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SEMICONDUCTOR FULL QUANTUM HYDRODYNAMIC MODEL WITH NON-FLAT DOPING PROFILE: (I) STABILITY OF STEADY STATE*

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Abstract. This is the first part of our series of studies concerning the full quantum hydrodynamic model for semiconductors with non-flat doping profile. In this paper, we are concerned with the existence, uniqueness and asymptotic stability of subsonic steady states to the model in a bounded interval, which is subject to physical boundary conditions. The main results are proved by Stampacchia's truncation method, the Leray-Schauder Fixed Point Theorem, Schauder's Fixed Point Theorem and intricate energy estimates.

Keywords. Full quantum hydrodynamic model; dispersive velocity term; non-flat doping profile; asymptotic stability; semiconductor.

AMS subject classifications. 35A01; 35B40; 35M33; 76Y05.

1. Introduction

In the mathematical modeling of the nano-size semiconductor devices (e.g. HEMTs, MOSFETs, RTDs and superlattice devices), the quantum effects (like particle tunneling through potential barriers and particle buildup in quantum wells) take place and can not be simulated by classical hydrodynamic models. Therefore, the quantum hydrodynamical (QHD) equations are important and dominative in the description of the motion of electrons or holes transport under the self-consistent electric field.

The QHD conservation laws have the same form as the classical hydrodynamic equations (for simplicity, we treat the flow of electrons in the self-consistent electric field for unipolar devices):

$$\begin{cases} \partial_t n + \partial_{x_k} j_k = 0, & (1.1a) \\ \partial_t j_l + \partial_{x_k}(u_k j_l - P_{kl}) = n\partial_{x_l}\phi - \dfrac{j_l}{\tau_m}, \quad l = 1,2,3, & (1.1b) \\ \partial_t e + \partial_{x_k}(u_k e - u_l P_{kl} + q_k) = j_k \partial_{x_k}\phi + C_e, & (1.1c) \\ \lambda^2 \Delta\phi = n - D(\boldsymbol{x}), & (1.1d) \end{cases}$$

where $n>0$ is the electron density, $\boldsymbol{u}=(u_1,u_2,u_3)$ is the velocity, $\boldsymbol{j}=(j_1,j_2,j_3)$ is the momentum density, $\boldsymbol{P}=(P_{kl})$ is the stress tensor, ϕ is the self-consistent electrostatic potential, e is the energy density, $\boldsymbol{q}=(q_1,q_2,q_3)$ is the heat flux. Indices k,l equal $1,2,3$, and repeated indices are summed over using the Einstein convention. Equation (1.1a) expresses conservation of electron number, (1.1b) expresses conservation of momentum, and (1.1c) expresses conservation of energy. The last terms in (1.1b) and (1.1c) represent electron scattering (the collision terms may include the effects of electron-phonon and electron-impurity collisions, intervalley and interband scattering), which is modeled by the standard relaxation time approximation with momentum and energy relaxation

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times $\tau_m > 0$ and $\tau_e > 0$. The energy relaxation term C_e is given by

$$C_e = -\frac{1}{\tau_e}\left(\frac{1}{2}n|\boldsymbol{u}|^2 + \frac{3}{2}n(\theta - \theta_L)\right),$$

where $\theta > 0$ is the electron temperature and $\theta_L > 0$ is the temperature of the semiconductor lattice in energy units. The transport Equations (1.1a)~(1.1c) are coupled to Poisson's Equation (1.1d) for the self-consistent electrostatic potential, where $\lambda > 0$ is the Debye length, $D = N_d - N_a$ is the doping profile, $N_d > 0$ is the density of donors, and $N_a > 0$ is the density of acceptors.

The QHD Equations (1.1a)~(1.1c) are derived as a set of nonlinear conservation laws by a moment expansion of the Wigner-Boltzmann equation [32] and an expansion of the thermal equilibrium Wigner distribution function to $O(\varepsilon^2)$, where $\varepsilon > 0$ is the scaled Planck's constant. However, to close the moment expansion at the first three moments, we must define, for example, $\boldsymbol{j}$, $\boldsymbol{P}$, e and $\boldsymbol{q}$ in terms of n, $\boldsymbol{u}$ and θ. According to the closure assumption [6], up to order $O(\varepsilon^2)$, we define the momentum density $\boldsymbol{j}$, the stress tensor $\boldsymbol{P} = (P_{kl})$, the energy density e and the heat flux $\boldsymbol{q}$ as follows:

$$\boldsymbol{j} = n\boldsymbol{u}, \qquad P_{kl} = -n\theta\delta_{kl} + \frac{\varepsilon^2}{2}n\partial_{x_k}\partial_{x_l}\ln n,$$
$$e = \frac{3}{2}n\theta + \frac{1}{2}n|\boldsymbol{u}|^2 - \frac{\varepsilon^2}{4}n\Delta\ln n, \qquad \boldsymbol{q} = -\kappa\nabla\theta - \frac{3\varepsilon^2}{4}n\Delta\boldsymbol{u},$$

with the Kronecker symbol δ_{kl} and the heat conductivity $\kappa > 0$. The quantum correction to the stress tensor was first stated in the semiconductor context by Ancona and Iafrate [1], and Ancona and Tiersten [2]. Since

$$\frac{\varepsilon^2}{2}\operatorname{div}(n(\nabla\otimes\nabla)\ln n) = \varepsilon^2 n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right),$$

it can be interpreted as a force including the Bohm potential $\varepsilon^2\Delta\sqrt{n}/\sqrt{n}$ [5]. The quantum correction to the energy density was first derived by Wigner [32]. The heat conduction term consists of a classical Fourier law $-\kappa\nabla\theta$ plus a new quantum contribution $-3\varepsilon^2 n\Delta\boldsymbol{u}/4$ which can be interpreted as a dispersive heat flux [7, 12]. For details on the more general quantum models for semiconductor devices, one can refer to the references [13, 23, 33].

Interestingly, most quantum terms cancel out in the energy Equation (1.1c). In fact, by substituting the above expressions for C_e, $\boldsymbol{j}$, $\boldsymbol{P}$, e and $\boldsymbol{q}$ into (1.1), a computation yields the multi-dimensional full quantum hydrodynamic (FQHD) model for semiconductors as follows.

$$\begin{cases} n_t + \operatorname{div}(n\boldsymbol{u}) = 0, & \text{(1.2a)} \\ (n\boldsymbol{u})_t + \operatorname{div}(n\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla(n\theta) - \varepsilon^2 n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) = n\nabla\phi - \frac{n\boldsymbol{u}}{\tau_m}, & \text{(1.2b)} \\ n\theta_t + n\boldsymbol{u}\cdot\nabla\theta + \frac{2}{3}n\theta\operatorname{div}\boldsymbol{u} - \frac{2}{3}\operatorname{div}(\kappa\nabla\theta) - \frac{\varepsilon^2}{3}\operatorname{div}(n\Delta\boldsymbol{u}) & \\ \qquad = \frac{2\tau_e - \tau_m}{3\tau_m\tau_e}n|\boldsymbol{u}|^2 - \frac{n(\theta - \theta_L)}{\tau_e}, & \text{(1.2c)} \\ \lambda^2\Delta\phi = n - D(\boldsymbol{x}). & \text{(1.2d)} \end{cases}$$

Compared with the classical full hydrodynamic (FHD) model, the new feature of the FQHD model is the Bohm potential term

$$-\varepsilon^2 n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right)$$

in the momentum Equation (1.2b) and the dispersive velocity term

$$-\frac{\varepsilon^2}{3}\mathrm{div}(n\Delta\boldsymbol{u})$$

in the energy Equation (1.2c). Both of them are called quantum correction terms (or dispersive terms) and belong to the third-order derivative terms of the system (1.2).

In the past two decades, the study concerning the semiconductor quantum models and the related quantum systems has become popular. For one-dimensional case, Jüngel and Li [15,16] investigated the unipolar isentropic QHD model with the Dirichlet-Neumann boundary condition and the flat doping profile; they proved the existence, uniqueness and exponential stability of the subsonic steady state for the quite general pressure-density function. Nishibata and Suzuki [24] reconsidered this QHD model with isothermal simplification and the vanishing Bohmenian-type boundary condition, and generalized Jüngel and Li's results to the non-flat doping profile case. Hu, Mei and Zhang [9] extended Nishibata and Suzuki's results to the bipolar QHD model in the non-constant doping profile setting. Huang, Li and Matsumura [11] proved the existence and exponential stability of steady state to the Cauchy problem for the isentropic unipolar QHD model. All the above-mentioned results addressed the quantum effect presented everywhere in the device. However, Di Michele, Mei, Rubino and Sampalmieri [20] derived a hybrid QHD model to account for the practical case, that is, the quantum effect is localized in small regions of the semiconductor device, whereas the other parts of the device can be treated classically; they further proved the existence and uniqueness of weak solutions to the hybrid QHD model. In recent times, Di Michele, Mei, Rubino and Sampalmieri [22] further extended their results in [20] to a new hybrid QHD model with discontinuous pressure function and relaxation time.

As for multi-dimensional case, Jüngel [14] first considered the unipolar steady-state isentropic QHD model for potential flows on a bounded domain, the existence of solutions was proved under the assumption that the electric energy was small compared to the thermal energy, where Dirichlet boundary conditions were addressed. This result was then generalized to the bipolar case by Liang and Zhang [18]. Unterreiter [31] proved the existence of the thermal equilibrium solution of the bipolar isentropic QHD model confined to a bounded domain by variational method. This result was heuristically developed by Di Michele, Mei, Rubino and Sampalmieri [21] to a new model of the bipolar isentropic hybrid quantum hydrodynamics; they considered two different kinds of hybrid behaviour. Regarding the unipolar QHD model for irrotational flow in spatially periodic domain, the global existence of the dynamic solutions and the exponential convergence to their equilibria were artfully proved by Li and Marcati in [17]. Furthermore, the weak solutions with large initial data for the quantum hydrodynamic system were obtained by Antonelli and Marcati in [3, 4]. Li, Zhang and Zhang [19] investigated the large-time behavior of solutions to the initial value problem of the isentropic QHD model in the whole space $\mathbb{R}^3$ and obtained the algebraic time-decay rate. Pu and Guo [27] studied the Cauchy problem of quantum hydrodynamic equations with viscosity and heat conduction in $\mathbb{R}^3$; the global existence around a constant steady-state was shown by the energy method. This result was developed by Pu and Xu [28], they obtained the optimal convergence rates to the constant equilibrium solution by the pure energy method and negative Sobolev space estimates. Pu and Li [29] further extended the result in [28] to the bounded domain with physical boundary conditions. Very recently, Ra and Hong [30] proved the existence, uniqueness and exponential decay for the Cauchy problem in $\mathbb{R}^3$ of the FQHD model in the semiconductor setting.

To the best of our knowledge, the initial-boundary value problem of the FQHD model with the physical boundary conditions is still open in the semiconductor setting. Therefore, in this paper, we study the one-dimensional version of the FQHD model (1.2) over a bounded interval $\Omega=(0,1)$ in assuming that $\tau_m=\tau_e=\kappa=\lambda=1$ for simplicity; namely, we consider

$$\begin{cases} n_t+j_x=0, & (1.3\text{a})\\ j_t+\left(\dfrac{j^2}{n}+n\theta\right)_x-\varepsilon^2 n\left[\dfrac{(\sqrt{n})_{xx}}{\sqrt{n}}\right]_x=n\phi_x-j, & (1.3\text{b})\\ n\theta_t+j\theta_x+\dfrac{2}{3}n\theta\left(\dfrac{j}{n}\right)_x-\dfrac{2}{3}\theta_{xx}-\dfrac{\varepsilon^2}{3}\left[n\left(\dfrac{j}{n}\right)_{xx}\right]_x=\dfrac{1}{3}\dfrac{j^2}{n}-n(\theta-\theta_L), & (1.3\text{c})\\ \phi_{xx}=n-D(x), \qquad t>0,\ x\in\Omega=(0,1), & (1.3\text{d}) \end{cases}$$

with initial conditions

$$(n,j,\theta)(0,x)=(n_0,j_0,\theta_0)(x), \tag{1.4}$$

and boundary conditions

$$n(t,0)=n_l, \qquad n(t,1)=n_r, \tag{1.5a}$$
$$\left(\sqrt{n}\right)_{xx}(t,0)=\left(\sqrt{n}\right)_{xx}(t,1)=0, \tag{1.5b}$$
$$\theta(t,0)=\theta_l, \qquad \theta(t,1)=\theta_r, \tag{1.5c}$$
$$\phi(t,0)=0, \qquad \phi(t,1)=\phi_r, \tag{1.5d}$$

where the boundary data $n_l,n_r,\theta_l,\theta_r$ and ϕ_r are positive constants. The vanishing Bohmenian-type boundary condition (1.5b) means that the quantum Bohm potential vanishes on the boundary, which is derived in [6, 26] and is also physically reasonable. The other boundary conditions in (1.5) are called Ohmic contact boundary conditions. In order to establish the existence of a classical solution, we further assume the initial data (n_0,j_0,θ_0) is compatible with the boundary data (1.5a)∼(1.5c) and $n_t(t,0)=n_t(t,1)=0$, namely,

$$\begin{gathered} n_0(0)=n_l, \quad n_0(1)=n_r, \quad \theta_0(0)=\theta_l, \quad \theta_0(1)=\theta_r,\\ j_{0x}(0)=j_{0x}(1)=\left(\sqrt{n_0}\right)_{xx}(0)=\left(\sqrt{n_0}\right)_{xx}(1)=0. \end{gathered} \tag{1.6}$$

In realistic semiconductor devices, the doping profile will be a non-flat function of the spatial variable. For instance, it has two steep slopes in n^+-n-n^+ diodes [6]. Therefore, we should only assume the continuity and positivity to cover the actual devices, that is,

$$D\in C(\overline{\Omega}), \qquad \inf_{x\in\overline{\Omega}} D(x)>0. \tag{1.7}$$

An explicit formula of the electrostatic potential

$$\phi(t,x)=\Phi[n](t,x):=\int_0^x\int_0^y\big(n(t,z)-D(z)\big)dzdy+\left(\phi_r-\int_0^1\int_0^y\big(n(t,z)-D(z)\big)dzdy\right)x, \tag{1.8}$$

follows from (1.3d) and (1.5d). Further consideration of the solvability of the initial-boundary value problem (IBVP) (1.3)∼(1.5) leads to the following properties

$$\inf_{x\in\Omega} n>0, \qquad \inf_{x\in\Omega}\theta>0, \tag{1.9a}$$

$$\inf_{x\in\Omega} S[n,j,\theta]>0, \qquad S[n,j,\theta]:=\theta-\frac{j^2}{n^2}, \tag{1.9b}$$

which attract our main interest. The condition (1.9a) represents the positivity of the electron density and temperature. The other one (1.9b) is called the subsonic condition. Needless to say, if we want to construct the solution in the physical region where the conditions (1.9) hold, then the initial data (1.4) must first satisfy the same conditions

$$\inf_{x\in\Omega} n_0>0, \qquad \inf_{x\in\Omega}\theta_0>0, \qquad \inf_{x\in\Omega} S[n_0,j_0,\theta_0]>0. \tag{1.10}$$

The strength of the boundary data defined by

$$\delta:=|n_l-n_r|+|\theta_l-\theta_L|+|\theta_r-\theta_L|+|\phi_r|, \tag{1.11}$$

plays a crucial role in the proofs of our main results in what follows.

The aim of this paper is to investigate the existence, uniqueness and asymptotic stability of subsonic steady states, solving the boundary value problem (BVP) below:

$$\begin{cases} \tilde{j}_x=0, & (1.12a)\\ S[\tilde{n},\tilde{j},\tilde{\theta}]\tilde{n}_x+\tilde{n}\tilde{\theta}_x-\varepsilon^2\tilde{n}\left[\dfrac{(\sqrt{\tilde{n}})_{xx}}{\sqrt{\tilde{n}}}\right]_x=\tilde{n}\tilde{\phi}_x-\tilde{j}, & (1.12b)\\ \tilde{j}\tilde{\theta}_x-\dfrac{2}{3}\tilde{j}\tilde{\theta}\big(\ln\tilde{n}\big)_x-\dfrac{2}{3}\tilde{\theta}_{xx}-\dfrac{\varepsilon^2}{3}\left[\tilde{n}\left(\dfrac{\tilde{j}}{\tilde{n}}\right)_{xx}\right]_x=\dfrac{1}{3}\dfrac{\tilde{j}^2}{\tilde{n}}-\tilde{n}(\tilde{\theta}-\theta_L), & (1.12c)\\ \tilde{\phi}_{xx}=\tilde{n}-D(x), \qquad \forall x\in\Omega, & (1.12d) \end{cases}$$

and

$$\tilde{n}(0)=n_l, \qquad \tilde{n}(1)=n_r, \tag{1.13a}$$

$$\big(\sqrt{\tilde{n}}\big)_{xx}(0)=\big(\sqrt{\tilde{n}}\big)_{xx}(1)=0, \tag{1.13b}$$

$$\tilde{\theta}(0)=\theta_l, \qquad \tilde{\theta}(1)=\theta_r, \tag{1.13c}$$

$$\tilde{\phi}(0)=0, \qquad \tilde{\phi}(1)=\phi_r. \tag{1.13d}$$

Throughout the rest of this paper, we will use the following notations. For a nonnegative integer $l\geq 0$, $H^l(\Omega)$ denotes the l-th order Sobolev space in the L^2 sense, equipped with the norm $\|\cdot\|_l$. In particular, $H^0=L^2$ and $\|\cdot\|:=\|\cdot\|_0$. For a nonnegative integer $k\geq 0$, $C^k(\overline{\Omega})$ denotes the k-times continuously differentiable function space, equipped with the norm $|f|_k:=\sum_{i=0}^k\sup_{x\in\overline{\Omega}}|\partial_x^i f(x)|$. The positive constants C, C_1, $\cdots$ only depend on n_l, θ_L and $|D|_0$. If the constants C, C_1, $\cdots$ additionally depend on some other quantities

α, β, $\cdots$, we write $C(\alpha,\beta,\cdots)$, $C_1(\alpha,\beta,\cdots)$, $\cdots$. The notations $\mathfrak{Y}^l_m$ and $\mathfrak{Z}$ denote the function spaces defined by

$$\mathfrak{Y}^l_m([0,T]) := \bigcap_{k=0}^{[m/2]} C^k([0,T];H^{l+m-2k}(\Omega)), \quad m=2,3,4, \text{and } l=0,2,$$
$$\mathfrak{Z}([0,T]) := C^2([0,T];H^2(\Omega)).$$

We are now in a position to formulate our main results.

Theorem 1.1 (Existence and uniqueness of steady states). *Suppose that the doping profile and the boundary data satisfy conditions (1.7) and (1.13). For arbitrary positive constants n_l and θ_L, there exist three positive constants δ_1, $\varepsilon_1(\le 1)$ and C such that if $\delta \le \delta_1$ and $0<\varepsilon\le\varepsilon_1$, then the BVP (1.12)∼(1.13) has a unique solution $(\tilde{n},\tilde{j},\tilde{\theta},\tilde{\phi}) \in H^4(\Omega)\times H^4(\Omega)\times H^3(\Omega)\times C^2(\overline{\Omega})$ satisfying the condition (1.9) and the uniform estimates*

$$0 < b^2 \le \tilde{n} \le B^2, \quad 0 < \frac{1}{2}\theta_L \le \tilde{\theta} \le \frac{3}{2}\theta_L, \tag{1.14a}$$
$$\|\tilde{n}\|_2 + \|(\varepsilon\partial_x^3\tilde{n}, \varepsilon^2\partial_x^4\tilde{n})\| + |\tilde{\phi}|_2 \le C, \tag{1.14b}$$
$$|\tilde{j}| + \|\tilde{\theta}-\theta_L\|_3 \le C\delta, \tag{1.14c}$$

where the positive constants B and b are defined as follows

$$B := \frac{3}{2}\sqrt{n_l}\, e^{2|D|_0/\theta_L}, \quad b := \frac{1}{2}\sqrt{n_l}\, e^{-(B^2+2|D|_0)/\theta_L}. \tag{1.15}$$

Theorem 1.2 (Asymptotic stability of steady states). *Assume that the doping profile and the boundary data satisfy conditions (1.7) and (1.5). Let the initial data $(n_0,j_0,\theta_0)\in H^4(\Omega)\times H^3(\Omega)\times H^2(\Omega)$ and satisfies the conditions (1.6) and (1.10). For arbitrary positive constants n_l and θ_L, there exist four positive constants δ_2, ε_2, γ and C such that if $0<\varepsilon\le\varepsilon_2$ and $\delta + \|(n_0-\tilde{n}, j_0-\tilde{j}, \theta_0-\tilde{\theta})\|_2 + \|(\varepsilon\partial_x^3(n_0-\tilde{n}), \varepsilon\partial_x^3(j_0-\tilde{j}), \varepsilon^2\partial_x^4(n_0-\tilde{n}))\| \le \delta_2$, then the IBVP (1.3)∼(1.5) has a unique global solution (n,j,θ,ϕ) satisfying the condition (1.9) in $\left[\mathfrak{Y}_4([0,\infty))\cap H^2_{loc}(0,\infty;H^1(\Omega))\right] \times \left[\mathfrak{Y}_3([0,\infty))\cap H^2_{loc}(0,\infty;L^2(\Omega))\right] \times \left[\mathfrak{Y}_2([0,\infty))\cap H^1_{loc}(0,\infty;H^1(\Omega))\right] \times \mathfrak{Z}([0,\infty))$. Moreover, the solution verifies the additional regularity $\phi-\tilde{\phi}\in\mathfrak{Y}^2_4([0,\infty))$ and the decay estimate*

$$\begin{aligned}
&\|(n-\tilde{n}, j-\tilde{j}, \theta-\tilde{\theta})(t)\|_2 \\
&\qquad + \|(\varepsilon\partial_x^3(n-\tilde{n}), \varepsilon\partial_x^3(j-\tilde{j}), \varepsilon^2\partial_x^4(n-\tilde{n}))(t)\| + \|(\phi-\tilde{\phi})(t)\|_4 \\
&\le C\Big(\|(n_0-\tilde{n}, j_0-\tilde{j}, \theta_0-\tilde{\theta})\|_2 \\
&\qquad + \|(\varepsilon\partial_x^3(n_0-\tilde{n}), \varepsilon\partial_x^3(j_0-\tilde{j}), \varepsilon^2\partial_x^4(n_0-\tilde{n}))\|\Big) e^{-\gamma t}, \quad \forall t\in[0,\infty).
\end{aligned} \tag{1.16}$$

Remark 1.1. It is worth mentioning that the uniform estimates (1.14) and (1.16) in Theorems 1.1 and 1.2 are fairly useful for further discussing the semi-classical limit ($\varepsilon \to 0$) of both steady states and global-in-time solutions. However, the rigorous verification of the semi-classical limit is somewhat lengthy and technical, whence we refer the reader to a sequel [10] to the present paper for details.

We conclude this section by illustrating the main ideas in the proofs of Theorems 1.1 and 1.2. Though both of the proofs are lengthy, the basic ideas are easily comprehensible.

Firstly, the proof of Theorem 1.1 is carried out in five steps. In Step1 we reduce the steady-state problem (1.12)∼(1.13) to an easily-handled one via the transformation $\tilde{w}=\sqrt{\tilde{n}}$. Step2 introduces a fixed point operator $\mathcal{T}$ by decoupling the reformulated problem into a semilinear problem $(P1)$ and a linear nonlocal problem $(P2)$. Thus Schauder's Fixed Point Theorem applies to $\mathcal{T}$, which a fixed point argument in Step5 guarantees that Theorem 1.1 holds. Step3 and Step4 are respectively devoted to proving the solvability of $(P1)$ and $(P2)$ via the Leray-Schauder Fixed Point Theorem.

As far as the proof of Theorem 1.2 is concerned, similarly to Theorem 1.1, the transformation $w=\sqrt{n}$ is introduced to simplify the original problem (1.3)∼(1.5). Subsequently, we solve the reformulated problem by the standard continuation principle. In the continuation argument, we first state the local existence, and then establish the uniform a priori estimate via energy methods.

The organization of this paper is as follows. The proof of Theorem 1.1 is given in Section 2. The proof of Theorem 1.2 is given in Section 3, including the local existence lemma, basic estimate, higher-order estimates and decay estimate.

2. Existence and uniqueness of steady states

In this section, we show Theorem 1.1. The proof is based on Schauder's Fixed Point Theorem (cf. Corollary 11.2 in [8]), the Leray-Schauder Fixed Point Theorem (cf. Theorem 11.3 in [8]), the truncation method and energy estimates.

Proof. **(Proof of Theorem 1.1.)** In order to make the proof clear, we will divide it into a sequence of steps.

Step1. Reformulation. The transformation $\tilde{w}:=\sqrt{\tilde{n}}$ and a standard calculation equivalently reduce the BVP (1.12)∼(1.13) to the following BVP with a constant subsonic current density $\tilde{j}$ determined shortly in (2.5):

$$\begin{cases} \varepsilon^2\tilde{w}_{xx}=h(\tilde{w},\tilde{\theta}), & (2.1a)\\ \frac{2}{3}\tilde{\theta}_{xx}-\tilde{j}\tilde{\theta}_x+\frac{2}{3}\tilde{j}\tilde{\theta}(\ln\tilde{w}^2)_x-\tilde{w}^2(\tilde{\theta}-\theta_L)=g(\tilde{w},\tilde{\theta};\varepsilon), \quad x\in\Omega, & (2.1b)\end{cases}$$

and boundary conditions

$$\tilde{w}(0)=w_l, \qquad \tilde{w}(1)=w_r, \tag{2.2a}$$

$$\tilde{\theta}(0)=\theta_l, \qquad \tilde{\theta}(1)=\theta_r, \tag{2.2b}$$

where

$$F(a_1,a_2,a_3):=\frac{a_2^2}{2a_1^2}+a_3+a_3\ln a_1, \quad w_l:=\sqrt{n_l}, \quad w_r:=\sqrt{n_r}, \tag{2.3a}$$

$$\tilde{\phi}(x)=G[\tilde{w}^2](x):=\int_0^1 G(x,y)(\tilde{w}^2-D)(y)dy+\phi_r x, \quad G(x,y):=\begin{cases} x(y-1), & x<y,\\ y(x-1), & x>y,\end{cases} \tag{2.3b}$$

$$h(\tilde{w},\tilde{\theta}):=\tilde{w}\left[F(\tilde{w}^2,\tilde{j},\tilde{\theta})-F(n_l,\tilde{j},\theta_l)-\tilde{\phi}-\int_0^x\tilde{\theta}_x\ln\tilde{w}^2dy+\tilde{j}\int_0^x\tilde{w}^{-2}dy\right], \tag{2.3c}$$

$$g(\tilde{w},\tilde{\theta};\varepsilon):=-\frac{1}{3}\frac{\tilde{j}^2}{\tilde{w}^2}+\frac{\varepsilon^2}{3}\tilde{j}\left(\frac{12\tilde{w}_x^3}{\tilde{w}^3}-\frac{14\tilde{w}_x\tilde{w}_{xx}}{\tilde{w}^2}+\frac{2\tilde{w}_{xxx}}{\tilde{w}}\right). \tag{2.3d}$$

Taking $x=1$ in Equation (2.1a), the boundary condition (1.13b) yields the current-voltage relationship

$$F(n_r,\tilde{j},\theta_r)-F(n_l,\tilde{j},\theta_l)-\phi_r-\int_0^1\tilde{\theta}_x\ln\tilde{w}^2dy+\tilde{j}\int_0^1\tilde{w}^{-2}dy=0, \tag{2.4}$$

which further implies that $\tilde{j}$ can be expressed by the following explicit formula

$$\tilde{j}=J[\tilde{w}^2,\tilde{\theta}]:=2\Big(\bar{b}+\int_0^1\tilde{\theta}_x\ln\tilde{w}^2dy\Big)K[\tilde{w}^2,\tilde{\theta}]^{-1}, \tag{2.5}$$

$$K[\tilde{w}^2,\tilde{\theta}]:=\int_0^1\tilde{w}^{-2}dy+\sqrt{\Big(\int_0^1\tilde{w}^{-2}dy\Big)^2+2\Big(\bar{b}+\int_0^1\tilde{\theta}_x\ln\tilde{w}^2dy\Big)\Big(n_r^{-2}-n_l^{-2}\Big)},$$

$$\bar{b}:=\phi_r-\theta_r+\theta_l-\theta_r\ln n_r+\theta_l\ln n_l.$$

The uniqueness of $\tilde{j}$ is guaranteed by the subsonic condition (1.9b).

Step2. Defining the fixed point operator. We will apply Schauder's Fixed Point Theorem in a closed convex subset

$$\mathcal{U}[N_1,N_2]:=\Big\{q\in C^2(\overline{\Omega})\,\Big|\,\|q-\theta_L\|_1\le N_1\delta,\quad \|q_{xx}\|\le N_2\delta,\quad q(0)=\theta_l,\ q(1)=\theta_r\Big\} \tag{2.6}$$

of the Banach space $C^2(\overline{\Omega})$ to solve the BVP (2.1)∼(2.2), where N_1 and N_2 are positive constants to be determined later (cf. (2.42)). The fixed point operator

$$\begin{aligned}\mathcal{T}:\ \mathcal{U}[N_1,N_2]&\longrightarrow H^3(\Omega)\\ q&\longmapsto Q\end{aligned} \tag{2.7}$$

is defined as follows. Given $q\in\mathcal{U}[N_1,N_2]$, we assert that the semilinear elliptic problem

$$(P1)\quad\begin{cases}\varepsilon^2u_{xx}=h(u,q),\quad x\in\Omega, & (2.8a)\\ u(0)=w_l,\quad u(1)=w_r, & (2.8b)\end{cases}$$

for the unknown u is uniquely solvable; namely, we have

Claim1. Given $q\in\mathcal{U}[N_1,N_2]$, if δ and ε are small enough, then $(P1)$ has a unique solution $u=u[q]\in H^4(\Omega)$ satisfying the uniform estimates (in ε)

$$0<b\le u(x)\le B, \tag{2.9a}$$

$$\|u\|_2+\|(\varepsilon\partial_x^3u,\varepsilon^2\partial_x^4u)\|\le C, \tag{2.9b}$$

where the positive constants b and B are given by (1.15), and the positive constant C only depends on n_l, θ_L and $|D|_0$.

Subsequently, let us define $\mathcal{T}q=Q$ whenever Q uniquely solves the linear nonlocal elliptic problem

$$(P2)\quad\begin{cases}\frac{2}{3}Q_{xx}-JQ_x+\frac{2}{3}J_*(\ln u^2)_x\theta_L\\ \quad+\frac{2}{3}J(\ln u^2)_x(Q-\theta_L)-u^2(Q-\theta_L)=g(u,q;\varepsilon),\quad x\in\Omega, & (2.10a)\\ Q(0)=\theta_l,\quad Q(1)=\theta_r, & (2.10b)\end{cases}$$

where $u=u[q]$ is derived from q via Claim1, and $J=J[u^2,q]$, $J_*=2\big(\bar{b}+\int_0^1Q_x\ln u^2dx\big)K[u^2,q]^{-1}$. As for $(P2)$, we have

Claim2. *Given (u,q) in Claim1, if δ is small enough, then $(P2)$ admits a unique solution $Q\in H^3(\Omega)$ satisfying the uniform estimate (in ε)*

$$\|Q-\theta_L\|_1 \le C_1\delta + C_2(b,B,N_1)\delta^2, \tag{2.11a}$$
$$\|Q_{xx}\| \le C_3(b,B,N_1)\delta, \tag{2.11b}$$
$$\|Q_{xxx}\| \le C_4(b,B,N_1,N_2)\delta, \tag{2.11c}$$

where the positive constant C_1 only depends on n_l, θ_L and $|D|_0$.

We will set about demonstrating Claims 1 and 2 by applying the Leray-Schauder Fixed Point Theorem, in Steps 3 and 4 respectively.

Step3. *Proof of Claim1.* To avoid singularity, we proceed by the truncation method to reduce $(P1)$ to the truncated problem

$$(tP) \quad \begin{cases} \varepsilon^2 u_{xx} = h(u_{\alpha\beta}, q), & x\in\Omega, \quad (2.12a)\\ u(0)=w_l, \quad u(1)=w_r, & \quad (2.12b)\end{cases}$$

where

$$u_{\alpha\beta} := \max\{\beta, \min\{\alpha, u\}\}, \quad 0<\frac{1}{2}b =: \beta < \alpha := 2B.$$

Firstly, we will employ the Leray-Schauder Fixed Point Theorem in $H^1(\Omega)$ to solve (tP). For all $r\in H^1(\Omega)$, the fixed point operator $\mathcal{T}_1$ is defined by letting $R=\mathcal{T}_1 r$ be the unique solution in $H^3(\Omega)$ of the linear problem

$$\begin{cases} \varepsilon^2 R_{xx} = h(r_{\alpha\beta}, q), & x\in\Omega, \quad (2.13a)\\ R(0)=w_l, \quad R(1)=w_r. & \quad (2.13b)\end{cases}$$

Standard arguments give that the operator $\mathcal{T}_1$ is continuous and compact. It only remains to show that there exists a positive constant M_1 such that $\|v\|_1 \le M_1$ for all $v\in H^1(\Omega)$ and $\lambda\in[0,1]$ satisfying $v=\lambda\mathcal{T}_1 v$. The equation $v=\lambda\mathcal{T}_1 v$ is equivalent to the semilinear problem

$$\begin{cases} \varepsilon^2 v_{xx} = \lambda h(v_{\alpha\beta}, q), & x\in\Omega, \quad (2.14a)\\ v(0)=\lambda w_l, \quad v(1)=\lambda w_r. & \quad (2.14b)\end{cases}$$

Multiplying the identity (2.14a) by $(v-\lambda\bar{w})$ and integrating over Ω, where $\bar{w}(x)=w_l(1-x)+w_r x$, yields the estimate

$$\|v\|_1 \le C\left(1+\frac{1}{\varepsilon}\right), \tag{2.15}$$

provided δ is small enough. We can now take $M_1=C(1+\varepsilon^{-1})$ because the constant C does not depend on $\lambda\in[0,1]$. Applying the Leray-Schauder Fixed Point Theorem in $H^1(\Omega)$, we conclude that $\mathcal{T}_1$ has a fixed point $u\in H^3(\Omega)$, which in turn solves (tP).

Secondly, we have to show that the solution u to (tP) has strictly positive lower and upper bounds so as to remove the truncation. We will apply a variant of Stampacchia's maximum argument to reach this target, which further implies the equivalence between (tP) and $(P1)$. Observe that for all $q\in\mathcal{U}[N_1,N_2]$, we have

$$0<\frac{1}{2}\theta_L \le q(x) \le \frac{3}{2}\theta_L, \quad \|q_x\|\le N_1\delta, \quad \|q_{xx}\|\le N_2\delta, \tag{2.16}$$

whenever δ is sufficiently small. Taking $-\Big(\big(\ln(u_{\alpha\beta}^2/\bar{n})\big)_+\Big)^k \in H_0^1(\Omega)$ as test function in Equation (2.12a), where $\bar{n}=\max\{n_l,n_r\}$, $(\cdot)_+=\max\{0,\cdot\}$ and $k=1,2,3,\cdots$, gives:

$$\int_0^1 2\varepsilon^2 k\frac{[(u_{\alpha\beta})_x]^2}{u_{\alpha\beta}}\left(\ln\frac{u_{\alpha\beta}^2}{\bar{n}}\right)_+^{k-1}dx=\int_0^1 -h(u_{\alpha\beta},q)\left(\ln\frac{u_{\alpha\beta}^2}{\bar{n}}\right)_+^{k}dx. \tag{2.17}$$

It is important to note that the left-hand side of the identity (2.17) is nonnegative. As for the right-hand side of (2.17), assuming a priori that $\delta\ll 1$, it can be calculated as follows:

$$\begin{aligned}
(2.17)_r=&\int_0^1 -u_{\alpha\beta}\Big[F(u_{\alpha\beta}^2,\bar{J},q)-q\ln\bar{n}+q\ln\bar{n}-F(n_l,\bar{J},\theta_l)-G[u_{\alpha\beta}^2]\\
&-\int_0^x q_x\ln u_{\alpha\beta}^2dy+\bar{J}\int_0^x u_{\alpha\beta}^{-2}dy\Big]\left(\ln\frac{u_{\alpha\beta}^2}{\bar{n}}\right)_+^k dx\\
=&-\int_0^1 u_{\alpha\beta}q\left(\ln\frac{u_{\alpha\beta}^2}{\bar{n}}\right)_+^{k+1}dx\\
&+\int_0^1\left(\phi_r x-\int_0^1 G(x,y)D(y)dy-\bar{J}\int_0^x u_{\alpha\beta}^{-2}dy+\frac{\bar{J}^2}{2n_l}\right)u_{\alpha\beta}\left(\ln\frac{u_{\alpha\beta}^2}{\bar{n}}\right)_+^k dx\\
&+\int_0^1\left(\theta_l\ln n_l-q\ln\bar{n}+\theta_l-q+\int_0^x q_x\ln u_{\alpha\beta}^2dy\right)u_{\alpha\beta}\left(\ln\frac{u_{\alpha\beta}^2}{\bar{n}}\right)_+^k dx\\
&+\int_0^1\left(\int_0^1 \underbrace{G(x,y)}_{\le 0}u_{\alpha\beta}^2(y)dy-\frac{\bar{J}^2}{2u_{\alpha\beta}^4}\right)u_{\alpha\beta}\left(\ln\frac{u_{\alpha\beta}^2}{\bar{n}}\right)_+^k dx\\
\le&-\int_0^1\frac{1}{2}\theta_L u_{\alpha\beta}\left(\ln\frac{u_{\alpha\beta}^2}{\bar{n}}\right)_+^{k+1}dx+\int_0^1(C(N_1)\delta+|D|_0)u_{\alpha\beta}\left(\ln\frac{u_{\alpha\beta}^2}{\bar{n}}\right)_+^k dx\\
&+\int_0^1 C(N_1)\delta u_{\alpha\beta}\left(\ln\frac{u_{\alpha\beta}^2}{\bar{n}}\right)_+^k dx\\
\le&-\int_0^1\frac{1}{2}\theta_L u_{\alpha\beta}\left(\ln\frac{u_{\alpha\beta}^2}{\bar{n}}\right)_+^{k+1}dx+\int_0^1\frac{1}{2}\theta_L u_{\alpha\beta}\underbrace{\left(\ln\frac{u_{\alpha\beta}^2}{\bar{n}}\right)_+^k\frac{4|D|_0}{\theta_L}}_{\text{Young's inequality}}dx\\
\le&-\int_0^1\frac{1}{2}\theta_L u_{\alpha\beta}\left(\ln\frac{u_{\alpha\beta}^2}{\bar{n}}\right)_+^{k+1}dx\\
&+\int_0^1\frac{1}{2}\theta_L u_{\alpha\beta}\left[\frac{k}{k+1}\left(\ln\frac{u_{\alpha\beta}^2}{\bar{n}}\right)_+^{k+1}+\frac{1}{k+1}\left(\frac{4|D|_0}{\theta_L}\right)^{k+1}\right]dx\\
=&-\frac{1}{k+1}\frac{1}{2}\theta_L\int_0^1 u_{\alpha\beta}\left(\ln\frac{u_{\alpha\beta}^2}{\bar{n}}\right)_+^{k+1}dx+\frac{1}{k+1}\frac{1}{2}\theta_L\left(\frac{4|D|_0}{\theta_L}\right)^{k+1}\int_0^1\underbrace{u_{\alpha\beta}}_{\le\alpha}dx\\
\le&\frac{\theta_L}{2(k+1)}\left[-\int_0^1 u_{\alpha\beta}\left(\ln\frac{u_{\alpha\beta}^2}{\bar{n}}\right)_+^{k+1}dx+\alpha\left(\frac{4|D|_0}{\theta_L}\right)^{k+1}\right].
\end{aligned}\tag{2.18}$$

where $\bar{J}=J[u_{\alpha\beta}^2,q]$ satisfies the estimate $|\bar{J}|\le C(\alpha,\beta,N_1)\delta$; we have also used the estimate (2.16), non-positivity of Green's function $G(x,y)$ and Young's inequality. Substituting (2.18) into (2.17) yields

$$\left\|\left(\ln\frac{u_{\alpha\beta}^2}{\bar{n}}\right)_+\right\|_{L^{k+1}(\Omega)}\le\left(\frac{\alpha}{\sqrt{\bar{n}}}\right)^{\frac{1}{k+1}}\frac{4|D|_0}{\theta_L}. \tag{2.19}$$

After passage to the limit as $k\to\infty$, it follows that

$$\left\|\left(\ln\frac{u_{\alpha\beta}^2}{\bar{n}}\right)_+\right\|_{L^{\infty}(\Omega)}\le\frac{4|D|_0}{\theta_L}, \tag{2.20}$$

which in turn implies

$$u_{\alpha\beta}\le\sqrt{\bar{n}}e^{2|D|_0/\theta_L}\le B. \tag{2.21}$$

Analyses similar to those in (2.17)∼(2.21) show that

$$\left\|\left(\ln\frac{u_{\alpha\beta}^2}{\underline{n}}\right)_-\right\|_{L^{2k}(\Omega)}\le\frac{2\left(B^2+2|D|_0\right)}{\theta_L},\quad k=1,2,3,\cdots, \tag{2.22}$$

provided $\delta\ll 1$. Taking the limit as $k\to\infty$ gives

$$\left\|\left(\ln\frac{u_{\alpha\beta}^2}{\underline{n}}\right)_-\right\|_{L^{\infty}(\Omega)}\le\frac{2\left(B^2+2|D|_0\right)}{\theta_L}. \tag{2.23}$$

Because $\left(\ln(u_{\alpha\beta}^2/\underline{n})\right)_-$ is nonpositive, it follows from (2.23) that

$$u_{\alpha\beta}\ge\sqrt{\underline{n}}e^{-(B^2+2|D|_0)/\theta_L}\ge b. \tag{2.24}$$

Effectively we have chosen $-u_{\alpha\beta}^{-1}\left(\left(\ln(u_{\alpha\beta}^2/\underline{n})\right)_-\right)^{2k-1}$ as test function in (2.12a) to establish the above lower bound, where $\underline{n}=\min\{n_l,n_r\}$ and $(\cdot)_-=\min\{0,\cdot\}$. Then (2.21) combined with (2.24) gives $u_{\alpha\beta}=u$, which in turn implies the existence result of $(P1)$ and the uniform estimate (2.9a).

Thirdly, the uniform estimate (2.9b) can be established by drawing on energy methods; a bootstrap argument gives the H^4-regularity of u. More precisely, performing the following procedures

$$\int_0^1\left[(2.8\text{a})\times\frac{1}{u}\right]_x\times u_x dx, \tag{2.25}$$

and

$$\int_0^1\left[(2.8\text{a})\times\frac{1}{u}\right]_x\times\left(\frac{u_{xx}}{u}\right)_x dx, \tag{2.26}$$

by integration by parts, yields

$$\|u_x\|_1,\ \|\varepsilon(u_{xx}/u)_x\|\le C. \tag{2.27}$$

Furthermore, noting that

$$\varepsilon\partial_x^3 u=\varepsilon u\left(\frac{u_{xx}}{u}\right)_x+\varepsilon\frac{u_x u_{xx}}{u},$$

and taking the L^2-norm gives

$$\|\varepsilon\partial_x^3 u\| \le C. \tag{2.28}$$

Similarly, applying the differential operator ∂_x^2 to the Equation (2.8a) and taking the L^2-norm yields

$$\|\varepsilon^2\partial_x^4 u\| \le C, \tag{2.29}$$

where the generic constant C in (2.27)∼(2.29) only depends on n_l, θ_L and $|D|_0$.

Finally, we can prove the uniqueness result of $(P1)$ based on the uniform estimate (2.9). To this end, we assume that u_1, u_2 are two solutions to $(P1)$, and introduce $z_i = \ln u_i^2$, $J_i = J[e^{z_i}, q]$, $S_i = q - J_i^2/e^{2z_i}$, $\varphi_i = G[e^{z_i}]$, $i = 1,2$, for simplicity. A straightforward calculation in terms of the procedure

$$\left[(2.8a)\times\frac{1}{u_1}\right]_x - \left[(2.8a)\times\frac{1}{u_2}\right]_x \tag{2.30}$$

forms the working equation for the difference $z = z_1 - z_2$, that is

$$-\left(\frac{J_1^2}{e^{2z_1}} - \frac{J_2^2}{e^{2z_2}}\right)z_{1x} + S_2 z_x - \frac{\varepsilon^2}{2}\left[z_{xx} + \frac{z_{1x}^2}{2} - \frac{z_{2x}^2}{2}\right]_x = (\varphi_1 - \varphi_2)_x - \left(\frac{J_1}{e^{z_1}} - \frac{J_2}{e^{z_2}}\right). \tag{2.31}$$

The following estimates are a matter of straightforward computation based on the L^∞ estimate (2.9a) and the explicit formula (2.5):

$$|J_i| \le C(b,B,N_1)\delta, \qquad |J_1 - J_2| \le C\delta\|z_x\|. \tag{2.32}$$

Multiplying (2.31) by z_x, integrating over Ω and using boundary conditions

$$z_i(0) = \ln n_l, \quad z_i(1) = \ln n_r, \quad \left(z_{ixx} + \frac{z_{ix}^2}{2}\right)(0) = \left(z_{ixx} + \frac{z_{ix}^2}{2}\right)(1) = 0, \tag{2.33}$$

we write

$$\begin{aligned}
&\int_0^1 S_2 z_x^2 dx + \int_0^1 \frac{\varepsilon^2}{2} z_{xx}^2 dx + \int_0^1 \underbrace{(e^{z_1} - e^{z_2})z}_{\ge 0} dx \\
&= \int_0^1 \left(\frac{J_1^2}{e^{2z_1}} - \frac{J_2^2}{e^{2z_2}}\right) z_{1x} z_x dx - \int_0^1 \frac{\varepsilon^2(z_{1x} + z_{2x})}{4} z_x z_{xx} dx - \int_0^1 \left(\frac{J_1}{e^{z_1}} - \frac{J_2}{e^{z_2}}\right) z_x dx.
\end{aligned} \tag{2.34}$$

The left-hand side of the identity (2.34) can easily be bounded below by $\frac{\theta_L}{4}\|z_x\|^2 + \frac{\varepsilon^2}{2}\|z_{xx}\|^2$. The right-hand side of (2.34) can be estimated by Hölder's inequality, Poincaré's inequality and the Cauchy-Schwarz inequality, together with estimates (2.9) and (2.32), as follows:

$$\begin{aligned}
(2.34)_r &\le C\Big(|J_1 - J_2|\|z_x\| + |J_2|\|z\|\|z_x\|\Big) + C\varepsilon^2\|z_x\|\|z_{xx}\| \\
&\le C(B,b,N_1)\delta\|z_x\|^2 + \frac{\varepsilon^2}{4}\|z_{xx}\|^2 + C\varepsilon^2\|z_x\|^2 \\
&= \Big(C(B,b,N_1)\delta + C\varepsilon^2\Big)\|z_x\|^2 + \frac{\varepsilon^2}{4}\|z_{xx}\|^2.
\end{aligned} \tag{2.35}$$

Thus if δ and ε are small enough, we have $\|z_x\|_1=0$, which implies $u_1=u_2$.

Step4. Proof of Claim2. The unique solvability of $(P2)$ can be proved in much the same way as in Step3. In particular, the Leray-Schauder Fixed Point Theorem in $H^1(\Omega)$ applies. Fix (u,q) derived from *Claim1*, for all $q_1\in H^1(\Omega)$, define the fixed point operator $\mathcal{T}_2$ by letting $Q_1=\mathcal{T}_2q_1$ be the unique solution in $H^3(\Omega)$ of the linear problem

$$\begin{cases} \frac{2}{3}Q_{1xx}-JQ_{1x}+\frac{2}{3}J_{1*}(\ln u^2)_x\theta_L \\ \quad +\frac{2}{3}J(\ln u^2)_x(q_1-\theta_L)-u^2(Q_1-\theta_L)=g(u,q;\varepsilon), \quad x\in\Omega, & (2.36a)\\ Q_1(0)=\theta_l, \quad Q_1(1)=\theta_r, & (2.36b)\end{cases}$$

where $J=J[u^2,q]$ and $J_{1*}=2\big(\bar{b}+\int_0^1 q_{1x}\ln u^2dx\big)K[u^2,q]^{-1}$. The same line of reasoning as $\mathcal{T}_1$ shows that $\mathcal{T}_2$ is continuous and compact. Thus it suffices to verify that there is a positive constant M_2 such that $\|\Theta\|_1\le M_2$ for all $\Theta\in H^1(\Omega)$ and $\lambda\in[0,1]$ satisfying the equation $\Theta=\lambda\mathcal{T}_2\Theta$, which is equivalent to the linear nonlocal problem

$$\begin{cases} \frac{2}{3}\Theta_{xx}-J\Theta_x+\frac{2}{3}\lambda J_*(\ln u^2)_x\theta_L \\ \quad +\frac{2}{3}\lambda J(\ln u^2)_x(\Theta-\theta_L)-u^2(\Theta-\lambda\theta_L)=\lambda g(u,q;\varepsilon), \quad x\in\Omega, & (2.37a)\\ \Theta(0)=\lambda\theta_l, \quad \Theta(1)=\lambda\theta_r, \qquad \forall\lambda\in[0,1], & (2.37b)\\ J:=J[u^2,q], \quad J_*:=2\left(\bar{b}+\int_0^1\Theta_x\ln u^2dx\right)K[u^2,q]^{-1}. \end{cases}$$

Rewrite the Equation (2.37a) in terms of $\Theta_\lambda=\Theta-\lambda\bar{\theta}$, where $\bar{\theta}(x)=\theta_l(1-x)+\theta_r x$, and then multiply by Θ_λ, integrate over Ω, to get

$$\begin{aligned} &\frac{2}{3}\|\Theta_{\lambda x}\|^2+b^2\|\Theta_\lambda\|^2 \\ \le& -\frac{2}{3}\lambda\theta_LJ_*\int_0^1\Theta_{\lambda x}\ln u^2dx-\int_0^1\left[\frac{2}{3}\lambda J\ln u^2\left(\Theta_\lambda^2\right)_x+J\Theta_{\lambda x}\Theta_\lambda\right]dx \\ &-\frac{2}{3}\lambda J\int_0^1\ln u^2\Big[(\lambda\bar{\theta}-\theta_L)\Theta_\lambda\Big]_x dx-\lambda\int_0^1\Big[J\bar{\theta}_x+u^2(\bar{\theta}-\theta_L)+g(u,q;\varepsilon)\Big]\Theta_\lambda dx \\ \le& -\frac{4\lambda\theta_L}{3K[u^2,q]}\left(\bar{b}+\int_0^1(\Theta_{\lambda x}+\lambda\bar{\theta}_x)\ln u^2dx\right)\int_0^1\Theta_{\lambda x}\ln u^2dx+C(b,B,N_1)\delta\|\Theta_\lambda\|_1^2 \\ &+\mu\|\Theta_\lambda\|_1^2+C(\mu,b,B,N_1)\delta^2\left(\delta^2+\|\lambda\bar{\theta}-\theta_L\|^2\right)+C(\mu,b,B)\delta^2 \\ \le& \underbrace{-\frac{4\lambda\theta_L}{3K[u^2,q]}\left(\int_0^1\Theta_{\lambda x}\ln u^2dx\right)^2}_{\le 0}+\big[\mu+C(b,B,N_1)\delta\big]\|\Theta_\lambda\|_1^2 \\ &+C(\mu,b,B,N_1)\delta^2\left(\delta^2+\|\lambda\bar{\theta}-\theta_L\|^2\right)+C(\mu,b,B,\theta_L)\delta^2 \\ \le& \big[\mu+C(b,B,N_1)\delta\big]\|\Theta_\lambda\|_1^2+C(\mu,b,B,N_1)\delta^2\left(\delta^2+\|\lambda\bar{\theta}-\theta_L\|^2\right) \\ &+C(\mu,b,B,\theta_L)\delta^2, \end{aligned} \tag{2.38}$$

where we have used estimates (2.9) and $|J|\le C(b,B,N_1)\delta$. Thus if μ and δ are sufficiently small, we obtain

$$\|\Theta_\lambda\|_1^2\le C(b,B,\theta_L)\delta^2+C(b,B,N_1)\delta^2\left(\delta^2+\|\lambda\bar{\theta}-\theta_L\|^2\right), \tag{2.39}$$

which further implies

$$\begin{aligned}\|\Theta\|_1 =& \|\Theta_\lambda+\lambda\bar{\theta}\|_1 \le \|\Theta_\lambda\|_1+\lambda\|\bar{\theta}\|_1\\ \le& \sqrt{C(b,B,\theta_L)+C(b,B,N_1)(1+\theta_L^2)}+2\theta_L = M_2.\end{aligned}\tag{2.40}$$

The Leray-Schauder Fixed Point Theorem and standard theory of elliptic regularity guarantee that $(P2)$ admits a solution $Q\in H^3(\Omega)$. Besides, a similar argument to that in (2.38) gives the uniqueness result of $(P2)$. And lastly, taking $\Theta=Q$ and $\lambda=1$ in (2.39) automatically gives

$$\begin{aligned}\|Q-\theta_L\|_1 \le& \|Q-\bar{\theta}\|_1+\|\bar{\theta}-\theta_L\|_1\\ \le& C(b,B,\theta_L)\delta + C(b,B,N_1)\delta^2\\ =& C_1\delta+C_2(b,B,N_1)\delta^2,\end{aligned}\tag{2.41}$$

which is exactly the estimate (2.11a). In light of estimates $|J|,|J_*|\le C(b,B,N_1)\delta$, (2.9) and (2.11a), it follows from taking L^2-norm of ∂_x^k(2.10a) for $k=0,1$ that the estimates (2.11b) and (2.11c) hold.

Step5. End of the proof. Based on the estimate (2.11) we can at last determine the constants N_1 and N_2. Precisely, to set

$$N_1 := 2C_1, \quad N_2 := C_3(b,B,2C_1),\tag{2.42}$$

it is easy to see that if $\delta\le \frac{C_1}{C_2(b,B,2C_1)}$, then $\mathcal{T}$ maps $\mathcal{U}[N_1,N_2]$ into itself. Similar arguments to those in Step3 and Step4 show that $\mathcal{T}$ is continuous and the image $\mathcal{T}\big(\mathcal{U}[N_1,N_2]\big)$ is precompact, if we note that the embedding $H^3(\Omega)\hookrightarrow C^2(\overline{\Omega})$ is compact. Thus, Schauder's Fixed Point Theorem applies. Denote by $\tilde{\theta}$ the fixed point of $\mathcal{T}$ in $\mathcal{U}[N_1,N_2]$ and consequently $(\tilde{n},\tilde{j},\tilde{\theta},\tilde{\phi})\in H^4(\Omega)\times H^4(\Omega)\times H^3(\Omega)\times C^2(\overline{\Omega})$ is a solution to the BVP (1.12)$\sim$(1.13), where $\tilde{n}=\tilde{w}^2=\big(u[\tilde{\theta}]\big)^2$, $\tilde{j}=J[\tilde{w}^2,\tilde{\theta}]$ and $\tilde{\phi}=G[\tilde{w}^2]$. Repeating the computations developed above shows that the solution to the BVP (1.12)$\sim$(1.13) is unique, and satisfies the properties (1.9) and uniform estimates (1.14). ∎

3. Asymptotic stability of steady states

In this section, our basic strategy for proving Theorem 1.2 is the standard continuation principle because we are merely concerned with the small amplitude solution around the steady state.

Similarly to Section 2, we also introduce the transformation $w:=\sqrt{n}$ to reduce the IBVP (1.3)$\sim$(1.5) to the equivalent one:

$$\begin{cases} 2ww_t+j_x=0, & (3.1\text{a})\\ j_t+2S[w^2,j,\theta]ww_x+\dfrac{2j}{w^2}j_x+w^2\theta_x-\varepsilon^2w^2\left(\dfrac{w_{xx}}{w}\right)_x=w^2\phi_x-j, & (3.1\text{b})\\ w^2\theta_t+j\theta_x+\frac{2}{3}w^2\theta\left(\frac{j}{w^2}\right)_x-\frac{2}{3}\theta_{xx}-\frac{\varepsilon^2}{3}\left[w^2\left(\frac{j}{w^2}\right)_{xx}\right]_x=\frac{1}{3}\frac{j^2}{w^2}-w^2(\theta-\theta_L), & (3.1\text{c})\\ \phi_{xx}=w^2-D(x), \qquad \forall t>0,\ \forall x\in\Omega, & (3.1\text{d})\end{cases}$$

with initial conditions

$$(w,j,\theta)(0,x)=(w_0,j_0,\theta_0)(x), \quad w_0:=\sqrt{n_0},\tag{3.2}$$

and boundary conditions

$$w(t,0)=w_l, \qquad w(t,1)=w_r, \tag{3.3a}$$
$$w_{xx}(t,0)=w_{xx}(t,1)=0, \tag{3.3b}$$
$$\theta(t,0)=\theta_l, \qquad \theta(t,1)=\theta_r, \tag{3.3c}$$
$$\phi(t,0)=0, \qquad \phi(t,1)=\phi_r. \tag{3.3d}$$

Based on this, we proceed to introduce the perturbed variables close to the steady state, that is,

$$\psi(t,x):=w(t,x)-\tilde{w}(x), \qquad \eta(t,x):=j(t,x)-\tilde{j},$$
$$\chi(t,x):=\theta(t,x)-\tilde{\theta}(x), \qquad \sigma(t,x):=\phi(t,x)-\tilde{\phi}(x), \tag{3.4}$$

thereby obtaining the working equations which are more amenable to energy estimates:

$$\begin{cases} 2(\psi+\tilde{w})\psi_t+\eta_x=0, & (3.5a)\\ \left[\dfrac{\eta+\tilde{j}}{(\psi+\tilde{w})^2}\right]_t+\dfrac{1}{2}\left\{\left[\dfrac{\eta+\tilde{j}}{(\psi+\tilde{w})^2}\right]^2-\left(\dfrac{\tilde{j}}{\tilde{w}^2}\right)^2\right\}_x+\chi\left[\ln(\psi+\tilde{w})^2\right]_x \\ \quad +\tilde{\theta}\left[\ln(\psi+\tilde{w})^2-\ln\tilde{w}^2\right]_x+\chi_x-\varepsilon^2\left[\dfrac{(\psi+\tilde{w})_{xx}}{\psi+\tilde{w}}-\dfrac{\tilde{w}_{xx}}{\tilde{w}}\right]_x \\ =\sigma_x-\left[\dfrac{\eta+\tilde{j}}{(\psi+\tilde{w})^2}-\dfrac{\tilde{j}}{\tilde{w}^2}\right], & (3.5b)\\ (\psi+\tilde{w})^2\chi_t-\dfrac{2}{3}\chi_{xx}+\dfrac{2}{3}\tilde{\theta}\eta_x-\dfrac{4\tilde{j}\tilde{\theta}}{3\tilde{w}}\psi_x \\ \quad -\dfrac{\varepsilon^2}{3}\left\{(\psi+\tilde{w})^2\left[\dfrac{\eta+\tilde{j}}{(\psi+\tilde{w})^2}\right]_{xx}-\tilde{w}^2\left(\dfrac{\tilde{j}}{\tilde{w}^2}\right)_{xx}\right\}_x=H(t,x), & (3.5c)\\ \sigma_{xx}=(\psi+2\tilde{w})\psi, & (3.5d)\end{cases}$$

with initial conditions

$$\psi(0,x)=\psi_0(x):=w_0(x)-\tilde{w}(x), \quad \eta(0,x)=\eta_0(x):=j_0(x)-\tilde{j},$$
$$\chi(0,x)=\chi_0(x):=\theta_0(x)-\tilde{\theta}(x), \tag{3.6}$$

and boundary conditions

$$\psi(t,0)=\psi(t,1)=0, \tag{3.7a}$$
$$\psi_{xx}(t,0)=\psi_{xx}(t,1)=0, \tag{3.7b}$$
$$\chi(t,0)=\chi(t,1)=0, \tag{3.7c}$$
$$\sigma(t,0)=\sigma(t,1)=0, \tag{3.7d}$$

where the right-hand side of the Equation (3.5c) is defined by

$$H(t,x):=\frac{4\tilde{\theta}(\psi+\tilde{w})_x}{3(\psi+\tilde{w})}\eta-\tilde{w}^2\chi+H_1(t,x), \tag{3.8}$$

and

$$H_1(t,x):=\frac{4\chi(\psi+\tilde{w})_x}{3(\psi+\tilde{w})}\eta-(\psi+2\tilde{w})(\chi+\tilde{\theta}-\theta_L)\psi$$

$$
\begin{aligned}
&-(\chi+\tilde{\theta})_x\eta-\tilde{j}\chi_x-\frac{2\eta_x}{3}\chi+\frac{4\tilde{j}\tilde{\theta}(\psi+\tilde{w})_x}{3\tilde{w}^2}\psi+\frac{4\tilde{j}(\psi+\tilde{w})_x}{3(\psi+\tilde{w})}\chi\\
&+\frac{4\tilde{j}\tilde{\theta}(\psi+2\tilde{w})(\psi+\tilde{w})_x}{3\tilde{w}^2(\psi+\tilde{w})}\psi+\frac{\eta+2\tilde{j}}{3(\psi+\tilde{w})^2}\eta-\frac{\tilde{j}^2(\psi+2\tilde{w})}{3\tilde{w}^2(\psi+\tilde{w})^2}\psi. \qquad (3.9)
\end{aligned}
$$

For simplicity of notation, we set hereafter for some $T>0$ that

$$
\begin{aligned}
\mathbf{X}(0,T):=\Big\{(\psi,\eta,\chi,\sigma)\ \big|\ &\psi\in\mathfrak{Y}_4([0,T])\cap H^2(0,T;H^1(\Omega)),\\
&\eta\in\mathfrak{Y}_3([0,T])\cap H^2(0,T;L^2(\Omega)),\\
&\chi\in\mathfrak{Y}_2([0,T])\cap H^1(0,T;H^1(\Omega)),\ \sigma\in\mathfrak{Y}_4^2([0,T])\Big\}, \qquad (3.10)
\end{aligned}
$$

and for all $t\in[0,T]$ that

$$
n_\varepsilon(t):=\|(\psi,\eta,\chi)(t)\|_2+\|(\varepsilon\partial_x^3\psi,\varepsilon\partial_x^3\eta,\varepsilon^2\partial_x^4\psi)(t)\|,\quad N_\varepsilon(T):=\sup_{t\in[0,T]}n_\varepsilon(t). \qquad (3.11)
$$

Before giving the proof of Theorem 1.2, we first state a local existence lemma for the IBVP (3.5)∼(3.7). Its proof is strongly reminiscent of that in [24,25], because the dispersive velocity term here will not bring about any essential trouble. Therefore, we omit the details.

Lemma 3.1 (Local existence). *Suppose the initial data* $(\psi_0,\eta_0,\chi_0)\in H^4(\Omega)\times H^3(\Omega)\times H^2(\Omega)$ *and set*

$$
\inf_{x\in\Omega}(\psi_0+\tilde{w})>0,\quad \inf_{x\in\Omega}(\chi_0+\tilde{\theta})>0,\quad \inf_{x\in\Omega}S\big[(\psi_0+\tilde{w})^2,\eta_0+\tilde{j},\chi_0+\tilde{\theta}\big]>0.
$$

Also assume that the initial data are compatible with the boundary conditions (3.7). Then there exists a constant $T_0>0$ such that the IBVP (3.5)∼(3.7) admits a unique solution $(\psi,\eta,\chi,\sigma)\in\mathbf{X}(0,T_0)$ satisfying

$$
\inf_{x\in\Omega}(\psi+\tilde{w})>0,\quad \inf_{x\in\Omega}(\chi+\tilde{\theta})>0,\quad \inf_{x\in\Omega}S\big[(\psi+\tilde{w})^2,\eta+\tilde{j},\chi+\tilde{\theta}\big]>0.
$$

We are now ready to proceed to present the uniform a priori estimate for the local solution of the IBVP (3.5)∼(3.7), which is the most complicated part in the continuation principle.

Proposition 3.1 (Uniform a priori estimate). *Suppose that for some $T>0$, $(\psi,\eta,\chi,\sigma)\in\mathbf{X}(0,T)$ is a solution to the IBVP (3.5)∼(3.7). Then there exist positive constants δ_0, C and γ such that if $N_\varepsilon(T)+\delta+\varepsilon\le\delta_0$, then it holds:*

$$
n_\varepsilon(t)+\|\sigma(t)\|_4\le Cn_\varepsilon(0)e^{-\gamma t},\quad \forall t\in[0,T], \qquad (3.12)
$$

where the constants δ_0, C and γ are independent of δ, ε and T.

Based on Lemma 3.1 and Proposition 3.1, we are now in a position to complete the proof of Theorem 1.2.

Proof. **(Proof of Theorem 1.2.)** Combining Lemma 3.1 with Proposition 3.1 directly yields the global existence and exponential decay estimate of the solution to the IBVP (3.5)∼(3.7) according to the standard continuation principle, which in turn implies manifestly Theorem 1.2 because of the transformations $n=w^2$ and $\tilde{n}=\tilde{w}^2$. ◻

Obviously, it only remains for us to show Proposition 3.1. Owing to the difficulties caused by the quantum terms, the proof of Proposition 3.1 is technical, sophisticated and rather lengthy. In the next lemma, we firstly state without proof some frequently-used estimates of the local solution, as the proof of this lemma is just a tedious but straightforward computation based on Sobolev's inequalities and the system (3.5) itself.

LEMMA 3.2. *Under the same assumptions as in Proposition 3.1, the following estimates hold for* $t\in[0,T]$,

$$|\tilde{w}|_1+\left|\left(\varepsilon^{1/2}\tilde{w}_{xx},\varepsilon^{3/2}\tilde{w}_{xxx}\right)\right|_0\le C,\quad |\tilde{j}|+|\tilde{\theta}-\theta_L|_2\le C\delta, \tag{3.13a}$$

$$|(\psi,\eta,\chi)(t)|_1+\left|\left(\varepsilon^{1/2}\psi_{xx},\varepsilon^{1/2}\eta_{xx},\varepsilon^{3/2}\psi_{xxx},\psi_t,\eta_t\right)(t)\right|_0\le CN_\varepsilon(T), \tag{3.13b}$$

$$\|\partial_t^i\sigma(t)\|_2\le C\left[\|\partial_t^i\psi(t)\|+\frac{i(i-1)}{2}N_\varepsilon(T)\|\psi_t(t)\|\right],\quad i=0,1,2, \tag{3.13c}$$

$$\|\sigma_{tx}(t)\|\le\|\eta(t)\|,\quad \|\sigma(t)\|_4\le C\|\psi(t)\|_2, \tag{3.13d}$$

$$\|\partial_x^l\eta_x(t)\|\le C\|\psi_t(t)\|_l,\quad \|\partial_x^l\psi_t(t)\|\le C\|\eta_x(t)\|_l,\quad l=0,1,2, \tag{3.13e}$$

$$\|\eta_{tx}(t)\|\le C\|(\psi_t,\psi_{tt})(t)\|,\quad \|\eta_{txx}(t)\|\le C\|(\psi_{tt},\psi_{tx},\psi_{ttx})(t)\|, \tag{3.13f}$$

where the positive constant C *is independent of* δ, ε *and* T.

And then, the rest of proof of Proposition 3.1 will be divided into three parts, including the basic estimate, higher-order estimates and decay estimate; each of them will be treated in details in Subsection 3.1, Subsection 3.2 and Subsection 3.3, respectively.

3.1. Basic estimate. To prove Proposition 3.1, we start with establishing the basic estimate.

LEMMA 3.3. *Suppose the same assumptions as in Proposition 3.1 hold. Then there exist positive constants* δ_0, c *and* C *such that if* $N_\varepsilon(T)+\delta+\varepsilon\le\delta_0$, *it holds that for* $t\in[0,T]$,

$$\frac{d}{dt}\Xi(t)+c\Pi(t)\le C\Gamma(t), \tag{3.14}$$

where

$$\Xi(t):=\int_0^1\left\{\left[\frac{1}{2w^2}\eta^2+\tilde{\theta}w^2\Psi\left(\frac{\tilde{w}^2}{w^2}\right)+\varepsilon^2\psi_x^2+\frac{1}{2}\sigma_x^2\right]+\frac{3w^2}{4\tilde{\theta}}\chi^2-\alpha\left(\frac{j}{w^2}-\frac{\tilde{j}}{\tilde{w}^2}\right)\sigma_x\right\}dx, \tag{3.15}$$

here $\alpha\in(0,1)$ *is a small constant which will be determined later and* $\Psi(s):=s-1-\ln s$ *for* $s>0$,

$$\Pi(t):=\|(\psi,\varepsilon\psi_x,\eta,\chi,\chi_x)(t)\|^2, \tag{3.16}$$

and

$$\Gamma(t):=\left(N_\varepsilon(T)+\delta+\varepsilon^{3/2}\right)\|(\psi_x,\eta_x)(t)\|^2+\varepsilon^3\|(\psi_{xx},\eta_{xx})(t)\|^2. \tag{3.17}$$

Furthermore, if α *is small enough, then the following equivalent relation holds true,*

$$c\|(\psi,\eta,\chi,\varepsilon\psi_x)(t)\|^2\le\Xi(t)\le C\|(\psi,\eta,\chi,\varepsilon\psi_x)(t)\|^2, \tag{3.18}$$

where the constants c *and* C *are independent of* δ, ε *and* T.

Proof. Multiply the Equation (3.5b) by η, together with Equations (3.5a) and (3.5d), we compute

$$\partial_t\left[\frac{1}{2w^2}\eta^2+\tilde{\theta}w^2\Psi\left(\frac{\tilde{w}^2}{w^2}\right)+\varepsilon^2\psi_x^2+\frac{1}{2}\sigma_x^2\right]+\frac{1}{\tilde{w}^2}\eta^2+\frac{2\tilde{w}_x}{w}\chi\eta+\chi_x\eta=\partial_x R_1(t,x)+R_2(t,x), \tag{3.19}$$

where

$$R_1(t,x):=\sigma\sigma_{tx}+\sigma\eta-\tilde{\theta}\left(\ln w^2-\ln\tilde{w}^2\right)\eta+\varepsilon^2\left[\left(\frac{w_{xx}}{w}-\frac{\tilde{w}_{xx}}{\tilde{w}}\right)\eta+2\psi_t\psi_x\right], \tag{3.20}$$

$$\begin{aligned}R_2(t,x):=&-\frac{\eta+2\tilde{j}}{2w^4}\eta\eta_x-\frac{1}{2}\left[\left(\frac{j}{w^2}\right)^2-\left(\frac{\tilde{j}}{\tilde{w}^2}\right)^2\right]_x\eta\\&-\frac{2\psi_x}{w}\chi\eta+\tilde{\theta}_x\left(\ln w^2-\ln\tilde{w}^2\right)\eta+\frac{(w+\tilde{w})j}{w^2\tilde{w}^2}\psi\eta+\varepsilon^2\frac{\tilde{w}_{xx}}{\tilde{w}w}\psi\eta_x.\end{aligned} \tag{3.21}$$

Applying the Cauchy-Schwartz inequality combined with estimates (3.13a) and (3.13b) gives

$$R_2(t,x)\le C\left(N_\varepsilon(T)+\delta+\varepsilon^{3/2}\right)|(\psi,\eta,\psi_x,\eta_x)(t,x)|^2. \tag{3.22}$$

Multiplying the Equation (3.5c) by $3\chi/(2\tilde{\theta})$, we calculate

$$\partial_t\left(\frac{3w^2}{4\tilde{\theta}}\chi^2\right)+\frac{3\tilde{w}^2}{2\tilde{\theta}}\chi^2+\frac{1}{\tilde{\theta}}\chi_x^2-\frac{2\tilde{w}_x}{w}\eta\chi-\eta\chi_x=\partial_x R_3(t,x)+R_4(t,x), \tag{3.23}$$

where

$$R_3(t,x):=\frac{1}{\tilde{\theta}}\chi\chi_x-\eta\chi+\frac{\varepsilon^2}{2\tilde{\theta}}\left[w^2\left(\frac{j}{w^2}\right)_{xx}-\tilde{w}^2\left(\frac{\tilde{j}}{\tilde{w}^2}\right)_{xx}\right]\chi, \tag{3.24}$$

$$\begin{aligned}R_4(t,x):=&\frac{2\psi_x}{w}\eta\chi-\frac{3\eta_x}{4\tilde{\theta}}\chi^2+\frac{\tilde{\theta}_x}{\tilde{\theta}^2}\chi\chi_x+\frac{2\tilde{j}}{\tilde{w}}\psi_x\chi+\frac{3}{2\tilde{\theta}}H_1(t,x)\chi\\&+\frac{\varepsilon^2\tilde{\theta}_x}{2\tilde{\theta}^2}\left[w^2\left(\frac{j}{w^2}\right)_{xx}-\tilde{w}^2\left(\frac{\tilde{j}}{\tilde{w}^2}\right)_{xx}\right]\chi-\frac{\varepsilon^2}{2\tilde{\theta}}\left[w^2\left(\frac{j}{w^2}\right)_{xx}-\tilde{w}^2\left(\frac{\tilde{j}}{\tilde{w}^2}\right)_{xx}\right]\chi_x.\end{aligned} \tag{3.25}$$

A similar argument to that in (3.22) gives

$$\begin{aligned}R_4(t,x)\le& C\left(N_\varepsilon(T)+\delta+\varepsilon^3\right)|(\psi,\eta,\psi_x,\eta_x)(t,x)|^2\\&+C\varepsilon|(\chi,\chi_x)(t,x)|^2+C\varepsilon^3|(\psi_{xx},\eta_{xx})(t,x)|^2.\end{aligned} \tag{3.26}$$

Note that the steady-state density $\tilde{w}$ is non-flat owing to the non-flat doping profile assumption, it is crucial to capture the dissipation rate of the perturbed density ψ in the basic estimate. Therefore, multiplying the Equation (3.5b) by $-\sigma_x$, we have

$$\begin{aligned}&-\partial_t\left[\left(\frac{j}{w^2}-\frac{\tilde{j}}{\tilde{w}^2}\right)\sigma_x\right]+\tilde{\theta}(w+\tilde{w})\left(\ln w^2-\ln\tilde{w}^2\right)\psi+\frac{w+\tilde{w}}{w}\varepsilon^2\psi_x^2+\sigma_x^2\\&=\partial_x R_5(t,x)+R_6(t,x),\end{aligned} \tag{3.27}$$

where

$$R_5(t,x):=\tilde{\theta}\Big(\ln w^2-\ln\tilde{w}^2\Big)\sigma_x-\varepsilon^2\Big(\frac{w_{xx}}{w}-\frac{\tilde{w}_{xx}}{\tilde{w}}\Big)\sigma_x+\varepsilon^2\frac{w+\tilde{w}}{w}\psi_x\psi, \tag{3.28}$$

$$\begin{aligned}R_6(t,x):=&-\Big(\frac{j}{w^2}-\frac{\tilde{j}}{\tilde{w}^2}\Big)\sigma_{tx}+\frac{1}{2}\Big[\Big(\frac{j}{w^2}\Big)^2-\Big(\frac{\tilde{j}}{\tilde{w}^2}\Big)^2\Big]_x\sigma_x+\Big(\ln w^2\Big)_x\chi\sigma_x\\&-\tilde{\theta}_x\Big(\ln w^2-\ln\tilde{w}^2\Big)\sigma_x+\chi_x\sigma_x+\Big(\frac{j}{w^2}-\frac{\tilde{j}}{\tilde{w}^2}\Big)\sigma_x+\varepsilon^2\frac{(w+\tilde{w})\psi}{w^2}\psi_x^2\\&-\varepsilon^2\frac{\tilde{w}_{xx}(w+\tilde{w})}{w\tilde{w}}\psi^2+\varepsilon^2\frac{(w+\tilde{w})\tilde{w}_x}{w^2}\psi_x\psi-\varepsilon^2\frac{(w+\tilde{w})_x}{w}\psi_x\psi,\end{aligned} \tag{3.29}$$

and $R_6(t,x)$ can be estimated below

$$\begin{aligned}R_6(t,x)\le&(\mu+C\delta)|\sigma_x(t,x)|^2+C\big(N_\varepsilon(T)+\delta\big)|(\psi,\psi_x,\eta_x)(t,x)|^2\\&+C_\mu|(\chi,\chi_x,\eta)(t,x)|^2+C|(\eta,\sigma_{tx})(t,x)|^2+C\big(N_\varepsilon(T)+\varepsilon\big)|(\psi,\varepsilon\psi_x)(t,x)|^2.\end{aligned} \tag{3.30}$$

The middle two terms on the left-hand side of the identity (3.27) can be bounded below by $c|(\psi,\varepsilon\psi_x)(t,x)|^2$, and

$$\Big|-\Big(\frac{j}{w^2}-\frac{\tilde{j}}{\tilde{w}^2}\Big)\sigma_x\Big|\le C|(\psi,\eta,\sigma_x)(t,x)|^2. \tag{3.31}$$

Let μ and $N_\varepsilon(T)+\delta+\varepsilon$ be small enough, we get

$$\begin{aligned}&-\partial_t\Big[\Big(\frac{j}{w^2}-\frac{\tilde{j}}{\tilde{w}^2}\Big)\sigma_x\Big]+c|(\psi,\varepsilon\psi_x)(t,x)|^2\\&\le\partial_xR_5(t,x)+C|(\sigma_{tx},\eta,\chi,\chi_x)(t,x)|^2+C\big(N_\varepsilon(T)+\delta\big)|(\psi_x,\eta_x)(t,x)|^2.\end{aligned} \tag{3.32}$$

From

$$\int_0^1\Big[(3.19)+(3.23)+\alpha(3.32)\Big]dx,$$

here the constant $\alpha>0$ will be determined shortly, we obtain

$$\begin{aligned}&\frac{d}{dt}\Xi(t)+\int_0^1\Big(\frac{1}{\tilde{w}^2}\eta^2+\frac{3\tilde{w}^2}{2\tilde{\theta}}\chi^2+\frac{1}{\tilde{\theta}}\chi_x^2\Big)dx+c\alpha\|(\psi,\varepsilon\psi_x)(t)\|^2\\&\le\int_0^1\Big[R_2(t,x)+R_4(t,x)\Big]dx+C\alpha\|(\sigma_{tx},\eta,\chi,\chi_x)(t)\|^2+C\alpha\big(N_\varepsilon(T)+\delta\big)\|(\psi_x,\eta_x)(t)\|^2,\end{aligned} \tag{3.33}$$

where we have used

$$\int_0^1\partial_x\Big[R_1(t,x)+R_3(t,x)+\alpha R_5(t,x)\Big]=0 \tag{3.34}$$

because of boundary conditions (3.7). In light of estimates (3.13a), (3.13d), (3.22), (3.26) and (3.31), the inequality (3.33) implies (3.14) and (3.18) by letting α and $N_\varepsilon(T)+\delta+\varepsilon$ be sufficiently small. ∎

3.2. Higher-order estimates. In order to close the uniform a priori estimate (3.12), we have to establish the higher-order estimates. To make the most of homogeneous boundary conditions (3.7), we focus attention on calculating the higher-order estimates of temporal and mixed derivatives. As for the higher-order estimates of spatial derivatives, we observe that they can be bounded by those of temporal and mixed derivatives via the working equations.

For simplicity of notation, we introduce

$$A_{-1}(t) := \|(\psi,\eta,\chi,\chi_x)(t)\|,$$

$$A_k(t) := A_{-1}(t) + \sum_{i=0}^{k} \|(\partial_t^i\psi_t, \partial_t^i\psi_x, \varepsilon\partial_t^i\psi_{xx})(t)\|, \quad k=0,1.$$

From

$$\partial_t^k\left[\frac{\partial_x(1.3b)}{w} - \frac{\partial_x(1.12b)}{\tilde{w}}\right], \quad k=0,1,$$

along with Equations (3.5a) and (3.5d), we have

$$\begin{aligned}
&2\partial_t^k\psi_{tt} - 2\tilde{\theta}\partial_t^k\psi_{xx} + \varepsilon^2\partial_t^k\psi_{xxxx} + 2\partial_t^k\psi_t - \tilde{w}\partial_t^k\chi_{xx} - 2\tilde{w}_{xx}\partial_t^k\chi \\
&= \frac{2(\eta+\tilde{j})}{(\psi+\tilde{w})^3}\partial_t^k\eta_{xx} - \frac{2(\eta+\tilde{j})^2}{(\psi+\tilde{w})^4}\partial_t^k\psi_{xx} + 2\chi\partial_t^k\psi_{xx} + \psi\partial_t^k\chi_{xx} \\
&\quad + \varepsilon^2\frac{(k+1)\psi_{xx}+2\tilde{w}_{xx}}{\psi+\tilde{w}}\partial_t^k\psi_{xx} + \partial_t^k P(t,x) + O_k(t,x), \quad k=0,1,
\end{aligned} \tag{3.35}$$

where

$$\begin{aligned}
P(t,x) :=& -(\psi+\tilde{w})(\psi+2\tilde{w})\psi - (\tilde{w}^2 - D)\psi - \frac{2(\psi+\tilde{w})_x^2(\chi+\tilde{\theta})}{(\psi+\tilde{w})\tilde{w}}\psi \\
&+ \frac{2\tilde{w}_x^2}{\tilde{w}}\chi + 4\tilde{w}_x\chi_x - 2(\psi+\tilde{w})_x\sigma_x \\
&+ \tilde{\theta}_{xx}\psi + \frac{6(\psi+\tilde{w})_x^2(\eta+\tilde{j})^2}{(\psi+\tilde{w})^5\tilde{w}^5}\left[\tilde{w}^5 - (\psi+\tilde{w})^5\right] + \frac{6\tilde{w}_x^2(\eta+2\tilde{j})}{\tilde{w}^5}\eta \\
&- \frac{2(\eta+\tilde{j})^2\left[\tilde{w}^4 - (\psi+\tilde{w})^4\right]}{(\psi+\tilde{w})^4\tilde{w}^4}\tilde{w}_{xx} - \frac{2(\eta+2\tilde{j})\eta}{\tilde{w}^4}\tilde{w}_{xx} - \frac{\varepsilon^2\tilde{w}_{xx}^2}{(\psi+\tilde{w})\tilde{w}}\psi \\
&+ \frac{6(\psi+2\tilde{w})_x(\eta+\tilde{j})^2}{\tilde{w}^5}\psi_x + \frac{2(\chi+\tilde{\theta})}{\tilde{w}}\psi_x^2 + \frac{2(\psi+2\tilde{w})_x\chi}{\tilde{w}}\psi_x + 4(\chi+\tilde{\theta})_x\psi_x \\
&- \frac{2}{\psi+\tilde{w}}\psi_t^2 + \frac{2}{(\psi+\tilde{w})^3}\eta_x^2 - \frac{8(\psi+\tilde{w})_x(\eta+\tilde{j})}{(\psi+\tilde{w})^4}\eta_x + 2\left(\frac{2\tilde{w}_x\tilde{\theta}}{\tilde{w}} - \tilde{\phi}_x\right)\psi_x,
\end{aligned}$$

and

$$\begin{aligned}
O_0(t,x) := 0, \quad O_1(t,x) :=& -\frac{6(\eta+\tilde{j})}{(\psi+\tilde{w})^4}\psi_t\eta_{xx} + \frac{2}{(\psi+\tilde{w})^3}\eta_t\eta_{xx} + \frac{8(\eta+\tilde{j})^2}{(\psi+\tilde{w})^5}\psi_t\psi_{xx} \\
&- \frac{4(\eta+\tilde{j})}{(\psi+\tilde{w})^4}\eta_t\psi_{xx} + 2\chi_t\psi_{xx} + \psi_t\chi_{xx} - \frac{\varepsilon^2(\psi_{xx}+2\tilde{w}_{xx})\psi_{xx}}{(\psi+\tilde{w})^2}\psi_t.
\end{aligned}$$

It follows from estimates (3.13a)∼(3.13f) that

$$
\begin{aligned}
&\|\partial_t^k P(t)\|+\|O_k(t)\|\le C\|(\partial_t^k\psi,\partial_t^k\chi,\partial_t^k\chi_x)(t)\|\\
&\qquad\qquad +C\big(N_\varepsilon(T)+\delta+\varepsilon^{1/2}\big)\|(\partial_t^k\psi_t,\partial_t^k\psi_x,\partial_t^k\eta)(t)\|,\quad k=0,1, \qquad (3.36)
\end{aligned}
$$

where we have used

$$
\left|2\left(\frac{2\tilde w_x\tilde\theta}{\tilde w}-\tilde\phi_x\right)\right|=\left|2\left[\frac{2\tilde j^2}{\tilde w^5}\tilde w_x-\tilde\theta_x+\varepsilon^2\left(\frac{\tilde w_{xx}}{\tilde w}\right)_x-\frac{\tilde j}{\tilde w^2}\right]\right|\le C\big(\delta+\varepsilon^{1/2}\big)
$$

to deal with the first factor in the last term of $P(t,x)$.

For $k=0,1$, taking ∂_t^k to the Equation (3.5c) gives

$$
\begin{aligned}
&(\psi+\tilde w)^2\partial_t^k\chi_t-\frac{2}{3}\partial_t^k\chi_{xx}+\frac{2}{3}\tilde\theta\partial_t^k\eta_x-\frac{4\tilde j\tilde\theta}{3\tilde w}\partial_t^k\psi_x\\
=&\partial_x\mathcal V_k(t,x)+\partial_t^kH(t,x)+L_k(t,x),\quad k=0,1, \qquad (3.37)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal V_k(t,x):=&\frac{\varepsilon^2}{3}\partial_t^k\left\{(\psi+\tilde w)^2\left[\frac{\eta+\tilde j}{(\psi+\tilde w)^2}\right]_{xx}-\tilde w^2\left(\frac{\tilde j}{\tilde w^2}\right)_{xx}\right\}\\
=&\frac{\varepsilon^2}{3}\partial_t^k\eta_{xx}-\frac{2\varepsilon^2\tilde j}{3\tilde w}\partial_t^k\psi_{xx}+\partial_t^k\mathcal K(t,x),\quad k=0,1, \qquad (3.38)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal K(t,x):=&\frac{\varepsilon^2}{3}\left[-\frac{4(\psi+\tilde w)_x}{\psi+\tilde w}\eta_x+\frac{6(\psi+\tilde w)_x^2}{(\psi+\tilde w)^2}\eta-\frac{6\tilde j(\psi+2\tilde w)(\psi+\tilde w)_x^2}{(\psi+\tilde w)^2\tilde w^2}\psi\right.\\
&\left.+\frac{6\tilde j(\psi+2\tilde w)_x}{\tilde w^2}\psi_x-\frac{2(\psi+\tilde w)_{xx}}{\psi+\tilde w}\eta+\frac{2\tilde j(\psi+\tilde w)_{xx}}{(\psi+\tilde w)\tilde w}\psi\right], \qquad (3.39)
\end{aligned}
$$

$$
L_0(t,x):=0,\quad L_1(t,x):=-2(\psi+\tilde w)\psi_t\chi_t. \qquad (3.40)
$$

Calculate $\partial_x\mathcal V_k(t,x)$ for later use,

$$
\partial_x\mathcal V_0(t,x)=\frac{\varepsilon^2}{3}\eta_{xxx}-\frac{2\varepsilon^2\tilde j}{3\tilde w}\psi_{xxx}+\underbrace{\frac{2\varepsilon^2\tilde j\tilde w_x}{\tilde w^2}\psi_{xx}+\partial_x\mathcal K(t,x)}_{=:\mathcal K_1(t,x)}, \qquad (3.41)
$$

and

$$
\partial_x\mathcal V_1(t,x)=\frac{\varepsilon^2}{3}\eta_{txxx}-\frac{2\varepsilon^2(\eta+\tilde j)}{3(\psi+\tilde w)}\psi_{txxx}+\mathcal K_2(t,x), \qquad (3.42)
$$

where

$$
\begin{aligned}
\mathcal K_2(t,x):=&\frac{\varepsilon^2}{3}\left[-\frac{4(\psi+\tilde w)_x}{\psi+\tilde w}\eta_{txx}+\frac{4(\psi+\tilde w)_x\eta_{xx}}{(\psi+\tilde w)^2}\psi_t-\frac{4\eta_{xx}}{\psi+\tilde w}\psi_{tx}\right.\\
&+\frac{10(\psi+\tilde w)_x^2}{(\psi+\tilde w)^2}\eta_{tx}-\frac{20(\psi+\tilde w)_x^2\eta_x}{(\psi+\tilde w)^3}\psi_t+\frac{20(\psi+\tilde w)_x\eta_x}{(\psi+\tilde w)^2}\psi_{tx}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{6(\psi+\tilde{w})_{xx}}{\psi+\tilde{w}}\eta_{tx}+\frac{6(\psi+\tilde{w})_{xx}\eta_x}{(\psi+\tilde{w})^2}\psi_t-\frac{6\eta_x}{\psi+\tilde{w}}\psi_{txx}\\
&-\frac{12(\psi+\tilde{w})_x^3}{(\psi+\tilde{w})^3}\eta_t+\frac{36(\eta+\tilde{j})(\psi+\tilde{w})_x^3}{(\psi+\tilde{w})^4}\psi_t-\frac{36(\eta+\tilde{j})(\psi+\tilde{w})_x^2}{(\psi+\tilde{w})^3}\psi_{tx}\\
&+\frac{14(\psi+\tilde{w})_x(\psi+\tilde{w})_{xx}}{(\psi+\tilde{w})^2}\eta_t-\frac{28(\eta+\tilde{j})(\psi+\tilde{w})_x(\psi+\tilde{w})_{xx}}{(\psi+\tilde{w})^3}\psi_t\\
&+\frac{14(\eta+\tilde{j})(\psi+\tilde{w})_{xx}}{(\psi+\tilde{w})^2}\psi_{tx}+\frac{14(\eta+\tilde{j})(\psi+\tilde{w})_x}{(\psi+\tilde{w})^2}\psi_{txx}\\
&-\frac{2(\psi+\tilde{w})_{xxx}}{\psi+\tilde{w}}\eta_t+\frac{2(\eta+\tilde{j})(\psi+\tilde{w})_{xxx}}{(\psi+\tilde{w})^2}\psi_t\Bigg].
\end{aligned}
\tag{3.43}
$$

It follows from estimates (3.13a)∼(3.13f) that

$$\|H(t)\|\le C\|(\eta,\chi)(t)\|+C\big(N_\varepsilon(T)+\delta\big)\|(\psi,\chi_x)(t)\|, \tag{3.44a}$$

$$\|\partial_t H(t)\|+\|L_1(t)\|\le C\|(\eta_t,\chi_t)(t)\|+C\big(N_\varepsilon(T)+\delta\big)\|(\psi_t,\psi_{tt},\psi_{tx},\chi_{tx})(t)\|, \tag{3.44b}$$

$$\|\mathcal{K}(t)\|\le C\varepsilon^{3/2}\|(\psi,\psi_t,\psi_x,\eta)(t)\|, \tag{3.44c}$$

$$\|\partial_t\mathcal{K}(t)\|\le C\varepsilon^{3/2}\|\psi_t(t)\|+C\varepsilon^2\|(\psi_{tt},\psi_{tx},\eta_t)(t)\|+C\big(N_\varepsilon(T)+\delta\big)\varepsilon\|\varepsilon\psi_{txx}(t)\|, \tag{3.44d}$$

$$\|\mathcal{K}_1(t)\|\le C\varepsilon^{1/2}\|(\psi,\eta)(t)\|+C\varepsilon^{3/2}\|(\psi_t,\psi_x)(t)\|+C\varepsilon^2\|(\psi_{tx},\psi_{xx})(t)\|, \tag{3.44e}$$

$$\|\mathcal{K}_2(t)\|\le C\varepsilon^{1/2}\|(\psi_t,\eta_t)(t)\|+C\varepsilon^{3/2}\|(\psi_{tt},\psi_{tx})(t)\|+C\varepsilon\|(\varepsilon\psi_{ttx},\varepsilon\psi_{txx})(t)\|. \tag{3.44f}$$

In the next lemma, we observe that the higher-order estimates of spatial derivatives can be controlled by those of temporal and mixed derivatives.

Lemma 3.4. *Under the same assumptions as in Proposition 3.1, the following equivalent relationship holds for* $t\in[0,T]$,

$$c\big(A_1(t)+\|\chi_t(t)\|\big)\le n_\varepsilon(t)\le C\big(A_1(t)+\|\chi_t(t)\|\big), \tag{3.45}$$

where the two positive constants c *and* C *are independent of* δ, ε *and* T.

Proof. It suffices to show the right side inequality in (3.45), because a similar and much easier argument guarantees the left side one. In light of estimates (3.13a)∼(3.13f), $(3.36)|_{k=0}$, (3.44a) and (3.44e), together with Equations $(3.35)|_{k=0}$ and $(3.37)|_{k=0}$, we compute

$$
\begin{aligned}
n_\varepsilon(t)=&\|(\psi,\eta,\chi)(t)\|_2+\|(\varepsilon\partial_x^3\psi,\varepsilon\partial_x^3\eta,\varepsilon^2\partial_x^4\psi)(t)\|\\
\le& CA_1(t)+\|\chi_{xx}(t)\|+\|\psi_{xx}(t)\|+\|\varepsilon\partial_x^3\psi(t)\|\\
&+C\|(\psi_t,\psi_{tt},\eta_{xx},P,\chi,\chi_x,\chi_{xx},\psi_{xx})(t)\|\\
\le& C\big(A_1(t)+\|\psi_{xx}(t)\|+\|\varepsilon\partial_x^3\psi(t)\|+\|\chi_{xx}(t)\|\big)\\
=&C\big(A_1(t)+\|\psi_{xx}(t)\|+\|\varepsilon\partial_x^3\psi(t)\|\big)\\
&+C\left\|-\frac{3}{2}\left[-(\psi+\tilde{w})^2\chi_t-\frac{2\tilde{\theta}}{3}\eta_x+\frac{4\tilde{j}\tilde{\theta}}{3\tilde{w}}\psi_x+\frac{\varepsilon^2}{3}\partial_x^3\eta-\frac{2\varepsilon^2\tilde{j}}{3\tilde{w}}\partial_x^3\psi+\mathcal{K}_1+H\right](t)\right\|\\
\le& C\big(A_1(t)+\|\chi_t(t)\|+\|\psi_{xx}(t)\|+\|\varepsilon\partial_x^3\psi(t)\|\big).
\end{aligned}
\tag{3.46}
$$

From

$$-\int_0^1 (3.35)|_{k=0}\psi_{xx}dx,$$

integration by parts yields

$$\begin{aligned}&\theta_L\|\psi_{xx}(t)\|^2+\|\varepsilon\partial_x^3\psi(t)\|^2\\ \le&\big[\mu+C\big(N_\varepsilon(T)+\delta+\varepsilon^{3/2}\big)\big]\|\psi_{xx}(t)\|^2+C_\mu\big(A_1^2(t)+\|\chi_t(t)\|^2+\varepsilon^2\|\varepsilon\partial_x^3\psi(t)\|^2\big),\end{aligned} \tag{3.47}$$

which further implies

$$\|\psi_{xx}(t)\|+\|\varepsilon\partial_x^3\psi(t)\|\le C\big(A_1(t)+\|\chi_t(t)\|\big) \tag{3.48}$$

if μ and $N_\varepsilon(T)+\delta+\varepsilon$ are small enough. Substitute (3.48) into (3.46), we complete the proof. □

For $k=0,1$, we proceed to estimate $\partial_t^k\eta_t$ as follows.

LEMMA 3.5. *Under the same assumptions as in Proposition 3.1, the following estimates hold for* $t\in[0,T]$,

$$\|\eta_t(t)\|\le C\|(\psi,\eta,\chi,\chi_x,\psi_x)(t)\|+C\big(N_\varepsilon(T)+\delta\big)\|\psi_t(t)\|+C\varepsilon^{1/2}\big(A_1(t)+\|\chi_t(t)\|\big), \tag{3.49}$$

and

$$\partial_t\eta_t=\varepsilon^2 w\psi_{txxx}+Y_1(t,x), \tag{3.50a}$$

$$\begin{aligned}\|Y_1(t)\|\le& C\|(\psi,\eta,\chi,\chi_x,\psi_t,\psi_x,\psi_{tx},\chi_{tx})(t)\|\\ &+C\big(N_\varepsilon(T)+\delta+\varepsilon^{1/2}\big)\|(\psi_{tt},\varepsilon\psi_{xx},\varepsilon\psi_{txx})(t)\|,\end{aligned} \tag{3.50b}$$

where $Y_1(t,x)$ *is given by* (3.53) *and the positive constant* C *is independent of* δ, ε *and* T.

Proof. From Equation (3.5b), we have

$$\eta_t=\varepsilon^2w^2\left(\frac{w_{xx}}{w}-\frac{\tilde w_{xx}}{\tilde w}\right)_x+Y(t,x), \tag{3.51}$$

where

$$\begin{aligned}Y(t,x):=&\frac{2j}{w}\psi_t-\frac{w^2}{2}\left[\left(\frac{j}{w^2}\right)^2-\left(\frac{\tilde j}{\tilde w^2}\right)^2\right]_x-w^2\chi\big(\ln w^2\big)_x\\ &-w^2\tilde\theta\big(\ln w^2-\ln\tilde w^2\big)_x-w^2\chi_x+w^2\sigma_x-w^2\left(\frac{j}{w^2}-\frac{\tilde j}{\tilde w^2}\right).\end{aligned} \tag{3.52}$$

Taking L^2-norm of (3.51) and combining estimates (3.13a), (3.13b), (3.13d) and (3.48) yield (3.49). Differentiating (3.51) in t, we obtain (3.50a),

$$\begin{aligned}Y_1(t,x):=&-\varepsilon^2w_x\psi_{txx}-\varepsilon^2w_{xxx}\psi_t+\frac{2\varepsilon^2w_{xx}w_x}{w}\psi_t\\ &-\varepsilon^2w_{xx}\psi_{tx}+2\varepsilon^2w\psi_t\left(\frac{w_{xx}}{w}-\frac{\tilde w_{xx}}{\tilde w}\right)_x+\partial_tY(t,x).\end{aligned} \tag{3.53}$$

The same argument as in calculating (3.49) gives (3.50b). □

We are now ready to calculate higher order estimates of temporal and mixed derivatives. The following lemma will tell us that the Bohm potential term is a dissipative term, which contributes the dissipation rate $\|\varepsilon\partial_t^k\psi_{xx}(t)\|^2$ at the quantum level.

LEMMA 3.6. *Suppose the same assumptions as in Proposition 3.1 hold. Then there exist positive constants* δ_0*, c and C such that if* $N_\varepsilon(T)+\delta+\varepsilon\le\delta_0$*, it holds that for* $t\in[0,T]$*,*

$$\frac{d}{dt}\Xi_1^{(k)}(t)+c\Pi_1^{(k)}(t)\le C\Gamma_1^{(k)}(t),\quad k=0,1, \tag{3.54}$$

where

$$\Xi_1^{(k)}(t):=\int_0^1\Big[\big(\partial_t^k\psi\big)^2+2\partial_t^k\psi_t\partial_t^k\psi\Big]dx,\quad \Pi_1^{(k)}(t):=\|(\partial_t^k\psi_x,\varepsilon\partial_t^k\psi_{xx})(t)\|^2,$$
$$\Gamma_1^{(k)}(t):=\|(\partial_t^k\psi_t,\partial_t^k\psi,\partial_t^k\chi,\partial_t^k\chi_x)(t)\|^2+\big(N_\varepsilon(T)+\delta+\varepsilon^{1/2}\big)A_k^2(t), \tag{3.55}$$

and the constants c and C are independent of δ*,* ε *and* T*.*

Proof. For $k=0,1$, from

$$\int_0^1(3.35)\partial_t^k\psi dx,$$

we compute

$$\frac{d}{dt}\Xi_1^{(k)}(t)+\int_0^1\Big[2\tilde\theta\big(\partial_t^k\psi_x\big)^2+\big(\varepsilon\partial_t^k\psi_{xx}\big)^2\Big]dx=\mathcal{I}_1^{(k)}(t), \tag{3.56}$$

where

$$\begin{aligned}\mathcal{I}_1^{(k)}(t):=\int_0^1\bigg\{&2\big(\partial_t^k\psi_t\big)^2-2\tilde\theta_x\partial_t^k\psi_x\partial_t^k\psi-\Big(\tilde w_x\partial_t^k\chi_x\partial_t^k\psi+\tilde w\partial_t^k\chi_x\partial_t^k\psi_x\Big)\\&+2\tilde w_{xx}\partial_t^k\chi\partial_t^k\psi+\bigg(\frac{6w_xj}{w^4}\partial_t^k\eta_x\partial_t^k\psi-\frac{2\eta_x}{w^3}\partial_t^k\eta_x\partial_t^k\psi-\frac{2j}{w^3}\partial_t^k\eta_x\partial_t^k\psi_x\bigg)\\&-\bigg[\frac{8w_xj^2}{w^5}\partial_t^k\psi_x\partial_t^k\psi-\frac{4j\eta_x}{w^4}\partial_t^k\psi_x\partial_t^k\psi-\frac{2j^2}{w^4}\big(\partial_t^k\psi_x\big)^2\bigg]\\&-\Big[2\chi_x\partial_t^k\psi_x\partial_t^k\psi+2\chi\big(\partial_t^k\psi_x\big)^2\Big]-\Big(\psi_x\partial_t^k\chi_x\partial_t^k\psi+\psi\partial_t^k\chi_x\partial_t^k\psi_x\Big)\\&+\varepsilon^2\frac{(k+1)\psi_{xx}+2\tilde w_{xx}}{w}\partial_t^k\psi_{xx}\partial_t^k\psi+\Big(\partial_t^kP+O_k\Big)\partial_t^k\psi\bigg\}dx.\end{aligned} \tag{3.57}$$

Young's inequality combined with estimates (3.13a), (3.13b), (3.13f), (3.36) and (3.49) yields

$$\int_0^1\Big[2\tilde\theta\big(\partial_t^k\psi_x\big)^2+\big(\varepsilon\partial_t^k\psi_{xx}\big)^2\Big]dx\ge c\Pi_1^{(k)}(t), \tag{3.58}$$

and

$$\begin{aligned}\mathcal{I}_1^{(k)}(t)\le&\mu\|\partial_t^k\psi_x(t)\|^2+C_\mu\|(\partial_t^k\psi_t,\partial_t^k\psi,\partial_t^k\chi,\partial_t^k\chi_x)(t)\|^2\\&+C\big(N_\varepsilon(T)+\delta+\varepsilon^{1/2}\big)A_k^2(t).\end{aligned} \tag{3.59}$$

Insert (3.58) and (3.59) into (3.56), let μ be small enough, to obtain (3.54). ∎

We are now at the sharp end of establishing higher-order estimates because of the dispersive velocity term. Precisely, the following lemma will reveal that the dispersive velocity term is another dissipative term at the quantum level, contributing the dissipation rate $\|\varepsilon\partial_t^k\psi_{tx}(t)\|^2$.

LEMMA 3.7. *Suppose the same assumptions as in Proposition 3.1 hold. Then there exist positive constants δ_0, c and C such that if $N_\varepsilon(T)+\delta+\varepsilon\le\delta_0$, it holds that for $t\in[0,T]$,*

$$\frac{d}{dt}\Xi_2^{(k)}(t)+c\Pi_2^{(k)}(t)\le C\Gamma_2^{(k)}(t),\quad k=0,1, \tag{3.60}$$

where

$$\Xi_2^{(k)}(t):=\int_0^1\left\{\left(\partial_t^k\psi_t\right)^2+\left(\theta-\frac{j^2}{w^4}\right)\left(\partial_t^k\psi_x\right)^2+\frac{1}{2}\left(\varepsilon\partial_t^k\psi_{xx}\right)^2\right.$$
$$\left.-\frac{3w^3}{2}\partial_t^k\chi\partial_t^k\psi_t-k\left[\frac{9w^5\varepsilon}{8}\chi_t\left(\varepsilon\psi_{txx}\right)+\frac{3w^4\varepsilon^2}{8}\left(\varepsilon\psi_{txx}\right)^2\right]\right\}dx,$$

$$\Pi_2^{(k)}(t):=\|(\partial_t^k\psi_t,\varepsilon\partial_t^k\psi_{tx})(t)\|^2,\quad \Gamma_2^{(k)}(t):=\left(\mu+N_\varepsilon(T)+\delta+\varepsilon^{1/2}\right)\|\partial_t^k\psi_x(t)\|^2+\Upsilon^{(k)}(t),$$
$$\Upsilon^{(0)}(t):=C_\mu\|(\psi,\eta,\chi,\chi_x)(t)\|^2+\left(N_\varepsilon(T)+\delta+\varepsilon^{1/2}\right)\|(\psi_{tt},\psi_{tx},\varepsilon\psi_{xx},\varepsilon\psi_{txx},\chi_t)(t)\|^2,$$
$$\Upsilon^{(1)}(t):=C_\mu\|(\chi_t,\chi_{tx})(t)\|^2+\|(\psi,\eta,\chi,\chi_x,\psi_t,\psi_x)(t)\|^2$$
$$+\left(N_\varepsilon(T)+\delta+\varepsilon^{1/2}\right)\|(\psi_{tt},\varepsilon\psi_{xx},\varepsilon\psi_{txx})(t)\|^2. \tag{3.61}$$

Here the constants c and C are independent of δ, ε and T; μ is an arbitrary positive constant to be determined and C_μ is a generic constant which only depends on μ.

Proof. For $k=0,1$, from

$$\int_0^1(3.35)\partial_t^k\psi_t dx,$$

we calculate

$$\frac{d}{dt}\int_0^1\left[\left(\partial_t^k\psi_t\right)^2+\left(\theta-\frac{j^2}{w^4}\right)\left(\partial_t^k\psi_x\right)^2+\frac{1}{2}\left(\varepsilon\partial_t^k\psi_{xx}\right)^2\right]dx$$
$$+2\|\partial_t^k\psi_t(t)\|^2-\int_0^1 w\partial_t^k\chi_{xx}\partial_t^k\psi_t dx=\mathcal{I}_2^{(k)}(t), \tag{3.62}$$

where

$$\mathcal{I}_2^{(k)}(t):=\int_0^1\left\{\left[-2\left(\theta-\frac{j^2}{w^4}\right)_x\partial_t^k\psi_x\partial_t^k\psi_t+\left(\theta-\frac{j^2}{w^4}\right)_t\left(\partial_t^k\psi_x\right)^2\right]\right.$$
$$+2\tilde{w}_{xx}\partial_t^k\chi\partial_t^k\psi_t+\frac{2j}{w^3}\partial_t^k\eta_{xx}\partial_t^k\psi_t+\varepsilon^2\frac{(k+1)\psi_{xx}+2\tilde{w}_{xx}}{w}\partial_t^k\psi_{xx}\partial_t^k\psi_t$$
$$\left.+\left[\partial_t^k P(t,x)+O_k(t,x)\right]\partial_t^k\psi_t\right\}dx, \tag{3.63}$$

and can be estimated below,

$$\begin{aligned}\mathcal{I}_2^{(k)}(t) \le & 2\mu\|\partial_t^k\psi_t(t)\|^2 + C_\mu\|(\partial_t^k\psi,\partial_t^k\chi,\partial_t^k\chi_x)(t)\|^2 \\ & + C_\mu\left(N_\varepsilon(T)+\delta+\varepsilon^{1/2}\right)\left(\|(\partial_t^k\eta,\partial_t^k\psi_t,\partial_t^k\psi_x,\varepsilon\partial_t^k\psi_{xx})(t)\|^2 + \|\chi_t(t)\|_k^2\right), \quad (3.64)\end{aligned}$$

with the aid of estimates (3.13a), (3.13b), (3.13f) and (3.36), and by virtue of Young's inequality.

The last integral on the left-hand side of (3.62) is heavily influenced by the dispersive velocity term. Thus we have to make a great effort to contend with it. Specifically, due to Equation (3.37), we compute

$$\begin{aligned}-\int_0^1 w\partial_t^k\chi_{xx}\partial_t^k\psi_t dx = & \int_0^1 w\frac{3}{2}\left[-w^2\partial_t^k\chi_t - \frac{2}{3}\tilde{\theta}\partial_t^k\eta_x + \frac{4\tilde{j}\tilde{\theta}}{3\tilde{w}}\partial_t^k\psi_x\right. \\ & \left. + \partial_x\mathcal{V}_k(t,x) + \partial_t^k H(t,x) + L_k(t,x)\right]\partial_t^k\psi_t dx \\ = & -\int_0^1 \frac{3}{2}w^3\partial_t^k\chi_t\partial_t^k\psi_t dx - \int_0^1 w\tilde{\theta}\partial_t^k\eta_x\partial_t^k\psi_t dx \\ & + \int_0^1 \frac{2w\tilde{j}\tilde{\theta}}{\tilde{w}}\partial_t^k\psi_x\partial_t^k\psi_t dx + \int_0^1 \frac{3}{2}w\partial_x\mathcal{V}_k(t,x)\partial_t^k\psi_t dx \\ & + \int_0^1 \frac{3}{2}w\left[\partial_t^k H(t,x) + L_k(t,x)\right]\partial_t^k\psi_t dx \\ = & \mathfrak{T}_1^{(k)}(t) + \mathfrak{T}_2^{(k)}(t) + \mathfrak{T}_3^{(k)}(t) + \mathfrak{T}_4^{(k)}(t) + \mathfrak{T}_5^{(k)}(t). \quad (3.65)\end{aligned}$$

Here we point out that $\mathfrak{T}_1^{(k)}(t)$, $\mathfrak{T}_4^{(k)}(t)$ are much harder to treat than $\mathfrak{T}_2^{(k)}(t)$, $\mathfrak{T}_3^{(k)}(t)$, $\mathfrak{T}_5^{(k)}(t)$, and the latter three integrals can be estimated as a whole:

$$\begin{aligned}\mathfrak{T}_2^{(k)}(t) + \mathfrak{T}_3^{(k)}(t) + \mathfrak{T}_5^{(k)}(t) \ge & c\|\partial_t^k\psi_t(t)\|^2 - C\|(\partial_t^k\psi,\partial_t^k\eta,\partial_t^k\chi,\partial_t^k\chi_x)(t)\|^2 \\ & - C(N_\varepsilon(T)+\delta)\|\partial_t^k\psi_x(t)\|^2. \quad (3.66)\end{aligned}$$

Before estimating the remaining two integrals, we still need to do some auxiliary calculations via Equation (3.5a):

$$\partial_t^k\psi_{tt} = -\frac{1}{2w}\partial_t^k\eta_{tx} + \mathcal{B}_k(t,x), \quad k=0,1, \quad (3.67)$$

where

$$\mathcal{B}_0(t,x) := \frac{1}{2w^2}\psi_t\eta_x, \quad \mathcal{B}_1(t,x) := \frac{1}{w^2}\psi_t\eta_{tx} - \frac{1}{w^3}\psi_t^2\eta_x + \frac{1}{2w^2}\psi_{tt}\eta_x,$$

satisfying

$$\|\mathcal{B}_k(t)\| \le CN_\varepsilon(T)\|(\partial_t^k\psi_t,\psi_t)(t)\|. \quad (3.68)$$

Based on (3.67) and (3.68), integration by parts gives

$$\begin{aligned}\mathfrak{T}_1^{(k)}(t) = & -\frac{d}{dt}\int_0^1 \frac{3w^3}{2}\partial_t^k\chi\partial_t^k\psi_t dx + \int_0^1 \frac{9w^2\psi_t}{2}\partial_t^k\chi\partial_t^k\psi_t dx + \int_0^1 \frac{3w^3}{2}\partial_t^k\chi\partial_t^k\psi_{tt} dx \\ \ge & -\frac{d}{dt}\int_0^1 \frac{3w^3}{2}\partial_t^k\chi\partial_t^k\psi_t dx - CN_\varepsilon(T)\|(\partial_t^k\psi_t,\psi_t,\partial_t^k\chi)(t)\|^2\end{aligned}$$

$$+\int_0^1 \frac{3w^2}{4}\partial_t^k\chi_x\partial_t^k\eta_t dx+\int_0^1 \frac{3ww_x}{2}\partial_t^k\chi\partial_t^k\eta_t dx. \tag{3.69}$$

The two integrals in the last line of (3.69) both have to be estimated case-by-case depending on $k=0$ and $k=1$.

Firstly, for $k=0$ we have

$$\int_0^1 \frac{3w^2}{4}\chi_x\eta_t dx \geq -\mu\|\eta_t(t)\|^2 - C_\mu\|\chi_x(t)\|^2, \tag{3.70a}$$

and for $k=1$ we compute via Equations (3.50a) and (3.37):

$$\begin{aligned}\int_0^1 \frac{3w^2}{4}\chi_{tx}\partial_t\eta_t dx =& -\int_0^1 \frac{3w^3\varepsilon^2}{4}\chi_{txx}\psi_{txx}dx\\ &-\int_0^1 \frac{9w^2w_x\varepsilon}{4}\chi_{tx}(\varepsilon\psi_{txx})dx+\int_0^1 \frac{3w^2}{4}\chi_{tx}Y_1 dx\\ \geq& \int_0^1 \frac{9w^3\varepsilon^2}{8}\left[-w^2\chi_{tt}-\frac{2}{3}\tilde{\theta}\eta_{tx}+\frac{4\tilde{j}\tilde{\theta}}{3\tilde{w}}\psi_{tx}+\partial_x\mathcal{V}_1+\partial_t H+L_1\right]\psi_{txx}dx\\ &-C\varepsilon\|(\varepsilon\psi_{txx},\chi_{tx})(t)\|^2-\mu\|Y_1(t)\|^2-C_\mu\|\chi_{tx}(t)\|^2\\ \geq& -\frac{d}{dt}\int_0^1 \frac{9w^5\varepsilon}{8}\chi_t(\varepsilon\psi_{txx})dx-\frac{d}{dt}\int_0^1 \frac{3w^4\varepsilon^2}{8}(\varepsilon\psi_{txx})^2dx\\ &-C\varepsilon\|(\eta_t,\psi_t,\psi_{tt},\psi_{tx},\varepsilon\psi_{ttx},\varepsilon\psi_{txx})(t)\|^2\\ &-\mu\|Y_1(t)\|^2-C_\mu\|\chi_{tx}(t)\|^2,\end{aligned} \tag{3.70b}$$

where we have used the following computations:

$$\begin{aligned}&-\int_0^1 \frac{9w^5\varepsilon^2}{8}\chi_{tt}\psi_{txx}dx\\ =&-\frac{d}{dt}\int_0^1 \frac{9w^5\varepsilon}{8}\chi_t(\varepsilon\psi_{txx})dx+\int_0^1 \frac{45w^4\psi_t\varepsilon}{8}\chi_t(\varepsilon\psi_{txx})dx+\int_0^1 \frac{9w^5\varepsilon^2}{8}\chi_t\psi_{ttxx}dx\\ \geq&-\frac{d}{dt}\int_0^1 \frac{9w^5\varepsilon}{8}\chi_t(\varepsilon\psi_{txx})dx-C\varepsilon\|(\varepsilon\psi_{ttx},\varepsilon\psi_{txx},\chi_{tx})(t)\|^2,\end{aligned} \tag{3.71}$$

and

$$\begin{aligned}\int_0^1 \frac{9w^3\varepsilon^2}{8}\partial_x\mathcal{V}_1\psi_{txx}dx =& \int_0^1 \frac{9w^3\varepsilon^2}{8}\left(\frac{\varepsilon^2}{3}\eta_{txxx}-\frac{2\varepsilon^2 j}{3w}\psi_{txxx}+\mathcal{K}_2\right)\psi_{txx}dx\\ \geq& -\frac{d}{dt}\int_0^1 \frac{3w^4\varepsilon^2}{8}(\varepsilon\psi_{txx})^2dx-C\varepsilon\|(\eta_t,\psi_t,\psi_{tt},\psi_{tx},\varepsilon\psi_{ttx},\varepsilon\psi_{txx})(t)\|^2,\end{aligned} \tag{3.72}$$

Similarly to (3.70) , for $k=0$ we have

$$\int_0^1 \frac{3ww_x}{2}\chi\eta_t dx \geq -\mu\|\eta_t(t)\|^2 - C_\mu\|\chi(t)\|^2, \tag{3.73a}$$

and for $k=1$ we obtain

$$\int_0^1 \frac{3ww_x}{2}\chi_t\partial_t\eta_t dx = \int_0^1 \frac{3ww_x}{2}\chi_t\left(\varepsilon^2 w\psi_{txxx}+Y_1\right)dx$$

$$\geq -\int_0^1 \left(\frac{3w^2 w_x \varepsilon^2}{2}\chi_t\right)_x \psi_{txx} dx - \mu\|Y_1(t)\|^2 - C_\mu\|\chi_t(t)\|^2$$
$$\geq -C\varepsilon^{1/2}\|(\varepsilon\psi_{txx}, \chi_{tx})(t)\|^2 - \mu\|Y_1(t)\|^2 - C_\mu\|\chi_t(t)\|^2. \quad (3.73b)$$

In addition, we proceed to compute via the expression (3.38) and Equation (3.5a):

$$\begin{aligned}\mathfrak{T}_4^{(k)}(t) =& -\int_0^1 \left(\frac{3}{2}w\partial_t^k\psi_t\right)_x \mathcal{V}_k dx\\ =& -\int_0^1 \frac{3}{2}w\partial_t^k\psi_{tx}\mathcal{V}_k dx - \int_0^1 \frac{3}{2}w_x\partial_t^k\psi_t\mathcal{V}_k dx\\ \geq& \int_0^1 \frac{w\varepsilon^2}{2}\partial_t^k\psi_{tx}\left(2w\partial_t^k\psi_{tx} + 2w_x\partial_t^k\psi_t + k4\psi_t\psi_{tx}\right)dx\\ &- C\left(N_\varepsilon(T)+\delta+\varepsilon^{1/2}\right)\|(\partial_t^k\eta, \partial_t^k\psi, \partial_t^k\psi_t, \partial_t^k\psi_x, \varepsilon\partial_t^k\psi_{tx}, \varepsilon\partial_t^k\psi_{xx})(t)\|^2\\ \geq& c\|\varepsilon\partial_t^k\psi_{tx}(t)\|^2 - C\left(N_\varepsilon(T)+\delta+\varepsilon^{1/2}\right)\|(\partial_t^k\eta, \partial_t^k\psi, \partial_t^k\psi_t, \partial_t^k\psi_x, \varepsilon\partial_t^k\psi_{xx})(t)\|^2,\end{aligned} \quad (3.74)$$

where we have also used estimates (3.44c) and (3.44d).

Substituting all the above estimates into (3.62) and letting μ, $N_\varepsilon(T)+\delta+\varepsilon$ be sufficiently small yield (3.60) and (3.61). □

Next, we shall pursue the higher-order estimates of χ, which is a relatively easier task than those of ψ. However, the dispersive velocity term still makes the computations more complicated.

LEMMA 3.8. *Suppose the same assumptions as in Proposition 3.1 hold. Then there exist positive constants δ_0, c and C such that if $N_\varepsilon(T)+\delta+\varepsilon \leq \delta_0$, it holds that for $t \in [0,T]$,*

$$\frac{d}{dt}\Xi_3(t) + c\Pi_3(t) \leq C\Gamma_3(t), \quad (3.75)$$

where

$$\Xi_3(t) := \int_0^1 \left(\frac{1}{3}\chi_x^2 + \frac{2\tilde{\theta}}{3}\eta_x\chi\right)dx, \quad \Pi_3(t) := \|\chi_t(t)\|^2,$$
$$\Gamma_3(t) := \mu\|\psi_x(t)\|^2 + C_\mu A_{-1}^2(t) + \left(N_\varepsilon(T)+\delta+\varepsilon\right)\left(A_1^2(t) + \|\chi_{tx}(t)\|^2\right), \quad (3.76)$$

and

$$\frac{d}{dt}\Xi_4(t) + c\Pi_4(t) \leq C\Gamma_4(t), \quad (3.77)$$

where

$$\Xi_4(t) := \int_0^1 \frac{w^2}{2}\chi_t^2 dx, \quad \Pi_4(t) := \|\chi_{tx}(t)\|^2,$$
$$\Gamma_4(t) := \|(\psi, \eta, \chi, \chi_x, \psi_x, \chi_t)(t)\|^2 + \left(N_\varepsilon(T)+\delta+\varepsilon\right)\left(A_1^2(t) + \|\varepsilon\psi_{ttx}(t)\|^2\right). \quad (3.78)$$

Here the constants c and C are independent of δ, ε and T; μ is an arbitrary positive constant to be determined and C_μ is a generic constant which only depends on μ.

Proof. From

$$\int_0^1 (3.37)|_{k=0}\chi_t dx,$$

we compute

$$\frac{d}{dt}\Xi_3(t)+\int_0^1 w^2\chi_t^2 dx=\mathcal{I}_3(t), \tag{3.79}$$

where

$$\mathcal{I}_3(t):=\int_0^1 \left[-\left(\frac{2\tilde{\theta}_x}{3}\eta_t\chi+\frac{2\tilde{\theta}}{3}\eta_t\chi_x\right)+\frac{4\tilde{j}\tilde{\theta}}{3\tilde{w}}\psi_x\chi_t+\partial_x\mathcal{V}_0(t,x)\chi_t+H(t,x)\chi_t\right]dx, \tag{3.80}$$

satisfying

$$\begin{aligned}\mathcal{I}_3(t)\leq&\int_0^1 \partial_x\mathcal{V}_0(t,x)\chi_t dx+(\mu+\delta+\varepsilon)\|\chi_t(t)\|^2\\&+\mu\|\psi_x(t)\|^2+C_\mu A_{-1}^2(t)+C\big(N_\varepsilon(T)+\delta+\varepsilon\big)A_1^2(t).\end{aligned} \tag{3.81}$$

Integration by parts gives

$$\begin{aligned}\int_0^1 \partial_x\mathcal{V}_0(t,x)\chi_t dx=&-\int_0^1 \mathcal{V}_0(t,x)\chi_{tx}dx\\=&-\int_0^1\left(\frac{\varepsilon^2}{3}\eta_{xx}-\frac{2\varepsilon^2\tilde{j}}{3\tilde{w}}\psi_{xx}+\mathcal{K}(t,x)\right)\chi_{tx}dx\\\leq&C\varepsilon\big(A_1^2(t)+\|\chi_{tx}(t)\|^2\big),\end{aligned} \tag{3.82}$$

by estimates (3.13e) and (3.44c). Inserting (3.82) into (3.81) yields

$$\mathcal{I}_3(t)\leq(\mu+\delta+\varepsilon)\|\chi_t(t)\|^2+C\Gamma_3(t). \tag{3.83}$$

Note that

$$\int_0^1 w^2\chi_t^2 dx\geq c\Pi_3(t), \tag{3.84}$$

whence substituting (3.83) and (3.84) into (3.79) produces (3.75).

From

$$\int_0^1 (3.37)|_{k=1}\chi_t dx,$$

we calculate

$$\frac{d}{dt}\Xi_4(t)+\frac{2}{3}\Pi_4(t)=\mathcal{I}_4(t), \tag{3.85}$$

where

$$\mathcal{I}_4(t):=\int_0^1\left[w\psi_t\chi_t^2+\left(\frac{2\tilde{\theta}_x}{3}\eta_t\chi_t+\frac{2\tilde{\theta}}{3}\eta_t\chi_{tx}\right)\right.$$

$$+\frac{4\tilde{j}\tilde{\theta}}{3\tilde{w}}\psi_{tx}\chi_t+\partial_x\mathcal{V}_1(t,x)\chi_t+\big(\partial_tH+L_1\big)(t,x)\chi_t\Big]dx, \tag{3.86}$$

satisfying

$$\begin{aligned}\mathcal{I}_4(t)\le&\int_0^1\partial_x\mathcal{V}_1(t,x)\chi_t dx+\big[\mu+C(N_\varepsilon(T)+\delta)\big]\|\chi_{tx}(t)\|^2\\&+C_\mu\|(\eta_t,\chi_t)(t)\|^2+C(N_\varepsilon(T)+\delta)\|(\psi_t,\psi_{tt},\psi_{tx})(t)\|^2.\end{aligned} \tag{3.87}$$

Integration by parts yields

$$\begin{aligned}\int_0^1\partial_x\mathcal{V}_1(t,x)\chi_t dx=&-\int_0^1\mathcal{V}_1(t,x)\chi_{tx}dx\\=&-\int_0^1\left(\frac{\varepsilon^2}{3}\eta_{txx}-\frac{2\varepsilon^2\tilde{j}}{3\tilde{w}}\psi_{txx}+\partial_t\mathcal{K}(t,x)\right)\chi_{tx}dx\\\le&C\varepsilon\big(\|\chi_{tx}(t)\|^2+\|\eta_t(t)\|^2+A_1^2(t)+\|\varepsilon\psi_{ttx}(t)\|^2\big),\end{aligned} \tag{3.88}$$

which in turn implies

$$\mathcal{I}_4(t)\le\big[\mu+C(N_\varepsilon(T)+\delta+\varepsilon)\big]\|\chi_{tx}(t)\|^2+C_\mu\Gamma_4(t). \tag{3.89}$$

Substituting (3.89) into (3.85) gives (3.77). ∎

3.3. Decay estimate. Based on Lemmas 3.3, 3.6, 3.7 and 3.8, we are able to show Proposition 3.1, that is, the decay estimate (3.12) at length.

Proof. **(Proof of Proposition 3.1.)** For positive constants α (which is the same as in (3.15)) and β, from

$$(3.14)+\beta\Big[\alpha(3.54)+(3.60)\Big]\Big|_{k=0}+\beta\Big[(3.75)+\beta(3.77)\Big]+\beta^3\Big[\alpha(3.54)+(3.60)\Big]\Big|_{k=1},$$

we have

$$\frac{d}{dt}\mathbb{E}(t)+\mathbb{D}(t)\le 0,\quad\forall t\in[0,T], \tag{3.90}$$

where

$$\mathbb{E}(t):=\Xi(t)+\beta\Big[\alpha\Xi_1^{(0)}(t)+\Xi_2^{(0)}(t)\Big]+\beta\Big[\Xi_3(t)+\beta\Xi_4(t)\Big]+\beta^3\Big[\alpha\Xi_1^{(1)}(t)+\Xi_2^{(1)}(t)\Big], \tag{3.91}$$

and

$$\begin{aligned}\mathbb{D}(t):=&\Big[c\Pi(t)-C\Gamma(t)\Big]+\beta\Big\{\alpha\Big[c\Pi_1^{(0)}(t)-C\Gamma_1^{(0)}(t)\Big]+\Big[c\Pi_2^{(0)}(t)-C\Gamma_2^{(0)}(t)\Big]\Big\}\\&+\beta\Big\{\Big[c\Pi_3(t)-C\Gamma_3(t)\Big]+\beta\Big[c\Pi_4(t)-C\Gamma_4(t)\Big]\Big\}\\&+\beta^3\Big\{\alpha\Big[c\Pi_1^{(1)}(t)-C\Gamma_1^{(1)}(t)\Big]+\Big[c\Pi_2^{(1)}(t)-C\Gamma_2^{(1)}(t)\Big]\Big\}.\end{aligned} \tag{3.92}$$

Substitute the expressions (3.15)∼(3.17), (3.55), (3.61), (3.76) and (3.78) into (3.91) and (3.92), and then take α, μ, β and $N_\varepsilon(T)+\delta+\varepsilon$ sufficiently small in the following order $0<N_\varepsilon(T)+\delta+\varepsilon\ll\beta^3\ll\beta^2\ll\beta\ll\mu\ll\alpha\ll 1$, to obtain

$$c\big(A_1^2(t)+\|\chi_t(t)\|^2\big)\le\mathbb{E}(t)\le C\big(A_1^2(t)+\|\chi_t(t)\|^2\big), \tag{3.93}$$

and

$$\begin{aligned}\mathbb{D}(t) \geq & c\left(A_1^2(t)+\|\chi_t(t)\|^2+\|(\chi_{tx},\varepsilon\psi_{ttx})(t)\|^2\right)\\ \geq & c\left(A_1^2(t)+\|\chi_t(t)\|^2\right),\\ \mathbb{D}(t) \leq & C\left(A_1^2(t)+\|\chi_t(t)\|^2+\|(\chi_{tx},\varepsilon\psi_{ttx})(t)\|^2\right),\end{aligned} \tag{3.94}$$

where the positive constants c and C are independent of δ, ε and T.

Applying (3.93) and (3.94) to (3.90), we have for some positive constant γ,

$$\frac{d}{dt}\mathbb{E}(t)+2\gamma\mathbb{E}(t)\leq 0, \quad \forall t\in[0,T], \tag{3.95}$$

which implies the decay estimate (3.12) by Gronwall's inequality. Here we have also used the elliptic estimate (3.13d) and the equivalent relationships (3.93), (3.45). The proof now is complete. ∎

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