

## A QUANTITATIVE VERSION OF THE TRANSVERSALITY THEOREM\*

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*Dedicated to Professor Duong Minh Duc on the occasion of his 70th birthday*

**Abstract.** The present paper studies a quantitative version of the transversality theorem. More precisely, given a continuous function  $g \in \mathcal{C}([0, 1]^d, \mathbb{R}^m)$  and a global smooth manifold  $W \subset \mathbb{R}^m$  of dimension  $p$ , we establish a quantitative estimate on the  $(d + p - m)$ -dimensional Hausdorff measure of the set  $\mathcal{Z}_W^g = \{x \in [0, 1]^d : g(x) \in W\}$ . The obtained result is applied to quantify the total number of shock curves in weak entropy solutions to scalar conservation laws with uniformly convex fluxes in one space dimension.

**Keywords.** Transversality lemma; Quantitative estimates; Conservation laws.

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### 1. Introduction

Given two smooth manifolds  $X$  of dimension  $d$  and  $Y$  of dimension  $m$ , let  $g : X \rightarrow Y$  be a  $\mathcal{C}^1$  map. For any smooth submanifold  $W$  of  $Y$ , we say that the function  $g$  is transverse to  $W$  and write  $g \bar{\cap} W$  if

$$(dg)_p(T_p X) + T_{g(p)}(W) = T_{g(p)}(Y) \quad \text{for all } p \in g^{-1}(W).$$

The transversality lemma, which is the key to studying Thom's transversality theorem [16, 18, 23], shows that the set of transverse maps is dense [6]. In particular, given a smooth manifold  $W \subset \mathbb{R}^m$  of dimension  $p$ , for any continuous function  $g : [0, 1]^d \mapsto \mathbb{R}^m$  and any  $\varepsilon > 0$ , there exists a  $\mathcal{C}^1$  function  $g_\varepsilon : [0, 1]^d \rightarrow \mathbb{R}^m$  such that

$$\|g_\varepsilon - g\|_{\mathcal{C}^1} \leq \varepsilon \quad \text{and} \quad g_\varepsilon \bar{\cap} W.$$

For every  $h \in \mathcal{C}([0, 1]^d, \mathbb{R}^m)$ , consider the set

$$\mathcal{Z}_W^h := \{x \in [0, 1]^d : h(x) \in W\}. \tag{1.1}$$

If  $h$  is smooth and transverse to  $W$ , then  $\mathcal{Z}_W^h$  is a  $(d + p - m)$ -dimensional smooth manifold. Hence, its  $(d + p - m)$ -dimensional Hausdorff measure is finite. In this paper, we perform a quantitative analysis of the measure of  $\mathcal{Z}_W^g$ . Namely, how small can we make this measure, by an  $\varepsilon$ -perturbation of  $g$ ? To formulate more precisely our result, given  $g \in \mathcal{C}([0, 1]^d, \mathbb{R}^m)$ , define

$$\mathcal{N}_W^g(\varepsilon) := \inf_{\|h-g\|_{\mathcal{C}^0} \leq \varepsilon} \mathcal{H}^{d+p-m}(\mathcal{Z}_W^h)$$

to be the smallest  $(d + p - m)$ -Hausdorff measure of  $\mathcal{Z}_W^h$  among all functions  $h \in \mathcal{C}([0, 1]^d, \mathbb{R}^m)$  with  $\|h - g\|_{\mathcal{C}^0} \leq \varepsilon$ . Relying on the concept of Kolmogorov  $\varepsilon$ -entropy [17], we will establish an upper bound on the number  $\mathcal{N}_W^g(\varepsilon)$  for a general continuous function  $g : [0, 1]^d \rightarrow \mathbb{R}^m$ . The result can be extended to the case of continuous functions  $g : X \rightarrow Y$

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where  $X, Y$  are smooth manifolds and  $W \subseteq Y$  is a smooth submanifold of  $Y$ . Specifically, we obtain the following estimate for the class of Hölder continuous functions.

**THEOREM 1.1.** *Assume that  $p + d \geq m$  and  $g \in C^\alpha([0, 1]^d, \mathbb{R}^m)$  is Hölder continuous with exponent  $\alpha \in ]0, 1]$ . Then for every  $\varepsilon > 0$  sufficiently small, it holds*

$$\mathcal{N}_W^g(\varepsilon) \leq C_W \cdot \left( \frac{\|g\|_{C^{0,\alpha}}}{\varepsilon} \right)^{\frac{m-p}{\alpha}}, \tag{1.2}$$

where the constant  $C_W > 0$  depends only on  $W$  and  $\|g\|_{C^{0,\alpha}}$  is the Hölder norm of  $g$ .

In the scalar case ( $d = m = 1$ ), the blow up rate  $(\frac{1}{\varepsilon})^{\frac{m-p}{\alpha}}$  with respect to  $\varepsilon$  is shown to be the best bound in terms of power function in Example 3.1. For the multi-dimensional cases ( $d \geq 2$ ), this should be still true but the situation becomes considerably more technical. We leave this open.

In the second part of the paper, we give an application to conservation laws. For several classes of hyperbolic PDEs, one can prove that there exists an open dense set of initial data for which the solution develops at most a finite number of singularities [5, 10, 11, 22]. A natural question is to provide a quantitative estimate on this number. For example, consider the scalar conservation laws in one space dimension

$$u_t(t, x) + f(u(t, x))_x = 0 \quad (t, x) \in [0, \infty[ \times \mathbb{R}, \tag{1.3}$$

with strictly convex flux  $f$ . In this case, there is a connection between (1.3) and the Hamilton-Jacobi equation which induces an explicit representation of solutions. Using this representation, Oleinik [19–21] shows that solutions of (1.3) are continuous, except on the union of an at most countable set of shock curves. Analogous results are also established for solutions to genuinely nonlinear hyperbolic systems of conservation laws in [7, 12, 14, 15]. The structure and smoothness of solutions to (1.3) were studied in [10], using the concept of generalized characteristics. For a dense set of initial data, a stronger regularity property holds. Namely, the total number of shock curves is finite [22]. Here, in the spirit of metric entropy, which was used in the study of the compactness estimates for solution sets [2–4, 8], we shall provide quantitative estimates on the number of shock curves in an entropy weak solution  $u$  to (1.3), which is a weak solution of (1.3) in the sense of distributions and satisfies an entropy criterion for admissibility

$$u(t, x-) \geq u(t, x+) \quad \text{for a.e. } t > 0, x \in \mathbb{R}.$$

More precisely, assuming that  $f \in C^4(\mathbb{R})$  is uniformly convex, i.e.,

$$f''(u) \geq \lambda > 0 \quad \text{for all } u \in \mathbb{R}. \tag{1.4}$$

For any given  $\varepsilon > 0$  and  $\bar{u} \in \mathbf{L}^1(\mathbb{R})$  with a compact support, we seek a perturbed initial datum  $\bar{v} \in C_c^3(\mathbb{R})$ , with  $\|\bar{v} - \bar{u}\|_{\mathbf{L}^1} \leq \varepsilon$  such that the solution  $v = v(t, x)$  of (1.3) with  $v(0, \cdot) = \bar{v}$  has the total number of shocks bounded in terms of  $\varepsilon^{-1}$ . The next simplified theorem provides an upper bound on this number of shocks.

**THEOREM 1.2.** *Let the flux function  $f$  be  $C^4$ -smooth and satisfy (1.4). Given constants  $V, V > 0$ , assume that  $\bar{u} \in \mathbf{L}^1(\mathbb{R})$  satisfies*

$$\text{Supp}(\bar{u}) \subseteq [-R, R] \quad \text{and} \quad \text{Tot. Var.}\{\bar{u}\} < V. \tag{1.5}$$

Then, for some constant  $C$ , the following holds. For every  $\varepsilon > 0$  sufficiently small, there exists  $\bar{v} \in C^3(\mathbb{R})$  with  $\text{Supp}(\bar{v}) \subseteq [-2R, 2R]$  and  $\|\bar{v} - \bar{u}\|_{L^1} \leq \varepsilon$ , such that the entropy weak solution  $v = v(t, x)$  of (1.3) with initial datum  $v(0, \cdot) = \bar{v}$  satisfies

$$[\text{Total number of shock curves of } v] \leq \frac{C}{\lambda} \cdot \frac{R^4 V^5}{\varepsilon^4} + 4. \tag{1.6}$$

The proof of Theorem 1.2 relies on Theorem 1.1 and the observation that the total number of shock curves arising in the solution  $v$  is bounded by the total number of inflection points of the function  $x \rightarrow f'(\bar{v}(x))$ . Finally, we remark that the constant  $C$  is explicitly computed in (4.13) and the result is proved for  $C^3$ -smooth  $f$  in Theorem 4.1.

The remainder of this paper is organized as follows. In Section 2, we recall basic concepts on the inverse of the minimal modulus of continuity and Kolmogorov  $\varepsilon$ -entropy, and also include a necessary result on the partition of the unit cube into polytopes in  $\mathbb{R}^d$ . Section 3 contains a general result on an upper estimate for  $\mathcal{N}_W^g(\varepsilon)$ , while Section 4 provides a brief review on the scalar conservation laws with uniformly convex fluxes in one space dimension and extends Theorem 1.2 to the case of  $C^3$ -smooth  $f$ .

**2. Notations and preliminaries**

Let  $d \geq 1$  and  $n \geq 1$  be integers and  $D$  be a measurable subset of  $\mathbb{R}^d$ . Throughout the paper we shall denote by:

- $|\cdot|$  the Euclidean norm of  $\mathbb{R}^d$ ;
- $B_d(a, r) = \{x \in \mathbb{R}^d : |x - a| < r\}$  the ball of radius  $r$  centered at  $a \in \mathbb{R}^d$  and

$$B_d(A, r) = \bigcup_{a \in A} B_d(a, r) \quad \text{for all } r \geq 0, A \subseteq \mathbb{R}^d;$$

- $\text{Int}(D)$  the interior of  $D$ ;
- $\text{Diam}(D) = \sup_{x, y \in D} |x - y|$ , the diameter of  $D$ ;
- $\chi_D = \begin{cases} 1 & \text{if } x \in D \\ 0 & \text{if } x \in \mathbb{R}^d \setminus D \end{cases}$  the characteristic function of  $D$ ;
- $\mathcal{H}^s(D)$  the  $s$ -dimensional Hausdorff measure of  $D$ ;
- $\#(S)$  the number of elements of any finite set  $S$ ;
- $L^1(\mathbb{R})$  the Lebesgue space of all (equivalence classes of) summable functions on  $\mathbb{R}$ , equipped with the usual norm  $\|\cdot\|_{L^1}$ ;
- $L^\infty(\mathbb{R})$  the space of all essentially bounded functions on  $\mathbb{R}$ , equipped with the usual norm  $\|\cdot\|_{L^\infty}$ ;
- $C^n(\mathbb{R})$ , the space of smooth functions on  $\mathbb{R}$  having continuous derivatives  $f', f'', \dots, f^{(n)}$ , equipped with the usual norm  $\|\cdot\|_{C^n}$ ;
- $\text{Tot.Var.}\{g, I\}$  total variation of a function  $g$  over an open interval  $I$  in  $\mathbb{R}$ ;
- $\text{Supp}(u)$  the essential support of a function  $u \in L^\infty(\mathbb{R})$ ;
- $[x] := \max\{z \in \mathbb{Z} : z \leq x\}$  the integer part of  $x$ .

In order to obtain the estimate (1.2) for general continuous functions, let us introduce the inverse of the minimal modulus of a continuity.

DEFINITION 2.1. *Given subsets  $U \subseteq \mathbb{R}^d$  and  $V \subseteq \mathbb{R}^m$ , let  $h : U \rightarrow V$  be continuous. The*

minimal modulus of continuity of  $h$  is given by

$$\omega_h(\delta) = \sup_{x,y \in U, |x-y| \leq \delta} |h(y) - h(x)| \quad \text{for all } \delta \in [0, \text{diam}(U)]. \tag{2.1}$$

The inverse of the minimal modulus of continuity of  $h$  is the map  $s \rightarrow \Psi_h(s)$  and is defined by

$$\Psi_h(s) := \sup \{ \delta \geq 0 : |h(x) - h(y)| \leq s \text{ for all } |x - y| \leq \delta, x, y \in U \} \tag{2.2}$$

for all  $s \geq 0$ .

From the above definition, it is clear that  $\Psi_h(s) = \infty$  for all  $s \in [M_h, \infty[$  with  $M_h := \sup_{x,y \in U} |h(x) - h(y)|$ . In particular, if  $h$  is a constant function then  $\Psi_h(s) = \infty$  for all  $s \geq 0$ . Otherwise, by the continuity of  $h$ , it holds

$$\Psi_h(0) = 0 \quad \text{and} \quad 0 < \Psi_h(s) \leq \text{diam}(U) \quad \text{for all } s \in ]0, M_h[.$$

Moreover,  $\Psi_h(\cdot) : ]0, \infty[ \rightarrow ]0, \infty[$  is increasing and superadditive

$$\Psi_h(s_1 + s_2) \geq \Psi_h(s_1) + \Psi_h(s_2) \quad \text{for all } s_1, s_2 \geq 0.$$

If the map  $\delta \rightarrow \omega_h(\delta)$  is strictly increasing in  $]0, \text{diam}(U)[$  then  $\Psi_h$  is the inverse of  $\omega_h$ , i.e.,

$$\Psi_h(s) = \omega_h^{-1}(s) \quad \text{for all } s \in ]0, M_h[.$$

In the case that  $h$  is Hölder continuous with an exponent  $\alpha \in ]0, 1]$ , for every  $s > 0$  it holds

$$\Psi_h(s) \geq \left( \frac{s}{\|h\|_{C^{0,\alpha}}} \right)^{\frac{1}{\alpha}} \quad \text{with} \quad \|h\|_{C^{0,\alpha}} = \sup_{x,y \in U, x \neq y} \frac{|h(x) - h(y)|}{|x - y|^\alpha}. \tag{2.3}$$

Toward a sharp estimate on  $\mathcal{N}_W^g(\varepsilon)$ , we recall the concept of Kolmogorov  $\varepsilon$ -entropy [17], which has been studied extensively in a variety of literature and disciplines. It plays a central role in various areas of information theory and statistics, including nonparametric function estimation, density information, empirical processes and machine learning. It provides a tool for characterizing the rate of mixing of sets of small measure.

**DEFINITION 2.2.** *Given a metric space  $(E, \rho)$ , let  $K$  be a totally bounded subset of  $E$ . For any  $\varepsilon > 0$ , let  $\mathbf{N}_\varepsilon(K|E)$  be the minimal number of sets in a covering of  $K$  by subsets of  $E$  having diameter no larger than  $2\varepsilon$ . Then the  $\varepsilon$ -entropy of  $K$  is defined as*

$$\mathbf{H}_\varepsilon(K|E) := \log_2 \mathbf{N}_\varepsilon(K|E).$$

To complete this section, let us prove a simple lemma of the decomposition of a unit cube in  $\mathbb{R}^d$  that will be used in the proof of Theorem 3.1.

**LEMMA 2.1.** *Let  $\square^d = [0, 1]^d$  be a unit cube in  $\mathbb{R}^d$ . Then,  $\square^d$  can be decomposed into  $2^{d-1}d!$  polytopes  $\Delta_k^d$  in  $\mathbb{R}^d$  such that  $\Delta_k^d$  has  $(d + 1)$  vertices for  $k \in \{0, 1, \dots, 2^{d-1}d! - 1\}$ .*

*Proof.* The decomposition of  $\square^d$  can be done by using the induction process:

- If  $d = 1$  then  $\square^1$  is an interval  $[0, 1]$ .

- For  $d \geq 2$ , assume that  $\square^{d-1}$  can be decomposed into  $2^{d-2}(d-1)!$  polytopes  $\Delta_\ell^{d-1}$  in  $\mathbb{R}^{d-1}$  such that  $\Delta_\ell^{d-1}$  has  $d$  vertices for  $\ell \in \{0, 1, \dots, 2^{d-2}(d-1)! - 1\}$ . Observe that  $\square^d$  has  $2d$  faces  $\square_h^{d-1} = \partial \square_h^d$  for  $h \in \{0, 1, \dots, 2d-1\}$  which are  $\mathbb{R}^{d-1}$ -cubes of side length 1. Thus, for each  $h \in \{0, 1, \dots, 2d-1\}$ , we can partition  $\square_h^{d-1}$  into  $2^{d-2}(d-1)!$  polytopes  $\Delta_{h,\ell}^{d-1}$  such that  $\Delta_{h,\ell}^{d-1}$  has  $d$  vertices for  $\ell \in \{0, 1, \dots, 2^{d-2}(d-1)! - 1\}$ . Then  $\square^d$  can be partitioned into  $2^{d-1}d!$  polytopes  $\Delta_k^d$  for  $k = \{1, 2, \dots, 2^{d-1}d!\}$  such that

$$\Delta_k^d = \left\{ \theta c + (1 - \theta) \cdot y : \theta \in [0, 1], y \in \Delta_{h,\ell}^{d-1} \right\}, \quad k = h \cdot 2^{d-2}(d-1)! + \ell,$$

with  $c$  being the center of  $\square^d$ .

The proof is complete. □

### 3. Upper estimates on $\mathcal{N}_W^g(\varepsilon)$

In this section, we provide a quantitative study on the Hausdorff measure of  $\mathcal{Z}_W^g$  for general continuous functions  $g \in \mathcal{C}([0, 1]^d, \mathbb{R}^m)$  and  $W \subseteq \mathbb{R}^m$  being a  $\mathcal{C}^1$  manifold with  $\dim(W) = p$ . More precisely, given the definition of the set  $\mathcal{Z}_W^h$  in (1.1), we establish an upper bound for

$$\mathcal{N}_W^g(\varepsilon) := \inf_{h \in \mathcal{C}([0, 1]^d, \mathbb{R}^m), \|h - g\|_{C^0} \leq \varepsilon} \mathcal{H}^{d+p-m}(\mathcal{Z}_W^h). \tag{3.1}$$

Recalling that  $B_m(W, r) = \bigcup_{w \in W} B_m(w, r)$  is a  $r$ -tubular neighbourhood of  $W$  in  $\mathbb{R}^m$ , we shall assume the following additional hypothesis on  $W$ :

- (A1) *There exists an open set  $U$  of  $\mathbb{R}^m$  and a  $\mathcal{C}^1$  diffeomorphism  $\varphi : B_m(W, r) \rightarrow U$  with  $\varphi(W) \subseteq \mathbb{R}^p \times \{0\} \subseteq \mathbb{R}^p \times \mathbb{R}^{m-p}$  for some  $r > 0$ .*

In this case, from (2.2), we denote by

$$\gamma_W := \frac{\lambda_1}{\lambda_2}, \quad \ell(s) = \frac{1}{2\sqrt{d}} \cdot \Psi_g(\gamma_W \cdot s) \quad \text{for all } s > 0, \tag{3.2}$$

with

$$0 < \lambda_1 := \inf_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|} \leq \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|} := \lambda_2 < \infty. \tag{3.3}$$

REMARK 3.1. When  $W$  is the graph of a  $\mathcal{C}^1$  function  $\phi : \mathbb{R}^p \rightarrow \mathbb{R}^{m-p}$  with  $\|\phi\|_{\mathcal{C}^1(\mathbb{R})} < \infty$ , then  $W$  satisfies the hypothesis (A1). Indeed, one can choose  $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^m$  such that

$$\varphi(x, y) = (x, y - \phi(x)) \quad \text{for all } x \in \mathbb{R}^p, y \in \mathbb{R}^{m-p}.$$

In this case, a direct computation yields

$$\min \left\{ \frac{1}{2}, \frac{1}{\sqrt{1 + 4\|\phi\|_{\mathcal{C}^1(\mathbb{R})}^2}} \right\} \leq \lambda_1 \leq \lambda_2 \leq \sqrt{2 + 2\|\phi\|_{\mathcal{C}^1(\mathbb{R})}^2}.$$

Introducing the constant which approximately measures the set  $g^{-1}(B_m(\overline{W}, \varepsilon)) \subseteq [0, 1]^d$  in terms of Kolmogorov  $\varepsilon$ -entropy

$$0 \leq \Lambda_\varepsilon := \min \left\{ (4\ell(\varepsilon))^d \cdot 2^{\mathbf{H}_{\ell(\varepsilon)}(g^{-1}(B_m(\overline{W}, \varepsilon))|_{\mathbb{R}^d})}, 1 \right\}, \tag{3.4}$$

we prove the following result.

**THEOREM 3.1.** *Assume that  $p+d \geq m$  and the  $C^1$  manifold  $W \subseteq \mathbb{R}^m$  satisfies the hypothesis **(A1)**. Then for every  $\varepsilon > 0$  sufficiently small such that  $\Psi_g(\gamma_W \cdot \varepsilon) \leq \sqrt{d}$ , it holds*

$$\mathcal{N}_{\bar{W}}^g(\varepsilon) \leq C \Lambda_\varepsilon \cdot \left( \frac{1}{\Psi_g(\gamma_W \cdot \varepsilon)} \right)^{m-p} \tag{3.5}$$

with the constant  $C = 2^{d+m-p-1} d! d^{d+\frac{p-m}{2}}$ .

*Proof.* The proof is divided into several steps:

**Step 1.** Fix  $0 < \varepsilon < \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot r$ , for every  $\delta > 0$  such that

$$\omega_g(\sqrt{d}\delta) \leq \frac{\lambda_1 \varepsilon}{\lambda_2} \quad \text{and} \quad \varepsilon + \omega_g(\delta) < r, \tag{3.6}$$

we divide  $[0, 1]^d$  into  $(K_\delta)^d$  closed cubes  $\square_\iota$  of side length  $\ell_\delta = \frac{1}{K_\delta} \leq \delta$  with  $K_\delta = \left\lceil \frac{1}{\delta} \right\rceil + 1$  and set

$$\mathcal{I}_\delta = \left\{ \iota \in \{1, \dots, (K_\delta)^d\} : \text{int}(\square_\iota) \cap g^{-1}(B_m(\bar{W}, \varepsilon)) \neq \emptyset \right\}, \quad \mathcal{O}_\delta = \bigcup_{\iota \in \mathcal{I}_\delta} \square_\iota. \tag{3.7}$$

From (2.1) and  $\varepsilon + \omega_g(\delta) < r$ , one has

$$g(\square_\iota) \subseteq B_m(\bar{W}, \varepsilon) + B_m(0, \omega_g(\delta)) \subseteq B_m(W, r) \quad \text{for all } \iota \in \mathcal{I}_\delta,$$

and this implies that  $g(\mathcal{O}_\delta) \subseteq B_m(W, r)$ . Therefore, one can define the function composition  $\tilde{g}: \mathcal{O}_\delta \rightarrow \mathbb{R}^m$  such that

$$\tilde{g}(x) = \varphi(g(x)) \quad \text{for all } x \in \mathcal{O}_\delta. \tag{3.8}$$

Since  $\text{dist}(g(x), W) \geq \varepsilon$  for all  $x \in \partial \mathcal{O}_\delta \setminus \partial [0, 1]^d$ , it holds

$$\inf_{x \in \partial \mathcal{O}_\delta \setminus \partial [0, 1]^d} \text{dist}(\tilde{g}(x), \varphi(W)) \geq \inf_{|x-y| \geq \varepsilon} |\varphi(x) - \varphi(y)| \geq \lambda_1 \varepsilon. \tag{3.9}$$

In the next two steps, we will approximate  $\tilde{g}$  by a function  $\tilde{h}_\delta: \mathcal{O}_\delta \rightarrow \mathbb{R}^m$  such that

(i)  $\tilde{h}_\delta$  is a piecewise continuous function with

$$\left\| \tilde{h}_\delta - \tilde{g} \right\|_{\mathbf{L}^\infty(\mathbb{R})} \leq \lambda_2 \cdot \omega_g(\sqrt{d}\ell_\delta), \quad \inf_{x \in \partial \mathcal{O}_\delta \setminus \partial [0, 1]^d} \text{dist}(\tilde{h}_\delta(x), \varphi(W)) > 0; \tag{3.10}$$

(ii) The  $(d+p-m)$ -Hausdorff measure of  $\mathcal{Z}_{\varphi(W)}^{\tilde{h}_\delta} = \left\{ x \in \mathcal{O}_\delta : \tilde{h}_\delta(x) \in \varphi(W) \right\}$  is bounded by

$$\mathcal{H}^{d+p-m} \left( \mathcal{Z}_{\varphi(W)}^{\tilde{h}_\delta} \right) \leq (2^{2d-1} d! d^{d+p-m}) \cdot \ell_\delta^{d+p-m} \cdot 2^{\mathbf{H}_{\ell_\delta}} \left( g^{-1}(B_m(\bar{W}, \varepsilon)) \Big| \mathbb{R}^d \right). \tag{3.11}$$

**Step 2.** For every  $\iota \in I_\delta$ , following the induction process in Lemma 2.1, we partition  $\square_\iota$  into  $2^{d-1}d!$  polytopes  $\Delta_\iota^k$  in  $\mathbb{R}^d$  such that the set of vertices  $\mathbf{V}_\iota^k$  of  $\Delta_\iota^k$  has  $(d+1)$  elements and is written by

$$\mathbf{V}_\iota^k = \{v_\iota^{k,j} \in \mathbb{R}^d : j \in \{1, 2, \dots, d+1\}\} \quad \text{for all } k \in \{1, 2, \dots, 2^{d-1}d!\}.$$

Set  $m_d := \min\{d, m\}$ . Observe that for any given  $\iota \in I_\delta$ ,  $k \in \{1, 2, \dots, 2^{d-1}d!\}$ , and  $s > 0$ , there are  $m_d$  linearly independent vectors  $z_1, z_2, \dots, z_{m_d}$  in  $\mathbb{R}^m$  with  $|z_j - \tilde{g}(v_\iota^{k,j})| < s$  for  $j \in \{1, 2, \dots, m_d\}$  such that the following subspace of  $\mathbb{R}^m$  has dimension  $m_d + p - m$

$$\text{span}\{z_1 - \tilde{g}(v_\iota^{k,d+1}), z_2 - \tilde{g}(v_\iota^{k,d+1}), \dots, z_{m_d} - \tilde{g}(v_\iota^{k,d+1})\} \cap \mathbb{R}^p \times \{0\}.$$

Thus, up to an arbitrarily small modification on  $\tilde{g}(v_\iota^{k,j})$ , we can assume that for every  $\iota \in I_\delta$  and  $k \in \{1, 2, \dots, 2^{d-1}d!\}$ , the subspace

$$\text{span}\{\tilde{g}(v_\iota^{k,1}) - \tilde{g}(v_\iota^{k,d+1}), \tilde{g}(v_\iota^{k,2}) - \tilde{g}(v_\iota^{k,d+1}), \dots, \tilde{g}(v_\iota^{k,m_d}) - \tilde{g}(v_\iota^{k,d+1})\} \cap \mathbb{R}^p \times \{0\} \tag{3.12}$$

has dimension  $m_d + p - m$ . Denote by  $\nabla^d = \left\{ \alpha \in \mathbb{R}^d : \alpha_j \geq 0, \sum_{j=1}^d \alpha_j \leq 1 \right\}$ , we have

$$\Delta_\iota^k = \left\{ \sum_{j=1}^d \alpha_j \cdot v_\iota^{k,j} + \left(1 - \sum_{j=1}^d \alpha_j\right) \cdot v_\iota^{k,d+1} : \alpha \in \nabla^d \right\}.$$

The piecewise linear continuous function  $\tilde{h}_\iota : \square_\iota \rightarrow \mathbb{R}^m$  is then defined as follows: For all  $k \in \{1, 2, \dots, 2^{d-1}d!\}$ ,  $x = \sum_{j=1}^d \alpha_j \cdot v_\iota^{k,j} + \left(1 - \sum_{j=1}^d \alpha_j\right) \cdot v_\iota^{k,d+1}$  with  $\alpha \in \nabla^d$ , we set

$$\tilde{h}_\iota(x) := \tilde{g}(v_\iota^{k,d+1}) + \sum_{j=1}^d \alpha_j \cdot [\tilde{g}(v_\iota^{k,j}) - \tilde{g}(v_\iota^{k,d+1})]. \tag{3.13}$$

From (3.8) and (3.3), one estimates

$$\begin{aligned} |\tilde{h}_\iota(x) - \tilde{g}(x)| &\leq \sup_{|y-z| \leq \text{diam}(\Delta_\iota^k)} |\tilde{g}(y) - \tilde{g}(z)| \\ &\leq \lambda_2 \cdot \sup_{|y-z| \leq \sqrt{d}\ell_\delta} |g(y) - g(z)| \leq \lambda_2 \cdot \omega_g(\sqrt{d}\ell_\delta). \end{aligned}$$

The function  $\tilde{h}_\delta : \mathcal{O}_\delta \rightarrow \mathbb{R}^m$  is defined by

$$\tilde{h}_\delta(x) = \tilde{h}_\iota(x) \quad \text{for all } x \in \square_\iota, \iota \in I_\delta$$

is continuous and satisfies

$$|\tilde{h}_\delta(x) - \tilde{g}(x)| \leq \lambda_2 \cdot \omega_g(\sqrt{d}\ell_\delta) \quad \text{for all } x \in \mathcal{O}_\delta.$$

Recalling (3.9) and (3.6), we have

$$\inf_{x \in \partial \mathcal{O}_\delta \setminus \partial [0,1]^d} \text{dist}(\tilde{h}_\delta(x), \varphi(W)) \geq \lambda_1 \varepsilon - \lambda_2 \cdot \omega_g(\sqrt{d}\ell_\delta) > 0. \tag{3.14}$$

**Step 3.** Let us show that  $\tilde{h}_\delta$  satisfies (ii). Fix  $\iota \in I_\delta$ , we consider the  $m \times d$  matrix

$$\mathbf{A}_\iota^k = [\tilde{g}(v_\iota^{k,1}) - \tilde{g}(v_\iota^{k,d+1}), \dots, \tilde{g}(v_\iota^{k,d}) - \tilde{g}(v_\iota^{k,d+1})].$$

By the rank-nullity theorem, one has that  $\text{rank}(\mathbf{A}_\iota^k) = m_d$  and

$$\{\alpha \in \mathbb{R}^d : \mathbf{A}_\iota^k \alpha = 0\} = \mathbf{Y}_\iota^k \quad \text{with} \quad \dim(\mathbf{Y}_\iota^k) = d - m_d.$$

Assume that  $\mathbf{X}_\iota^k \oplus \mathbf{Y}_\iota^k = \mathbb{R}^d$ . The linear map  $\alpha \rightarrow \mathbf{A}_\iota^k \alpha$  is injective from  $\mathbf{X}_\iota^k$  to  $\mathbb{R}^m$  and  $\dim(\mathbf{X}_\iota^k) = m_d$ . Thus, from (3.12), the following set is a  $(m_d + p - m)$ -dimensional hyperplane

$$\Gamma_\iota^k := \{\alpha \in \mathbf{X}_\iota^k : \mathbf{A}_\iota^k \alpha \in \mathbb{R}^p \times \{0\} - \tilde{g}(v_\iota^{k,d+1}) \subset \mathbb{R}^p \times \mathbb{R}^{m-p}\}.$$

For every  $k \in \{1, 2, \dots, 2^{d-1}d!\}$ , we set

$$\nabla_k^d := \left\{ \alpha \in \nabla^d : \tilde{h}_\iota \left( \sum_{j=1}^d \alpha_j \cdot v_\iota^{k,j} + \left( 1 - \sum_{j=1}^d \alpha_j \right) \cdot v_\iota^{k,d+1} \right) \in \varphi(W) \subset \mathbb{R}^p \times \{0\} \right\}.$$

From (3.13), it holds

$$\begin{aligned} \nabla_k^d &= \left\{ \alpha \in \nabla^d : \tilde{g}(v_\iota^{k,d+1}) + \sum_{j=1}^d \alpha_j \cdot [\tilde{g}(v_\iota^{k,j}) - \tilde{g}(v_\iota^{k,d+1})] \in \varphi(W) \right\} \\ &= \left\{ \alpha \in \nabla^d : \sum_{j=1}^d \alpha_j \cdot [\tilde{g}(v_\iota^{k,j}) - \tilde{g}(v_\iota^{k,d+1})] \in \varphi(W) - \tilde{g}(v_\iota^{k,d+1}) \right\} \\ &\subseteq \{\alpha \in \mathbb{R}^d : \mathbf{A}_\iota^k \alpha \in \mathbb{R}^p \times \{0\} - \tilde{g}(v_\iota^{k,d+1})\} = \mathbf{Y}_\iota^k + \Gamma_\iota^k. \end{aligned}$$

Observe that  $\mathbf{Y}_\iota^k + \Gamma_\iota^k$  is a  $(d + p - m)$ -dimensional hyperplane. Again recalling (3.13), we obtain

$$\{x \in \Delta_\iota^k : \tilde{h}_\iota(x) \in \varphi(W)\} = \left\{ \sum_{j=1}^d \alpha_j \cdot v_\iota^{k,j} + \left( 1 - \sum_{j=1}^d \alpha_j \right) \cdot v_\iota^{k,d+1} : \alpha \in \nabla_k^d \right\},$$

and

$$\begin{aligned} \mathcal{H}^{d-m+p} \left( \{x \in \Delta_\iota^k : \tilde{h}_\iota(x) \in \varphi(W)\} \right) &\leq \left( \sup_{j \in \{1, \dots, d\}} |v_\iota^{k,j} - v_\iota^{k,d+1}| \right)^{d-m+p} \cdot \mathcal{H}^{d-m+p}(\nabla_k^d) \\ &\leq (\sqrt{d}\ell_\delta)^{d-m+p} \cdot \mathcal{H}^{d-m+p}(\nabla_k^d) \leq (d\ell_\delta)^{d-m+p}. \end{aligned}$$

Thus, for all  $\iota \in \mathcal{I}_\delta$ , it holds

$$\mathcal{H}^{d-m+p} \left( \mathcal{Z}_{\varphi(W)}^{\tilde{h}_\iota} \right) \leq \sum_{k=1}^{2^{d-1}d!} \mathcal{H}^{d-m+p} \left( \{x \in \Delta_\iota^k : \tilde{h}_\iota(x) \in \varphi(W)\} \right) \leq 2^{d-1}d!(d\ell_\delta)^{d-m+p}.$$

By the concept of  $\varepsilon$ -entropy in Definition 2.2, we have

$$g^{-1}(B_m(\overline{W}, \varepsilon)) \subseteq \bigcup_{j=1}^J D_j, \quad J \leq 2^{\mathbf{H}_{\ell_\delta} \left( g^{-1}(B_m(\overline{W}, \varepsilon)) \Big| \mathbb{R}^d \right)},$$



for some  $D_j \subset \mathbb{R}^d$  with  $\text{diam}(D_j) \leq 2\ell_\delta$ . For any  $j \in \{1, \dots, J\}$ , it holds

$$\#\left\{ \iota \in \left\{ 1, \dots, (K_\delta)^d \right\} : \text{int}(\square_\iota) \cap D_j \neq \emptyset \right\} \leq 2^d.$$

Hence,

$$\#(\mathcal{I}_\delta) \leq \min \left\{ 2^d \cdot 2^{\mathbf{H}_{\ell_\delta} \left( g^{-1}(B_m(\overline{W}, \varepsilon)) \middle| \mathbb{R}^d \right)}, (K_\delta)^d = \left( \frac{1}{\ell_\delta} \right)^d \right\}, \tag{3.15}$$

and this yields (3.11) by

$$\mathcal{H}^{d+p-m} \left( \mathcal{Z}_{\varphi(W)}^{\tilde{h}_\delta} \right) \leq \sum_{\iota \in \mathcal{I}_\delta} \mathcal{H}^{d-m+p} \left( \mathcal{Z}_{\varphi(W)}^{\tilde{h}_\iota} \right) \leq \#(\mathcal{I}_\delta) \cdot 2^{d-1} d! (d\ell_\delta)^{d-m+p}. \tag{3.16}$$

**Step 4.** To complete the proof, we first approximate  $g$  by the continuous function  $g_\delta : \mathcal{O}_\delta \rightarrow \mathbb{R}^m$  which is defined by

$$g_\delta(x) = \varphi^{-1}(\tilde{h}_\delta(x)) \quad \text{for all } x \in \mathcal{O}_\delta.$$

From (3.10), it holds

$$|g_\delta(x) - g(x)| = \left| \varphi^{-1}(\tilde{h}_\delta(x)) - \varphi^{-1}(\tilde{g}(x)) \right| \leq \frac{|\tilde{h}_\delta(x) - \tilde{g}(x)|}{\lambda_1} \leq \frac{\lambda_2}{\lambda_1} \cdot \omega_g \left( \sqrt{d}\ell_\delta \right),$$

and

$$\begin{aligned} \inf_{\partial \mathcal{O}_\delta \setminus \partial[0,1]^d} \text{dist}(g_\delta(x), W) &= \inf_{\partial \mathcal{O}_\delta \setminus \partial[0,1]^d} \text{dist}(\varphi^{-1}(\tilde{h}_\delta(x)), \varphi^{-1}(\varphi(W))) \\ &\geq \frac{1}{\lambda_2} \cdot \text{dist}(\tilde{h}_\delta(x), \varphi(W)) > 0. \end{aligned}$$

Since  $\text{dist}(g(x), W) \geq \varepsilon$  for all  $x \in ([0, 1]^d \setminus \mathcal{O}_\delta)$ , we can extend  $g_\delta$  to  $[0, 1]^d$  such that  $g_\delta$  is still continuous with  $\|g_\delta - g\|_{\mathbf{L}^\infty(\mathbb{R})} \leq \frac{\lambda_2}{\lambda_1} \cdot \omega_g \left( \sqrt{d}\ell_\delta \right)$  and  $g_\delta(x)$  does not belong to  $W$  for every  $x \in [0, 1]^d \setminus \mathcal{O}_\delta$ . Thus, (3.16) and (3.15) yield

$$\begin{aligned} \mathcal{H}^{d+p-m}(\mathcal{Z}_W^{g_\delta}) &= \mathcal{H}^{d+p-m}(\{x \in \mathcal{O}_\delta : g_\delta(x) \in W\}) \\ &= \mathcal{H}^{d+p-m}(\{x \in \mathcal{O}_\delta : \tilde{h}_\delta \in \varphi(W)\}) = \mathcal{H}^{d+p-m}(\mathcal{Z}_{\varphi(W)}^{\tilde{h}_\delta}) \\ &\leq \frac{2^{d-1} d! d^{d-m+p}}{\ell_\delta^{m-p}} \cdot \min \left\{ 2^d \ell_\delta^d \cdot 2^{\mathbf{H}_{\ell_\delta} \left( g^{-1}(B_m(\overline{W}, \varepsilon)) \middle| \mathbb{R}^d \right)}, 1 \right\}. \end{aligned}$$

Recalling (2.1) and (2.2), we choose  $\delta = \frac{1}{\sqrt{d}} \cdot \Psi_g \left( \frac{\lambda_1 \varepsilon}{\lambda_2} \right)$  such that  $\omega_g \left( \sqrt{d}\delta \right) \leq \frac{\lambda_1 \varepsilon}{\lambda_2}$ , the condition (3.6) on  $\delta$  holds, and

$$\|g_\delta - g\|_{\mathbf{L}^\infty(\mathbb{R})} \leq \frac{\lambda_2}{\lambda_1} \cdot \omega_g \left( \sqrt{d}\ell_\delta \right) \leq \varepsilon,$$

and

$$\frac{1}{2\sqrt{d}} \cdot \Psi_g \left( \frac{\lambda_1 \varepsilon}{\lambda_2} \right) = \frac{\delta}{2} \leq \ell_\delta = \frac{1}{\left\lfloor \frac{1}{\delta} \right\rfloor + 1} \leq \delta = \frac{1}{\sqrt{d}} \cdot \Psi_g \left( \frac{\lambda_1 \varepsilon}{\lambda_2} \right).$$

Hence, (3.2)-(3.4) yields (3.5) and the proof is complete.  $\square$

REMARK 3.2. In addition, if  $g \in \mathcal{C}^\alpha([0,1]^d, \mathbb{R}^m)$  is Hölder continuous with exponent  $\alpha \in ]0,1[$  then from (2.3) it holds

$$\Psi_g(\gamma_W \cdot \varepsilon) \geq \left( \frac{\gamma_W \cdot \varepsilon}{\|g\|_{\mathcal{C}^{0,\alpha}}} \right)^{\frac{1}{\alpha}} \quad \text{for all } \varepsilon \geq 0.$$

Recalling (3.4)-(3.5), we obtain Theorem 1.1 with  $C_W = \left( 2^{d+m-p-1} d^{d+\frac{p-m}{2}} d! \right) \cdot \left( \frac{1}{\gamma_W} \right)^{\frac{m-p}{\alpha}}$ .

To conclude this section, let us provide an example to show that the blow up rate  $\left( \frac{1}{\varepsilon} \right)^{\frac{m-p}{\alpha}}$  with respect to  $\varepsilon$  is the best bound in terms of power function in the case  $d=m=1, p=0, W=\{0\}$ , and  $\alpha=1$ .

EXAMPLE 3.1. Consider the Lipschitz function  $g: [0,1] \rightarrow \mathbb{R}$  with the Lipschitz constant 1 such that

$$g(x) = \sum_{n=1}^{\infty} u_n(x) \cdot \chi_{[s_n, s_{n+1}]}(x) \quad \text{with} \quad s_1 = 0, s_n = \sum_{j=1}^{n-1} 2^{-j} \quad \text{for all } n \geq 2.$$

Here, the function  $u_n: [0,1] \rightarrow \mathbb{R}$  is defined as follows:  $u_n = 0$  on  $[0,1] \setminus [s_n, s_{n+1}]$  and for all  $x \in [s_n, s_{n+1}]$

$$u_n(x) = 2^{-\ell_n} \cdot \sum_{k=0}^{2^{n^2}-1} u \left( 2^{\ell_n} \cdot \left[ (x - s_n) - k 2^{-\ell_n} \right] \right) \cdot \chi_{[s_n + k 2^{-\ell_n}, s_n + (k+1) 2^{-\ell_n}]}$$

with

$$u(x) = \left( \frac{1}{4} - \left| x - \frac{1}{4} \right| \right) \cdot \chi_{[0,1/2]} + \left( \left| x - \frac{3}{4} \right| - \frac{1}{4} \right) \cdot \chi_{[1/2,1]}, \quad \ell_n = n^2 + n.$$

Given any  $\varepsilon \in [2^{-\ell_{n+1}}, 2^{-\ell_n}[$ , for any  $h \in \mathcal{C}([0,1], \mathbb{R})$  with  $\|h - g\|_{\mathcal{C}^0([0,1])} \leq \varepsilon$ , we have

$$\mathcal{H}^0 \left( \mathcal{Z}_{\{0\}}^h \right) = \#\{x \in [0,1] : h(x) = 0\} \geq \#\{x \in ]s_n, s_{n+1}[ : u_n(x) = 0\} \geq 2^{n^2} \geq \left( \frac{1}{\varepsilon} \right)^{\frac{1}{1+o(\varepsilon)}}$$

with  $\lim_{\varepsilon \rightarrow 0^+} o(\varepsilon) = 0$ . Thus,

$$\mathcal{N}_{\{0\}}^g(\varepsilon) = \inf_{\|h-g\|_{\mathcal{C}^0} \leq \varepsilon} \mathcal{H}^0 \left( \mathcal{Z}_{\{0\}}^h \right) \geq \left( \frac{1}{\varepsilon} \right)^{\frac{1}{1+o(\varepsilon)}},$$

and the blow up rate  $\left( \frac{1}{\varepsilon} \right)^{\frac{m-p}{\alpha}} = \frac{1}{\varepsilon}$  in Theorem 1.1 is optimal in terms of power function in the case  $d=m=1, p=0$  and  $\alpha=1$ . In this scalar case, one can follow the same construction to show that the rate is optimal for  $\alpha \in ]0,1[$ .

For the multi-dimensional cases ( $d \geq 2$ ), the blow up rate  $\left( \frac{1}{\varepsilon} \right)^{\frac{m-p}{\alpha}}$  in Theorem 1.1 should be still optimal in terms of power function but the situation becomes considerably more technical. We leave this open.

**4. A quantitative bound on the total number of shock curves**

In this section, we shall use Theorem 3.1 to prove Theorem 1.2. In general, the scalar conservation laws (1.3) do not possess classical solutions since discontinuities arise in finite time even if the initial data are smooth. Hence, it is natural to consider weak solutions in the sense of distributions that, for sake of uniqueness, satisfy an entropy criterion for admissibility

$$u(t, x-) \geq u(t, x+) \quad \text{for a.e. } t > 0, x \in \mathbb{R}.$$

Under the convexity Assumption (1.4), it is well known (see e.g. in [7]) that for every  $\bar{u} \in \mathbf{L}^\infty(\mathbb{R}) \cap \mathbf{L}^1(\mathbb{R})$ , the Cauchy problem (1.3) with  $u(0, \cdot) = \bar{u}$  admits a unique entropy solution  $u(t, x)$  which satisfies the Oleinik’s estimate

$$u(t, y) - u(t, x) \leq \frac{1}{\lambda t} \cdot (y - x) \quad \text{for all } t > 0, y > x.$$

Moreover, the solution is continuous except on the union of an at most countable set of Lipschitz continuous curves (shocks). To be precise, we recall the definition and theory of generalized characteristic curves associated to (1.3). For a more in depth theory of generalized characteristics, we direct the readers to [9, 13].

DEFINITION 4.1. *A Lipschitz continuous curve  $\xi(t)$  defined on an interval  $[0, \infty[$  is called a generalized characteristic if for a.e.  $t$  in the interval*

$$\dot{\xi}(t) \in [f'(u(t, \xi(t)+)), f'(u(t, \xi(t)-))]. \tag{4.1}$$

Moreover, we say that

- $\xi$  on  $[a, b]$  is genuine if  $u(t, \xi(t)+) = u(t, \xi(t)-)$  for a.e.  $t \in [a, b]$ .
- $\xi$  on an interval  $[\bar{t}, \sigma[$  for some  $\bar{t} < \sigma \leq +\infty$  is a shock if

$$u(t, \xi(t)-) > u(t, \xi(t)+) \quad \text{for all } t \in [\bar{t}, \sigma[.$$

- A point  $(\bar{t}, \bar{x}) \in ]0, \infty[ \times \mathbb{R}$  is called a shock generation point if the forward characteristic through  $(\bar{t}, \bar{x})$  is a shock, while every backward characteristic through  $(\bar{t}, \bar{x})$  is genuine.

The existence of backward (forward) characteristics was studied by Fillipov. As in [9], the speed of the characteristic curves are determined and genuine characteristics are essentially classical characteristics.

PROPOSITION 4.1. *Let  $\xi: [a, b] \rightarrow \mathbb{R}$  be a generalized characteristic curve of (1.3), associated with an entropy weak solution  $u$ . Then for almost every time  $t \in [a, b]$ , it holds that*

$$\dot{\xi}(t) = \begin{cases} f'(u(t, \xi(t))) & \text{if } u(t, \xi(t)+) = u(t, \xi(t)-), \\ \frac{f(u(t, \xi(t)+)) - f(u(t, \xi(t)-))}{u(t, \xi(t)+) - u(t, \xi(t)-)} & \text{if } u(t, \xi(t)+) < u(t, \xi(t)-). \end{cases} \tag{4.2}$$

In addition, if  $\xi$  is genuine on  $[a, b]$ , then  $(t, \xi(\cdot))$  is a straight line and the solution  $u$  is constant along this line.

Given  $(t, x) \in ]0, +\infty[ \times \mathbb{R}$ , all backward characteristics  $\xi$  are confined between maximal and minimal backward characteristics, denoted by  $\xi_{(t,x)}^+$  and  $\xi_{(t,x)}^-$ . Moreover, we

recall properties of generalized characteristics, including the non-crossing property of two genuine characteristics.

PROPOSITION 4.2. *Let  $u$  be an entropy weak solution to (1.3). Then for any  $(t, x) \in ]0, +\infty[ \times \mathbb{R}$ , the followings hold:*

- (i) *The maximal and minimal backward characteristics  $\xi_{(t,x)}^\pm$  are genuine.*
- (ii) *There is a unique forward characteristic, denoted by  $\xi^{(t,x)}$ , which passes through  $(t, x)$ . If  $u(t, \cdot)$  is discontinuous at a point  $x$ , then*

$$u\left(\tau, \xi^{(t,x)}(\tau)-\right) > u\left(\tau, \xi^{(t,x)}(\tau)+\right) \quad \text{for all } \tau \geq t.$$

- (iii) *Two genuine characteristics may intersect only at their endpoints.*

From the above proposition, one can easily obtain the following lemma.

LEMMA 4.1. *For any given initial data  $\bar{v} \in C^2(\mathbb{R})$  with  $\text{supp}(\bar{v}) \subseteq [-R, R]$  such that*

$$\#\{x \in [-R, R] : [f'(\bar{v})]''(x) = 0\} < \infty \tag{4.3}$$

*The total number of shock curves of the entropy weak solution  $v$  of (1.3) with  $u(0, \cdot) = \bar{v}$  is at most the total number of inflection points of  $f'(\bar{v})$ .*

*Proof.* From Proposition 4.2, we have that the total number of shock curves of  $v$  is bounded by the total number of shock generation points. Given a shock generation point  $(\bar{t}, \bar{x})$ , let  $\mathbf{d} : \mathbb{R} \rightarrow \mathbb{R}$  be a  $C^2$  such that

$$\mathbf{d}(\beta) = \beta + f'(\bar{v}(\beta)) \cdot \bar{t} \quad \text{for all } \beta \in \mathbb{R}.$$

Two cases are considered:

- If  $v(\bar{t}, \bar{x}-) = v(\bar{t}, \bar{x}+)$  then let  $\xi_{(\bar{t}, \bar{x})}(\cdot)$  be the backward characteristic starting from  $(\bar{t}, \bar{x})$ . Set  $\bar{\beta} := \xi_{(\bar{t}, \bar{x})}(0)$ . From ([9, Lemma 5.2]), it holds

$$\mathbf{d}'(\bar{\beta}) = 0 \quad \implies \quad [f'(\bar{v})]'(\bar{\beta}) = -\frac{1}{\bar{t}}. \tag{4.4}$$

For every  $\delta > 0$ , there exist  $\bar{x} - \delta < x_\delta^- < \bar{x} < x_\delta^+ < \bar{x} + \delta$  such that  $v(\bar{t}, \cdot)$  is continuous at  $x_\delta^\pm$ . By the non-crossing property (iii) in Proposition 4.2 and the continuity of  $\bar{v}(\bar{t}, \cdot)$  at  $\bar{x}$ , we have

$$\xi_{(\bar{t}, x_\delta^-)}(0) := \beta_\delta^- < \bar{\beta} < \beta_\delta^+ := \xi_{(\bar{t}, x_\delta^+)}(0), \quad \lim_{\delta \rightarrow 0^+} \beta_\delta^- = \lim_{\delta \rightarrow 0^+} \beta_\delta^+ = \bar{\beta},$$

and

$$x_\delta^- = \mathbf{d}(\beta_\delta^-) < \mathbf{d}(\bar{\beta}) = \bar{x} < \mathbf{d}(\beta_\delta^+) = x_\delta^+.$$

This implies that there exist  $\tilde{\beta}_\delta^- \in ]\beta_\delta^-, \bar{\beta}[$  and  $\tilde{\beta}_\delta^+ \in ]\bar{\beta}, \beta_\delta^+[$  such that

$$\mathbf{d}'(\tilde{\beta}_\delta^\pm) > 0 \quad \implies \quad [f'(\bar{v})]'(\tilde{\beta}_\delta^\pm) > -\frac{1}{\bar{t}}.$$

Hence, (4.4) and the Assumption (4.3) imply that  $\bar{\beta}$  is an inflection point of  $f'(\bar{v})$ .

- Otherwise, if  $v(\bar{t}, \bar{x}-) > v(\bar{t}, \bar{x}+)$  then  $(\bar{t}, \bar{x})$  is a center of a centered compression wave, i.e., there are two genuine backward characteristics  $\xi_1$  and  $\xi_2$  through  $(\bar{t}, \bar{x})$  so that every backward characteristic through  $(\bar{t}, \bar{x})$  is contained in the funnel confined between  $\xi_1$  and  $\xi_2$ . In this case, one has that

$$[f'(v)]'(\beta) = -\frac{1}{\bar{t}} \quad \text{for all } \beta \in ]\xi_1(0), \xi_2(0)[$$

and this contradicts (4.3).

Therefore, the total number of shock generation points is at most the total number of inflection points of  $f'(\bar{v})$ . □

From the above lemma and Theorem 3.1, we now extend Theorem 1.2 to the case of  $C^3$ -smooth  $f$ . In order to do so, given constants  $R, V > 0$ , we denote by

$$\Phi_{f,V,R}(\varepsilon) = 2^{12} \cdot \max \left\{ 45 \left( 1 + \frac{1}{V} \right) \|f\|_{C^3} \Big|_{-\frac{V}{2}, \frac{V}{2}} \Big|, \frac{4\beta_\varepsilon}{\Psi_f^V(\beta_\varepsilon)} \right\} \tag{4.5}$$

with  $\beta_\varepsilon = \frac{5\lambda\varepsilon^3}{2^9V^4R^3}$  and  $\Psi_{f^{(3)}}^V$  being the inverse of the minimal modulus of continuity of the restriction of  $f^{(3)}$  to the interval  $]-\frac{V}{2}, \frac{V}{2}[$  which is defined in (2.2).

**THEOREM 4.1.** *Given constants  $R, V > 0$ , assume that  $f \in C^3(\mathbb{R})$  and  $\bar{u} \in \mathbf{L}^1(\mathbb{R})$  satisfies (1.5). Then, for every  $\varepsilon > 0$  sufficiently small, there exists  $\bar{v} \in C^2(\mathbb{R})$  with  $\text{Supp}(\bar{v}) \subseteq [-2R, 2R]$  and  $\|\bar{v} - \bar{u}\|_{\mathbf{L}^1} \leq \varepsilon$ , such that the entropy weak solution  $v = v(t, x)$  of (1.3) with initial datum  $v(0, \cdot) = \bar{v}$  satisfies*

$$[\text{Total number of shock curves of } v] \leq \frac{\Phi_{f,V,R}(\varepsilon)}{\lambda} \cdot \frac{R^4V^5}{\varepsilon^4} + 4. \tag{4.6}$$

*Proof. Step 1.* Let  $\bar{u} \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})$  be such that

$$\text{Supp}(\bar{u}) \subseteq [-R, R] \quad \text{and} \quad \text{Tot.Var.}(\bar{u}, ]-\infty, \infty[) \leq V.$$

For every  $\delta > 0$ , we first approximate  $\bar{u}$  by the smooth function  $u_\delta \in C^3(\mathbb{R})$  with  $\text{Supp}(u_\delta) \subseteq [-R - \delta, R + \delta]$  which is defined by

$$u_\delta(x) := [\bar{u} * \rho_\delta](x) = \int_{-\infty}^{\infty} \bar{u}(y)\rho_\delta(x-y)dy \quad \text{for all } x \in \mathbb{R}$$

where the mollifier

$$\rho_\delta(x) = \frac{315}{256 \cdot \delta} \cdot \left( 1 - \frac{x^2}{\delta^2} \right)^4 \cdot \chi_{[-\delta, \delta]}(x)$$

is a  $C^4(\mathbb{R})$  function with  $\text{Supp}(\rho_\delta) \subseteq [-\delta, \delta]$  and  $\int_{-\infty}^{\infty} \rho_\delta(x)dx = 1$ . From [1, Lemma 3.24], the  $\mathbf{L}^1$ -distance between  $\bar{u}$  and  $u_\delta$  is bounded by

$$\|u_\delta - \bar{u}\|_{\mathbf{L}^1(\mathbb{R})} \leq \delta \cdot \text{Tot.Var.}(\bar{u}, ]-\infty, \infty[) \leq V\delta. \tag{4.7}$$

Moreover, a direct computation yields

$$\|u_\delta\|_{\mathbf{L}^\infty(\mathbb{R})} \leq \|u\|_{\mathbf{L}^\infty(\mathbb{R})} \cdot \|\rho_\delta\|_{\mathbf{L}^1(\mathbb{R})} = \|u\|_{\mathbf{L}^\infty(\mathbb{R})} \leq \frac{1}{2} \cdot \text{Tot.Var.}(\bar{u}, ]-\infty, \infty[) \leq \frac{V}{2},$$

$$\|u'_\delta\|_{\mathbf{L}^\infty(\mathbb{R})} \leq \|u\|_{\mathbf{L}^\infty(\mathbb{R})} \cdot \|\rho'_\delta\|_{\mathbf{L}^1(\mathbb{R})} \leq \frac{V}{2} \cdot \|\rho'_\delta\|_{\mathbf{L}^1(\mathbb{R})} = \frac{315V}{256 \cdot \delta},$$

and

$$\|u''_\delta\|_{\mathbf{L}^\infty(\mathbb{R})} \leq \frac{V}{2} \cdot \|\rho''_\delta\|_{\mathbf{L}^1(\mathbb{R})} = \frac{1215V}{98\sqrt{7} \cdot \delta^2}, \quad \|u'''_\delta\|_{\mathbf{L}^\infty(\mathbb{R})} \leq \frac{V}{2} \cdot \|\rho'''_\delta\|_{\mathbf{L}^1(\mathbb{R})} < \frac{5085V}{224 \cdot \delta^3}.$$

Consider the continuous function  $h_\delta := [f'(u_\delta)]''$  with  $\text{Supp}(h_\delta) \subseteq [-R - 2\delta, R + 2\delta]$ . For every  $x, y \in \mathbb{R}$ , we can roughly estimate

$$\begin{aligned} |h_\delta(y) - h_\delta(x)| &= \left| [f'''(u_\delta)[u'_\delta]^2 + f''(u_\delta)u''_\delta](y) - [f'''(u_\delta)[u'_\delta]^2 + f''(u_\delta)u''_\delta](x) \right| \\ &\leq \frac{45V(1+V)\|f\|_{C^3] - \frac{V}{2}, \frac{V}{2}[}}{2\delta^3} \cdot |x - y| + \|u'_\delta\|_{\mathbf{L}^\infty(\mathbb{R})}^2 \cdot |f^{(3)}(u_\delta(x)) - f^{(3)}(u_\delta(y))| \\ &\leq \frac{45V(1+V)\|f\|_{C^3] - \frac{V}{2}, \frac{V}{2}[}}{2\delta^3} \cdot |x - y| + \frac{8V^2}{5\delta^2} \cdot \omega_{f^{(3)}} \left( \frac{5V}{4\delta} \cdot |x - y| \right) \end{aligned}$$

and (2.1) yields

$$\omega_{h_\delta}(\tau) \leq \frac{45V(1+V)\|f\|_{C^3] - \frac{V}{2}, \frac{V}{2}[}}{2\delta^3} \cdot \tau + \frac{8V^2}{5\delta^2} \cdot \omega_{f^{(3)}} \left( \frac{5V\tau}{4\delta} \right) \quad \text{for all } \tau \geq 0.$$

Recalling (2.2), we then derive an upper bound on the inverse of the minimal modulus of continuity of  $h_\delta$

$$\Psi_{h_\delta}(s) \geq \min \left\{ \frac{\delta^3 s}{45V(V+1)\|f\|_{C^3] - \frac{V}{2}, \frac{V}{2}[}}, \frac{4\delta}{5V} \cdot \Psi_f^V \left( \frac{5\delta^2 s}{16V^2} \right) \right\}, \tag{4.8}$$

where  $\Psi_{f^{(3)}}^V$  is the inverse of the minimal modulus of continuity of the restriction of  $f^{(3)}$  on  $] -V/2, V/2[$ .

On the other hand, applying Theorem 3.1 for  $m = d = 1, p = 0, W = \{0\} \in \mathbb{R}$ , and  $\Lambda_\varepsilon \leq 1$ , we get that for any given  $\sigma > 0$  sufficiently small, there exists a continuous function  $\tilde{h}_{\sigma, \delta}$  such that

$$\text{Supp}(\tilde{h}_{\sigma, \delta}) \subseteq ] -R - 2\delta, R + 2\delta[, \quad \|\tilde{h}_{\sigma, \delta} - h_\delta\|_{C^0(\mathbb{R})} \leq \sigma \tag{4.9}$$

and

$$\#\{x \in ] -R - 2\delta, R + 2\delta[: \tilde{h}_{\sigma, \delta}(x) = 0\} \leq \frac{4(R + \delta)}{\Psi_{h_\delta}(\sigma)}. \tag{4.10}$$

**Step 2.** Set  $R_1 := \max\{R + \delta, \sup\{x \in \mathbb{R} : \tilde{h}_{\sigma, \delta} \neq 0\}\} \in ]R + \delta, R + 2\delta[$  and

$$\alpha_1 = \int_{-R-2\delta}^{R_1} \tilde{h}_{\sigma, \delta}(z) dz, \quad \alpha_0 = \int_{-R-2\delta}^{R_1} \left( \int_{-R-2\delta}^y \tilde{h}_{\sigma, \delta}(z) dz \right) dy.$$

We approximate  $f'(u_\delta)$  by a function  $F_{\sigma, \delta}$  defined by

$$F_{\sigma, \delta}(x) = \begin{cases} f'(0), & x \in \mathbb{R} \setminus [-R - 2\delta, R + 2\delta], \\ f'(0) + \int_{-R-2\delta}^x \left( \int_{-R-2\delta}^y \tilde{h}_{\sigma, \delta}(z) dz \right) dy, & x \in ] -R - 2\delta, R_1[, \\ f'(0) + \alpha_0 \cdot \left( \frac{R + 2\delta - x}{R + 2\delta - R_1} \right)^3 + G_\theta(x)\chi_{[R_1, R_1 + \theta]}, & x \in [R_1, R + 2\delta[, \end{cases} \tag{4.11}$$

with

$$\begin{cases} G_\theta(x) = (x - R_1)(x - R_2)^3 \cdot \left( \frac{\alpha_2}{(R_1 - R_2)^3} + (x - R_1) \cdot \left( \frac{\alpha_3}{2(R_1 - R_2)^3} - \frac{3\alpha_2}{(R_1 - R_2)^4} \right) \right), \\ \alpha_2 = \alpha_1 + \frac{3\beta_1}{R + 2\delta - R_1}, \quad \alpha_3 = \frac{-6\beta_1}{R + 2\delta - R_1}, \quad R_2 = R_1 + \theta, \end{cases}$$

for some  $\theta > 0$  sufficiently small. One computes that

$$G_\theta(R_1) = G_\theta(R_2) = G'_\theta(R_2) = G''_\theta(R_2) = 0, \quad G'_\theta(R_1) = \alpha_2, \quad G''_\theta(R_1) = \alpha_3,$$

and this yields

$$\begin{cases} F_{\sigma,\delta}(R_1 \pm) = f'(0) + \alpha_0, \quad F'_{\sigma,\delta}(R_1 \pm) = \alpha_1, \quad F''_{\sigma,\delta}(R_1 \pm) = \tilde{h}_{\sigma,\delta}(R_1) = 0, \\ F_{\sigma,\delta}(R + 2\delta) = F'_{\sigma,\delta}(R + 2\delta) = F''_{\sigma,\delta}(R + 2\delta) = 0. \end{cases}$$

Hence,  $F_{\sigma,\delta}$  is a  $\mathcal{C}^2$ -function. Moreover, observe that the number of inflection points of  $F_{\sigma,\delta}$  in  $[R_1, R + 2\delta]$  is less than 5, we have from (4.10) that

$$\#\{x \in \mathbb{R} : x \text{ is an inflection point of } F_{\sigma,\delta}\} \leq \frac{4(R + \delta)}{\Psi_{h_\delta}(\sigma)} + 4. \tag{4.12}$$

Recalling (4.9), we estimate

$$\begin{aligned} |F_{\sigma,\delta}(x) - f'(u_\delta)(x)| &\leq \int_{-R-2\delta}^{R_1} \left( \int_{-R-2\delta}^y |\tilde{h}_{\sigma,\delta}(z) - h_\delta(z)| dz \right) dy \\ &\leq \frac{(R_1 + R + 2\delta)^2}{2} \cdot \sigma \quad \text{for all } x \in ]-\infty, R_1]. \end{aligned}$$

This also implies that

$$|\beta_1| = |F_{\sigma,\delta}(R_1) - f'(0)| = |F_{\sigma,\delta}(R_1) - f'(u_\delta)(R_1)| \leq \frac{(R_1 + R + 2\delta)^2}{2} \cdot \sigma.$$

Since  $|G_\theta(x)| \leq \theta \cdot \left( 4|\alpha_2| + \frac{|\alpha_3|}{2} \right)$  for all  $x \in [R_1, R_1 + \theta]$ , one gets from (4.11) that

$$|F_{\sigma,\delta}(x) - f'(u_\delta)(x)| = |F_{\sigma,\delta}(x) - f'(0)| \leq \frac{(R_1 + R + 2\delta)^2}{2} \cdot \sigma + \theta \cdot \left( 4|\alpha_2| + \frac{|\alpha_3|}{2} \right)$$

for all  $x \in [R_1, \infty[$ . Thus, we can choose  $\theta > 0$  sufficiently small such that

$$\|F_{\sigma,\delta} - f'(u_\delta)\|_{\mathbf{L}^\infty(\mathbb{R})} \leq 2(R + 2\delta)^2 \cdot \sigma.$$

**Step 3.** Let  $v_{\sigma,\delta} : \mathbb{R} \rightarrow \mathbb{R}$  be a  $\mathcal{C}^2$  function such that

$$v_{\sigma,\delta}(x) = (f')^{-1}(F_\delta(x)) \quad \text{for all } x \in \mathbb{R}.$$

By the uniform convexity of  $f$  in (1.4), we get

$$|v_{\sigma,\delta}(x) - u_\delta(x)| \leq \frac{1}{\lambda} \cdot |F_{\sigma,\delta}(x) - f'(u_\delta)(x)| \leq \frac{2(R + 2\delta)^2 \cdot \sigma}{\lambda},$$

and (4.7) yields

$$\|v_{\sigma,\delta} - \bar{u}\|_{L^1(\mathbb{R})} \leq \|u_\delta - \bar{u}\|_{L^1(\mathbb{R})} + \|v_{\sigma,\delta} - u_\delta\|_{L^1(\mathbb{R})} \leq V\delta + \frac{4(R+2\delta)^3 \cdot \sigma}{\lambda}.$$

Given  $\varepsilon > 0$ , if we choose

$$\delta = \frac{\varepsilon}{2V} \quad \text{and} \quad \sigma = \frac{\lambda\varepsilon}{2^3(R+2\delta)^3},$$

then the function  $\bar{v} := v_{\sigma,\delta}$  has  $\text{Supp}(\bar{v}) \subseteq [-2R, 2R]$  and  $\|\bar{v} - \bar{u}\|_{L^1(\mathbb{R})} \leq \varepsilon$ . In the case, recalling (4.8), we have

$$\Psi_{h_\delta}(\sigma) = \min \left\{ \frac{\lambda\varepsilon^4}{2^6 45 V^4 (V+1)(R+2\delta)^3 \|f\|_{C^3]_{-\frac{V}{2}, \frac{V}{2}}]}, \frac{2\varepsilon}{5V} \cdot \Psi_f^V \left( \frac{5\lambda\varepsilon^3}{2^9 V^4 (R+2\delta)^3} \right) \right\}.$$

Thus, (4.12) yields

$$\begin{aligned} & \#\{x \in \mathbb{R} : x \text{ is an inflection point of } f'[\bar{v}]\} \\ & \leq \max \left\{ \frac{2^8 45 V^4 (V+1)(R+2\delta)^4 \|f\|_{C^3]_{-\frac{V}{2}, \frac{V}{2}}]}{\lambda\varepsilon^4}, \frac{10V(R+2\delta)}{\varepsilon \cdot \Psi_f^V \left( \frac{5\lambda\varepsilon^3}{2^9 V^4 (R+2\delta)^3} \right)} \right\} + 4 \\ & = 2^8 \cdot \frac{V^5 (R+2\delta)^4}{\lambda\varepsilon^4} \cdot \max \left\{ 45 \left( 1 + \frac{1}{V} \right) \|f\|_{C^3]_{-\frac{V}{2}, \frac{V}{2}}}, \frac{4\beta_\varepsilon}{\Psi_f^V(\beta_\varepsilon)} \right\} + 4 \end{aligned}$$

with  $\beta_\varepsilon = \frac{5\lambda\varepsilon^3}{2^9 V^4 R^3}$ . In particular, for  $0 < \varepsilon \leq \frac{RV}{4}$  such that  $2\delta \leq R$ , it holds

$$\#\{x \in \mathbb{R} : x \text{ is an inflection point of } f'[\bar{v}]\} \leq \frac{\Phi_{f,V,R}(\varepsilon)}{\lambda} \cdot \frac{R^4 V^5}{\varepsilon^4} + 4$$

with  $\Phi_{f,V,R}(\varepsilon)$  defined in (4.5).

**Step 4.** To complete the proof, recalling Lemma 4.1, we obtain that the total number of shock curves in the weak entropy solution  $v$  of (1.3) with initial data  $u(0, \cdot) = \bar{v}$  is bounded as in (4.6). □

REMARK 4.1. If we assume  $f \in C^4(\mathbb{R})$  as in Theorem 1.2, then from (2.3) it holds that

$$\Psi_{f^{(3)}}^V(s) \geq \frac{s}{\|f\|_{C^4]_{-\frac{V}{2}, \frac{V}{2}}]} \quad \text{for all } s > 0.$$

Thus, the function  $\Phi_{f,V,R}$  is bounded by

$$\Phi_{f,V,R} \leq C := 2^{12} 45 \cdot \left( 1 + \frac{1}{V} \right) \cdot \|f\|_{C^4]_{-\frac{V}{2}, \frac{V}{2}}} \tag{4.13}$$

and (4.6) yields (1.6).

Finally, in the spirit of approximation theory, we state the following corollary of Theorem 1.2.



COROLLARY 4.1. *Under the same assumptions in Theorem 1.2, given an integer  $N > 4$  and an initial datum  $\bar{u} \in \mathbf{L}^1(\mathbb{R})$  satisfying (1.5), there exists  $\bar{v} \in \mathcal{C}^3(\mathbb{R})$  with  $\text{supp}(\bar{v}) \subseteq ]-2R, 2R[$  and*

$$\|\bar{v} - \bar{u}\|_{\mathbf{L}^1(\mathbb{R})} \leq 2^3(45)^{1/4} \cdot \left[ \frac{V+1}{\lambda} \cdot \|f\|_{C^4}]_{-\frac{V}{2}, \frac{V}{2}} \right]^{1/4} \cdot \frac{RV}{(N-4)^{1/4}}$$

*such that the entropy weak solution  $v = v(t, x)$  of (1.3) with initial datum  $v(0, \cdot) = \bar{v}$  contains at most  $N$  shocks.*

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