrete matrix and adhesive sites;
- The spreading of matrix with the convection of ECs and adhesive sites with it.

That is, the movement of ECs is governed by a combination of random motility, chemotaxis and convection.

Before proceeding to our motivation and main results, we first recall some most relevant results to the system (1.1) under homogeneous Neumann boundary conditions and nonnegative initial data (IBVP).

- (i) $\xi_1 = \xi_2 = 0$: In this case, the first two components of (1.1) reduce to the well-known classical (minimal) Keller-Segel chemotaxis model:

$$\begin{cases} u_t = d_1 \Delta u - \chi \nabla \cdot (u \nabla v), \\ v_t = d_2 \Delta v + \alpha u - \beta v, \end{cases} \quad (1.2)$$

whose solution behaviors have been extensively studied in various perspectives in the past five decades including boundedness, blow-up, large-time behavior and pattern formation. One can find more details from survey articles [5, 6, 10] and the references therein. More precisely, the boundedness and blowup of solutions for (1.2) have been established in two or higher dimensions [11, 31, 41, 43] based on the following Lyapunov energy functional:

$$\mathcal{E}_1(u, v) = d_1 \int_{\Omega} u \ln u - \chi \int_{\Omega} uv + \frac{\beta \chi}{2\alpha} \int_{\Omega} v^2 + \frac{\chi d_2}{2\alpha} \int_{\Omega} |\nabla v|^2.$$

- (ii) $\xi_2 = 0$: This case means that the convection effect of matrix on the adhesive sites is neglected, and then the system (1.1) reduces to the following widely studied attraction-repulsion Keller-Segel (ARKS) model

$$\begin{cases} u_t = d_1 \Delta u - \chi \nabla \cdot (u \nabla v) + \xi_1 \nabla \cdot (u \nabla w), \\ v_t = d_2 \Delta v + \alpha u - \beta v, \\ w_t = d_3 \Delta w + \gamma u - \delta w. \end{cases} \quad (1.3)$$

The ARKS model (1.3) has been proposed to describe the aggregation of *Microglia* in Alzheimer's disease in [28] and to describe quorum effect in chemotaxis [33]. In one-dimensional space, the existence of global boundedness of classical solution [14, 27] and time-periodic patterns as well as steady states patterns [25] were established. In high dimensional spaces ($n \geq 2$), it has been found that the sign of $\Theta := d_2 \xi_1 \gamma - d_3 \chi \alpha$ plays an important role in determining the solution behavior of (1.3). More precisely, if $\Theta \geq 0$ (i.e., repulsion dominates or cancels attraction), the 2D fully parabolic ARKS model [13, 26] or higher D parabolic-elliptic-elliptic simplification of the ARKS model [40] admits only global bounded solutions. However, if $\Theta < 0$ (i.e., attraction prevails over repulsion), based on the availability of Lyapunov functional, 2D critical and 3D generic mass blow-up phenomenon have been detected (see [9, 15, 20, 21] for more details). Recently, in the repulsion dominated case, i.e., $\Theta \geq 0$, the global stability of constant steady state has been studied in [16, 23].

- (iii) $\xi_1, \xi_2 > 0$: Due to the strong coupling of chemotaxis and convection in a cascade-like manner, which increases the complexity of mathematical analysis, the Lyapunov functional as constructed for the system (1.3) does not work anymore. To the best of our knowledge, the existing results on the system (1.1) seem at a rather rudimentary stage: in one-dimensional space, the global existence of classical solution was very recently established in [22] based on semigroup estimate technique. Furthermore, based on an appropriate energy functional, the 1D bounded solution was shown to converge to constant steady state under some restrictions on the model parameters like ξ_2 being small [18]. Very recently, for a parabolic-parabolic-elliptic simplified model in a bounded *convex* domain $\Omega \subset \mathbb{R}^n (1 \leq n \leq 3)$, Tao & Winkler [39] used a Moser-type iteration to derive the global boundedness of classical solution for large ξ_1 without qualitative information.

In summary of the above related results, some interesting questions naturally arise:

- (Q1) It follows from [39] that large repulsive convection prevents blow-up phenomenon in ≤ 3 D (lower dimensional) *convex* domains. Hence, it is natural to ask whether or not large repulsive convection can prevent blow-up of solution in any dimensional bounded smooth but not necessarily convex domains.
- (Q2) In one-dimensional space, due to nice Sobolev embeddings, a small $\xi_2 (\in (0, 1])$ -independent upper bound of solution is available. This makes the small ξ_2 -global stability toward constant equilibrium possible [18]. However, in higher dimensions, solution bounds typically depend on ξ_2 (actually with a complex relation containing ξ_2 and its inverse ξ_2^{-1} , cf. (1.7) and (1.8) for instance), and thus the method used in [18] does not work anymore. Hence, it is challenging to study long-time dynamics of bounded solutions in higher dimensions.
- (Q3) Although there may be no significant increase in the rate of ECs mitosis during the first stages of angiogenesis, the mitosis occurs after the first spouts have formed [32]. Moreover, cell division also plays an essential role when repairing and remodelling of large wounds [34]. Thus, it is interesting and practically needed to explore the effect of mitosis for the system (1.1).

To study the impact of convection more deeply and to provide relatively complete answers for the three questions above, for simplicity and clarity, based on the assumption that matrix diffuses much faster than adhesive sites and endothelial cells, we shall use a quasi-stationary approximation procedure as in [12, 42] ($\tilde{w} = w - \bar{w}$, and then the w -equation becomes $d_3^{-1} \tilde{w}_t = \Delta \tilde{w} + \gamma d_3^{-1} (u - \bar{u}) - \delta d_3^{-1} \tilde{w}$; then assuming γ has the same order as d_3 and δ has lower order than d_3 , and finally, sending $d_3 \rightarrow \infty$ and dropping the tilde notation) to arrive at the following version of parabolic-parabolic-elliptic problem:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi_1 \nabla \cdot (u \nabla w) + u(a - \mu u^\theta), & x \in \Omega, t > 0, \\ v_t = d \Delta v + \xi_2 \nabla \cdot (v \nabla w) + u - v, & x \in \Omega, t > 0, \\ 0 = \Delta w + u - \bar{u}, \quad \int_\Omega w = 0, \quad \bar{u} := \frac{1}{|\Omega|} \int_\Omega u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Here we keep the parameters χ, ξ_1, ξ_2 as above and simplify other parameters in an obvious way, for convenience. The kinetic term $u(a - \mu u^\theta)$ with $a, \mu \geq 0, \theta > 0$ is incorporated to exhibit the effect of ECs

mitosis. We shall henceforth assume that

$$(u_0, v_0) \in C^0(\bar{\Omega}) \times W^{1,\infty} \quad \text{with } u_0 \geq 0, v_0 \geq 0, \text{ and } u_0 \not\equiv 0. \tag{1.5}$$

Then our main findings on qualitative boundedness and convergence are stated as follows.

THEOREM 1.1 (Qualitative boundedness). *Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a bounded and smooth domain, the model parameters $\xi_1, \xi_2, d, \theta > 0, a, \chi, \mu \geq 0$, and, let the initial datum (u_0, v_0) satisfy the regularity (1.5). Then there exist positive constants ξ_0 and μ_0 (cf. (3.61) and (3.62) for details) depending on $d, \xi_2, n, u_0, v_0, \Omega$ such that the IBVP (1.4) admits a unique global classical solution provided one of the following conditions holds:*

- (1) $\xi_1 \geq \xi_0 \chi^2$; (2) $\theta = 1, \mu \geq \max \left\{ 1, \chi^{\frac{8+2n}{5+n}} \right\} \mu_0 \chi^{\frac{2}{5+n}}$; (3) $\theta > 1, \mu > 0$.

Moreover, the global solution is qualitatively bounded in the following way:

$$\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,\infty}} + \|w(\cdot, t)\|_{W^{1,\infty}} \leq M, \quad \forall t > 0, \tag{1.6}$$

where, up to a multiplier depending only on n, u_0, v_0 and Ω , the positive constant M is explicitly expressible in terms of the model parameters $d, a, \theta, \chi_1, \xi_1, \xi_2, \mu$, see Lemmas 3.1, 3.5, 3.6 and 3.7. In particular, the qualitative bounds for $\|v\|_{L^\infty}$ and $\|\nabla w\|_{L^\infty}$, crucial to derive large-time behaviors of bounded solutions for (1.4), are bounded as follows:

$$\begin{aligned} \|v(\cdot, t)\|_{L^\infty} &\leq K_1 \begin{cases} \left(1 + \frac{1}{\xi_2}\right) \left(1 + \xi_2 + \left(\frac{1}{d}\right)^{\frac{n}{2}} \xi_2^{1+\frac{n}{2}}\right), & \text{if } \mu = 0, \\ \left(1 + \left(\frac{1}{\mu}\right)^{\frac{1}{\theta}} + \frac{1}{\xi_2}\right) \left(1 + \left(\frac{1}{\mu}\right)^{\frac{1}{\theta}} \xi_2 + \left(\frac{1}{d}\right)^{\frac{n}{2}} \left[\left(\frac{1}{\mu}\right)^{\frac{1}{\theta}} \xi_2\right]^{1+\frac{n}{2}}\right), & \text{if } \mu > 0, \end{cases} \\ &:= M_0, \end{aligned} \tag{1.7}$$

and

$$\|\nabla w(\cdot, t)\|_{L^\infty} \leq K_2 \left(1 + \left(1 + d_\Omega M_0^{2(n+1)}\right) d^{-1} \chi^2 M_0^{1-n} + M_1^c(n)\right)^{\frac{1}{n+1}}, \tag{1.8}$$

where $K_i (i=1,2)$ depend only on n, u_0, v_0 and Ω , $M_1^c(n) = M_1^c$ is defined by

$$M_1^c(n) = \begin{cases} \xi_1, & \text{if } \mu = 0, \xi_1 \geq \xi_0 \chi^2, \\ M_\mu(1), & \text{if } \theta = 1, \mu > \max \left\{ 1, \chi^{\frac{8+2n}{5+n}} \right\} \mu_0 \chi^{\frac{2}{5+n}}, \\ M_\mu(\theta) + \frac{(\theta-1)}{\mu^{\frac{n+2}{\theta-1}}} \left[\left(1 + \frac{1}{d^{n+2}}\right) (1 + M_0 \xi_2) M_0 \chi^2\right]^{\frac{n+1+\theta}{\theta-1}}, & \text{if } \theta > 1, \mu > 0, \end{cases} \tag{1.9}$$

the function M_μ and the symbol $d_\Omega = d1_\Omega$, with 1_Ω being the indicator whether Ω is non-convex, are defined by

$$M_\mu(\theta) = \left(1 + \xi_1 \left(\frac{1}{\mu}\right)^{\frac{1}{\theta}} + \left(\frac{1}{\mu}\right)^{\frac{1}{\theta}}\right) \left(\frac{1}{\mu}\right)^{\frac{n+1}{\theta}}, \quad d_\Omega = d1_\Omega = \begin{cases} 0, & \text{if } \Omega \text{ is convex,} \\ d, & \text{if } \Omega \text{ is non-convex.} \end{cases} \tag{1.10}$$

REMARK 1.1. When $\Omega \subset \mathbb{R}^1$ is an open interval, using simpler arguments than those of [18, 22], one can easily obtain global boundedness without any parameter restrictions. For fixed parameters and initial data, we also mention that the infimums of ξ_0 and μ_0

depend indeed on $\frac{n}{2}$ instead of n , see Lemma 3.6; this is comparable to the widely known $L^{\frac{n}{2}+}$ -criterion [6,44]. Moreover, we note that the upper bounds for $\|v\|_{L^\infty}$ and $\|\nabla w\|_{L^\infty}$, cf. (1.7) and (1.8), are bounded for large d and, in particular, it can be non-increasing in d in the case that Ω is convex. This makes the study of global stability possible in the case of $a = \mu = 0$. Finally, we remark, when $\theta = 1$ (resp. $\theta > 1$), our boundedness holds for any $\xi_1 \geq 0$ as long as (2) is satisfied (resp. $\mu > 0$).

Thanks in particular to the qualitative bounds for $\|v\|_{L^\infty}$ and $\|\nabla w\|_{L^\infty}$ in (1.7) and (1.8), upon successfully deriving higher order gradient estimates, we are able to prove convergence and exponential convergence rate of bounded solutions in the following ways.

THEOREM 1.2 (Global stability). *The global bounded classical solution (u, v, w) obtained from Theorem 1.1 enjoys the following convergence properties:*

- (C1) *In the case of $a = \mu = 0$, there exists $d_0(\chi) \geq 0$ with $d_0(0) = 0$ depending on $n, u_0, v_0, \xi_1, \xi_2, \chi$ such that whenever $d \geq d_0(\chi)$ with $\chi > 0$, the global bounded solution (u, v, w) converges uniformly to $(\bar{u}_0, \bar{u}_0, 0)$:*

$$\|u(\cdot, t) - \bar{u}_0\|_{L^\infty} + \|v(\cdot, t) - \bar{u}_0\|_{L^\infty} + \|w(\cdot, t)\|_{W^{1,\infty}} \rightarrow 0 \text{ as } t \rightarrow \infty. \tag{1.11}$$

If, in addition, $d > d_0(\chi)$, the above convergence is exponential: for some $K_3, \zeta > 0$,

$$\|u(\cdot, t) - \bar{u}_0\|_{L^\infty} + \|v(\cdot, t) - \bar{u}_0\|_{L^\infty} + \|w(\cdot, t)\|_{W^{1,\infty}} \leq K_3 e^{-\zeta t}, \quad \forall t > 0. \tag{1.12}$$

- (C2) *In the case of $a, \mu > 0$, let C_p denote the Poincaré constant defined by*

$$C_p^{-2} = \inf \left\{ \int_\Omega |\nabla w|^2 : \int_\Omega w = 0, \int_\Omega w^2 = 1 \right\} \tag{1.13}$$

and let

$$\Lambda(z) = \frac{d\chi^2 + d^2 C_p^2 \xi_1^2 + C_p^2 \xi_2^2 z}{2d^2 a^{\frac{\theta-2}{\theta}}}. \tag{1.14}$$

Then, whenever

$$\mu > \max \left\{ \Lambda^{\frac{\theta}{2}} \left(\sup_{\mu > 1} M_0^2 \right), \Lambda^{\frac{\theta}{6+n}} \left(\sup_{0 < \mu \leq 1} \left(M_0^2 \mu^{\frac{4+n}{\theta}} \right) \right) \right\}, \tag{1.15}$$

the solution (u, v, w) converges exponentially to $((\frac{a}{\mu})^{\frac{1}{\theta}}, (\frac{a}{\mu})^{\frac{1}{\theta}}, 0)$: for some $K_4, \eta > 0$,

$$\|u(\cdot, t) - (\frac{a}{\mu})^{\frac{1}{\theta}}\|_{L^\infty} + \|v(\cdot, t) - (\frac{a}{\mu})^{\frac{1}{\theta}}\|_{L^\infty} + \|w(\cdot, t)\|_{W^{1,\infty}} \leq K_4 e^{-\eta t}, \quad \forall t > 0. \tag{1.16}$$

In the above texts, the positive constants $K_i (i = 3, 4), \zeta, \eta$ depend on $n, u_0, v_0, \Omega, d, a, \theta, \chi_1, \xi_1, \xi_2, \mu$; we also have used the following short notations like

$$\|f(\cdot, t)\|_{L^p} = \left(\int_\Omega |f(x, t)|^p dx \right)^{\frac{1}{p}} = \left(\int_\Omega |f(\cdot, t)|^p \right)^{\frac{1}{p}}.$$

REMARK 1.2.

- (i) Compared to the global boundedness in [39], we first remove the technical assumption that Ω is convex, and then we provide more qualitative information by digging out the dependence of solution upper bounds on the involving model parameters. Lastly, we extend lower dimensional spaces to any dimensional ones.
- (ii) Even though we cannot apply the arguments in [18] to obtain large-time behaviors of bounded solutions in that fashion, due to especially the qualitative bounds for $\|v\|_{L^\infty}$ and $\|\nabla w\|_{L^\infty}$ in (1.7) and (1.8), upon successfully establishing higher order gradient estimates, we are able to show convergence and exponential convergence of bounded solutions in qualitative ways.

As a direct consequence of our main results, we obtain unconditional boundedness and convergence to constant steady states of (1.4) with $\chi = a = \mu = 0$.

COROLLARY 1.1. *All solutions (u, v, w) to the IBVP (1.4) with $\chi = 0$ are globally bounded. Moreover, if $a = \mu = 0$, they converge exponentially to $(\bar{u}_0, \bar{u}_0, 0)$ as $t \rightarrow \infty$.*

It can be easily seen from Theorem 1.1 that the dynamics of (1.4) with small $\xi_1 > 0$ or simply $\xi_1 = 0$ remain largely open. We shall leave this as a future project.

In the rest of this section, we outline the plan as well as main ideas of this article.

In Section 2, we first state the local well-posedness and extensibility criterion and then derive some basic properties for (1.4). Next, we extend the interpolation-type inequalities in [39, Lemma 4.2] to general setting in Lemma 2.4. Finally, we collect some abstract functional inequalities including well-known smoothing L^p - L^q estimates of the Neumann heat group in Ω , etc.

In Section 3, we aim to show the proof of qualitative boundedness in Theorem 1.1. Our subtle analysis, inspired from [39], begins with the qualitative control of $\|v\|_{L^\infty}$ via Moser-iteration in a careful manner, cf. Lemma 3.1. Then, thanks to the generalized interpolation-type inequalities in Lemma 2.4, under one of the conditions in Theorem 1.1 and upon some skillful treatments, we successfully establish a key ODI (ordinary differential inequality) for the time derivative of the coupled quantity $\|u\|_{L^k}^k + \|\nabla v\|_{L^{2k}}^{2k}$ for some $k > n/2$, which allows us to derive qualitative bounds for $\|u\|_{L^k} + \|\nabla v\|_{L^{2k}}$, cf. Lemma 3.5. Thereafter, with subtle analysis via semigroup estimates, elliptic estimates and Sobolev embeddings, we finally conclude the desired uniform-in-time qualitative bounds as stated in (1.6).

In Section 4, we proceed further to derive Schauder-type estimates of u and L^{2n} -estimate of ∇u so as to study long-time dynamics of bounded solutions to (1.4). Since both the u - and v -equation in (1.4) have cross-diffusions, the derivation of boundedness of $\|\nabla u\|_{L^{2n}}$ becomes lengthy and technical. Roughly speaking, motivated by [24, Section 3.3], we first establish an ODI for the time evolution of the coupled quantity $\|u\|_{L^2}^2 + \|\Delta v\|_{L^2}^2$ to obtain a bound for $\|\Delta v\|_{L^2}$. And then, we directly derive an ODI for $\|\nabla u\|_{L^2}^2$ so as to obtain a bound of $\|\nabla u\|_{L^2}$. This enables us to handle the emerging boundary integral and thus allows us to derive an ODI involving $\|\Delta v\|_{L^{2n}}^{2n}$ for the time derivative of $\|\nabla u\|_{L^{2n}}^{2n}$. Finally, applying the widely known maximal Sobolev regularity to the v -equation in (1.4), we obtain the boundedness of $\|\nabla u\|_{L^{2n}}$, see Lemma 4.2.

With qualitative bounds in Section 3 and enhanced regularity properties in Section 4, in Section 5, we aim to study the large-time behavior of bounded solutions to (1.4). In the absence of ECs mitosis ($a = \mu = 0$), based on the important fact that upper bounds of $\|v\|_{L^\infty}$ and $\|\nabla w\|_{L^\infty}$ are bounded for d away from zero (cf. (1.7) and (1.8)), for d

suitably large, we deduce a Lyapunov functional for the coupled quantity

$$\int_{\Omega} u \ln \frac{u}{\bar{u}} + \frac{\chi}{2} \int_{\Omega} |\nabla v|^2,$$

which enables us to derive the L^2 -convergence and decay rate of $(u - \bar{u}_0, \nabla v)$ to $(0, 0)$. Then using the Gagliardo-Nirenberg inequality together with the enhanced regularity of (u, v) and standard elliptic estimates, we achieve the convergence properties of bounded solution (u, v, w) as in (1.11) and (1.12) of Theorem 1.2, see details in Section 5.1.

The convergence analysis for the case $a, \mu > 0$ follows in a similar manner. Indeed, for μ satisfying (1.15), we are able to derive a Lyapunov functional for the coupled quantity

$$\int_{\Omega} (u - b - b \ln \frac{u}{b}) + \frac{b\chi^2}{2d} \int_{\Omega} (v - b)^2, \quad b := (\frac{a}{\mu})^{\frac{1}{\theta}},$$

which yields the key starting L^2 -convergence of $(u - b, v - b)$ to $(0, 0)$. Then we can easily use the Gagliardo-Nirenberg interpolation inequality to lift this L^2 -convergence to L^∞ -convergence. Finally, the convergence property of w follows from the elliptic estimate applied to the w -equation in (1.4), thus achieving (1.16) of Theorem 1.2, see Section 5.2 for details.

2. Local existence and preliminaries

In the sequel, the integral $\int_{\Omega} f(x)dx$ and $\|f\|_{L^p(\Omega)}$ will be abbreviated as $\int_{\Omega} f$ and $\|f\|_{L^p}$, respectively. The generic constants c_i (defined within the proofs of lemmas) or C_i (defined in the statements of lemmas) for $i = 1, 2, \dots$, depending on n, Ω and the initial data u_0, v_0 but they are independent of t and of the model parameters $\chi, \xi_1, \xi_2, d, \mu$, will vary line-by-line. The existence and uniqueness of local solutions of (1.4) can be easily shown in a fixed-point theorem framework by means of the Amann’s theorems [3, 4] and the parabolic/elliptic regularity theory, as similarly demonstrated in [18, 22, 39, 42].

LEMMA 2.1 (Local existence). *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded and smooth domain, the model parameters $\chi, \xi_1, \xi_2, d > 0, a, \mu, \theta \geq 0$, and, let the initial data (u_0, v_0) satisfy (1.5). Then there exist a maximal time $T_{max} \in (0, \infty]$ and a unique triple (u, v, w) of functions with u and v positive which solves (1.4) classically on $\bar{\Omega} \times (0, T_{max})$, and satisfies*

$$\begin{cases} u \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})), \\ v \in \cap_{p>n} C^0([0, T_{max}); W^{1,p}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})), \\ w \in C^{2,0}(\bar{\Omega} \times (0, T_{max})). \end{cases}$$

Moreover, if $T_{max} < \infty$, then, for any $p > \max\{n, 2\}$,

$$\limsup_{t \nearrow T_{max}} \{ \|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,p}} \} = \infty. \tag{2.1}$$

LEMMA 2.2 (Young’s inequality with ε). *Let $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$XY \leq \varepsilon X^p + \frac{Y^q}{q(\varepsilon p)^{\frac{q}{p}}} \quad (X, Y > 0, \varepsilon > 0).$$

LEMMA 2.3. *Let (u, v, w) be a solution of (1.4) obtained in Lemma 2.1. Then*

$$|\Omega| \bar{u} = \|u(\cdot, t)\|_{L^1} \leq m_1, \quad \forall t \in (0, T_{max}), \tag{2.2}$$

where

$$m_1 := \begin{cases} \|u_0\|_{L^1}, & \text{if } a = \mu = 0, \\ \|u_0\|_{L^1} + (1+a)^{\frac{1+\theta}{\theta}} \left(\frac{1}{\mu}\right)^{\frac{1}{\theta}} \left(\frac{2}{\theta+1}\right)^{\frac{1}{\theta}} \frac{\theta}{\theta+1} |\Omega|, & \text{if } \mu > 0, \end{cases}$$

and

$$\|v(\cdot, t)\|_{L^1} \leq m_1 + \|v_0\|_{L^1}. \tag{2.3}$$

Proof. Integrating the first and second equations of (1.4) over Ω respectively, one has

$$\frac{d}{dt} \int_{\Omega} u + \mu \int_{\Omega} u^{\theta+1} = a \int_{\Omega} u, \tag{2.4}$$

and

$$\frac{d}{dt} \int_{\Omega} v + \int_{\Omega} v = \int_{\Omega} u. \tag{2.5}$$

If $a = \mu = 0$, one can easily check that (2.2) and (2.3) hold by integrating (2.4) and (2.5).

Next, if $\mu > 0$, it follows from the Young’s inequality with ε in Lemma 2.2 that

$$(1+a) \int_{\Omega} u \leq \frac{\mu}{2} \int_{\Omega} u^{\theta+1} + (1+a)^{\frac{1+\theta}{\theta}} \left(\frac{1}{\mu}\right)^{\frac{1}{\theta}} \left(\frac{2}{\theta+1}\right)^{\frac{1}{\theta}} \frac{\theta}{\theta+1} |\Omega|,$$

which, upon being substituted into (2.4), gives

$$\frac{d}{dt} \int_{\Omega} u + \int_{\Omega} u + \frac{\mu}{2} \int_{\Omega} u^{\theta+1} \leq (1+a)^{\frac{1+\theta}{\theta}} \left(\frac{1}{\mu}\right)^{\frac{1}{\theta}} \left(\frac{2}{\theta+1}\right)^{\frac{1}{\theta}} \frac{\theta}{\theta+1} |\Omega|. \tag{2.6}$$

Then multiplying (2.6) by the factor e^t and then integrating, we simply get

$$\int_{\Omega} u \leq (1+a)^{\frac{1+\theta}{\theta}} \left(\frac{1}{\mu}\right)^{\frac{1}{\theta}} \left(\frac{2}{\theta+1}\right)^{\frac{1}{\theta}} \frac{\theta}{\theta+1} |\Omega| + \int_{\Omega} u_0,$$

which entails (2.2). Then substituting (2.2) into (2.5), one has

$$\frac{d}{dt} \int_{\Omega} v + \int_{\Omega} v \leq (1+a)^{\frac{1+\theta}{\theta}} \left(\frac{1}{\mu}\right)^{\frac{1}{\theta}} \left(\frac{2}{\theta+1}\right)^{\frac{1}{\theta}} \frac{\theta}{\theta+1} |\Omega| + \int_{\Omega} u_0. \tag{2.7}$$

Solving this ODI or using Grönwall’s inequality again, then (2.7) implies (2.3). \square

For our later boundedness purpose, let us generalize the interpolation-type inequalities in [39, Lemma 4.2] to the general case as follows.

LEMMA 2.4. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded and smooth domain and let $g, h \in C^2(\bar{\Omega})$ with $\frac{\partial g}{\partial \nu}|_{\partial\Omega} = \frac{\partial h}{\partial \nu}|_{\partial\Omega} = 0$. Then, for all $p \geq 1$,*

$$\begin{aligned} & \left| \int_{\Omega} |\nabla g|^{2p-2} \nabla g \cdot \nabla (\nabla g \cdot \nabla h) \right| \\ & \leq \left(\frac{\sqrt{n}}{2p} + 1 \right) \|\nabla g\|_{L^{2(p+1)}}^{2p} \|D^2 h\|_{L^{p+1}}, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} & \left| \int_{\Omega} g \Delta h \nabla \cdot (|\nabla g|^{2p-2} \nabla g) \right| \\ & \leq (2(p-1) + \sqrt{n}) \|g\|_{L^\infty} \|\nabla g\|_{L^{2(p+1)}}^{p-1} \|\Delta h\|_{L^{p+1}} \left(\int_{\Omega} |\nabla g|^{2p-2} |D^2 g|^2 \right)^{\frac{1}{2}}, \end{aligned} \tag{2.9}$$

as well as

$$\int_{\Omega} |\nabla g|^{2(p+1)} \leq (2p + \sqrt{n})^2 \|g\|_{L^\infty}^2 \int_{\Omega} |\nabla g|^{2(p-1)} |D^2 g|^2. \tag{2.10}$$

Proof. Using the following two facts

$$\nabla(\nabla g \cdot \nabla h) = D^2 g \cdot \nabla h + D^2 h \cdot \nabla g,$$

and

$$\nabla |\nabla g|^{2p} = p |\nabla g|^{2(p-1)} \nabla |\nabla g|^2 = 2p |\nabla g|^{2(p-1)} D^2 g \cdot \nabla g, \tag{2.11}$$

we thus use the symmetry of $D^2 g$ and integration by parts formula to derive that

$$\begin{aligned} & \int_{\Omega} |\nabla g|^{2p-2} \nabla g \cdot \nabla(\nabla g \cdot \nabla h) \\ & = \int_{\Omega} |\nabla g|^{2p-2} \nabla g \cdot (D^2 g \cdot \nabla h + D^2 h \cdot \nabla g) \\ & = \int_{\Omega} |\nabla g|^{2p-2} (D^2 g \cdot \nabla g) \cdot \nabla h + \int_{\Omega} |\nabla g|^{2p-2} \nabla g \cdot (D^2 h \cdot \nabla g) \\ & = \frac{1}{2p} \int_{\Omega} \nabla |\nabla g|^{2p} \cdot \nabla h + \int_{\Omega} |\nabla g|^{2p-2} \nabla g \cdot (D^2 h \cdot \nabla g) \\ & = -\frac{1}{2p} \int_{\Omega} |\nabla g|^{2p} \cdot \Delta h + \int_{\Omega} |\nabla g|^{2p-2} \nabla g \cdot (D^2 h \cdot \nabla g). \end{aligned} \tag{2.12}$$

Noting the fact $|\Delta h| \leq \sqrt{n} |D^2 h|$ and using Hölder's inequality, from (2.12) we have

$$\begin{aligned} \left| \int_{\Omega} |\nabla g|^{2p-2} \nabla g \cdot \nabla(\nabla g \cdot \nabla h) \right| & \leq \frac{\sqrt{n}}{2p} \int_{\Omega} |\nabla g|^{2p} |D^2 h| + \int_{\Omega} |\nabla g|^{2p} |D^2 h| \\ & = \left(\frac{\sqrt{n}}{2p} + 1\right) \int_{\Omega} |\nabla g|^{2p} |D^2 h| \\ & \leq \left(\frac{\sqrt{n}}{2p} + 1\right) \|\nabla g\|_{L^{2(p+1)}}^{2p} \|D^2 h\|_{L^{p+1}}, \end{aligned}$$

which is our desired estimate (2.8). Similarly, we infer that

$$\begin{aligned} & \left| \int_{\Omega} g \Delta h \nabla \cdot (|\nabla g|^{2p-2} \nabla g) \right| \\ & = \left| \int_{\Omega} g \Delta h \nabla (|\nabla g|^{2p-2}) \cdot \nabla g + \int_{\Omega} g \Delta h |\nabla g|^{2p-2} \Delta g \right| \\ & = \left| 2(p-1) \int_{\Omega} g \Delta h |\nabla g|^{2(p-2)} \nabla g \cdot (D^2 g \cdot \nabla g) + \int_{\Omega} g \Delta h |\nabla g|^{2p-2} \Delta g \right| \end{aligned}$$

$$\begin{aligned} &\leq (2(p-1) + \sqrt{n}) \int_{\Omega} |g| \cdot |\Delta h| \cdot |\nabla g|^{2p-2} \cdot |D^2 g| \\ &\leq (2(p-1) + \sqrt{n}) \|g\|_{L^\infty} \|\Delta h\|_{L^{p+1}} \|\nabla g\|_{L^{2(p+1)}}^{p-1} \| |\nabla g|^{p-1} |D^2 g| \|_{L^2}, \end{aligned}$$

which gives rise to (2.9).

Finally, we use (2.11) to estimate the term $\int_{\Omega} |\nabla g|^{2(p+1)}$ as follows:

$$\begin{aligned} \int_{\Omega} |\nabla g|^{2(p+1)} &= \int_{\Omega} |\nabla g|^{2p} \nabla g \cdot \nabla g \\ &= - \int_{\Omega} g |\nabla g|^{2p} \Delta g - \int_{\Omega} g \nabla (|\nabla g|^{2p}) \cdot \nabla g \\ &= - \int_{\Omega} g |\nabla g|^{2p} \Delta g - 2p \int_{\Omega} g |\nabla g|^{2(p-1)} \nabla g \cdot (D^2 g \cdot \nabla g) \\ &\leq (2p + \sqrt{n}) \int_{\Omega} |g| \cdot |\nabla g|^{2p} \cdot |D^2 g| \\ &\leq (2p + \sqrt{n}) \|g\|_{L^\infty} \left(\int_{\Omega} |\nabla g|^{2(p+1)} \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla g|^{2(p-1)} |D^2 g|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

which upon simple algebraic manipulations entails (2.10). □

Now, for convenience of reference, we collect the well-known smoothing L^p - L^q estimates of the Neumann heat group in Ω , which can be found in [7, 41].

LEMMA 2.5. *Let $(e^{t\Delta})_{t \geq 0}$ be the Neumann heat semigroup in Ω , and let $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then there exist some positive constants c_i ($i = 1, 2, 3, 4$) depending only on Ω such that:*

(i) *If $1 \leq q \leq p \leq \infty$, then*

$$\|e^{t\Delta} f\|_{L^p} \leq c_1 \left(1 + t^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\right) e^{-\lambda_1 t} \|f\|_{L^q} \text{ for all } t > 0 \tag{2.13}$$

holds for all $f \in L^q(\Omega)$ satisfying $\int_{\Omega} f = 0$.

(ii) *If $1 \leq q \leq p \leq \infty$, then*

$$\|\nabla e^{t\Delta} f\|_{L^p} \leq c_2 \left(1 + t^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\right) e^{-\lambda_1 t} \|f\|_{L^q} \text{ for all } t > 0$$

is valid for all $f \in L^q(\Omega)$.

(iii) *If $2 \leq q \leq p < \infty$, then*

$$\|\nabla e^{t\Delta} f\|_{L^p} \leq c_3 \left(1 + t^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\right) e^{-\lambda_1 t} \|\nabla f\|_{L^q} \text{ for all } t > 0$$

is true for all $f \in W^{1,p}(\Omega)$.

(iv) *If $1 < q \leq p \leq \infty$, then*

$$\|e^{t\Delta} \nabla \cdot f\|_{L^p} \leq c_4 \left(1 + t^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\right) e^{-\lambda_1 t} \|f\|_{L^q} \text{ for all } t > 0 \tag{2.14}$$

is valid for all $f \in (W^{1,p}(\Omega))^n$.

LEMMA 2.6 ([16]). *Let $f(x, t)$ be a positive function for $(x, t) \in \Omega \times (0, \infty)$ and define $\bar{f} = \frac{1}{|\Omega|} \int_{\Omega} f$. Then it holds that*

$$0 \leq \frac{1}{2\bar{f}} \|f - \bar{f}\|_{L^1}^2 \leq \int_{\Omega} f \ln \frac{f}{\bar{f}} \leq \frac{1}{\bar{f}} \|f - \bar{f}\|_{L^2}^2.$$

3. Qualitative boundedness: Proof of Theorem 1.1

In this section, we are devoted to proving the qualitative boundedness in terms of model parameters as stated in Theorem 1.1. To this end, we first use the convection effect to show the boundedness of $\|v\|_{L^\infty}$ based on some ideas in [39].

LEMMA 3.1. *Let (u, v, w) be the solution of (1.4) obtained from Lemma 2.1. Then*

$$\begin{aligned} & \|v(\cdot, t)\|_{L^\infty} \\ & \leq C_0 \max \left\{ m_1 + \|v_0\|_{L^1}, \frac{1}{\xi_2}, \|v_0\|_{L^\infty} \right\} \left(1 + m_1 \xi_2 + \left(\frac{1}{d}\right)^{\frac{n}{2}} (m_1 \xi_2)^{1+\frac{n}{2}} \right) := M_0, \end{aligned} \tag{3.1}$$

for all $t \in [0, T_{max})$; here $C_0 > 0$ is defined in (3.13), which depends only on n and Ω .

Proof. For $p > 1$, multiplying the second equation of (1.4) by v^{p-1} , integrating the result over Ω by parts and combining the equation $\Delta w = -u + \bar{u}$, we obtain that

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_{\Omega} v^p &= d \int_{\Omega} \Delta v \cdot v^{p-1} + \xi_2 \int_{\Omega} \nabla \cdot (v \nabla w) v^{p-1} + \int_{\Omega} u v^{p-1} - \int_{\Omega} v^p \\ &= -(p-1)d \int_{\Omega} v^{p-2} |\nabla v|^2 - \xi_2 (p-1) \int_{\Omega} v^{p-1} \nabla v \cdot \nabla w + \int_{\Omega} u v^{p-1} - \int_{\Omega} v^p \\ &= -(p-1)d \int_{\Omega} v^{p-2} |\nabla v|^2 + \frac{\xi_2 (p-1)}{p} \int_{\Omega} v^p \Delta w + \int_{\Omega} u v^{p-1} - \int_{\Omega} v^p \\ &= -(p-1)d \int_{\Omega} v^{p-2} |\nabla v|^2 + \frac{\xi_2 (p-1)}{p} \int_{\Omega} v^p (-u + \bar{u}) + \int_{\Omega} u v^{p-1} - \int_{\Omega} v^p, \end{aligned}$$

which upon using the fact $v^{p-2} |\nabla v|^2 = \frac{4}{p^2} |\nabla v^{\frac{p}{2}}|^2$ gives

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} v^p + \frac{4d(p-1)}{p} \int_{\Omega} |\nabla v^{\frac{p}{2}}|^2 + \xi_2 (p-1) \int_{\Omega} u v^p + p \int_{\Omega} v^p \\ &= \xi_2 (p-1) \bar{u} \int_{\Omega} v^p + p \int_{\Omega} u v^{p-1}. \end{aligned} \tag{3.2}$$

Applying Young’s inequality with ε (cf. Lemma 2.2) and the facts $\|u(\cdot, t)\|_{L^1} \leq m_1$ and $\bar{u} = \frac{1}{|\Omega|} \int_{\Omega} u \leq \frac{m_1}{|\Omega|}$, we infer

$$\begin{aligned} p \int_{\Omega} u v^{p-1} &\leq \xi_2 (p-1) \int_{\Omega} u v^p + \xi_2^{-(p-1)} \int_{\Omega} u \\ &\leq \xi_2 (p-1) \int_{\Omega} u v^p + m_1 \xi_2^{1-p}, \end{aligned} \tag{3.3}$$

and

$$\xi_2 (p-1) \bar{u} \int_{\Omega} v^p \leq \frac{\xi_2 m_1 (p-1)}{|\Omega|} \int_{\Omega} v^p. \tag{3.4}$$

A substitution of (3.3) and (3.4) into (3.2) shows that

$$\frac{d}{dt} \int_{\Omega} v^p + \frac{4d(p-1)}{p} \int_{\Omega} |\nabla v^{\frac{p}{2}}|^2 + p \int_{\Omega} v^p \leq \frac{\xi_2 m_1 (p-1)}{|\Omega|} \int_{\Omega} v^p + m_1 \xi_2^{1-p}. \tag{3.5}$$

For $\varepsilon > 0$, there exists $c_1 > 0$ depending only on n and Ω such that (cf. [40, 44])

$$\|U\|_{L^2}^2 \leq \varepsilon \|\nabla U\|_{L^2}^2 + c_1 (1 + \varepsilon^{-\frac{n}{2}}) \|U\|_{L^1}^2. \tag{3.6}$$

We choose $U = v^{\frac{p}{2}}$ and $\varepsilon = \frac{4d|\Omega|}{p\xi_2 m_1}$ in (3.6) to derive that

$$\begin{aligned} & \frac{\xi_2 m_1 (p-1)}{|\Omega|} \int_{\Omega} v^p \\ & \leq \frac{4(p-1)d}{p} \int_{\Omega} |\nabla v^{\frac{p}{2}}|^2 + \frac{\xi_2 m_1 c_1 (p-1)}{|\Omega|} \left(1 + \left(\frac{\xi_2 m_1 p}{4d|\Omega|} \right)^{\frac{n}{2}} \right) \left(\int_{\Omega} v^{\frac{p}{2}} \right)^2 \\ & \leq \frac{4(p-1)d}{p} \int_{\Omega} |\nabla v^{\frac{p}{2}}|^2 + c_2 p (1+p^{\frac{n}{2}}) \left(\int_{\Omega} v^{\frac{p}{2}} \right)^2, \end{aligned} \tag{3.7}$$

where $c_2 := \frac{\xi_2 m_1 c_1}{|\Omega|} \max\{1, \left(\frac{\xi_2 m_1}{4d|\Omega|}\right)^{\frac{n}{2}}\}$. Then substituting (3.7) into (3.5) and noting the fact $(1+p^{\frac{n}{2}}) \leq 2(1+p)^{\frac{n}{2}}$, we obtain

$$\frac{d}{dt} \int_{\Omega} v^p + p \int_{\Omega} v^p \leq 2c_2 p (1+p)^{\frac{n}{2}} \left(\int_{\Omega} v^{\frac{p}{2}} \right)^2 + m_1 \xi_2^{1-p},$$

which implies that

$$\frac{d}{dt} \left(e^{pt} \int_{\Omega} v^p \right) \leq 2e^{pt} c_2 p (1+p)^{\frac{n}{2}} \left(\int_{\Omega} v^{\frac{p}{2}} \right)^2 + e^{pt} m_1 \xi_2^{1-p}. \tag{3.8}$$

For any $T \in (0, T_{\max})$, we integrate (3.8) over $[0, t]$ for $0 \leq t \leq T$ to obtain

$$\int_{\Omega} v^p(\cdot, t) \leq \int_{\Omega} v_0^p + 2c_2 (1+p)^{\frac{n}{2}} \sup_{0 \leq t \leq T} \left(\int_{\Omega} v^{\frac{p}{2}}(\cdot, t) \right)^2 + \frac{m_1 \xi_2^{1-p}}{p},$$

which immediately yields

$$\begin{aligned} & \left(\int_{\Omega} v^p(\cdot, t) \right)^{\frac{1}{p}} \\ & \leq \left[2c_2 (1+p)^{\frac{n}{2}} \sup_{0 \leq t \leq T} \left(\int_{\Omega} v^{\frac{p}{2}}(\cdot, t) \right)^2 + m_1 \xi_2^{1-p} + |\Omega| \|v_0\|_{L^\infty}^p \right]^{\frac{1}{p}} \\ & \leq (\max\{2c_2, m_1 \xi_2, |\Omega|\})^{\frac{1}{p}} (1+p)^{\frac{n}{2p}} \left\{ \sup_{0 \leq t \leq T} \left(\int_{\Omega} v^{\frac{p}{2}}(\cdot, t) \right)^{\frac{2}{p}} + \frac{1}{\xi_2} + \|v_0\|_{L^\infty} \right\}. \end{aligned}$$

Therefore, it follows with $c_3 := 3 \max\{2c_2, m_1 \xi_2, |\Omega|, 1\}$ that

$$\begin{aligned} & \max \left\{ \sup_{0 \leq t \leq T} \left(\int_{\Omega} v^p(\cdot, t) \right)^{\frac{1}{p}}, \frac{1}{\xi_2}, \|v_0\|_{L^\infty} \right\} \\ & \leq c_3^{\frac{1}{p}} (1+p)^{\frac{n}{2p}} \max \left\{ \sup_{0 \leq t \leq T} \left(\int_{\Omega} v^{\frac{p}{2}}(\cdot, t) \right)^{\frac{2}{p}}, \frac{1}{\xi_2}, \|v_0\|_{L^\infty} \right\}. \end{aligned} \tag{3.9}$$

Upon setting

$$H(p) := \max \left\{ \sup_{0 \leq t \leq T} \left(\int_{\Omega} v^p(\cdot, t) \right)^{\frac{1}{p}}, \frac{1}{\xi_2}, \|v_0\|_{L^\infty} \right\},$$

then (3.9) becomes

$$H(p) \leq c_3^{\frac{1}{p}} (1+p)^{\frac{n}{2p}} H\left(\frac{p}{2}\right), \quad \forall p \geq 2.$$

Taking $p = 2^j$ ($j = 1, 2, \dots$), we obtain inductively that

$$\begin{aligned} H(2^j) &\leq c_3^{2^{-j}} (1+2^j)^{\frac{n}{2} \cdot 2^{-j}} H(2^{j-1}) \\ &\leq c_3^{2^{-j} + 2^{-(j-1)}} (1+2^j)^{\frac{n}{2} \cdot 2^{-j}} (1+2^{j-1})^{\frac{n}{2} \cdot 2^{-(j-1)}} H(2^{j-2}) \\ &\leq c_3^{\sum_{i=0}^2 2^{-(j-i)}} \cdot \prod_{k=j-2}^j (1+2^k)^{\frac{1}{2^k} \cdot \frac{n}{2}} H(2^{j-3}) \\ &\dots\dots \\ &\leq c_3^{\sum_{i=0}^{j-1} 2^{-(j-i)}} \cdot \prod_{k=1}^j (1+2^k)^{\frac{1}{2^k} \cdot \frac{n}{2}} H(1) \\ &= c_3^{1 - \frac{1}{2^j}} \prod_{k=1}^j (1+2^k)^{\frac{1}{2^k} \cdot \frac{n}{2}} H(1). \end{aligned} \tag{3.10}$$

On the one hand, using the fact that $\ln(1+z) \leq \sqrt{z}$ for all $z \geq 0$, we have

$$\ln \prod_{k=1}^j (1+2^k)^{\frac{1}{2^k}} = \sum_{k=1}^j \frac{\ln(1+2^k)}{2^k} \leq \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^k \leq 6, \quad \forall j = 1, 2, \dots,$$

and so,

$$\lim_{j \rightarrow \infty} \prod_{k=1}^j (1+2^k)^{\frac{1}{2^k} \cdot \frac{n}{2}} \leq e^{3n}. \tag{3.11}$$

Then the combination of (3.10), (3.11) and (2.3) with $c_2 := \frac{\xi_2 m_1 c_1}{|\Omega|} \max\{1, (\frac{\xi_2 m_1}{4d|\Omega|})^{\frac{n}{2}}\}$ gives

$$\begin{aligned} \|v(\cdot, t)\|_{L^\infty} &\leq \lim_{j \rightarrow \infty} H(2^j) \\ &\leq c_3 e^{3n} H(1) \\ &= 3 \max\{2c_2, m_1 \xi_2, |\Omega|, 1\} e^{3n} \max\left\{ \sup_{0 \leq t \leq T} \int_{\Omega} v(\cdot, t), \frac{1}{\xi_2}, \|v_0\|_{L^\infty} \right\} \\ &\leq 3e^{3n} (1 + |\Omega| + m_1 \xi_2 + 2c_2) \max\left\{ m_1 + \|v_0\|_{L^1}, \frac{1}{\xi_2}, \|v_0\|_{L^\infty} \right\} \\ &\leq C_0 \max\left\{ m_1 + \|v_0\|_{L^1}, \frac{1}{\xi_2}, \|v_0\|_{L^\infty} \right\} \left((1 + m_1 \xi_2 + (\frac{1}{d})^{\frac{n}{2}} (m_1 \xi_2)^{1 + \frac{n}{2}}) \right) \end{aligned} \tag{3.12}$$

for all $t \in [0, T]$, where

$$C_0 := 3e^{3n} \max\left\{ 1 + |\Omega|, 1 + \frac{2c_1}{|\Omega|}, \frac{2c_1}{(4|\Omega|)^{\frac{n}{2}} |\Omega|} \right\}. \tag{3.13}$$

Since $T \in (0, T_{\max})$ is arbitrary and the upper bound in (3.12) is independent of T , the desired estimate (3.1) follows from (3.12) and (3.13). \square

3.1. Qualitative L^p -estimates. In this subsection, by means of the key estimate on $\|v\|_{L^\infty}$ in (3.12), we shall establish further coupled L^p -energy estimates with dependence on key parameters. Here, we shall stress that our arguments are applicable to both cases: $a, \mu > 0$ and $a = \mu = 0$.

LEMMA 3.2. For $p > 1$, the local-in-time solution (u, v, w) of (1.4) obtained in Lemma 2.1 satisfies

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} u^p + \frac{p(p-1)}{2} \int_{\Omega} u^{p-2} |\nabla u|^2 + \frac{\xi_1(p-1)}{2} \int_{\Omega} u^{p+1} + \frac{p\mu}{2} \int_{\Omega} u^{\theta+p} \\ & \leq \frac{\chi^2 p(p-1)}{2} \int_{\Omega} u^p |\nabla v|^2 + M_1(p), \quad \text{for } t \in (0, T_{max}), \end{aligned} \tag{3.14}$$

where

$$M_1(p) := \begin{cases} \xi_1 m_1^{p+1} \frac{(p-1)}{p+1} \cdot \left(\frac{2p}{|\Omega|(p+1)} \right)^p + \left(\frac{1}{\mu} \right)^{\frac{p}{\theta}} \left(\frac{2ap}{p+\theta} \right)^{\frac{p}{\theta}} \frac{ap\theta}{p+\theta} |\Omega|, & a, \mu > 0, \\ \xi_1 m_1^{p+1} \frac{(p-1)}{p+1} \cdot \left(\frac{2p}{|\Omega|(p+1)} \right)^p, & a = \mu = 0. \end{cases} \tag{3.15}$$

Proof. For $p > 1$, multiplying the first equation of (1.4) by u^{p-1} and integrating the result over Ω by parts, we conclude that

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + (p-1) \int_{\Omega} u^{p-2} |\nabla u|^2 + \mu \int_{\Omega} u^{\theta+p} \\ & = \chi(p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v - \xi_1(p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w + a \int_{\Omega} u^p \\ & = \chi(p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v + \frac{\xi_1(p-1)}{p} \int_{\Omega} u^p \Delta w + a \int_{\Omega} u^p, \end{aligned}$$

which together with the fact $\Delta w = \bar{u} - u$ gives

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + (p-1) \int_{\Omega} u^{p-2} |\nabla u|^2 + \mu \int_{\Omega} u^{\theta+p} + \frac{\xi_1(p-1)}{p} \int_{\Omega} u^{p+1} \\ & = \chi(p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v + \frac{\xi_1(p-1)}{p} \bar{u} \int_{\Omega} u^p + a \int_{\Omega} u^p. \end{aligned} \tag{3.16}$$

Applications of Hölder’s inequality and Young’s inequality yield that

$$\chi(p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v \leq \frac{(p-1)}{2} \int_{\Omega} u^{p-2} |\nabla u|^2 + \frac{\chi^2(p-1)}{2} \int_{\Omega} u^p |\nabla v|^2, \tag{3.17}$$

and noting from $\bar{u} = \frac{1}{|\Omega|} \int_{\Omega} u \leq \frac{m_1}{|\Omega|}$ that

$$\begin{aligned} \frac{\xi_1(p-1)}{p} \bar{u} \int_{\Omega} u^p & \leq \frac{\xi_1(p-1)m_1}{p|\Omega|} \int_{\Omega} u^p \\ & \leq \frac{\xi_1(p-1)}{2p} \int_{\Omega} u^{p+1} + \xi_1 m_1^{p+1} \frac{(p-1)}{p(p+1)} \left(\frac{2p}{|\Omega|(p+1)} \right)^p, \end{aligned} \tag{3.18}$$

as well as, in the case of $a, \mu > 0$,

$$a \int_{\Omega} u^p \leq \frac{\mu}{2} \int_{\Omega} u^{p+\theta} + \left(\frac{2ap}{\mu(p+\theta)} \right)^{\frac{p}{\theta}} \frac{a\theta}{p+\theta} |\Omega|. \tag{3.19}$$

Substituting (3.17), (3.18) and (3.19) into (3.16) and noting the definition of $M_1(p)$ in (3.15), we end up with (3.14) directly, thus proving this lemma. \square

LEMMA 3.3. For $k \geq 1$, the local-in-time classical solution of (1.4) obtained in Lemma 2.1 satisfies, for $t \in [0, T_{max})$, that

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |\nabla v|^{2k} + 2k \int_{\Omega} |\nabla v|^{2k} + 2kd \int_{\Omega} |\nabla v|^{2k-2} |D^2 v|^2 + k(k-1) d \int_{\Omega} |\nabla v|^{2k-4} |\nabla |\nabla v|^2|^2 \\ & \leq 2k\xi_2 \int_{\Omega} |\nabla v|^{2k-2} \nabla v \cdot \nabla (\nabla v \cdot \nabla w) - 2k\xi_2 \int_{\Omega} v \Delta w \nabla \cdot (|\nabla v|^{2k-2} \nabla v) \\ & \quad + 2k(2(k-1) + \sqrt{n}) \int_{\Omega} u |\nabla v|^{2k-2} |D^2 v| + dk \int_{\partial\Omega} |\nabla v|^{2k-2} \frac{\partial |\nabla v|^2}{\partial \nu}. \end{aligned} \tag{3.20}$$

Proof. For $k \geq 1$, differentiating the second equation of (1.4) and multiplying the result by $|\nabla v|^{2k-2} \nabla v$, we obtain via integration by parts that

$$\begin{aligned} \frac{1}{2k} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2k} &= \int_{\Omega} |\nabla v|^{2k-2} \nabla v \cdot \nabla (d\Delta v + \xi_2 \nabla \cdot (v \nabla w) - v + u) \\ &= d \int_{\Omega} |\nabla v|^{2k-2} \nabla v \cdot \nabla \Delta v + \xi_2 \int_{\Omega} |\nabla v|^{2k-2} \nabla v \cdot \nabla (\nabla \cdot (v \nabla w)) \\ & \quad + \int_{\Omega} |\nabla v|^{2k-2} \nabla v \cdot \nabla u - \int_{\Omega} |\nabla v|^{2k} \\ &=: I_1 + I_2 + I_3 - \int_{\Omega} |\nabla v|^{2k}. \end{aligned} \tag{3.21}$$

Using the identity $\nabla v \cdot \nabla \Delta v = \frac{1}{2} \Delta |\nabla v|^2 - |D^2 v|^2$ and integrating by parts, we compute

$$\begin{aligned} I_1 &= \frac{d}{2} \int_{\Omega} |\nabla v|^{2k-2} \Delta |\nabla v|^2 - d \int_{\Omega} |\nabla v|^{2k-2} |D^2 v|^2 \\ &= \frac{d}{2} \int_{\partial\Omega} |\nabla v|^{2k-2} \frac{\partial |\nabla v|^2}{\partial \nu} - \frac{(k-1)d}{2} \int_{\Omega} |\nabla v|^{2k-4} |\nabla |\nabla v|^2|^2 - d \int_{\Omega} |\nabla v|^{2k-2} |D^2 v|^2. \end{aligned} \tag{3.22}$$

Similarly, we use integration by parts to rewrite I_2 as follows:

$$\begin{aligned} I_2 &= \xi_2 \int_{\Omega} |\nabla v|^{2k-2} \nabla v \cdot \nabla (\nabla v \cdot \nabla w + v \Delta w) \\ &= \xi_2 \int_{\Omega} |\nabla v|^{2k-2} \nabla v \cdot \nabla (\nabla v \cdot \nabla w) + \xi_2 \int_{\Omega} |\nabla v|^{2k-2} \nabla v \cdot \nabla (v \Delta w) \\ &= \xi_2 \int_{\Omega} |\nabla v|^{2k-2} \nabla v \cdot \nabla (\nabla v \cdot \nabla w) - \xi_2 \int_{\Omega} v \Delta w \nabla \cdot (|\nabla v|^{2k-2} \nabla v). \end{aligned} \tag{3.23}$$

As for I_3 , using the fact $|\Delta v| \leq \sqrt{n} |D^2 v|$ and the identity (2.11), we have

$$\begin{aligned} I_3 &= \int_{\Omega} |\nabla v|^{2k-2} \nabla v \cdot \nabla u \\ &= - \int_{\Omega} u \nabla |\nabla v|^{2k-2} \cdot \nabla v - \int_{\Omega} u |\nabla v|^{2k-2} \Delta v \\ &= -2(k-1) \int_{\Omega} u |\nabla v|^{2k-4} \nabla v \cdot (D^2 v \cdot \nabla v) - \int_{\Omega} u |\nabla v|^{2k-2} \Delta v \\ &\leq (2(k-1) + \sqrt{n}) \int_{\Omega} u |\nabla v|^{2k-2} |D^2 v|. \end{aligned} \tag{3.24}$$

A substitution of (3.22), (3.23) and (3.24) into (3.21) shows (3.20). □

Next, we use the L^∞ -bound of v provided by Lemma 3.1 to control the terms on the right-hand side of (3.20).

LEMMA 3.4. *For $k \geq 1$, the local-in-time classical solution of (1.4) obtained in Lemma 2.1 satisfies, for $t \in [0, T_{max})$, that*

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |\nabla v|^{2k} + 2k \int_{\Omega} |\nabla v|^{2k} + \frac{k}{2} (3d - d_{\Omega}) \int_{\Omega} |\nabla v|^{2k-2} |D^2 v|^2 \\ & \quad + k(k-1)(d - d_{\Omega}) \int_{\Omega} |\nabla v|^{2k-4} |\nabla |\nabla v|^2|^2 \\ & \leq \frac{C_1}{d^k} (1 + M_0 \xi_2)^{k+1} M_0^{k-1} \int_{\Omega} u^{k+1} + C_1 d_{\Omega} M_0^{2k}, \end{aligned} \tag{3.25}$$

where $C_1 > 0$ depends only on n, k, u_0, v_0 and Ω and d_{Ω} is defined by (1.10).

Proof. By (2.8) and (2.10) and the fact $\|v(\cdot, t)\|_{L^\infty} \leq M_0$ in (3.1), one obtains that

$$\begin{aligned} J_1 & := 2k\xi_2 \int_{\Omega} |\nabla v|^{2k-2} \nabla v \cdot \nabla (|\nabla v|^2) \\ & \leq 2k\xi_2 \left(\frac{\sqrt{n}}{2k} + 1\right) \|\nabla v\|_{L^{2(k+1)}}^{2k} \|D^2 w\|_{L^{k+1}} \\ & = 2k\xi_2 \left(\frac{\sqrt{n}}{2k} + 1\right) \left(\int_{\Omega} |\nabla v|^{2(k+1)}\right)^{\frac{k}{k+1}} \|D^2 w\|_{L^{k+1}} \\ & \leq 2k\xi_2 \left(\frac{\sqrt{n}}{2k} + 1\right) (2k + \sqrt{n})^{\frac{2k}{k+1}} \|v\|_{L^\infty}^{\frac{2k}{k+1}} \left(\int_{\Omega} |\nabla v|^{2(k-1)} |D^2 v|^2\right)^{\frac{k}{k+1}} \|D^2 w\|_{L^{k+1}} \\ & \leq 2k\xi_2 \left(\frac{\sqrt{n}}{2k} + 1\right) (2k + \sqrt{n})^{\frac{2k}{k+1}} M_0^{\frac{2k}{k+1}} \left(\int_{\Omega} |\nabla v|^{2(k-1)} |D^2 v|^2\right)^{\frac{k}{k+1}} \|D^2 w\|_{L^{k+1}} \end{aligned} \tag{3.26}$$

for all $t \in (0, T_{max})$. On the other hand, by the uniqueness of the elliptic problem

$$-\Delta w = u - \bar{u} \text{ in } \Omega, \quad \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega, \quad \int_{\Omega} w = 0, \tag{3.27}$$

the well-known $W^{2,p}$ -elliptic estimate (cf. [1, 2, 19], indeed, $\Delta^{-1} : L^p \rightarrow \bar{W}_0^{2,p}$, the subspace of $W^{2,p}$ with average zero, is a homeomorphism) shows, for some $c_5 = c_5(n, k, \Omega) > 0$, that

$$\| \Delta w \|_{L^{k+1}} \leq \sqrt{n} \| D^2 w \|_{L^{k+1}} \leq c_4 \sqrt{n} \| u - \bar{u} \|_{L^{k+1}} \leq 2c_4 \sqrt{n} \| u \|_{L^{k+1}} := c_5 \| u \|_{L^{k+1}}. \tag{3.28}$$

Then we substitute (3.28) into (3.26) to get

$$J_1 \leq 2c_5 k \xi_2 \left(\frac{1}{2k} + \frac{1}{\sqrt{n}}\right) (2k + \sqrt{n})^{\frac{2k}{k+1}} M_0^{\frac{2k}{k+1}} \left(\int_{\Omega} |\nabla v|^{2(k-1)} |D^2 v|^2\right)^{\frac{k}{k+1}} \|u\|_{L^{k+1}}. \tag{3.29}$$

Furthermore, using (2.9), (2.10) and the fact (3.28), one can derive

$$\begin{aligned} J_2 & := -2k\xi_2 \int_{\Omega} v \Delta w \nabla \cdot (|\nabla v|^{2k-2} \nabla v) \\ & \leq 2k\xi_2 (2(k-1) + \sqrt{n}) \|v\|_{L^\infty} \|\nabla v\|_{L^{2(k+1)}}^{k-1} \left(\int_{\Omega} |\nabla v|^{2k-2} |D^2 v|^2\right)^{\frac{1}{2}} \| \Delta w \|_{L^{k+1}} \end{aligned}$$

$$\begin{aligned}
 &\leq 2k\xi_2(2(k-1) + \sqrt{n})(2k + \sqrt{n})^{\frac{k-1}{k+1}} \|v\|_{L^\infty}^{\frac{2k}{k+1}} \left(\int_{\Omega} |\nabla v|^{2k-2} |D^2 v|^2 \right)^{\frac{k}{k+1}} \|\Delta w\|_{L^{k+1}} \\
 &\leq 2c_5 k \xi_2 (2(k-1) + \sqrt{n})(2k + \sqrt{n})^{\frac{k-1}{k+1}} M_0^{\frac{2k}{k+1}} \left(\int_{\Omega} |\nabla v|^{2k-2} |D^2 v|^2 \right)^{\frac{k}{k+1}} \|u\|_{L^{k+1}}.
 \end{aligned} \tag{3.30}$$

In addition, applying Hölder’s inequality together with (2.10), we have

$$\begin{aligned}
 J_3 &:= 2k(2(k-1) + \sqrt{n}) \int_{\Omega} u |\nabla v|^{2k-2} |D^2 v| \\
 &\leq 2k(2(k-1) + \sqrt{n}) \left(\int_{\Omega} u^2 |\nabla v|^{2(k-1)} \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla v|^{2(k-1)} |D^2 v|^2 \right)^{\frac{1}{2}} \\
 &\leq 2k(2(k-1) + \sqrt{n}) \|u\|_{L^{k+1}} \|\nabla v\|_{L^{2(k+1)}}^{k-1} \left(\int_{\Omega} |\nabla v|^{2(k-1)} |D^2 v|^2 \right)^{\frac{1}{2}} \\
 &\leq 2k(2(k-1) + \sqrt{n})(2k + \sqrt{n})^{\frac{k-1}{k+1}} M_0^{\frac{k-1}{k+1}} \left(\int_{\Omega} |\nabla v|^{2(k-1)} |D^2 v|^2 \right)^{\frac{k}{k+1}} \cdot \|u\|_{L^{k+1}}.
 \end{aligned} \tag{3.31}$$

Now, summing over (3.29), (3.30) and (3.31), we infer

$$J_1 + J_2 + J_3 \leq c_6 \left(\int_{\Omega} |\nabla v|^{2(k-1)} |D^2 v|^2 \right)^{\frac{k}{k+1}} \|u\|_{L^{k+1}}, \tag{3.32}$$

where

$$\begin{aligned}
 c_6 &= 2c_5 k \xi_2 \left(\frac{1}{2k} + \frac{1}{\sqrt{n}} \right) (2k + \sqrt{n})^{\frac{2k}{k+1}} M_0^{\frac{2k}{k+1}} \\
 &\quad + 2c_5 k \xi_2 (2(k-1) + \sqrt{n})(2k + \sqrt{n})^{\frac{k-1}{k+1}} M_0^{\frac{2k}{k+1}} \\
 &\quad + 2k(2(k-1) + \sqrt{n})(2k + \sqrt{n})^{\frac{k-1}{k+1}} M_0^{\frac{k-1}{k+1}} \\
 &\leq 6c_5 k \xi_2 (2k + \sqrt{n})^{\frac{2k}{k+1}} M_0^{\frac{2k}{k+1}} + 2k(2k + \sqrt{n})^{\frac{2k}{k+1}} M_0^{\frac{k-1}{k+1}} \\
 &= 2k(1 + 3c_5 \xi_2 M_0) (2k + \sqrt{n})^{\frac{2k}{k+1}} M_0^{\frac{k-1}{k+1}}.
 \end{aligned} \tag{3.33}$$

Therefore, by means of Young’s inequality, we infer from (3.32) and (3.33) that

$$\begin{aligned}
 &J_1 + J_2 + J_3 \\
 &\leq \frac{k}{2} d \int_{\Omega} |\nabla v|^{2(k-1)} |D^2 v|^2 + \frac{c_6^{k+1}}{k+1} \left(\frac{2}{d(k+1)} \right)^k \int_{\Omega} u^{k+1} \\
 &\leq \frac{k}{2} d \int_{\Omega} |\nabla v|^{2(k-1)} |D^2 v|^2 + \frac{2^{2k+1}}{d^k} (1 + 3c_5 \xi_2 M_0)^{k+1} (2k + \sqrt{n})^{2k} M_0^{k-1} \int_{\Omega} u^{k+1}.
 \end{aligned} \tag{3.34}$$

To control the boundary integral on (3.20), we first state the following well-known fact due to homogeneous Neumann conditions:

$$\frac{\partial |\nabla v|^2}{\partial \nu} \leq 2\sigma_{\Omega} |\nabla v|^2 \text{ on } \partial\Omega, \tag{3.35}$$

where $\sigma_\Omega = \sigma 1_\Omega$ with σ being the maximum curvature of $\partial\Omega$ and 1_Ω being the indicator whether Ω is non-convex defined by

$$1_\Omega = \begin{cases} 0, & \text{if } \Omega \text{ is convex [29, Lemma 5.3],} \\ 1, & \text{if } \Omega \text{ is non-convex [30, Lemma 4.2].} \end{cases}$$

Now, combining (3.35) and using the trace inequality $\|\psi\|_{L^2(\partial\Omega)} \leq \varepsilon \|\nabla\psi\|_{L^2(\Omega)} + C_\varepsilon \|\psi\|_{L^2(\Omega)}$ for any $\varepsilon > 0$ (cf. [37, Remark 52.9] and [45, (3.19)]), we have

$$\begin{aligned} kd \int_{\partial\Omega} |\nabla v|^{2k-2} \frac{\partial |\nabla v|^2}{\partial\nu} &\leq 2kd\sigma_\Omega \|\nabla v\|_{L^2(\partial\Omega)}^2 \\ &\leq k(k-1)d_\Omega \int_\Omega |\nabla v|^{2(k-2)} |\nabla |\nabla v|^2|^2 + c_7 d_\Omega \int_\Omega |\nabla v|^{2k}, \end{aligned} \quad (3.36)$$

where d_Ω is defined by (1.10). By (2.10), Young's inequality with ε and the fact $\|v(\cdot, t)\|_{L^\infty} \leq M_0$ in (3.1), we infer that

$$\begin{aligned} c_7 d_\Omega \int_\Omega |\nabla v|^{2k} &\leq \frac{kd_\Omega}{2(2k + \sqrt{n})^2 M_0^2} \int_\Omega |\nabla v|^{2(k+1)} + c_8 d_\Omega M_0^{2k} \\ &\leq \frac{k}{2} d_\Omega \int_\Omega |\nabla v|^{2(k-1)} |D^2 v|^2 + c_8 d_\Omega M_0^{2k}. \end{aligned} \quad (3.37)$$

Finally, substituting (3.34), (3.36) and (3.37) into (3.20) and then keeping key parameters like ξ_2 and M_0 , we accomplish our desired estimate (3.25). \square

LEMMA 3.5. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) and (u, v, w) be the solution of (1.4) obtained in Lemma 2.1. Then, for any $k \geq 1$, there exist two positive constants $\xi_*(k)$ and $\mu_*(k)$ defined respectively by (3.52) and (3.57) such that whenever one of the following conditions holds:*

$$(1) \xi_1 \geq \xi_*(k)\chi^2; \quad (2) \theta = 1, \mu \geq \max\left\{1, \chi^{\frac{8+2n}{5+n}}\right\} \mu_*(k)\chi^{\frac{2}{5+n}}; \quad (3) \theta > 1, \mu > 0, \quad (3.38)$$

there exist three positive constants C_2, C_3, C_4 independent of t, χ, μ, d and ξ_i ($i = 1, 2$) but depending on $u_0, v_0, \theta, a, k, n$ and Ω such that, for $t \in (0, T_{max})$,

$$\int_\Omega u^k \leq C_2 (1 + m_1^k + M_1(k) + (1 + d_\Omega M_0^{2k}) d^{-1} \chi^2 M_0^{2-k} + M_2^c(k)) := C_2 M_2(k), \quad (3.39)$$

and

$$\int_\Omega |\nabla v|^{2k} \leq \frac{C_3 M_2(k)}{(1 + M_0 \xi_2)^{-k} d^{-1} \chi^2 M_0^{2-k}}, \quad (3.40)$$

as well as, if $k > n$,

$$\|w(\cdot, t)\|_{W^{1,\infty}} \leq C_4 M_2^{\frac{1}{k}}(k), \quad (3.41)$$

where M_2^c is defined by

$$M_2^c(k) := \begin{cases} 0, & \text{if } \xi_1 \geq \xi_*(k)\chi^2, \\ 0, & \theta = 1, \text{ if } \mu \geq \max\left\{1, \chi^{\frac{8+2n}{5+n}}\right\} \mu_*(k)\chi^{\frac{2}{5+n}}, \\ \frac{(\theta-1)}{\mu^{\frac{k+1}{\theta-1}}} \left[\left(1 + \frac{1}{d^{k+1}}\right) (1 + M_0 \xi_2) M_0 \chi^2 \right]^{\frac{k+\theta}{\theta-1}}, & \text{if } \theta > 1, \mu > 0. \end{cases} \quad (3.42)$$

Proof. We apply the estimate (2.10) and the fact $\|v(\cdot, t)\|_{L^\infty} \leq M_0$ to deduce that

$$\begin{aligned} \frac{\chi^{2k(k-1)}}{2} \int_{\Omega} u^k |\nabla v|^2 &\leq \varepsilon_1 \int_{\Omega} u^{k+1} + c_9 \chi^{2(k+1)} \varepsilon_1^{-k} \int_{\Omega} |\nabla v|^{2(k+1)} \\ &\leq \varepsilon_1 \int_{\Omega} u^{k+1} + c_{10} \chi^{2(k+1)} M_0^2 \varepsilon_1^{-k} \int_{\Omega} |\nabla v|^{2(k-1)} |D^2 v|^2, \end{aligned} \tag{3.43}$$

where $\varepsilon_1 > 0$ to be chosen in (3.47) below. On the one hand, using the widely known Gagliardo-Nirenberg inequality (cf. [40, Section 3]), we can find a constant $c_{11} > 0$ only depending on k, n and Ω and $\alpha = \frac{\frac{k}{2} - \frac{1}{2}}{\frac{k}{2} - \frac{1}{2} + \frac{1}{n}} \in (0, 1)$ such that

$$\begin{aligned} \int_{\Omega} u^k &= \|u^{\frac{k}{2}}\|_{L^2}^2 \leq c_{11} (\|\nabla u^{\frac{k}{2}}\|_{L^2}^{2\alpha} \|u^{\frac{k}{2}}\|_{L^{\frac{2}{k}}}^{2(1-\alpha)} + \|u^{\frac{k}{2}}\|_{L^{\frac{2}{k}}}^2) \\ &\leq c_{11} \left(\frac{k^2}{4} \int_{\Omega} u^{k-2} |\nabla u|^2 \right)^\alpha m_1^{k(1-\alpha)} + c_{11} m_1^k \\ &\leq \frac{k(k-1)}{2} \int_{\Omega} u^{k-2} |\nabla u|^2 + c_{12} m_1^k. \end{aligned} \tag{3.44}$$

Then substituting (3.43) and (3.44) into (3.14) with $p = k$, one has

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u^k + \int_{\Omega} u^k + \frac{\xi_1(k-1)}{2} \int_{\Omega} u^{k+1} + \frac{k\mu}{2} \int_{\Omega} u^{\theta+k} \\ \leq \varepsilon_1 \int_{\Omega} u^{k+1} + c_{10} \chi^{2(k+1)} M_0^2 \varepsilon_1^{-k} \int_{\Omega} |\nabla v|^{2(k-1)} |D^2 v|^2 + c_{12} m_1^k + M_1(k). \end{aligned} \tag{3.45}$$

On the other hand, it follows from Lemma 3.4 and the fact $d_\Omega \leq d$ that

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2k} + 2k \int_{\Omega} |\nabla v|^{2k} + kd \int_{\Omega} |\nabla v|^{2k-2} |D^2 v|^2 \\ \leq \frac{C_1}{d^k} (1 + M_0 \xi_2)^{k+1} M_0^{k-1} \int_{\Omega} u^{k+1} + C_1 d_\Omega M_0^{2k}. \end{aligned} \tag{3.46}$$

Let us first set

$$c_{13} = \frac{C_1 c_{10}}{k}, \quad \varepsilon_1 = (1 + M_0 \xi_2) M_0 \chi^2 (k c_{13})^{\frac{1}{k+1}}, \tag{3.47}$$

and then we put

$$\delta(\varepsilon_1) = \frac{c_{10} \chi^{2(k+1)} M_0^2 \varepsilon_1^{-k}}{kd} = \frac{c_{10}}{k} (1 + M_0 \xi_2)^{-k} d^{-1} \chi^2 M_0^{2-k} (k c_{13})^{-\frac{k}{k+1}}, \tag{3.48}$$

and

$$y(t) := \int_{\Omega} u^k + \delta(\varepsilon_1) \int_{\Omega} |\nabla v|^{2k}. \tag{3.49}$$

Then we multiply (3.46) by $\delta(\varepsilon_1)$ and add the result to (3.45) to obtain

$$\begin{aligned} y'(t) + y(t) + \frac{\xi_1(k-1)}{2} \int_{\Omega} u^{k+1} + \frac{k\mu}{2} \int_{\Omega} u^{\theta+k} \\ \leq \left\{ c_{13} \left(\frac{1}{d} (1 + M_0 \xi_2) M_0 \chi^2 \right)^{k+1} \varepsilon_1^{-k} + \varepsilon_1 \right\} \int_{\Omega} u^{k+1} + c_{14} \end{aligned}$$

$$= (kc_{13})^{\frac{1}{k+1}} \left(1 + \frac{1}{kd^{k+1}} \right) (1 + M_0\xi_2)M_0\chi^2 \int_{\Omega} u^{k+1} + c_{14}, \tag{3.50}$$

where

$$c_{14} = c_{12}m_1^k + M_1(k) + \delta(\varepsilon_1)C_1d_{\Omega}M_0^{2k}. \tag{3.51}$$

To show that the first term on the right-hand side of (3.50) can be absorbed by the terms involving $\int_{\Omega} u^{k+1}$ or $\int_{\Omega} u^{\theta+k}$, we first define

$$\xi_*(k) = \frac{2(kc_{13})^{\frac{1}{k+1}}}{k-1} \left(1 + \frac{1}{kd^{k+1}} \right) (1 + M_0\xi_2)M_0. \tag{3.52}$$

Case 1: $\xi_1 \geq \xi_*(k)\chi^2$. In this case, notice that M_0 is independent of ξ_1 by (3.1) and (2.2), in view of (3.50) along with (3.51) and (3.52), we deduce that

$$y'(t) + y(t) \leq c_{12}m_1^k + M_1(k) + \delta(\varepsilon_1)C_1d_{\Omega}M_0^{2k}. \tag{3.53}$$

Solving this ODI and using (3.48) and (3.49), we get that

$$\begin{aligned} & \int_{\Omega} u^k + \frac{c_{10}}{k} (1 + M_0\xi_2)^{-k} d^{-1} \chi^2 M_0^{2-k} (kc_{13})^{-\frac{k}{k+1}} \int_{\Omega} |\nabla v|^{2k} \\ & \leq \|u_0\|_{L^k}^k + c_{15} (1 + M_0\xi_2)^{-k} d^{-1} \chi^2 M_0^{2-k} \|\nabla v_0\|_{L^{2k}}^{2k} \\ & \quad + c_{12}m_1^k + M_1(k) + c_{15} (1 + M_0\xi_2)^{-k} d^{-1} \chi^2 M_0^{k+2} d_{\Omega} \\ & \leq c_{16} [1 + m_1^k + M_1(k) + (1 + d_{\Omega}M_0^{2k})d^{-1} \chi^2 M_0^{2-k}]. \end{aligned} \tag{3.54}$$

Case 2: $\theta=1$ and $\mu > 0$ is suitably large alone. In this case, we first observe from (3.1) and (2.2) that M_0 is bounded by $O(1)(1 + \mu^{-(2+n/2)})$, and therefore,

$$(1 + M_0\xi_2)M_0 \leq O(1) \left(1 + \left(\frac{1}{\mu} \right)^{4+n} \right).$$

This enables us to infer that

$$\hat{\mu}_*(k) = \frac{2(kc_{13})^{\frac{1}{k+1}}}{k} \left(1 + \frac{1}{kd^{k+1}} \right) \sup_{0 < \mu < 1} \{ \mu^{4+n} (1 + M_0\xi_2)M_0 \} < +\infty \tag{3.55}$$

and

$$\tilde{\mu}_*(k) = \frac{2(kc_{13})^{\frac{1}{k+1}}}{k} \left(1 + \frac{1}{kd^{k+1}} \right) \sup_{\mu \geq 1} \{ (1 + M_0\xi_2)M_0 \} < +\infty. \tag{3.56}$$

Now, we define

$$\mu_*(k) = \max \left\{ (\hat{\mu}_*(k))^{\frac{1}{5+n}}, \tilde{\mu}_*(k) \right\} < +\infty. \tag{3.57}$$

Then, under (2) of (3.38), we see from (3.57) that

$$\mu \geq \max \left\{ 1, \chi^{\frac{8+2n}{5+n}} \right\} \mu_*(k) \chi^{\frac{2}{5+n}} \geq \max \left\{ (\hat{\mu}_*(k))^{\frac{1}{5+n}} \chi^{\frac{2}{5+n}}, \tilde{\mu}_*(k) \chi^2 \right\},$$

and so (3.55) together with (3.56) implies

$$\mu \geq \frac{2(kc_{13})^{\frac{1}{k+1}}}{k} \left(1 + \frac{1}{kd^{k+1}}\right) (1 + M_0\xi_2)M_0\chi^2. \tag{3.58}$$

Then in light of (3.50) along with (3.51) and (3.58), we derive an identical ODI as (3.53) for any $\xi_1 \geq 0$, and then we get the same estimate as (3.54).

Case 3: $\theta > 1$ and $\mu > 0$. Let

$$\begin{aligned} c_{17} &:= \sup_{s>0} \left\{ (kc_{13})^{\frac{1}{k+1}} \left(1 + \frac{1}{kd^{k+1}}\right) (1 + M_0\xi_2)M_0\chi^2 s^{k+1} - \frac{k\mu}{2} s^{k+\theta} \right\} \\ &= \frac{(\theta - 1)}{k + \theta} \left(\frac{2(k+1)}{k(k+\theta)\mu} \right)^{\frac{k+1}{\theta-1}} \left[(kc_{13})^{\frac{1}{k+1}} \left(1 + \frac{1}{kd^{k+1}}\right) (1 + M_0\xi_2)M_0\chi^2 \right]^{\frac{k+\theta}{\theta-1}}. \end{aligned} \tag{3.59}$$

Then, combining (3.59) and (3.50) along with (3.51), we end up with

$$y'(t) + y(t) \leq c_{12}m_1^k + M_1(k) + \delta(\varepsilon_1)C_1d_\Omega M_0^{2k} + c_{17}|\Omega|,$$

which in conjunction with (3.54), (3.59) and (3.48) allows us to deduce that

$$\begin{aligned} &\int_\Omega u^k + \frac{c_{10}}{k} (1 + M_0\xi_2)^{-k} d^{-1} \chi^2 M_0^{2-k} (kc_{13})^{-\frac{k}{k+1}} \int_\Omega |\nabla v|^{2k} \\ &\leq c_{16} [1 + m_1^k + M_1(k) + (1 + d_\Omega M_0^{2k})d^{-1} \chi^2 M_0^{2-k}] \\ &\quad + c_{18} \frac{(\theta - 1)}{\mu^{\frac{k+1}{\theta-1}}} \left[\left(1 + \frac{1}{d^{k+1}}\right) (1 + M_0\xi_2)M_0\chi^2 \right]^{\frac{k+\theta}{\theta-1}}. \end{aligned} \tag{3.60}$$

The desired qualitative (L^k, L^{2k}) -bounds of $(u, \nabla v)$ in (3.39) and (3.40) result from (3.54) and (3.60) upon simple algebraic manipulations. Finally, if $k > n$, then the $W^{2,k}$ -estimate (3.28) and the Sobolev embedding $W^{2,k} \hookrightarrow W^{1,\infty}$ together imply (3.41). □

3.2. Qualitative $(L^\infty, W^{1,\infty})$ -boundedness of (u, v) . Before proceeding, based on (3.52) and (3.57), we first define

$$\xi_0 = \inf_{k>\frac{n}{2}} \xi_*(k) = \inf_{k>\frac{n}{2}} \left\{ \frac{2(kc_{13})^{\frac{1}{k+1}}}{k-1} \left(1 + \frac{1}{kd^{k+1}}\right) \right\} (1 + M_0\xi_2)M_0 \tag{3.61}$$

and, with $\hat{\mu}_*(k)$ and $\tilde{\mu}_*(k)$ defined by (3.55) and (3.56),

$$\mu_0 = \inf_{k>\frac{n}{2}} \mu_*(k) = \inf_{k>\frac{n}{2}} \left\{ \max \left\{ (\hat{\mu}_*(k))^{\frac{1}{5+n}}, \tilde{\mu}_*(k) \right\} \right\}. \tag{3.62}$$

LEMMA 3.6. *Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ and (u, v, w) be the local solution of (1.4) obtained in Lemma 2.1. Suppose that one of the following conditions holds:*

$$(1) \xi_1 > \xi_0\chi^2; \quad (2) \theta = 1, \mu > \max \left\{ 1, \chi^{\frac{8+2n}{5+n}} \right\} \mu_0\chi^{\frac{2}{5+n}}; \quad (3) \theta > 1, \mu > 0. \tag{3.63}$$

Then the L^∞ -norm of the u -solution component is uniformly bounded on $(0, T_{max})$. In particular, if one of the following conditions holds:

$$(1) \xi_1 \geq \xi_*(3n)\chi^2; \quad (2) \theta = 1, \mu \geq \max \left\{ 1, \chi^{\frac{8+2n}{5+n}} \right\} \mu_*(3n)\chi^{\frac{2}{5+n}}; \quad (3) \theta > 1, \mu > 0, \tag{3.64}$$

then there exists $C_5 > 0$ independent of t, χ, μ, d and $\xi_i (i = 1, 2)$ but depending on n, u_0, v_0 and Ω such that the local solution (u, v, w) of (1.4) obtained in Lemma 2.1 satisfies, for $t \in (0, T_{max})$, that

$$\begin{aligned} & \|u(t)\|_{L^\infty} \leq C_5 M_3(3n) \\ & := C_5 \begin{cases} 1 + M_2^{\frac{1}{3n}}(3n) + \xi_1 M_2^{\frac{2}{3n}}(3n) + \frac{\chi M_2^{\frac{2}{3n}}(3n)}{((1 + M_0 \xi_2)^{-3n} d^{-1} \chi^2 M_0^{2-3n})^{\frac{1}{3n}}}, & \text{if } a = \mu = 0, \\ 1 + \left(\frac{1}{\mu}\right)^{\frac{1}{\theta}} + \xi_1 M_2^{\frac{2}{3n}}(3n) + \frac{\chi M_2^{\frac{2}{3n}}(3n)}{((1 + M_0 \xi_2)^{-3n} d^{-1} \chi^2 M_0^{2-3n})^{\frac{1}{3n}}}, & \text{if } a, \mu > 0, \end{cases} \end{aligned} \tag{3.65}$$

where $M_2(\cdot)$ is defined by (3.42).

Proof. By the definitions of ξ_0 and μ_0 respectively in (3.61) and (3.62), it is easy to see that the corresponding case of (3.63) implies that of (3.38) for some $k > \frac{n}{2}$. Then an application of Lemma 3.5 shows

$$\|u(\cdot, t)\|_{L^k} \leq c_{19} M_2^{\frac{1}{k}}(k), \quad \|\nabla v(\cdot, t)\|_{L^{2k}} \leq c_{20} \left(\frac{M_2(k)}{(1 + M_0 \xi_2)^{-k} d^{-1} \chi^2 M_0^{2-k}} \right)^{\frac{1}{2k}}. \tag{3.66}$$

Therefore, the $W^{2,k}$ -estimate (3.28) yields

$$\|w(\cdot, t)\|_{W^{2,k}} \leq c_{20} \|u(\cdot, t) - \bar{u}\|_{L^k} \leq 2c_{20} \|u(\cdot, t)\|_{L^k} \leq 2c_{19} c_{20} M_2^{\frac{1}{k}}(k),$$

and so the Sobolev embedding $W^{2,k} \hookrightarrow W^{1,q}$ for some $q > n$ shows

$$\|w(\cdot, t)\|_{W^{1,q}} \leq c_{21} M_2^{\frac{1}{k}}(k), \quad q = \frac{4nk}{k + 3(n - k)^+}. \tag{3.67}$$

We now use similar spirits as used in [17, 45] to derive the L^∞ -bound of u . To that purpose, we use the variation-of-constants formula to the u -equation in (1.4) to write

$$\begin{aligned} u(\cdot, t) &= e^{t(\Delta-1)} u_0 + \int_0^t e^{(\Delta-1)(t-s)} \nabla \cdot \{u(\cdot, s)(\xi_1 \nabla w(\cdot, s) - \chi \nabla v(\cdot, s))\} ds \\ &\quad + \int_0^t e^{(\Delta-1)(t-s)} [(a+1)u(\cdot, s) - \mu u^{\theta+1}(\cdot, s)] ds \\ &=: u_1(\cdot, t) + u_2(\cdot, t) + u_3(\cdot, t). \end{aligned}$$

We next estimate u_1, u_2 and u_3 . First, the nonnegativity of u shows, for $t \in (0, T_{max})$,

$$\|u(\cdot, t)\|_{L^\infty} = \sup_{x \in \Omega} u(x, t) \leq \sup_{x \in \Omega} u_1(x, t) + \sup_{x \in \Omega} u_2(x, t) + \sup_{x \in \Omega} u_3(x, t). \tag{3.68}$$

By the order property of the Neumann heat semigroup $(e^{t\Delta})_{t>0}$ due to the maximum principle, we estimate u_1 and u_3 in the following ways:

$$\|u_1(\cdot, t)\|_{L^\infty} = \|e^{t(\Delta-1)} u_0\|_{L^\infty} \leq e^{-t} \|u_0\|_{L^\infty} \leq \|u_0\|_{L^\infty}, \tag{3.69}$$

and, the semigroup estimate (2.13) in Lemma 2.5 and (3.66) yield

$$u_3(\cdot, t) = \int_0^t e^{(\Delta-1)(t-s)} [(a+1)u(\cdot, s) - \mu u^{\theta+1}(\cdot, s)] ds$$

$$\begin{aligned}
 &\leq \begin{cases} \int_0^t e^{(\Delta-1)(t-s)} u(\cdot, s) ds, & \text{if } a = \mu = 0, \\ \int_0^t e^{(\Delta-1)(t-s)} \left(\frac{a+1}{(1+\theta)\mu}\right)^{\frac{1}{\theta}} \frac{(a+1)\theta}{1+\theta} ds, & \text{if } a \geq 0, \mu > 0, \end{cases} \\
 &\leq c_{21} \begin{cases} M_2^{\frac{1}{k}}(k) \int_0^t (1+(t-s)^{-\frac{n}{2k}}) e^{-(t-s)} ds, & \text{if } a = \mu = 0, \\ \int_0^t e^{-(t-s)} e^{\Delta(t-s)} \left(\frac{1}{\mu}\right)^{\frac{1}{\theta}} ds, & \text{if } a \geq 0, \mu > 0, \end{cases} \\
 &\leq c_{22} \begin{cases} M_2^{\frac{1}{k}}(k), & \text{if } a = \mu = 0, \\ \left(\frac{1}{\mu}\right)^{\frac{1}{\theta}}, & \text{if } a \geq 0, \mu > 0. \end{cases} \tag{3.70}
 \end{aligned}$$

For convenience of reference, we list two Hölder-type interpolation inequalities:

$$\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q} \text{ with } p, q, r \geq 1, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q},$$

and

$$\|f\|_{L^r} \leq \|f\|_{L^\infty}^{\frac{(r-k)^+}{r}} \|f\|_{L^k}^{\frac{k+(r-k)^+}{k}} |\Omega|^{\frac{(k-r)^+}{kr}} \text{ with } k, r \geq 1.$$

Employing the semigroup estimate (2.14) in Lemma 2.5, Hölder’s interpolation inequalities above and (3.67), we deduce, for $k > \frac{n}{2}$, that

$$\begin{aligned}
 &\left\| \xi_1 \int_0^t e^{(\Delta-1)(t-s)} \nabla \cdot (u \nabla w) ds \right\|_{L^\infty} \\
 &\leq c_{23} \xi_1 \int_0^t \left(1+(t-s)^{-1+\frac{k-(n-k)^+}{8k}}\right) e^{-(t-s)} \|u \nabla w\|_{L^{\frac{4nk}{3k+(n-k)^+}}} ds \\
 &\leq c_{24} \xi_1 \int_0^t \left(1+(t-s)^{-1+\frac{k-(n-k)^+}{8k}}\right) e^{-(t-s)} \|u\|_{L^{\frac{2nk}{k-(n-k)^+}}} \|\nabla w\|_{L^{\frac{4nk}{k+3(n-k)^+}}} ds \\
 &\leq c_{25} \xi_1 M_2^{\frac{2k-2(n-k)^++(2n-k+(n-k)^+)^+}{[k-(n-k)^++(2n-k+(n-k)^+)^+]^k}}(k) \left(\sup_{s \in (0,t)} \|u(s)\|_{L^\infty}\right)^{\frac{(2n-k+(n-k)^+)^+}{2n}} \\
 &= c_{25} \xi_1 \begin{cases} M_2^{\frac{2n+k-(n-k)^+}{2nk}}(k) \left(\sup_{s \in (0,t)} \|u(s)\|_{L^\infty}\right)^{\frac{2n-k+(n-k)^+}{2n}}, & \text{if } k < 2n, \\ M_2^{\frac{2}{k}}(k), & \text{if } k \geq 2n, \end{cases} \\
 &\leq \begin{cases} \frac{1}{4} \sup_{s \in (0,t)} \|u(s)\|_{L^\infty} + c_{26} \xi_1^{\frac{2n}{k-(n-k)^+}} M_2^{\frac{2n+k-(n-k)^+}{[k-(n-k)^+]^k}}(k), & \text{if } k < 2n, \\ c_{25} \xi_1 M_2^{\frac{2}{k}}(k), & \text{if } k \geq 2n, \end{cases} \tag{3.71}
 \end{aligned}$$

where we used the fact that

$$\begin{aligned}
 &\|u\|_{L^{\frac{2nk}{k-(n-k)^+}}} \|\nabla w\|_{L^{\frac{4nk}{k+3(n-k)^+}}} \\
 &\leq c_{27} \|u\|_{L^k}^{\frac{k-(n-k)^+}{k-(n-k)^++(2n-k+(n-k)^+)^+}} \|u\|_{L^\infty}^{\frac{(2n-k+(n-k)^+)^+}{2n}} \|\nabla w\|_{L^{\frac{4nk}{k+3(n-k)^+}}}
 \end{aligned}$$

$$\leq c_{28} M_2^{\frac{2k-2(n-k)^+ + (2n-k+(n-k)^+)^+}{k-(n-k)^+ + (2n-k+(n-k)^+)^+}} \left\| u \right\|_{L^\infty}^{\frac{(2n-k+(n-k)^+)^+}{2n}}$$

and the finiteness of gamma integral due to the fact that $k - (n - k)^+ > 0$:

$$\int_0^t \left(1 + (t-s)^{-1 + \frac{k-(n-k)^+}{8k}} \right) e^{-(t-s)} ds = \int_0^t \left(1 + \tau^{-1 + \frac{k-(n-k)^+}{8k}} \right) e^{-\tau} d\tau < \infty.$$

Similarly, using the boundedness information in (3.66), we infer

$$\begin{aligned} & \left\| -\chi \int_0^t e^{(\Delta-1)(t-s)} \nabla \cdot (u \nabla v)(\cdot, s) ds \right\|_{L^\infty} \\ & \leq c_{29} \chi \int_0^t \left(1 + (t-s)^{-1 + \frac{2k-n}{8k}} \right) e^{-(t-s)} \left\| (u \nabla v)(\cdot, s) \right\|_{L^{\frac{4nk}{2k+n}}} ds \\ & \leq c_{30} \chi \int_0^t \left(1 + (t-s)^{-1 + \frac{2k-n}{8k}} \right) e^{-(t-s)} \left\| u(\cdot, s) \right\|_{L^{\frac{4nk}{2k-n}}} \left\| \nabla v(\cdot, s) \right\|_{L^{2k}} ds \\ & \leq c_{31} \frac{\chi M_2^{\frac{4k-2n+(5n-2k)^+}{k[2k-n+(5n-2k)^+]}}(k)}{\left((1+M_0\xi_2)^{-k} d^{-1} \chi^2 M_0^{2-k} \right)^{\frac{1}{k}}} \left(\sup_{s \in (0,t)} \left\| u(\cdot, s) \right\|_{L^\infty} \right)^{\frac{(5n-2k)^+}{4n}} \\ & \leq \begin{cases} \frac{1}{4} \sup_{s \in (0,t)} \left\| u(s) \right\|_{L^\infty} + c_{32} \frac{\chi^{\frac{4n}{2k-n}} M_2^{\frac{2k+3n}{k(2k-n)}}(k)}{\left((1+M_0\xi_2)^{-k} d^{-1} \chi^2 M_0^{2-k} \right)^{\frac{4n}{k(2k-n)}}}, & \text{if } k < \frac{5n}{2}, \\ c_{31} \frac{\chi M_2^{\frac{2}{k}}(k)}{\left((1+M_0\xi_2)^{-k} d^{-1} \chi^2 M_0^{2-k} \right)^{\frac{1}{k}}}, & \text{if } k \geq \frac{5n}{2}. \end{cases} \end{aligned} \tag{3.72}$$

In the case of $k \geq \frac{5n}{2}$, substituting (3.69), (3.70), (3.71) and (3.72) into (3.68), we accomplish that

$$\left\| u(t) \right\|_{L^\infty} \leq c_{32} \begin{cases} 1 + M_2^{\frac{1}{k}}(k) + \xi_1 M_2^{\frac{2}{k}}(k) + \frac{\chi M_2^{\frac{2}{k}}(k)}{\left((1+M_0\xi_2)^{-k} d^{-1} \chi^2 M_0^{2-k} \right)^{\frac{1}{k}}}, & \text{if } \mu = 0, \\ 1 + \left(\frac{1}{\mu} \right)^{\frac{1}{\theta}} + \xi_1 M_2^{\frac{2}{k}}(k) + \frac{\chi M_2^{\frac{2}{k}}(k)}{\left((1+M_0\xi_2)^{-k} d^{-1} \chi^2 M_0^{2-k} \right)^{\frac{1}{k}}}, & \text{if } \mu > 0. \end{cases} \tag{3.73}$$

Similarly, in the case of $k \in [2n, \frac{5n}{2})$, we have

$$\left\| u(t) \right\|_{L^\infty} \leq c_{33} \begin{cases} 1 + M_2^{\frac{1}{k}}(k) + \xi_1 M_2^{\frac{2}{k}}(k) + \frac{\chi^{\frac{4n}{2k-n}} M_2^{\frac{2k+3n}{k(2k-n)}}(k)}{\left((1+M_0\xi_2)^{-k} d^{-1} \chi^2 M_0^{2-k} \right)^{\frac{4n}{k(2k-n)}}}, & \text{if } \mu = 0, \\ 1 + \left(\frac{1}{\mu} \right)^{\frac{1}{\theta}} + \xi_1 M_2^{\frac{2}{k}}(k) + \frac{\chi^{\frac{4n}{2k-n}} M_2^{\frac{2k+3n}{k(2k-n)}}(k)}{\left((1+M_0\xi_2)^{-k} d^{-1} \chi^2 M_0^{2-k} \right)^{\frac{4n}{k(2k-n)}}}, & \text{if } \mu > 0. \end{cases} \tag{3.74}$$

And in the case of $k \in (\frac{n}{2}, 2n)$, we get

$$\begin{aligned} \left\| u(t) \right\|_{L^\infty} & \leq c_{34} \left(1 + \frac{\chi^{\frac{4n}{2k-n}} M_2^{\frac{2k+3n}{k(2k-n)}}(k)}{\left((1+M_0\xi_2)^{-k} d^{-1} \chi^2 M_0^{2-k} \right)^{\frac{4n}{k(2k-n)}}} \right) \\ & \quad + c_{34} \xi_1^{\frac{2n}{k-(n-k)^+}} M_2^{\frac{2n+k-(n-k)^+}{[k-(n-k)^+]k}}(k) + c_{34} \begin{cases} M_2^{\frac{1}{k}}(k), & \text{if } \mu = 0, \\ \left(\frac{1}{\mu} \right)^{\frac{1}{\theta}}, & \text{if } \mu > 0. \end{cases} \end{aligned} \tag{3.75}$$

Combining (3.73), (3.74) and (3.75), we see either one of (3.63) implies that $\|u(\cdot, t)\|_{L^\infty}$ is uniformly bounded for $t \in (0, T_{\max})$. In particular, if one of (3.64) holds, then the desired qualitative bound of u in (3.65) follows upon setting $k = 3n$ in (3.73). \square

LEMMA 3.7. *Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ and (u, v, w) be the solution of (1.4) obtained in Lemma 2.1. If one of (3.63) holds, then $\|\nabla v(\cdot, t)\|_{L^\infty}$ is uniformly bounded on $(0, T_{\max})$. In particular, if one of the following conditions holds:*

$$(1) \xi_1 \geq \xi_*(3n)\chi^2; \quad (2) \theta = 1, \mu \geq \max\left\{1, \chi^{\frac{8+2n}{5+n}}\right\} \mu_*(3n)\chi^{\frac{2}{5+n}}; \quad (3) \theta > 1, \mu > 0, \quad (3.76)$$

then, for $t \in (0, T_{\max})$, there exists $C_6 > 0$ depending only on n, u_0, v_0 and Ω such that

$$\begin{aligned} & \|\nabla v(\cdot, t)\|_{L^\infty} \\ & \leq C_6 \left(1 + \frac{\xi_2 M_2^{\frac{1}{2n}}(3n)}{d((1 + M_0 \xi_2)^{-3n} d^{-1} \chi^2 M_0^{2-3n})^{\frac{1}{6n}}} + \frac{\xi_2}{d} M_0 M_2^{\frac{1}{3n}}(3n) + \frac{M_2^{\frac{1}{3n}}(3n)}{d} \right), \end{aligned} \quad (3.77)$$

where M_0 and $M_2(\cdot)$ are defined by (3.1) and (3.39), respectively.

Proof. Using the time scaling $\tilde{t} = dt$, we rewrite the second equation in (1.4) as follows:

$$v_{\tilde{t}} = \Delta v - \frac{1}{d}v + \frac{\xi_2}{d} \nabla w \cdot \nabla v + \frac{\xi_2}{d} v(\bar{u} - u) + \frac{1}{d}u.$$

Then an application of the variation-of-constants formula yields

$$\begin{aligned} v(\cdot, \tilde{t}) &= e^{(\Delta - \frac{1}{d})\tilde{t}} v_0 + \frac{\xi_2}{d} \int_0^{\tilde{t}} e^{(\Delta - \frac{1}{d})(\tilde{t}-s)} (\nabla w \cdot \nabla v)(\cdot, s) ds \\ &+ \frac{\xi_2}{d} \int_0^{\tilde{t}} e^{(\Delta - \frac{1}{d})(\tilde{t}-s)} v(\cdot, s) (\bar{u}(s) - u(\cdot, s)) ds + \frac{1}{d} \int_0^{\tilde{t}} e^{(\Delta - \frac{1}{d})(\tilde{t}-s)} u(\cdot, s) ds. \end{aligned} \quad (3.78)$$

Under one of (3.63), we first see from Lemma 3.6 and its proof, for some $k > \frac{n}{2}$, that

$$\|u(\cdot, \tilde{t})\|_{L^\infty} + \|\nabla v(\cdot, \tilde{t})\|_{L^{2k}} \leq c_{35}, \quad \forall \tilde{t} > 0. \quad (3.79)$$

Then the elliptic estimate applied to $\Delta w = \bar{u} - u$ along with Lemma 3.1 shows that

$$\|v(\cdot, \tilde{t})\|_{L^\infty} + \|\nabla w(\cdot, \tilde{t})\|_{L^\infty} \leq c_{36}, \quad \forall \tilde{t} > 0. \quad (3.80)$$

Then, with (3.79) and (3.80) at hand, we can apply the smoothing L^p - L^q semigroup estimates in Lemma 2.5 to (3.78) to infer that $\|\nabla v(\cdot, t)\|_{L^\infty}$ is uniformly bounded. We next aim to derive a qualitative bound for $\|\nabla v(\cdot, t)\|_{L^\infty}$ under one of (3.76). In this circumstance, Lemma 3.5 allows us to see, for $\tilde{t} \in (0, dT_{\max})$, that

$$\|u(\cdot, \tilde{t})\|_{L^{3n}} \leq c_{37} M_2^{\frac{1}{3n}}(3n), \quad \|\nabla v(\cdot, \tilde{t})\|_{L^{6n}} \leq \frac{c_{38} M_2^{\frac{1}{6n}}(3n)}{((1 + M_0 \xi_2)^{-3n} d^{-1} \chi^2 M_0^{2-3n})^{\frac{1}{6n}}}. \quad (3.81)$$

Next, with the aid of (3.81) and (3.41) with $k = 3n$, and keeping in mind that $\|v(\cdot, \tilde{t})\|_{L^\infty} \leq M_0$ by Lemma 3.1, we utilize the Hölder inequality and the semigroup estimates in Lemma 2.5 to (3.78) to deduce that

$$\|\nabla v(\cdot, \tilde{t})\|_{L^\infty} \leq \|\nabla e^{(\Delta - \frac{1}{d})\tilde{t}} v_0\|_{L^\infty} + \frac{\xi_2}{d} \int_0^{\tilde{t}} \|\nabla e^{(\Delta - \frac{1}{d})(\tilde{t}-s)} (\nabla w \cdot \nabla v)(\cdot, s)\|_{L^\infty} ds$$

$$\begin{aligned}
 & + \frac{\xi_2}{d} \int_0^{\tilde{t}} \|\nabla e^{(\Delta - \frac{1}{d})(\tilde{t}-s)} v(\cdot, s) (\bar{u}(s) - u(\cdot, s))\|_{L^\infty} ds \\
 & + \frac{1}{d} \int_0^{\tilde{t}} \|\nabla e^{(\Delta - \frac{1}{d})(\tilde{t}-s)} u(\cdot, s)\|_{L^\infty} ds \\
 & \leq c_{40} \|\nabla v_0\|_{L^\infty} + \frac{c_{40} \xi_2}{d} \int_0^{\tilde{t}} (1 + (\tilde{t} - s)^{-\frac{2}{3}}) e^{-\lambda_1(\tilde{t}-s)} \|\nabla v\|_{L^{6n}} \|\nabla w\|_{L^{6n}} ds \\
 & + \frac{c_{40} \xi_2}{d} \int_0^{\tilde{t}} (1 + (\tilde{t} - s)^{-\frac{2}{3}}) e^{-\lambda_1(\tilde{t}-s)} \|v\|_{L^\infty} \|\bar{u} - u\|_{L^{3n}} ds \\
 & + \frac{c_{40}}{d} \int_0^{\tilde{t}} (1 + (\tilde{t} - s)^{-\frac{2}{3}}) e^{-\lambda_1(\tilde{t}-s)} \|u\|_{L^{3n}} ds \\
 & \leq c_{41} \left(1 + \frac{\xi_2 M_2^{\frac{1}{3n}} (3n)}{d ((1 + M_0 \xi_2)^{-3n} d^{-1} \chi^2 M_0^{2-3n})^{\frac{1}{6n}}} + \frac{(1 + \xi_2 M_0)}{d} M_2^{\frac{1}{3n}} (3n) \right),
 \end{aligned}$$

which is our desired qualitative bound in (3.77) since $\tilde{t} \in (0, dT_{\max})$ was arbitrary. \square

Proof of Theorem 1.1. Based on the extensibility criterion (2.1) in Lemma 2.1, the qualitative boundedness described in Theorem 1.1 can be traced out from Lemma 3.1, Lemma 3.6 and Lemma 3.7. The qualitative bound for $\|v\|_{L^\infty}$ in (1.7) can be easily seen from Lemma 3.1. The bounds for $\|\nabla w\|_{L^\infty}$ in (1.8) come mainly from Lemma 3.5 with $k = n + 1$.

4. Higher order regularity of solutions

In Section 3, we have established the qualitative boundedness and thus global existence of solutions. To study large-time behavior of global bounded solutions, we need further to enhance regularity properties of bounded solutions.

LEMMA 4.1. *Let (u, v, w) be the global and bounded classical solution of (1.4) obtained in Theorem 1.1. Then there exist $\sigma \in (0, 1)$ and $C_7 > 0$ such that*

$$\|u(\cdot, t)\|_{C^{\sigma, \frac{\sigma}{2}}(\Omega \times [t, t+1])} \leq C_7, \quad \forall t \geq 1. \tag{4.1}$$

Proof. In light of Theorem 1.1, the global classical solution (u, v, w) satisfies

$$u > 0, v > 0, u + v + |\nabla v| + |\nabla w| \leq M \text{ on } \Omega \times (0, \infty). \tag{4.2}$$

We now rewrite the first equation of (1.4) in the following form:

$$u_t = \nabla \cdot D(x, t, \nabla u) + R(x, t), \quad (x, t) \in \Omega \times (0, \infty),$$

where $D(x, t, \eta) = \eta - \chi(u \nabla v)(x, t) + \xi_1(u \nabla w)(x, t)$ and $R(x, t) = (au - \mu u^{\theta+1})(x, t)$.

In view of the boundedness in (4.2) and the Young inequality, we readily deduce

$$\begin{cases} D(x, t, \eta) \cdot \eta \geq \frac{1}{2} |\eta|^2 - \frac{(\chi + \xi_1)^2 M^4}{2}, & |D(x, t, \eta)| \leq |\eta| + (\chi + \xi_1) M^2, \\ |R(x, t)| \leq aM + \mu M^{\theta+1}, & \forall (x, t, \eta) \in \Omega \times (0, \infty) \times \mathbb{R}^n. \end{cases}$$

Then (4.1) follows from the Hölder regularity for parabolic equations [35, Theorem 1.3]. \square

LEMMA 4.2. *There exists a constant $C_8 > 0$ such that the global bounded solution of (1.4) obtained in Theorem 1.1 fulfills*

$$\|\nabla u(\cdot, t)\|_{L^{2n}} \leq C_8, \quad \forall t \geq 1. \tag{4.3}$$

Proof. Inspired from [24, Section 3.3], we first establish an L^2 -bound for Δv . To this end, we first recall from (4.2) and (3.14) with $p=2$ of Lemma 3.2 that

$$\frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} u^2 + \int_{\Omega} |\nabla u|^2 \leq c_{42}. \tag{4.4}$$

Next, we take ∇ to the v -equation, and then take dot product with $-\nabla \Delta v$, and finally use the w -equation and repeatedly use integration by parts to derive from (1.4) that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\Delta v|^2 + \int_{\Omega} |\Delta v|^2 + d \int_{\Omega} |\nabla \Delta v|^2 \\ &= -\xi_2 \int_{\Omega} \nabla(\nabla v \cdot \nabla w + \bar{u}v - uv) \cdot \nabla \Delta v - \int_{\Omega} \nabla u \cdot \nabla \Delta v \\ &= -\xi_2 \int_{\Omega} (D^2 v \cdot \nabla w + D^2 w \cdot \nabla v) \cdot \nabla \Delta v + \xi_2 \int_{\Omega} (u - \bar{u}) \nabla v \cdot \nabla \Delta v \\ & \quad + \int_{\Omega} (\xi_2 v - 1) \nabla u \cdot \nabla \Delta v. \end{aligned} \tag{4.5}$$

Based on (4.4) and (4.5), by the boundedness (4.2), the elliptic estimate $\|D^2 w\|_{L^2} \leq c_{43} \|u\|_{L^2} \leq c_{44}$, the Gagliardo-Nirenberg interpolation inequality and the H^3 -elliptic estimate (cf. [44, (4.19)]), we obtain that

$$\begin{aligned} & -\xi_2 \int_{\Omega} (D^2 v \cdot \nabla w + D^2 w \cdot \nabla v) \nabla \Delta v \\ & \leq M \xi_2 \|D^2 v\|_{L^2} \|\nabla \Delta v\|_{L^2} + \frac{d}{7} \|\nabla \Delta v\|_{L^2}^2 + c_{45} \\ & \leq c_{46} \left(\|D^3 v\|_{L^2}^{\frac{2}{3}} \|v\|_{L^2}^{\frac{1}{3}} + \|v\|_{L^2} \right) \|\nabla \Delta v\|_{L^2} + \frac{d}{7} \|\nabla \Delta v\|_{L^2}^2 + c_{45} \\ & \leq c_{47} \|v\|_{H^3}^{\frac{2}{3}} \|\nabla \Delta v\|_{L^2} + \frac{d}{6} \|\nabla \Delta v\|_{L^2}^2 + c_{47} \\ & \leq c_{48} (\|\Delta v\|_{H^1} + \|v\|_{L^2})^{\frac{2}{3}} \|\nabla \Delta v\|_{L^2} + \frac{d}{6} \|\nabla \Delta v\|_{L^2}^2 + c_{47} \\ & \leq c_{49} (\|\nabla \Delta v\|_{L^2} + \|v\|_{L^2})^{\frac{2}{3}} \|\nabla \Delta v\|_{L^2} + \frac{d}{6} \|\nabla \Delta v\|_{L^2}^2 + c_{47} \\ & \leq \frac{d}{5} \|\nabla \Delta v\|_{L^2}^2 + c_{50}. \end{aligned} \tag{4.6}$$

In easier ways, we again use the boundedness (4.2) to estimate

$$\begin{aligned} & \xi_2 \int_{\Omega} (u - \bar{u}) \nabla v \cdot \nabla \Delta v + \int_{\Omega} (\xi_2 v - 1) \nabla u \cdot \nabla \Delta v \\ & \leq \frac{3d}{5} \int_{\Omega} |\nabla \Delta v|^2 + \frac{(1 + M \xi_2)^2}{2d} \int_{\Omega} |\nabla u|^2 + c_{51}. \end{aligned} \tag{4.7}$$

Substituting (4.6) and (4.7) into (4.5), we conclude that

$$\frac{d}{dt} \int_{\Omega} |\Delta v|^2 + 2 \int_{\Omega} |\Delta v|^2 \leq \frac{(1 + M \xi_2)^2}{d} \int_{\Omega} |\nabla u|^2 + 2c_{50} + 2c_{51}. \tag{4.8}$$

An obvious combination from (4.4) and (4.8) enables one to derive that

$$\frac{d}{dt} \int_{\Omega} \left(\frac{(1 + M\xi_2)^2}{d} u^2 + |\Delta v|^2 \right) + \int_{\Omega} \left(\frac{(1 + M\xi_2)^2}{d} u^2 + |\Delta v|^2 \right) \leq c_{52},$$

which along with H^2 -elliptic estimate yields a uniform H^2 -bound for v :

$$\|D^2 v(\cdot, t)\|_{L^2} + \|\Delta v(\cdot, t)\|_{L^2} \leq c_{53}, \quad \forall t \geq 1. \tag{4.9}$$

Next, we proceed to derive an L^2 -bound for ∇u . For this, we compute from (1.4) that

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |\nabla u|^2 + 2 \int_{\Omega} |\Delta u|^2 \\ &= 2\chi \int_{\Omega} (\nabla u \cdot \nabla v + u \Delta v) \Delta u - 2\xi_1 \int_{\Omega} (\nabla u \cdot \nabla w + u \bar{u} - u^2) \Delta u \\ & \quad - 2 \int_{\Omega} (au - \mu u^{\theta+1}) \Delta u + \int_{\Omega} |\nabla u|^2. \end{aligned} \tag{4.10}$$

Using the boundedness (4.2), (4.9), the Gagliardo-Nirenberg inequality and the H^2 -elliptic estimate, we deduce that

$$\begin{aligned} & 2\chi \int_{\Omega} (\nabla u \nabla v + u \Delta v) \Delta u - 2\xi_1 \int_{\Omega} (\nabla u \nabla w + u \bar{u} - u^2) \Delta u \\ & \quad - 2 \int_{\Omega} (au - \mu u^{\theta+1}) \Delta u + \int_{\Omega} |\nabla u|^2 \\ & \leq \int_{\Omega} |\Delta u|^2 + [1 + 4M^2(\chi^2 + \xi_1^2)] \int_{\Omega} |\nabla u|^2 + 4M^2\chi^2 \int_{\Omega} |\Delta v|^2 + c_{54} \\ & \leq \int_{\Omega} |\Delta u|^2 + c_{55} \left(\|D^2 u\|_{L^2}^{\frac{1}{2}} \|u\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^2} \right)^2 + c_{55} \\ & \leq \int_{\Omega} |\Delta u|^2 + c_{56} \|D^2 u\|_{L^2} + c_{56} \\ & \leq \int_{\Omega} |\Delta u|^2 + c_{57} (\|\Delta u\|_{L^2} + \|u\|_{L^2}) + c_{57} \\ & \leq 2 \int_{\Omega} |\Delta u|^2 + c_{58}. \end{aligned}$$

Substituting this into (4.10) entails a simple ODI as follows:

$$\frac{d}{dt} \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |\nabla u|^2 \leq c_{58},$$

yielding immediately a uniform L^2 -bound for ∇u :

$$\|\nabla u(\cdot, t)\|_{L^2} \leq c_{59}, \quad \forall t \geq 1. \tag{4.11}$$

We are now ready to show the uniform boundedness of $\|\nabla u(\cdot, t)\|_{L^{2n}}$. Again, we apply integration by parts to compute from (1.4) that

$$\frac{1}{2n} \frac{d}{dt} \int_{\Omega} |\nabla u|^{2n} = \int_{\Omega} |\nabla u|^{2n-2} \nabla u \cdot \nabla \Delta u - \chi \int_{\Omega} |\nabla u|^{2n-2} \nabla u \cdot \nabla (\nabla \cdot (u \nabla v))$$

$$\begin{aligned}
 & + \xi_1 \int_{\Omega} |\nabla u|^{2n-2} \nabla u \cdot \nabla (\nabla \cdot (u \nabla w)) \\
 & + \int_{\Omega} |\nabla u|^{2n-2} \nabla u \cdot \nabla (au - \mu u^{\theta+1}) =: \sum_{i=1}^4 J_i.
 \end{aligned} \tag{4.12}$$

For J_1 , noticing the fact that $2\nabla u \cdot \nabla \Delta u = \Delta |\nabla u|^2 - 2|D^2 u|^2$ and the L^2 -boundedness of ∇u in (4.11), we employ the way to control boundary integral as done in (3.36) and the Gagliardo-Nirenberg inequality to estimate that

$$\begin{aligned}
 J_1 &= \frac{1}{2} \int_{\Omega} |\nabla u|^{2n-2} \Delta |\nabla u|^2 - \int_{\Omega} |\nabla u|^{2n-2} |D^2 u|^2 \\
 &= \frac{1}{2} \int_{\partial\Omega} |\nabla u|^{2n-2} \frac{\partial |\nabla u|^2}{\partial \nu} - \frac{(n-1)}{2} \int_{\Omega} |\nabla u|^{2n-4} |\nabla |\nabla u|^2|^2 - \int_{\Omega} |\nabla u|^{2n-2} |D^2 u|^2 \\
 &\leq -\frac{(n-1)}{3} \int_{\Omega} |\nabla u|^{2n-4} |\nabla |\nabla u|^2|^2 - \frac{1}{2} \int_{\Omega} |\nabla u|^{2n-2} |D^2 u|^2 + c_{60}.
 \end{aligned} \tag{4.13}$$

By the inequality $|\Delta u| \leq \sqrt{n} |D^2 u|$ and the boundedness (4.2) and (4.9), we use integration by parts to bound J_2 as follows:

$$\begin{aligned}
 J_2 &= \chi \int_{\Omega} \nabla |\nabla u|^{2n-2} \cdot \nabla u \nabla \cdot (u \nabla v) + \chi \int_{\Omega} |\nabla u|^{2n-2} \Delta u \nabla \cdot (u \nabla v) \\
 &= \chi(n-1) \int_{\Omega} |\nabla u|^{2(n-2)} \nabla |\nabla u|^2 \cdot \nabla u (\nabla u \cdot \nabla v + u \Delta v) \\
 &\quad + \chi \int_{\Omega} |\nabla u|^{2n-2} \Delta u (\nabla u \cdot \nabla v + u \Delta v) \\
 &\leq M\chi(n-1) \int_{\Omega} |\nabla u|^{2n-3} |\nabla |\nabla u|^2| (|\nabla u| + |\Delta v|) \\
 &\quad + M\chi\sqrt{n} \int_{\Omega} |\nabla u|^{2n-2} |D^2 u| (|\nabla u| + |\Delta v|) \\
 &\leq \frac{(n-1)}{8} \int_{\Omega} |\nabla u|^{2n-4} |\nabla |\nabla u|^2|^2 + \frac{1}{8} \int_{\Omega} |\nabla u|^{2n-2} |D^2 u|^2 \\
 &\quad + 4(2n-1)M^2\chi^2 \int_{\Omega} |\nabla u|^{2n} + 4(2n-1)M^2\chi^2 \int_{\Omega} |\nabla u|^{2n-2} |\Delta v|^2.
 \end{aligned} \tag{4.14}$$

In a similar way, since $|\Delta w| = |u - \bar{u}| \leq 2M$, and using Young’s inequality, we estimate J_3 as follows:

$$\begin{aligned}
 J_3 &= -\xi_1 \int_{\Omega} \nabla |\nabla u|^{2n-2} \cdot \nabla u \nabla \cdot (u \nabla w) - \xi_1 \int_{\Omega} |\nabla u|^{2n-2} \Delta u \cdot \nabla (u \nabla w) \\
 &\leq M\xi_1(n-1) \int_{\Omega} |\nabla u|^{2n-3} |\nabla |\nabla u|^2| (|\nabla u| + |\Delta w|) + M\xi_1\sqrt{n} \int_{\Omega} |\nabla u|^{2n-2} |D^2 u| (|\nabla u| + |\Delta w|) \\
 &\leq \frac{(n-1)}{8} \int_{\Omega} |\nabla u|^{2n-4} |\nabla |\nabla u|^2|^2 + \frac{1}{8} \int_{\Omega} |\nabla u|^{2n-2} |D^2 u|^2 \\
 &\quad + 4(2n-1)M^2\xi_1^2 \int_{\Omega} |\nabla u|^{2n} + 16(2n-1)M^4\xi_1^2 \int_{\Omega} |\nabla u|^{2n-2} \\
 &\leq \frac{(n-1)}{8} \int_{\Omega} |\nabla u|^{2n-4} |\nabla |\nabla u|^2|^2 + \frac{1}{8} \int_{\Omega} |\nabla u|^{2n-2} |D^2 u|^2 + 2(4n-1)M^2\xi_1^2 \int_{\Omega} |\nabla u|^{2n} + c_{61}.
 \end{aligned} \tag{4.15}$$

At last, the term J_4 can be easily computed:

$$J_4 = a \int_{\Omega} |\nabla u|^{2n} - \mu(\theta + 1) \int_{\Omega} u^\theta |\nabla u|^{2n}. \tag{4.16}$$

Substituting (4.13), (4.14), (4.15) and (4.16) into (4.12) and using Young’s inequality, one has a differential inequality of the form:

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |\nabla u|^{2n} + 2n \int_{\Omega} |\nabla u|^{2n} + \frac{(n-1)n}{6} \int_{\Omega} |\nabla u|^{2n-4} |\nabla |\nabla u|^2|^2 \\ & \quad + \frac{n}{2} \int_{\Omega} |\nabla u|^{2n-2} |D^2 u|^2 \\ & \leq c_{62} \int_{\Omega} |\nabla u|^{2n} + c_{62} \int_{\Omega} |\nabla u|^{2n-2} |\Delta v|^2 + c_{62} \\ & \leq c_{63} \int_{\Omega} |\nabla u|^{2n} + c_{63} \int_{\Omega} |\Delta v|^{2n} + c_{63}. \end{aligned} \tag{4.17}$$

Since $\|u(\cdot, t)\|_{L^\infty} \leq M$ in (4.2), Young’s inequality enables one to deduce that

$$\begin{aligned} 2c_{63} \int_{\Omega} |\nabla u|^{2n} &= 2c_{63} \left(-(n-1) \int_{\Omega} u |\nabla u|^{2n-4} \nabla |\nabla u|^2 \cdot \nabla u - \int_{\Omega} u |\nabla u|^{2n-2} \Delta u \right) \\ &\leq 2c_{63} M \left((n-1) \int_{\Omega} |\nabla u|^{2n-3} |\nabla |\nabla u|^2| + \sqrt{n} \int_{\Omega} |\nabla u|^{2n-2} |D^2 u| \right) \\ &\leq \frac{n(n-1)}{6} \int_{\Omega} |\nabla u|^{2n-4} |\nabla |\nabla u|^2|^2 + \frac{n}{2} \int_{\Omega} |\nabla u|^{2n-2} |D^2 u|^2 + c_{64} \int_{\Omega} |\nabla u|^{2n-2} \\ &\leq \frac{n(n-1)}{6} \int_{\Omega} |\nabla u|^{2n-4} |\nabla |\nabla u|^2|^2 + \frac{n}{2} \int_{\Omega} |\nabla u|^{2n-2} |D^2 u|^2 \\ &\quad + c_{63} \int_{\Omega} |\nabla u|^{2n} + c_{65}, \end{aligned}$$

which, upon being substituted into (4.17), allows us to conclude

$$\frac{d}{dt} \int_{\Omega} |\nabla u|^{2n} + 2n \int_{\Omega} |\nabla u|^{2n} \leq c_{63} \int_{\Omega} |\Delta v|^{2n} + c_{66}.$$

Solving this simple differential inequality via integrating factor method, we get

$$\int_{\Omega} |\nabla u(\cdot, t)|^{2n} \leq \frac{c_{66}}{2n} + \int_{\Omega} |\nabla u(\cdot, 1)|^{2n} + c_{63} e^{-2nt} \int_1^t e^{2ns} \int_{\Omega} |\Delta v(\cdot, s)|^{2n} ds, \quad \forall t \geq 1. \tag{4.18}$$

Next, we apply the widely known maximal Sobolev regularity to the v -equation to bound the third term on the right. To this end, with the help of the w -equation, we first rewrite the v -equation in (1.4) in the following form:

$$v_t = d\Delta v - v + \xi_2 \nabla v \cdot \nabla w + \xi_2 v(\bar{u} - u) + u. \tag{4.19}$$

Let $h(x, t) := \xi_2 \nabla v \cdot \nabla w + \xi_2 v(\bar{u} - u) + u$. Then it follows from the boundedness (4.2) that

$$|h| \leq M^2 \xi_2 + 2M^2 \xi_2 + M = 3M^2 \xi_2 + M \text{ on } \Omega \times (0, \infty). \tag{4.20}$$

Now, an application of the maximal Sobolev regularity to (4.19) with (4.20) shows that

$$\begin{aligned} \int_1^t e^{2ns} \int_{\Omega} |\Delta v(\cdot, t)|^{2n} &\leq c_{67} \left(e^{2n} \int_{\Omega} |\Delta v(\cdot, 1)|^{2n} + \int_1^t e^{2ns} \int_{\Omega} |h|^{2n} \right) \\ &\leq c_{67} \left(e^{2n} \int_{\Omega} |\Delta v(\cdot, 1)|^{2n} + \frac{(3M^2\xi_2 + M)^{2n} |\Omega|}{2n} e^{2nt} \right), \end{aligned}$$

which, substituted into (4.18), gives, for $\forall t \geq 1$,

$$\int_{\Omega} |\nabla u(\cdot, t)|^{2n} \leq \frac{c_{66}}{2n} + \int_{\Omega} |\nabla u(\cdot, 1)|^{2n} + c_{63}c_{67} \left(\int_{\Omega} |\Delta v(\cdot, 1)|^{2n} + \frac{(3M^2\xi_2 + M)^{2n} |\Omega|}{2n} \right),$$

yielding our desired gradient estimate (4.3). □

5. Global stability: Proof of Theorem 1.2

Given the enhanced regularity properties in the preceding section, we shall examine the long-time dynamics of bounded solutions as obtained in Theorem 1.1. Under certain conditions, we shall show convergence and exponential convergence of bounded solutions to the unique constant steady state as time goes to infinity.

5.1. Case 1: $a = \mu = 0$. In this case, the system (1.4) becomes

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi_1 \nabla \cdot (u \nabla w), & x \in \Omega, t > 0, \\ v_t = d \Delta v + \xi_2 \nabla \cdot (v \nabla w) + u - v, & x \in \Omega, t > 0, \\ 0 = \Delta w + u - \bar{u}_0, \quad \int_{\Omega} w = 0, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \Omega. \end{cases} \tag{5.1}$$

LEMMA 5.1. *The global classical solution of (5.1) satisfies the following identity:*

$$\begin{aligned} &\frac{d}{dt} \left(\int_{\Omega} u \ln \frac{u}{\bar{u}} + \frac{\chi}{2} \int_{\Omega} |\nabla v|^2 \right) + \int_{\Omega} \frac{|\nabla u|^2}{u} + d\chi \int_{\Omega} |\Delta v|^2 + \chi \int_{\Omega} |\nabla v|^2 + \xi_1 \int_{\Omega} |\Delta w|^2 \\ &= 2\chi \int_{\Omega} \Delta w \cdot \Delta v - \chi \xi_2 \int_{\Omega} \nabla \cdot (v \nabla w) \Delta v. \end{aligned} \tag{5.2}$$

Proof. Multiplying the first equation of (5.1) by $\ln u - \ln \bar{u}$, we have

$$\frac{d}{dt} \int_{\Omega} u \ln \frac{u}{\bar{u}} + \int_{\Omega} \frac{|\nabla u|^2}{u} = \chi \int_{\Omega} \nabla u \cdot \nabla v - \xi_1 \int_{\Omega} \nabla u \cdot \nabla w. \tag{5.3}$$

We multiply the second equation of (5.1) by $-\Delta v$ to obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla v|^2 + d \int_{\Omega} |\Delta v|^2 + \int_{\Omega} |\nabla v|^2 = \int_{\Omega} \nabla u \cdot \nabla v - \xi_2 \int_{\Omega} \nabla \cdot (v \nabla w) \Delta v. \tag{5.4}$$

Then multiplying (5.4) by χ and adding it to (5.3), and using the facts $u = \bar{u}_0 - \Delta w$ and $\int_{\Omega} \Delta v = 0 = \int_{\Omega} \Delta w$ due to homogeneous boundary conditions, we obtain

$$\begin{aligned} &\frac{d}{dt} \left(\int_{\Omega} u \ln \frac{u}{\bar{u}} + \frac{\chi}{2} \int_{\Omega} |\nabla v|^2 \right) + \int_{\Omega} \frac{|\nabla u|^2}{u} + d\chi \int_{\Omega} |\Delta v|^2 + \chi \int_{\Omega} |\nabla v|^2 \\ &= 2\chi \int_{\Omega} \nabla u \cdot \nabla v - \xi_1 \int_{\Omega} \nabla u \cdot \nabla w - \chi \xi_2 \int_{\Omega} \nabla \cdot (v \nabla w) \Delta v \\ &= 2\chi \int_{\Omega} \Delta w \cdot \Delta v - \xi_1 \int_{\Omega} |\Delta w|^2 - \chi \xi_2 \int_{\Omega} \nabla \cdot (v \nabla w) \Delta v, \end{aligned}$$

which gives rise to (5.2). □

In the limiting case of $\chi=0$, there is no bad term to be estimated by (5.2), we can directly jump to (5.15) below, and hence its proof is simpler. We proceed to treat the hard case of $\chi > 0$, for that purpose, we denote

$$A = A(d) = \sup_{(x,t) \in \Omega \times (0,\infty)} v(x,t), \quad B = B(d) = \sup_{(x,t) \in \Omega \times (0,\infty)} |\nabla w(x,t)|. \quad (5.5)$$

Then by the qualitative bounds for v and ∇w in (1.7) and (1.8) with (1.9) and (1.10), we see that they are bounded for large d and hence

$$d_0(\chi) = \inf \left\{ \hat{d}_0 > 0 : \left(d - \frac{(2+A\xi_2)^2 \chi}{4\xi_1} - \frac{B^2 \xi_2^2}{4} \right) \chi \geq 0 \text{ for all } d \geq \hat{d}_0 \right\} < +\infty. \quad (5.6)$$

This allows us to include the limiting case of $\chi=0$, which corresponds to $d_0(0)=0$.

LEMMA 5.2. *Assume $\xi_1 \geq \xi_0 \chi^2$ and $d \geq d_0(\chi)$. Then the global classical solution (u, v, w) of (5.1) satisfies the following estimate:*

$$\int_1^t \int_{\Omega} \frac{|\nabla u|^2}{u} \leq C_9, \quad \forall t > 1. \quad (5.7)$$

In addition, if $d > d_0(\chi)$, then there exists $\zeta_1 > 0$ such that

$$\|u(\cdot, t) - \bar{u}_0\|_{L^1} + \|\nabla v(\cdot, t)\|_{L^2} \leq C_{10} e^{-\zeta_1 t}, \quad \forall t > 0. \quad (5.8)$$

Proof. Using the definitions of A and B in (5.5), we infer from (5.2) of Lemma 5.1 that

$$\begin{aligned} & \frac{d}{dt} \left(\int_{\Omega} u \ln \frac{u}{\bar{u}} + \frac{\chi}{2} \int_{\Omega} |\nabla v|^2 \right) + \int_{\Omega} \frac{|\nabla u|^2}{u} + d\chi \int_{\Omega} |\Delta v|^2 + \chi \int_{\Omega} |\nabla v|^2 + \xi_1 \int_{\Omega} |\Delta w|^2 \\ &= 2\chi \int_{\Omega} \Delta w \cdot \Delta v - \chi \xi_2 \int_{\Omega} v \Delta w \cdot \Delta v - \xi_2 \chi \int_{\Omega} \nabla v \cdot \nabla w \cdot \Delta v \\ &\leq (2 + A\xi_2) \chi \int_{\Omega} |\Delta w| |\Delta v| + B\chi \xi_2 \int_{\Omega} |\nabla v| |\Delta v|. \end{aligned} \quad (5.9)$$

Using Young's inequality, we readily derive

$$(2 + A\xi_2) \chi \int_{\Omega} |\Delta w| |\Delta v| \leq \xi_1 \int_{\Omega} |\Delta w|^2 + \frac{(2 + A\xi_2)^2 \chi^2}{4\xi_1} \int_{\Omega} |\Delta v|^2, \quad (5.10)$$

and

$$B\chi \xi_2 \int_{\Omega} |\nabla v| |\Delta v| \leq (1 - \varepsilon_1) \chi \int_{\Omega} |\nabla v|^2 + \frac{B^2 \chi \xi_2^2}{4(1 - \varepsilon_1)} \int_{\Omega} |\Delta v|^2, \quad (5.11)$$

where, due to $d \geq d_0(\chi)$ with $d_0(\chi)$ defined by (5.6), $\varepsilon_1 \in [0, 1)$ is defined by

$$\varepsilon_1 = \begin{cases} 0, & \text{if } d = d_0(\chi), \\ \frac{1}{2} \left(d - \frac{(2+A\xi_2)^2 \chi}{4\xi_1} - \frac{B^2 \xi_2^2}{4} \right) \left(d - \frac{(2+A\xi_2)^2 \chi}{4\xi_1} \right)^{-1}, & \text{if } d > d_0(\chi). \end{cases} \quad (5.12)$$

A substitution of (5.10) and (5.11) into (5.9) gives

$$\frac{d}{dt} \left(\int_{\Omega} u \ln \frac{u}{\bar{u}} + \frac{\chi}{2} \int_{\Omega} |\nabla v|^2 \right) + \int_{\Omega} \frac{|\nabla u|^2}{u} + \varepsilon_1 \chi \int_{\Omega} |\nabla v|^2$$

$$+ \left(d - \frac{(2 + A\xi_2)^2 \chi}{4\xi_1} - \frac{B^2 \xi_2^2}{4(1 - \varepsilon_1)} \right) \chi \int_{\Omega} |\Delta v|^2 \leq 0. \tag{5.13}$$

Since $d \geq d_0(\chi)$, using the definitions of d_0 and ε_1 in (5.6) and (5.12), one has

$$\left(d - \frac{(2 + A\xi_2)^2 \chi}{4\xi_1} - \frac{B^2 \xi_2^2}{4(1 - \varepsilon_1)} \right) \chi \geq 0,$$

which, substituted into (5.13), yields

$$\frac{d}{dt} \left(\int_{\Omega} u \ln \frac{u}{\bar{u}} + \frac{\chi}{2} \int_{\Omega} |\nabla v|^2 \right) + \int_{\Omega} \frac{|\nabla u|^2}{u} + \varepsilon_1 \chi \int_{\Omega} |\nabla v|^2 \leq 0. \tag{5.14}$$

Then an integration with respect to t over $(1, t)$ shows that

$$\int_{\Omega} u \ln \frac{u}{\bar{u}} + \frac{\chi}{2} \int_{\Omega} |\nabla v|^2 + \int_1^t \int_{\Omega} \frac{|\nabla u|^2}{u} \leq \int_{\Omega} u(\cdot, 1) \ln \frac{u(\cdot, 1)}{\bar{u}_0} + \frac{\chi}{2} \int_{\Omega} |\nabla v(\cdot, 1)|^2. \tag{5.15}$$

A use of Lemma 2.6 along with the fact $\bar{u} = \bar{u}_0$ entails

$$\int_{\Omega} u \ln \frac{u}{\bar{u}} \geq \frac{1}{2\bar{u}_0} \|u - \bar{u}_0\|_{L^1}^2 \geq 0, \tag{5.16}$$

and

$$\int_{\Omega} u(\cdot, 1) \ln \frac{u(\cdot, 1)}{\bar{u}_0} \leq \frac{1}{\bar{u}_0} \|u(\cdot, 1) - \bar{u}_0\|_{L^2}^2. \tag{5.17}$$

Substituting (5.16) and (5.17) into (5.15), we obtain (5.7) directly.

On the other hand, using Lemma 2.6 again and noting the fact that $\|u\|_{L^\infty} \leq M$, we use the Poincaré inequality to derive that

$$\int_{\Omega} u \ln \frac{u}{\bar{u}} \leq \frac{1}{\bar{u}} \|u - \bar{u}\|_{L^2}^2 \leq \frac{c_{68}}{\bar{u}_0} \|\nabla u\|_{L^2}^2 \leq \frac{c_{68}M}{\bar{u}_0} \int_{\Omega} \frac{|\nabla u|^2}{u},$$

which yields

$$\frac{\bar{u}_0}{c_{68}M} \int_{\Omega} u \ln \frac{u}{\bar{u}} \leq \int_{\Omega} \frac{|\nabla u|^2}{u}. \tag{5.18}$$

Substituting (5.18) into (5.14), and recalling the fact $\varepsilon_1 > 0$ due to $d > d_0(\chi)$, then we find a positive constant $c_{69} := \min\{\frac{\bar{u}_0}{c_{68}M}, 2\varepsilon_1\}$ such that

$$\frac{d}{dt} \left(\int_{\Omega} u \ln \frac{u}{\bar{u}} + \frac{\chi}{2} \int_{\Omega} |\nabla v|^2 \right) + c_{69} \left(\int_{\Omega} u \ln \frac{u}{\bar{u}} + \frac{\chi}{2} \int_{\Omega} |\nabla v|^2 \right) \leq 0,$$

which immediately entails, for $t \geq 1$,

$$\int_{\Omega} u \ln \frac{u}{\bar{u}} + \frac{\chi}{2} \int_{\Omega} |\nabla v|^2 \leq \left(\int_{\Omega} u(\cdot, 1) \ln \frac{u(\cdot, 1)}{\bar{u}_0} + \frac{\chi}{2} \int_{\Omega} |\nabla v(\cdot, 1)|^2 \right) e^{-c_{69}(t-1)}. \tag{5.19}$$

Then the combination of (5.16), (5.17) and (5.19) implies (5.8). □

LEMMA 5.3. *Under Lemma 5.2, the u, w -components of the global bounded classical solution of (5.1) fulfill the following properties:*

(uc1) If $d \geq d_0(\chi)$, then (u, w) decays to $(\bar{u}_0, 0)$ uniformly:

$$\|u(\cdot, t) - \bar{u}_0\|_{L^\infty} + \|w(\cdot, t)\|_{W^{1,\infty}} \rightarrow 0 \text{ as } t \rightarrow \infty. \tag{5.20}$$

(uc2) If $d > d_0(\chi)$, then (u, w) decays exponentially to $(\bar{u}_0, 0)$: for some $\zeta_2 > 0$,

$$\|u(\cdot, t) - \bar{u}_0\|_{L^\infty} + \|w(\cdot, t)\|_{W^{1,\infty}} \leq C_{11}e^{-\zeta_2 t}, \quad \forall t > 0. \tag{5.21}$$

Proof. Notice from Theorem 1.1 that $\|u(\cdot, t)\|_{L^\infty}$ is uniformly bounded, and thus the space-time estimate (5.7) along with Poincaré’s inequality ensures

$$\int_1^\infty \int_\Omega |u - \bar{u}_0|^2 \leq c_{69} \int_1^\infty \int_\Omega |\nabla u|^2 \leq c_{70}. \tag{5.22}$$

Then the uniform continuity of $\|u - \bar{u}_0\|_{L^2}^2$ implied by (4.1) shows that

$$\|u - \bar{u}_0\|_{L^2} \rightarrow 0 \text{ as } t \rightarrow \infty. \tag{5.23}$$

Thus, with the boundedness of $\|\nabla u\|_{L^{2n}}$ in (4.3), the Gagliardo-Nirenberg inequality gives

$$\begin{aligned} \|u(\cdot, t) - \bar{u}_0\|_{L^\infty} &\leq c_{71} \left(\|\nabla u(\cdot, t)\|_{L^{2n}}^{\frac{n}{n+1}} \|u(\cdot, t) - \bar{u}_0\|_{L^2}^{\frac{1}{n+1}} + \|u(\cdot, t) - \bar{u}_0\|_{L^2} \right) \\ &\leq c_{72} \left(\|u(\cdot, t) - \bar{u}_0\|_{L^2}^{\frac{1}{n+1}} + \|u(\cdot, t) - \bar{u}_0\|_{L^2} \right) \rightarrow 0 \text{ as } t \rightarrow \infty. \end{aligned} \tag{5.24}$$

This establishes the u -convergence in (5.20). We also provide here an alternative short proof based on [38, Lemma 3.10]. Indeed, assume to the contrary, then there would exist some sequences $(x_j)_{j \in \mathbb{N}} \subset \Omega$ and $(t_j)_{j \in \mathbb{N}} \subset (0, \infty)$ satisfying $t_j \rightarrow \infty$ as $j \rightarrow \infty$ such that

$$|u(x_j, t_j) - \bar{u}_0| \geq c_{73}, \quad \forall j \in \mathbb{N}.$$

The uniform continuity of u due to Lemma 4.1 warrants there exist $r > 0$ and $\delta > 0$ such that, for any $j \in \mathbb{N}$,

$$|u - \bar{u}_0| \geq \frac{c_{73}}{2} \text{ on } B_r(x_j) \cap \Omega \times (t_j, t_j + \delta). \tag{5.25}$$

The smoothness of $\partial\Omega$ shows that

$$|B_r(x_j) \cap \Omega| \geq c_{74}, \quad \forall x_j \in \Omega. \tag{5.26}$$

Therefore, for all $j \in \mathbb{N}$, it follows from (5.25) and (5.26) that

$$\int_{t_j}^{t_j+\delta} \int_\Omega |u - \bar{u}_0|^2 \geq \int_{t_j}^{t_j+\delta} \int_{B_r(x_j) \cap \Omega} |u - \bar{u}_0|^2 \geq \frac{c_{73}^2 c_{74} \delta}{4}, \tag{5.27}$$

which clearly contradicts the following fact due to (5.22):

$$\int_{t_j}^{t_j+\delta} \int_\Omega |u - \bar{u}_0|^2 \leq \int_{t_j}^\infty \int_\Omega |u - \bar{u}_0|^2 \rightarrow 0 \text{ as } j \rightarrow \infty.$$

This contradiction gives rise to (5.21).

In the case of $d > d_0(\chi)$, we shall apply Lemma 4.2 and (5.8) to show the L^∞ -exponential decay (5.21). Indeed, employing the Gagliardo-Nirenberg inequality, the L^1 -exponential decay (5.8) and the boundedness of $\|\nabla u\|_{L^{2n}}$ in Lemma 4.2, we conclude that

$$\begin{aligned} \|u(\cdot, t) - \bar{u}_0\|_{L^\infty} &\leq c_{75} \left(\|\nabla u(\cdot, t)\|_{L^{2n}}^{\frac{2n}{2n+1}} \|u(\cdot, t) - \bar{u}_0\|_{L^1}^{\frac{1}{2n+1}} + \|u(\cdot, t) - \bar{u}_0\|_{L^1} \right) \\ &\leq c_{76} e^{-\frac{\zeta_1}{2n+1}t}, \end{aligned}$$

which shows our u -exponential decay estimate (5.20).

Finally, by the $W^{2,n+1}$ -estimate applied to (3.27) and the embedding $W^{2,n+1} \hookrightarrow W^{1,\infty}$, we obtain that

$$\|w(\cdot, t)\|_{W^{1,\infty}} \leq c_{77} \|w(\cdot, t)\|_{W^{2,n+1}} \leq c_{78} \|u(\cdot, t) - \bar{u}_0\|_{L^\infty},$$

hence, the convergence for w follows directly from the convergence of u . □

LEMMA 5.4. *Under Lemma 5.2, the v -component of the global bounded classical solution of (5.1) enjoys the following convergence properties:*

(vc1) *If $d \geq d_0(\chi)$, then v decays to \bar{u}_0 uniformly:*

$$\|v(\cdot, t) - \bar{u}_0\|_{L^\infty} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \tag{5.28}$$

(vc2) *If $d > d_0(\chi)$, then v decays exponentially to \bar{u}_0 uniformly: for some $\zeta_3 > 0$,*

$$\|v(\cdot, t) - \bar{u}_0\|_{L^\infty} \leq C_{12} e^{-\zeta_3 t}, \quad \forall t > 0. \tag{5.29}$$

Proof. We first rewrite the v -equation in (5.1) as

$$(v - \bar{u}_0)_t = d\Delta(v - \bar{u}_0) + \xi_2 \nabla \cdot ((v - \bar{u}_0) \nabla w) + \xi_2 \bar{u}_0 \Delta w + u - \bar{u}_0 - (v - \bar{u}_0). \tag{5.30}$$

Multiplying (5.30) by $v - \bar{u}_0$, integrating over Ω by parts and using the fact $\Delta w = \bar{u}_0 - u$ by (5.1), we compute that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\Omega} (v - \bar{u}_0)^2 + d \int_{\Omega} |\nabla v|^2 + \int_{\Omega} (v - \bar{u}_0)^2 \\ &= \frac{\xi_2}{2} \int_{\Omega} (v - \bar{u}_0)^2 \Delta w + \xi_2 \bar{u}_0 \int_{\Omega} (v - \bar{u}_0) \Delta w + \int_{\Omega} (u - \bar{u}_0)(v - \bar{u}_0) \\ &= \frac{\xi_2}{2} \int_{\Omega} (v - \bar{u}_0)^2 (\bar{u}_0 - u) + \xi_2 \bar{u}_0 \int_{\Omega} (v - \bar{u}_0) (\bar{u}_0 - u) + \int_{\Omega} (u - \bar{u}_0)(v - \bar{u}_0). \end{aligned} \tag{5.31}$$

The fact $\|u - \bar{u}_0\|_{L^\infty} \rightarrow 0$ as $t \rightarrow \infty$ by (5.20) shows there exists $t_1 > 0$ such that

$$\|u(\cdot, t) - \bar{u}_0\|_{L^\infty} \leq \frac{1}{2\xi_2}, \quad t \geq t_1. \tag{5.32}$$

On the other hand, using Young's inequality, we derive

$$\xi_2 \bar{u}_0 \int_{\Omega} (v - \bar{u}_0) (\bar{u}_0 - u) \leq \frac{1}{8} \int_{\Omega} (v - \bar{u}_0)^2 + 2\xi_2^2 \bar{u}_0^2 \int_{\Omega} (u - \bar{u}_0)^2, \tag{5.33}$$

and

$$\int_{\Omega} (u - \bar{u}_0)(v - \bar{u}_0) \leq \frac{1}{8} \int_{\Omega} (v - \bar{u}_0)^2 + 2 \int_{\Omega} (u - \bar{u}_0)^2. \tag{5.34}$$

Combining (5.32),(5.33), (5.34) and (5.31), we conclude, for all $t \geq t_1$, that

$$\frac{d}{dt} \int_{\Omega} (v - \bar{u}_0)^2 + 2d \int_{\Omega} |\nabla v|^2 + \int_{\Omega} (v - \bar{u}_0)^2 \leq 4(1 + \xi_2^2 \bar{u}_0^2) \int_{\Omega} (u - \bar{u}_0)^2,$$

which, upon being solved and the fact (5.23), implies

$$\|v(\cdot, t) - \bar{u}_0\|_{L^2}^2 \leq \|v(\cdot, 1) - \bar{u}_0\|_{L^2}^2 e^{-(t-1)} + 4(1 + \xi_2^2 \bar{u}_0^2) \int_1^t e^{-(t-s)} \int_{\Omega} (u(\cdot, s) - \bar{u}_0)^2 \rightarrow 0$$

as $t \rightarrow \infty$. (5.35)

Then, with the L^2 -convergence (5.35) at hand, replacing v with u in the Gagliardo-Nirenberg inequality (5.24), we readily obtain (5.28).

In the case of $d > d_0(\chi)$, to show v decays exponentially, we first see from (5.1) that

$$\bar{v} - \bar{u}_0 = (\bar{v}_0 - \bar{u}_0) e^{-t}, \quad \forall t > 0. \tag{5.36}$$

Now, we use Poincaré’s inequality and the exponential decays (5.8) and (5.36) to deduce

$$\begin{aligned} \|v(\cdot, t) - \bar{u}_0\|_{L^2} &\leq \|v - \bar{v}\|_{L^2} + \|\bar{v} - \bar{u}_0\|_{L^2} \\ &\leq c_{79} \|\nabla v\|_{L^2} + |\bar{v}_0 - \bar{u}_0| |\Omega|^{\frac{1}{2}} e^{-t} \\ &\leq c_{80} e^{-\min\{\zeta_1, 1\}t}, \quad \forall t > 0. \end{aligned}$$

Now, since $\|\nabla v\|_{L^{2n}}$ is uniformly-in-time bounded by Theorem 1.1, replacing v with u in the Gagliardo-Nirenberg inequality (5.24), we readily improve this L^2 -exponential decay to L^∞ -exponential decay of v as in (5.29). □

5.2. Case 2: $a, \mu > 0$ In this subsection, we shall study the large-time behavior of solution for the complete system (1.4) with $a, \mu > 0$. For convenience, we rewrite it here:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi_1 \nabla \cdot (u \nabla w) + u(a - \mu u^\theta), & x \in \Omega, t > 0, \\ v_t = d \Delta v + \xi_2 \nabla \cdot (v \nabla w) + u - v, & x \in \Omega, t > 0, \\ 0 = \Delta w + u - \bar{u}, \quad \int_{\Omega} w = 0, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \Omega. \end{cases} \tag{5.37}$$

LEMMA 5.5. *Let C_p denote the Poincaré constant defined by (1.13) and (u, v, w) be the solution of the system (5.37). Then*

$$\|\nabla w\|_{L^2} \leq C_p \|u - b\|_{L^2}, \quad \|\Delta w\|_{L^2} \leq \|u - b\|_{L^2}, \quad \|w\|_{W^{2,2}} \leq C_{13} \|u - b\|_{L^2}, \quad b = \left(\frac{a}{\mu}\right)^{\frac{1}{\theta}}. \tag{5.38}$$

Proof. By the w -equation in (5.37) and Poincaré’s inequality due to $\int_{\Omega} w = 0$, we infer

$$\int_{\Omega} |\nabla w|^2 = \int_{\Omega} (u - b)w + (b - \bar{u}) \int_{\Omega} w \leq \|u - b\|_{L^2} \|w\|_{L^2} \leq C_p \|u - b\|_{L^2} \|\nabla w\|_{L^2},$$

which shows

$$\|\nabla w\|_{L^2} \leq C_p \|u - b\|_{L^2}. \tag{5.39}$$

Next, we deduce from the third equation in (5.37) and the fact that $\int_{\Omega} \Delta w = 0$ that

$$\begin{aligned} \int_{\Omega} |\Delta w|^2 &= \int_{\Omega} (b-u)\Delta w + (\bar{u}-b) \int_{\Omega} \Delta w \\ &= - \int_{\Omega} (u-b)\Delta w \leq \left(\int_{\Omega} (u-b)^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} |\Delta w|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

which along with the $W^{2,2}$ -elliptic estimate applied to (3.27) yields

$$\|w\|_{W^{2,2}} \leq c_{81} \|\Delta w\|_{L^2} \leq c_{82} \|u-b\|_{L^2}.$$

This together with (5.39) gives rise to (5.38). □

LEMMA 5.6. *The classical solution of (5.37) satisfies*

$$\frac{d}{dt} \int_{\Omega} (u-b-b \ln \frac{u}{b}) + \mu \int_{\Omega} (u-b)(u^\theta - b^\theta) \leq \frac{b\chi^2}{2} \int_{\Omega} |\nabla v|^2 + \frac{b\xi_1^2 C_p^2}{2} \int_{\Omega} (u-b)^2. \tag{5.40}$$

Proof. We use integration by parts to deduce from (5.37) that

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega} (u-b-b \ln \frac{u}{b}) + b \int_{\Omega} \frac{|\nabla u|^2}{u^2} + \mu \int_{\Omega} (u-b)(u^\theta - b^\theta) \\ &= b\chi \int_{\Omega} \frac{\nabla u \cdot \nabla v}{u} - b\xi_1 \int_{\Omega} \frac{\nabla u \cdot \nabla w}{u}. \end{aligned} \tag{5.41}$$

Using Young’s inequality, one has

$$b\chi \int_{\Omega} \frac{\nabla u \cdot \nabla v}{u} \leq \frac{b}{2} \int_{\Omega} \frac{|\nabla u|^2}{u^2} + \frac{b\chi^2}{2} \int_{\Omega} |\nabla v|^2, \tag{5.42}$$

and, due to (5.38) of Lemma 5.5,

$$-b\xi_1 \int_{\Omega} \frac{\nabla u \cdot \nabla w}{u} \leq \frac{b}{2} \int_{\Omega} \frac{|\nabla u|^2}{u^2} + \frac{b\xi_1^2}{2} \int_{\Omega} |\nabla w|^2 \leq \frac{b}{2} \int_{\Omega} \frac{|\nabla u|^2}{u^2} + \frac{b\xi_1^2 C_p^2}{2} \int_{\Omega} (u-b)^2. \tag{5.43}$$

Then we substitute (5.42) and (5.43) into (5.41) to obtain (5.40). □

LEMMA 5.7. *The global bounded classical solution of (5.37) fulfils*

$$\frac{d}{dt} \int_{\Omega} (v-b)^2 + \int_{\Omega} (v-b)^2 + d \int_{\Omega} |\nabla v|^2 \leq \left(1 + \frac{\xi_2^2 M_0^2 C_p^2}{d} \right) \int_{\Omega} (u-b)^2. \tag{5.44}$$

Proof. From (5.37), we use integration by parts, the fact $\|v(\cdot, t)\|_{L^\infty} \leq M_0$ in (3.1), the Hölder inequality and Young’s inequality to estimate

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega} (v-b)^2 + 2 \int_{\Omega} (v-b)^2 + 2d \int_{\Omega} |\nabla v|^2 \\ &= -2\xi_2 \int_{\Omega} v \nabla v \cdot \nabla w + 2 \int_{\Omega} (u-b)(v-b) \\ &\leq 2\xi_2 M_0 \|\nabla v\|_{L^2} \|\nabla w\|_{L^2} + \int_{\Omega} (u-b)^2 + \int_{\Omega} (v-b)^2 \\ &\leq d \int_{\Omega} |\nabla v|^2 + \frac{\xi_2^2 M_0^2}{d} \int_{\Omega} |\nabla w|^2 + \int_{\Omega} (u-b)^2 + \int_{\Omega} (v-b)^2, \end{aligned}$$

which in conjunction with (5.38) shows (5.44). □

We are now ready to show the exponential decay of global bounded solutions.

LEMMA 5.8. *Let (u, v, w) be the global solution of (5.37) obtained in Theorem 1.1. If $\mu > 0$ satisfies (1.15), then (u, v, w) decays exponentially to $(b, b, 0)$: for some $\sigma > 0$,*

$$\|u(\cdot, t) - b\|_{L^\infty} + \|v(\cdot, t) - b\|_{L^\infty} + \|w(\cdot, t)\|_{W^{1,\infty}} \leq C_{14} e^{-\frac{\sigma}{2(n+1)}t}, \quad t > 0. \tag{5.45}$$

Proof. By the estimate for M_0 in (3.1) or (1.7), we first infer that

$$M_0^2 \leq O(1) \left(1 + \left(\frac{1}{\mu} \right)^{\frac{4+n}{\theta}} \right),$$

and then, we find that our assumption on μ in (1.15) along with (1.14) implies

$$\mu > \frac{\left[\left(1 + \frac{C_p^2 \xi_2^2 M_0^2}{d} \right) \frac{\chi^2}{d} + C_p^2 \xi_1^2 \right]}{2b^{\theta-2}} \iff \mu > \left(\frac{d\chi^2 + d^2 C_p^2 \xi_1^2 + C_p^2 \xi_2^2 M_0^2}{2d^2 a^{\frac{\theta-2}{\theta}}} \right)^{\frac{\theta}{2}}. \tag{5.46}$$

Now, multiplying (5.44) by $\frac{b\chi^2}{2d}$ and adding the resulting inequality to (5.40), we obtain

$$\begin{aligned} & \frac{d}{dt} \left(\int_{\Omega} (u - b - b \ln \frac{u}{b}) + \frac{b\chi^2}{2d} \int_{\Omega} (v - b)^2 \right) + \frac{b\chi^2}{2d} \int_{\Omega} (v - b)^2 \\ & \leq -\mu \int_{\Omega} (u - b)(u^\theta - b^\theta) + \frac{b}{2} \left[\left(1 + \frac{C_p^2 \xi_2^2 M_0^2}{d} \right) \frac{\chi^2}{d} + C_p^2 \xi_1^2 \right] \int_{\Omega} (u - b)^2. \end{aligned} \tag{5.47}$$

Notice, since $\theta \geq 1$, by examining monotonicity, one easily sees that

$$\sup_{z \in (0, b) \cup (b, \infty)} \frac{(z - b)^2}{(z - b)(z^\theta - b^\theta)} = \frac{1}{b^{\theta-1}} \iff -(z - b)(z^\theta - b^\theta) \leq -b^{\theta-1}(z - b)^2, \quad \forall z > 0.$$

Applying this inequality to (5.47), we immediately arrive at

$$\begin{aligned} & \frac{d}{dt} \left(\int_{\Omega} (u - b - b \ln \frac{u}{b}) + \frac{b\chi^2}{2d} \int_{\Omega} (v - b)^2 \right) + \frac{b\chi^2}{2d} \int_{\Omega} (v - b)^2 \\ & + b^{\theta-1} \left[\mu - \frac{\left[\left(1 + \frac{C_p^2 \xi_2^2 M_0^2}{d} \right) \frac{\chi^2}{d} + C_p^2 \xi_1^2 \right]}{2b^{\theta-2}} \right] \int_{\Omega} (u - b)^2 \leq 0. \end{aligned} \tag{5.48}$$

We first integrate (5.48) from 1 to t and then we use our derived condition (5.46) and the elementary algebraic fact that $z - b - b \ln \frac{z}{b} \geq 0$ for $z > 0$ to obtain that

$$\int_1^t \int_{\Omega} (u - b)^2 < c_{83}, \quad \forall t \geq 1. \tag{5.49}$$

Given (5.49), using the same arguments as in Lemma 5.3 for u , one can easily show that

$$\|u(\cdot, t) - b\|_{L^\infty} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \tag{5.50}$$

Next, observing the following elementary fact that

$$\lim_{z \rightarrow b} \frac{(z-b)^2}{z-b-b \ln \frac{z}{b}} = 2b,$$

we infer from (5.50) there exists $t_2 > 0$ such that

$$b \left[u(\cdot, t) - b - b \ln \frac{u(\cdot, t)}{b} \right] \leq [u(\cdot, t) - b]^2 \leq 3b \left[u(\cdot, t) - b - b \ln \frac{u(\cdot, t)}{b} \right], \quad \forall t \geq t_2. \quad (5.51)$$

Substituting (5.51) into (5.48), we end up with an important ODI:

$$\frac{d}{dt} \left(\int_{\Omega} (u - b - b \ln \frac{u}{b}) + \frac{b\chi^2}{2d} \int_{\Omega} (v - b)^2 \right) + \sigma \left(\int_{\Omega} (u - b - b \ln \frac{u}{b}) + \frac{b\chi^2}{2d} \int_{\Omega} (v - b)^2 \right) \leq 0, \quad \forall t \geq t_2, \quad (5.52)$$

where, due to (5.46),

$$\sigma := \min \left\{ 1, \quad b^\theta \left[\mu - \frac{\left[\left(1 + \frac{C_p^2 \xi_2^2 M_0^2}{d} \right) \frac{\chi^2}{d} + C_p^2 \xi_1^2 \right]}{2b^{\theta-2}} \right] \right\} > 0.$$

Solving the ODI (5.52), we simply get the following exponential decay estimate:

$$\int_{\Omega} (u - b - b \ln \frac{u}{b}) + \int_{\Omega} (v - b)^2 \leq c_{84} e^{-\sigma(t-t_2)}, \quad \forall t \geq t_2,$$

which combined with (5.51) gives

$$\|u(\cdot, t) - b\|_{L^2} + \|v(\cdot, t) - b\|_{L^2} \leq c_{85} e^{-\frac{\sigma}{2}(t-t_2)}, \quad \forall t \geq t_2. \quad (5.53)$$

Now, it is easy to improve this L^2 -convergence to L^∞ -convergence as in Lemma 5.3. Indeed, by the boundedness of $\|\nabla u\|_{L^{2n}}$ in (4.3) and the Gagliardo-Nirenberg inequality, (5.53) entails

$$\begin{aligned} \|u(\cdot, t) - b\|_{L^\infty} &\leq c_{86} \left(\|\nabla u(\cdot, t)\|_{L^{2n}}^{\frac{n}{n+1}} \|u(\cdot, t) - b\|_{L^2}^{\frac{1}{n+1}} + \|u(\cdot, t) - b\|_{L^2} \right) \\ &\leq c_{87} e^{-\frac{\sigma}{2(n+1)}t}, \quad \forall t \geq t_2. \end{aligned} \quad (5.54)$$

Similarly, using the $W^{1,\infty}$ -boundedness of v guaranteed by Theorem 1.1, we use the Gagliardo-Nirenberg inequality to improve the L^2 -convergence of v in (5.53) to its L^∞ -convergence as follows:

$$\|v(\cdot, t) - b\|_{L^\infty} \leq c_{88} \|v(\cdot, t)\|_{W^{1,\infty}}^{\frac{n}{n+2}} \|v(\cdot, t) - b\|_{L^2}^{\frac{2}{n+2}} \leq c_{89} e^{-\frac{\sigma}{n+2}(t-t_2)}, \quad \forall t \geq t_2. \quad (5.55)$$

As for the $W^{1,\infty}$ -convergence of w , applying the $W^{2,n+1}$ -estimate to (3.27) along the embedding $W^{2,n+1} \hookrightarrow W^{1,\infty}$ and using (5.54), we see that w decays to zero exponentially:

$$\begin{aligned} \|w(\cdot, t)\|_{W^{1,\infty}} &\leq c_{90} \|w(\cdot, t)\|_{W^{2,n+1}} \leq c_{91} \|\Delta w\|_{L^{n+1}} \leq c_{92} (\|u(\cdot, t) - b\|_{L^\infty} + |b - \bar{u}|) \\ &\leq 2c_{92} \|u(\cdot, t) - b\|_{L^\infty} \leq c_{93} e^{-\frac{\sigma}{2(n+1)}t}, \quad \forall t \geq t_2. \end{aligned} \quad (5.56)$$

Our desired exponential convergence (5.45) follows from (5.54), (5.55) and (5.56). \square

Proof of Theorem 1.2. The respective convergence and exponential convergence in (C1) and (C2) of Theorem 1.2 have been fully contained in Lemmas 5.3, 5.4, and 5.8.

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