# EXISTENCE AND UNIQUENESS FOR "GOOD" BOUSSINESQ EQUATIONS WITH QUASI-PERIODIC INITIAL DATA*

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**Abstract.** This paper studies the local well-posedness for the "good" Boussinesq equation subject to quasi-periodic initial conditions. By constructing a delicately and subtly iterative process together with an explicit combinatorial analysis, we show that there exists a unique solution for such a model in a small region of time. The size of this region depends on both the given data and the frequency vector. Moreover the local solution has an expansion with exponentially decaying Fourier coefficients.

**Keywords.** Boussinesq equations; Quasi-periodic initial data; Existence; Uniqueness; Exponential decay.

**AMS subject classifications.** 35Q35; 35A01; 35B15.

## 1. Introduction

The aim of this paper is to investigate the existence and uniqueness for the "good" Boussinesq equation

$$u_{tt} + u_{xxxx} - u_{xx} - (u^2)_{xx} = 0, \quad x \in \mathbb{R}, \ t > 0 \tag{1.1}$$

with respect to quasi-periodic initial data

$$u(0,x) = u_0(x) = \sum_{\boldsymbol{n} \in \mathbb{Z}^\nu} c(0,\boldsymbol{n}) \exp(ix\boldsymbol{n} \cdot \boldsymbol{\omega}) := \sum_{\boldsymbol{n} \in \mathbb{Z}^\nu} c(\boldsymbol{n}) \exp(ix\boldsymbol{n} \cdot \boldsymbol{\omega}), \tag{1.2}$$

$$\partial_t u(0,x) = u_1(x) = \sum_{\boldsymbol{n} \in \mathbb{Z}^\nu} \partial_t c(0,\boldsymbol{n}) \exp(ix\boldsymbol{n} \cdot \boldsymbol{\omega}) := \sum_{\boldsymbol{n} \in \mathbb{Z}^\nu} c'(\boldsymbol{n}) \exp(ix\boldsymbol{n} \cdot \boldsymbol{\omega}), \tag{1.3}$$

where

$$\boldsymbol{n} = (n_1, \cdots, n_\nu) \in \mathbb{Z}^\nu, \quad \boldsymbol{\omega} = (\omega_1, \cdots, \omega_\nu) \in \mathbb{R}^\nu.$$

Equation (1.1) governs small nonlinear oscillations in an elastic beam, which is also known as the "nonlinear string equation" (see [9]).

When investigating the bidirectional propagation of small amplitude and long wavelength capillary-gravity waves on the surface of shallow water, in 1872 Boussinesq [4] gave the classical Boussinesq equation

$$v_{tt} - gh_0 v_{xx} = gh_0 \left( \frac{3}{2} \frac{v^2}{h_0} + \frac{h_0^2}{3} v_{xx} \right)_{xx}, \quad x \in \mathbb{R}, \ t > 0, \tag{1.4}$$

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where $v(t,x)$ is the perturbation of free surface, $h_0$ is the mean depth, and $g$ is the gravitational constant. In nondimensional units, Equation (1.4) can be reduced to

$$u_{tt} - u_{xxxx} - u_{xx} - (u^2)_{xx} = 0, \quad x \in \mathbb{R}, \ t > 0, \tag{1.5}$$

which is called "bad" Boussinesq equation. This was the first mathematical model for the phenomenon of solitary waves observed by Scott–Russell [25]. It admits the special travelling-wave solutions

$$u(x,t) = \frac{2}{3}(c^2 - 1)\mathrm{sech}^2\left(\frac{\sqrt{c^2 - 1}}{2}(x - ct)\right),$$

where the constant $c$ stands for velocity of the wave. Such solutions are also called solitary waves. However the "bad" Boussinesq Equation (1.5) is ill-posed because of the exponential growth of the Fourier components. In order to investigate the initial value problem, Deift et al. [8] imposed exponentially decaying of the initial functions and applied the techniques of inverse scattering theory to the following Boussinesq equation

$$u_{tt} - 3u_{xxxx} + 12(u^2)_{xx} = 0.$$

The other way to solve the ill-posed problems is that we can change the sign of the fourth order derivative term in Equation (1.5) from negative to positive, i.e., the $u_{tt}$ and $u_{xxxx}$ terms have the same sign, and is called "good" Boussinesq equation and is linearly well-posed. The "good" Boussinesq equation was suggested by Zakharov [33] as a model of nonlinear vibrations along a string, and also by Turitsyn [30] for describing electromagnetic waves in nonlinear dielectric materials; some more details can be found in [18, 24, 26, 32] and the references therein.

The local well-posedness of the Cauchy problem for the "good" Boussinesq Equation (1.1) has a relatively recent history. Bona and Sachs [3] considered the following Cauchy problem associated with Boussinesq equations

$$\begin{cases} u_{tt} + u_{xxxx} - u_{xx} + (f(u))_{xx} = 0, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x). \end{cases} \tag{1.6}$$

By using Kato's abstract theory for quasi-linear evolution equation, they concluded local well-posedness with respect to initial data $(u_0, u_1) \in H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})$ for $s > \frac{5}{2}$. At the same time, they also showed that the solution with initial data close to isolated wave ones is orbital stable and always exists. Tsutsumi and Matahashi [29] established local and global well-posedness of the Cauchy problem (1.6) with $(u_0, u_1) \in H^1(\mathbb{R}) \times H^{-1}(\mathbb{R})$. Linares [19] further applied Strichartz-type estimates to investigate local well-posedness of the Cauchy problem (1.6) when initial data $(u_0, u_1) \in L^2(\mathbb{R}) \times H^{-1}(\mathbb{R})$. Farah [11] improved the local well-posedness results above by proving that the Cauchy problem (1.6) is locally well-posed when $(u_0, u_1)$ belong to $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > -\frac{1}{4}$. The main proof is based on defining suitable Bourgain-type spaces to the linear part of the equation, and using them to derive the appropriate bilinear estimates. Moreover, Constantin and Molinet [6] demonstrated the existence and uniqueness of local solutions of the generalized Boussinesq equation for initial data of low regularity. They also discussed the existence of global solutions and the occurrence of blow-up phenomena. Taniuchi [27] showed that a two-dimensional Boussinesq equation with non-decaying initial data admits a unique global solution on the whole plane. In addition, we refer

the readers to the articles [5, 12, 14, 17] for well-posedness associated with the "good" Boussinesq equation.

In recent years there has been extensive interest in nonlinear partial differential equations with respect to either periodic or quasi-periodic or almost periodic initial data. Venakides [31] calculated weak limit of solutions of the following KdV equation

$$u_t + \epsilon^2 u_{xxx} - 6uu_x = 0$$

for the periodic initial value if $\epsilon$ tends to 0. In the neighborhood of a point $(x,t)$, he obtained that the solution $u(x,t,\epsilon)$ could be approximated either by a constant or a periodic or a quasi-periodic solution for such a model. Tadahiro [21, 22], respectively, studied the Cauchy problem of a class of nonlinear Schrödinger equations with the limit periodic function and almost periodic function as initial value. For the almost periodic case, under a set of frequencies $\boldsymbol{\omega} = \{\omega_j\}_{j=1}^{\infty}$, he presented that the corresponding Cauchy problem was locally well-posed in the algebras $\mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R})$ consisting of almost periodic functions with absolutely convergent Fourier series. Moreover, he also provided the first example of blasting solutions for such a model with general almost periodic initial values in finite time. In addition, Tsugawa [28] gave well-posedness of the KdV equation with quasi-periodic initial value by using the Fourier restriction norm method introduced by Bourgain. Recently, applying the Diophantine conditions and an exponential decay assumption on the generalized Fourier components, Damanik and Goldstein [7] constructed local and global solutions to the KdV equation corresponding to quasi-periodic initial data.

Let us review the achievements related to the "good" Boussinesq equation subject to periodic initial data. In spite of the "good" Boussinesq Equation (1.1) having the Lax pair formula and being linear stable, Kalantarov and Ladyzhenskaya proved [13] that in the periodic case and Dirichlet boundary case solutions may blow-up in a finite time. Given minimal regularity assumptions on periodic initial data, Fang and Grillakis [10] established local and global existence results (using the conservation of energy) for the Cauchy problem (1.6) by using Fourier series and a fixed-point argument. Later, Oh and Stefanov [20] considered local well-posedness of the Cauchy problem (1.6) with periodic initial data and $f(u) = u^p$. They reduced the Sobolev index to $s > -\frac{3}{8}$. Recently, Barostichi [1] also studied local well-posedness for initial data in Gevrey spaces on the circle. Different with the case in [10] and [1], the energy is indefinite in our case and the solution may blow up. To the best of our knowledge, there are few results on well-posedness for the "good" Boussinesq equation under quasi-periodic initial data. In this work, we intend to prove the existence and uniqueness for "good" Boussinesq equation with quasi-periodic initial data.

More precisely, we have the following theorem.

THEOREM 1.1.    *Let $\boldsymbol{\omega} \in \mathbb{R}^\nu$ be a Diophantine wave vector. Suppose that the Fourier coefficients associated with initial data (1.2)–(1.3) satisfy*

$$|c(\boldsymbol{n})| \leq B\exp(-\frac{\kappa|\boldsymbol{n}|}{2}), \quad |c'(\boldsymbol{n})| \leq B\exp(-\frac{\kappa|\boldsymbol{n}|}{2}),$$

*where $B, \kappa$ are two positive constants. Then there exists $t_0 > 0$ such that for $t \in (0, t_0)$, $x \in \mathbb{R}$, one can construct a function*

$$u(t,x) = \sum_{\boldsymbol{n} \in \mathbb{Z}^\nu} c(t, \boldsymbol{n}) \exp(\mathrm{i} x \boldsymbol{n} \cdot \boldsymbol{\omega}),$$

*which satisfies Equation* (1.1) *and initial conditions* (1.2)–(1.3). *Moreover,*

$$c(t,\boldsymbol{n}) = \frac{1}{2}c(\boldsymbol{n})\left(\exp\left(\mathrm{i}t\lambda\right) + \exp\left(-\mathrm{i}t\lambda\right)\right) - \frac{\mathrm{i}}{2\lambda}c'(\boldsymbol{n})\left(\exp\left(\mathrm{i}t\lambda\right) - \exp\left(-\mathrm{i}t\lambda\right)\right)$$
$$- \frac{\mathrm{i}}{\lambda}\int_0^t \exp\left(\mathrm{i}(\tau-t)\lambda\right) - \exp\left(\mathrm{i}(t-\tau)\lambda\right) \sum_{\boldsymbol{m}\in\mathbb{Z}^\nu} (\boldsymbol{m}\cdot\boldsymbol{\omega})(\boldsymbol{n}\cdot\boldsymbol{\omega})c(t,\boldsymbol{m})c(t,\boldsymbol{n}-\boldsymbol{m})\mathrm{d}\tau$$

*with* $\lambda = \left((\boldsymbol{n}\cdot\boldsymbol{\omega})^2 + (\boldsymbol{n}\cdot\boldsymbol{\omega})^4\right)^{\frac{1}{2}}$, *and*

$$|c(t,\boldsymbol{n})| \le 2B\exp(-\frac{\kappa|\boldsymbol{n}|}{4}).$$

*Furthermore, if*

$$v(t,x) = \sum_{\boldsymbol{n}\in\mathbb{Z}^\nu} h(t,\boldsymbol{n})\exp(\mathrm{i}x\boldsymbol{n}\cdot\boldsymbol{\omega})$$

*is also a solution of Equation* (1.1) *with initial conditions* (1.2)–(1.3) *satisfying that for some positive constants* $C_1, \rho$,

$$|h(t,\boldsymbol{n})| \le C_1\exp(-\rho|\boldsymbol{n}|),$$

*then there exists* $t_1 > 0$ *such that* $v(t,x) = u(t,x)$ *for* $0 \le t \le t_1$, $x \in \mathbb{R}$.

Contrast with the global result for KdV equation in [7], Damanik and Goldstein can apply the fundamental property of the Schrödinger operators (conservation of spectrum) by Lax [15] to extend the local result to global result. The Boussinesq equation doesn't possess these properties. In fact, using the method of Levine [16], Kalantarov and Ladyzhenskaya [13] showed that for a large set of initial values there is no smooth solution of Equation (1.1) for all time. This nonexistence is generally referred to blow-up rather than collapse, while the blow-up for "good" Boussinesq was proved by Sachs [23] in $H^{-1}$ for certain initial data (the energy is indefinite).

The nonlinear interaction between high- and very-low-frequency parts of solutions make the well-posedness problem difficult in the study of the Boussinesq equation. To avoid this difficulty, in the periodic data case, one can apply the conservation law:$\int_\mathbb{T} u\mathrm{d}x = c$ for any solution of the Boussinesq equation. It is not enough for the quasi-periodic case, while the main difficulty with quasi-periodic initial data is in the complicated nature of the conservation laws. Furthermore, the spectrum in the quasi-periodic case is known to typically have a dense set of gaps. In our analysis, the major difficulty is to keep the Fourier coefficients of local solutions to have exponential decay. In order to overcome this problem, we apply an explicit combinatorial analysis of the iteration of the integral transformation.

This paper is organized as follows. Section 2 shows the exponential decay of Fourier coefficients of local solutions for the "good" Boussinesq equation. An integral transform is introduced to reduce the different equation for the Fourier coefficients into integral equations. A Picard iteration sequence for the Fourier coefficients is constructed. Due to the complex representation of iterative terms, we introduce inductive tree branches, and attach an appropriate lattice to each tree branch for keeping the terms in iterative equations. Another step is to define some weights which enable us to develop estimation techniques for iterative equations. Finally, we make a combination analysis of the explicit iteration of integral transformation. Although the derivation process is quite

complicated, the exponential decay of the Fourier coefficients is in good agreement with the combined growth factor produced in the iterative process. There is no small denominator problem in the estimation as well. Therefore our derivation does not involve any Diophantine condition. The aim of Section 3 is to present that the Fourier coefficients of solutions for the "good" Boussinesq equation indeed exist and are unique. In Section 4, we give the proof of Theorem 1.1. More precisely, we prove the existence and uniqueness of local solutions for the "good" Boussinesq equation with quasi-periodic initial data.

Before ending this section, let us mention that Binder et al. [2] recently investigate the Cauchy problem for the KdV equation with almost periodic initial data and obtained the existence, uniqueness, and almost periodicity in time of solutions. Their result can also apply to all small analytic quasi-periodic initial data with Diophantine frequency vector. However, it is not clear whether it is valid for the general quasi-periodic initial data for the Boussinesq equation.

## 2. Exponential decay of Fourier coefficients

Suppose that the function

$$u(t,x) = \sum_{\boldsymbol{n}\in\mathbb{Z}^\nu} c(t,\boldsymbol{n})\exp(\mathrm{i}x\boldsymbol{n}\cdot\boldsymbol{\omega}) \tag{2.1}$$

is a solution of Equation (1.1) with respect to initial conditions (1.2)–(1.3). Meanwhile we assume that $(u^2)_{xx}$ has the following expansion

$$(u^2)_{xx} = \sum_{\boldsymbol{n}\in\mathbb{Z}^\nu} A(t,\boldsymbol{n})\exp(\mathrm{i}x\boldsymbol{n}\cdot\boldsymbol{\omega}). \tag{2.2}$$

The main purpose of this section is to establish the exponential decay of the Fourier coefficients $c(t,\boldsymbol{n})$ under some assumptions. Moreover we denote by $|\cdot|$ the $\ell^1$-norm on $\mathbb{R}^\nu$ as follows

$$|\boldsymbol{y}| = \sum |y_j|, \quad \boldsymbol{y} = (y_1,\ldots,y_\nu)\in\mathbb{R}^\nu.$$

The following lemma gives the expressions of the Fourier coefficients $c(t,\boldsymbol{n})$.

LEMMA 2.1.    *For some constant $t_0 > 0$, let $c(t,\boldsymbol{n})$, $A(t,\boldsymbol{n})$ be continuous functions of $t\in[0,t_0)$, $\boldsymbol{n}\in\mathbb{Z}^\nu$. Assume that*

$$\sup_t \sum_{\boldsymbol{n}\in\mathbb{Z}^\nu} (1+|\boldsymbol{n}|^2+|\boldsymbol{n}|^4)(|c(t,\boldsymbol{n})|+|A(t,\boldsymbol{n})|) < \infty. \tag{2.3}$$

*Then the Fourier coefficients $c(t,\boldsymbol{n})$ associated with the ansatz (2.1) can be expressed as the following integral forms*

$$c(t,\boldsymbol{n}) = \left(\frac{1}{2}c(\boldsymbol{n}) - \frac{\mathrm{i}}{2\lambda}c'(\boldsymbol{n})\right)\exp(\mathrm{i}\lambda t) + \left(\frac{1}{2}c(\boldsymbol{n}) + \frac{\mathrm{i}}{2\lambda}c'(\boldsymbol{n})\right)\exp(-\mathrm{i}\lambda t)$$

$$+ \int_0^t \frac{\exp(\mathrm{i}\lambda(\tau-t)) - \exp(\mathrm{i}\lambda(t-\tau))}{-2\lambda\mathrm{i}} A(\tau,\boldsymbol{n})\mathrm{d}\tau, \tag{2.4}$$

*where $\lambda = ((\boldsymbol{n}\cdot\boldsymbol{\omega})^2 + (\boldsymbol{n}\cdot\boldsymbol{\omega})^4)^{\frac{1}{2}}$ with $\boldsymbol{\omega}\in\mathbb{R}^\nu$. Moreover the functions $u_{tt}$, $-u_{xxxx}$, $-u_{xx}$, $(u^2)_{xx}$ are continuous with respect to $(t,x)\in[0,t_0)\times\mathbb{R}$.*

*Proof.*    Substituting the ansatz (2.1)–(2.2) into Equation (1.1) yields

$$\sum_{\boldsymbol{n}\in\mathbb{Z}^\nu}\frac{\mathrm{d}^2 c(t,\boldsymbol{n})}{\mathrm{d}t^2}\exp(\mathrm{i}x\boldsymbol{n}\cdot\boldsymbol{\omega})+\sum_{\boldsymbol{n}\in\mathbb{Z}^\nu}(\mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\omega})^4 c(t,\boldsymbol{n})\exp(\mathrm{i}x\boldsymbol{n}\cdot\boldsymbol{\omega})$$

$$-\sum_{\boldsymbol{n}\in\mathbb{Z}^\nu}(\mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\omega})^2 c(t,\boldsymbol{n})\exp(\mathrm{i}x\boldsymbol{n}\cdot\boldsymbol{\omega})-\sum_{\boldsymbol{n}\in\mathbb{Z}^\nu}A(t,\boldsymbol{n})\exp(\mathrm{i}x\boldsymbol{n}\cdot\boldsymbol{\omega})=0.$$

This is equivalent to

$$\frac{\mathrm{d}^2 c(t,\boldsymbol{n})}{\mathrm{d}t^2}+\big((\boldsymbol{n}\cdot\boldsymbol{\omega})^4+(\boldsymbol{n}\cdot\boldsymbol{\omega})^2\big)c(t,\boldsymbol{n})-A(t,\boldsymbol{n})=0. \tag{2.5}$$

The corresponding characteristic equation for the homogeneous equation of (2.5) is

$$\eta^2+(\boldsymbol{n}\cdot\boldsymbol{\omega})^4+(\boldsymbol{n}\cdot\boldsymbol{\omega})^2=0.$$

Thus the homogeneous equation has two solutions $\exp(\mathrm{i}\lambda t)$ and $\exp(-\mathrm{i}\lambda t)$. By variation of constants formula, we obtain

$$c(t,\boldsymbol{n})=c_1\exp(\mathrm{i}\lambda t)+c_2\exp(-\mathrm{i}\lambda t)+\int_0^t\frac{\Delta(t,\tau)}{W(\tau)}A(\tau,\boldsymbol{n})\mathrm{d}\tau,$$

where

$$\Delta(t,\tau):=\det\begin{pmatrix}\exp(\mathrm{i}\lambda\tau)\ \exp(-\mathrm{i}\lambda\tau)\\ \exp(\mathrm{i}\lambda t)\ \exp(-\mathrm{i}\lambda t)\end{pmatrix}=\exp(\mathrm{i}\lambda(\tau-t))-\exp(\mathrm{i}\lambda(t-\tau)),$$

$$W(\tau):=\det\begin{pmatrix}\exp(\mathrm{i}\lambda\tau)&\exp(-\mathrm{i}\lambda\tau)\\ \mathrm{i}\lambda\exp(\mathrm{i}\lambda\tau)&-\mathrm{i}\lambda\exp(-\mathrm{i}\lambda\tau)\end{pmatrix}=-2\mathrm{i}\lambda.$$

Using initial conditions (1.2)–(1.3), we have

$$c(t,\boldsymbol{n})=c_1\exp(\mathrm{i}\lambda t)+c_2\exp(-\mathrm{i}\lambda t)+\int_0^t\frac{\exp(\mathrm{i}\lambda(\tau-t))-\exp(\mathrm{i}\lambda(t-\tau))}{-2\lambda\mathrm{i}}A(\tau,\boldsymbol{n})\mathrm{d}\tau,$$

where

$$c_1=\frac{1}{2}c(\boldsymbol{n})-\frac{\mathrm{i}}{2\lambda}c'(\boldsymbol{n}),\quad c_2=\frac{1}{2}c(\boldsymbol{n})+\frac{\mathrm{i}}{2\lambda}c'(\boldsymbol{n}).$$

Moreover, all series involved converge absolutely and uniformly under condition (2.3). This completes the proof.    □

In the following lemma, we further present forms clearer than (2.4) for the Fourier coefficients $c(t,\boldsymbol{n})$ associated with the ansatz (2.1).

LEMMA 2.2.    *Let $c(t,\boldsymbol{n})$ be the Fourier coefficients associated with the ansatz (2.1). Then for $\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}^\nu$, one has the following integral equations*

$$c(t,\boldsymbol{n})=\frac{1}{2}c(\boldsymbol{n})\left(\exp\left(\mathrm{i}t\lambda\right)+\exp\left(-\mathrm{i}t\lambda\right)\right)-\frac{\mathrm{i}}{2\lambda}c'(\boldsymbol{n})\left(\exp\left(\mathrm{i}t\lambda\right)-\exp\left(-\mathrm{i}t\lambda\right)\right)$$

$$-\tfrac{\mathrm{i}}{\lambda}\int_0^t\left(\exp\left(\mathrm{i}(\tau-t)\lambda\right)-\exp\left(\mathrm{i}(t-\tau)\lambda\right)\right)\sum_{\boldsymbol{m}\in\mathbb{Z}^\nu}(\boldsymbol{m}\cdot\boldsymbol{\omega})(\boldsymbol{n}\cdot\boldsymbol{\omega})c(t,\boldsymbol{m})c(t,\boldsymbol{n}-\boldsymbol{m})\mathrm{d}\tau, \tag{2.6}$$

*where $\lambda=((\boldsymbol{n}\cdot\boldsymbol{\omega})^2+(\boldsymbol{n}\cdot\boldsymbol{\omega})^4)^{\frac{1}{2}}$ with $\boldsymbol{\omega}\in\mathbb{R}^\nu$.*

*Proof.*    The key of the proof is to give the expression of $A(t,\boldsymbol{n})$ in (2.2). Observe that

$$(u^2)_{xx} = 2(u_x u_x + u_{xx} u).$$

Moreover, it follows from (2.1) that

$$u_x = \sum_{\boldsymbol{n}} c(t,\boldsymbol{n})(\mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\omega})\exp(\mathrm{i}x\boldsymbol{n}\cdot\boldsymbol{\omega}),$$

and

$$u_{xx} = \sum_{\boldsymbol{n}} c(t,\boldsymbol{n})(\mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\omega})^2\exp(\mathrm{i}x\boldsymbol{n}\cdot\boldsymbol{\omega}) = -\sum_{\boldsymbol{n}} c(t,\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{\omega})^2\exp(\mathrm{i}x\boldsymbol{n}\cdot\boldsymbol{\omega}).$$

Hence,

$$u_x u_x = \sum_{\boldsymbol{n}}\sum_{\boldsymbol{m}}((\boldsymbol{m}\cdot\boldsymbol{\omega})^2 - (\boldsymbol{m}\cdot\boldsymbol{\omega})(\boldsymbol{n}\cdot\boldsymbol{\omega}))c(t,\boldsymbol{m})c(t,\boldsymbol{n}-\boldsymbol{m})\exp(\mathrm{i}x\boldsymbol{n}\cdot\boldsymbol{\omega}),$$

$$u_{xx} u = \sum_{\boldsymbol{n}}\sum_{\boldsymbol{m}} -(\boldsymbol{m}\cdot\boldsymbol{\omega})^2 c(t,\boldsymbol{m})c(t,\boldsymbol{n}-\boldsymbol{m})\exp(\mathrm{i}x\boldsymbol{n}\cdot\boldsymbol{\omega}).$$

Consequently, we get

$$(u^2)_{xx} = 2\sum_{\boldsymbol{n}}\sum_{\boldsymbol{m}} -(\boldsymbol{m}\cdot\boldsymbol{\omega})(\boldsymbol{n}\cdot\boldsymbol{\omega})c(t,\boldsymbol{m})c(t,\boldsymbol{n}-\boldsymbol{m})\exp(\mathrm{i}x\boldsymbol{n}\cdot\boldsymbol{\omega}).$$

This shows that

$$A(t,\boldsymbol{n}) = -2\sum_{\boldsymbol{m}}(\boldsymbol{m}\cdot\boldsymbol{\omega})(\boldsymbol{n}\cdot\boldsymbol{\omega})c(t,\boldsymbol{m})c(t,\boldsymbol{n}-\boldsymbol{m}).$$

The proof is completed.                                                                       □

By Lemma 2.2, we can obtain the integral Equation (2.6). In order to prove the existence and uniqueness of solutions for Equation (2.6), we will construct the Picard iteration sequence for $c(t,\boldsymbol{n})$. Moreover we have to assume that the Fourier coefficients $c(\boldsymbol{n}),c'(\boldsymbol{n})$ associated with initial data (1.2)–(1.3) have exponential decay. Namely, there exist two constants $B>0$, $0<\kappa\leq 1$ such that for all $\boldsymbol{n}\in\mathbb{Z}^\nu$,

$$|c(\boldsymbol{n})|\leq B\exp\left(-\frac{\kappa|\boldsymbol{n}|}{2}\right), \quad |c'(\boldsymbol{n})|\leq B\exp\left(-\frac{\kappa|\boldsymbol{n}|}{2}\right) \tag{2.7}$$

with $\boldsymbol{\omega}\in\mathbb{R}^\nu$. Thus we have to start the iteration from an exponentially decaying collection of Fourier coefficients and keep this property in check.

Let $\lambda=((\boldsymbol{n}\cdot\boldsymbol{\omega})^2+(\boldsymbol{n}\cdot\boldsymbol{\omega})^4)^{\frac{1}{2}}$. Construct a sequence $\{c_k(t,\boldsymbol{n})\},k\geq 0$ as follows:

$$c_0(t,\boldsymbol{n}) = \frac{1}{2}c(\boldsymbol{n})(\exp(\mathrm{i}t\lambda)+\exp(-\mathrm{i}t\lambda)) - \frac{\mathrm{i}}{2\lambda}c'(\boldsymbol{n})(\exp(\mathrm{i}t\lambda)-\exp(-\mathrm{i}t\lambda)), \tag{2.8}$$

and for $k=1,2,\cdots$,

$$c_k(t,\boldsymbol{n}) = \left(\frac{1}{2}c(\boldsymbol{n})-\frac{\mathrm{i}}{2\lambda}c'(\boldsymbol{n})\right)\left(e^{\mathrm{i}t\lambda}-e^{-\mathrm{i}t\lambda}\right) - \frac{\mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\omega}}{2\sqrt{1+(\boldsymbol{n}\cdot\boldsymbol{\omega})^2}}$$

$$\cdot \int_0^t \left( e^{\mathrm{i}t\lambda} - e^{-\mathrm{i}t\lambda} \right) \sum_{\substack{\boldsymbol{m}_1,\boldsymbol{m}_2 \in \mathbb{Z}^\nu \\ \boldsymbol{m}_1 + \boldsymbol{m}_2 = \boldsymbol{n}}} c_{k-1}(\tau,\boldsymbol{m}_1) c_{k-1}(\tau,\boldsymbol{m}_2) \mathrm{d}\tau. \tag{2.9}$$

For some constant $t_0 > 0$, we will show inductively that the functions $c_k(t,\boldsymbol{n})$ are well-defined and continuous with respect to $t \in [0,t_0)$. On the other hand, we need to prove that the sequence $\{c_k(t,\boldsymbol{n})\}$ converges absolutely and uniformly on the interval $0 \leq t < t_0$. However it is very difficult to prove the absolute and uniform convergence of the sequence $\{c_k(t,\boldsymbol{n})\}$. Through the observation of $c_k(t,\boldsymbol{n})$, we find that it has 6 terms for $k=1$, 38 terms for $k=2$, 1446 terms for $k=3$. In fact, let $N_0 = 2$ and $N_k = 2 + N_{k-1}^2$ for all $k \geq 1$. It follows from (2.9) that $N_k$ stands for the number of the terms appearing in $c_k$, from which one can see that there are infinite number of terms as $k$ tends to $\infty$.

As a result, we intend to represent $c_k(t,\boldsymbol{n})$. By virtue of the summation in (2.9), we first label these terms of the iterative equation via points on a tree. The branches of the tree originate from points on the lattice $\mathbb{Z}^\nu$ and split under the condition $\boldsymbol{m}_1 + \boldsymbol{m}_2 = constant$. Our next goal is to introduce the branches $\boldsymbol{\gamma}$ by induction, and then attach an appropriate lattice $\mathbb{Z}^\nu$ to each branch for keeping the terms of the iterative equation. Finally, we define some weights which enable us to develop estimation techniques for iterative equations. Although the definition of these objects seems to be quite complicated, it is naturally generated by the induction of the number of iterations of the equation.

Now let us introduce some definitions. Denote by "×" the cartesian product. We set

$$\begin{aligned} \mathscr{D}^{(1)} &= \{0,1\}, \\ \mathscr{D}^{(2)} &= \mathscr{D}^{(1)} \cup \mathscr{D}^{(1)} \times \mathscr{D}^{(1)} = \{0,1\} \cup \{(0,0),(0,1),(1,0),(1,1)\} \\ &= \{0,1,(0,0),(0,1),(1,0),(1,1)\}, \\ \mathscr{D}^{(k)} &= \mathscr{D}^{(1)} \cup \mathscr{D}^{(k-1)} \times \mathscr{D}^{(k-1)}, \quad k=3,4,\cdots, \end{aligned} \tag{2.10}$$

and

$$\mathfrak{M}^{(k,\gamma)} = \begin{cases} \mathbb{Z}^\nu & \text{if } \boldsymbol{\gamma} = 0 \text{ or } 1 \in \mathscr{D}^{(k)}, \\ \mathbb{Z}^\nu \times \mathbb{Z}^\nu & \text{if } \boldsymbol{\gamma} \in \mathscr{D}^{(2)}, \boldsymbol{\gamma} = (0,0) \text{ or } (0,1) \text{ or} \\ & \quad (1,0) \text{ or } (1,1), \\ \mathfrak{M}^{(k-1,\gamma_1^{(k-1)})} \times \mathfrak{M}^{(k-1,\gamma_2^{(k-1)})} & \text{if } \boldsymbol{\gamma} \in \mathscr{D}^{(k)}, k \geq 3, \\ & \quad \boldsymbol{\gamma} = (\boldsymbol{\gamma}_1^{(k-1)},\boldsymbol{\gamma}_2^{(k-1)}) \in \mathscr{D}^{(k-1)} \times \mathscr{D}^{(k-1)}. \end{cases} \tag{2.11}$$

For $\boldsymbol{m}^{(k)} \in \mathfrak{M}^{(k,\gamma)}$, we further define

$$\mathfrak{C}(\boldsymbol{m}^{(k)}) = \begin{cases} c(\boldsymbol{m}) & \text{if } \boldsymbol{\gamma} = 0 \in \mathscr{D}^{(k)}, \boldsymbol{m}^{(k)} = \boldsymbol{m} \in \mathfrak{M}^{(k,\gamma)}, \\ c'(\boldsymbol{m}) & \text{if } \boldsymbol{\gamma} = 1 \in \mathscr{D}^{(k)}, \boldsymbol{m}^{(k)} = \boldsymbol{m} \in \mathfrak{M}^{(k,\gamma)}, \\ c(\boldsymbol{m}_1)c(\boldsymbol{m}_2) & \text{if } \boldsymbol{\gamma} = (0,0) \in \mathscr{D}^{(2)}, \boldsymbol{m}^{(2)} = (\boldsymbol{m}_1,\boldsymbol{m}_2) \in \mathfrak{M}^{(2,\gamma)}, \\ c(\boldsymbol{m}_1)c'(\boldsymbol{m}_2) & \text{if } \boldsymbol{\gamma} = (0,1) \in \mathscr{D}^{(2)}, \boldsymbol{m}^{(2)} = (\boldsymbol{m}_1,\boldsymbol{m}_2) \in \mathfrak{M}^{(2,\gamma)}, \\ c'(\boldsymbol{m}_1)c(\boldsymbol{m}_2) & \text{if } \boldsymbol{\gamma} = (1,0) \in \mathscr{D}^{(2)}, \boldsymbol{m}^{(2)} = (\boldsymbol{m}_1,\boldsymbol{m}_2) \in \mathfrak{M}^{(2,\gamma)}, \\ c'(\boldsymbol{m}_1)c'(\boldsymbol{m}_2) & \text{if } \boldsymbol{\gamma} = (1,1) \in \mathscr{D}^{(2)}, \boldsymbol{m}^{(2)} = (\boldsymbol{m}_1,\boldsymbol{m}_2) \in \mathfrak{M}^{(2,\gamma)}, \\ \mathfrak{C}(\boldsymbol{m}_1^{(k-1)})\mathfrak{C}(\boldsymbol{m}_2^{(k-1)}) & \text{if } \boldsymbol{m}^{(k)} \in \mathfrak{M}^{(k,\gamma)}, k \geq 3, \\ & \quad \boldsymbol{\gamma} = (\boldsymbol{\gamma}_1^{(k-1)},\boldsymbol{\gamma}_2^{(k-1)}) \in \mathscr{D}^{(k-1)} \times \mathscr{D}^{(k-1)}, \boldsymbol{m}^{(k)} \\ & \quad = (\boldsymbol{m}_1^{(k-1)},\boldsymbol{m}_2^{(k-1)}) \in \mathfrak{M}^{(k-1,\gamma_1^{(k-1)})} \times \mathfrak{M}^{(k-1,\gamma_2^{(k-1)})}, \end{cases}$$
$$\tag{2.12}$$

and

$$
f(\boldsymbol{m}^{(k)}) = \begin{cases}
1 & \text{if } \boldsymbol{\gamma} = 0 \text{ or } 1 \in \mathscr{D}^{(1)}, \boldsymbol{m}^{(1)} \in \mathfrak{M}^{(1,\boldsymbol{\gamma})}, \\[2mm]
\dfrac{-\mathrm{i}\mu(\boldsymbol{m}^{(2)}) \cdot \boldsymbol{\omega}}{2\sqrt{1 + (\mu(\boldsymbol{m}^{(2)}) \cdot \boldsymbol{\omega})^2}} & \text{if } \boldsymbol{\gamma} \in \mathscr{D}^{(2)}, \boldsymbol{\gamma} = (0,0) \text{ or } (0,1) \\
& \quad \text{or } (1,0) \text{ or } (1,1), \boldsymbol{m}^{(2)} \in \mathfrak{M}^{(2,\boldsymbol{\gamma})}, \\[2mm]
\dfrac{-\mathrm{i}\mu(\boldsymbol{m}^{(k)}) \cdot \boldsymbol{\omega}}{2\sqrt{1 + (\mu(\boldsymbol{m}^{(k)}) \cdot \boldsymbol{\omega})^2}} f(\boldsymbol{m}_1^{(k-1)}) f(\boldsymbol{m}_2^{(k-1)}) & \text{if } \boldsymbol{\gamma} \in \mathscr{D}^{(k)}, \gamma = (\boldsymbol{\gamma}_1^{(k-1)}, \boldsymbol{\gamma}_2^{(k-1)}) \\
& \quad \in \mathscr{D}^{(k-1)} \times \mathscr{D}^{(k-1)}, k \geq 3, \\
& \quad \boldsymbol{m}^{(k)} = (\boldsymbol{m}_1^{(k-1)}, \boldsymbol{m}_2^{(k-1)}) \\
& \quad \in \mathfrak{M}^{(k-1,\boldsymbol{\gamma}_1^{(k-1)})} \times \mathfrak{M}^{(k-1,\boldsymbol{\gamma}_2^{(k-1)})},
\end{cases}
\tag{2.13}
$$

where

$$
\mu(\boldsymbol{m}) = \sum_j \boldsymbol{m}_j, \quad \boldsymbol{m} = (\boldsymbol{m}_1, \cdots, \boldsymbol{m}_N), \quad \boldsymbol{m}_j \in \mathbb{Z}^{n\nu}, n, \ N \in \mathbb{N}_+.
\tag{2.14}
$$

Moreover, for $t > 0$, we also define

$$
I(t, \boldsymbol{m}^{(k)}) = \begin{cases}
\dfrac{1}{2}\exp(\mathrm{i}t\lambda_{\boldsymbol{m}^{(k)}}) + \dfrac{1}{2}\exp(-\mathrm{i}t\lambda_{\boldsymbol{m}^{(k)}}) \\
\quad \text{if } \boldsymbol{\gamma} = 0 \in \mathscr{D}^{(k)}, \ \boldsymbol{m}^{(k)} \in \mathfrak{M}^{(k,\boldsymbol{\gamma})}, \\[2mm]
\dfrac{-\mathrm{i}}{2\lambda_{\boldsymbol{m}^{(k)}}}\exp(\mathrm{i}t\lambda_{\boldsymbol{m}^{(k)}}) + \dfrac{\mathrm{i}}{2\lambda_{\boldsymbol{m}^{(k)}}}\exp(-\mathrm{i}t\lambda_{\boldsymbol{m}^{(k)}}) \\
\quad \text{if } \boldsymbol{\gamma} = 1 \in \mathscr{D}^{(k)}, \ \boldsymbol{m}^{(k)} \in \mathfrak{M}^{(k,\boldsymbol{\gamma})}, \\[2mm]
\dfrac{1}{4}\displaystyle\int_0^t (\exp(\mathrm{i}(\tau - t)\lambda_{\boldsymbol{m}^{(2)}}) - \exp(\mathrm{i}(t - \tau)\lambda_{\boldsymbol{m}^{(2)}})) \\
\quad \cdot (\exp(\mathrm{i}\tau\lambda_{\boldsymbol{m}_1}) + \exp(-\mathrm{i}\tau\lambda_{\boldsymbol{m}_1}))(\exp(\mathrm{i}\tau\lambda_{\boldsymbol{m}_2}) + \exp(-\mathrm{i}\tau\lambda_{\boldsymbol{m}_2}))\mathrm{d}\tau \\
\quad \text{if } \boldsymbol{\gamma} \in \mathscr{D}^{(2)}, \boldsymbol{\gamma} = (0,0), \ \boldsymbol{m}^{(2)} = (\boldsymbol{m}_1, \boldsymbol{m}_2) \in \mathfrak{M}^{(2,\boldsymbol{\gamma})}, \\[2mm]
\dfrac{-\mathrm{i}}{4\lambda_{\boldsymbol{m}_2}}\displaystyle\int_0^t (\exp(\mathrm{i}(\tau - t)\lambda_{\boldsymbol{m}^{(2)}}) - \exp(\mathrm{i}(t - \tau)\lambda_{\boldsymbol{m}^{(2)}})) \\
\quad \cdot (\exp(\mathrm{i}\tau\lambda_{\boldsymbol{m}_1}) + \exp(-\mathrm{i}\tau\lambda_{\boldsymbol{m}_1}))(\exp(\mathrm{i}\tau\lambda_{\boldsymbol{m}_2}) + \exp(-\mathrm{i}\tau\lambda_{\boldsymbol{m}_2}))\mathrm{d}\tau \\
\quad \text{if } \boldsymbol{\gamma} \in \mathscr{D}^{(2)}, \boldsymbol{\gamma} = (0,1), \ \boldsymbol{m}^{(2)} = (\boldsymbol{m}_1, \boldsymbol{m}_2) \in \mathfrak{M}^{(2,\boldsymbol{\gamma})}, \\[2mm]
\dfrac{-\mathrm{i}}{4\lambda_{\boldsymbol{m}_1}}\displaystyle\int_0^t (\exp(\mathrm{i}(\tau - t)\lambda_{\boldsymbol{m}^{(2)}}) - \exp(\mathrm{i}(t - \tau)\lambda_{\boldsymbol{m}^{(2)}})) \\
\quad \cdot (\exp(\mathrm{i}\tau\lambda_{\boldsymbol{m}_1}) + \exp(-\mathrm{i}\tau\lambda_{\boldsymbol{m}_1}))(\exp(\mathrm{i}\tau\lambda_{\boldsymbol{m}_2}) + \exp(-\mathrm{i}\tau\lambda_{\boldsymbol{m}_2}))\mathrm{d}\tau \\
\quad \text{if } \boldsymbol{\gamma} \in \mathscr{D}^{(2)}, \boldsymbol{\gamma} = (1,0), \ \boldsymbol{m}^{(2)} = (\boldsymbol{m}_1, \boldsymbol{m}_2) \in \mathfrak{M}^{(2,\boldsymbol{\gamma})}, \\[2mm]
\dfrac{1}{4\lambda_{\boldsymbol{m}_1}\lambda_{\boldsymbol{m}_2}}\displaystyle\int_0^t (\exp(\mathrm{i}(\tau - t)\lambda_{\boldsymbol{m}^{(2)}}) - \exp(\mathrm{i}(t - \tau)\lambda_{\boldsymbol{m}^{(2)}})) \\
\quad \cdot (\exp(\mathrm{i}\tau\lambda_{\boldsymbol{m}_1}) + \exp(-\mathrm{i}\tau\lambda_{\boldsymbol{m}_1}))(\exp(\mathrm{i}\tau\lambda_{\boldsymbol{m}_2}) + \exp(-\mathrm{i}\tau\lambda_{\boldsymbol{m}_2}))\mathrm{d}\tau \\
\quad \text{if } \boldsymbol{\gamma} \in \mathscr{D}^{(2)}, \boldsymbol{\gamma} = (1,1), \ \boldsymbol{m}^{(2)} = (\boldsymbol{m}_1, \boldsymbol{m}_2) \in \mathfrak{M}^{(2,\boldsymbol{\gamma})}, \\[2mm]
\displaystyle\int_0^t (\exp(\mathrm{i}(\tau - t)\lambda_{\boldsymbol{m}^{(k)}}) - \exp(\mathrm{i}(t - \tau)\lambda_{\boldsymbol{m}^{(k)}}))I(\tau, \boldsymbol{m}_1^{(k-1)}) \times I(\tau, \boldsymbol{m}_2^{(k-1)})\mathrm{d}\tau \\
\quad \text{if } \gamma \in \mathscr{D}^{(k)}, k \geq 3, \boldsymbol{\gamma} = (\boldsymbol{\gamma}_1^{(k-1)}, \boldsymbol{\gamma}_2^{(k-1)}) \in \mathscr{D}^{(k-1)} \times \mathscr{D}^{(k-1)}, \\
\quad \boldsymbol{m}^{(k)} = (\boldsymbol{m}_1^{(k-1)}, \boldsymbol{m}_2^{(k-1)}) \in \mathfrak{M}^{(k-1,\boldsymbol{\gamma}_1^{(k-1)})} \times \mathfrak{M}^{(k-1,\boldsymbol{\gamma}_2^{(k-1)})},
\end{cases}
\tag{2.15}
$$

where

$$\lambda_{\boldsymbol{m}^{(k)}} = ((\mu(\boldsymbol{m}^{(k)})\cdot\boldsymbol{\omega})^2 + (\mu(\boldsymbol{m}^{(k)})\cdot\boldsymbol{\omega})^4)^{\frac{1}{2}}, \quad \lambda_{\boldsymbol{m}^{(2)}} = ((\mu(\boldsymbol{m}^{(2)})\cdot\boldsymbol{\omega})^2 + (\mu(\boldsymbol{m}^{(2)})\cdot\boldsymbol{\omega})^4)^{\frac{1}{2}},$$
$$\lambda_{\boldsymbol{m}_1} = ((\boldsymbol{m}_1\cdot\boldsymbol{\omega})^2 + (\boldsymbol{m}_1\cdot\boldsymbol{\omega})^4)^{\frac{1}{2}}, \quad \lambda_{\boldsymbol{m}_2} = ((\boldsymbol{m}_2\cdot\boldsymbol{\omega})^2 + (\boldsymbol{m}_2\cdot\boldsymbol{\omega})^4)^{\frac{1}{2}}.$$

The following lemma addresses that $c_k(t,\boldsymbol{n})$ can be expressed by the functions $\mathfrak{C}^{(k,\boldsymbol{\gamma})}(\boldsymbol{m}^{(k)})$, $f^{(k,\boldsymbol{\gamma})}(\boldsymbol{m}^{(k)})$ and $I^{(k,\boldsymbol{\gamma})}(t,\boldsymbol{m}^{(k)})$ defined above.

LEMMA 2.3. *For $k=1,2,\cdots$, the function $c_{k-1}(t,\boldsymbol{n})$ defined in* (2.8)–(2.9) *is the $k$-th term of the sequence $\{d_k(t,\boldsymbol{n})\}$, that is,*

$$d_k(t,\boldsymbol{n}) = \sum_{\boldsymbol{\gamma}\in\mathscr{D}^{(k)}} \sum_{\substack{\boldsymbol{m}^{(k)}\in\mathfrak{M}^{(k,\boldsymbol{\gamma})} \\ \mu(\boldsymbol{m}^{(k)})=\boldsymbol{n}}} \mathfrak{C}^{(k,\boldsymbol{\gamma})}(\boldsymbol{m}^{(k)})f^{(k,\boldsymbol{\gamma})}(\boldsymbol{m}^{(k)})I^{(k,\boldsymbol{\gamma})}(t,\boldsymbol{m}^{(k)}) = c_{k-1}(t,\boldsymbol{n}). \quad (2.16)$$

*Proof.* In view of definitions (2.8)–(2.9) and (2.10)–(2.15), we will prove the lemma by an inductive argument.

For $k=1$, it follows that for $\boldsymbol{n}\in\mathbb{Z}^\nu$,

$$d_1(t,\boldsymbol{n}) = \sum_{\boldsymbol{\gamma}\in\mathscr{D}^{(1)}} \sum_{\substack{\boldsymbol{m}^{(1)}\in\mathfrak{M}^{(1,\boldsymbol{\gamma})} \\ \mu(\boldsymbol{m}^{(1)})=\boldsymbol{n}}} \mathfrak{C}^{(1,\boldsymbol{\gamma})}(\boldsymbol{m}^{(1)})f^{(1,\boldsymbol{\gamma})}(\boldsymbol{m}^{(1)})I^{(1,\boldsymbol{\gamma})}(t,\boldsymbol{m}^{(1)})$$

$$= \sum_{\mu(\boldsymbol{m}^{(1)})=\boldsymbol{n}} \mathfrak{C}^{(1,0)}(\boldsymbol{m}^{(1)})f^{(1,0)}(\boldsymbol{m}^{(1)})I^{(1,0)}(t,\boldsymbol{m}^{(1)})$$

$$+ \sum_{\mu(\boldsymbol{m}^{(1)})=\boldsymbol{n}} \mathfrak{C}^{(1,1)}(\boldsymbol{m}^{(1)})f^{(1,1)}(\boldsymbol{m}^{(1)})I^{(1,1)}(t,\boldsymbol{m}^{(1)})$$

$$= \frac{1}{2}c(\boldsymbol{n})(\exp(\mathrm{i}t\lambda) + \exp(-\mathrm{i}t\lambda)) - \frac{\mathrm{i}}{2\lambda}c'(\boldsymbol{n})(\exp(\mathrm{i}t\lambda) - \exp(-\mathrm{i}t\lambda))$$

$$= c_0(t,\boldsymbol{n}). \quad (2.17)$$

It is clear that (2.16) holds for $k=1$.

Suppose that (2.16) could hold for $k=\ell$, with $\ell\in\mathbb{N}$ and $\ell\geq 2$. For $k=\ell+1$, one has

$$d_{\ell+1}(t,\boldsymbol{n}) = \sum_{\boldsymbol{\gamma}\in\mathscr{D}^{(\ell+1)}} \sum_{\substack{\boldsymbol{m}^{(\ell+1)}\in\mathfrak{M}^{(\ell+1,\boldsymbol{\gamma})} \\ \mu(\boldsymbol{m}^{(\ell+1)})=\boldsymbol{n}}} \mathfrak{C}^{(\ell+1,\boldsymbol{\gamma})}(\boldsymbol{m}^{(\ell+1)})f^{(\ell+1,\boldsymbol{\gamma})}(\boldsymbol{m}^{(\ell+1)})I^{(\ell+1,\boldsymbol{\gamma})}(t,\boldsymbol{m}^{(\ell+1)})$$

$$= \sum_{\boldsymbol{\gamma}\in\mathscr{D}^{(1)}} \sum_{\substack{\boldsymbol{m}^{(\ell+1)}\in\mathfrak{M}^{(\ell+1,\boldsymbol{\gamma})} \\ \mu(\boldsymbol{m}^{(\ell+1)})=\boldsymbol{n}}} \mathfrak{C}^{(\ell+1,\boldsymbol{\gamma})}(\boldsymbol{m}^{(\ell+1)})f^{(\ell+1,\boldsymbol{\gamma})}(\boldsymbol{m}^{(\ell+1)})I^{(\ell+1,\boldsymbol{\gamma})}(t,\boldsymbol{m}^{(\ell+1)})$$

$$+ \sum_{\boldsymbol{\gamma}\in\mathscr{D}^{(\ell)}\times\mathscr{D}^{(\ell)}} \sum_{\substack{\boldsymbol{m}^{(\ell+1)}\in\mathfrak{M}^{(\ell+1,\boldsymbol{\gamma})} \\ \mu(\boldsymbol{m}^{(\ell+1)})=\boldsymbol{n}}} \mathfrak{C}^{(\ell+1,\boldsymbol{\gamma})}(\boldsymbol{m}^{(\ell+1)})f^{(\ell+1,\boldsymbol{\gamma})}(\boldsymbol{m}^{(\ell+1)})I^{(\ell+1,\boldsymbol{\gamma})}(t,\boldsymbol{m}^{(\ell+1)})$$

$$= \sum_{\boldsymbol{\gamma}\in\mathscr{D}^{(1)}} \sum_{\substack{\boldsymbol{m}^{(1)}\in\mathfrak{M}^{(1,\boldsymbol{\gamma})} \\ \mu(\boldsymbol{m}^{(1)})=\boldsymbol{n}}} \mathfrak{C}^{(1,\boldsymbol{\gamma})}(\boldsymbol{m}^{(1)})f^{(1,\boldsymbol{\gamma})}(\boldsymbol{m}^{(1)})I^{(1,\boldsymbol{\gamma})}(t,\boldsymbol{m}^{(1)})$$

$$- \frac{\mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\omega}}{2\sqrt{1+(\boldsymbol{n}\cdot\boldsymbol{\omega})^2}} \int_0^t (\exp(\mathrm{i}(\tau-t)\lambda) - \exp(\mathrm{i}(t-\tau)\lambda))$$

$$\cdot \sum_{\substack{\boldsymbol{m}_1,\boldsymbol{m}_2\in\mathbb{Z}^\nu \\ \boldsymbol{m}_1+\boldsymbol{m}_2=\boldsymbol{n}}} \left( \sum_{\boldsymbol{\gamma}_1^{(\ell)}\in\mathscr{D}^{(\ell)}} \sum_{\substack{\boldsymbol{m}_1^{(\ell)}\in\mathfrak{M}^{(\ell,\boldsymbol{\gamma}_1^{(\ell)})} \\ \mu(\boldsymbol{m}_1^{(\ell)})=\boldsymbol{m}_1}} \mathfrak{C}^{(\ell,\boldsymbol{\gamma}_1^{(\ell)})}(\boldsymbol{m}_1^{(\ell)})f^{(\ell+1,\boldsymbol{\gamma})}(\boldsymbol{m}_1^{(\ell)})I^{(\ell+1,\boldsymbol{\gamma}_1^{(\ell)})}(\tau,\boldsymbol{m}_1^{(\ell)}) \right)$$

$$\cdot\left(\sum_{\substack{\boldsymbol{\gamma}_2^{(\ell)}\in\mathscr{D}^{(\ell)}}}\sum_{\substack{\boldsymbol{m}_2^{(\ell)}\in\mathfrak{M}^{(\ell,\boldsymbol{\gamma}_2^{(\ell)})}\\\mu(\boldsymbol{m}_2^{(\ell)})=\boldsymbol{m}_2}}\mathfrak{C}^{(\ell,\boldsymbol{\gamma}_2^{(\ell)})}(\boldsymbol{m}_2^{(\ell)})f^{(\ell+1,\boldsymbol{\gamma}_2^{(\ell)})}(\boldsymbol{m}_2^{(\ell)})I^{(\ell+1,\boldsymbol{\gamma}_2^{(\ell)})}(\tau,\boldsymbol{m}_2^{(\ell)})\right)\mathrm{d}\tau$$

$$=\frac{1}{2}c(\boldsymbol{n})\left(\exp(\mathrm{i}t\lambda)+\exp(-\mathrm{i}t\lambda)\right)-\frac{\mathrm{i}}{2\lambda}c'(\boldsymbol{n})\left(\exp(\mathrm{i}t\lambda)-\exp(-\mathrm{i}t\lambda)\right)-\frac{\mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\omega}}{2\sqrt{1+(\boldsymbol{n}\cdot\boldsymbol{\omega})^2}}$$

$$\cdot\int_0^t\left(\exp(\mathrm{i}(\tau-t)\lambda)-\exp(\mathrm{i}(t-\tau)\lambda)\right)\cdot\sum_{\substack{\boldsymbol{m}_1,\boldsymbol{m}_2\in\mathbb{Z}^\nu\\\boldsymbol{m}_1+\boldsymbol{m}_2=\boldsymbol{n}}}d_\ell(\tau,\boldsymbol{m}_1)d_\ell(\tau,\boldsymbol{m}_2)\mathrm{d}\tau$$

$$=c_\ell(t,\boldsymbol{n}),\quad \boldsymbol{n}\in\mathbb{Z}^\nu.\tag{2.18}$$

This completes the proof of Lemma 2.3. □

By Lemma 2.3, if we want to prove the absolute and uniform convergence of the sequence $\{c_k(t,\boldsymbol{n})\}$, we just need to consider the absolute and uniform convergence of the sequence $\{d_k(t,\boldsymbol{n})\}$ given by (2.16). Equivalently, we may verify that the series $\sum_{k=1}^\infty(d_{k+1}(t,\boldsymbol{n})-d_k(t,\boldsymbol{n}))$ converges absolutely and uniformly on the interval $t\in[0,t_0)$. For this, we have to give the upper bounds on $|d_{k+1}(t,\boldsymbol{n})-d_k(t,\boldsymbol{n})|,k\in\mathbb{N}_+$.

The term $|d_k(t,\boldsymbol{n})|$ will be bounded from above in the following corollary.

COROLLARY 2.1.     Let $B>0,\ 0<\kappa\le1$ be two constants and $\boldsymbol{\omega}\in\mathbb{R}^\nu$. If $0\le t\le\frac{\kappa^\nu}{32B(48)^\nu|\boldsymbol{\omega}|}$, then

$$|d_k(t,\boldsymbol{n})|\le2B\exp(-\frac{\kappa|\boldsymbol{n}|}{4}).$$

*Proof.*    The proof will be divided into the following three steps.

**Step 1:**    Formula (2.16) shows that $d_k(t,\boldsymbol{n})$ consists of the functions $\mathfrak{C}^{(k,\boldsymbol{\gamma})}(\boldsymbol{m}^{(k)})$, $f^{(k,\boldsymbol{\gamma})}(\boldsymbol{m}^{(k)})$ and $I^{(k,\boldsymbol{\gamma})}(t,\boldsymbol{m}^{(k)})$. Thus we must introduce the following functions for estimating the above-mentioned functions.

Let us define

$$\sigma(\boldsymbol{\gamma})=\begin{cases}1 & \text{if }\boldsymbol{\gamma}=0\text{ or }1\in\mathscr{D}^{(k)},\\2 & \text{if }\boldsymbol{\gamma}\in\mathscr{D}^{(2)},\boldsymbol{\gamma}=(0,0)\text{ or }(0,1)\text{ or }(1,0)\text{ or }(1,1),\\\sigma(\boldsymbol{\gamma}_1^{(k-1)})+\sigma(\boldsymbol{\gamma}_2^{(k-1)}) & \text{if }\boldsymbol{\gamma}\in\mathscr{D}^{(k)},k\ge3,\\ & \boldsymbol{\gamma}=(\boldsymbol{\gamma}_1^{(k-1)},\boldsymbol{\gamma}_2^{(k-1)})\in\mathscr{D}^{(k-1)}\times\mathscr{D}^{(k-1)},\end{cases}\tag{2.19}$$

$$\ell(\boldsymbol{\gamma})=\begin{cases}0 & \text{if }\boldsymbol{\gamma}=0\text{ or }1\in\mathscr{D}^{(k)},\\1 & \text{if }\boldsymbol{\gamma}\in\mathscr{D}^{(2)},\boldsymbol{\gamma}=(0,0)\text{ or }(0,1)\text{ or }(1,0)\text{ or }(1,1),\\\ell(\boldsymbol{\gamma}_1^{(k-1)})+\ell(\boldsymbol{\gamma}_2^{(k-1)})+1 & \text{if }\boldsymbol{\gamma}\in\mathscr{D}^{(k)},k\ge3,\\ & \boldsymbol{\gamma}=(\boldsymbol{\gamma}_1^{(k-1)},\boldsymbol{\gamma}_2^{(k-1)})\in\mathscr{D}^{(k-1)}\times\mathscr{D}^{(k-1)},\end{cases}\tag{2.20}$$

$$|\boldsymbol{m}^{(k)}|=\begin{cases}|\boldsymbol{m}| & \text{if }\boldsymbol{\gamma}=0\text{ or }1\in\mathscr{D}^{(k)},\boldsymbol{m}^{(k)}=\boldsymbol{m}\in\mathbb{Z}^\nu,\\|\boldsymbol{m}_1|+|\boldsymbol{m}_2| & \text{if }\boldsymbol{\gamma}\in\mathscr{D}^{(2)},\boldsymbol{\gamma}=(0,0)\text{ or }(0,1)\text{ or }(1,0)\text{ or }(1,1),\\ & \boldsymbol{m}^{(2)}=(\boldsymbol{m}_1,\boldsymbol{m}_2),\\|\boldsymbol{m}_1^{(k-1)}|+|\boldsymbol{m}_2^{(k-1)}| & \text{if }\boldsymbol{\gamma}\in\mathscr{D}^{(k)},k\ge3,\\ & \boldsymbol{\gamma}=(\boldsymbol{\gamma}_1^{(k-1)},\boldsymbol{\gamma}_2^{(k-1)})\in\mathscr{D}^{(k-1)}\times\mathscr{D}^{(k-1)},\\ & \boldsymbol{m}^{(k)}=(\boldsymbol{m}_1^{(k-1)},\boldsymbol{m}_2^{(k-1)})\\ & \in\mathfrak{M}^{(k-1,\boldsymbol{\gamma}_1^{(k-1)})}\times\mathfrak{M}^{(k-1,\boldsymbol{\gamma}_2^{(k-1)})},\end{cases}\tag{2.21}$$

$$\mathfrak{P}(\boldsymbol{m}^{(k)}) = \begin{cases} 1 & \text{if } \boldsymbol{m}^{(k)} \in \mathfrak{M}^{(k,\boldsymbol{\gamma})}, \boldsymbol{\gamma} = 0 \text{ or } 1 \in \mathscr{D}^{(k)}, \\ |\mu(\boldsymbol{m}^{(2)}| & \text{if } k = 2, \boldsymbol{m}^{(k)} \in \mathfrak{M}^{(k,\boldsymbol{\gamma})}, \boldsymbol{\gamma} \in \mathscr{D}^{(2)}, \\ |\mu(\boldsymbol{m}^{(k)})| \mathfrak{P}(\boldsymbol{m}_1^{(k-1)}) \mathfrak{P}(\boldsymbol{m}_2^{(k-1)}) & \text{if } \boldsymbol{m}^{(k)} \in \mathfrak{M}^{(k,\boldsymbol{\gamma})}, k \geq 3, \boldsymbol{\gamma} = (\boldsymbol{\gamma}_1^{(k-1)}, \\ & \quad \boldsymbol{\gamma}_2^{(k-1)}) \in \mathscr{D}^{(k-1)} \times \mathscr{D}^{(k-1)}, \\ & \quad \boldsymbol{m}^{(k)} = (\boldsymbol{m}_1^{(k-1)}, \boldsymbol{m}_2^{(k-1)}) \\ & \quad \in \mathfrak{M}^{(k-1,\boldsymbol{\gamma}_1^{(k-1)})} \times \mathfrak{M}^{(k-1,\boldsymbol{\gamma}_2^{(k-1)})}, \end{cases} \tag{2.22}$$

and

$$\mathfrak{F}(\boldsymbol{\gamma}) = \begin{cases} 1 & \text{if } \boldsymbol{\gamma} = 0 \text{ or } 1 \in \mathscr{D}^{(k)} \\ & \quad \boldsymbol{\gamma} = (0,0) \text{ or } (0,1) \text{ or } (1,0) \text{ or } (1,1) \in \mathscr{D}^{(2)}, \\ \ell(\boldsymbol{\gamma}) \mathfrak{F}(\boldsymbol{\gamma}_1^{(k-1)}) \mathfrak{F}(\boldsymbol{\gamma}_2^{(k-1)}) & \text{if } k \geq 3, \boldsymbol{\gamma} = (\boldsymbol{\gamma}_1^{(k-1)}, \boldsymbol{\gamma}_2^{(k-1)}) \in \mathscr{D}^{(k-1)} \times \mathscr{D}^{(k-1)}. \end{cases} \tag{2.23}$$

In the following lemma, we will give the upper bounds on the functions $\mathfrak{C}^{(k,\boldsymbol{\gamma})}(\boldsymbol{m}^{(k)})$, $f^{(k,\boldsymbol{\gamma})}(\boldsymbol{m}^{(k)})$ and $I^{(k,\boldsymbol{\gamma})}(t, \boldsymbol{m}^{(k)})$.

LEMMA 2.4.    Let $\boldsymbol{m}^{(k)} \in \mathfrak{M}^{(k,\boldsymbol{\gamma})}$. One has

(I)

$$|\mathfrak{C}(\boldsymbol{m}^{(k)})| \leq B^{\sigma(\boldsymbol{\gamma})} \exp\left(-\frac{\kappa |\boldsymbol{m}^{(k)}|}{2}\right).$$

(II)

$$|f(\boldsymbol{m}^{(1)})| = 1 \qquad \text{if } \boldsymbol{\gamma} = 0 \text{ or } 1 \in \mathscr{D}^{(1)}, \boldsymbol{m}^{(1)} \in \mathfrak{M}^{(1,\boldsymbol{\gamma})},$$
$$|f(\boldsymbol{m}^{(2)})| \leq |\boldsymbol{\omega}||\mu(\boldsymbol{m}^{(2)})| \qquad \text{if } \boldsymbol{\gamma} \in \mathscr{D}^{(2)}, \boldsymbol{\gamma} = (0,0) \text{ or } (0,1) \text{ or } (1,0)$$
$$\qquad \text{or } (1,1), \boldsymbol{m}^{(2)} \in \mathfrak{M}^{(2,\boldsymbol{\gamma})},$$
$$|f(\boldsymbol{m}^{(k)})| = 1 \qquad \text{if } \boldsymbol{\gamma} = 0 \text{ or } 1 \in \mathscr{D}^{(k)}, k \geq 3, \boldsymbol{m}^{(k)} \in \mathfrak{M}^{(k,\boldsymbol{\gamma})},$$
$$|f(\boldsymbol{m}^{(k)})| \leq |\boldsymbol{\omega}|^{\ell(\boldsymbol{\gamma})} \mathfrak{P}(\boldsymbol{m}^{(k)}) \qquad \text{if } \boldsymbol{\gamma} = (\boldsymbol{\gamma}_1^{(k-1)}, \boldsymbol{\gamma}_2^{(k-1)}) \in \mathscr{D}^{(k-1)} \times \mathscr{D}^{(k-1)},$$
$$\qquad k \geq 3, \boldsymbol{m}^{(k)} \in \mathfrak{M}^{(k,\boldsymbol{\gamma})}.$$

(III) *Assume that $\boldsymbol{\omega}$ is Diophantine, that is*

$$|\boldsymbol{n} \cdot \boldsymbol{\omega}| \geq \theta |n|^{-\delta}, \quad \forall \boldsymbol{n} \in \mathbb{Z}^{\nu} \backslash \{0\}.$$

*Under the condition of $\mu(\boldsymbol{m}^{(k)}) = \boldsymbol{n}$, we have*

$$|I(t, \boldsymbol{m}^{(1)})| \leq 1 \qquad \text{if } \boldsymbol{\gamma} \in \mathscr{D}^{(1)}, \boldsymbol{\gamma} = 0, \boldsymbol{m}^{(1)} \in \mathfrak{M}^{(1,\boldsymbol{\gamma})},$$
$$|I(t, \boldsymbol{m}^{(1)})| \leq \theta^{-1} |\boldsymbol{n}|^{\delta} \qquad \text{if } \boldsymbol{\gamma} \in \mathscr{D}^{(1)}, \boldsymbol{\gamma} = 1, \boldsymbol{m}^{(1)} \in \mathfrak{M}^{(1,\boldsymbol{\gamma})},$$
$$|I(t, \boldsymbol{m}^{(2)})| \leq 2t \qquad \text{if } \boldsymbol{\gamma} \in \mathscr{D}^{(2)}, \boldsymbol{\gamma} = (0,0), \boldsymbol{m}^{(2)} = (\boldsymbol{m}_1, \boldsymbol{m}_2) \in \mathfrak{M}^{(2,\boldsymbol{\gamma})},$$
$$|I(t, \boldsymbol{m}^{(2)})| \leq 2t \cdot \theta^{-1} |\boldsymbol{n}|^{\delta} \qquad \text{if } \boldsymbol{\gamma} \in \mathscr{D}^{(2)}, \boldsymbol{\gamma} = (0,1), \boldsymbol{m}^{(2)} = (\boldsymbol{m}_1, \boldsymbol{m}_2) \in \mathfrak{M}^{(2,\boldsymbol{\gamma})},$$
$$|I(t, \boldsymbol{m}^{(2)})| \leq 2t \cdot \theta^{-1} |\boldsymbol{n}|^{\delta} \qquad \text{if } \boldsymbol{\gamma} \in \mathscr{D}^{(2)}, \boldsymbol{\gamma} = (1,0), \boldsymbol{m}^{(2)} = (\boldsymbol{m}_1, \boldsymbol{m}_2) \in \mathfrak{M}^{(2,\boldsymbol{\gamma})},$$
$$|I(t, \boldsymbol{m}^{(2)})| \leq 2t \cdot (\theta^{-1} |\boldsymbol{n}|^{\delta})^2 \qquad \text{if } \boldsymbol{\gamma} \in \mathscr{D}^{(2)}, \boldsymbol{\gamma} = (1,1), \boldsymbol{m}^{(2)} = (\boldsymbol{m}_1, \boldsymbol{m}_2) \in \mathfrak{M}^{(2,\boldsymbol{\gamma})},$$
$$|I(t, \boldsymbol{m}^{(k)})| \leq 1 \qquad \text{if } \boldsymbol{\gamma} = 0 \in \mathscr{D}^{(k)}, k \geq 3, \boldsymbol{\gamma} = 0, \boldsymbol{m}^{(k)} \in \mathfrak{M}^{(k,\boldsymbol{\gamma})},$$

$$|I(t, \boldsymbol{m}^{(k)})| \leq \theta^{-1} |\boldsymbol{n}|^{\delta} \qquad \text{if } \boldsymbol{\gamma} = 1 \in \mathscr{D}^{(k)}, k \geq 3, \boldsymbol{\gamma} = 1, \boldsymbol{m}^{(k)} \in \mathfrak{M}^{(k,\boldsymbol{\gamma})},$$

$$|I(t, \boldsymbol{m}^{(k)})| \leq \frac{(2t)^{\ell(\gamma)}}{\mathfrak{F}(\gamma)} (\theta^{-1} |\boldsymbol{n}|^{\delta})^{\sigma(\gamma)} \quad \text{if } \boldsymbol{\gamma} = (\boldsymbol{\gamma}_1^{(k-1)}, \boldsymbol{\gamma}_2^{(k-1)}) \in \mathscr{D}^{(k-1)} \times \mathscr{D}^{(k-1)},$$

$$k \geq 3, \boldsymbol{m}^{(k)} \in \mathfrak{M}^{(k,\boldsymbol{\gamma})}.$$

*Proof.*

(I) The proof is based on (2.10)–(2.12), (2.19), (2.21) and the decay assumption (2.7). Let us consider the following three cases.

<u>*Case 1*</u>: $k = 1$. For $\boldsymbol{m}^{(1)} = \boldsymbol{m} \in \mathfrak{M}^{(1,\boldsymbol{\gamma})}$, it follows that

$$|\mathfrak{C}(\boldsymbol{m}^{(1)})| = |c(\boldsymbol{m})| \leq B \exp(-\frac{\kappa |\boldsymbol{m}|}{2}) \quad \text{if } \boldsymbol{\gamma} = 0 \in \mathscr{D}^{(1)},$$

$$|\mathfrak{C}(\boldsymbol{m}^{(1)})| = |c'(\boldsymbol{m})| \leq B \exp(-\frac{\kappa |\boldsymbol{m}|}{2}) \quad \text{if } \boldsymbol{\gamma} = 1 \in \mathscr{D}^{(1)}.$$

<u>*Case 2*</u>: $k = 2$. If $\boldsymbol{m}^{(2)} = (\boldsymbol{m}_1, \boldsymbol{m}_2) \in \mathfrak{M}^{(2,\boldsymbol{\gamma})}, \boldsymbol{\gamma} = 0$ or $1 \in \mathscr{D}^{(2)}$, we have the same estimations as the case $k = 1$. Moreover,

$$|\mathfrak{C}(\boldsymbol{m}^{(2)})| = |c(\boldsymbol{m}_1)||c(\boldsymbol{m}_2)| \leq B \exp(-\frac{\kappa |\boldsymbol{m}_1|}{2}) B \exp(-\frac{\kappa |\boldsymbol{m}_2|}{2})$$

$$\leq B^2 \exp(-\frac{\kappa |\boldsymbol{m}^{(2)}|}{2}) \quad \text{if } \boldsymbol{\gamma} = (0,0) \in \mathscr{D}^{(2)},$$

$$|\mathfrak{C}(\boldsymbol{m}^{(2)})| = |c(\boldsymbol{m}_1)||c'(\boldsymbol{m}_2)| \leq B \exp(-\frac{\kappa |\boldsymbol{m}_1|}{2}) B \exp(-\frac{\kappa |\boldsymbol{m}_2|}{2})$$

$$\leq B^2 \exp(-\frac{\kappa |\boldsymbol{m}^{(2)}|}{2}) \quad \text{if } \boldsymbol{\gamma} = (0,1) \in \mathscr{D}^{(2)},$$

$$|\mathfrak{C}(\boldsymbol{m}^{(2)})| = |c'(\boldsymbol{m}_1)||c(\boldsymbol{m}_2)| \leq B \exp(-\frac{\kappa |\boldsymbol{m}_1|}{2}) B \exp(-\frac{\kappa |\boldsymbol{m}_2|}{2})$$

$$\leq B^2 \exp(-\frac{\kappa |\boldsymbol{m}^{(2)}|}{2}) \quad \text{if } \boldsymbol{\gamma} = (1,0) \in \mathscr{D}^{(2)},$$

$$|\mathfrak{C}(\boldsymbol{m}^{(2)})| = |c'(\boldsymbol{m}_1)||c'(\boldsymbol{m}_2)| \leq B \exp(-\frac{\kappa |\boldsymbol{m}_1|}{2}) B |\boldsymbol{\omega}| \exp(-\frac{\kappa |\boldsymbol{m}_2|}{2})$$

$$\leq B^2 \exp(-\frac{\kappa |\boldsymbol{m}^{(2)}|}{2}) \quad \text{if } \boldsymbol{\gamma} = (1,1) \in \mathscr{D}^{(2)}.$$

<u>*Case 3*</u>: $k \geq 3$. If $\boldsymbol{\gamma} = 0$ or $1$ or $(0,0)$ or $(0,1)$ or $(1,0)$ or $(1,1) \in \mathscr{D}^{(k)}$, the upper bounds of $|\mathfrak{C}(\boldsymbol{m}^{(k)})|$ are the same as the case $k = 2$. Let $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_1^{(k-1)}, \boldsymbol{\gamma}_2^{(k-1)}) \in \mathscr{D}^{(k-1)} \times \mathscr{D}^{(k-1)}, \boldsymbol{m}^{(k)} = (\boldsymbol{m}_1^{(k-1)}, \boldsymbol{m}_2^{(k-1)}) \in \mathfrak{M}^{(k-1,\boldsymbol{\gamma}_1^{(k-1)})} \times \mathfrak{M}^{(k-1,\boldsymbol{\gamma}_2^{(k-1)})}$. By a inductive argument, we conclude

$$|\mathfrak{C}(\boldsymbol{m}^{(k)})| = |\mathfrak{C}(\boldsymbol{m}_1^{(k-1)})||\mathfrak{C}(\boldsymbol{m}_2^{(k-1)})|$$

$$\leq B^{\sigma(\boldsymbol{\gamma}_1^{(k-1)})} \exp(-\frac{\kappa |\boldsymbol{m}_1^{(k-1)}|}{2}) B^{\sigma(\boldsymbol{\gamma}_2^{(k-1)})} \exp(-\frac{\kappa |\boldsymbol{m}_2^{(k-1)}|}{2})$$

$$= B^{\sigma(\boldsymbol{\gamma}_1^{(k-1)}) + \sigma(\boldsymbol{\gamma}_2^{(k-1)})} \exp(-\frac{\kappa (|\boldsymbol{m}_1^{(k-1)}| + |\boldsymbol{m}_2^{(k-1)}|)}{2})$$

$$= B_0^{\sigma(\boldsymbol{\gamma})} \exp(-\frac{\kappa |\boldsymbol{m}^{(k)}|}{2}).$$

(II) The terms $|f(\boldsymbol{m}^{(k)})|$ can be bounded from above by (2.13)–(2.14), (2.20) and (2.22). We consider the following three cases.

*Case 1′*: $k=1$. It is clear that

$$|f(\boldsymbol{m}^{(1)})|=1 \quad \text{if } \boldsymbol{m}^{(1)}\in\mathfrak{M}^{(1,\gamma)},\boldsymbol{\gamma}=0 \text{ or } 1\in\mathscr{D}^{(1)}.$$

*Case 2′*: $k=2$. For $\boldsymbol{m}^{(2)}\in\mathfrak{M}^{(2,\gamma)},\boldsymbol{\gamma}=0$ or $1\in\mathscr{D}^{(2)}$, one has $|f(\boldsymbol{m}^{(2)})|=1$. Moreover, if $\boldsymbol{m}^{(2)}\in\mathfrak{M}^{(2,\gamma)},\boldsymbol{\gamma}=(0,0)$ or $(0,1)$ or $(1,0)$ or $(1,1)\in\mathscr{D}^{(2)}$, then

$$|f(\boldsymbol{m}^{(2)})|=\left|-\frac{\mathrm{i}\mu(\boldsymbol{m}^{(2)})\cdot\boldsymbol{\omega}}{2\sqrt{1+(\mu(\boldsymbol{m}^{(2)})\cdot\boldsymbol{\omega})^2}}\right|\leq|\mathrm{i}\mu(\boldsymbol{m}^{(2)})\cdot\boldsymbol{\omega}|\leq|\boldsymbol{\omega}||\mu(\boldsymbol{m}^{(2)})|.$$

*Case 3′*: $k\geq3$. For $\boldsymbol{\gamma}=0$ or $1$ or $(0,0)$ or $(0,1)$ or $(1,0)$ or $(1,1)\in\mathscr{D}^{(k)}$, we can obtain the same estimations as the case $k=2$.

For $\boldsymbol{\gamma}=(\boldsymbol{\gamma}_1^{(k-1)},\boldsymbol{\gamma}_2^{(k-1)})\in\mathscr{D}^{(k-1)}\times\mathscr{D}^{(k-1)}$, $\boldsymbol{m}^{(k)}=(\boldsymbol{m}_1^{(k-1)},\boldsymbol{m}_2^{(k-1)})\in\mathfrak{M}^{(k-1,\gamma_1^{(k-1)})}\times\mathfrak{M}^{(k-1,\gamma_2^{(k-1)})}$, by induction, we have

$$\begin{aligned}|f(\boldsymbol{m}^{(k)})|&\leq|\boldsymbol{\omega}||\mu(\boldsymbol{m}^{(k)})||f(\boldsymbol{m}_1^{(k-1)})||f(\boldsymbol{m}_2^{(k-1)})|\\&\leq|\boldsymbol{\omega}||\mu(\boldsymbol{m}^{(k)})||\boldsymbol{\omega}|^{\ell(\gamma_1^{(k-1)})}\mathfrak{P}(\boldsymbol{m}_1^{(k-1)})|\boldsymbol{\omega}|^{\ell(\gamma_2^{(k-1)})}\mathfrak{P}(\boldsymbol{m}_2^{(k-1)})\\&=|\boldsymbol{\omega}|^{\ell(\gamma)}\mathfrak{P}(\boldsymbol{m}^{(k)}).\end{aligned}$$

(III) We apply (2.15), (2.20) and (2.23) to estimate the upper bound of $|I(t,\boldsymbol{m}^{(k)})|$. The following cases can be considered.

*Case 1″*: $k=1$. Obviously, it follows that

$$|I(t,\boldsymbol{m}^{(1)})|\leq\frac{1}{2}(|1|+|1|)=1 \qquad \text{if } \boldsymbol{m}^{(1)}\in\mathfrak{M}^{(1,\gamma)},\boldsymbol{\gamma}=0\in\mathscr{D}^{(1)},$$

$$\begin{aligned}|I(t,\boldsymbol{m}^{(1)})|&\leq\frac{1}{2}\frac{|\mathrm{i}|(|1|+|1|)}{|((\boldsymbol{n}\cdot\boldsymbol{\omega})^2+(\boldsymbol{n}\cdot\boldsymbol{\omega})^4)^{\frac{1}{2}}|}\leq\frac{1}{|\boldsymbol{n}\cdot\boldsymbol{\omega}|\sqrt{1+(\boldsymbol{n}\cdot\boldsymbol{\omega})^2}}\\&\leq\theta|\boldsymbol{n}|^{-\delta} \qquad \text{if } \boldsymbol{m}^{(1)}\in\mathfrak{M}^{(1,\gamma)},\boldsymbol{\gamma}=1\in\mathscr{D}^{(1)}.\end{aligned}$$

*Case 2″*: $k=2$. If $\boldsymbol{\gamma}=0$ or $1\in\mathscr{D}^{(2)}$, then we can get the same estimations as the case $k=1$. Moreover,

$$|I(t,\boldsymbol{m}^{(2)})|\leq\frac{1}{4}\int_0^t|2||2||2|\mathrm{d}\tau=2t \quad \text{if } \boldsymbol{\gamma}=(0,0)\in\mathscr{D}^{(2)},$$

$$|I(t,\boldsymbol{m}^{(2)})|\leq\frac{\theta^{-1}|\boldsymbol{n}|^\delta}{4}\int_0^t|2||2||2|\mathrm{d}\tau=2t\cdot\theta^{-1}|\boldsymbol{n}|^\delta \quad \text{if } \boldsymbol{\gamma}=(0,1)\in\mathscr{D}^{(2)},$$

$$|I(t,\boldsymbol{m}^{(2)})|\leq\frac{\theta^{-1}|\boldsymbol{n}|^\delta}{4}\int_0^t|2||2||2|\mathrm{d}\tau=2t\cdot\theta^{-1}|\boldsymbol{n}|^\delta \quad \text{if } \boldsymbol{\gamma}=(1,0)\in\mathscr{D}^{(2)},$$

$$|I(t,\boldsymbol{m}^{(2)})|\leq\frac{(\theta^{-1}|\boldsymbol{n}|^\delta)^2}{4}\int_0^t|2||2||2|\mathrm{d}\tau=2t\cdot(\theta^{-1}|\boldsymbol{n}|^\delta)^2 \quad \text{if } \boldsymbol{\gamma}=(1,1)\in\mathscr{D}^{(2)}.$$

*Case 3″*: $k\geq3$. For $\boldsymbol{\gamma}=0$ or $1$ or $(0,0)$ or $(0,1)$ or $(1,0)$ or $(1,1)\in\mathscr{D}^{(k)}$, the same estimations can be shown as the case $k=2$. If $\boldsymbol{\gamma}=(\boldsymbol{\gamma}_1^{(k-1)},\boldsymbol{\gamma}_2^{(k-1)})\in\mathscr{D}^{(k-1)}\times\mathscr{D}^{(k-1)}$,

$\boldsymbol{m}^{(k)} = (\boldsymbol{m}_1^{(k-1)}, \boldsymbol{m}_2^{(k-1)}) \in \mathfrak{M}^{(k-1,\boldsymbol{\gamma}_1^{(k-1)})} \times \mathfrak{M}^{(k-1,\boldsymbol{\gamma}_2^{(k-1)})}$, an inductive argument yields that

$$|I(t,\boldsymbol{m}^{(k)})| \leq 2\int_0^t |I(\tau,\boldsymbol{m}_1^{(k-1)})||I(\tau,\boldsymbol{m}_2^{(k-1)})|\mathrm{d}\tau$$

$$\leq 2\int_0^t \frac{(2\tau)^{\ell(\boldsymbol{\gamma}_1^{(k-1)})}(\theta^{-1}|\mu(\boldsymbol{m}_1^{(k-1)})|^\delta)^{\sigma(\boldsymbol{\gamma}_1^{(k-1)})}}{\mathfrak{F}(\boldsymbol{\gamma}_1^{(k-1)})} \frac{(2\tau)^{\ell(\boldsymbol{\gamma}_2^{(k-1)})}(\theta^{-1}|\mu(\boldsymbol{m}_2^{(k-1)})|^\delta)^{\sigma(\boldsymbol{\gamma}_2^{(k-1)})}}{\mathfrak{F}(\boldsymbol{\gamma}_2^{(k-1)})}\mathrm{d}\tau$$

$$= 2^{\ell(\boldsymbol{\gamma}_1^{(k-1)})+\ell(\boldsymbol{\gamma}_2^{(k-1)})+1}\int_0^t \frac{\tau^{\ell(\boldsymbol{\gamma}_1^{(k-1)})}}{\mathfrak{F}(\boldsymbol{\gamma}_1^{(k-1)})}\frac{\tau^{\ell(\boldsymbol{\gamma}_2^{(k-1)})}}{\mathfrak{F}(\boldsymbol{\gamma}_2^{(k-1)})}\mathrm{d}\tau \cdot (\theta^{-1}|\boldsymbol{n}|^\delta)^{\sigma_1(\boldsymbol{\gamma}_1^{(k-1)})+\sigma_2(\boldsymbol{\gamma}_1^{(k-1)})}$$

$$= \frac{(2t)^{\ell(\boldsymbol{\gamma})}}{\mathfrak{F}(\boldsymbol{\gamma})}(\theta^{-1}|\boldsymbol{n}|^\delta)^{\sigma(\boldsymbol{\gamma})}.$$

The proof is now completed.                                                      □

**Step 2:** Our next goal is to establish an estimation of the sums involving the functions $\mathfrak{C}^{(k,\boldsymbol{\gamma})}(\boldsymbol{m}^{(k)})$, $f^{(k,\boldsymbol{\gamma})}(\boldsymbol{m}^{(k)})$ and $I^{(k,\boldsymbol{\gamma})}(t,\boldsymbol{m}^{(k)})$. The main difficulty comes from the complicated combinatorics of the summation process. To overcome this difficulty, we "change variables" in the summations. Now we need to define the following set and isomorphic mapping.

Denote by

$$\mathfrak{B}^{(k,\boldsymbol{\gamma})} = \begin{cases} \mathbb{Z} & \text{if } \boldsymbol{\gamma}=0 \text{ or } 1 \in \mathscr{D}^{(k)}, \\ \mathbb{Z}\times\mathbb{Z} & \text{if } \boldsymbol{\gamma}\in\mathscr{D}^{(2)}, \boldsymbol{\gamma}=(0,0) \\ & \text{or } (0,1) \text{ or } (1,0) \text{ or } (1,1), \\ \mathfrak{B}^{(k-1,\boldsymbol{\gamma}_1^{(k-1)})}\times\mathfrak{B}^{(k-1,\boldsymbol{\gamma}_2^{(k-1)})} & \text{if } \boldsymbol{\gamma}\in\mathscr{D}^{(k)}, k\geq 3, \\ & \boldsymbol{\gamma}=(\boldsymbol{\gamma}_1^{(k-1)},\boldsymbol{\gamma}_2^{(k-1)})\in\mathscr{D}^{(k-1)}\times\mathscr{D}^{(k-1)}. \end{cases}$$

DEFINITION 2.1.   *Define inductively the isomorphism* $\varphi_{\boldsymbol{\gamma}}^{(k)}:\mathfrak{M}^{(k,\boldsymbol{\gamma})}\longrightarrow \prod_{j=1}^{\sigma(\boldsymbol{\gamma})}\mathbb{Z}^v$ *by*

$$\varphi_{\boldsymbol{\gamma}}^{(k)}(\boldsymbol{m}^{(k)}) = \begin{cases} \boldsymbol{m} & \text{if } \boldsymbol{\gamma}=0 \;\; or \; 1\in\mathscr{D}^{(k)}, \\ & \boldsymbol{m}^{(k)}=\boldsymbol{m}\in\mathbb{Z}^\nu=\mathfrak{M}^{(k,\boldsymbol{\gamma})}, \\ (\boldsymbol{m}_1,\boldsymbol{m}_2) & \text{if } k=2, \boldsymbol{\gamma}=(0,0) \;\; or \;(0,1) \\ & or \;(1,0) \;\; or \;(1,1)\in\mathscr{D}^{(2)}, \\ & \boldsymbol{m}^{(2)}=(\boldsymbol{m}_1,\boldsymbol{m}_2)\in\mathfrak{M}^{(2,\boldsymbol{\gamma})}, \\ (\varphi_{\boldsymbol{\gamma}_1^{(k-1)}}^{(k-1)}(\boldsymbol{m}_1^{(k-1)}),\varphi_{\boldsymbol{\gamma}_2^{(k-1)}}^{(k-1)}(\boldsymbol{m}_2^{(k-1)})) & \text{if } k\geq 3, \boldsymbol{\gamma}=(\boldsymbol{\gamma}_1^{(k-1)},\boldsymbol{\gamma}_2^{(k-1)}) \\ & \in\Gamma^{(k-1)}\times\Gamma^{(k-1)}, \\ & \boldsymbol{m}^{(k)}=(\boldsymbol{m}_1^{(k-1)},\boldsymbol{m}_2^{(k-1)})\in \\ & \mathfrak{M}^{(k-1,\boldsymbol{\gamma}_1^{(k-1)})}\times\mathfrak{M}^{(k-1,\boldsymbol{\gamma}_2^{(k-1)})}. \end{cases}$$

*Moreover we also define inductively the isomorphism $\phi_{\boldsymbol{\gamma}}^{(k)} : \mathfrak{B}^{(k,\boldsymbol{\gamma})} \longrightarrow \prod_{j=1}^{\sigma(\boldsymbol{\gamma})} \mathbb{Z}$ by*

$$\phi_{\boldsymbol{\gamma}}^{(k)}(\boldsymbol{\alpha}^{(k)}) = \begin{cases} \alpha & \text{if } \boldsymbol{\gamma} = 0 \text{ or } 1 \in \mathscr{D}^{(k)}, \\ & \qquad \boldsymbol{\alpha}^{(k)} = \alpha \in \mathbb{Z} = \mathfrak{B}^{(k,\boldsymbol{\gamma})}, \\ (\alpha_1, \alpha_2) & \text{if } k = 2, \boldsymbol{\gamma} = (0,0) \quad \text{or } (0,1) \\ & \qquad \text{or } (1,0) \text{ or } (1,1) \in \mathscr{D}^{(2)}, \\ & \qquad \boldsymbol{\alpha}^{(2)} = (\alpha_1, \alpha_2) \in \mathfrak{B}^{(2,\boldsymbol{\gamma})}, \\ (\phi_{\boldsymbol{\gamma}_1^{(k-1)}}^{(k-1)}(\boldsymbol{\alpha}_1^{(k-1)}), \phi_{\boldsymbol{\gamma}_2^{(k-1)}}^{(k-1)}(\boldsymbol{\alpha}_2^{(k-1)})) & \text{if } k \geq 3, \boldsymbol{\alpha}^{(k)} \in \mathfrak{B}^{(k,\boldsymbol{\gamma})}, \boldsymbol{\gamma} = (\boldsymbol{\gamma}_1^{(k-1)}, \\ & \qquad \boldsymbol{\gamma}_2^{(k-1)}) \in \mathscr{D}^{(k-1)} \times \mathscr{D}^{(k-1)}, \\ & \qquad \boldsymbol{\alpha}^{(k)} = (\boldsymbol{\alpha}_1^{(k-1)}, \boldsymbol{\alpha}_2^{(k-1)}) \\ & \qquad \in \mathfrak{B}^{(k-1, \boldsymbol{\gamma}_1^{(k-1)})} \times \mathfrak{B}^{(k-1, \boldsymbol{\gamma}_2^{(k-1)})}. \end{cases}$$

Remark that these isomorphisms defined above induce an ordering of the components of the corresponding vectors. More precisely, for $1 \leq i \leq \sigma(\boldsymbol{\gamma})$, we denote the $i$-th component of $\varphi_{\boldsymbol{\gamma}}^{(k)}(\boldsymbol{m}^{(k)}) \in \prod_{j=1}^{\sigma(\boldsymbol{\gamma})} \mathbb{Z}^{\nu}$ by $(\boldsymbol{m}^{(k)})_i$, where

$$(\boldsymbol{m}^{(k)})_i = (\boldsymbol{m}_1^{(k-1)})_i \quad \text{if } 1 \leq i \leq \sigma(\boldsymbol{\gamma}_1^{(k-1)}),$$
$$(\boldsymbol{m}^{(k)})_{i+\sigma(\boldsymbol{\gamma}_1^{(k-1)})} = (\boldsymbol{m}_2^{(k-1)})_i \quad \text{if } 1 \leq i \leq \sigma(\boldsymbol{\gamma}_2^{(k-1)}).$$

By Definition 2.1, we need to introduce the following sets

$$\mathscr{T}^{(k,\boldsymbol{\gamma})} = \left\{ \boldsymbol{\alpha} \in \mathbb{Z}^{\sigma(\boldsymbol{\gamma})} : \sum_j \alpha_j = 1, \alpha_j \geq 0 \right\}, \tag{2.24}$$

and

$$\mathscr{A}^{(k,\boldsymbol{\gamma})} = \begin{cases} \{0 \in \mathbb{Z}\} & \text{if } \boldsymbol{\gamma} = 0 \text{ or } 1 \in \mathscr{D}^{(k)}, \\ \{(\alpha_1, \alpha_2) \in \mathbb{Z}^2 : \alpha_1 + \alpha_2 = 1, \alpha_j \geq 0\} & \text{if } \boldsymbol{\gamma} \in \mathscr{D}^{(2)}, \boldsymbol{\gamma} = (0,0) \text{ or } (0,1) \\ & \qquad \text{or } (1,0) \text{ or } (1,1), \\ \mathscr{A}^{(k-1, \boldsymbol{\gamma}_1^{(k-1)})} \times \mathscr{A}^{(k-1, \boldsymbol{\gamma}_2^{(k-1)})} + \mathscr{T}^{(k,\boldsymbol{\gamma})} & \text{if } \boldsymbol{\gamma} \in \mathscr{D}^{(k)}, k \geq 3, \boldsymbol{\gamma} = (\boldsymbol{\gamma}_1^{(k-1)}, \\ & \qquad \boldsymbol{\gamma}_2^{(k-1)}) \in \mathscr{D}^{(k-1)} \times \mathscr{D}^{(k-1)}. \end{cases}$$
$$\tag{2.25}$$

Because of Lemma 2.4, the terms $|f(\boldsymbol{m}^{(k)})|$ can be bounded from above by $\mathfrak{P}(\boldsymbol{m}^{(k)})$. The following lemma addresses that the functions $\mathfrak{P}(\boldsymbol{m}^{(k)})$ can be estimated by the "new variables" $\alpha_j$'s.

LEMMA 2.5.   *For $\boldsymbol{\gamma} \in \mathscr{D}^{(k)}$, $\boldsymbol{m}^{(k)} \in \mathfrak{M}^{(k,\boldsymbol{\gamma})}$, one has*

$$\mathfrak{P}(\boldsymbol{m}^{(k)}) \leq \sum_{\boldsymbol{\alpha} = (\alpha_i)_{1 \leq i \leq \sigma(\boldsymbol{\gamma})} \in \mathscr{A}^{(k,\boldsymbol{\gamma})}} \prod_i |(\boldsymbol{m}^{(k)})_i|^{\alpha_i}. \tag{2.26}$$

*Proof.*   The statement follows from (2.19), (2.22) and (2.24)–(2.25). We need to consider the following three cases.

*Case 1*: $k=1$. For $\boldsymbol{m}^{(1)} \in \mathfrak{M}^{(1,\gamma)}, \boldsymbol{\gamma}=0$ or $1 \in \mathscr{D}^{(1)}$, one has $\mathfrak{P}(\boldsymbol{m}^{(1)})=1$. Moreover, in the right-hand side of (2.26), we have

$$\sum_{\boldsymbol{\alpha}=(\alpha_i)_{1\leq i\leq\sigma(\boldsymbol{\gamma})}\in\mathscr{A}^{(k,\boldsymbol{\gamma})}}\prod_i|(\boldsymbol{m}^{(k)})_i|^{\alpha_i}=\sum_{\boldsymbol{\alpha}=(\alpha_i)_{1\leq i\leq1}\in\{0\in\mathbb{Z}\}}\prod_i|(\boldsymbol{m}^{(k)})_i|^{\alpha_i}$$
$$=\prod_i|(\boldsymbol{m}^{(k)})_i|^0=1.$$

*Case 2*: $k=2$. In the left-hand side of (2.26), we can obtain

$$\mathfrak{P}(\boldsymbol{m}^{(2)})=\sum_{\boldsymbol{\alpha}=(\alpha_i)_{1\leq i\leq1}\in\{0\in\mathbb{Z}\}}\prod_i|(\boldsymbol{m}^{(2)})_i|^0=1\quad\text{if }\boldsymbol{m}^{(2)}\in\mathfrak{M}^{(2,\boldsymbol{\gamma})},\boldsymbol{\gamma}=0\text{ or }1\in\mathscr{D}^{(2)},$$

$$\mathfrak{P}(\boldsymbol{m}^{(2)})=|\boldsymbol{m}_1+\boldsymbol{m}_2|\leq|\boldsymbol{m}_1|+|\boldsymbol{m}_2|\quad\text{if }\boldsymbol{m}^{(2)}\in\mathfrak{M}^{(2,\boldsymbol{\gamma})},$$
$$\boldsymbol{\gamma}=(0,0)\text{ or }(0,1)\text{ or }(1,0)\text{ or }(1,1)\in\mathscr{D}^{(2)}.$$

On the other hand, it follows that

$$\sum_{\boldsymbol{\alpha}=(\alpha_i)_{1\leq i\leq\sigma(\boldsymbol{\gamma})}\in\mathscr{A}^{(2,\boldsymbol{\gamma})}}\prod_i|(\boldsymbol{m}^{(k)})_i|^{\alpha_i}=\sum_{\{(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2)\in\mathbb{Z}^2:\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2=1,\boldsymbol{\alpha}_j\geq0\}}\prod_i|(\boldsymbol{m}^{(2)})_i|^{\alpha_i}$$
$$=\sum_{\boldsymbol{\alpha}=(\alpha_i)_{1\leq i\leq2}\in\{(1,0),(0,1)\}}\prod_i|(\boldsymbol{m}^{(2)})_i|^{\alpha_i}$$
$$=|(\boldsymbol{m}^{(2)})_1|^1|(\boldsymbol{m}^{(2)})_2|^0+|(\boldsymbol{m}^{(2)})_1|^0|(\boldsymbol{m}^{(2)})_2|^1$$
$$=|\boldsymbol{m}_1|+|\boldsymbol{m}_2|.$$

*Case 3*: $k\geq3$. In the left-hand side of (2.26), it is clear that

$$\mathfrak{P}(\boldsymbol{m}^{(k)})=\sum_{\boldsymbol{\alpha}=(\alpha_i)_{1\leq i\leq1}\in\{0\in\mathbb{Z}\}}\prod_i|(\boldsymbol{m}^{(k)})_i|^0=1\quad\text{if }\boldsymbol{\gamma}=0\text{ or }1\in\mathscr{D}^{(k)},$$

$$\mathfrak{P}(\boldsymbol{m}^{(k)})=|\boldsymbol{m}_1+\boldsymbol{m}_2|\leq|\boldsymbol{m}_1|+|\boldsymbol{m}_2|=\sum_{\boldsymbol{\alpha}=(\alpha_i)_{1\leq i\leq2}\in\{(1,0),(0,1)\}}\prod_i|(\boldsymbol{m}^{(k)})_i|^{\alpha_i}$$
$$\text{if }\boldsymbol{\gamma}=(0,0)\text{ or }(0,1)\text{ or }(1,0)\text{ or }(1,1)\in\mathscr{D}^{(k)}.$$

Suppose that (2.26) could hold for any $\boldsymbol{\gamma}'\in\mathscr{D}^{(k')}$ with $k'<k$. Let $\boldsymbol{\gamma}=(\boldsymbol{\gamma}_1^{(k-1)},\boldsymbol{\gamma}_2^{(k-1)})\in\mathscr{D}^{(k-1)}\times\mathscr{D}^{(k-1)}$, $\boldsymbol{m}^{(k)}\in\mathfrak{M}^{(k,\boldsymbol{\gamma})}$ and $\boldsymbol{m}^{(k)}=(\boldsymbol{m}_1^{(k-1)},\boldsymbol{m}_2^{(k-1)})\in\mathfrak{M}^{(k-1,\boldsymbol{\gamma}_1^{(k-1)})}\times\mathfrak{M}^{(k-1,\boldsymbol{\gamma}_1^{(k-1)})}$. Then it follows from the above assumption, the definition of $\mathfrak{P}(\boldsymbol{m}^{(k)})$, $\mathscr{A}^{(k,\boldsymbol{\gamma})}$ (recall (2.22) and (2.25)) and Definition 2.1 that

$$\mathfrak{P}(\boldsymbol{m}^{(k)})=\left|\sum_{i=1}^{\sigma(\boldsymbol{\gamma}_1^{(k-1)})}(\boldsymbol{m}^{(k)})_i+\sum_{i=1}^{\sigma(\boldsymbol{\gamma}_2^{(k-1)})}(\boldsymbol{m}^{(k)})_{i+\sigma(\boldsymbol{\gamma}_1^{(k-1)})}\right|\prod_{j=1}^2\mathfrak{P}(\boldsymbol{m}_j^{(k-1)})$$

$$=\left|\sum_{i=1}^{\sigma(\boldsymbol{\gamma}_1^{(k-1)})}(\boldsymbol{m}_1^{(k-1)})_i+\sum_{i=1}^{\sigma(\boldsymbol{\gamma}_2^{(k-1)})}(\boldsymbol{m}_2^{(k-1)})_i\right|\prod_{j=1}^2\mathfrak{P}(\boldsymbol{m}_j^{(k-1)})$$

$$=\left|\sum_{j=1}^2\sum_{i=1}^{\sigma(\boldsymbol{\gamma}_j^{(k-1)})}(\boldsymbol{m}_j^{(k-1)})_i\right|\prod_{j=1}^2\mathfrak{P}(\boldsymbol{m}_j^{(k-1)})$$

$$\leq \left( \sum_{j=1}^{2} \sum_{i=1}^{\sigma(\boldsymbol{\gamma}_j^{(k-1)})} |(\boldsymbol{m}_j^{(k-1)})_i| \right) \prod_{j=1}^{2} \sum_{\boldsymbol{\alpha}_j \in \mathscr{A}^{(k-1,\boldsymbol{\gamma}_j^{(k-1)})}} \prod_i |(\boldsymbol{m}_j^{(k-1)})_i|^{\alpha_{i,j}}$$

$$= \sum_{\boldsymbol{\alpha}=(\alpha_i)_{1\leq i\leq\sigma(\boldsymbol{\gamma})}\in\mathscr{A}^{(k,\boldsymbol{\gamma})}} \prod_i |(\boldsymbol{m}^{(k)})_i|^{\alpha_i},$$

where $\boldsymbol{\alpha}_j = (\alpha_{1,j}, \alpha_{2,j}, \cdots, \alpha_{\sigma(\boldsymbol{\gamma}_j^{(k-1)}),j}) \in \mathscr{A}^{(k-1,\boldsymbol{\gamma}_j^{(k-1)})}$. This ends the proof of the lemma.

$\square$

In the following lemma, we evaluate the sums involving the functions $\mathfrak{P}(\boldsymbol{m}^{(k)})$ and $\exp(-\frac{\kappa|\boldsymbol{m}^{(k)}|}{2})$. As a result, we can give an estimation of the sums involving the functions $\mathfrak{C}^{(k,\boldsymbol{\gamma})}(\boldsymbol{m}^{(k)})$ and $f^{(k,\boldsymbol{\gamma})}(\boldsymbol{m}^{(k)})$.

LEMMA 2.6.   *Denote by* $\boldsymbol{\alpha}! := \prod_j \alpha_j!$. *One has that for any* $0 < \kappa \leq 1$,

(I)

$$\sum_{\boldsymbol{m}^{(k)}\in\mathfrak{M}^{(k,\boldsymbol{\gamma})}} \exp(-\frac{\kappa|\boldsymbol{m}^{(k)}|}{2})\mathfrak{P}(\boldsymbol{m}^{(k)}) \leq (\frac{24}{\kappa})^{\sigma(\boldsymbol{\gamma})\nu} \sum_{\boldsymbol{\alpha}=(\alpha_i)_{1\leq i\leq\sigma(\boldsymbol{\gamma})}\in\mathbb{Z}_+^{\sigma(\boldsymbol{\gamma})}} \prod_i \alpha_i!. \qquad (2.27)$$

(II)

$$\sum_{\substack{\boldsymbol{m}^{(k)}\in\mathfrak{M}^{(k,\boldsymbol{\gamma})}\\ \mu(\boldsymbol{m}^{(k)})=\boldsymbol{n}}} \exp(-\frac{\kappa|\boldsymbol{m}^{(k)}|}{2})\mathfrak{P}(\boldsymbol{m}^{(k)}) \leq (\frac{48}{\kappa})^{\sigma(\boldsymbol{\gamma})\nu}\exp(-\frac{\kappa}{4}|\boldsymbol{n}|) \sum_{\boldsymbol{\alpha}=(\alpha_i)_{1\leq i\leq\sigma(\boldsymbol{\gamma})}\in\mathbb{Z}_+^{\sigma(\boldsymbol{\gamma})}} \prod_i \alpha_i!.$$

$$(2.28)$$

*Proof.*
(I) It follows from Lemma 2.5 that

$$\sum_{\boldsymbol{m}^{(k)}\in\mathfrak{M}^{(k,\boldsymbol{\gamma})}} \exp(-\frac{\kappa|\boldsymbol{m}^{(k)}|}{2})\mathfrak{P}(\boldsymbol{m}^{(k)})$$

$$\leq \sum_{\boldsymbol{m}^{(k)}\in\mathfrak{M}^{(k,\boldsymbol{\gamma})}} \exp(-\frac{\kappa|\boldsymbol{m}^{(k)}|}{2}) \sum_{\boldsymbol{\alpha}=(\alpha_i)_1^{\sigma(\boldsymbol{\gamma})}\in\mathscr{A}^{(k,\boldsymbol{\gamma})}} \prod_i |(\boldsymbol{m}^{(k)})_i|^{\alpha_i}$$

$$\leq \sum_{\boldsymbol{\alpha}=(\alpha_i)_1^{\sigma(\boldsymbol{\gamma})}\in\mathscr{A}^{(k,\boldsymbol{\gamma})}} \sum_{((\boldsymbol{m}^{(k)})_i)_{1\leq i\leq\sigma(\boldsymbol{\gamma})}\in\mathfrak{M}^{(k,\boldsymbol{\gamma})}} \prod_i |(\boldsymbol{m}^{(k)})_i|^{\alpha_i}\exp(-\frac{\kappa|(\boldsymbol{m}^{(k)})_i|}{2})$$

$$\leq \sum_{\boldsymbol{\alpha}=(\alpha_i)_1^{\sigma(\boldsymbol{\gamma})}\in\mathscr{A}^{(k,\boldsymbol{\gamma})}} \sum_{((\boldsymbol{m}^{(k)})_i)_{1\leq i\leq\sigma(\boldsymbol{\gamma})}\in\mathfrak{M}^{(k,\boldsymbol{\gamma})}} \prod_i \alpha_i!(\frac{4}{\kappa})^{\alpha_i}\frac{(\frac{\kappa|(\boldsymbol{m}^{(k)})_i|}{4})^{\alpha_i}}{\alpha_i!}\exp(-\frac{\kappa|(\boldsymbol{m}^{(k)})_i|}{2})$$

$$\leq \sum_{\boldsymbol{\alpha}=(\alpha_i)_1^{\sigma(\boldsymbol{\gamma})}\in\mathscr{A}^{(k,\boldsymbol{\gamma})}} \sum_{((\boldsymbol{m}^{(k)})_i)_{1\leq i\leq\sigma(\boldsymbol{\gamma})}\in\mathfrak{M}^{(k,\boldsymbol{\gamma})}} \boldsymbol{\alpha}!\prod_i (\frac{4}{\kappa})^{\alpha_i}\exp(\frac{\kappa}{4}|(\boldsymbol{m}^{(k)})_i|)\exp(-\frac{\kappa|(\boldsymbol{m}^{(k)})_i|}{2})$$

$$\leq \sum_{\boldsymbol{\alpha}=(\alpha_i)_1^{\sigma(\boldsymbol{\gamma})}\in\mathscr{A}^{(k,\boldsymbol{\gamma})}} \boldsymbol{\alpha}!(\frac{4}{\kappa})^{|\boldsymbol{\alpha}|} \sum_{((\boldsymbol{m}^{(k)})_i)_{1\leq i\leq\sigma(\boldsymbol{\gamma})}\in\mathfrak{M}^{(k,\boldsymbol{\gamma})}} \prod_i \exp(-\frac{\kappa}{4}|(\boldsymbol{m}^{(k)})_i|)$$

$$= \sum_{\boldsymbol{\alpha}=(\alpha_i)_1^{\sigma(\boldsymbol{\gamma})}\in\mathscr{A}^{(k,\boldsymbol{\gamma})}} \boldsymbol{\alpha}!(\frac{4}{\kappa})^{|\boldsymbol{\alpha}|} \sum_{((\boldsymbol{m}^{(k)})_1,\ldots,(\boldsymbol{m}^{(k)})_{\sigma(\boldsymbol{\gamma})})\in\underbrace{\mathbb{Z}^\nu\times\mathbb{Z}^\nu\times\cdots\times\mathbb{Z}^\nu}} \prod_i \exp(-\frac{\kappa}{4}|(\boldsymbol{m}^{(k)})_i|)$$

$$= \sum_{\boldsymbol{\alpha}=(\alpha_i)_1^{\sigma(\boldsymbol{\gamma})}\in\mathscr{A}(k,\boldsymbol{\gamma})} \boldsymbol{\alpha}!(\frac{4}{\kappa})^{|\boldsymbol{\alpha}|}\left(\sum_{\boldsymbol{m}\in\mathbb{Z}^\nu}\exp(-\frac{\kappa}{4}|\boldsymbol{m}|)\right)^{\sigma(\boldsymbol{\gamma})}$$

$$= \sum_{\boldsymbol{\alpha}=(\alpha_i)_1^{\sigma(\boldsymbol{\gamma})}\in\mathscr{A}(k,\boldsymbol{\gamma})} \boldsymbol{\alpha}!(\frac{4}{\kappa})^{|\boldsymbol{\alpha}|}\left(\sum_{\boldsymbol{m}\in\mathbb{Z}}\exp(-\frac{\kappa}{4}|\boldsymbol{m}|)\right)^{\sigma(\boldsymbol{\gamma})\nu}$$

$$= \sum_{\boldsymbol{\alpha}=(\alpha_i)_1^{\sigma(\boldsymbol{\gamma})}\in\mathscr{A}(k,\boldsymbol{\gamma})} \boldsymbol{\alpha}!(\frac{4}{\kappa})^{|\boldsymbol{\alpha}|}\left(1+2\sum_{\boldsymbol{m}\in\mathbb{N}_+}\exp(-\frac{\kappa}{4}\boldsymbol{m})\right)^{\sigma(\boldsymbol{\gamma})\nu}$$

$$= \sum_{\boldsymbol{\alpha}=(\alpha_i)_1^{\sigma(\boldsymbol{\gamma})}\in\mathscr{A}(k,\boldsymbol{\gamma})} \boldsymbol{\alpha}!(\frac{4}{\kappa})^{|\boldsymbol{\alpha}|}\left(1+2\frac{\exp(-\frac{\kappa}{4})}{1-\exp(-\frac{\kappa}{4})}\right)^{\sigma(\boldsymbol{\gamma})\nu}$$

$$\leq \sum_{\boldsymbol{\alpha}=(\alpha_i)_1^{\sigma(\boldsymbol{\gamma})}\in\mathscr{A}(k,\boldsymbol{\gamma})} \boldsymbol{\alpha}!(\frac{4}{\kappa})^{|\boldsymbol{\alpha}|}\left(\frac{2}{1-\exp(-\frac{\kappa}{4})}\right)^{\sigma(\boldsymbol{\gamma})\nu}$$

$$\leq \sum_{\boldsymbol{\alpha}=(\alpha_i)_1^{\sigma(\boldsymbol{\gamma})}\in\mathscr{A}(k,\boldsymbol{\gamma})} \boldsymbol{\alpha}!(\frac{4}{\kappa})^{|\boldsymbol{\alpha}|}(\frac{24}{\kappa})^{\sigma(\boldsymbol{\gamma})\nu}$$

$$\leq \sum_{\boldsymbol{\alpha}=(\alpha_i)_1^{\sigma(\boldsymbol{\gamma})}\in\mathscr{A}(k,\boldsymbol{\gamma})} \boldsymbol{\alpha}!(\frac{24}{\kappa})^{|\boldsymbol{\alpha}|+\sigma(\boldsymbol{\gamma})\nu}$$

$$\leq (\frac{24}{\kappa})^{\sigma(\boldsymbol{\gamma})\nu} \sum_{\boldsymbol{\alpha}=(\alpha_i)_1^{\sigma(\boldsymbol{\gamma})}\in\mathbb{Z}_+^{\sigma(\boldsymbol{\gamma})}} \prod_i \alpha_i!.$$

(II) Let $\mu(\boldsymbol{m}^{(k)})=\boldsymbol{n}$. Observe that $|\boldsymbol{m}^{(k)}|\geq|\boldsymbol{n}|$. Then combining this with (2.27) gives that

$$\sum_{\substack{\boldsymbol{m}^{(k)}\in\mathfrak{M}(k,\boldsymbol{\gamma})\\ \mu(\boldsymbol{m}^{(k)})=\boldsymbol{n}}} \exp(-\frac{\kappa|\boldsymbol{m}^{(k)}|}{2})\mathfrak{P}(\boldsymbol{m}^{(k)}) \leq \sum_{\substack{\boldsymbol{m}^{(k)}\in\mathfrak{M}(k,\boldsymbol{\gamma})\\ \mu(\boldsymbol{m}^{(k)})=\boldsymbol{n}}} \exp(-\frac{\kappa}{4}|\boldsymbol{m}^{(k)}|)\mathfrak{P}(\boldsymbol{m}^{(k)})\exp(-\frac{\kappa}{4}|\boldsymbol{n}|)$$

$$\leq (\frac{48}{\kappa})^{\sigma(\boldsymbol{\gamma})\nu}\exp(-\frac{\kappa}{4}|\boldsymbol{n}|) \sum_{\boldsymbol{\alpha}=(\alpha_i)_{1\leq i\leq\sigma(\boldsymbol{\gamma})}\in\mathbb{Z}_+^{\sigma(\boldsymbol{\gamma})}} \prod_i \alpha_i!.$$

Thus we arrive at the conclusion of the lemma. □

It remains to estimate the upper bounds of the terms $|I(t,\boldsymbol{m}^{(k)})|$. In view of Lemma 2.4, $|I(t,\boldsymbol{m}^{(k)})|$ can be bounded from above by $\frac{(2t)^{\ell(\boldsymbol{\gamma})}}{\mathfrak{F}(\boldsymbol{\gamma})}$. In the next lemma, we will provide an estimation of the terms $\frac{(2t)^{\ell(\boldsymbol{\gamma})}}{\mathfrak{F}(\boldsymbol{\gamma})}$ and $\prod_i \alpha_i!$. In fact, this gives an ideal estimation for the total sum of these terms when $t$ belongs to a given interval.

LEMMA 2.7.   *If $0<t\leq 1/16$, then*

$$\sum_{\boldsymbol{\gamma}\in\mathscr{D}^{(k)}} \frac{(2t)^{\ell(\boldsymbol{\gamma})}}{\mathfrak{F}(\boldsymbol{\gamma})} \sum_{\boldsymbol{\alpha}=(\alpha_i)_{1\leq i\leq\sigma(\boldsymbol{\gamma})}\in\mathscr{A}(k,\boldsymbol{\gamma})} \prod_i \alpha_i!\leq 2. \tag{2.29}$$

*Proof.*   The proof is based on formulae (2.19)–(2.20) and (2.23)–(2.25). Let us consider the following three cases.

_Case 1_: $k=1$. Since $\gamma=0$ or $1\in\mathscr{D}^{(1)}$, we obtain

$$\ell(\gamma)=0, \quad \mathfrak{F}(\gamma)=1, \quad \mathscr{A}^{(1,\gamma)}=\{0\in\mathbb{Z}\}.$$

Therefore,

$$\sum_{\gamma\in\mathscr{D}^{(1)}} \frac{(2t)^0}{1} \sum_{\boldsymbol{\alpha}=(\alpha_i)_{1\le i\le\sigma(\gamma)}\in\{0\in\mathbb{Z}\}} \prod 0!=1<2. \tag{2.30}$$

_Case 2_: $k=2$. If $\gamma=0$ or $1\in\mathscr{D}^{(2)}$, then the same estimation can be obtained as the case $k=1$. For $\gamma=(0,0)$ or $(0,1)$ or $(1,0)$ or $(1,1)\in\mathscr{D}^{(2)}$, we can carry out

$$\ell(\gamma)=1, \quad \mathfrak{F}(\gamma)=1, \quad \mathscr{A}^{(2,\gamma)}=\{(1,0),(0,1)\in\mathbb{Z}^2\}.$$

Hence,

$$\sum_{\gamma\in\mathscr{D}^{(2)}\setminus\mathscr{D}^{(1)}} \frac{(2t)^1}{1} \sum_{\boldsymbol{\alpha}=(\alpha_i)_{1\le i\le\sigma(\gamma)}\in\{(1,0),(0,1)\in\mathbb{Z}^2\}} \prod_i \alpha_i!=2t.$$

This implies that

$$\sum_{\gamma\in\mathscr{D}^{(2)}} \frac{(2t)^{\ell(\gamma)}}{\mathfrak{F}(\gamma)} \sum_{\boldsymbol{\alpha}=(\alpha_i)_{1\le i\le\sigma(\gamma)}\in\mathscr{A}^{(2,\gamma)}} \prod_i \alpha_i!$$

$$=\sum_{\gamma\in\mathscr{D}^{(2)}\setminus\mathscr{D}^{(1)}} \frac{(2t)^1}{1} \sum_{\boldsymbol{\alpha}=(\alpha_i)_{1\le i\le\sigma(\gamma)}\in\{(1,0),(0,1)\in\mathbb{Z}^2\}} \prod_i \alpha_i!$$

$$+\sum_{\gamma\in\mathscr{D}^{(1)}} \frac{(2t)^0}{1} \sum_{\boldsymbol{\alpha}=(\alpha_i)_{1\le i\le\sigma(\gamma)}\in\{0\in\mathbb{Z}\}} \prod 0!$$

$$<1+2t\le2.$$

_Case 3_: $k\ge3$. If $\gamma=0$ or $1$ or $(0,0)$ or $(0,1)$ or $(1,0)$ or $(1,1)\in\mathscr{D}^{(k)}$, we can derive the same estimation as the case $k=2$.

Moreover, for any $\gamma$, it is evident that $\sigma(\gamma)=\ell(\gamma)+1$. We arrive at

$$\sum_{j_0=1,2} \frac{\ell(\gamma_{j_0}^{(k-1)})+\sigma(\gamma_{j_0}^{(k-1)})}{\ell(\gamma_1^{(k-1)})+\ell(\gamma_2^{(k-1)})+1} = \frac{\ell(\gamma_1^{(k-1)})+\sigma(\gamma_1^{(k-1)})+\ell(\gamma_2^{(k-1)})+\sigma(\gamma_2^{(k-1)})}{\ell(\gamma_1^{(k-1)})+\ell(\gamma_2^{(k-1)})+1}=2$$

for any $\gamma_1,\gamma_2$.

Let us consider $\gamma=(\gamma_1^{(k-1)},\gamma_2^{(k-1)})\in\mathscr{D}^{(k-1)}\times\mathscr{D}^{(k-1)}$. As a consequence,

$$\sum_{\gamma\in\mathscr{D}^{(k)}\setminus\mathscr{D}^{(1)}} \frac{(2t)^{\ell(\gamma)}}{\mathfrak{F}(\gamma)} \sum_{\boldsymbol{\alpha}\in\mathscr{A}^{(k,\gamma)}} \prod_{i=1}^{\sigma(\gamma)} \alpha_i!$$

$$=\sum_{\gamma=(\gamma_1^{(k-1)},\gamma_2^{(k-1)})\in\mathscr{D}^{(k-1)}\times\mathscr{D}^{(k-1)}} \frac{2t}{\ell(\gamma_1^{(k-1)})+\ell(\gamma_2^{(k-1)})+1} \prod_{j=1,2} \frac{(2t)^{\ell(\gamma_j^{(k-1)})}}{\mathfrak{F}(\gamma_j^{(k-1)})}$$

$$\cdot \sum_{\boldsymbol{\alpha}=(\boldsymbol{\alpha}^{(1)},\boldsymbol{\alpha}^{(2)})+\boldsymbol{\beta},\boldsymbol{\alpha}^{(j)}\in\mathscr{A}^{(k-1,\gamma_j^{(k-1)})},\boldsymbol{\beta}\in\mathscr{T}^{(k,(\gamma_1^{(k-1)},\gamma_2^{(k-1)}))}} \prod_{i=1}^{\sigma(\gamma)} \alpha_i!$$

$$\leq \sum_{\boldsymbol{\gamma}=(\boldsymbol{\gamma}_1^{(k-1)},\boldsymbol{\gamma}_2^{(k-1)})\in\mathscr{D}^{(k-1)}\times\mathscr{D}^{(k-1)}} \frac{2t}{\ell(\boldsymbol{\gamma}_1^{(k-1)})+\ell(\boldsymbol{\gamma}_2^{(k-1)})+1} \sum_{\boldsymbol{\alpha}^{(j)}\in\mathscr{A}^{(k-1,\boldsymbol{\gamma}_j^{(k-1)})},j=1,2}$$

$$\sum_{j_0=1,2}\sum_{i_0=1}^{\sigma(\boldsymbol{\gamma}_{j_0}^{(k-1)})}\prod_{j=1,2}\frac{(2t)^{\ell(\boldsymbol{\gamma}_j^{(k-1)})}}{\mathfrak{F}(\boldsymbol{\gamma}_j^{(k-1)})}\prod_{i=1}^{\sigma(\boldsymbol{\gamma}_j^{(k-1)})}((\boldsymbol{\alpha}^{(j)})_i+\delta_{i,i_0}\delta_{j,j_0})!$$

$$= \sum_{\boldsymbol{\gamma}=(\boldsymbol{\gamma}_1^{(k-1)},\boldsymbol{\gamma}_2^{(k-1)})\in\mathscr{D}^{(k-1)}\times\mathscr{D}^{(k-1)}} \frac{2t}{\ell(\boldsymbol{\gamma}_1^{(k-1)})+\ell(\boldsymbol{\gamma}_2^{(k-1)})+1} \sum_{\boldsymbol{\alpha}^{(j)}\in\mathscr{A}^{(k-1,\boldsymbol{\gamma}_j^{(k-1)})},j=1,2}$$

$$\sum_{j_0=1,2}\sum_{i_0=1}^{\sigma(\boldsymbol{\gamma}_{j_0}^{(k-1)})}((\alpha^{(j_0)})_{i_0}+1)\prod_{j=1,2}\frac{(2t)^{\ell(\boldsymbol{\gamma}_j^{(k-1)})}}{\mathfrak{F}(\boldsymbol{\gamma}_j^{(k-1)})}\prod_{i=1}^{\sigma(\boldsymbol{\gamma}_j^{(k-1)})}((\boldsymbol{\alpha}^{(j)})_i)!$$

$$= \sum_{\boldsymbol{\gamma}=(\boldsymbol{\gamma}_1^{(k-1)},\boldsymbol{\gamma}_2^{(k-1)})\in\mathscr{D}^{(k-1)}\times\mathscr{D}^{(k-1)}} \frac{2t}{\ell(\boldsymbol{\gamma}_1^{(k-1)})+\ell(\boldsymbol{\gamma}_2^{(k-1)})+1} \sum_{\boldsymbol{\alpha}^{(j)}\in\mathscr{A}^{(k-1,\boldsymbol{\gamma}_j^{(k-1)})},j=1,2}$$

$$\sum_{j_0=1,2}(\ell(\boldsymbol{\gamma}_{j_0}^{(k-1)})+\sigma(\boldsymbol{\gamma}_{j_0}^{(k-1)}))\prod_{j=1,2}\frac{(2t)^{\ell(\boldsymbol{\gamma}_j^{(k-1)})}}{\mathfrak{F}(\boldsymbol{\gamma}_j^{(k-1)})}\prod_{i=1}^{\sigma(\boldsymbol{\gamma}_j^{(k-1)})}((\boldsymbol{\alpha}^{(j)})_i)!$$

$$=4t\sum_{\boldsymbol{\gamma}=(\boldsymbol{\gamma}_1^{(k-1)},\boldsymbol{\gamma}_2^{(k-1)})\in\mathscr{D}^{(k-1)}\times\mathscr{D}^{(k-1)}}\sum_{\boldsymbol{\alpha}^{(j)}\in\mathscr{A}^{(k-1,\boldsymbol{\gamma}_j^{(k-1)})},j=1,2}$$

$$\prod_{j=1,2}\frac{(2t)^{\ell(\boldsymbol{\gamma}_j^{(k-1)})}}{\mathfrak{F}(\boldsymbol{\gamma}_j^{(k-1)})}\prod_{i=1}^{\sigma(\boldsymbol{\gamma}_j^{(k-1)})}((\boldsymbol{\alpha}^{(j)})_i)!$$

$$=4t\prod_{j=1,2}\frac{(2t)^{\ell(\boldsymbol{\gamma}_j^{(k-1)})}}{\mathfrak{F}(\boldsymbol{\gamma}_j^{(k-1)})}\sum_{\boldsymbol{\gamma}_j^{(k-1)}\in\mathscr{D}^{(k-1)}}\sum_{\boldsymbol{\alpha}^{(j)}\in\mathscr{A}^{(k-1,\boldsymbol{\gamma}_j^{(k-1)})}}\prod_{i=1}^{\sigma(\boldsymbol{\gamma}_j^{(k-1)})}((\boldsymbol{\alpha}^{(j)})_i)!$$

$$\leq 4t\cdot 2\cdot 2=16t.$$

Combining above estimate with (2.30) yields that

$$\sum_{\boldsymbol{\gamma}\in\mathscr{D}^{(k)}}\frac{(2t)^{\ell(\boldsymbol{\gamma})}}{\mathfrak{F}(\boldsymbol{\gamma})}\sum_{\boldsymbol{\alpha}=(\alpha_i)_{1\leq i\leq\sigma(\boldsymbol{\gamma})}\in\mathscr{A}^{(k,\boldsymbol{\gamma})}}\prod\alpha_i!=\sum_{\boldsymbol{\gamma}\in\mathscr{D}^{(1)}}\frac{(2t)^{\ell(\boldsymbol{\gamma})}}{\mathfrak{F}(\boldsymbol{\gamma})}\sum_{\boldsymbol{\alpha}=(\alpha_i)_{1\leq i\leq\sigma(\boldsymbol{\gamma})}\in\mathscr{A}^{(k,\boldsymbol{\gamma})}}\prod\alpha_i!$$

$$+\sum_{\boldsymbol{\gamma}\in\mathscr{D}^{(k)}\setminus\mathscr{D}^{(1)}}\frac{(2t)^{\ell(\boldsymbol{\gamma})}}{\mathfrak{F}(\boldsymbol{\gamma})}\sum_{\boldsymbol{\alpha}=(\alpha_i)_{1\leq i\leq\sigma(\boldsymbol{\gamma})}\in\mathscr{A}^{(k,\boldsymbol{\gamma})}}\prod\alpha_i!$$

$$<1+16t\leq 2.$$

Hence we complete the proof of Lemma 2.7. $\square$

COROLLARY 2.2. If $0\leq t\leq \frac{\kappa^\nu}{32B(24)^\nu|\boldsymbol{\omega}|}$, then

$$|d_k(t,\boldsymbol{n})|\leq \sum_{\substack{\boldsymbol{\gamma}\in\mathscr{D}^{(k)}}}\sum_{\substack{\boldsymbol{m}^{(k)}\in\mathfrak{M}^{(k,\boldsymbol{\gamma})}\\\mu(\boldsymbol{m}^{(k)})=\boldsymbol{n}}}B^{\sigma(\boldsymbol{\gamma})}\exp(-\frac{\kappa|\boldsymbol{m}^{(k)}|}{2})\mathfrak{P}(\boldsymbol{m}^{(k)})\frac{(|\boldsymbol{\omega}|2t)^{\ell(\boldsymbol{\gamma})}}{\mathfrak{F}(\boldsymbol{\gamma})}(\theta^{-1}|\boldsymbol{n}|^\delta)^{\sigma(\boldsymbol{\gamma})}\leq 2B$$

for some constant $B>0$.

*Proof.*        In the proof, we will apply the identity $\sigma(\boldsymbol{\gamma}) = \ell(\boldsymbol{\gamma}) + 1$. By virtue of formulae (2.27) and (2.29), we derive

$$
|d_k(t,\boldsymbol{n})| = \left| \sum_{\boldsymbol{\gamma} \in \mathscr{D}^{(k)}} \sum_{\substack{\boldsymbol{m}^{(k)} \in \mathfrak{M}^{(k,\boldsymbol{\gamma})} \\ \mu(\boldsymbol{m}^{(k)}) = \boldsymbol{n}}} \mathfrak{C}^{(k,\boldsymbol{\gamma})}(\boldsymbol{m}^{(k)}) f^{(k,\boldsymbol{\gamma})}(\boldsymbol{m}^{(k)}) I^{(k,\boldsymbol{\gamma})}(t,\boldsymbol{m}^{(k)}) \right|
$$

$$
\leq \sum_{\boldsymbol{\gamma} \in \mathscr{D}^{(k)}} \sum_{\substack{\boldsymbol{m}^{(k)} \in \mathfrak{M}^{(k,\boldsymbol{\gamma})} \\ \mu(\boldsymbol{m}^{(k)}) = \boldsymbol{n}}} |\mathfrak{C}^{(k,\boldsymbol{\gamma})}(\boldsymbol{m}^{(k)})| |f^{(k,\boldsymbol{\gamma})}(\boldsymbol{m}^{(k)})| |I^{(k,\boldsymbol{\gamma})}(t,\boldsymbol{m}^{(k)})|
$$

$$
\leq \sum_{\boldsymbol{\gamma} \in \mathscr{D}^{(k)}} \sum_{\substack{\boldsymbol{m}^{(k)} \in \mathfrak{M}^{(k,\boldsymbol{\gamma})} \\ \mu(\boldsymbol{m}^{(k)}) = \boldsymbol{n}}} B^{\sigma(\boldsymbol{\gamma})} \exp\left(-\frac{\kappa|\boldsymbol{m}^{(k)}|}{2}\right) \mathfrak{P}(\boldsymbol{m}^{(k)}) \frac{(2t)^{\ell(\boldsymbol{\gamma})}}{\mathfrak{F}(\boldsymbol{\gamma})} (\theta^{-1}|\boldsymbol{n}|^\delta)^{\sigma(\boldsymbol{\gamma})}
$$

$$
\leq 2B.
$$

Thus this ends the proof.                                                    □

**Step 3:**   For $0 \leq t \leq \frac{\kappa^\nu}{(32B(48)^\nu |\boldsymbol{\omega}|}$, it follows from (2.28), Lemma 2.7 and Corollary 2.2 that

$$
|d_k(t,\boldsymbol{n})| \leq 2B \exp\left(-\frac{\kappa|\boldsymbol{n}|}{4}\right).
$$

Hence we have completed the proof of Corollary 2.1.                          □

### 3. Existence and uniqueness of the Fourier coefficients

This section is devoted to showing the existence and uniqueness of the Fourier coefficients $c(t,\boldsymbol{n})$ associated with the ansatz (2.1). In Section 2, by the Picard successive approximation method, we have constructed the corresponding iteration sequence $\{c_k(t,\boldsymbol{n})\}, k \geq 0$ defined by (2.8)–(2.9). What we need to do next is the convergence analysis of this Picard sequence. To this end, we need to consider two problems. One is to give the upper bounds of $|d_{k+1}(t,\boldsymbol{n}) - d_k(t,\boldsymbol{n})|$. The other is to prove that the infinite series $\sum_{k=1}^\infty (d_{k+1}(t,\boldsymbol{n}) - d_k(t,\boldsymbol{n}))$ converges absolutely and uniformly.

We first introduce the following sets

$$
\mathbb{I}^{(k)} = \left\{ \boldsymbol{\alpha} \in \mathbb{Z}^{k+1} : \sum_j \alpha_j = 1, \alpha_j \geq 0 \right\},
$$

and

$$
\mathbb{B}^{(k)} = \begin{cases} \{(\alpha_1, \alpha_2) \in \mathbb{Z}^2 : \alpha_1 + \alpha_2 = 1, \quad \alpha_j \geq 0\}, & k = 1, \\ \mathbb{B}^{(k-1)} \times \{0 \in \mathbb{Z}\} + \mathbb{I}^{(k)}, & k \geq 2. \end{cases}
$$

Notice that for any $\boldsymbol{\alpha} \in \mathbb{B}^{(k)}$,

$$
\boldsymbol{\alpha} \in \mathbb{R}^{k+1}, \quad \sum_j \alpha_j = k. \tag{3.1}
$$

COROLLARY 3.1.    *If* $0 \leq t \leq \frac{\kappa^\nu}{(32B(192)^\nu |\boldsymbol{\omega}|)}$, *then*

$$
|d_{k+1}(t,\boldsymbol{n}) - d_k(t,\boldsymbol{n})| \leq B^{k+1}(8e)^k (96)^{k\nu} (\kappa^{-\nu} |\boldsymbol{\omega}| t)^k \exp\left(-\frac{\kappa}{8}|\boldsymbol{n}|\right) \to 0
$$

*as k tends to ∞.*

**Proof.** The proof will be divided into the following four steps.

**Step 1:** We first show that the terms $|d_{k+1}(t,\boldsymbol{n})-d_k(t,\boldsymbol{n})|$ can be bounded from above by the variables $\boldsymbol{m}_j$.

LEMMA 3.1. *For $0 \le t \le \frac{\kappa^\nu}{32B(48)^\nu|\boldsymbol{\omega}|}$, one has*

$$|d_{k+1}(t,\boldsymbol{n})-d_k(t,\boldsymbol{n})| \le \frac{2^{2k}B^{k+1}(|\boldsymbol{\omega}|t)^k}{k!} \sum_{\substack{\boldsymbol{m}=(\boldsymbol{m}_1,\cdots,\boldsymbol{m}_{k+1})\in\mathbb{Z}^{(k+1)\nu} \\ \sum_j \boldsymbol{m}_j=\boldsymbol{n}}} \sum_{\boldsymbol{\alpha}\in\mathbb{B}^{(k)}}$$
$$\prod_j |\boldsymbol{m}_j|^{\alpha_j}\exp(-\frac{\kappa}{4}|\boldsymbol{m}_j|). \qquad (3.2)$$

**Proof.** According to Corollary 2.1 and (2.17)–(2.18), we will prove the lemma by induction.

For $k=1$, we derive

$$|d_2(t,\boldsymbol{n})-d_1(t,\boldsymbol{n})| \le |\boldsymbol{n}||\boldsymbol{\omega}| \int_0^t \sum_{\substack{\boldsymbol{m}_1,\boldsymbol{m}_2\in\mathbb{Z}^\nu \\ \boldsymbol{m}_1+\boldsymbol{m}_2=\boldsymbol{n}}} |d_1(\tau,\boldsymbol{m}_1)||d_1(\tau,\boldsymbol{m}_2)|\mathrm{d}\tau$$
$$\le 4B^2 t|\boldsymbol{\omega}| \sum_{\substack{\boldsymbol{m}_1,\boldsymbol{m}_2\in\mathbb{Z}^\nu \\ \boldsymbol{m}_1+\boldsymbol{m}_2=\boldsymbol{n}}} |\sum_j \boldsymbol{m}_j|\exp(-\frac{\kappa(|\boldsymbol{m}_1|+|\boldsymbol{m}_2|)}{4}).$$

Hence (3.2) holds for $k=1$.

Suppose that (3.2) could hold for any $1\le k'\le k-1$ with $k\ge 2$. Note that

$$|d_{k+1}(t,\boldsymbol{n})-d_k(t,\boldsymbol{n})|$$
$$\le |\boldsymbol{n}||\boldsymbol{\omega}| \int_0^t \sum_{\substack{\boldsymbol{m}_1,\boldsymbol{m}_2\in\mathbb{Z}^\nu \\ \boldsymbol{m}_1+\boldsymbol{m}_2=\boldsymbol{n}}} |d_k(\tau,\boldsymbol{m}_1)d_k(\tau,\boldsymbol{m}_2)-d_{k-1}(\tau,\boldsymbol{m}_1)d_{k-1}(\tau,\boldsymbol{m}_2)|\mathrm{d}\tau$$
$$\le |\boldsymbol{n}||\boldsymbol{\omega}| \int_0^t \sum_{\substack{\boldsymbol{m}_1,\boldsymbol{m}_2\in\mathbb{Z}^\nu \\ \boldsymbol{m}_1+\boldsymbol{m}_2=\boldsymbol{n}}} |d_k(\tau,\boldsymbol{m}_1)-d_{k-1}(\tau,\boldsymbol{m}_1)||d_k(\tau,\boldsymbol{m}_2)|\mathrm{d}\tau$$
$$+|\boldsymbol{n}||\boldsymbol{\omega}| \int_0^t \sum_{\substack{\boldsymbol{m}_1,\boldsymbol{m}_2\in\mathbb{Z}^\nu \\ \boldsymbol{m}_1+\boldsymbol{m}_2=\boldsymbol{n}}} |d_k(\tau,\boldsymbol{m}_2)-d_{k-1}(\tau,\boldsymbol{m}_2)||d_{k-1}(\tau,\boldsymbol{m}_1)|\mathrm{d}\tau.$$

Using the inductive assumption yields

$$|\boldsymbol{n}||\boldsymbol{\omega}| \int_0^t \sum_{\substack{\boldsymbol{n}_1,\boldsymbol{n}_2\in\mathbb{Z}^\nu \\ \boldsymbol{n}_1+\boldsymbol{n}_2=\boldsymbol{n}}} |d_k(\tau,\boldsymbol{n}_1)-d_{k-1}(\tau,\boldsymbol{n}_1)||d_k(\tau,\boldsymbol{n}_2)|\mathrm{d}\tau$$
$$\le |\boldsymbol{\omega}| \int_0^t \sum_{\substack{\boldsymbol{n}_1,\boldsymbol{n}_2\in\mathbb{Z}^\nu \\ \boldsymbol{n}_1+\boldsymbol{n}_2=\boldsymbol{n}}} \frac{2^{2(k-1)}B^k(|\boldsymbol{\omega}|\tau)^{k-1}}{(k-1)!} \sum_{\substack{\boldsymbol{m}=(\boldsymbol{m}_1,\cdots,\boldsymbol{m}_k)\in\mathbb{Z}^{k\nu} \\ \sum_j \boldsymbol{m}_j=\boldsymbol{n}_1}} |(\sum_j \boldsymbol{m}_j)+\boldsymbol{n}_2|$$
$$\cdot \sum_{\boldsymbol{\alpha}\in\mathbb{B}^{(k-1)}} \prod_j |\boldsymbol{m}_j|^{\alpha_j}\exp(-\frac{\kappa}{4}|\boldsymbol{m}_j|)(2B\exp(-\frac{\kappa}{4}|\boldsymbol{n}_2|))\mathrm{d}\tau$$

$$\leq \frac{2^{(2k-1)} B^{k+1} (|\boldsymbol{\omega}|t)^k}{k!} \sum_{\substack{\boldsymbol{m}=(\boldsymbol{m}_1,\cdots,\boldsymbol{m}_{k+1}) \in \mathbb{Z}^{(k+1)\nu} \\ \sum_j \boldsymbol{m}_j = \boldsymbol{n}}} \sum_{\boldsymbol{\alpha} \in \mathbb{B}^k} \prod_j |\boldsymbol{m}_j|^{\alpha_j} \exp(-\frac{\kappa}{4}|\boldsymbol{m}_j|).$$

Similarly, we conclude

$$|\boldsymbol{n}||\boldsymbol{\omega}| \int_0^t \sum_{\substack{\boldsymbol{n}_1, \boldsymbol{n}_2 \in \mathbb{Z}^\nu \\ \boldsymbol{n}_1 + \boldsymbol{n}_2 = \boldsymbol{n}}} |d_k(\tau, \boldsymbol{n}_2) - d_{k-1}(\tau, \boldsymbol{n}_2)||d_{k-1}(\tau, \boldsymbol{n}_1)|\mathrm{d}\tau$$

$$\leq \frac{2^{(2k-1)} B^{k+1} (|\boldsymbol{\omega}|t)^k}{k!} \sum_{\substack{\boldsymbol{m}=(\boldsymbol{m}_1,\cdots,\boldsymbol{m}_{k+1}) \in \mathbb{Z}^{(k+1)\nu} \\ \sum_j \boldsymbol{m}_j = \boldsymbol{n}}} \sum_{\boldsymbol{\alpha} \in \mathbb{B}^{(k)}} \prod_j |\boldsymbol{m}_j|^{\alpha_j} \exp(-\frac{\kappa}{4}|\boldsymbol{m}_j|).$$

Thus we can get (3.2). The proof of the lemma is now completed. □

**Step 2:** The next goal is to give the upper bounds of $|d_{k+1}(t,\boldsymbol{n}) - d_k(t,\boldsymbol{n})|$ with respect to the new variables $\alpha_j$.

COROLLARY 3.2. *For $0 \leq t \leq \frac{\kappa^\nu}{32B(48)^\nu |\boldsymbol{\omega}|}$, one has*

$$|d_{k+1}(t,\boldsymbol{n}) - d_k(t,\boldsymbol{n})| \leq \frac{B^{k+1} 4^k (96)^{\nu k} (\kappa^{-\nu} |\boldsymbol{\omega}|t)^k}{k!} \exp(-\frac{\kappa}{8}|\boldsymbol{n}|) \sum_{\boldsymbol{\alpha} \in \mathbb{B}^{(k)}} \prod_j \alpha_j!. \qquad (3.3)$$

*Proof.* Due to Lemma 3.1, we have

$$|d_{k+1}(t,\boldsymbol{n}) - d_k(t,\boldsymbol{n})|$$

$$\leq \frac{2^{2k} B^{k+1} (|\boldsymbol{\omega}|t)^k}{k!} \sum_{\substack{\boldsymbol{m}=(\boldsymbol{m}_1,\cdots,\boldsymbol{m}_{k+1}) \in \mathbb{Z}^{(k+1)\nu} \\ \sum_j \boldsymbol{m}_j = \boldsymbol{n}}} \sum_{\boldsymbol{\alpha} \in \mathbb{B}^{(k)}} \prod_j |\boldsymbol{m}_j|^{\alpha_j} \exp(-\frac{\kappa}{4}|\boldsymbol{m}_j|)$$

$$\leq \frac{2^{2k} B^{k+1} (|\boldsymbol{\omega}|t)^k}{k!} \exp(-\frac{\kappa}{8}|\boldsymbol{n}|) \sum_{\substack{\boldsymbol{m}=(\boldsymbol{m}_1,\cdots,\boldsymbol{m}_{k+1}) \in \mathbb{Z}^{(k+1)\nu} \\ \sum_j \boldsymbol{m}_j = \boldsymbol{n}}} \sum_{\boldsymbol{\alpha} \in \mathbb{B}^{(k)}} \prod_j |\boldsymbol{m}_j|^{\alpha_j} \exp(-\frac{\kappa}{8}|\boldsymbol{m}_j|)$$

$$\leq \frac{2^{2k} B^{k+1} (|\boldsymbol{\omega}|t)^k}{k!} \exp(-\frac{\kappa}{8}|\boldsymbol{n}|) \sum_{\substack{\boldsymbol{m}=(\boldsymbol{m}_1,\cdots,\boldsymbol{m}_{k+1}) \in \mathbb{Z}^{(k+1)\nu} \\ |\boldsymbol{m}| \geq \boldsymbol{n}}} \sum_{\boldsymbol{\alpha} \in \mathbb{B}^{(k)}} \prod_j |\boldsymbol{m}_j|^{\alpha_j} \exp(-\frac{\kappa}{8}|\boldsymbol{m}_j|).$$

By the proving procedure of Lemma 2.6, combining this with (3.1) shows that (3.3) holds. □

**Step 3:** We further to estimate the sum $\sum_{\boldsymbol{\alpha} \in \mathbb{B}^{(k)}} \prod_j \alpha_j!$ in the right-hand side of (3.3). For this, let us introduce some notation.

For any $N, l$, we define

$$\mathfrak{A}_N(l) = \left\{ \boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_N) \in \mathbb{Z}^N : \alpha_j \geq 0, \sum_j \alpha_j = l \right\},$$

$$\mathfrak{A}_N(l-1) = \left\{ \boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_N) \in \mathbb{Z}^N : \alpha_j \geq 0, \sum_j \alpha_j = l-1 \right\}.$$

Given $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_N) \in \mathbb{Z}^N$, denote by $\Psi$ the following mapping

$$\Psi(\boldsymbol{\alpha}) = (\psi_1(\boldsymbol{\alpha}), \cdots, \psi_N(\boldsymbol{\alpha})),$$

where

$$\psi_j(\boldsymbol{\alpha}) = \begin{cases} \alpha_j & \text{if } j \neq j_1(\boldsymbol{\alpha}), \\ \alpha_{j_1(\boldsymbol{\alpha})} - 1 & \text{if } j = j_1(\boldsymbol{\alpha}). \end{cases}$$

Note that $j_1(\boldsymbol{\alpha})$ is the subscript of the smallest component of $\Psi(\boldsymbol{\alpha})$. Therefore $\Psi$ maps from $\mathfrak{A}_N(l)$ into $\mathfrak{A}_N(l-1)$. Moreover, set

$$\mathfrak{A}'_N(l) = \left\{ \boldsymbol{\alpha} \in \mathfrak{A}_N(l) : \alpha_{j_1(\boldsymbol{\alpha})} > 1 \right\},$$
$$\mathfrak{A}''_N(l) = \mathfrak{A}_N(l) \backslash \mathfrak{A}'_N(l) = \left\{ \boldsymbol{\alpha} \in \mathfrak{A}_N(l) : 0 \leq \alpha_{j_1(\boldsymbol{\alpha})} \leq 1 \right\}.$$

It is clear that $\Psi$ is an injective mapping on $\mathfrak{A}'_N(l)$ and $\operatorname{card}(\Psi^{-1}(\boldsymbol{\beta})) \leq N$ for any $\boldsymbol{\beta}$. Hence we have the following fact.

LEMMA 3.2.    *For any $l \leq N$, one has*

$$\sum_{\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_N) \in \mathfrak{A}_N(l)} \prod_i \alpha_i! < (2N)^l.$$

*Proof.*    It can be seen that

$$\begin{aligned}
\sum_{\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_N) \in \mathfrak{A}_N(l)} \prod_i \alpha_i! &= \sum_{\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_N) \in \mathfrak{A}_N(l)} \alpha_{j_1(\boldsymbol{\alpha})}! \prod_{i \neq j_1(\boldsymbol{\alpha})} \alpha_i! \\
&= \sum_{\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_N) \in \mathfrak{A}_N(l)} \alpha_{j_1(\boldsymbol{\alpha})} (\alpha_{j_1(\boldsymbol{\alpha})} - 1)! \prod_{i \neq j_1(\boldsymbol{\alpha})} \alpha_i! \\
&= \sum_{\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_N) \in \mathfrak{A}_N(l)} \alpha_{j_1(\boldsymbol{\alpha})} \prod_i \phi_i(\boldsymbol{\alpha})! \\
&= \sum_{\boldsymbol{\alpha} \in \mathfrak{A}'_N(l)} \alpha_{j_1(\boldsymbol{\alpha})} \prod_i \phi_i(\boldsymbol{\alpha})! + \sum_{\boldsymbol{\alpha} \in \mathfrak{A}''_N(l)} \alpha_{j_1(\boldsymbol{\alpha})} \prod_i \phi_i(\boldsymbol{\alpha})! \\
&= \sum_{\boldsymbol{\alpha} \in \mathfrak{A}'_N(l)} \alpha_{j_1(\boldsymbol{\alpha})} \prod_i \phi_i(\boldsymbol{\alpha})! + \sum_{\boldsymbol{\alpha} \in \mathfrak{A}''_N(l)} \prod_i \phi_i(\boldsymbol{\alpha})! \\
&\leq l \sum_{\boldsymbol{\beta} \in \mathfrak{A}_N(l-1)} \prod_i \beta_i! + N \sum_{\boldsymbol{\beta} \in \mathfrak{A}_N(l-1)} \prod_i \beta_i! \\
&\leq (l+N) \sum_{\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathfrak{A}_N(l-1)} \prod_i \alpha_i!.
\end{aligned}$$

By induction, we obtain that for $l \leq N$,

$$\begin{aligned}
\sum_{\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_N) \in \mathfrak{A}_N(l-1)} \prod_i \alpha_i! &\leq (l-1+N) \sum_{\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_N) \in \mathfrak{A}_N(l-2)} \prod_i \alpha_i! \\
&\leq (l-1+N)(l-2+N) \sum_{\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_N) \in \mathfrak{A}_N(l-3)} \prod_i \alpha_i! \\
&\leq \cdots \\
&\leq (l-1+N)(l-2+N)\cdots(N+1) \sum_{\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_N) \in \mathfrak{A}_N(0)} \prod_i \alpha_i!
\end{aligned}$$

$$\leq (l-1+N)(l-2+N)\cdots(N+1)$$
$$\leq (2N)^{l-1}.$$

This shows that

$$\sum_{\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_N)\in\mathfrak{A}_N(l)}\prod_i \alpha_i! \leq (l+N)(2N)^{l-1} \leq (2N)^l.$$

This ends the proof of the lemma. $\qquad\square$

**Step 4:** We are now turning to the proof of Corollary 3.1. It follows from Corollary 3.2 that

$$|d_{k+1}(t,\boldsymbol{n})-d_k(t,\boldsymbol{n})| \leq \frac{B^{k+1}4^k(96)^{k\nu}(\kappa^{-\nu}|\boldsymbol{\omega}|t)^k}{k!}\exp(-\frac{\kappa}{8}|\boldsymbol{n}|)\sum_{\boldsymbol{\alpha}\in\mathbb{B}^{(k)}}\prod_j \alpha_j!.$$

Consequently, according to formula (3.1) and Lemma 3.2 with $N=k+1$, $l=k$, we can obtain

$$\sum_{\boldsymbol{\alpha}\in\mathbb{B}^{(k)}}\prod_j \alpha_j! \leq \sum_{\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_N)\in\mathfrak{A}_N(N)}\prod_i \alpha_i! < (2N)^k. \tag{3.4}$$

Moreover, using Stirling's formulae

$$k! \gtrsim k^k e^{-k}, \qquad (k!)^{-1}(2N)^k \lesssim (2e)^k,$$

one has

$$|d_{k+1}(t,\boldsymbol{n})-d_k(t,\boldsymbol{n})| \leq \frac{B^{k+1}4^k(96)^{\nu k}(\kappa^{-\nu}|\boldsymbol{\omega}|t)^k}{k!}\exp(-\frac{\kappa}{8}|\boldsymbol{n}|)\sum_{\boldsymbol{\alpha}\in\mathbb{B}^{(k)}}\prod_j \alpha_j!$$

$$\leq \frac{B^{k+1}4^k(96)^{\nu k}(\kappa^{-\nu}|\boldsymbol{\omega}|t)^k}{k!}\exp(-\frac{\kappa}{8}|\boldsymbol{n}|)(2N)^k$$

$$\leq B^{k+1}(8e)^k(96)^{\nu k}(\kappa^{-\nu}|\boldsymbol{\omega}|t)^k\exp(-\frac{\kappa}{8}|\boldsymbol{n}|).$$

Hence we complete the proof. $\qquad\square$

In view of Corollary 3.1, the infinite series $\sum_{k=1}^\infty (d_{k+1}(t,\boldsymbol{n})-d_k(t,\boldsymbol{n}))$ converges absolutely and uniformly for any $0 \leq t \leq \frac{\kappa^\nu}{32B2^\nu(96)^\nu|\boldsymbol{\omega}|}$.

The following corollary will give uniqueness of the Fourier coefficients $c(t,\boldsymbol{n})$ associated with the ansatz (2.1). Suppose that there could be two solutions for the "good" Boussinesq Equation (1.1) with quasi-periodic initial data (1.2)–(1.3). We will compare the corresponding Fourier coefficients. For this, we need a priori exponential decay estimate for the decay of the coefficients. This is why we invoke the estimation of the sums of the "new variables" $\alpha_j$, which require exponential decay.

We assume that $u, v$ could be two solutions for the "good" Boussinesq Equation (1.1). Moreover suppose that $u, v$ could have the following expansions

$$u(t,x) = \sum_{\boldsymbol{n}\in\mathbb{Z}^\nu} c(t,\boldsymbol{n})\exp(ix\boldsymbol{n}\cdot\boldsymbol{\omega}), \quad v(t,x) = \sum_{\boldsymbol{n}\in\mathbb{Z}^\nu} h(t,\boldsymbol{n})\exp(ix\boldsymbol{n}\cdot\boldsymbol{\omega}).$$

COROLLARY 3.3.    *For some constant $t_0 > 0$, let $c(t, \boldsymbol{n}), h(t, \boldsymbol{n})$ be functions of $t \in [0, t_0)$, $\boldsymbol{n} \in \mathbb{Z}^\nu$ satisfying that for $C_1, \rho > 0$,*

$$|c(t, \boldsymbol{n})| \leq C_1 \exp(-\rho|\boldsymbol{n}|), \quad |h(t, \boldsymbol{n})| \leq C_1 \exp(-\rho|\boldsymbol{n}|) \quad \forall \boldsymbol{n} \in \mathbb{Z}^\nu,$$

*and*

$$c(\boldsymbol{n}) = c(0, \boldsymbol{n}) = h(0, \boldsymbol{n}) = h(\boldsymbol{n}), \quad c'(\boldsymbol{n}) = c'(0, \boldsymbol{n}) = h'(0, \boldsymbol{n}) = h'(\boldsymbol{n}) \quad \forall \boldsymbol{n} \in \mathbb{Z}^\nu.$$

*If we assume*

$$c(t, \boldsymbol{n}) = \frac{1}{2} c(\boldsymbol{n}) \left( \exp(it\lambda) + \exp(-it\lambda) \right) - \frac{i}{2\lambda} c'(\boldsymbol{n}) \left( \exp(it\lambda) - \exp(-it\lambda) \right)$$
$$- \frac{1}{\sqrt{1 + (\boldsymbol{n} \cdot \boldsymbol{\omega})^2}} \sum_{\boldsymbol{m}} \int_0^t \left( \exp(i(\tau - t)\lambda) - \exp(i(t - \tau)\lambda) \right)$$
$$\cdot (i\boldsymbol{m} \cdot \boldsymbol{\omega}) c(\tau, \boldsymbol{m}) c(\tau, \boldsymbol{n} - \boldsymbol{m}) d\tau, \quad \boldsymbol{n} \in \mathbb{Z}^\nu, \qquad (3.5)$$

$$h(t, \boldsymbol{n}) = \frac{1}{2} h(\boldsymbol{n}) \left( \exp(it\lambda) + \exp(-it\lambda) \right) - \frac{i}{2\lambda} h'(\boldsymbol{n}) \left( \exp(it\lambda) - \exp(-it\lambda) \right)$$
$$- \frac{1}{\sqrt{1 + (\boldsymbol{n} \cdot \boldsymbol{\omega})^2}} \sum_{\boldsymbol{m}} \int_0^t \left( \exp(i(\tau - t)\lambda) - \exp(i(t - \tau)\lambda) \right)$$
$$\cdot (i\boldsymbol{m} \cdot \boldsymbol{\omega}) h(\tau, \boldsymbol{m}) h(\tau, \boldsymbol{n} - \boldsymbol{m}) d\tau, \quad \boldsymbol{n} \in \mathbb{Z}^\nu, \qquad (3.6)$$

*where*    $\lambda = \left( (\boldsymbol{n} \cdot \boldsymbol{\omega})^2 + (\boldsymbol{n} \cdot \boldsymbol{\omega})^4 \right)^{\frac{1}{2}}$    *with*    $\boldsymbol{\omega} \in \mathbb{R}^\nu$,    *then*    *there*    *exists*    $t_1 = \min(t_0, \frac{\rho^\nu}{C_1 2^{\nu+1} (288)^\nu |\boldsymbol{\omega}|})$ *such that for all $\boldsymbol{n} \in \mathbb{Z}^\nu$, $0 < t \leq t_1$, one has $c(t, \boldsymbol{n}) = h(t, \boldsymbol{n})$.*

   *Proof.*    The proof will be divided into the following three steps.

   **Step 1:**    Let us consider the upper bounds on $|h(t, \boldsymbol{n}) - c(t, \boldsymbol{n})|$ with respect to the variables $\boldsymbol{m}_j$.

LEMMA 3.3.    *Let $c(t, \boldsymbol{n}), h(t, \boldsymbol{n})$ be as seen in Corollary 3.3. There is a positive constant $C_1$ such that for $k = 1, 2, \cdots$,*

$$|h(t, \boldsymbol{n}) - c(t, \boldsymbol{n})|$$
$$\leq \frac{2(C_1)^{k+1} (|\boldsymbol{\omega}| t)^k}{k!} \sum_{\substack{\boldsymbol{m} = (\boldsymbol{m}_1, \cdots, \boldsymbol{m}_{k+1}) \in \mathbb{Z}^{(k+1)\nu} \\ \sum_j \boldsymbol{m}_j = \boldsymbol{n}}} \sum_{\boldsymbol{\alpha} \in \mathbb{B}^{(k)}} \prod_j |\boldsymbol{m}_j|^{\alpha_j} \exp(-\rho|\boldsymbol{m}_j|). \qquad (3.7)$$

   *Proof.*    Obviously, formulas (3.5)–(3.6) can be rewritten as

$$c(t, \boldsymbol{n}) = \frac{1}{2} c(\boldsymbol{n}) \left( \exp(it\lambda) + \exp(-it\lambda) \right) - \frac{i}{2\lambda} c'(\boldsymbol{n}) \frac{-i}{\lambda} \left( \exp(it\lambda) - \exp(-it\lambda) \right)$$
$$- \frac{i\boldsymbol{n} \cdot \boldsymbol{\omega}}{2\sqrt{1 + (\boldsymbol{n} \cdot \boldsymbol{\omega})^2}} \int_0^t \left( \exp(i(\tau - t)\lambda) - \exp(i(t - \tau)\lambda) \right)$$
$$\cdot \sum_{\substack{\boldsymbol{m}_1, \boldsymbol{m}_2 \in \mathbb{Z}^\nu \\ \boldsymbol{m}_1 + \boldsymbol{m}_2 = \boldsymbol{n}}} c(\tau, \boldsymbol{m}_1) c(\tau, \boldsymbol{m}_2) d\tau, \quad \boldsymbol{n} \in \mathbb{Z}^\nu, \qquad (3.8)$$

$$h(t, \boldsymbol{n}) = \frac{1}{2} h(\boldsymbol{n}) \left( \exp(it\lambda) + \exp(-it\lambda) \right) - \frac{i}{2\lambda} h'(\boldsymbol{n}) \frac{-i}{\lambda} \left( \exp(it\lambda) - \exp(-it\lambda) \right)$$

$$-\frac{\mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\omega}}{2\sqrt{1+(\boldsymbol{n}\cdot\boldsymbol{\omega})^2}}\int_0^t\left(\exp\left(\mathrm{i}(\tau-t)\lambda\right)-\exp\left(\mathrm{i}(t-\tau)\lambda\right)\right)$$

$$\cdot\sum_{\substack{\boldsymbol{m}_1,\boldsymbol{m}_2\in\mathbb{Z}^\nu\\\boldsymbol{m}_1+\boldsymbol{m}_2=\boldsymbol{n}}}h(\tau,\boldsymbol{m}_1)h(\tau,\boldsymbol{m}_2)\mathrm{d}\tau,\quad\boldsymbol{n}\in\mathbb{Z}^\nu.\qquad(3.9)$$

The difference between (3.8) and (3.9) is bounded from above by

$$|h(t,\boldsymbol{n})-c(t,\boldsymbol{n})|\le|\boldsymbol{n}||\boldsymbol{\omega}|\sum_{\substack{\boldsymbol{m}_1,\boldsymbol{m}_2\in\mathbb{Z}^\nu\\\boldsymbol{m}_1+\boldsymbol{m}_2=\boldsymbol{n}}}\int_0^t|h(\tau,\boldsymbol{m}_1)h(\tau,\boldsymbol{m}_2)-c(\tau,\boldsymbol{m}_1)c(\tau,\boldsymbol{m}_2)|\mathrm{d}\tau$$

$$\le|\boldsymbol{n}||\boldsymbol{\omega}|\sum_{\substack{\boldsymbol{m}_1,\boldsymbol{m}_2\in\mathbb{Z}^\nu\\\boldsymbol{m}_1+\boldsymbol{m}_2=\boldsymbol{n}}}\int_0^t(|h(\tau,\boldsymbol{m}_1)||h(\tau,\boldsymbol{m}_2)|+|c(\tau,\boldsymbol{m}_1)||c(\tau,\boldsymbol{m}_2)|)\mathrm{d}\tau$$

$$\le|\boldsymbol{n}||\boldsymbol{\omega}|\sum_{\substack{\boldsymbol{m}_1,\boldsymbol{m}_2\in\mathbb{Z}^\nu\\\boldsymbol{m}_1+\boldsymbol{m}_2=\boldsymbol{n}}}\int_0^t((2C_1)^2\exp(-\rho(|\boldsymbol{m}_1|+|\boldsymbol{m}_2|)))\mathrm{d}\tau$$

$$\le2(C_1)^2t|\boldsymbol{\omega}|\sum_{\substack{\boldsymbol{m}_1,\boldsymbol{m}_2\in\mathbb{Z}^\nu\\\boldsymbol{m}_1+\boldsymbol{m}_2=\boldsymbol{n}}}|\sum_j\boldsymbol{m}_j|\exp(-\rho(|\boldsymbol{m}_1|+|\boldsymbol{m}_2|)).$$

Hence (3.7) holds for $k=1$.

We prove (3.7) by induction. Suppose that (3.7) could hold for $k-1$. Observe that

$$|h(t,\boldsymbol{n})-c(t,\boldsymbol{n})|\le|\boldsymbol{n}||\boldsymbol{\omega}|\sum_{\substack{\boldsymbol{m}_1,\boldsymbol{m}_2\in\mathbb{Z}^\nu\\\boldsymbol{m}_1+\boldsymbol{m}_2=\boldsymbol{n}}}\int_0^t|h(\tau,\boldsymbol{m}_1)h(\tau,\boldsymbol{m}_2)-c(\tau,\boldsymbol{m}_1)c(\tau,\boldsymbol{m}_2)|\mathrm{d}\tau$$

$$\le|\boldsymbol{n}||\boldsymbol{\omega}|\sum_{\substack{\boldsymbol{m}_1,\boldsymbol{m}_2\in\mathbb{Z}^\nu\\\boldsymbol{m}_1+\boldsymbol{m}_2=\boldsymbol{n}}}\int_0^t(|h(\tau,\boldsymbol{m}_1)-c(\tau,\boldsymbol{m}_1)||h(\tau,\boldsymbol{m}_2)|$$

$$+|h(\tau,\boldsymbol{m}_2)-c(\tau,\boldsymbol{m}_2)||c(\tau,\boldsymbol{m}_1)|)\mathrm{d}\tau.$$

By using the inductive assumption, we have

$$|\boldsymbol{n}||\boldsymbol{\omega}|\int_0^t\sum_{\substack{\boldsymbol{n}_1,\boldsymbol{n}_2\in\mathbb{Z}^\nu\\\boldsymbol{n}_1+\boldsymbol{n}_2=\boldsymbol{n}}}|h(\tau,\boldsymbol{n}_1)-c(\tau,\boldsymbol{n}_1)||h(\tau,\boldsymbol{n}_2)|\mathrm{d}\tau$$

$$\le|\boldsymbol{\omega}|\int_0^t\sum_{\substack{\boldsymbol{n}_1,\boldsymbol{n}_2\in\mathbb{Z}^\nu\\\boldsymbol{n}_1+\boldsymbol{n}_2=\boldsymbol{n}}}\frac{(C_1)^k(|\boldsymbol{\omega}|\tau)^{k-1}}{(k-1)!}\sum_{\substack{\boldsymbol{m}=(\boldsymbol{m}_1,\cdots,\boldsymbol{m}_k)\in\mathbb{Z}^{k\nu}\\\sum_j\boldsymbol{m}_j=\boldsymbol{n}_1}}|(\sum_j\boldsymbol{m}_j)+\boldsymbol{n}_2|\mathrm{d}\tau$$

$$\cdot\sum_{\boldsymbol{\alpha}\in\mathbb{B}^{(k-1)}}\prod_j|\boldsymbol{m}_j|^{\alpha_j}\exp(-\rho|\boldsymbol{m}_j|)(C_1\exp(-\rho|\boldsymbol{n}_2|))$$

$$\le\frac{(C_1)^{k+1}(|\boldsymbol{\omega}|t)^k}{k!}\sum_{\substack{\boldsymbol{m}=(\boldsymbol{m}_1,\cdots,\boldsymbol{m}_{k+1})\in\mathbb{Z}^{(k+1)\nu}\\\sum_j\boldsymbol{m}_j=\boldsymbol{n}}}\sum_{\boldsymbol{\alpha}\in\mathbb{B}^{(k)}}\prod_j|\boldsymbol{m}_j|^{\alpha_j}\exp(-\rho|\boldsymbol{m}_j|).$$

Similarly, one has

$$|\boldsymbol{n}||\boldsymbol{\omega}|\int_0^t\sum_{\substack{\boldsymbol{n}_1,\boldsymbol{n}_2\in\mathbb{Z}^\nu\\\boldsymbol{n}_1+\boldsymbol{n}_2=\boldsymbol{n}}}|h(\tau,\boldsymbol{n}_2)-c(\tau,\boldsymbol{n}_2)||c(\tau,\boldsymbol{n}_2)|\mathrm{d}\tau$$

$$\leq \frac{(C_1)^{k+1}(|\boldsymbol{\omega}|t)^k}{k!} \sum_{\substack{\boldsymbol{m}=(\boldsymbol{m}_1,\cdots,\boldsymbol{m}_{k+1})\in\mathbb{Z}^{(k+1)\nu} \\ \sum_j \boldsymbol{m}_j=\boldsymbol{n}}} \sum_{\boldsymbol{\alpha}\in\mathbb{B}^{(k)}} \prod_j |\boldsymbol{m}_j|^{\alpha_j}\exp(-\rho|\boldsymbol{m}_j|).$$

Thus we can get (3.7). Consequently, we arrive at the conclusion of the lemma. □

**Step 2:** Our next goal is to give an estimation of $|h(t,\boldsymbol{n})-c(t,\boldsymbol{n})|$ with respect to the variables $\alpha_j$.

COROLLARY 3.4. *Let $c(t,\boldsymbol{n}),h(t,\boldsymbol{n})$ be given in Corollary 3.3. There exists a positive constant $C_1$ such that for $k=1,2,\cdots,$*

$$|h(t,\boldsymbol{n})-c(t,\boldsymbol{n})| \leq \frac{2(C_1)^{k+1}(288)^{k\nu}(\rho^{-\nu}|\boldsymbol{\omega}|t)^k}{k!} \sum_{\boldsymbol{\alpha}\in\mathbb{B}^{(k)}} \prod_j \alpha_j!.$$

*Proof.* It follows from Lemma 3.3 that

$$|h(t,\boldsymbol{n})-c(t,\boldsymbol{n})| \leq \frac{2(C_1)^{k+1}(|\boldsymbol{\omega}|t)^k}{k!} \sum_{\substack{\boldsymbol{m}=(\boldsymbol{m}_1,\cdots,\boldsymbol{m}_{k+1})\in\mathbb{Z}^{(k+1)\nu} \\ \sum_j \boldsymbol{m}_j=\boldsymbol{n}}} \sum_{\boldsymbol{\alpha}\in\mathbb{B}^{(k)}} \prod_j |\boldsymbol{m}_j|^{\alpha_j}\exp(-\rho|\boldsymbol{m}_j|)$$

$$\leq \frac{2(C_1)^{k+1}(|\boldsymbol{\omega}|t)^k}{k!} \sum_{\substack{\boldsymbol{m}=(\boldsymbol{m}_1,\cdots,\boldsymbol{m}_{k+1})\in\mathbb{Z}^{(k+1)\nu} \\ \sum_j \boldsymbol{m}_j=\boldsymbol{n}}} \prod_j \exp(-\frac{\rho}{2}|\boldsymbol{m}_j|)$$

$$\times \sum_{\boldsymbol{\alpha}\in\mathbb{B}^{(k)}} \prod_j |\boldsymbol{m}_j|^{\alpha_j}\exp(-\frac{\rho}{2}|\boldsymbol{m}_j|).$$

Combining this with the proving procedure of Lemma 2.6, we obtain the conclusion of the lemma. □

**Step 3:** Finally, combining Corollary 3.4 with formula (3.4) yields that

$$|h(t,\boldsymbol{n})-c(t,\boldsymbol{n})| \leq \frac{2(C_1)^{k+1}(288^\nu\rho^{-\nu}|\boldsymbol{\omega}|t)^k}{k!}(2N)^k$$

with $N=k+1$. Due to Stirling's formulae

$$k! \gtrsim k^k e^{-k}, \qquad (k!)^{-1}(2N)^k \lesssim (2e)^k,$$

for $0<t\leq\min(t_0,\frac{\rho^\nu}{C_1 2^{\nu+1}(288)^\nu|\boldsymbol{\omega}|})$, we have

$$\lim_{k\to\infty} \frac{2(C_1)^{k+1}(288^\nu\rho^{-\nu}|\boldsymbol{\omega}|t)^k}{k!}(2N)^k = 0.$$

This implies that $c(t,\boldsymbol{n})=h(t,\boldsymbol{n})$ for all $\boldsymbol{n}\in\mathbb{Z}^\nu$ and $0<t\leq t_1$.

Hence we complete the proof of Corollary 3.3. □

## 4. Proof of the main results

The remainder of this paper is to give the proof of the main results.

*Proof.* (**Proof of Theorem 1.1.**) We first show the existence of local solutions for the "good" Boussinesq Equation (1.1) subject to quasi-periodic initial data (1.2)–(1.3).

_Existence_. It follows from Corollary 3.1 that for all $0 \leq t \leq \frac{\kappa^\nu}{32B(192)^\nu|\boldsymbol{\omega}|}$ and $\boldsymbol{n} \in \mathbb{Z}^\nu$, the following limit

$$d^{(0)}(t,\boldsymbol{n}) = \lim_{k \to \infty} d_k(t,\boldsymbol{n}) = \lim_{k \to \infty} c_{k-1}(t,\boldsymbol{n})$$

exists with

$$|d^{(0)}(t,\boldsymbol{n}) - d_{k-1}(t,\boldsymbol{n})| \leq B^{k+1}(8e)^k(96)^{k\nu}(\kappa^{-\nu}|\boldsymbol{\omega}|t)^k \exp(-\frac{\kappa}{8}|\boldsymbol{n}|).$$

Moreover, using Corollary 2.1 yields that

$$|d^{(0)}(t,\boldsymbol{n})| \leq 2B \exp(-\frac{\kappa|\boldsymbol{n}|}{4}).$$

Based on the above estimations, $d^{(0)}(t,\boldsymbol{n})$ satisfies the following system coming from (2.9)

$$d^{(0)}(t,\boldsymbol{n}) = \frac{1}{2}c(\boldsymbol{n})(\exp(\mathrm{i}t\lambda) + \exp(-\mathrm{i}t\lambda)) - \frac{\mathrm{i}}{2\lambda}c'(\boldsymbol{n})(\exp(\mathrm{i}t\lambda) - \exp(-\mathrm{i}t\lambda))$$

$$- \frac{\mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\omega}}{2\sqrt{1+(\boldsymbol{n}\cdot\boldsymbol{\omega})^2}} \int_0^t (\exp(\mathrm{i}(\tau-t)\lambda) - \exp(\mathrm{i}(t-\tau)\lambda))$$

$$\cdot \sum_{\substack{\boldsymbol{m}_1,\boldsymbol{m}_2 \in \mathbb{Z}^\nu \\ \boldsymbol{m}_1+\boldsymbol{m}_2=\boldsymbol{n}}} d^{(0)}(\tau,\boldsymbol{m}_1)d^{(0)}(\tau,\boldsymbol{m}_2)\mathrm{d}\tau.$$

Due to Lemma 2.2, the function

$$u(t,x) = \sum_{\boldsymbol{n} \in \mathbb{Z}^\nu} d^{(0)}(t,\boldsymbol{n})\exp(\mathrm{i}x\boldsymbol{n}\cdot\boldsymbol{\omega})$$

satisfies the "good" Boussinesq Equation (1.1) with quasi-periodic initial conditions (1.2)–(1.3).

It remains to prove uniqueness of local solutions for the "good" Boussinesq Equation (1.1) subject to quasi-periodic initial data (1.2)–(1.3).

_Uniqueness_. Let $u,v$ be two local solutions for the "good" Boussinesq Equation (1.1) subject to quasi-periodic initial data (1.2)–(1.3). Namely, both $u$ and $v$ satisfy that for $0 \leq t \leq t_0$, $x \in \mathbb{R}$,

$$\partial_t^2 u + \partial_x^4 u - \partial_x^2 u - \partial_x^2(u^2) = 0, \quad \partial_t^2 v + \partial_x^4 v - \partial_x^2 v - \partial_x^2(v^2) = 0$$

with

$$u(0,x) = v(0,x), \quad \partial_t u(0,x) = \partial_t v(0,x) \quad \forall x \in \mathbb{R}.$$

Moreover, $u,v$ have the following expansions

$$u(t,x) = \sum_{\boldsymbol{n} \in Z^\nu} c(t,\boldsymbol{n})\exp(\mathrm{i}x\boldsymbol{n}\cdot\boldsymbol{\omega}), \quad v(t,x) = \sum_{\boldsymbol{n} \in Z^\nu} h(t,\boldsymbol{n})\exp(\mathrm{i}x\boldsymbol{n}\cdot\boldsymbol{\omega}),$$

where the Fourier coefficients $|c(t,\boldsymbol{n})|$, $|h(t,\boldsymbol{n})|$ satisfy that for some constants $C_1 > 0, \rho > 0$,

$$|c(t,\boldsymbol{n})| \leq C_1 \exp(-\rho|\boldsymbol{n}|), \quad |h(t,\boldsymbol{n})| \leq C_1 \exp(-\rho|\boldsymbol{n}|), \quad \boldsymbol{n} \in \mathbb{Z}^\nu.$$

From Lemma 2.2, we have equations (3.5)–(3.6). Then $c(t, \boldsymbol{n})$ and $h(t, \boldsymbol{n})$ obey the conditions of Corollary 3.3. In view of Corollary 3.3, one has $u(t, x) = v(t, x)$ for $0 < t \leq \min(t_0, \frac{\rho^\nu}{C_1 2^{\nu+1} (288)^\nu |\boldsymbol{\omega}|})$ and $x \in \mathbb{R}$.

Hence we have completed the proof of Theorem 1.1.                                        □

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