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THE DELAYED CUCKER-SMALE MODEL WITH ATTRACTIVE POWER-LAW POTENTIALS*

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Abstract. We study the large-time behavior of the delayed Cucker−Smale model with attractive power-law potentials. By making full use of the energy fluctuation and another Lyapunov functional involving the communication function, it is proved that this model achieves consensus, i.e. velocity difference and space diameter converge to zero. More importantly, the precise convergence rate is established.

Keywords. Cucker−Smale model; time delay; consensus.

AMS subject classifications. 35F25; 35J05; 35Q83; 82C40.

1. Introduction

In this paper, we are interested in the large-time behavior of the delayed Cucker−Smale (C−S) model with pairwise attractive potentials. Let $(x_i(t),v_i(t))\in\mathbb{R}^{2d}$ be the position and velocity of i-th agent at the time t, $i=1,2,\cdots,N$, and let V denote the pairwise attractive potential. Then, this model is described by the following dynamical system

$$\begin{cases} \dfrac{dx_i(t)}{dt}=v_i(t),\\ \dfrac{dv_i(t)}{dt}=-\left[\dfrac{1}{N}\sum_{j\neq i}\phi(|x_i-x_j|)(v_i-v_j)+\dfrac{1}{N}\sum_{j\neq i}\nabla_{x_i}V(|x_i-x_j|)\right](t-\tau), \end{cases} \tag{1.1}$$

where $\tau>0$ is the reaction delay, and $\phi\geq 0$ is the communication weight function. The first term on the right-hand side of $(1.1)_2$ represents the velocity alignment force, and the second term serves as the attractive force through the potential V. The initial configuration is given by

$$(x_i(t),v_i(t))=(x_i^0(t),v_i^0(t)),\quad \forall\, t\in[-\tau,0], \tag{1.2}$$

where $x_i^0,v_i^0\in C([-\tau,0];\mathbb{R}^d)$.

Without reaction delays and attractive potentials, model (1.1), (1.2) is the classical C−S model, which was introduced by Cucker and Smale in [14, 15] in 2007. Its large-time behavior was fully discussed in [14,19,20]. More precisely, when the communication weight ϕ has a long range, this model exhibits flocking behavior for any initial data. Moreover, the convergence rate is exponential. But when ϕ has a short range, flocking behavior only appears for a restricted class of initial configurations. Then, this seminal model was quickly extended in many directions, such as adding stochastic noises (see e.g. [16, 18]), to include singular communication functions (see e.g. [3, 19, 27–29]), the kinetic description this model (see e.g. [4, 9, 20]) and so on.

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In real systems of interacting agents-animals, humans or robots, there exist reaction delays, which might have a significant effect on their flocking behavior. Without potentials V, the exponential asymptotic flocking of the C−S model with fixed reaction delays and distributed reaction delays, which are small enough, were established in [23] and [24], respectively. The flocking of two agents with large reaction delays was also considered in [11]. For flocking results on the C−S model with transmission-type delays and other delays, we refer to [12, 13, 21, 22, 26, 30–32] and the references therein.

In many situations, agents are driven not only by the alignment rule, but also subject to other pairwise potentials. Thus, it is natural to consider the C−S model coupled with pairwise potentials. When V are some typical attractive-repulsive potentials, flocking behavior was established for this model for any smooth and positive communication weight in [2], but convergence rates were not obtained. Recently, Shu and Tadmor considered the hydrodynamic C-S model with the quadratic potential, i.e., $V(r)=\frac{1}{2}r^2$. When the positive communication function decays slow enough at infinity, according to a key Lyapunov functional combining the energy and the longitudinal momentum, the consensus behavior and the exponential convergence rate were both established in [33]. Then, similar Lyapunov functionals were successfully used to deal with a Cucker-Smale type system with matrix communications in [34], where the pairwise potential can cover some low order power-law functions. In Subsection 3.3 of [35], with nice modifications and simplifications it was proved that the flocking behavior holds for any attractive power-law potential. But for the high order power-law potential, it needs acting from a scale $L>0$, i.e., $V(r)=r^\alpha, r>L, \alpha>2$.

In [8], by constructing a new Lyapunov functional $\mathcal{L}(t)$ involving the communication function ϕ, we established the large-time behavior of the kinetic C−S model with $V(r)=r^\alpha, \alpha>2$ and improved the convergence rate for the case of $\alpha\in[1,2)$. In this paper, we mainly generalize this idea to the delayed model (1.1). To establish the large-time behavior, the boundedness of space diameter should be proved firstly. When $\tau=0$, it can be easily obtained from the energy dispassion. When there exists a delay, the boundedness of space diameter is not obvious since the energy dispassion does not hold. Actually, we firstly use the decreasing of $\mathcal{L}(t)$ to get the boundedness of space diameter, where α should be no less than 2. Then, by the estimates of $\mathcal{L}(t)$ and its derivative again, we obtain the large-time behavior.

Theorem 1.1. *Let $V(r)=r^\alpha, \alpha\geq 2$. Let ϕ be smooth and strictly positive. Then there exists $\tau_0>0$ depending upon the initial data such that the global solution $\{(x_i,v_i)\}_{i=1}^N$ of model (1.1), (1.2) achieves consensus if $\tau\leq\tau_0$. Furthermore, for any $t\geq 0$ and $i\neq j$,*

$$\sum_{i,j=1}^N \left(|v_i-v_j|^2+|x_i-x_j|^2\right)\leq\begin{cases} C\exp\{-Ct\}, & \alpha=2,\\ C(t+1)^{-\frac{2}{\alpha-2}}, & \alpha>2,\end{cases} \tag{1.3}$$

where the constants C depend upon the initial data, α and N.

With a simpler calculation, the above estimates also hold for $\tau=0$.

Remark 1.1. Let $V(r)=r^\alpha, \alpha>0$. When $\tau=0$, the global solution $\{(x_i,v_i)\}_{i=1}^N$ of model (1.1), (1.2) achieves consensus and satisfies

$$\sum_{i,j=1}^N \left(|v_i-v_j|^2+|x_i-x_j|^2\right)\leq\begin{cases} C\exp\{-Ct\}, & \alpha\in(0,2],\\ C(t+1)^{-\frac{2}{\alpha-2}}, & \alpha>2.\end{cases} \tag{1.4}$$

where the constants C depend upon the initial data, α and N.

In this paper, we assume that the particle number N is not very large. Thus, it is allowed that the constants in (1.3) and (1.4) depend upon N. For the case of $\tau=0$, it can be improved according to the particle energy and $\mathcal{L}(t)$.

REMARK 1.2. When ϕ decays slow enough, the constants in (1.4) can be independent of N, please see [8, 33–35] for the details.

When V is other popular potentials such as Morse potentials or more general power-law potentials, flock and mill solutions were investigated in [1, 5–7]. We also refer to [17, 25, 36–38] for the research on other multi-agent models.

The paper is organized as follows: In Section 2, we first give a basic estimate of the energy fluctuation. Then, another Lyapunov functional is introduced to show the boundedness of space diameter. In Section 3, to show the consensus of this model we are devoted to establishing the exponential and polynomial decay of this Lyapunov functional for $\alpha=2$ and $\alpha>2$, respectively.

2. Boundedness of velocity difference and space diameter

Firstly, there exists a local unique solution $\{(x_i,v_i)\}_{i=1}^N$ to model (1.1), (1.2) since the right-hand side of $(1.1)_2$ is locally Lipschitz continuous as a function of $(x_i,v_i)(t-\tau)$. Note that the solution is actually global since the boundedness of $|x_i-x_j|,|v_i-v_j|$ will be proved in this section. Then, we can consider the large-time behavior of this solution. In this section, we are devoted to proving the uniform boundedness of velocity difference and space diameter.

2.1. Energy fluctuation.

Now, we give some basic properties of this model. Firstly, from (1.1) we have that

$$\frac{d}{dt}\left(\sum_{i=1}^N v_i(t)\right)=0, \quad \forall\, t\geq 0. \tag{2.1}$$

Then, we consider the energy fluctuation defined as

$$\mathcal{E}(t)=\frac{1}{2}\sum_{i,j=1}^N |v_i-v_j|^2+\sum_{i,j=1}^N V(|\tilde{x}_i-\tilde{x}_j|), \tag{2.2}$$

where $\tilde{x}_i:=x_i(t-\tau)$, $\tilde{v}_i:=v_i(t-\tau)$. We also denote that $\tilde{\phi}_{ji}:=\phi(|\tilde{x}_j-\tilde{x}_i|)$ and assume that $\phi\leq 1$ for simplicity in the following subsections.

By (1.1) and (2.1), a basic calculation gives that for any $t\geq 0$,

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(t)&=\frac{d}{dt}\left(N\sum_{i=1}^N v_i^2+\sum_{i,j=1}^N |\tilde{x}_i-\tilde{x}_j|^\alpha\right)\\
&=-2N\sum_{i=1}^N v_i\dot{v}_i+\alpha\sum_{i,j=1}^N |\tilde{x}_i-\tilde{x}_j|^{\alpha-2}(\tilde{x}_i-\tilde{x}_j)(\tilde{v}_i-\tilde{v}_j)\\
&=-\sum_{i,j=1}^N \tilde{\phi}_{ji}(\tilde{v}_i-\tilde{v}_j)(v_i-v_j)-\alpha\sum_{i,j=1}^N |\tilde{x}_i-\tilde{x}_j|^{\alpha-2}(\tilde{x}_i-\tilde{x}_j)[(v_i-v_j)-(\tilde{v}_i-\tilde{v}_j)]\\
&=-\sum_{i,j=1}^N \tilde{\phi}_{ji}(\tilde{v}_i-\tilde{v}_j)(v_i-v_j)+2\alpha\sum_{i,j=1}^N |\tilde{x}_i-\tilde{x}_j|^{\alpha-2}(\tilde{x}_i-\tilde{x}_j)(\tilde{v}_i-v_i).
\end{aligned} \tag{2.3}$$

When $\tau=0$, (2.3) becomes $\frac{d}{dt}\mathcal{E}(t)=-\sum_{i,j=1}^N\phi_{ji}|v_i-v_j|^2\le 0$, from which we can immediately obtain the uniform boundedness of $|x_i-x_j|$ and $|v_i-v_j|$. However, when the time delay is considered, the dissipation of energy fluctuation is broken. For now, we complete the calculation of $\mathcal{E}'(t)$.

LEMMA 2.1. *Let $\{(x_i,v_i)\}_{i=1}^N$ be a global solution to model (1.1), (1.2), then*

$$\frac{d}{dt}\mathcal{E}(t)\le -\frac{1}{4}\sum_{i,j=1}^N\tilde\phi_{ji}|\tilde v_j-\tilde v_i|^2-\frac{1}{4}\sum_{i,j=1}^N\tilde\phi_{ji}|v_j-v_i|^2+\tau\alpha^2\sum_{i,j=1}^N|\tilde x_i-\tilde x_j|^{2\alpha-2}$$
$$+(4\tau+2)\int_{t-\tau}^t\sum_{i,j=1}^N|\tilde v_j-\tilde v_i|^2ds+(4\tau+2)\alpha^2\int_{t-\tau}^t\sum_{i,j=1}^N|\tilde x_i-\tilde x_j|^{2\alpha-2}ds$$

for $t\ge\tau$.

Proof. Following from (2.3), we can obtain that for any $t\ge 0$,

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t)&\le-\sum_{i,j=1}^N\tilde\phi_{ji}(\tilde v_j-\tilde v_i)(v_j-v_i)+2\alpha\sum_{i,j=1}^N|\tilde x_i-\tilde x_j|^{\alpha-1}|\tilde v_i-v_i|\\&\le-\sum_{i,j=1}^N\tilde\phi_{ji}|v_j-v_i|^2-2\sum_{i,j=1}^N\tilde\phi_{ji}(v_i-v_j)(\tilde v_i-v_i)+2\alpha\sum_{i,j=1}^N|\tilde x_i-\tilde x_j|^{\alpha-1}|\tilde v_i-v_i|\\&\le-\frac{1}{2}\sum_{i,j=1}^N\tilde\phi_{ji}|v_j-v_i|^2+\left(2+\frac{1}{\tau}\right)\sum_{i,j=1}^N|\tilde v_i-v_i|^2+\tau\alpha^2\sum_{i,j=1}^N|\tilde x_i-\tilde x_j|^{2\alpha-2},\end{aligned}$$

where the last inequality is obtained from the Young inequality. Similarly, we can find that

$$\frac{d}{dt}\mathcal{E}(t)\le-\frac{1}{2}\sum_{i,j=1}^N\tilde\phi_{ji}|\tilde v_j-\tilde v_i|^2+\left(2+\frac{1}{\tau}\right)N\sum_{i=1}^N|\tilde v_i-v_i|^2+\tau\alpha^2\sum_{i=1}^N|\tilde x_i-\tilde x_j|^{2\alpha-2}.$$

Hence, combining the above two inequalities we have that

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t)\le&-\frac{1}{4}\sum_{i,j=1}^N\tilde\phi_{ji}|\tilde v_j-\tilde v_i|^2-\frac{1}{4}\sum_{i,j=1}^N\tilde\phi_{ji}|v_j-v_i|^2\\&+\left(2+\frac{1}{\tau}\right)N\sum_{i=1}^N|\tilde v_i-v_i|^2+\tau\alpha^2\sum_{i=1}^N|\tilde x_i-\tilde x_j|^{2\alpha-2}.\end{aligned}\tag{2.4}$$

Based on (1.1) and Cauchy's inequality, we obtain that for any $t\ge\tau$,

$$\begin{aligned}|\tilde v_i-v_i|^2=&\left|\int_{t-\tau}^t-\frac{1}{N}\sum_{j\ne i}\tilde\phi_{ji}(\tilde v_i-\tilde v_j)-\frac{1}{N}\sum_{j\ne i}\nabla_{\tilde x_i}V(|\tilde x_i-\tilde x_j|)ds\right|^2\\\le&\frac{2\tau}{N}\int_{t-\tau}^t\sum_{j\ne i}|\tilde v_j-\tilde v_i|^2ds+\frac{2\tau\alpha^2}{N}\int_{t-\tau}^t\sum_{j\ne i}|\tilde x_i-\tilde x_j|^{2\alpha-2}ds.\end{aligned}\tag{2.5}$$

From (2.4) and (2.5) we can get that for any $t\ge\tau$,

$$\frac{d}{dt}\mathcal{E}(t)\le(4\tau+2)\int_{t-\tau}^t\sum_{i,j=1}^N|\tilde v_j-\tilde v_i|^2ds+(4\tau+2)\alpha^2\int_{t-\tau}^t\sum_{i,j=1}^N|\tilde x_i-\tilde x_j|^{2\alpha-2}ds$$

$$-\frac{1}{4}\sum_{i,j=1}^{N}\tilde{\phi}_{ji}|\tilde{v}_j-\tilde{v}_i|^2-\frac{1}{4}\sum_{i,j=1}^{N}\tilde{\phi}_{ji}|v_j-v_i|^2+\tau\alpha^2\sum_{i,j=1}^{N}|\tilde{x}_i-\tilde{x}_j|^{2\alpha-2},$$

which completes the proof. □

2.2. Another Lyapunov functional. To get the boundedness of velocity difference and space diameter, we need to neutralize the term $\sum_{i,j=1}^{N}|\tilde{x}_i-\tilde{x}_j|^{2\alpha-2}$ in Lemma 2.1. Inspired by [8], we now define another Lyapunov functional containing the energy fluctuation, that is,

$$\mathcal{L}(t)=\mathcal{E}(t)+\varepsilon\left(\sum_{i,j=1}^{N}(\tilde{x}_i-\tilde{x}_j)(v_i-v_j)+\sum_{i,j=1}^{N}\int_0^{|\tilde{x}_i-\tilde{x}_j|}r\phi(r)dr\right), \tag{2.6}$$

where $\varepsilon\le 1/4$ is an undetermined parameter. The new element in $\mathcal{L}(t)$ is the term $\sum_{i,j=1}^{N}\int_0^{|\tilde{x}_i-\tilde{x}_j|}r\phi(r)dr$, which is essential to deal with general power-law potentials.

In the following lemma, we establish the relationship between $\mathcal{L}(t)$ and $\mathcal{E}(t)$. For convenience, we define velocity difference and space diameter as follows:

$$\begin{cases}R_x(t)=\sup\limits_{s\in[0,t]}\max\limits_{1\le i,j\le N}|x_i(s)-x_j(s)|,\\ R_v(t)=\sup\limits_{s\in[0,t]}\max\limits_{1\le i,j\le N}|v_i(s)-v_j(s)|.\end{cases}$$

LEMMA 2.2. *Let $\{(x_i,v_i)\}_{i=1}^{N}$ be a global solution to model (1.1), (1.2). Assume that $\tilde{R}_x(t_0)<\infty$. If $\varepsilon\le\frac{\phi(\tilde{R}_x(t_0))}{4}$, then*

$$\mathcal{L}(t)\ge\frac{1}{4}\sum_{i,j=1}^{N}|v_i-v_j|^2+\sum_{i,j=1}^{N}|\tilde{x}_i-\tilde{x}_j|^{\alpha}+3\varepsilon^2\sum_{i,j=1}^{N}|\tilde{x}_i-\tilde{x}_j|^2,\ \forall\ t\in[0,t_0].$$

Proof. On the one hand, by the Young inequality we have that

$$\varepsilon\sum_{i,j=1}^{N}(\tilde{x}_i-\tilde{x}_j)(v_i-v_j)\ge-\frac{1}{4}\sum_{i,j=1}^{N}|v_i-v_j|^2-\varepsilon^2\sum_{i,j=1}^{N}|\tilde{x}_i-\tilde{x}_j|^2.$$

On the other hand, from the decreasing of ϕ we get that

$$\varepsilon\sum_{i,j=1}^{N}\int_0^{|\tilde{x}_i-\tilde{x}_j|}r\phi(r)dr\ge\varepsilon\phi(\tilde{R}_x(t))\sum_{i,j=1}^{N}|\tilde{x}_i-\tilde{x}_j|^2\ge\varepsilon\phi(\tilde{R}_x(t_0))\sum_{i,j=1}^{N}|\tilde{x}_i-\tilde{x}_j|^2.$$

Combining the above two inequalities with the definition of $\mathcal{L}(t)$, we complete the proof. □

Conversely, it is easy to obtain the following lemma, whose proof is omitted.

LEMMA 2.3. *Let $\{(x_i,v_i)\}_{i=1}^{N}$ be a global solution to model (1.1), (1.2), then*

$$\mathcal{L}(t)\le\frac{1+\varepsilon}{2}\sum_{i,j=1}^{N}|v_i-v_j|^2+\sum_{i,j=1}^{N}|\tilde{x}_i-\tilde{x}_j|^{\alpha}+\varepsilon\sum_{i,j=1}^{N}|\tilde{x}_i-\tilde{x}_j|^2,\ \forall\ t\ge 0.$$

Then, we compute $\mathcal{L}'(t)$.

LEMMA 2.4. *Let* $\{(x_i,v_i)\}_{i=1}^N$ *be a global solution to model* (1.1), (1.2), *then*

$$\begin{aligned}&\frac{d}{dt}\mathcal{L}(t)\\ \le&-\left[\frac{\phi\left(\tilde{R}_x(t)\right)}{4}-\frac{\varepsilon}{2}\right]\sum_{i,j=1}^N\left(|\tilde{v}_i-\tilde{v}_j|^2+|v_i-v_j|^2\right)-\left(\alpha\varepsilon-\tau\alpha^2\tilde{R}_x(t)^{\alpha-2}\right)\sum_{i,j=1}^N|\tilde{x}_i-\tilde{x}_j|^\alpha\\ &+(4\tau+2)\int_{t-\tau}^t\sum_{i,j=1}^N|\tilde{v}_i-\tilde{v}_j|^2ds+(4\tau+2)\alpha^2\int_{t-\tau}^t\sum_{i,j=1}^N|\tilde{x}_i-\tilde{x}_j|^{2\alpha-2}ds\end{aligned}$$

for any $t\ge\tau$.

Proof. Following from (1.1) and (2.1), we have that for any $t\ge\tau$,

$$\begin{aligned}&\frac{d}{dt}\left(\sum_{i,j=1}^N(\tilde{x}_i-\tilde{x}_j)(v_i-v_j)+\sum_{i,j=1}^N\int_0^{|\tilde{x}_i-\tilde{x}_j|}r\phi(r)dr\right)\\ =&\frac{d}{dt}\left(2N\sum_{i=1}^N\tilde{x}_iv_i-2\sum_{i=1}^N\tilde{x}_i\sum_{i=1}^Nv_i+\sum_{i,j=1}^N\int_0^{|\tilde{x}_i-\tilde{x}_j|}r\phi(r)dr\right)\\ =&2N\sum_{i=1}^N\tilde{v}_iv_i-2\sum_{i=1}^N\tilde{v}_i\sum_{i=1}^Nv_i+2N\sum_{i=1}^N\tilde{x}_i\dot{v}_i+\sum_{i,j=1}^N\tilde{\phi}_{ji}(\tilde{x}_i-\tilde{x}_j)(\tilde{v}_i-\tilde{v}_j)\\ =&\sum_{i,j=1}^N(\tilde{v}_i-\tilde{v}_j)(v_i-v_j)-\alpha\sum_{i,j=1}^N|\tilde{x}_i-\tilde{x}_j|^\alpha\\ \le&\frac{1}{2}\sum_{i,j=1}^N|\tilde{v}_i-\tilde{v}_j|^2+\frac{1}{2}\sum_{i,j=1}^N|v_i-v_j|^2-\alpha\sum_{i,j=1}^N|\tilde{x}_i-\tilde{x}_j|^\alpha.\end{aligned}\tag{2.7}$$

Combining Lemma 2.1 with (2.7), we can get that

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t)\le&-\frac{1}{4}\sum_{i.j=1}^N\tilde{\phi}_{ji}|\tilde{v}_j-\tilde{v}_i|^2-\frac{1}{4}\sum_{i,j=1}^N\tilde{\phi}_{ji}|v_j-v_i|^2+\tau\alpha^2\sum_{i,j=1}^N|\tilde{x}_i-\tilde{x}_j|^{2\alpha-2}\\ &+(4\tau+2)\int_{t-\tau}^t\sum_{i,j=1}^N|\tilde{v}_j-\tilde{v}_i|^2ds+(4\tau+2)\alpha^2\int_{t-\tau}^t\sum_{i,j=1}^N|\tilde{x}_i-\tilde{x}_j|^{2\alpha-2}ds\\ &+\frac{\varepsilon}{2}\sum_{i,j=1}^N|\tilde{v}_i-\tilde{v}_j|^2+\frac{\varepsilon}{2}\sum_{i,j=1}^N|v_i-v_j|^2-\alpha\varepsilon\sum_{i,j=1}^N|\tilde{x}_i-\tilde{x}_j|^\alpha.\end{aligned}\tag{2.8}$$

By the decreasing of ϕ and the definition of $\tilde{R}_x(t)$, we can obtain that

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t)\le&-\left[\frac{\phi(\tilde{R}_x(t))}{4}-\frac{\varepsilon}{2}\right]\sum_{i,j=1}^N\left(|\tilde{v}_i-\tilde{v}_j|^2+|v_i-v_j|^2\right)-\left[\alpha\varepsilon-\tau\alpha^2\tilde{R}_x(t)^{\alpha-2}\right]\sum_{i,j=1}^N|\tilde{x}_i-\tilde{x}_j|^\alpha\\ &+(4\tau+2)\int_{t-\tau}^t\sum_{i,j=1}^N|\tilde{v}_i-\tilde{v}_j|^2ds+(4\tau+2)\alpha^2\int_{t-\tau}^t\sum_{i,j=1}^N|\tilde{x}_i-\tilde{x}_j|^{2\alpha-2}ds.\end{aligned}$$

This completes the proof. ∎

To show the boundedess of $\mathcal{L}(t)$, we integrate on both sides of the inequality in Lemma 2.4 from τ to t.

LEMMA 2.5. *Let $\{(x_i,v_i)\}_{i=1}^N$ be a global solution to model (1.1), (1.2), then there exists a positive constant C_3 depending upon the initial data such that for any $t\geq\tau$,*

$$\mathcal{L}(t)\leq C_3^\alpha-\left[\alpha\varepsilon-(4\tau+3)\tau\alpha^2\tilde{R}_x^{\alpha-2}(t)\right]\int_\tau^t\sum_{i,j=1}^N|\tilde{x}_i-\tilde{x}_j|^\alpha ds$$
$$-\left[\frac{\phi(\tilde{R}_x(t))}{4}-\frac{\varepsilon}{2}-(4\tau+2)\tau\right]\int_\tau^t\left(\sum_{i,j\neq1}^N|\tilde{v}_i-\tilde{v}_j|^2+\sum_{i,j=1}^N|v_i-v_j|^2\right)ds.$$

Proof. Exchanging the order of integrals, we can obtain that

$$\begin{aligned}&\int_\tau^t\int_{s-\tau}^s\sum_{i,j=1}^N|\tilde{x}_i-\tilde{x}_j|^{2\alpha-2}duds\\
=&\int_0^t\left(\int_{\max\{u,\tau\}}^{\min\{t,u+\tau\}}ds\right)\sum_{i,j=1}^N|\tilde{x}_i-\tilde{x}_j|^{2\alpha-2}du\\
\leq&\tau\int_0^t\sum_{i,j=1}^N|\tilde{x}_i-\tilde{x}_j|^{2\alpha-2}du\\
\leq&\tau\int_0^\tau\sum_{i,j=1}^N|\tilde{x}_i-\tilde{x}_j|^{2\alpha-2}du+\tau\tilde{R}_x(t)^{\alpha-2}\int_\tau^t\sum_{i,j=1}^N|\tilde{x}_i-\tilde{x}_j|^\alpha du\end{aligned}\tag{2.9}$$

for $t\geq\tau$. By Lemma 2.4 and the computations in (2.9), we can obtain that

$$\begin{aligned}\mathcal{L}(t)\leq&\mathcal{L}(\tau)+(4\tau+2)\tau\int_0^\tau\sum_{i,j=1}^N|\tilde{v}_i-\tilde{v}_j|^2ds+(4\tau+2)\tau\alpha^2\int_0^\tau\sum_{i,j=1}^N|\tilde{x}_i-\tilde{x}_j|^{2\alpha-2}ds\\
&-\left[\alpha\varepsilon-(4\tau+3)\tau\alpha^2\tilde{R}_x(t)^{\alpha-2}\right]\int_\tau^t\sum_{i,j=1}^N|\tilde{x}_i-\tilde{x}_j|^\alpha ds\\
&-\left[\frac{\phi\left(\tilde{R}_x(t)\right)}{4}-\frac{\varepsilon}{2}-(4\tau+2)\tau\right]\int_\tau^t\sum_{i,j\neq1}^N|\tilde{v}_i-\tilde{v}_j|^2ds\\
&-\left[\frac{\phi(\tilde{R}_x(t))}{4}-\frac{\varepsilon}{2}\right]\int_\tau^t\sum_{i,j=1}^N|v_i-v_j|^2ds.\end{aligned}\tag{2.10}$$

Note that $\tilde{x}_i,\tilde{v}_i$ are given on $[0,\tau]$, so

$$(4\tau+2)\tau\int_0^\tau\sum_{i,j=1}^N|\tilde{v}_i-\tilde{v}_j|^2ds+(4\tau+2)\tau\alpha^2\int_0^\tau\sum_{i,j=1}^N|\tilde{x}_i-\tilde{x}_j|^{2\alpha-2}ds\leq C_1,$$

where $C_1>0$ depends only upon the initial data (if we assume that $\tau\leq1$). Then, we consider $\mathcal{L}(\tau)$. By (1.1), we know that $\dot{v}_i$ is given on $[0,\tau]$. Consequently, $v_i(\tau)$ can be easily computed. Thus, following from the definition of $\mathcal{L}(\tau)$, there exists a positive constant C_2 depending only upon the initial data such that $\mathcal{L}(\tau)\le C_2$. Let $C_3=\max\{(C_1+C_2)^{1/\alpha},R_x(0)\}$, then we complete the proof. □

For convenience, we further require that $C_3>R_x(0)$. Now, the precise value of ε in $\mathcal{L}(t)$ can be given, i.e.,

$$\varepsilon=\frac{1}{4}\phi(C_3). \tag{2.11}$$

With the above preparation, we can prove the boundedness of velocity difference and space diameter.

THEOREM 2.1. *Let $\{(x_i,v_i)\}_{i=1}^N$ be a global solution to model (1.1), (1.2), then for any $t\ge 0$,*

$$\begin{cases} R_x(t)\le C_3,\\ R_v(t)\le 2(C_3)^{\alpha/2}, \end{cases} \tag{2.12}$$

if τ is small enough such that

$$\begin{cases} \dfrac{\phi(C_3)}{8}-(4\tau+2)\tau\ge 0\\ \dfrac{\phi(C_3)}{8}-(4\tau+3)\tau\alpha C_3^{\alpha-2}\ge 0. \end{cases} \tag{2.13}$$

Proof. Since $\tilde{R}_x(\tau)=R_x(0)<C_3$, by the continuity of $R_x(t)$ we obtain that $\tilde{R}_x(t)<C_3$ holds in a time interval. Then, we define that

$$t_0=\sup\left\{t\ge\tau;\tilde{R}_x(t)<C_3 \text{ holds on } [\tau,t)\right\}. \tag{2.14}$$

We now claim $t_0=\infty$. If not, we have that $\tilde{R}_x(t_0)=C_3$ and $\tilde{R}_x(t)<C_3$ for any $t\in[\tau,t_0)$. Following from the decreasing of ϕ and (2.13), we have that

$$\begin{cases} \dfrac{\phi\left(\tilde{R}_x(t)\right)}{4}-\dfrac{\phi(C_3)}{8}-(4\tau+2)\tau\ge 0\\ \dfrac{\phi(C_3)}{8}-(4\tau+3)\tau\alpha\tilde{R}_x(t)^{\alpha-2}\ge 0 \end{cases} \tag{2.15}$$

for any $t\in[\tau,t_0]$. Combining (2.15) with Lemma 2.5, we obtain that $\mathcal{L}(t)\le C_3^\alpha$. Then, according to $\varepsilon=\phi(C_3)/4=\phi(\tilde{R}_x(t_0))/4$, we can use Lemma 2.2 to get that for any $t\in[\tau,t_0]$

$$C_3^\alpha\ge\frac{1}{4}\sum_{i,j=1}^N|v_i-v_j|^2+\sum_{i,j=1}^N|\tilde{x}_i-\tilde{x}_j|^\alpha+3(\phi(C_3)/4)^2\sum_{i,j=1}^N|\tilde{x}_i-\tilde{x}_j|^2.$$

Thus, for any $i\ne j$,

$$\begin{cases} |\tilde{x}_i-\tilde{x}_j|<C_3,\ \forall\ t\in[\tau,t_0]\\ |v_i-v_j|\le 2(C_3)^{\alpha/2},\ \forall\ t\in[\tau,t_0], \end{cases}$$

which conflicts with $\tilde{R}_x(t_0)=C_3$. Thus, $t_0=\infty$ and the desired estimates can be obtained. □

REMARK 2.1. Combining the above theorem with Lemma 2.3, we can obtain that

$$\mathcal{L}(t) \le C\left(\sum_{i,j=1}^{N}|v_i-v_j|^2+\sum_{i,j=1}^{N}|\tilde{x}_i-\tilde{x}_j|^{\alpha}\right)^{\min\{2/\alpha,1\}} \le C\mathcal{E}(t)^{\min\{2/\alpha,1\}},$$

where $C>0$ depends upon initial data.

REMARK 2.2. The Lyapunov functional $\mathcal{L}(t)$ can not obtain the boundedness of $R_x(t), R_v(t)$ for the case of $\alpha<2$ because of the delay.

3. The Lyapunov functional decays

In the previous section, the Lyapunov functional $\mathcal{L}(t)$ was used to deduce the boundedness of velocity difference and space diameter. More importantly, this functional is also the key to prove the consensus of model (1.1), (1.2). Now, we give the precise definition of consensus.

DEFINITION 3.1.

(1) *Model* (1.1), (1.2) *exhibits flocking iff* $\lim_{t\to\infty}|v_i-v_j|=0$ *and* $\sup_{t\ge 0}|x_i-x_j|<\infty$ *for any* i,j.

(2) *Model* (1.1), (1.2) *achieves consensus iff it exhibits flocking and* $\lim_{t\to\infty}|x_i-x_j|=0$ *for any* i,j.

Actually, we are devoted to proving that the Lyapunov functional decays, where the estimate of $\mathcal{L}'(t)$ in Lemma 2.4 will be fully used. Before that, we need the following lemma to establish the relationship between $\mathcal{E}(t)$ and $\mathcal{E}(s)$ for $s\in[t-\tau,t]$.

LEMMA 3.1. *Let* $v\ge 0$ *satisfy that*

$$v'(t) \ge -a \sup_{s\in[t-\tau,t]} v(s),$$

where $a>0$. *Let* $v(0)\ne 0$ *and* $k_0=\sup_{s\in[-\tau,0]} v(s)/v(0)$. *If* $\tau>0$ *satisfies* $e^{2ak_0\tau}\le 2$, *then for any* $t\ge 0$,

$$v(s)\le k_0 e^{2ak_0(t-s)}v(t), \ -\tau\le s<t.$$

This lemma and its proof are only slightly different from Lemma 2.2 in [10], so we omit the proof.

LEMMA 3.2. *Let* $\{(x_i,v_i)\}_{i=1}^N$ *be a global solution of* (1.1). *There exists* $\tau_0>0$ *depending upon the initial data such that*

$$\frac{d}{dt}\mathcal{L}(t)\le -\frac{1}{16}\phi(C_3)\mathcal{E}(t), \quad \forall\ t\ge\tau,$$

if $\tau\le\tau_0$.

Proof. Combining (2.12) with Lemma 2.4, we have that

$$\frac{d}{dt}\mathcal{L}(t)\le -\frac{1}{8}\phi(C_3)\sum_{i,j=1}^{N}|v_i-v_j|^2-\left(\frac{\alpha}{4}\phi(C_3)-\tau\alpha^2C_3^{\alpha-2}\right)\sum_{i,j=1}^{N}|\tilde{x}_i-\tilde{x}_j|^{\alpha}$$
$$+(4\tau+2)\int_{t-\tau}^{t}\sum_{i,j=1}^{N}|\tilde{v}_i-\tilde{v}_j|^2ds+(4\tau+2)\alpha^2C_3^{\alpha-2}\int_{t-\tau}^{t}\sum_{i,j=1}^{N}|\tilde{x}_i-\tilde{x}_j|^{\alpha}ds.$$

Consequently, when τ satisfies (2.13), we have that

$$\frac{d}{dt}\mathcal{L}(t)\le -\frac{1}{8}\phi(C_3)\mathcal{E}(t)+3\left(\alpha^2 C_3^{\alpha-2}+1\right)\int_{t-\tau}^{t}(\tilde{\mathcal{E}}(s)+\mathcal{E}(s))ds. \tag{3.1}$$

Note that

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t)&=\frac{d}{dt}\left(N\sum_{i=1}^{N}v_i^2+\sum_{i,j=1}^{N}|\tilde{x}_i-\tilde{x}_j|^2\right)\\&=-\sum_{i,j=1}^{N}\tilde{\phi}_{ji}(\tilde{v}_i-\tilde{v}_j)(v_i-v_j)-2\sum_{i,j=1}^{N}(\tilde{x}_i-\tilde{x}_j)(v_i-v_j)+2\sum_{i,j=1}^{N}(\tilde{x}_i-\tilde{x}_j)(\tilde{v}_i-\tilde{v}_j)\\&\ge -3\mathcal{E}(t)-3\tilde{\mathcal{E}}(t)\ge -6\max_{s\in[t-\tau,t]}\mathcal{E}(s).\end{aligned} \tag{3.2}$$

According to and (3.2), when τ is sufficiently small, there exists a positive constant C depending upon initial data such that

$$\mathcal{E}(s)+\tilde{\mathcal{E}}(s)\le C\mathcal{E}(t),\quad \forall\ s\in[t-\tau,t].$$

Combining the above inequality with (3.1), we can get that $\frac{d}{dt}\mathcal{L}(t)\le -\frac{1}{8}\phi(C_3)\mathcal{E}(t)+C\tau\mathcal{E}(t)$, which yields the conclusion. □

Proof. **(Proof of Theorem 1.1.)** Combining Lemma 3.2 with Remark 2.1, we have that

$$\frac{d}{dt}\mathcal{L}(t)\le -C\mathcal{L}(t)^{\max\{\alpha/2,1\}},\quad \forall\ t\ge\tau.$$

Then,

$$\mathcal{L}(t)\le\begin{cases}\mathcal{L}(\tau)\exp\{-Ct\}, & \alpha=2,\\ C(t+1)^{-\frac{2}{\alpha-2}}, & \alpha>2.\end{cases}$$

Using the above inequality and Lemma 2.2, we complete the proof. □

Proof. **(Proof of Remark 1.1.)** When $\tau=0$, (2.3) yields that

$$\frac{d}{dt}\mathcal{E}(t)\le -\phi(R_x(t))\sum_{i,j=1}^{N}|v_i-v_j|^2\le 0.$$

Then, following from the proof of Lemma 2.4 we can obtain that

$$\begin{aligned}\frac{d\mathcal{L}}{dt}&=-[\phi(R_x(t))-\epsilon]\sum_{i,j=1}^{N}|v_i-v_j|^2-\epsilon\sum_{i,j=1}^{N}|x_i-x_j|V'(|x_i-x_j|)\\&\le -[\phi(R_x(t))-\epsilon]\sum_{i,j=1}^{N}|v_i-v_j|^2-C\epsilon\sum_{i,j=1}^{N}V(|x_i-x_j|)\end{aligned}$$

since $rV'(r)\ge CV(r)$ for some positive constant C. Note that $R_x(t)$ is bounded, by choosing a sufficiently small ϵ there exists $C>0$ depending upon the initial data such that

$$\frac{d\mathcal{L}}{dt}\le -C\mathcal{E}(t). \tag{3.3}$$

Secondly, when $\tau=0$ we have from the definition of $\mathcal{L}(t)$ that

$$\mathcal{L}(t)=\mathcal{E}(t)+\varepsilon\left(\sum_{i,j=1}^{N}(x_i-x_j)(v_i-v_j)+\sum_{i,j=1}^{N}\int_0^{|x_i-x_j|}r\phi(r)dr\right)$$
$$\leq 2\mathcal{E}(t)+C\sum_{i,j=1}^{N}|x_i-x_j|^2.$$

Consequently, if $V(r)\geq Cr^{\alpha}$ for a $C>0$, by the above inequality we get that

$$\mathcal{L}(t)\leq C\mathcal{E}(t)^{\min\{2/\alpha,1\}}. \tag{3.4}$$

Combining (3.3), (3.4) with Lemma 2.2, we can obtain the desired estimates. ◻

REMARK 3.1. From the above proof, we actually obtain (1.4) when V only satisfies $V(0)=0$,

$$V(r)\geq Cr^{\alpha},\ \alpha>0 \quad \text{and} \quad rV'(r)\geq CV(r).$$

The above assumption of V is rather general. For examples, $V(r)=r^2+r^3$, $V(r)=e^r-1$ and $V(r)=r^{\alpha}+V_0(r)$, where $V_0\geq 0$ is a convex function passing through the origin. But, such easy generalization is impossible when there is a delay.

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