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A WASSERSTEIN NORM FOR SIGNED MEASURES, WITH APPLICATION TO NON-LOCAL TRANSPORT EQUATION WITH SOURCE TERM*

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Abstract. We introduce an optimal transportation interpretation of the Kantorovich norm on the space of signed Radon measures with finite mass, based on the generalized Wasserstein distance for measures with different masses. With this new interpretation, we obtain new topological properties for this norm. We use these tools to prove existence and uniqueness for solutions to non-local transport equations with source terms, when the initial condition is a signed measure.

Keywords. Wasserstein distance; Transport equation; Signed measures; Kantorovich duality.

AMS subject classifications. 28A33; 35A01.

1. Introduction

The problem of optimal transportation, also called Monge-Kantorovich problem, has been intensively studied in the mathematical community. Related to this problem, Wasserstein distances in the space of probability measures have revealed to be powerful tools, in particular for dealing with dynamics of measures like the transport Partial Differential Equation (PDE in the following), see e.g. [2,3]. For a complete introduction to Wasserstein distances, see [27,28].

The main limit of this approach, at least for its application to dynamics of measures, is that the Wasserstein distances $W_p(\mu,\nu)$ $(p\geq 1)$ are defined only if the two positive measures μ,ν have the same mass. For this reason, the generalized Wasserstein distances $W_p^{a,b}(\mu,\nu)$ are introduced in [24,25]: they combine the standard Wasserstein and total variation distances. In rough words, for $W_p^{a,b}(\mu,\nu)$ an infinitesimal mass $\delta\mu$ of μ can either be removed at cost $a|\delta\mu|$, or moved from μ to ν at cost $bW_p(\delta\mu,\delta\nu)$. An optimal transportation problem between densities with different masses has been studied in [9,15], where only a given fraction m of each density is transported. These works were motivated by a modeling issue: Using the example of a resource that is extracted and that we want to distribute in factories, one aims to use only a certain given fraction of production and consumption capacity. In this approach and contrarily to the generalized Wasserstein distance [23], the mass that is leftover has no impact on the distance between the measures μ and ν. In another context, for the purpose of interpreting some reaction-diffusion equations not preserving masses as gradient flows, the authors of [16] define the distance Wb_2 between measures with different masses on a bounded domain. Further generalizations for positive measures with different masses, based on the Wasserstein distance and its Benamou-Brenier formulation, are introduced in [11,19,20]. See [10] for a unifying framewok for unbalanced optimal transport.

Such generalizations still have a drawback: both measures need to be positive. In the present paper we introduce a norm, parametrized by two positive numbers (a,b), on the space of signed Radon measures with finite mass on $\mathbb{R}^d$. Such norm, based

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on an optimal transportation approach, induces a distance generalizing the Wasserstein distance to signed measures. We then prove that for $(a,b)=(1,1)$ this norm corresponds to the extension of the so-called Kantorovich distance or Bounded-Lipschitz norm (BL norm) for finite signed Radon measures presented in [17] in the dual form

$$\|\mu\|_{BL}=\sup_{\|f\|_\infty\leq 1,\,\|f\|_{Lip}\leq 1}\int_{\mathbb{R}^d} f\,d\mu. \tag{1.1}$$

This formulation, already given in [17], enables us to propose new proofs for the main topological properties and characterizations of the BL norm.

The main contribution of the paper is the statement of Theorem (1.1) and lies in the use of the (a,b) norm to guarantee well-posedness of the following nonlocal transport equation

$$\partial_t\mu_t(x)+\mathrm{div}\,(v[\mu_t](x)\mu_t(x))=h[\mu_t](x),\qquad \mu_{|t=0}(x)=\mu_0(x), \tag{1.2}$$

for $x\in\mathbb{R}^d$ and $\mu_0\in\mathcal{M}^s(\mathbb{R}^d)$, $h[\mu]\in\mathcal{M}^s(\mathbb{R}^d)$, where $\mathcal{M}^s(\mathbb{R}^d)$ is the space of signed Radon measures with finite mass on $\mathbb{R}^d$. We recall some instances of (1.2):

- The equation has already been studied in the framework of positive measures, for modeling several different phenomena; see a review in [26]. In particular:
  - in crowd modeling, the presence of negative sources allows to describe pedestrians exiting a door [26];
  - in opinion dynamics, it allows to describe increase/decrease of populations associated to a given opinion or role, such as in the leader-follower models proposed in [1, 12];
  - in mathematical biology, it models morphogen and tissue dynamics. Even though in [22] the dynamics is based on a second-order operator, methods based on optimal transportation can be adapted to this setting too, see e.g. [13].
- Again for dynamics of positive measures, it is still useful to generalize to signed measures, either for computations around steady states or for approximations of the dynamics in which (small) negative masses can appear, see e.g. [24].
- Signed measures appear in a model coming from the hydrodynamic equations of Ginzburg-Landau vortices, where the vortex density μ_t (which can be positive or negative depending on the local topological degree) in a two-dimensional domain occupied by a superconducting sample inducing the magnetic field $v[\mu_t]$ satisfies (1.2) (see [4, 21]) with

  $$\begin{cases} h[\mu]=0,\\ v[\mu]=-(\nabla\Delta^{-1}\mu)(x)n(x),\end{cases}$$

  where n is the Radon-Nikodym derivative of $|\mu|$ with respect to μ.
- Another motivation to study Equation (1.2) in the framework of signed measures is the interpretation of μ_t as the spatial derivative of the entropy solution $\rho(x,t)$ to a scalar conservation law. A link between scalar conservation laws and nonlocal transport equation has been initiated in [6, 18], but until now, studies are restricted to convex fluxes and monotonous initial conditions, so that the spatial derivative μ_t is a positive measure for all $t>0$.

To deal with generic scalar conservation laws, one needs a space of signed measures equipped with a metric of Wasserstein type, see e.g. [5].

Formally, the spatial derivative μ_t of the one-dimensional scalar conservation law associated with flux $f \in \mathcal{C}^1(\mathbb{R})$ satisfies (1.2) with

$$\begin{cases} h[\mu] = 0, \\ v[\mu] = f'\left(\int_{(-\infty,x]} d\mu(s)\right). \end{cases}$$

Motivated by Ginzburg-Landau vortices, the authors of [4] suggested to extend the usual Wasserstein distance W_1 to the couples of signed measures $\mu = \mu^+ - \mu^-$ and $\nu = \nu^+ - \nu^-$ such that $|\mu^+| + |\nu^-| = |\mu^-| + |\nu^+|$ by the formula $\mathbb{W}_1(\mu,\nu) = W_1(\mu^+ + \nu^-, \mu^- + \nu^+)$. This procedure fails for $p \neq 1$, since triangular inequality is lost. A counter-example to the triangular inequality is provided in [4] for $d = 1$ and $p = 2$: Taking $\mu = \delta_0$, $\nu = \delta_4$, $\eta = \delta_1 - \delta_2 + \delta_3$, we obtain $\mathbb{W}_2(\mu,\nu) = 4$ whereas $\mathbb{W}_2(\mu,\eta) + \mathbb{W}_2(\eta,\nu) = \sqrt{2} + \sqrt{2}$.

We use the same trick from [4] to turn the generalized Wasserstein distance $W_1^{a,b}$ into a norm for signed measures, by setting

$$\|\mu\|^{a,b} := W_1^{a,b}(\mu^+,\mu^-) = \inf_{\substack{\tilde{\eta},\tilde{\nu} \in \mathcal{M}(\mathbb{R}^d) \\ |\tilde{\eta}| = |\tilde{\nu}|}} \left(a \left(|\mu^+ - \tilde{\eta}| + |\mu^- - \tilde{\nu}| \right) + b W_1(\tilde{\eta},\tilde{\nu}) \right),$$

where μ^+, μ^- are any positive finite measures such that $\mu = \mu^+ - \mu^-$, and where $|\mu|$ is the total variation of μ. For the reason mentioned above, this construction only defines a norm for $p = 1$.

Notice that we need to restrict ourselves to Radon measures μ with finite mass. The regularity assumptions on the vector field and on the source term are the following:

(H-1) There exists K such that for all $\mu,\nu \in \mathcal{M}^s(\mathbb{R}^d)$ it holds

$$\|v[\mu] - v[\nu]\|_{\mathcal{C}^0(\mathbb{R}^d)} \le K \|\mu - \nu\|^{a,b}. \tag{1.3}$$

(H-2) There exist L, M such that for all $x,y \in \mathbb{R}^d$, for all $\mu \in \mathcal{M}^s(\mathbb{R}^d)$ it holds

$$|v[\mu](x) - v[\mu](y)| \le L|x-y|, \qquad |v[\mu](x)| \le M. \tag{1.4}$$

(H-3) There exist Q, P, R such that for all $\mu,\nu \in \mathcal{M}^s(\mathbb{R}^d)$ it holds

$$\|h[\mu] - h[\nu]\|^{a,b} \le Q\|\mu - \nu\|^{a,b}, \qquad |h[\mu]| \le P, \qquad \operatorname{supp}(h[\mu]) \subset B_0(R). \tag{1.5}$$

The main result of the paper is the following.

Theorem 1.1 (Existence, uniqueness and stability in $(\mathcal{M}^s(\mathbb{R}^d), \|.\|^{a,b})$). *Let v and h satisfy (H-1)-(H-2)-(H-3) and $\mu_0 \in \mathcal{M}^s(\mathbb{R}^d)$ compactly supported be given. Then, there exists a unique distributional solution to (1.2). In addition, for μ_0 and ν_0 in $\mathcal{M}^s(\mathbb{R}^d)$, denoting by μ_t and ν_t the corresponding solutions, we have the following property for $t \in [0,1]$ of continuous dependence with respect to initial data:*

$$\|\mu_t - \nu_t\|^{a,b} \le \|\mu_0 - \nu_0\|^{a,b} \exp(K_1 t), \qquad K_1 = 2L + 2bK(P + \min\{|\mu_0|, |\nu_0|\}) + Q,$$

the following estimates on the mass and support:

$$|\mu_t| \le |\mu_0| + Pt, \quad supp\{\mu_t\} \subset B(0, R' + tM) \ for\ R'\ s.\ t.\ (supp\{\mu_0\} \cup B_0(R)) \subset B_0(R').$$

Moreover, the solution is Lipschitz in time:

$$\|\mu_{t+\tau}-\mu_t\|^{a,b}\leq K_2\tau,\quad K_2=aP+bM(P+|\mu_0|),\quad \tau\geq 0,\qquad t+\tau\leq 1.$$

REMARK 1.1. We emphasize that the assumptions (H-1)-(H-2) are incompatible with a direct interpretation of the solution of (1.2) as the spatial derivative of a conservation law and need to be relaxed in a future work. Indeed, to draw a parallel between conservation laws and non-local equations, discontinuous vector fields need to be considered.

The structure of the article is the following. In Section 2, we state and prove results of measure theory, which are needed for the rest of the paper. We also recall the definition of generalized Wasserstein distance. In Section 3, we to define the norm $\|\cdot\|^{a,b}$ on the space of signed Radon measures with finite mass, state a Kantorovich-Rubinstain type duality for this norm, and we end Section 3 by proving some topological properties for this norm. Section 4 is devoted the proof of Theorem 1.1.

2. Generalized Wasserstein distance

In this section, we introduce the notations and state preliminary results. Throughout the paper, $\mathcal{B}(\mathbb{R}^d)$ is the space of Borel sets on $\mathbb{R}^d$, $\mathcal{M}(\mathbb{R}^d)$ is the space of Radon measures with finite mass (i.e. Borel regular, positive, and finite on every set). We denote with $\mathcal{M}^s(\mathbb{R}^d)$ the space of such signed Radon measures, that are measures μ that can be written as $\mu=\mu_+-\mu_-$ with $\mu_+,\mu_-\in\mathcal{M}(\mathbb{R}^d)$. For $\mu\in\mathcal{M}^s(\mathbb{R}^d)$, we define $|\mu|=|\mu_+^J|+|\mu_-^J|$ where (μ_+^J,μ_-^J) is the unique Jordan decomposition of μ, i.e. $\mu=\mu_+^J-\mu_-^J$ with $\mu_+^J\perp\mu_-^J$. Observe that $|\mu|$ is always finite, since $\mu_+^J,\mu_-^J\in\mathcal{M}(\mathbb{R}^d)$.

For a sequence of probability measures, different notions of weak convergences are equivalent. It is not the case for signed measures and we precise here what we call narrow and vague convergence. In the present paper, $\mathcal{C}^0(\mathbb{R}^d;\mathbb{R})$ is the set of continuous functions, $\mathcal{C}_b^0(\mathbb{R}^d;\mathbb{R})$ is the set of bounded continuous functions, and $\mathcal{C}_c^0(\mathbb{R}^d;\mathbb{R})$ is the set of continuous functions with compact support on $\mathbb{R}^d$.

DEFINITION 2.1 (Narrow and vague convergence for signed measures).

- *A sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures in $\mathcal{M}^s(\mathbb{R}^d)$ is said to converge narrowly to μ if the following holds: for all $\varphi\in C_b^0(\mathbb{R}^d;\mathbb{R})$, $\int_{\mathbb{R}^d}\varphi(x)d\mu_n(x)\to\int_{\mathbb{R}^d}\varphi(x)d\mu(x)$.*
- *A sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures in $\mathcal{M}^s(\mathbb{R}^d)$ is said to converge vaguely to μ if the following holds: for all $\varphi\in C_c^0(\mathbb{R}^d;\mathbb{R})$, $\int_{\mathbb{R}^d}\varphi(x)d\mu_n(x)\to\int_{\mathbb{R}^d}\varphi(x)d\mu(x)$.*

Notice that in [14], vague convergence is called weak convergence. In [17, 27] however, weak convergence refers to what we define here as narrow convergence. Notice that if a sequence of positive measures μ_n converges vaguely to μ and if $(\mu_n)_n$ is tight, then μ_n converges narrowly to μ.

We now recall definition and key properties of the standard and generalized Wasserstein distance. For more details on these topics, see [25, 27].

DEFINITION 2.2 (Transference plan). *A transference plan between two positive measures μ and ν of same mass is a measure $\pi\in\mathcal{M}(\mathbb{R}^d,\mathbb{R}^d)$ which satisfies for all $A,B\in\mathcal{B}(\mathbb{R}^d)$*

$$\pi(A\times\mathbb{R}^d)=\mu(A),\quad \pi(\mathbb{R}^d\times B)=\nu(B).$$

Note that transference plans are *not* probability measures in general, as their mass is $|\mu|=|\nu|$, the common mass of both marginals. We denote by $\Pi(\mu,\nu)$ the set of transference plans between μ and ν. The p-Wasserstein distance for positive Radon measures of same mass is defined for $p\geq 1$ as

$$W_p(\mu,\nu)=\left(\min_{\pi\in\Pi(\mu,\nu)}\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^p d\pi(x,y)\right)^{\frac{1}{p}}.$$

It was extended to positive measures having possibly different mass in [24,25], where the authors introduce the distance $W_p^{a,b}$ on the space $\mathcal{M}(\mathbb{R}^d)$ of Radon measures with finite mass. The formal definition is the following.

DEFINITION 2.3 (Generalized Wasserstein distance [24]). *Let μ,ν be two positive measures in $\mathcal{M}(\mathbb{R}^d)$. The generalized Wasserstein distance between μ and ν is given for $p\geq 1$, $a>0$ and $b>0$ by*

$$W_p^{a,b}(\mu,\nu)=\left(\inf_{\substack{\tilde\mu,\tilde\nu\in\mathcal{M}(\mathbb{R}^d)\\|\tilde\mu|=|\tilde\nu|}} a^p(|\mu-\tilde\mu|+|\nu-\tilde\nu|)^p+b^pW_p^p(\tilde\mu,\tilde\nu)\right)^{1/p}. \tag{2.1}$$

PROPOSITION 2.1 (Scaling and dilation formulae for the generalized Wasserstein distance). *Consider $a>0$ and $b>0$ and let μ,ν be two measures.*

(1) *The following scaling formula holds for $p\geq 1$*

$$W_p^{\lambda a,\lambda b}(\mu,\nu)=\lambda W_p^{a,b}(\mu,\nu),\quad \lambda>0. \tag{2.2}$$

(2) *Define $D_\ell: x\to \ell x$ with $\ell>0$ the dilation in $\mathbb{R}^n$, then it holds*

$$W_1^{a,b}(D_\ell\#\mu,D_\ell\#\nu)=W_1^{a,\ell b}(\mu,\nu). \tag{2.3}$$

Proof. The first statement is directly deduced from Definition 2.3. For the second statement, define

$$C^{a,b}(\bar\mu,\bar\nu,\pi;\mu,\nu):=a(|\mu-\bar\mu|+|\nu-\bar\nu|)+b\int|x-y|\,d\pi(x,y),$$

where π is a transference plan in $\Pi(\bar\mu,\bar\nu)$. It holds

$$\begin{aligned}&C^{a,b}(D_\ell\#\bar\mu,D_\ell\#\bar\nu,(D_\ell\times D_\ell)\#\pi;D_\ell\#\mu,D_\ell\#\nu)\\&=a(|D_\ell\#\mu-D_\ell\#\bar\mu|+|D_\ell\#\nu-D_\ell\#\bar\nu|)+b\int|x-y|\,d(D_\ell\times D_\ell)\pi(x,y),\\&=a(|\mu-\bar\mu|+|\nu-\bar\nu|)+b\int|\ell x-\ell y|\,d\pi(x,y)=C^{a,\ell b}(\bar\mu,\bar\nu,\pi;\mu,\nu).\end{aligned}$$

As a consequence, (2.3) holds. □

Notice that the first statement of Proposition 2.1 implies in particular that

$$W_p^{a,b}=\frac{b}{b'}W_p^{a',b'},\qquad \text{for } \frac{a}{b}=\frac{a'}{b'}. \tag{2.4}$$

The following lemma is useful to derive properties for the generalized Wasserstein distance.

LEMMA 2.1. *The infimum in (2.1) is always attained. Moreover, there always exists a minimizer that satisfies the additional constraint $\tilde\mu \le \mu$, $\tilde\nu \le \nu$.*

The proof can be found in [24].

For $f \in \mathcal{C}_c^0(\mathbb{R}^d;\mathbb{R})$, we define

$$\|f\|_\infty = \sup_{x\in\mathbb{R}^d} |f(x)|, \qquad \|f\|_{Lip} = \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|}.$$

We also denote by $\mathcal{C}_c^{0,Lip}(\mathbb{R}^d;\mathbb{R})$ the subset of functions $f \in \mathcal{C}_c^0(\mathbb{R}^d;\mathbb{R})$ for which it holds $\|f\|_{Lip} < +\infty$.

The following result is stated in [25, Theorem 13].

LEMMA 2.2 (Kantorovitch-Rubinstein duality for $W_1^{1,1}$). *For μ,ν in $\mathcal{M}(\mathbb{R}^d)$, it holds*

$$W_1^{1,1}(\mu,\nu) = \sup\left\{\int_{\mathbb{R}^d} \varphi\, d(\mu-\nu);\ \varphi \in \mathcal{C}_c^{0,Lip}, \|\varphi\|_\infty \le 1, \|\varphi\|_{Lip} \le 1\right\}.$$

LEMMA 2.3 (Properties of the generalized Wasserstein distance). *Let $\mu,\nu,\eta,\mu_1,\mu_2,\nu_1,\nu_2$ be some positive measures in $\mathcal{M}(\mathbb{R}^d)$. The following properties hold for $p\ge 1$, $a>0$ and $b>0$*

(1) $W_p^{a,b}(\mu_1+\mu_2,\nu_1+\nu_2) \le W_p^{a,b}(\mu_1,\nu_1) + W_p^{a,b}(\mu_2,\nu_2)$,

(2) $W_1^{a,b}(\mu+\eta,\nu+\eta) = W_1^{a,b}(\mu,\nu)$.

Proof. The first property is taken from [24, Proposition 11]. For $a=b=1$, the second statement is a direct consequence of the Kantorovitch-Rubinstein duality in Lemma 2.2 for $W_1^{1,1}$. For general $a>0, b>0$, we use the results of Proposition 2.1. Indeed, also applying Kantorovitch-Rubinstein duality for $W_1^{1,1}$ and setting $a'=1, b'=\ell = \frac{b}{a}$, it holds

$$W_1^{a,b}(\mu+\eta,\nu+\eta) = aW_1^{1,\frac{b}{a}}(\mu+\eta,\nu+\eta) = aW_1^{1,1}(D_\ell\#\mu + D_\ell\#\eta, D_\ell\#\nu + D_\ell\#\eta) =$$
$$= aW_1^{1,1}(D_\ell\#\mu, D_\ell\#\nu) = aW_1^{1,\frac{b}{a}}(\mu,\nu) = W_1^{a,b}(\mu,\nu).$$

□

DEFINITION 2.4 (Image of a measure under a plan). *Let μ and ν two measures in $\mathcal{M}(\mathbb{R}^d)$ of same mass and $\pi \in \Pi(\mu,\nu)$. For $\eta \le \mu$, we denote by f the Radon-Nikodym derivative of η with respect to μ and by π_f the transference plan defined by $\pi_f(x,y) = f(x)\pi(x,y)$. Then, we define the image of η under π as the second marginal η' of π_f.*

Observe that the second marginal satisfies $\eta' \le \nu$. Indeed, since $\eta \le \mu$, it holds $f \le 1$. Thus, for all Borel sets B of $\mathbb{R}^d$ we have

$$\eta'(B) = \pi_f(\mathbb{R}^d \times B) \le \pi(\mathbb{R}^d \times B) = \nu(B).$$

3. Generalized Wasserstein norm for signed measures

In this section, we define the generalized Wasserstein norm for signed measures, by generalizing [17]. We introduce general weight parameters a and b (Definition 3.2), that might be useful for modeling issues. We also prove the Kantorovich-Rubinstein duality for $\|.\|^{1,1}$, adapting similar results in [17] and [24, Lemma 14], that allows to interpret the norm in the framework of optimal transportation. We finally list the main topological properties of the normed space of signed Radon measures.

3.1. Definitions.

DEFINITION 3.1 (Generalized Wasserstein distance extended to signed measures). *For μ,ν two signed measures in $\mathcal{M}^s(\mathbb{R}^d)$, we define*

$$\mathbb{W}_1^{a,b}(\mu,\nu)=W_1^{a,b}(\mu_++\nu_-,\mu_-+\nu_+),$$

where μ_+,μ_-,ν_+ and ν_- are any measures in $\mathcal{M}(\mathbb{R}^d)$ such that $\mu=\mu_+-\mu_-$ and $\nu=\nu_+-\nu_-$.

PROPOSITION 3.1. *The operator $\mathbb{W}_1^{a,b}$ is a distance on the space $\mathcal{M}^s(\mathbb{R}^d)$ of signed measures with finite mass on $\mathbb{R}^d$.*

Proof. First, we prove that the definition does not depend on the decomposition. Indeed, if we consider two distinct decompositions, $\mu=\mu_+-\mu_-=\mu_+^J-\mu_-^J$, and $\nu=\nu_+-\nu_-=\nu_+^J-\nu_-{}^J$, with the second one being the Jordan decomposition, then we have $(\mu_++\nu_-)-(\mu_+^J+\nu_-^J)=(\mu_-+\nu_+)-(\mu_-^J+\nu_+^J)$, and this is a positive measure since $\mu_+\geq\mu_+^J$ and $\nu_+\geq\nu_+^J$. The second property of Lemma 2.3 then gives

$$\begin{aligned}&W_1^{a,b}(\mu_+^J+\nu_-^J,\mu_-^J+\nu_+^J)\\&=W_1^{a,b}(\mu_+^J+\nu_-^J+(\mu_++\nu_-)-(\mu_+^J+\nu_-^J),\mu_-^J+\nu_+^J+(\mu_-+\nu_+)-(\mu_-^J+\nu_+^J))\\&=W_1^{a,b}(\mu_++\nu_-,\mu_-+\nu_+).\end{aligned}$$

We now prove that $\mathbb{W}_1^{a,b}(\mu,\nu)=0$ implies $\mu=\nu$. By choosing the Jordan decomposition for both μ and ν and observing that $W_1^{a,b}$ is a distance, we obtain $\mu_++\nu_-=\mu_-+\nu_+$, thus $\mu=\nu$.

We now prove the triangle inequality. We have $\mathbb{W}_1^{a,b}(\mu,\eta)=W_1^{a,b}(\mu_++\eta_-,\mu_-+\eta_+)$. Using Lemma 2.3, we have

$$\begin{aligned}\mathbb{W}_1^{a,b}(\mu,\eta)&=W_1^{a,b}(\mu_++\eta_-+\nu_++\nu_-,\mu_-+\eta_++\nu_++\nu_-)\\&\leq W_1^{a,b}(\mu_++\nu_-,\mu_-+\nu_+)+W_1^{a,b}(\eta_-+\nu_+,\eta_++\nu_-)\\&=\mathbb{W}_1^{a,b}(\mu,\nu)+\mathbb{W}_1^{a,b}(\nu,\eta).\end{aligned}$$

∎

We also state the following lemma about adding and removing masses.

LEMMA 3.1. *Let $\mu,\nu,\eta,\mu_1,\mu_2,\nu_1,\nu_2$ be in $\mathcal{M}^s(\mathbb{R}^d)$, then the following properties hold*

- $\mathbb{W}_1^{a,b}(\mu+\eta,\nu+\eta)=\mathbb{W}_1^{a,b}(\mu,\nu)$,
- $\mathbb{W}_1^{a,b}(\mu_1+\mu_2,\nu_1+\nu_2)\leq\mathbb{W}_1^{a,b}(\mu_1,\nu_1)+\mathbb{W}_1^{a,b}(\mu_2,\nu_2)$.

Proof. The proof is direct. For the first item, it holds

$$\begin{aligned}\mathbb{W}_1^{a,b}(\mu+\eta,\nu+\eta)&=W_1^{a,b}([\mu_++\eta_+]+[\nu_-+\eta_-],[\nu_++\eta_+]+[\mu_-+\eta_-])\\&=W_1^{a,b}(\mu_++\nu_-+[\eta_++\eta_-],\nu_++\mu_-+[\eta_++\eta_-])\end{aligned}$$

which by Lemma 2.3 is equal to $W_1^{a,b}(\mu_++\nu_-,\mu_-+\nu_+)=\mathbb{W}_1^{a,b}(\mu,\nu)$.

For the second item, it holds

$$\begin{aligned}\mathbb{W}_1^{a,b}(\mu_1+\mu_2,\nu_1+\nu_2)&=W_1^{a,b}(\mu_{1,+}+\mu_{2,+}+\nu_{1,-}+\nu_{2,-},\nu_{1,+}+\nu_{2,+}+\mu_{1,-}+\mu_{2,-})\\&\leq W_1^{a,b}(\mu_{1,+}+\nu_{1,-},\nu_{1,+}+\mu_{1,-})+W_1^{a,b}(\mu_{2,+}+\nu_{2,-},\nu_{2,+}+\mu_{2,-})\\&=\mathbb{W}_1^{a,b}(\mu_1,\nu_1)+\mathbb{W}_1^{a,b}(\mu_2,\nu_2),\end{aligned}$$

where the inequality comes from Lemma 2.3. ∎

DEFINITION 3.2. *For $\mu \in \mathcal{M}^s(\mathbb{R}^d)$ and $a>0, b>0$, we define*

$$\|\mu\|^{a,b} = \mathbb{W}_1^{a,b}(\mu,0) = W_1^{a,b}(\mu_+,\mu_-),$$

where μ_+ and μ_- are any measures of $\mathcal{M}(\mathbb{R}^d)$ such that $\mu = \mu_+ - \mu_-$.

It is clear that the definition of $\|\mu\|^{a,b}$ does not depend on the choice of μ_+,μ_- as a consequence of the corresponding property for $W_1^{a,b}$.

PROPOSITION 3.2. *The space of signed measures $(\mathcal{M}^s(\mathbb{R}^d), \|.\|^{a,b})$ is a normed vector space.*

Proof. First, we notice that $\|\mu\|^{a,b} = 0$ implies that $W_1^{a,b}(\mu_+,\mu_-) = 0$, which is $\mu_+ = \mu_-$ so that $\mu = \mu_+ - \mu_- = 0$. For triangular inequality, using the second property of Lemma 3.1, we have that for $\mu, \eta \in \mathcal{M}^s(\mathbb{R}^d)$,

$$\|\mu+\eta\|^{a,b} = \mathbb{W}_1^{a,b}(\mu+\eta,0) \le \mathbb{W}_1^{a,b}(\mu,0) + \mathbb{W}_1^{a,b}(\eta,0) = \|\mu\|^{a,b} + \|\eta\|^{a,b}.$$

Homogeneity is obtained by writing for $\lambda > 0$, $\|\lambda\mu\|^{a,b} = \mathbb{W}_1^{a,b}(\lambda\mu,0) = W_1^{a,b}(\lambda\mu_+,\lambda\mu_-)$ where $\mu = \mu_+ - \mu_-$. Using Lemma 2.2 combined with Definition 2.3 and the notation of Proposition 2.1 we have

$$\begin{aligned}
&W_1^{a,b}(\lambda\mu_+,\lambda\mu_-) = aW_1^{1,\frac{b}{a}}(\lambda\mu_+,\lambda\mu_-)\\
&= aW_1^{1,1}\left(D_{\frac{b}{a}}\#\lambda\mu_+, D_{\frac{b}{a}}\#\lambda\mu_-\right)\\
&= a\sup\left\{\int_{\mathbb{R}^d}\varphi\, d\left(D_{\frac{b}{a}}\#\lambda\mu_+, D_{\frac{b}{a}}\#\lambda\mu_-\right); \varphi\in\mathcal{C}_c^{0,Lip}, \|\varphi\|_\infty \le 1, \|\varphi\|_{Lip}\le 1\right\}\\
&= \lambda\, a\sup\left\{\int_{\mathbb{R}^d}\varphi\, d\left(D_{\frac{b}{a}}\#\mu_+, D_{\frac{b}{a}}\#\mu_-\right); \varphi\in\mathcal{C}_c^{0,Lip}, \|\varphi\|_\infty \le 1, \|\varphi\|_{Lip}\le 1\right\}\\
&= \lambda W_1^{a,b}(\mu_+,\mu_-).
\end{aligned}$$

∎

We provide here an example that illustrates the competition between cancellation and transportation. This example is used later in the paper.

EXAMPLE 3.1. *Take $\mu = \delta_x - \delta_y$. Then*

$$\|\mu\|^{a,b} = W_1^{a,b}(\delta_x,\delta_y) = \inf_{\substack{\tilde\mu,\tilde\nu\in\mathcal{M}(\mathbb{R}^d)\\ |\tilde\mu| = |\tilde\nu|}} \{a(|\delta_x - \tilde\mu| + |\delta_y - \tilde\nu|) + bW_1(\tilde\mu,\tilde\nu)\}.$$

Using Lemma 2.1, the minimum is attained and it can be written as $\tilde\mu = \epsilon\delta_x$ and $\tilde\nu = \epsilon\delta_y$ for some $0 \le \epsilon \le 1$. Then

$$\|\mu\|^{a,b} = \min_{0\le\epsilon\le 1}\{2a(1-\epsilon) + b\epsilon|x-y|\}.$$

The minimizers depend on the distance between x and y. For $b|x-y| < 2a$, the minimum is attained for $\epsilon = 1$ and $\|\mu\|^{a,b} = b|x-y|$. In that case, we say that all the mass is transported. On the contrary, for $b|x-y| > 2a$, the minimum is attained for $\epsilon = 0$ and $\|\mu\|^{a,b} = 2a$, and we say that all the mass is cancelled (or removed). For $b|x-y| = 2a$, we can both transport and cancel.

3.2. Topological properties. In this section, we study the topological properties of the norm introduced above. In particular, we prove that it admits a duality formula, that indeed coincides with (1.1). We first prove that the topology of $\|.\|^{a,b}$ does not depend on $a,b>0$.

PROPOSITION 3.3. *For $a>0, b>0$, the norm $\|.\|^{a,b}$ is equivalent to $\|.\|^{1,1}$.*

Proof. For $\mu\in\mathcal{M}^s(\mathbb{R}^d)$ denote by $(m_+^{a,b},m_-^{a,b})$ the positive measures such that

$$\|\mu\|^{a,b}=a|\mu_+-m_+^{a,b}|+a|\mu_- -m_-^{a,b}|+bW_1(m_+^{a,b},m_-^{a,b}),$$

and similarly define $(m_+^{1,1},m_-^{1,1})$. Their existence is guaranteed by Lemma 2.1. By definition of the minimizers, we have

$$\begin{aligned}\|\mu\|^{a,b}&=a|\mu_+-m_+^{a,b}|+a|\mu_- -m_-^{a,b}|+bW_1(m_+^{a,b},m_-^{a,b})\\&\le a|\mu_+-m_+^{1,1}|+a|\mu_- -m_-^{1,1}|+bW_1(m_+^{1,1},m_-^{1,1})\le\max\{a,b\}\|\mu\|^{1,1}.\end{aligned}$$

In the same way, we obtain

$$\min\{a,b\}\|\mu\|^{1,1}\le\|\mu\|^{a,b}\le\max\{a,b\}\|\mu\|^{1,1}.$$

□

We now give an equivalent Kantorovich-Rubinstein duality formula for the new distance. For $f\in\mathcal{C}_b^0(\mathbb{R}^d;\mathbb{R})$, similarly to $\mathcal{C}_c^0(\mathbb{R}^d;\mathbb{R})$, we define the following

$$\|f\|_\infty=\sup_{x\in\mathbb{R}^d}|f(x)|,\qquad \|f\|_{Lip}=\sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|},$$

and we introduce

$$\mathcal{C}_b^{0,Lip}=\{f\in\mathcal{C}_b^0(\mathbb{R}^d;\mathbb{R})\,|\,\|f\|_{Lip}<\infty\}.$$

In the next proposition, we express the Kantorovich-Rubinstein duality for the norm $\mathbb{W}_1^{1,1}$. This shows that $\mathbb{W}_1^{1,1}$ coincides with the bounded Lipschitz norm introduced in [17], also called Fortet Mourier distance in [28].

PROPOSITION 3.4 (Kantorovitch-Rubinstein duality for $\|.\|^{1,1}$). *The signed generalized Wasserstein norm $\|.\|^{1,1}$ coincides with the bounded Lipschitz norm: For μ in $\mathcal{M}^s(\mathbb{R}^d)$, it holds*

$$\|\mu\|^{1,1}=\|\mu\|_{BL}.$$

We emphasize that Proposition 3.4 does not coincide with Lemma 2.2, since it involves non-compactly supported test functions.

Proof. By using Lemma 2.2 we have

$$\|\mu\|^{1,1}=W_1^{1,1}(\mu_+,\mu_-)=\sup\left\{\int_{\mathbb{R}^d}\varphi\,d\mu;\,\varphi\in\mathcal{C}_c^{0,Lip},\|\varphi\|_\infty\le1,\|\varphi\|_{Lip}\le1\right\}.$$

We denote by

$$S=\sup\left\{\int_{\mathbb{R}^d}\varphi\,d\mu;\,\varphi\in\mathcal{C}_b^{0,Lip},\|\varphi\|_\infty\le1,\|\varphi\|_{Lip}\le1\right\}.$$

First observe that $S<+\infty$. Indeed, it holds $\int_{\mathbb{R}^d}\varphi\,d\mu\le\|\varphi\|_\infty|\mu|<+\infty$. Denote with φ_n a sequence of functions of $\mathcal{C}_b^{0,Lip}$ such that $\int_{\mathbb{R}^d}\varphi_n\,d\mu\to S$ as $n\to\infty$. Consider a sequence of functions ρ_n in $\mathcal{C}_c^{0,Lip}$ such that $\rho_n(x)=1$ for $x\in B_0(n)$, $\rho_n(x)=0$ for $x\notin B_0(n+1)$ and $\|\rho_n\|_\infty\le 1$. For the sequence $\psi_n=\varphi_n\rho_n$ of functions of $\mathcal{C}_c^{0,Lip}$, it holds

$$\left|\int_{\mathbb{R}^d}\psi_n\,d\mu-S\right|\le\left|\int_{\mathbb{R}^d}(\psi_n-\varphi_n)\,d\mu\right|+\left|\int_{\mathbb{R}^d}\varphi_n\,d\mu-S\right|$$
$$\le 2\left|\int_{\mathbb{R}^d\setminus B_0(n)}d\mu\right|+\left|\int_{\mathbb{R}^d}\varphi_n\,d\mu-S\right|$$

since $\|\varphi_n\|_\infty\le 1$. The first term goes to zero with n, since μ, being of finite mass, is tight, and the second term goes to zero with n, by definition of S and φ_n. Then

$$S=\sup\left\{\int_{\mathbb{R}^d}\varphi\,d\mu;\,\varphi\in\mathcal{C}_c^{0,Lip},\|\varphi\|_\infty\le 1,\|\varphi\|_{Lip}\le 1\right\},$$

and Proposition 3.4 is proved. ∎

In the rest of the section, we state topological properties for the norm $\|.\|^{a,b}$.

Remark 3.1. We observe that a sequence μ_n of $\mathcal{M}^s(\mathbb{R})$ which satisfies $\|\mu_n\|^{a,b}\underset{n\to\infty}{\to}0$ is not necessarily tight, and its mass is not necessarily bounded. For instance, we have that

$$\nu_n=\delta_n-\delta_{n+\frac{1}{n}}$$

is not tight, whereas it satisfies for n sufficiently large

$$\|\nu_n\|^{a,b}=\frac{b}{n}\underset{n\to\infty}{\to}0.$$

See Example 3.1 for the details of the calculation. Now take the sequence

$$\mu_n=n\,\delta_{\frac{1}{n^2}}-n\,\delta_{-\frac{1}{n^2}}.$$

As explained in Example 3.1, depending on the sign of $2a-\frac{2b}{n^2}$, we either cancel the mass or transport it. For n large enough, $2a\ge\frac{2b}{n^2}$, so we transport the mass. Thus for n sufficiently large it holds

$$\|\mu_n\|^{a,b}=\frac{2bn}{n^2}\underset{n\to\infty}{\to}0$$

whereas $|\mu_n|=2n$ is not bounded.

We now show that norm $\|.\|^{1,1}$ does not metrize narrow convergence, via a counterexample based on unboundedness of the space. This is in line with the classical fact that weak*-convergence cannot be metrized [8, p. 76, Remark 20].

Remark 3.2. Take $\mu_n=\delta_{\sqrt{2\pi n+\frac{\pi}{2}}}-\delta_{\sqrt{2\pi n+\frac{3\pi}{2}}}$. We have

$$\|\mu_n\|^{1,1}\le\left|\sqrt{2\pi n+\frac{\pi}{2}}-\sqrt{2\pi n+\frac{3\pi}{2}}\right|\underset{n\to\infty}{\to}0,$$

even though for $\varphi(x)=\sin(x^2)$ in $\mathcal{C}_b^0(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \varphi d\mu_n = 2, \quad n\in\mathbb{N}.$$

REMARK 3.3. We have as a direct consequence of Proposition 3.4 that

$$\|\mu_n-\mu\|^{a,b} \underset{n\to\infty}{\to} 0 \qquad \Rightarrow \quad \forall\varphi\in\mathcal{C}_b^{0,Lip}(\mathbb{R}^d), \int_{\mathbb{R}^d}\varphi d\mu_n \underset{n\to\infty}{\to} \int_{\mathbb{R}^d}\varphi d\mu. \tag{3.1}$$

However, the reciprocal statement of (3.1) is false: define

$$\mu_n := n\cos(nx)\chi_{[0,\pi]}.$$

For

$$\varphi_n := \frac{1}{n}\cos(nx),$$

it is clear that

$$\int_{\mathbb{R}} \varphi_n \, d\mu_n = \int_0^\pi \cos^2(nx)\, dx = \frac{\pi}{2} \not\to 0.$$

In particular,

$$\sup_{\varphi\in\mathcal{C}_b^{0,Lip}(\mathbb{R})} \int_{\mathbb{R}} \varphi d(\mu_n - 0) \geq \frac{\pi}{2},$$

hence by Proposition 3.4, $\|\mu_n - 0\| \geq \frac{\pi}{2}$ does not converge to zero. We now prove that, for each φ in $\mathcal{C}_b^{0,Lip}(\mathbb{R})$, it holds $\int_{\mathbb{R}}\varphi d\mu_n \to 0$. Given $\varphi\in\mathcal{C}_b^{0,Lip}(\mathbb{R})$, define

$$f(x) := \begin{cases} \varphi(-x), & \text{when} \quad x\in[-\pi,0], \\ \varphi(x), & \text{when} \quad x\in[0,\pi], \end{cases}$$

and extend f as a 2π-periodic function on $\mathbb{R}$. We have

$$\int_{\mathbb{R}} \varphi d\mu_n = \int_{\mathbb{R}} f\, d\mu_n.$$

Since f is a 2π-periodic function, it also holds $\int f\, d\mu_n = na_n$, where a_n is the n-th cosine coefficient in the Fourier series expansion of f. We then prove $na_n \to 0$ for any 2π-periodic Lipschitz function f, following the ideas of [29, p. 46, last line]. Since f is Lipschitz, then its distributional derivative is in $L^\infty[-\pi,\pi]$ and thus in $L^1[-\pi,\pi]$. Then

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx = -\frac{1}{2n\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)\, dx = -\frac{b_n'}{n},$$

where b_n' is the n-th sine coefficient of f'. As a consequence of the Riemann-Lebesgue lemma, $b_n' \to 0$, and this implies $na_n \to 0$.

We recall from [25] that the space $(\mathcal{M}(\mathbb{R}^d), W_p^{a,b})$ is a complete metric space. The proof is based on the fact that a Cauchy sequence of positive measures is both uniformly bounded in mass and tight. This is not true anymore for a Cauchy sequence of signed measures.

Remark 3.4. *Observe that $(\mathcal{M}^s(\mathbb{R}^d),\|.\|^{a,b})$ is not a Banach space. Indeed, take the sequence*

$$\mu_n=\sum_{i=1}^{n}\left(\delta_{i+\frac{1}{2^i}}-\delta_{i-\frac{1}{2^i}}\right).$$

It is a Cauchy sequence in $(\mathcal{M}^s(\mathbb{R}^d),\|.\|^{a,b})$: Indeed, by choosing to transport all the mass from $\mu_n^+ + \mu_{n+k}^-$ onto $\mu_{n+k}^+ + \mu_n^-$ with the cost b, it holds

$$\mathbb{W}_1^{a,b}(\mu_n,\mu_{n+k})\le 2b\sum_{i=n+1}^{n+k}\frac{1}{2^i}\le 2b\sum_{i=n+1}^{+\infty}\frac{1}{2^i}\underset{n\to\infty}{\to}0.$$

However, the sequence $(\mu_n)_n$ does not converge in $(\mathcal{M}^s(\mathbb{R}^d),\|.\|^{a,b})$. As seen in Remark 3.3, the convergence for the norm $\|.\|^{a,b}$ implies the convergence in the sense of distributions. In the sense of distributions we have

$$\mu_n\rightharpoonup\mu^*:=\sum_{i=1}^{+\infty}\left(\delta_{i+\frac{1}{2^i}}-\delta_{i-\frac{1}{2^i}}\right)\notin\mathcal{M}^s(\mathbb{R}).$$

Indeed, for all $\varphi\in\mathcal{C}_c^\infty(\mathbb{R})$, since φ is compactly supported, it holds

$$\int_{\mathbb{R}}\varphi(x)\,(d\mu_n(x)-d\mu^*(x))=\sum_{i=n+1}^{+\infty}\left(\varphi\left(i+\frac{1}{2^i}\right)-\varphi\left(i-\frac{1}{2^i}\right)\right)\underset{n\to\infty}{\to}0.$$

The measure μ^* does not belong to $\mathcal{M}^s(\mathbb{R})$, as it has infinite mass.

Nevertheless, we have the following convergence result.

Theorem 3.1. *Let μ_n be a Cauchy sequence in $(\mathcal{M}^s(\mathbb{R}^d),\|.\|^{a,b})$. If μ_n is tight (in the sense of Definition A.3) and has uniformly bounded mass, then it converges in $(\mathcal{M}^s(\mathbb{R}^d),\|.\|^{a,b})$.*

Proof. Take a tight Cauchy sequence $(\mu_n)_n\in\mathcal{M}^s(\mathbb{R}^d)$ such that the sequences given by the Jordan decomposition $|\mu_n^+|$ and $|\mu_n^-|$ are uniformly bounded. Then, by Lemma A.2, there exists μ^+ and μ^- in $\mathcal{M}(\mathbb{R}^d)$ and φ_1 nondecreasing such that, $\mu^+_{\varphi_1(n)}\underset{n\to\infty}{\rightharpoonup}\mu^+$ vaguely. Then, $|\mu_n^-|$ being uniformly bounded, there exists φ_2 nondecreasing such that for $\varphi=\varphi_1\circ\varphi_2$ it holds

$$\mu^-_{\varphi(n)}\underset{n\to\infty}{\rightharpoonup}\mu^-\quad\text{vaguely.}$$

Since μ_n^+ and μ_n^- are assumed to be tight, the sequences $\mu^-_{\varphi(n)}$ and $\mu^+_{\varphi(n)}$ also converge to μ^- and μ^+ narrowly, and it holds $W_1^{a,b}(\mu^+_{\varphi(n)},\mu^+)\underset{n\to\infty}{\to}0$ and $W_1^{a,b}(\mu^-_{\varphi(n)},\mu^-)\underset{n\to\infty}{\to}0$, due to [24, Theorem 13]. We then have

$$\begin{aligned}\|\mu_n-(\mu^+-\mu^-)\|^{a,b}&\le\|\mu_n-\mu_{\varphi(n)}\|^{a,b}+\|\mu_{\varphi(n)}-(\mu^+-\mu^-)\|^{a,b}\\&\le\|\mu_n-\mu_{\varphi(n)}\|^{a,b}+W_1^{a,b}(\mu^+_{\varphi(n)}+\mu^-,\mu^-_{\varphi(n)}+\mu^+)\\&\le\|\mu_n-\mu_{\varphi(n)}\|^{a,b}+W_1^{a,b}(\mu^+_{\varphi(n)},\mu^+)+W_1^{a,b}(\mu^-_{\varphi(n)},\mu^-)\underset{n\to\infty}{\to}0.\end{aligned}$$

Here, we used the fact that $(\mu_n)_n$ is a Cauchy sequence. ∎

We end this section with a characterization of the convergence for the norm. If a sequence μ_n of signed measures converges toward $\mu \in \mathcal{M}^s(\mathbb{R}^d)$, then for any decomposition of μ_n into two positive measures $\mu_n = \mu_n^+ - \mu_n^-$ (not necessarily the Jordan decomposition), we have that each μ_n^+, μ_n^- is the sum of two positive measures: m_n^+, z_n^+ and m_n^-, z_n^-, respectively. The measures m_n^+ and m_n^- are the parts that converge respectively to μ^+ and μ^-. Both m_n^+ and m_n^- are uniformly bounded and tight. The measures z_n^+ and z_n^- are the residual terms that may be unbounded and not tight. They compensate each other in the sense that $W_1^{a,b}(z_n^+, z_n^-)$ vanishes for large n.

4. Application to the transport equation with source term

This section is devoted to the use of the norm introduced in Definition 3.2 to guarantee existence, uniqueness, and stability with respect to initial condition for the transport Equation (1.2). We denote the set of test functions (i.e. C^∞ with compact support) on a given space X by $\mathcal{D}(X)$.

Definition 4.1 (Measure-valued weak solution). *A measure-valued weak solution to (1.2) is a continuous map with respect to the weak-* topology of measures (i.e $\mu \in \mathcal{C}^0([0,1];\mathcal{M}^s(\mathbb{R}^d))$ such that for all $\varphi \in \mathcal{D}([0,1)\times\mathbb{R}^d)$ it holds*

$$\int_0^1 dt \int_{\mathbb{R}^d} (d\mu_t(\partial_t\varphi(t,x) + v[\mu_t]\cdot\nabla\varphi(t,x)) + dh[\mu_t]\varphi(t,x)) = -\int_{\mathbb{R}^d} d\mu_0\varphi(0,\cdot). \tag{4.1}$$

Equivalently, μ satisfies $\mu_{t=0} = \mu_0$ and for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ it holds

$$\frac{d}{dt}\langle \mu_t, \varphi\rangle = \langle \mu_t, v[\mu_t]\cdot\nabla\varphi\rangle + \langle h[\mu_t], \varphi\rangle, \tag{4.2}$$

where

$$\langle \mu_t, \varphi\rangle := \int_{\mathbb{R}^d} \varphi(x)d\mu_t(x).$$

The equivalence of definitions is classical, see e.g. [3, Chap 8]. We will also use the following classical fact: For μ_t solving the transport equation with source (1.2), any interval $[t_1,t_2]\subset[0,1)$ and all $\varphi \in \mathcal{D}([t_1,t_2]\times\mathbb{R}^d)$, it holds

$$\begin{aligned}&\int_{t_1}^{t_2} dt \int_{\mathbb{R}^d} d\mu_t(\partial_t\varphi(t,x) + v[\mu_t]\cdot\nabla\varphi(t,x)) + dh[\mu_t], \varphi(t,x)\\ &= \int_{\mathbb{R}^d} d\mu_{t_2}\varphi(t_2,\cdot) - \int_{\mathbb{R}^d} d\mu_{t_1}\varphi(t_1,\cdot).\end{aligned} \tag{4.3}$$

4.1. Estimates of the norm under flow action.

In this section, we extend the action of flows on probability measures to signed measures, and state some estimates about the evaluation of $\|\mu\|^{a,b}$ under a flow action on μ. Notice that for $\mu \in \mathcal{M}^s(\mathbb{R}^d)$ and T a map, we have $T\#\mu = T\#\mu^+ - T\#\mu^-$, where $\mu = \mu^+ - \mu^-$ is any decomposition of μ. Observe that in general, given $\mu \in \mathcal{M}^s(\mathbb{R}^d)$ and $T:\mathbb{R}^d \mapsto \mathbb{R}^d$ a Borel map, it only holds

$$|T\#\mu| \le |\mu|, \tag{4.4}$$

even by choosing the Jordan decomposition for (μ^+,μ^-), since it may hold that $T\#\mu^+$ and $T\#\mu^-$ are not orthogonal. However, if T is injective (as it will be in the rest of the paper), it holds $T\#\mu^+ \perp T\#\mu^-$, hence $|T\#\mu| = |\mu|$.

Lemma 4.1. *For $v(t,x)$ measurable in time, uniformly Lipschitz in space, and uniformly bounded, we denote by Φ_t^v the flow it generates, i.e. the unique solution to*

$$\frac{d}{dt}\Phi_t^v = v(t,\Phi_t^v), \qquad \Phi_0^v = I_d.$$

Given $\mu_0 \in \mathcal{M}^s(\mathbb{R}^d)$, then, $\mu_t = \Phi_t^v \# \mu_0$ is the unique solution of the linear transport equation

$$\begin{cases} \dfrac{\partial}{\partial t}\mu_t + \nabla\cdot(v(t,x)\mu_t) = 0, \\ \mu_{|t=0} = \mu_0 \end{cases}$$

in $\mathcal{C}([0,T],\mathcal{M}^s(\mathbb{R}^d))$.

Proof. The proof is a direct consequence of [27, Theorem 5.34] combined with [7, Theorem 2.1.1]. ∎

Lemma 4.2. *Let v and w be two vector fields, both satisfying for all $t\in[0,1]$ and $x,y\in\mathbb{R}^d$ the following properties:*

$$|v(t,x)-v(t,y)| \le L|x-y|, \qquad |v(t,x)| \le M.$$

Let μ and ν be two measures of $\mathcal{M}^s(\mathbb{R}^d)$. Then

- $\|\Phi_t^v\#\mu\|^{a,b} \le e^{Lt}\|\mu\|^{a,b}$
- $\|\mu-\Phi_t^v\#\mu\|^{a,b} \le b\,t\,M|\mu|$,
- $\|\Phi_t^v\#\mu-\Phi_t^w\#\mu\|^{a,b} \le b|\mu|\frac{(e^{Lt}-1)}{L}\|v-w\|_{L^\infty(0,1;\mathcal{C}^0)}$
- $\|\Phi_t^v\#\mu-\Phi_t^w\#\nu\|^{a,b} \le e^{Lt}\|\mu-\nu\|^{a,b} + b\min\{|\mu|,|\nu|\}\frac{(e^{Lt}-1)}{L}\|v-w\|_{L^\infty(0,1;\mathcal{C}^0)}$

Proof. The first three inequalities follow from [25, Proposition 10]. For the first inequality, we write

$$\begin{aligned}\|\Phi_t^v\#\mu\|^{a,b} &= W_1^{a,b}(\Phi_t^v\#\mu^+,\Phi_t^v\#\mu^-)\\ &\le e^{Lt}W_1^{a,b}(\mu^+,\mu^-) \quad \text{by [25, Prop. 10]}\\ &= e^{Lt}\|\mu\|^{a,b}.\end{aligned}$$

For the second inequality,

$$\begin{aligned}\|\mu-\Phi_t^v\#\mu\|^{a,b} &= W_1^{a,b}(\mu^+ + \Phi_t^v\#\mu^-,\mu^- + \Phi_t^v\#\mu^+)\\ &\le W_1^{a,b}(\mu^+,\Phi_t^v\#\mu^+) + W_1^{a,b}(\mu^-,\Phi_t^v\#\mu^-) \quad \text{(Lemma 2.3)}\\ &\le b\,t\,\|v\|_{\mathcal{C}^0}(|\mu^+|+|\mu^-|) \quad \text{by [25, Prop. 10]}\\ &= b\,t\,\|v\|_{L^\infty(0,1;\mathcal{C}^0(\mathbb{R}))}|\mu| \quad \text{since } (\mu^+,\mu^-) \text{ is the Jordan decomposition.}\end{aligned}$$

The third inequality is given by

$$\begin{aligned}\|\Phi_t^v\#\mu-\Phi_t^w\#\mu\|^{a,b} &= W_1{}^{a,b}(\Phi_t^v\#\mu^+ + \Phi_t^w\#\mu^-,\Phi_t^w\#\mu^+ + \Phi_t^v\#\mu^-)\\ &\le W_1^{a,b}(\Phi_t^v\#\mu^+,\Phi_t^w\#\mu^+) + W_1^{a,b}(\Phi_t^w\#\mu^-,\Phi_t^v\#\mu^-)\\ &\le b\big(W_1(\Phi_t^v\#\mu^+,\Phi_t^w\#\mu^+) + W_1(\Phi_t^w\#\mu^-,\Phi_t^v\#\mu^-)\big)\\ &\le b\,(|\mu^+|+|\mu^-|)\frac{(e^{Lt}-1)}{L}\|v-w\||_{L^\infty(0,1;\mathcal{C}^0(\mathbb{R}))}\end{aligned}$$

by using [25, Prop. 10] with $\mu=\nu$.

The last inequality is deduced from the first and the third ones using triangular inequality. ∎

4.2. A scheme for computing solutions of the transport equation. In this section, we define an approximation scheme for solutions to (1.2). This will be useful to prove existence of solutions. We then prove Theorem 1.1.

Fix $\mu_0 \in \mathcal{M}^s(\mathbb{R}^d)$ such that $\mathrm{supp}(\mu_0) \subset \mathcal{K}$, with $\mathcal{K}$ compact. Let $v \in \mathcal{C}^{0,Lip}(\mathcal{M}^s(\mathbb{R}^d), \mathcal{C}^{0,Lip}(\mathbb{R}^d))$ and $h \in \mathcal{C}^{0,Lip}(\mathcal{M}^s(\mathbb{R}^d), \mathcal{M}^s(\mathbb{R}^d))$ satisfy (H-1)-(H-2)-(H-3). We now define a sequence $(\mu_t^k)_k$ of approximated solutions for (1.2) through the following Euler-explicit-type iteration scheme. For simplicity of notations, we define a solution on the time interval $[0,1]$ only.

SCHEME

Initialization. Fix $k \in \mathbb{N}$. Define $\Delta t = \dfrac{1}{2^k}$. Set $\mu_0^k = \mu_0$.

Induction. Given $\mu_{i\Delta t}^k$ for $i \in \{0, 1, \ldots, 2^k - 1\}$, define $v_{i\Delta t}^k := v[\mu_{i\Delta t}^k]$ and

$$\mu_t^k = \Phi_{t-i\Delta t}^{v_{i\Delta t}^k} \# \mu_{i\Delta t}^k + (t - i\Delta t) h[\mu_{i\Delta t}^k], \qquad t \in [i\Delta t, (i+1)\Delta t]. \tag{4.5}$$

REMARK 4.1. The flow $\Phi_{t-i\Delta t}$ encodes the transport part $\partial_t \mu + \mathrm{div}(v\mu) = 0$ while $(t - i\Delta t)h$ encodes the reaction $\partial_t \mu = h$.

We now prove equi-Lipschitz continuity of the sequence $(\mu_t^k)_k$. We also define the following sup-norm on curves in $\mathcal{C}^0([0,1], \mathcal{M}^s(\mathbb{R}^d))$ by writing

$$\|\mu\| := \sup_{t \in [0,1]} \|\mu_t\|^{a,b}.$$

PROPOSITION 4.1. *The sequence $(\mu_t^k)_k \in \mathcal{C}^0([0,1], \mathcal{M}^s(\mathbb{R}^d), \|.\|)$ is equi-Lipschitz with respect to time, i.e. there exists $L' = aP + bM(P + |\mu_0|)$ independent of k such that for all $t, s \in [0,1]$ it holds*

$$\|\mu_t^k - \mu_s^k\|^{a,b} \le L'|t - s|. \tag{4.6}$$

Moreover, the sequence is uniformly bounded in mass and compactly supported, i.e.

$$|\mu_t^k| \le Pt + |\mu_0|, \qquad supp\{\mu_t\} \subset B(0, R' + M) \tag{4.7}$$

for R' such that $(supp\{\mu_0\} \cup B_0(R)) \subset B_0(R')$.

REMARK 4.2. Estimates (4.7) are expected at the discrete level from the PDE (1.2) with the assumptions (H-1), (H-2), (H-3). Indeed, the transport part preserves mass, while the reaction term gives a mass growth that is at most linear. Likewise, the support estimate is expected from the PDE since h has support in $B_0(R)$ (no mass created out of this ball) and transport cannot expand the support with a speed faster than $|v| \le M$.

Proof. We first prove (4.7). The sequence built by the scheme satisfies

$$|\mu_t^k| \le Pt + |\mu_0|, \quad t \in [0,1], \tag{4.8}$$

where P is such that $|h[\mu]| \le P$ by (H-3). Indeed, it holds directly from (4.5) and from (H-3) that

$$|\mu_{(i+1)\Delta t}^k| \le |\Phi_{\Delta t}^{v_{i\Delta t}^k} \# \mu_{i\Delta t}^k| + \Delta t |h[\mu_{i\Delta t}^k]| \le |\mu_{i\Delta t}^k| + \Delta t P,$$

then by induction on i (for k fixed), we have

$$|\mu^k_{i\Delta t}| \leq Pi\Delta t + |\mu_0|. \tag{4.9}$$

Thus for $t \in [i\Delta t, (i+1)\Delta t]$, using again (4.5) and (H-3)

$$|\mu^k_t| \leq |\Phi^{v^k_{i\Delta t}}_{t-i\Delta t} \# \mu^k_{i\Delta t}| + (t-i\Delta t)|h[\mu^k_{i\Delta t}]| \leq |\mu^k_{i\Delta t}| + (t-i\Delta t)P \leq |\mu_0| + Pt,$$

using (4.9) for the last inequality. This proves the first estimate of (4.7), as $t \leq 1$. We now prove the second statement of (4.7). First observe that $\mathrm{supp}\{\mu\} = \mathrm{supp}\{\mu^+\} \cup \mathrm{supp}\{\mu^-\}$, where (μ^+, μ^-) is the Jordan decomposition of μ. Choose $\mathcal{K}$ such that $\mathrm{supp}\{\mu_0\} \subset \mathcal{K}$ and use (4.5) and (H-2)-(H-3) to write

$$\mathrm{supp}\{\mu^k_t\} \subseteq \mathcal{K}_{t,M,R},$$

with

$$\mathcal{K}_{t,M,R} := \{x \in \mathbb{R}^d,\, x = x_{\mathcal{K},R} + x',\, x_{\mathcal{K},R} \in \mathcal{K} \cup B_0(R),\, \|x'\| \leq tM\}.$$

Take now R' such that $\mathcal{K} \cup B_0(R) \subset B_0(R')$. Then, it holds $\mathcal{K}_{t,M,R} \subset B(0, R' + tM)$. Again by recalling $t \leq 1$, we have the second statement of (4.7).

We now prove that $(\mu^k_t)_k$ is Lipschitz with respect to time. We have two cases:

- Let $t, s \in [i\Delta t, (i+1)\Delta t]$ for some $i \in \{0, 1, \ldots, 2^k - 1\}$. By applying (4.5), the triangular inequality and Lemma 4.2, we have

$$\begin{aligned}
\|\mu^k_t - \mu^k_s\|^{a,b} &\leq \|\Phi^{v^k_{i\Delta t}}_{t-i\Delta t} \# \mu^k_{i\Delta t} - \Phi^{v^k_{i\Delta t}}_{s-i\Delta t} \# \mu^k_{i\Delta t}\|^{a,b} \\
&\quad + \|(t-i\Delta t)h[\mu^k_{i\Delta t}] - (s-i\Delta t)h[\mu^k_{i\Delta t}]\|^{a,b} \\
&= \|\Phi^{v^k_{i\Delta t}}_{t-s} \# \nu_1 - \nu_1\|^{a,b} + \|(t-s)h[\mu^k_{i\Delta t}] + \nu_2 - \nu_2\|^{a,b} \\
&\leq |t-s| bM |\nu_1| + a|t-s|\,|h[\mu^k_{i\Delta t}]| \leq |t-s|(bM|\mu^k_{i\Delta t}| + aP) \\
&\leq |t-s|(bM(Ps + |\mu_0|) + aP),
\end{aligned}$$

  where $\nu_1 = \Phi^{v^k_{i\Delta t}}_{s-i\Delta t} \# \mu^k_{i\Delta t}$ and $\nu_2 = (s - i\Delta t)h[\mu^k_{i\Delta t}]$. Recall that $s \in [0,1]$ and observe that this implies Lipschitz continuity on $[i\Delta t, (i+1)\Delta t]$.
- Choose now any $t, s \in [0,1]$ and assume $t < s$ with no loss of generality. Then choose $i, j \in \{0, 1, \ldots, 2^k - 1\}$ the unique indexes so that

$$i\Delta t \leq t < (i+1)\Delta t < \ldots < (j-1)\Delta t < s \leq j\Delta t.$$

  By applying triangular inequality and the estimate of the previous case on each term, it holds

$$\begin{aligned}
\|\mu^k_t - \mu^k_s\|^{a,b} &\leq \|\mu^k_t - \mu^k_{(i+1)\Delta t}\|^{a,b} + \|\mu^k_{(i+1)\Delta t} - \mu^k_{(i+2)\Delta t}\|^{a,b} + \ldots \\
&\quad + \|\mu^k_{(j-1)\Delta t} - \mu^k_s\|^{a,b} \\
&\leq L'((i+1)\Delta t - t + (i+2)\Delta t - (i+1)\Delta t + \ldots \\
&\quad + s - (j-1)\Delta t) = L'(s-t).
\end{aligned}$$

This proves uniform Lipschitz continuity. ∎

We now prove that μ^k_t is an approximated solution of (1.2).

PROPOSITION 4.2. *There exists L'' such that, for each k and $\varphi \in \mathcal{D}([0,1)\times\mathbb{R}^d)$ satisfying*

$$\|\varphi(t,\cdot)\|_\infty \le 1, \qquad \|\varphi(t,\cdot)\|_{Lip} \le 1, \qquad \textit{for all} \quad t\in[0,1], \tag{4.10}$$

it holds

$$\left|\int_0^1 dt\int_{\mathbb{R}^d}\left(d\mu_t^k(\partial_t\varphi(t,\cdot)+v[\mu_t^k]\cdot\nabla\varphi(t,\cdot))+dh[\mu_t^k]\varphi(t,\cdot)\right)+\int_{\mathbb{R}^d}d\mu_0\varphi(0,\cdot)\right|\le\frac{L''}{2^k}. \tag{4.11}$$

Proof. By using the formulation (4.3), for each interval $[i\Delta t,(i+1)\Delta t]$ and $\varphi\in\mathcal{D}([i\Delta t,(i+1)\Delta t]\times\mathbb{R}^d)$ it holds

$$\begin{aligned}&\int_{i\Delta t}^{(i+1)\Delta t}dt\int_{\mathbb{R}^d}d\Phi_{t-i\Delta t}^{v[\mu_{i\Delta t}^k]}\#\mu_{i\Delta t}^k(\partial_t\varphi(t,\cdot)+v[\mu_{i\Delta t}^k]\cdot\nabla\varphi(t,\cdot))\\&=\int_{\mathbb{R}^d}d\Phi_{\Delta t}^{v[\mu_{i\Delta t}^k]}\#\mu_{i\Delta t}^k\varphi((i+1)\Delta t,\cdot)-\int_{\mathbb{R}^d}d\mu_{i\Delta t}^k\varphi(i\Delta t,\cdot)\end{aligned}$$

and

$$\begin{aligned}&\int_{i\Delta t}^{(i+1)\Delta t}dt\int_{\mathbb{R}^d}\left(d((t-i\Delta t)h[\mu_{i\Delta t}^k])(\partial_t\varphi(t,\cdot)+dh[\mu_{i\Delta t}^k]\varphi(t,\cdot)\right)\\&=\int_{\mathbb{R}^d}d(\Delta t\, h[\mu_{i\Delta t}^k])\varphi((i+1)\Delta t,\cdot)-\int_{\mathbb{R}^d}d0\,\varphi(i\Delta t,\cdot).\end{aligned}$$

By adding on both sides and recalling the definition of μ^k in (4.5), it holds

$$\begin{aligned}&\int_{i\Delta t}^{(i+1)\Delta t}dt\int_{\mathbb{R}^d}\left(d\mu_t^k\partial_t\varphi(t,\cdot)+d\Phi_{t-i\Delta t}^{v[\mu_{i\Delta t}^k]}\#\mu_{i\Delta t}^k v[\mu_{i\Delta t}^k]\cdot\nabla\varphi(t,\cdot))+dh[\mu_{i\Delta t}^k]\varphi(t,\cdot)\right)\\&=\int_{\mathbb{R}^d}d\mu_{(i+1)\Delta t}^k\varphi((i+1)\Delta t,\cdot)-\int_{\mathbb{R}^d}d\mu_{i\Delta t}^k\varphi(i\Delta t,\cdot).\end{aligned}$$

Recall that $\mu_t=\Phi_{t-i\Delta t}^{v[\mu_{i\Delta t}^k]}\#\mu_{i\Delta t}^k+(t-i\Delta t)h[\mu_{i\Delta t}^k]$ and sum all terms $i=0,\ldots,2^k-1$. By recalling that $\varphi(1,\cdot)=0$, we have

$$\begin{aligned}&\sum_{i=0}^{2^k-1}\int_{i\Delta t}^{(i+1)\Delta t}dt\int_{\mathbb{R}^d}\left(d\mu_t^k(\partial_t\varphi(t,\cdot)+v[\mu_{i\Delta t}^k]\cdot\nabla\varphi(t,\cdot))+dh[\mu_{i\Delta t}^k]\varphi(t,\cdot)\right)\\&=-\sum_{i=0}^{2^k-1}\int_{i\Delta t}^{(i+1)\Delta t}dt\int_{\mathbb{R}^d}(t-i\Delta t)h[\mu_{i\Delta t}^k]v[\mu_{i\Delta t}^k]\cdot\nabla\varphi(t,\cdot))-\int_{\mathbb{R}^d}d\mu_0\varphi(0,\cdot).\end{aligned}\tag{4.12}$$

Recall that (4.6) implies that for each $t\in[i\Delta t,(i+1)\Delta t]$ it holds $\|\mu_t^k-\mu_{i\Delta t}^k\|^{a,b}\le L'\Delta t$, hence by (H-1)-(H-3) it holds

$$\|v[\mu_{i\Delta t}^k]-v[\mu_t^k]\|_{\mathcal{C}^0(\mathbb{R}^d)}\le KL'\Delta t, \qquad \|h[\mu_{i\Delta t}^k]-h[\mu_t^k]\|^{a,b}\le QL'\Delta t.$$

By using (4.12), we have

$$\left|\int_0^1 dt\int_{\mathbb{R}^d}\left(d\mu_t^k(\partial_t\varphi(t,\cdot)+v[\mu_t^k]\cdot\nabla\varphi(t,\cdot))+dh[\mu_t^k]\varphi(t,\cdot)\right)+\int_{\mathbb{R}^d}d\mu_0\varphi(0,\cdot)\right|$$

$$\leq \sum_{i=0}^{2^k-1} \left| \int_{i\Delta t}^{(i+1)\Delta t} dt \int_{\mathbb{R}^d} (t-i\Delta t) h[\mu^k_{i\Delta t}] v[\mu^k_{i\Delta t}] \cdot \nabla\varphi(t,\cdot)) \right|$$
$$+ \sum_{i=0}^{2^k-1} \left| \int_{i\Delta t}^{(i+1)\Delta t} dt \int_{\mathbb{R}^d} \left(d\mu^k_t (v[\mu^k_t] - v[\mu^k_{i\Delta t}]) \cdot \nabla\varphi(t,\cdot)) + d(h[\mu^k_t] - h[\mu^k_{i\Delta t}]) \varphi(t,\cdot)) \right| \right.$$
$$\leq \int_0^1 dt \Delta t P M \|\nabla\varphi(t,\cdot)\|_{C^0(\mathbb{R}^d)}$$
$$+ \int_0^1 dt \int_{\mathbb{R}^d} d|\mu^k_t| L' K \Delta t \|\nabla\varphi(t,\cdot)\|_{C^0(\mathbb{R}^d)} + L' Q \Delta t \|\varphi\|_{C^0(\mathbb{R}^d)}$$
$$\leq \Delta t P M + \Delta t L'((P + |\mu_0|) K + Q).$$

Here we used that $h[\mu^k_{i\Delta t}]$ and μ^k_t have bounded mass, see (H-3)-(4.7), as well as bounded $\mathcal{C}^0$ norm of v, due to (H-2). Observe that $\|\nabla\varphi(t,\cdot)\|_{C^0(\mathbb{R}^d)} = \|\varphi(t,\cdot)\|_{Lip}$ and recall $\Delta t = \frac{1}{2^k}$. By choosing $L'' := PM + L'((P+|\mu_0|)K + Q)$ not depending on k, we have the result. ∎

4.3. Proof of Theorem 1.1. In this section, we prove Theorem 1.1, stating existence and uniqueness of the solution to the Cauchy problem associated to (1.2). The proof is based on the proof of the same result for positive measures written in [25]. We first focus on existence.

Step 1. Existence. Recall Proposition 4.1: The sequence given by the scheme $(\mu^k_t)_k$ is uniformly Lipschitz continuous, uniformly bounded in mass and tight. Since $\mu^k_0 = \mu_0$ for all k, this implies that the sequence is also uniformly bounded and equicontinuous. By Ascoli-Arzela theorem, this implies that the sequence is relatively compact in $\mathcal{C}^0\left([0,1], \mathcal{M}^s(\mathbb{R}^d)\right)$, hence there exist converging sub-sequences. By passing to one of such subsequences, for which we use the index j, we define

$$\mu_t := \lim_{j\to\infty} \mu^j_t. \tag{4.13}$$

We now prove that μ_t satisfies (4.1) for all $\varphi \in \mathcal{D}([0,1) \times \mathbb{R}^d)$. Observe that each φ satisfies $\sup_{t\in[0,1]} \|\varphi(t,\cdot)\|_{\mathcal{C}^0(\mathbb{R}^d)}, \|\varphi(t,\cdot)\|_{Lip} < +\infty$. Moreover, by homogeneity of (4.1) and of $\|\varphi(t,\cdot)\|_{\mathcal{C}^0(\mathbb{R}^d)}$, $\|\varphi(t,\cdot)\|_{Lip}$, $\|\partial_t\varphi(t,\cdot)\|_{\mathcal{C}^0(\mathbb{R}^d)}$, $\|\partial_t\varphi(t,\cdot)\|_{Lip}$ with respect to $\varphi \to \lambda\varphi$, it is sufficient to prove that μ_t satisfies (4.1) for all $\varphi \in \mathcal{D}([0,1) \times \mathbb{R}^d)$ with the additional constraint

$$\|\varphi(t,\cdot)\|_{\mathcal{C}^0(\mathbb{R}^d)}, \|\varphi(t,\cdot)\|_{Lip}, \|\partial_t\varphi(t,\cdot)\|_{\mathcal{C}^0(\mathbb{R}^d)}, \|\partial_t\varphi(t,\cdot)\|_{Lip} \leq 1 \quad \text{for all } t \in [0,1). \tag{4.14}$$

Observe that for each $\varphi \in \mathcal{D}([0,1) \times \mathbb{R}^d)$ satisfying (4.14) it holds

$$C := \left| \int_0^1 dt \int_{\mathbb{R}^d} (d\mu_t(\partial_t\varphi(t,x) + v[\mu_t]\cdot\nabla\varphi(t,x)) + dh[\mu_t]\varphi(t,x)) + \int_{\mathbb{R}^d} d\mu_0 \varphi(0,\cdot) \right|$$
$$\leq \tfrac{L''}{2^j} + \left| \int_0^1 dt \int_{\mathbb{R}^d} d(\mu_t - \mu^j_t)\partial_t\varphi(t,\cdot) + d\mu_t v[\mu_t]\cdot\nabla\varphi(t,\cdot) - d\mu^j_t v[\mu^j_t]\cdot\nabla\varphi(t,\cdot)) + d(h[\mu_t] - h[\mu^j_t])\varphi(t,\cdot) \right|.$$

Since such estimate holds for any j, it is sufficient to prove that the right-hand side tends to zero for $j \to +\infty$. We have

$$C \leq \tfrac{L''}{2^j} + \|\mu - \mu^j\| \sup_{t\in[0,1]} \max\{\|\partial_t\varphi(t,\cdot)\|_\infty, \|\partial_t\varphi(t,\cdot)\|_{Lip}\} + \left| \int_0^1 dt \int_{\mathbb{R}^d} d(\mu_t - \mu^j_t) v[\mu_t]\cdot\nabla\varphi(t,\cdot) \right|$$
$$+ \left| \int_0^1 dt \int_{\mathbb{R}^d} d\mu^j_t (v[\mu_t] - v[\mu^j_t])\cdot\nabla\varphi(t,\cdot)) \right| + \|h[\mu] - h[\mu^j]\| \leq \frac{L''}{2^j} + \|\mu - \mu^j\|$$

$$+\|\mu-\mu^j\|M\sup_{t\in[0,1]}\|\varphi(t,\cdot)\|_{Lip}+\int_0^1 dt\int_{\mathbb{R}^d} d|\mu_t^j|L\|\mu_t-\mu_t^j\|^{a,b}\|\varphi(t,\cdot)\|_{Lip}+Q\|\mu-\mu^j\|$$

$$\leq \frac{L''}{2^j}+\|\mu-\mu^j\|(1+M+(P+|\mu_0|)L+Q).$$

By letting $j\to+\infty$ and recalling that (4.13) is equivalent to $\|\mu-\mu^j\|\to 0$, we have the result.

REMARK 4.3. From this construction, we do not prove uniqueness of the limit for the sequence μ^k. Yet, we will prove uniqueness of the solution to (1.2) in Step 4, that will in turn ensure uniqueness of the limit.

Step 2. Any weak solution to (1.2) is Lipschitz in time. In this step, we prove that any weak solution in the sense of Definition 4.1 to the transport Equation (1.2) is Lipschitz with respect to time, since it satisfies

$$\|\mu_{t+\tau}-\mu_t\|^{a,b}\leq L'\tau, \qquad t\geq 0, \tau\geq 0, \tag{4.15}$$

with L' defined in Proposition (4.1). To do so, we consider a solution μ_t to (1.2). We define the vector field $w(t,x):=v[\mu_t](x)$ and the signed measure $b_t=h[\mu_t]$. The vector field w is uniformly Lipschitz and uniformly bounded with respect to x, since v is so. The field w is also measurable in time, since by definition, μ_t is continuous in time. Then, μ_t is the unique solution of

$$\partial_t\mu_t(x)+\mathrm{div}.(w(t,x)\mu_t(x))=b_t(x), \qquad \mu_{|t=0}(x)=\mu_0(x). \tag{4.16}$$

Uniqueness of the solution of the linear Equation (4.16) is a direct consequence of Lemma 4.1. Moreover, the scheme presented in Section 4.2 can be rewritten for the vector field w and the source b, in which dependence with respect to time is added and dependence with respect to the measure is dropped. Thus, the unique solution μ to (4.16) can be obtained as the limit of this scheme μ^k.

For each $k\geq 0$ it holds

$$\|\mu_{t+\tau}-\mu_t\|^{a,b}\leq\|\mu_t-\mu_t^k\|^{a,b}+\|\mu_t^k-\mu_{t+\tau}^k\|^{a,b}+\|\mu_{t+\tau}^k-\mu_{t+\tau}\|^{a,b}\leq 2\|\mu-\mu^k\|+L'\tau,$$

where we used (4.6) for Lipschitz continuity of μ^k. By letting $k\to+\infty$ we have $\|\mu-\mu^k\|\to 0$, thus (4.15) holds.

Step 3. Any weak solution to (1.2) satisfies the operator splitting estimate: There exist $K',\tau'>0$ such that for all $t\in[0,1)$ and $\tau\in(0,\tau')$ saisfying $t+\tau\leq 1$, it holds

$$\|\mu_{t+\tau}-(\Phi_\tau^{v[\mu_t]}\#\mu_t+\tau h[\mu_t])\|^{a,b}\leq K'\tau^2. \tag{4.17}$$

To this end, consider μ_t as the solution to the non-autonomous linear Equation (4.16), as in Step 2: This allows to define the trajectory μ_t^k given by the scheme presented in Section 4.2 and to prove that μ_t is the limit of μ_t^k. We now prove an estimate similar to (4.17) for μ_t^k.

Fix $k\in\mathbb{N}^*$ and let $t=i\Delta t$, $\tau=n\Delta t<\log(2)/L$ for some $i\in\{0,\ldots,2^k-1\}$, $n\in\mathbb{N}$ and $t+\tau\leq 1$ This ensures $e^{L\tau}\leq 1+2L\tau$. Define

$$a_n:=\|\mu_{(i+n)\Delta t}^k-(\Phi_{n\Delta t}^{v[\mu_{i\Delta t}^k]}\#\mu_{i\Delta t}^k+n\Delta t h[\mu_{i\Delta t}^k])\|^{a,b}.$$

Observe that it holds $a_1=0$, while for $n\geq 2$ it holds

$$\begin{aligned}
a_n &\leq \|\Phi_{\Delta t}^{v[\mu^k_{(i+n-1)\Delta t}]}\#\mu^k_{(i+n-1)\Delta t}-(\Phi_{\Delta t}^{v[\mu^k_{i\Delta t}]}\#(\Phi_{(n-1)\Delta t}^{v[\mu^k_{i\Delta t}]}\#\mu^k_{i\Delta t}+(n-1)\Delta t h[\mu^k_{i\Delta t}]))\|^{a,b}\\
&\quad+\|\Delta t h[\mu^k_{(i+n-1)\Delta t}]-\Delta t h[\mu^k_{i\Delta t}]\|^{a,b}+\|\Phi_{\Delta t}^{v[\mu^k_{i\Delta t}]}\#(n-1)\Delta t h[\mu^k_{i\Delta t}]-(n-1)\Delta t h[\mu^k_{i\Delta t}]\|^{a,b}\\
&\leq e^{L\Delta t}\|\mu^k_{(i+n-1)\Delta t}-(\Phi_{(n-1)\Delta t}^{v[\mu^k_t]}\#\mu^k_{i\Delta t}+(n-1)\Delta t h[\mu^k_{i\Delta t}])\|^{a,b}\\
&\quad+b|\mu^k_{i\Delta t}|\frac{e^{L\Delta t}-1}{L}\|v[\mu^k_{(i+n-1)\Delta t}]-v[\mu^k_{i\Delta t}]\|_{\mathcal{C}^0}+\Delta t Q\|\mu^k_{(i+n-1)\Delta t}-\mu^k_{i\Delta t}\|^{a,b}\\
&\quad+(n-1)\Delta t(\Delta t b M|h[\mu^k_{i\Delta t}]|\\
&\leq (1+2L\Delta t)a_{n-1}+(|\mu_0|+P)2\Delta t a_{n-1}+\Delta t Q a_{n-1}+(n-1)\Delta t^2 MP\\
&\leq (1+K_1'\Delta t)a_{n-1}+K_2'\tau\Delta t.
\end{aligned}$$

Here, we used (H-1)-(H-2)-(H-3), Lemma (4.2) as well as Lipschitz continuity and boundedness of mass proved in Proposition 4.1. Observe that K_1',K_2' do not depend on n or k, with $K_1>L$. Thus, choose $\tau'=\log(2)/K_1$ independent of k and observe that for all $\tau\in(0,\tau')$ it holds

$$\begin{aligned}
&\|\mu^k_{(i+n)\Delta t}-(\Phi_{n\Delta t}^{v[\mu^k_{i\Delta t}]}\#\mu^k_{i\Delta t}+n\Delta t h[\mu^k_{i\Delta t}])\|^{a,b}\\
&\leq K_2\tau\Delta t\frac{(1+K_1'\Delta t)^n-1}{K_1'\Delta t}=\frac{K_2'}{K_1'}\tau(e^{K_1'\tau}-1)\\
&\leq 2K_2'\tau^2.
\end{aligned}\tag{4.18}$$

We are now ready to prove (4.17). For $t\in[0,1)$ fixed, build the sequence $i_k\in\{0,\dots 2^k-1\}$ such that $|t-i_k2^{-k}|<2^{-k}$. Similarly, for $\tau\in(0,\tau')$ fixed, build the sequence $n_k\in\{0,\dots 2^k\}$ such that $|t-(i_k+n_k)2^{-k}|<2^{-k}$. Observe that $\lim_{k\to+\infty}\|\mu-\mu^k\|=0$, together with (4.6), implies

$$\begin{aligned}
\|\mu_{t+\tau}-\mu^k_{(i_k+n_k)2^{-k}}\|^{a,b}&\leq\|\mu_{t+\tau}-\mu^k_{t+\tau}\|^{a,b}+\|\mu^k_{t+\tau}-\mu^k_{(i_k+n_k)2^{-k}}\|^{a,b}\\
&\leq\|\mu-\mu^k\|+L'2^{-k}\to 0.
\end{aligned}$$

and similarly $\|\mu_t-\mu^k_{i_k2^{-k}}\|^{a,b}\to 0$. By using (H-1)-(H-2)-(H-3) and Lemma 4.2, this in turn ensures

$$\begin{aligned}
&\|\Phi_{n_k2^{-k}}^{v[\mu^k_{i_k2^{-k}}]}\#\mu^k_{i_k2^{-k}}-\Phi_\tau^{v[\mu_t]}\#\mu_t\|^{a,b}\\
&\leq\|\Phi_{n_k2^{-k}}^{v[\mu^k_{i_k2^{-k}}]}\#\mu^k_{i_k2^{-k}}-\Phi_\tau^{v[\mu^k_{i_k2^{-k}}]}\#\mu^k_{i_k2^{-k}}\|^{a,b}\\
&\quad+\|\Phi_\tau^{v[\mu^k_{i_k2^{-k}}]}\#\mu^k_{i_k2^{-k}}-\Phi_\tau^{v[\mu_t]}\#\mu^k_{i_k2^{-k}}\|^{a,b}+\|\Phi_\tau^{v[\mu_t]}\#\mu^k_{i_k2^{-k}}-\Phi_\tau^{v[\mu_t]}\#\mu_t\|^{a,b}\\
&\leq|\mu^k_{i_k2^{-k}}|M|n_k2^{-k}-\tau|+|\mu^k_{i_k2^{-k}}|\frac{e^{L\tau}-1}{L}K\|\mu_t-\mu^k_{i_k2^{-k}}\|^{a,b}+e^{L\tau}\|\mu_t-\mu^k_{i_k2^{-k}}\|^{a,b}\to 0
\end{aligned}$$

as well as

$$\begin{aligned}
&\|n_k2^{-k}h[\mu^k_{i_k2^{-k}}]-\tau h[\mu_t]\|^{a,b}\\
&\leq\|n_k2^{-k}h[\mu^k_{i_k2^{-k}}]-\tau h[\mu^k_{i_k2^{-k}}]\|^{a,b}+\tau\|h[\mu^k_{i_k2^{-k}}]-\tau h[\mu_t]\|^{a,b}\\
&\leq P|n_k2^{-k}-\tau|+\tau Q\|\mu^k_{i_k2^{-k}}-\mu_t\|^{a,b}\to 0.
\end{aligned}$$

Since $n_k 2^{-k} \to \tau \in (0,\tau')$, for k sufficiently large one can apply (4.18), thus

$$\begin{aligned}
&\lim_{k\to+\infty} \|\mu_{t+\tau} - (\Phi_\tau^{v[\mu_t]}\#\mu_t + \tau h[\mu_t])\|^{a,b} \\
&\le \lim_{k\to+\infty} \Big(\|\mu_{t+\tau} - \mu^k_{(i_k+n_k)2^{-k}}\|^{a,b} \\
&\quad + \|\mu^k_{(i_k+n_k)2^{-k}} - (\Phi_{n_k 2^{-k}}^{v[\mu^k_{i_k 2^{-k}}]}\#\mu^k_{i_k 2^{-k}} + n_k 2^{-k} h[\mu^k_{i_k 2^{-k}}])\|^{a,b} \\
&\quad + \|\Phi_{n_k 2^{-k}}^{v[\mu^k_{i_k 2^{-k}}]}\#\mu^k_{i_k 2^{-k}} - \Phi_\tau^{v[\mu_t]}\#\mu_t\|^{a,b} + \|n_k 2^{-k} h[\mu^k_{i_k 2^{-k}}] - \tau h[\mu_t]\|^{a,b} \Big) \\
&\le 0 + 2K_2'\tau^2 + 0 + 0.
\end{aligned}$$

that is (4.17) for $K' = 2K_2'$.

Step 4. Uniqueness of the solution to (1.2) and continuous dependence. Assume that μ_t and ν_t are two solutions to (1.2) with initial condition μ_0, ν_0, respectively. Define $\varepsilon(t) := \|\mu_t - \nu_t\|^{a,b}$, that is a Lipschitz function by Step 2. We denote

$$R_\mu(t,\tau) = \mu_{t+\tau} - (\Phi_\tau^{v[\mu_t]}\#\mu_t + \tau h[\mu_t]), \quad R_\nu(t,\tau) = \nu_{t+\tau} - (\Phi_\tau^{v[\nu_t]}\#\nu_t + \tau h[\nu_t]).$$

Fix $\tau < \tau' < \log(2)/L$, that ensures $e^{L\tau} \le 1 + 2L\tau$. By Step 3, it holds

$$\begin{aligned}
\varepsilon(t+\tau) &= \|\mu_{t+\tau} - \nu_{t+\tau}\|^{a,b} = \|\Phi_\tau^{v[\mu_t]}\#\mu_t + \tau h[\mu_t] + R_\mu(t,\tau) - \Phi_\tau^{v[\nu_t]}\#\nu_t - \tau h[\nu_t] - R_\nu(t,\tau)\|^{a,b} \\
&\le \|\Phi_\tau^{v[\mu_t]}\#\mu_t - \Phi_\tau^{v[\nu_t]}\#\nu_t\|^{a,b} + \tau\|h[\mu_t] - h[\nu_t]\|^{a,b} + \|R_\mu(t,\tau)\|^{a,b} + \|R_\nu(t,\tau)\|^{a,b} \\
&\le e^{L\tau}\|\mu_t - \nu_t\|^{a,b} + (P + \min\{|\mu_0|,|\nu_0|\})\tfrac{e^{L\tau}-1}{L}\|v[\mu_t] - v[\nu_t]\|_{\mathcal{C}^0} + \tau Q\|\mu_t - \nu_t\|^{a,b} + 2K'\tau^2 \\
&\le \left(e^{L\tau} + b(P + \min\{|\mu_0|,|\nu_0|\})2\tau K + \tau Q\right)\varepsilon(t) + 2K'\tau^2 \le (1+\tau K_3)\varepsilon(t) + 2K'\tau^2
\end{aligned}$$

for $K_2 = 2L + 2Kb(P + \min\{|\mu_0|,|\nu_0|\}) + Q$. By letting $\tau \to 0$, we deduce $\varepsilon'(t) \le K_3\varepsilon(t)$ almost everywhere. Then, $\varepsilon(t) \le \varepsilon(0)\exp(K_2 t)$, that implies continuous dependence with respect to the initial data.

Moreover, if $\mu_0 = \nu_0$, then $\varepsilon(0) = 0$, thus $\varepsilon(t) = 0$ for all t. Since $\|.\|^{a,b}$ is a norm, this implies $\mu_t = \nu_t$ for all t, that corresponds to uniqueness of the solution.

Appendix. Measure theory and signed measures. In this appendix, we recall standard results of measure theory and signed measures. In this section, μ and ν are either in $\mathcal{M}(\mathbb{R}^d)$, i.e. they are unsigned measures, or in $\mathcal{M}^s(\mathbb{R}^d)$, i.e. they are signed measures.

Definition A.1. *We say that $\mu,\nu \in \mathcal{M}(\mathbb{R}^d)$ satisfy*

- $\mu << \nu$ *if* $\forall A \in \mathcal{B}(\mathbb{R}^d)$, $(\nu(A) = 0) \Rightarrow (\mu(A) = 0)$
- $\mu \le \nu$ *if* $\forall A \in \mathcal{B}(\mathbb{R}^d)$, $\mu(A) \le \nu(A)$
- $\mu \perp \nu$ *if there exists* $E \in \mathcal{B}(\mathbb{R}^d)$ *such that* $\mu(\mathbb{R}^d) = \mu(E)$ *and* $\nu(E) = 0$

The concept of largest common measure between measures is now recalled.

Lemma A.1. *Let $\mu,\nu \in \mathcal{M}(\mathbb{R}^d)$. Then, there exists a unique measure $\mu \wedge \nu$ which satisfies*

$$\mu \wedge \nu \le \mu, \quad \mu \wedge \nu \le \nu, \quad (\eta \le \mu \text{ and } \eta \le \nu) \Rightarrow \eta \le \mu \wedge \nu. \tag{A.1}$$

We refer to $\mu \wedge \nu$ as the largest common measure to μ and ν. Moreover, denoting by f the Radon Nikodym derivative of μ with respect to ν, i.e. the unique measurable function f such that $\mu = f\nu + \nu_\perp$, with $\nu_\perp \perp \nu$, we have

$$\mu \wedge \nu = \min\{f, 1\}\nu. \tag{A.2}$$

Proof. The uniqueness is clear using (A.1). Existence is given by formula (A.2) as follows. First, it is obvious that $\min\{f,1\}\nu\le\nu$ and using $\mu=f\nu+\nu_\perp$, it is also clear that $\min\{f,1\}\nu\le\mu$. Let us now assume by contradiction the existence of a measure η and of $A\in\mathcal{B}(\mathbb{R}^d)$ such that

$$\eta\le\mu,\quad \eta\le\nu,\quad \eta(A)>\int_A \min\{f,1\}d\nu. \tag{A.3}$$

Since $\nu_\perp\perp\nu$, there exists $E\in\mathcal{B}(\mathbb{R}^d)$ such that $\nu(\mathbb{R}^d)=\nu(E)$ and $\nu_\perp(E)=0$ (see Definition A.1), then $\nu(A)=\nu(A\cap E)$ and $\nu_\perp(A)=\nu_\perp(A\cap E^c)$. Since $\eta\le\nu$, we have $\eta(A\cap E^c)=0$, thus $\eta(A\cap E)=\eta(A)$, and using (A.3)

$$\eta(A\cap E)>\int_{A\cap E}\min\{f,1\}d\nu.$$

We define

$$B=A\cap E\cap\{f>1\}.$$

Then

$$\begin{aligned}\eta(B)+\eta((A\cap E)\setminus B)&=\eta(A\cap E)\\&>\int_{A\cap E}\min\{f,1\}d\nu(x)\\&=\int_B\min\{f,1\}d\nu(x)+\int_{(A\cap E)\setminus B}\min\{f,1\}d\nu\\&=\int_B 1d\nu+\int_{(A\cap E)\setminus B}fd\nu\\&=\nu(B)+\mu((A\cap E)\setminus B)\end{aligned}$$

which contradicts the fact that both $\eta\le\nu$ and $\eta\le\mu$. This implies that η satisfying (A.3) does not exist, and then Lemma A.1 holds. ∎

LEMMA A.2 (Weak compactness for positive measures). *Let μ_n be a sequence of measures in $\mathcal{M}(\mathbb{R}^d)$ that are uniformly bounded in mass. We can then extract a subsequence $\mu_{\phi(n)}$ such that $\mu_{\phi(n)}$ converges vaguely to μ for some $\mu\in\mathcal{M}(\mathbb{R}^d)$.*

A proof can be found in [14, Theorem 1.41].

We finally recall two definitions for signed measures.

DEFINITION A.2 (Push-forward). *For $\mu\in\mathcal{M}^s(\mathbb{R}^d)$ and $T:\mathbb{R}^d\to\mathbb{R}^d$ a Borel map, the push-forward $T\#\mu$ is the measure on $\mathbb{R}^d$ defined by $T\#\mu(B)=\mu(T^{-1}(B))$ for any Borel set $B\subset\mathbb{R}^d$.*

DEFINITION A.3 (Tightness). *A sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures in $\mathcal{M}(\mathbb{R}^d)$ is tight if for each $\varepsilon>0$, there is a compact set $K\subset\mathbb{R}^d$ such that for all $n\ge0$, $\mu_n(\mathbb{R}^d\setminus K)<\varepsilon$. A sequence $(\mu_n)_{n\in\mathbb{N}}$ of signed measures of $\mathcal{M}^s(\mathbb{R}^d)$ is tight if the sequences $(\mu_n^+)_{n\in\mathbb{N}}$ and $(\mu_n^-)_{n\in\mathbb{N}}$ given by the Jordan decomposition are both tight.*

REFERENCES

[1] G. Albi, M. Bongini, F. Rossi, and F. Solombrino, *Leader formation with mean-field birth and death models*, Math. Model. Meth. Appl. Sci., 29(04):633–679, 2019. 1

[2] L. Ambrosio and W. Gangbo, *Hamiltonian ODEs in the Wasserstein space of probability measures*, Commun. Pure Appl. Math., 61(1):18–53, 2008. 1

[3] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient Flows: In Metric Spaces and in the Space of Probability Measures*, Springer Science & Business Media, 2008. 1, 4

[4] L. Ambrosio, E. Mainini, and S. Serfaty, *Gradient flow of the Chapman-Rubinstein-Schatzman model for signed vortices*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 28(2):217–246, 2011. 1

[5] F. Bolley, Y. Brenier, and G. Loeper, *Contractive metrics for scalar conservation laws*, J. Hyperbolic Differ. Equ., 2(1):91–107, 2005. 1

[6] G.A. Bonaschi, J.A. Carrillo, M.D. Francesco, and M.A. Peletier, *Equivalence of gradient flows and entropy solutions for singular nonlocal interaction equations in 1D*, ESAIM Control Optim. Calc. Var., 21(2):414–441, 2015. 1

[7] A. Bressan and B. Piccoli, *Introduction to the Mathematical Theory of Control*, AIMS Series on Appl. Math., AIMS, 2, 2007. 4.1

[8] H. Brezis, *Functional Analysis*, Collection of Applied Mathematics for the Master's Degree, Masson, Paris, 1983. 3.2

[9] L.A. Caffarelli and R.J. McCann, *Free boundaries in optimal transport and Monge-Ampère obstacle problems*, Ann. Math., 171(2):673–730, 2010. 1

[10] L. Chizat, G. Peyré, B. Schmitzer, and F.-X. Vialard, *Unbalanced optimal transport: Dynamic and Kantorovich formulations*, J. Funct. Anal., 274(11):3090–3123, 2018. 1

[11] L. Chizat, G. Peyré, B. Schmitzer, and F.-X. Vialard, *An interpolating distance between optimal transport and Fisher–Rao metrics*, Found. Comput. Math., 18(1):1–44, 2018. 1

[12] B. Düring, P. Markowich, J.-F. Pietschmann, and M.-T. Wolfram, *Boltzmann and Fokker-Planck equations modelling opinion formation in the presence of strong leaders*, Proc. Roy. Soc. A Math. Phys. Eng. Sci., 465(2112):3687–3708, 2009. 1

[13] M. Erbar, *The heat equation on manifolds as a gradient flow in the Wasserstein space*, Ann. Inst. H. Poincaré Probab. Statist., 46(1):1–23, 2010. 1

[14] L.C. Evans and R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015. 2, 4.3

[15] A. Figalli, *The optimal partial transport problem*, Arch. Ration. Mech. Anal., 195(2):533–560, 2010. 1

[16] A. Figalli and N. Gigli, *A new transportation distance between non-negative measures, with applications to gradients flows with Dirichlet boundary conditions*, J. Math. Pures Appl., 94(2):107–130, 2010. 1

[17] L.G. Hanin, *An extension of the Kantorovich norm*, in L.A. Caffarelli and M. Milman (eds.), Monge Ampère equation: Applications to Geometry and Optimization, Contemp. Math., Amer. Math. Soc., Providence, RI, 226:113–130, 1999. 1, 1, 2, 3, 3.2

[18] F. James and N. Vauchelet, *Equivalence between duality and gradient flow solutions for one-dimensional aggregation equations*, Discrete Contin. Dyn. Syst., 36(3):1355–1382, 2016. 1

[19] S. Kondratyev, L. Monsaingeon, and D. Vorotnikov, *A new optimal transport distance on the space of finite Radon measures*, Adv. Differ. Equ., 21(11-12):1117–1164, 2016. 1

[20] M. Liero, A. Mielke, and G. Savaré, *Optimal transport in competition with reaction: The Hellinger-Kantorovich distance and geodesic curves*, SIAM J. Math. Anal., 48(4):2869–2911, 2016. 1

[21] E. Mainini, *A description of transport cost for signed measures*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov., 390:147–181, 2011. 1

[22] M.D. Multerer, L.D. Wittwer, A. Stopka, D. Barac, C. Lang, and D. Iber, *Simulation of morphogen and tissue dynamics*, in J. Dubrulle (eds.), Morphogen Gradients. Methods in Molecular Biology, Vol. 1863. Humana Press, New York, NY, 223–250, 2018. 1

[23] B. Piccoli and F. Rossi, *Transport equation with nonlocal velocity in Wasserstein spaces: convergence of numerical schemes*, Acta Appl. Math., 124:73–105, 2013. 1

[24] B. Piccoli and F. Rossi, *Generalized Wasserstein distance and its application to transport equations with source*, Arch. Ration. Mech. Anal., 211:335–358, 2014. 1, 1, 2, 2.3, 2, 2, 3, 3.2

[25] B. Piccoli and F. Rossi, *On properties of the generalized Wasserstein distance*, Arch. Ration. Mech. Anal., 222(3):1339–1365, 2016. 1, 2, 2, 2, 3.2, 4.1, 4.3

[26] B. Piccoli and F. Rossi, *Measure-theoretic models for crowd dynamics*, in L. Gibelli and N. Bellomo (eds.), Crowd Dynamics, Volume 1. Modeling and Simulation in Science, Engineering and Technology, Springer, 1:137–165, 2018. 1

[27] C. Villani, *Topics in Optimal Transportation*, Graduate Studies in Mathematics, Amer. Math. Soc., Providence, RI, 58, 2003. 1, 2, 4.1

[28] C. Villani, *Optimal Transport: Old and New*, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 338, 2009. 1, 3.2

[29] A. Zygmund, *Trigonometric Series. Vol. I, II*, Cambridge University Press, 2002. 3.3