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GLOBAL EXISTENCE AND STABILITY OF TIME-PERIODIC SOLUTION TO ISENTROPIC COMPRESSIBLE EULER EQUATIONS WITH SOURCE TERM*

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Abstract. In this paper, we study the initial-boundary value problem of one-dimensional isentropic compressible Euler equations with the source term $\beta\rho|u|^{\alpha}u$. By means of wave decomposition and the uniform a-priori estimates, we prove the global existence of smooth solutions under small perturbations around the supersonic Fanno flow. Then, by Gronwall's inequality, we get that the smooth solution is time-periodic.

Keywords. Isentropic compressible Euler equations; global existence; time-periodic solutions; supersonic Fanno flow; wave decomposition.

AMS subject classifications. 35B10; 35A01; 35Q31.

1. Introduction

In this paper, we are concerned with isentropic compressible Euler equations with a nonlinear source term:

$$\begin{cases}\partial_t\rho+\partial_x(\rho u)=0,\\ \partial_t(\rho u)+\partial_x(\rho u^2+p)=\beta\rho|u|^{\alpha}u,\end{cases}\quad (t,x)\in[0,+\infty)\times[0,L], \tag{1.1}$$

where ρ,u and p are the density, velocity and pressure of gas, respectively. The pressure $p(\rho)$ is governed by $p(\rho)=a\rho^{\gamma}$, here the adiabatic exponent $\gamma>1$ and the parameter a is scaled to unity for mathematical convenience. The sound speed $c\geq 0$ is defined by $c^2=\partial p/\partial\rho$. And the term $\beta\rho|u|^{\alpha}u$ represents the friction with $\alpha,\beta\in\mathbb{R}$.

In this paper, we assume the initial data are

$$(\rho,u)^{\top}|_{t=0}=(\rho_0(x),u_0(x))^{\top}. \tag{1.2}$$

The boundary conditions are

$$(\rho,u)^{\top}|_{x=0}=(\rho_l(t),u_l(t))^{\top} \tag{1.3}$$

and $\rho_l(t),u_l(t)$ are periodic functions with a period $P>0$, i.e.

$$\rho_l(t+P)=\rho_l(t),u_l(t+P)=u_l(t).$$

In order to obtain the C^1 solution, the initial and boundary data should satisfy the following compatibility conditions at the point $(0,0)$

$$\begin{cases}\rho_l'(0)+\rho_0'(0)u_0(0)+\rho_0(0)u_0'(0)=0,\\ \rho_l'(0)u_l(0)+\rho_l(0)u_l'(0)+\rho_0'(0)u_0^2(0)+2\rho_0(0)u_0(0)u_0'(0)\\ \qquad+p_0'(0)-\beta\rho_0(0)u_0^{\alpha+1}(0)=0,\\ \rho_0(0)=\rho_l(0),u_0(0)=u_l(0),\end{cases} \tag{1.4}$$

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where

$$p_0'(0) = \gamma \rho_0^{\gamma-1}(0)\rho_0'(0).$$

Because of the widespread application background, the compressible Euler equation with several kinds of source terms have been studied extensively and there are fruitful results. For example, we can refer [6, 7, 16, 21] for the research on the existence and stability of the small smooth solution, [2,3,5,8,9,23,26] for the singularity formation of the smooth solution and the results on the weak solution. In this paper, we are interested in the time-periodic solution of problem (1.1)-(1.3). As far as we know, there are many works on the studies of time-periodic solutions of the partial differential equations such as the viscous fluids equations [1, 11, 15, 17, 18] and the hyperbolic conservation laws [4, 19, 20, 24, 25]. All of the studies mentioned above discuss the time-periodic solutions which are derived by the time-periodic external forces or the piston motion. But there are few works on the time-periodic solutions of the hyperbolic conservation laws derived by the time-periodic boundary condition. In [29], Yuan studied time-periodic supersonic solutions for the isentropic compressible Euler equation (i.e. $\beta = 0$) triggered by the periodic supersonic boundary condition. For the quasilinear hyperbolic system with more general time-periodic boundary conditions, Qu showed the existence and stability of the time-periodic solutions around a small neighborhood of $u \equiv 0$ in [22]. Recently, Wei et al. [28] studied the global stability problem for supersonic flows in one dimensional compressible Euler equations with a friction term $-\mu\rho|u|u, \mu > 0$.

In this paper, we would like to show global existence and uniqueness of time-periodic supersonic solutions of initial-boundary value problem (1.1)-(1.3) with the general friction term $\beta\rho|u|^\alpha u$ by perturbing some supersonic Fanno flow. Different from [28], we consider (1.1)-(1.3) in the form of sound speed and fluid speed. Then the Fanno flow is considered for some upstream with positive constants state (c_-, u_-) at the left side. After analyzing the ODEs carefully, we get the maximal duct length L_m, exceeding which the flow will get choked. Based on the supersonic Fanno flow, we prove the existence of time periodic solutions by wave decomposition.

The main results of this paper are:

THEOREM 1.1. *For any fixed non-sonic upstream state (ρ_-, u_-) satisfying $0 < u_- \neq \sqrt{\gamma}\rho_-^{\frac{\gamma-1}{2}}$, there exists a maximal duct length L_m, which only depends on α, β, γ and $(\rho_-, u_-)^\top$, such that the steady solution $\tilde{V} = (\tilde{\rho}(x), \tilde{u}(x))^\top$ of problem (1.1) exists in $[0, L_m]$ and keeps the upstream supersonic/subsonic state.*

THEOREM 1.2. *Suppose the duct length $L < L_m$ and the upstream state (ρ_-, u_-) is supersonic, i.e. $u_- > \sqrt{\gamma}\rho_-^{\frac{\gamma-1}{2}}$. Then there exists a $\varepsilon_0 > 0$ such that for any fixed ε with $0 < \varepsilon \leq \varepsilon_0$, if*

$$\|(\rho_0(x) - \tilde{\rho}(x),\ u_0(x) - \tilde{u}(x))\|_{C^1([0,L])} < \varepsilon, \tag{1.5}$$

$$\|(\rho_l(t) - \rho_-,\ u_l(t) - u_-)\|_{C^1([0,+\infty))} < \varepsilon, \tag{1.6}$$

then the mixed initial-boundary value problem (1.1)-(1.3) has a unique C^1 solution $V = (\rho(t,x), u(t,x))^\top$ in the domain $E = \{(t,x) | t > 0, x \in (0, L]\}$, satisfying

$$\|V - \tilde{V}\|_{C^1(E)} < C\varepsilon$$

for some constant $C > 0$ and

$$V(t+P, x) = V(t,x), \quad \forall t > T_1, x \in [0, L],$$

where $\tilde{V} = (\tilde{\rho}(x), \tilde{u}(x))^\top$ *is the supersonic Fanno flow obtained in Theorem 1.1 and*

$$T_1 = \max_{\substack{t \geq 0, x \in [0,L] \\ i=1,2}} \frac{L}{\lambda_i(V(t,x))}. \tag{1.7}$$

REMARK 1.1. For the supersonic flow, the flow at $x = L$ is completely determined by the initial data at $x \in [0,L]$ and boundary conditions at $x = 0$, so we only need to give the boundary condition at $x = 0$.

The rest of the paper is organised as follows. In Section 2, we construct the Fanno flow. In Section 3, we present a reformulation of the problem by perturbing the solution around the supersonic Fanno flow and introduce a wave decomposition for the perturbed solution. In Section 4, we prove the global existence and uniqueness of the solution with the help of uniform a-priori estimates. In Section 5, we prove time-periodicity of solutions by Gronwall's inequality.

2. Fanno flow

The Fanno flow refers to the adiabatic flow through a constant area duct where the effect of friction ($i.e. \beta < 0$) is considered. The friction causes the flow properties to change along the duct. For the completeness of our results, we also consider the case $\beta > 0$ in this section.

We rewrite the initial-boundary problem (1.1)-(1.3) in terms of the sound speed $c = \sqrt{\gamma}\rho^{\frac{\gamma-1}{2}}$ and the fluid velocity u as follows

$$\begin{cases} c_t + c_x u + \dfrac{\gamma-1}{2} c u_x = 0, \\ u_t + u u_x + \dfrac{2}{\gamma-1} c c_x = \beta |u|^\alpha u, \\ (c,u)^\top|_{t=0} = (c_0(x), u_0(x))^\top, \\ (c,u)^\top|_{x=0} = (c_l(t), u_l(t))^\top, \end{cases} \tag{2.1}$$

where $c_0(x) = \sqrt{\gamma}\rho_0^{\frac{\gamma-1}{2}}(x), c_l(t) = \sqrt{\gamma}\rho_l^{\frac{\gamma-1}{2}}(t)$.

Now, we consider the positive solution $(\tilde{c}, \tilde{u})^\top$ of the steady flow of system (2.1) which satisfies

$$\begin{cases} \tilde{c}'\tilde{u} + \dfrac{\gamma-1}{2}\tilde{c}\tilde{u}' = 0, \\ \tilde{u}\tilde{u}' + \dfrac{2}{\gamma-1}\tilde{c}\tilde{c}' = \beta \tilde{u}^{1+\alpha}, \\ (\tilde{c}, \tilde{u})^\top|_{x=0} = (c_-, u_-)^\top, \end{cases} \tag{2.2}$$

where u_- and c_- are two positive constants.

First, by $(2.2)_1$, we get

$$\tilde{c} = c_- u_-^{\frac{\gamma-1}{2}} \tilde{u}^{-\frac{\gamma-1}{2}}. \tag{2.3}$$

Substituting (2.3) into $(2.2)_2$, we have

$$\tilde{u}^{-\alpha}\tilde{u}' - c_-^2 u_-^{\gamma-1} \tilde{u}^{-\gamma-\alpha-1}\tilde{u}' = \beta. \tag{2.4}$$

We consider (2.4) by classifying α and β.

Case 1: $\alpha \neq 1$ and $\alpha \neq -\gamma$.

In this case, (2.4) becomes

$$\frac{1}{-\alpha+1}(\tilde{u}^{-\alpha+1})' + \frac{1}{\gamma+\alpha} c_-^2 u_-^{\gamma-1} (\tilde{u}^{-\gamma-\alpha})' = \beta. \tag{2.5}$$

Integrating (2.5) from 0 to x, we get

$$\frac{1}{-\alpha+1}\tilde{u}^{-\alpha+1} + \frac{1}{\gamma+\alpha} c_-^2 u_-^{\gamma-1} \tilde{u}^{-\gamma-\alpha} = \frac{1}{-\alpha+1} u_-^{-\alpha+1} + \frac{1}{\gamma+\alpha} c_-^2 u_-^{-1-\alpha} + \beta x. \tag{2.6}$$

Denote the left-hand-side function of (2.6) as $h(s)$, i.e.

$$h(s) = \frac{1}{-\alpha+1} s^{-\alpha+1} + \frac{1}{\gamma+\alpha} c_-^2 u_-^{\gamma-1} s^{-\gamma-\alpha},$$

then we deduce

$$\begin{aligned} &h'(s) < 0, \quad \text{for } 0 < s < s_c; \\ &h'(s) > 0, \quad \text{for } s > s_c, \end{aligned}$$

where $s_c = c_-^{\frac{2}{\gamma+1}} u_-^{\frac{\gamma-1}{\gamma+1}}$. This means that $h(s)$ gets its minimum at the point $s = s_c$. On the other hand, from (2.3), we have $\tilde{c} = c_-^{\frac{2}{\gamma+1}} u_-^{\frac{\gamma-1}{\gamma+1}}$ when $\tilde{u} = s_c = c_-^{\frac{2}{\gamma+1}} u_-^{\frac{\gamma-1}{\gamma+1}}$. That is, the flow speed equals to the sound speed ($i.e. M = 1$) at the choked point $(s_c, h(s_c))$. See Figure 2.1 below.

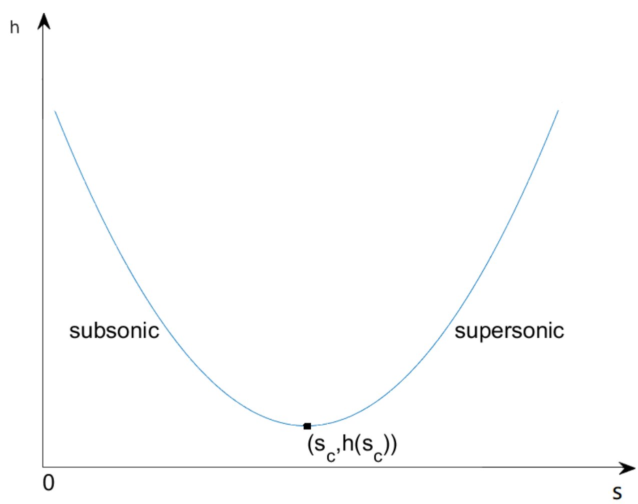

FIG. 2.1.

If $\beta > 0$ and the upstream is supersonic (i.e. $u_- > c_-$), $\tilde{u}$ is monotonically increasing by considering (2.6) and $\tilde{u} > u_-$. By (2.3), $\tilde{c}$ is monotonically decreasing and $\tilde{c} < c_-$. Then, $\tilde{u} > \tilde{c}$. If $\beta > 0$ and the upstream is subsonic (i.e. $u_- < c_-$), $\tilde{u}$ is monotonically decreasing and $\tilde{c}$ is monotonically increasing. Then $\tilde{u} < \tilde{c}$.

When $\beta<0$, from (2.6), $h(s)$ decreases with respect to the length of the duct till arriving at its minimum. Therefore, we can get the maximal length of the duct L_m for a supersonic or subsonic flow before it gets choked, which is

$$L_m=\frac{1}{\beta}\Big(\frac{1}{-\alpha+1}(s_c^{-\alpha+1}-u_-^{-\alpha+1})+\frac{1}{\gamma+\alpha}c_-^2(u_-^{\gamma-1}s_c^{-\gamma-\alpha}-u_-^{-1-\alpha})\Big). \tag{2.7}$$

Case 2: $\alpha=1$ or $\alpha=-\gamma$.

Now, (2.4) is turned into

$$(\ln\tilde{u})'+\frac{1}{\gamma+1}c_-^2u_-^{\gamma-1}(\tilde{u}^{-\gamma-1})'=\beta,\quad \text{for}\ \ \alpha=1, \tag{2.8}$$

and

$$\frac{1}{\gamma+1}(\tilde{u}^{\gamma+1})'-c_-^2u_-^{\gamma-1}(\ln\tilde{u})'=\beta,\quad \text{for}\ \ \alpha=-\gamma. \tag{2.9}$$

Integrating (2.8) and (2.9) from 0 to x, we get

$$\ln\tilde{u}+\frac{1}{\gamma+1}c_-^2u_-^{\gamma-1}\tilde{u}^{-\gamma-1}=\ln u_-+\frac{1}{\gamma+1}c_-^2u_-^{-2}+\beta x,\quad \text{for}\ \ \alpha=1, \tag{2.10}$$

and

$$\frac{1}{\gamma+1}\tilde{u}^{\gamma+1}-c_-^2u_-^{\gamma-1}\ln\tilde{u}=\frac{1}{\gamma+1}u_-^{\gamma+1}-c_-^2u_-^{\gamma-1}\ln u_-+\beta x,\quad \text{for}\ \ \alpha=-\gamma. \tag{2.11}$$

Define

$$f(s)=\ln s+\frac{1}{\gamma+1}c_-^2u_-^{\gamma-1}s^{-\gamma-1},$$

and

$$g(s)=\frac{1}{\gamma+1}s^{\gamma+1}-c_-^2u_-^{\gamma-1}\ln s.$$

The functions $f(s)$ and $g(s)$ get their minimums at the point $s=s_c=c_-^{\frac{2}{\gamma+1}}u_-^{\frac{\gamma-1}{\gamma+1}}$. Furthermore, we get the maximal length of the duct L_m for $\beta<0$:

$$L_m=\frac{1}{\beta}\Big(\frac{1}{\gamma+1}c_-^2(u_-^{\gamma-1}s_c^{-\gamma-1}-u_-^{-2})+\ln\frac{s_c}{u_-}\Big),\quad \text{for}\ \ \alpha=1 \tag{2.12}$$

and

$$L_m=\frac{1}{\beta}\Big(\frac{1}{\gamma+1}(s_c^{\gamma+1}-u_-^{\gamma+1})-c_-^2u_-^{\gamma-1}\ln\frac{s_c}{u_-}\Big),\quad \text{for}\ \ \alpha=-\gamma. \tag{2.13}$$

We can get similar results as in case 1, we omit the details here.

From the above discussion, we have the following lemma.

LEMMA 2.1. *If $u_->0,c_->0$ and the duct length $L<L_m$, where L_m is a positive constant only depending on α,β,γ,c_- and u_- (See (2.7), (2.12), (2.13)), then the Cauchy problem (2.2) admits a unique smooth positive solution $(\tilde{c}(x),\tilde{u}(x))^\top$ which satisfies the following properties:*

(1) $0<\tilde{u}(x)<u_-<c_-<\tilde{c}(x)$, *if* $\beta>0$ *and* $c_->u_-$;
(2) $0<\tilde{c}(x)<c_-<u_-<\tilde{u}(x)$, *if* $\beta>0$ *and* $c_-<u_-$;
(3) $0<u_-<\tilde{u}(x)<\tilde{c}(x)<c_-$, *if* $\beta<0$ *and* $c_->u_-$;
(4) $0<c_-<\tilde{c}(x)<\tilde{u}(x)<u_-$, *if* $\beta<0$ *and* $c_-<u_-$.

This result means that a subsonic flow entering a duct with friction ($\beta<0$) will have an increase in its Mach number until the flow is choked at $M=1$, i.e. $\tilde{u}=\tilde{c}$. Conversely, the Mach number of a supersonic flow will decrease until the flow is choked. However, if a flow entering a duct with acceleration ($\beta>0$), the Mach number of a subsonic flow will decrease and the Mach number of a supersonic flow will increase (i.e. accelerating the initial subsonic or supersonic state). It is worthy to be pointed out that the theoretical calculations are consistent with the experiment. Different from the calculations in [28], where the authors consider a differential equation that relates the change in Mach number with respect to the length of the duct $\frac{dM}{dx}$, we rewrite the dominating equations in terms of the relations between the sound speed and flow speeds. Fortunately, the resulting equations can be decoupled easily. Therefore, we can show the maximal duct length which makes the flow choke assuming the upstream Mach number is supersonic or subsonic. Thus by Lemma 2.1 and $\tilde{c}=\sqrt{\gamma}\tilde{\rho}^{\frac{\gamma-1}{2}}$, we can directly get Theorem 1.1.

3. Reformulation of problem and wave decomposition

For the supersonic flow, we should have $u>0$. Then, we can write the system (1.1) as

$$\begin{cases}\rho_t+\rho_x u+\rho u_x=0,\\ u_t+uu_x+\gamma\rho^{\gamma-2}\rho_x=\beta u^{\alpha+1}.\end{cases}\tag{3.1}$$

Letting

$$\rho(t,x)=\bar{\rho}(t,x)+\tilde{\rho}(x),\quad u(t,x)=\bar{u}(t,x)+\tilde{u}(x),\tag{3.2}$$

where $(\bar{\rho}(t,x),\bar{u}(t,x))^\top$ is the perturbation of the supersonic Fanno flow. Substituting (3.2) into (3.1), we get

$$\begin{cases}\bar{\rho}_t+\bar{\rho}_x u+\rho\bar{u}_x+\tilde{\rho}'\bar{u}+\bar{\rho}\tilde{u}'+\tilde{\rho}'\tilde{u}+\tilde{\rho}\tilde{u}'=0,\\ \bar{u}_t+u\bar{u}_x+\bar{u}\tilde{u}'+\tilde{u}\tilde{u}'+\gamma\rho^{\gamma-2}\bar{\rho}_x+\gamma\rho^{\gamma-2}\tilde{\rho}'=\beta(\bar{u}+\tilde{u})^{\alpha+1}.\end{cases}\tag{3.3}$$

Moreover, the system (3.3) can be further written into

$$\begin{cases}\bar{\rho}_t+\bar{\rho}_x u+\rho\bar{u}_x=-\tilde{\rho}'\bar{u}-\bar{\rho}\tilde{u}',\\ \bar{u}_t+u\bar{u}_x+\gamma\rho^{\gamma-2}\bar{\rho}_x=-F(\rho,\tilde{\rho})\bar{\rho}\tilde{\rho}'-\bar{u}\tilde{u}'-G(u,\tilde{u})\bar{u},\end{cases}\tag{3.4}$$

where $F(\rho,\tilde{\rho})\bar{\rho}=\gamma(\rho^{\gamma-2}-\tilde{\rho}^{\gamma-2})$, $G(u,\tilde{u})\bar{u}=-\beta[u^{\alpha+1}-\tilde{u}^{\alpha+1}]$ and $F(\rho,\tilde{\rho})$ and $G(u,\tilde{u})$ can take the following expressions

$$F(\rho,\tilde{\rho})=\gamma(\gamma-2)\int_0^1(\theta\bar{\rho}+\tilde{\rho})^{\gamma-3}d\theta,\quad G(u,\tilde{u})=-\beta(\alpha+1)\int_0^1(\theta\bar{u}+\tilde{u})^{\alpha}d\theta.\tag{3.5}$$

We also consider the perturbations of the initial and boundary conditions. The initial data is reformulated as

$$t=0:\begin{cases}\rho_0(x)=\bar{\rho}_0(x)+\tilde{\rho}(x),\quad x\in[0,L],\\ u_0(x)=\bar{u}_0(x)+\tilde{u}(x),\quad x\in[0,L],\end{cases}\tag{3.6}$$

where $L < L_m$, and boundary condition is

$$x=0:\begin{cases}\rho_l(t)=\bar{\rho}_l(t)+\tilde{\rho}(0), & t\geq 0,\\ u_l(t)=\bar{u}_l(t)+\tilde{u}(0), & t\geq 0,\end{cases} \tag{3.7}$$

where $\bar{\rho}_0,\bar{u}_0,\bar{\rho}_l,\bar{u}_l$ are C^1 functions.

Let $\bar{V}=(\bar{\rho},\bar{u})^\top$, the system (3.4) can be rewritten as the following quasi-linear form

$$\bar{V}_t+A(V)\bar{V}_x+D(\tilde{V},V)\bar{V}=0 \tag{3.8}$$

with the initial data

$$\bar{V}|_{t=0}=\bar{V}_0=(\bar{\rho}_0,\bar{u}_0)^\top, \tag{3.9}$$

and the boundary condition

$$V|_{x=0}=V_l=(\rho_l,u_l)^\top, \tag{3.10}$$

where $V(t,x)=\bar{V}(t,x)+\tilde{V}(x)$, and

$$A(V)=\begin{pmatrix} u & \rho\\ \gamma\rho^{\gamma-2} & u\end{pmatrix},\quad D(\tilde{V},V)=\begin{pmatrix}\tilde{u}' & \tilde{\rho}'\\ F(\rho,\tilde{\rho})\tilde{\rho}' & \tilde{u}'+G(u,\tilde{u})\end{pmatrix}.$$

We next introduce a wave decomposition of the solution $\bar{V}$ to the system (3.8). We can easily get the following two eigenvalues of the coefficient matrix $A(V)$

$$\lambda_1(V)=u-c,\quad \lambda_2(V)=u+c,$$

where $c=\sqrt{\gamma}\rho^{\frac{\gamma-1}{2}}$. The corresponding two right eigenvectors $r_i, i=1,2$ are

$$r_1(V)=\frac{1}{\sqrt{\rho^2+c^2}}(\rho,-c)^\top,\quad r_2(V)=\frac{1}{\sqrt{\rho^2+c^2}}(\rho,c)^\top. \tag{3.11}$$

The left eigenvectors $l_i(V), i=1,2$ are determined by

$$l_i(V)r_j(V)\equiv\delta_{ij},\quad i,j=1,2, \tag{3.12}$$

where δ_{ij} stands for the Kronecker's symbol. Then, $l_i, i=1,2$ have the following expressions

$$l_1(V)=\frac{\sqrt{\rho^2+c^2}}{2}(\rho^{-1},-c^{-1}),\quad l_2(V)=\frac{\sqrt{\rho^2+c^2}}{2}(\rho^{-1},c^{-1}), \tag{3.13}$$

which have the same regularity as $r_i(V)$.

Let

$$m_i=l_i(V)\bar{V},\quad n_i=l_i(V)\bar{V}_x,\quad m=(m_1,m_2)^\top,\quad n=(n_1,n_2)^\top, \tag{3.14}$$

then

$$\bar{V}=\sum_{k=1}^{2}m_kr_k(V),\quad \frac{\partial\bar{V}}{\partial x}=\sum_{k=1}^{2}n_kr_k(V), \tag{3.15}$$

$$\frac{\partial \bar{V}}{\partial t} = -D(\tilde{V},V)\bar{V} - \sum_{k=1}^{2}\lambda_k(V)n_k r_k(V). \tag{3.16}$$

Thus, we have

$$\begin{aligned}\frac{d\bar{V}}{d_i t} &= \frac{\partial \bar{V}}{\partial t} + \lambda_i(V)\frac{\partial \bar{V}}{\partial x}\\ &= \sum_{k=1}^{2}(\lambda_i(V)-\lambda_k(V))n_k r_k(V) - D(\tilde{V},V)\bar{V}.\end{aligned} \tag{3.17}$$

By (3.12)-(3.17), one has

$$\begin{aligned}\frac{dm_i}{d_i t} &= \frac{\partial m_i}{\partial t} + \lambda_i(V)\frac{\partial m_i}{\partial x}\\ &= \sum_{j,k=1}^{2}\Psi_{ijk}(V)n_j m_k + \sum_{j,k=1}^{2}\tilde{\Psi}_{ijk}(V)m_j m_k - \sum_{k=1}^{2}\tilde{\tilde{\Psi}}_{ik}(V)m_k,\end{aligned} \tag{3.18}$$

where

$$\Psi_{ijk}(V) = (\lambda_j(V)-\lambda_i(V))l_i(V)r_j(V)\cdot\nabla_V r_k(V), \tag{3.19}$$

$$\tilde{\Psi}_{ijk}(V) = l_i(V)D(\tilde{V},V)r_j(V)\cdot\nabla_V r_k(V), \tag{3.20}$$

$$\tilde{\tilde{\Psi}}_{ik}(V) = \lambda_i(V)l_i(V)\tilde{V}'\cdot\nabla_V r_k(V) + l_i(V)D(\tilde{V},V)r_k(V). \tag{3.21}$$

Similarly, using (3.8) and (3.12)-(3.17), we also get

$$\begin{aligned}\frac{dn_i}{d_i t} &= \frac{\partial n_i}{\partial t} + \lambda_i(V)\frac{\partial n_i}{\partial x}\\ &= \sum_{j,k=1}^{2}\Phi_{ijk}(V)n_j n_k + \sum_{j,k=1}^{2}\tilde{\Phi}_{ijk}(V)n_k - \sum_{k=1}^{2}l_i(V)D_x(\tilde{V},V)r_k(V)m_k,\end{aligned} \tag{3.22}$$

where the term $D_x(\tilde{V},V)$ makes sense if $\tilde{V}$ is a C^2 function, and

$$\begin{aligned}\Phi_{ijk}(V) = &(\lambda_j(V)-\lambda_k(V))l_i(V)r_j(V)\cdot\nabla_V r_k(V)\\ &-r_j(V)\cdot\nabla_V\lambda_k(V)\delta_{ik},\end{aligned} \tag{3.23}$$

$$\begin{aligned}\tilde{\Phi}_{ijk}(V) = &-\lambda_k(V)l_i(V)\tilde{V}'\cdot\nabla_V r_k(V) + l_i(V)D(\tilde{V},V)r_j(V)\cdot\nabla_V r_k(V)m_j(V)\\ &-l_i(V)D(\tilde{V},V)r_k(V) - \tilde{V}'\cdot\nabla_V\lambda_k(V)\delta_{ik}.\end{aligned} \tag{3.24}$$

For later use, we rewrite the system (3.4) by exchanging the variable t and x as follows

$$\bar{V}_x + A^{-1}(V)\bar{V}_t + A^{-1}(V)D(\tilde{V},V)\bar{V} = 0.$$

Denote $\hat{\lambda}_i(V), i=1,2$ are eigenvalues of the matrix $A^{-1}(V)$, $\hat{l}_i(V), i=1,2$ and $\hat{r}_i(V), i=1,2$ are the left and right eigenvectors respectively. They can be determined in terms of $\lambda_i(V), r_i(V)$ and $l_i(V)$ as follows

$$\hat{\lambda}_i(V) = \lambda_i(V)^{-1}, \quad \hat{r}_i(V) = r_i(V), \quad \hat{l}_i(V) = l_i(V). \tag{3.25}$$

Therefore, $\hat{r}_i(V)$ and $\hat{l}_i(V)$ also satisfy (3.12).

Let

$$\hat{m}_i = \hat{l}_i(V)\bar{V}, \quad \hat{n}_i = \hat{l}_i(V)\bar{V}_t, \quad \hat{m} = (\hat{m}_1, \hat{m}_2)^\top, \quad \hat{n} = (\hat{n}_1, \hat{n}_2)^\top. \tag{3.26}$$

By applying similar arguments as in (3.18)-(3.24), we can get

$$\begin{aligned}\frac{d\hat{m}_i}{d_i x} =& \frac{\partial \hat{m}_i}{\partial x} + \hat{\lambda}_i(V)\frac{\partial \hat{m}_i}{\partial t}\\ =& \sum_{j,k=1}^{2} \hat{\Psi}_{ijk}(V)\hat{n}_j\hat{m}_k + \sum_{j,k=1}^{2} \hat{\tilde{\Psi}}_{ijk}(V)\hat{m}_j\hat{m}_k - \sum_{k=1}^{2} \hat{\tilde{\tilde{\Psi}}}_{ik}(V)\hat{m}_k\end{aligned} \tag{3.27}$$

with

$$\hat{\Psi}_{ijk}(V) = (\hat{\lambda}_j(V) - \hat{\lambda}_i(V))\hat{l}_i(V)\hat{r}_j(V)\cdot\nabla_V\hat{r}_k(V), \tag{3.28}$$

$$\hat{\tilde{\Psi}}_{ijk}(V) = \hat{\lambda}_i(V)\hat{l}_i(V)D(\tilde{V},V)\hat{r}_j(V)\cdot\nabla_V\hat{r}_k(V), \tag{3.29}$$

$$\hat{\tilde{\tilde{\Psi}}}_{ik}(V) = \hat{l}_i(V)\tilde{V}'\cdot\nabla_V\hat{r}_k(V) + \hat{\lambda}_i(V)\hat{l}_i(V)D(\tilde{V},V)\hat{r}_k(V), \tag{3.30}$$

and

$$\begin{aligned}\frac{d\hat{n}_i}{d_i x} =& \frac{\partial \hat{n}_i}{\partial x} + \hat{\lambda}_i(V)\frac{\partial \hat{n}_i}{\partial t}\\ =& \sum_{j,k=1}^{2} \hat{\Phi}_{ijk}(V)\hat{n}_j\hat{n}_k + \sum_{j,k=1}^{2} \hat{\tilde{\Phi}}_{ijk}(V)\hat{n}_k - \sum_{k=1}^{2} \hat{l}_i(V)(A^{-1}(V)D(\tilde{V},V))_t\hat{r}_k(V)\hat{m}_k(V)\end{aligned} \tag{3.31}$$

with

$$\begin{aligned}\hat{\Phi}_{ijk}(V) =& (\hat{\lambda}_j(V) - \hat{\lambda}_k(V))\hat{l}_i(V)\hat{r}_j(V)\cdot\nabla_V\hat{r}_k(V) - \hat{r}_j(V)\cdot\nabla_V\hat{\lambda}_k(V)\delta_{ik},\\ \hat{\tilde{\Phi}}_{ijk}(V) =& -\hat{l}_i(V)\tilde{V}'\cdot\nabla_V\hat{r}_k(V) + \hat{\lambda}_i(V)\hat{l}_i(V)D(\tilde{V},V)\hat{r}_j(V)\cdot\nabla_V\hat{r}_k(V)\hat{m}_j(V)\\ & - \hat{\lambda}_i(V)\hat{l}_i(V)D(\tilde{V},V)\hat{r}_k(V).\end{aligned}$$

We also provide the wave decomposition of the initial and boundary data as follows

$$m_0 = (m_{10}, m_{20})^\top, \quad n_0 = (n_{10}, n_{20})^\top \tag{3.32}$$

with

$$m_{i0} = l_i(V_0)\bar{V}_0, \quad n_{i0} = l_i(V_0)\bar{V}_0',$$

and

$$\hat{m}_l = (\hat{m}_{1l}, \hat{m}_{2l})^\top, \quad \hat{n}_l = (\hat{n}_{1l}, \hat{n}_{2l})^\top, \tag{3.33}$$

with

$$\hat{m}_{il} = \hat{l}_i(V_l)\bar{V}_l, \quad \hat{n}_{il} = \hat{l}_i(V_l)\bar{V}_l',$$

where $\bar{V}_0$ and $\bar{V}_l$ are defined by (3.9) and (3.10) respectively, and

$$V_0 = (\rho_0, u_0)^\top, \quad \bar{V}_0' = (\bar{\rho}_0', \bar{u}_0')^\top, \tag{3.34}$$

$$V_l = (\rho_l, u_l)^\top, \quad \bar{V}_l' = (\bar{\rho}_l', \bar{u}_l')^\top. \tag{3.35}$$

4. Existence of global solutions

In this section, we will prove the existence of the global solution $\bar{V}=(\bar{\rho}(t,x),\bar{u}(t,x))^\top$ to the initial-boundary value problem (3.8) and (3.9)-(3.10) in the domain $E=\{(t,x)|t>0,x\in(0,L]\}$.

The local existence and uniqueness of the C^1 solution to the mixed initial-boundary value problem (3.8) and (3.9)-(3.10) is guaranteed by the classical theory in [13], which can be extended globally in terms of a uniform a-priori estimate of the global C^1 solutions (see [10–12, 14, 27, 28]).

Next we will establish a uniform a-priori estimate of the classical solution to help us to extend globally the local solution. Let us first give the following assumption

$$|m_i(t,x)|,|n_i(t,x)|\leq C\varepsilon,\quad \forall i=1,2,\quad (t,x)\in E \tag{4.1}$$

for a suitably small positive constant ε, which will be determined later.

From (3.11), (3.15) and (4.1), we have

$$|\bar{V}(t,x)|,|\frac{\partial\bar{V}}{\partial x}(t,x)|\leq C\varepsilon,\quad \forall (t,x)\in E. \tag{4.2}$$

Combining Lemma 2.1 with (4.2), we obtain the following results. The details of the proof are omitted here.

LEMMA 4.1. *For sufficiently small ε, it holds that*

$$|D(\tilde{V},V)(t,x)|,|\partial_x D(\tilde{V},V)(t,x)|,|\nabla_V r_i(V)(t,x)|,|\tilde{V}'|,T_1\leq C, \tag{4.3}$$

$$C^{-1}\leq|\lambda_i(V)(t,x)|,|\nabla_V\hat{\lambda}_i(V)(t,x)|,|l_i(V)(t,x)|\leq C, \tag{4.4}$$

$$|\frac{\partial\bar{V}}{\partial t}(t,x)|,|\partial_t A^{-1}(V)(t,x)|,|\partial_t D(\tilde{V},V)(t,x)|\leq C\varepsilon \tag{4.5}$$

for any $(t,x)\in E$, where the positive constant C only depends on $c_-,u_-,\tilde{c}(L)$, $\tilde{u}(L),\gamma,\alpha$ and β.

We observe from (4.2) and (4.4) that it suffices to prove (4.1) for a uniform a-priori estimate of the global C^1 solution.

Write $x=x_i^*(t),i=1,2$ as the characteristic curve of λ_i passing through a point $(0,0)$, which satisfy

$$\frac{dx_i^*(t)}{dt}=\lambda_i(V(t,x_i^*(t))),\quad x_i^*(0)=0.$$

Note that $x=x_2^*(t)$ lies below $x=x_1^*(t)$ since $\lambda_2(V)>\lambda_1(V)$.

We divide the region E into three small regions and discuss the uniform a-priori estimate of classical solutions in each small region separately.

Region 1: the region $E_1=\{(t,x)|0\leq t\leq T_1,0\leq x\leq L,x\geq x_2^*(t)\}$.

For any point $(t,x)\in E_1$, integrating the i-th equation in (3.18) along the i-characteristic curve with respect to τ from 0 to t which intersects the x-axis at a point $(0,b_i)$, we obtain from (3.18), (3.19)-(3.21), (4.1), (4.3) and (4.4) that

$$\begin{aligned}|m_i(t,x(t))|\leq&|m_i(0,b_i)|+\int_0^t\sum_{j,k=1}^2|\Psi_{ijk}(V)n_jm_k|d\tau\\&+\int_0^t\sum_{j,k=1}^2|\tilde{\Psi}_{ijk}(V)m_jm_k|d\tau+\int_0^t\sum_{k=1}^2|\tilde{\tilde{\Psi}}_{ik}(V)m_k|d\tau\end{aligned}$$

$$\leq |m_{i0}(b_i)| + C\int_0^t |m(\tau, x(\tau))| d\tau. \tag{4.6}$$

Applying the same procedures as above for (3.22), from (3.23), (3.24), (4.1), (4.3) and (4.4), we have

$$\begin{aligned} |n_i(t,x(t))| \leq & |n_i(0,b_i)| + \int_0^t \sum_{j,k=1}^2 |\Phi_{ijk}(V) n_j n_k| d\tau \\ & + \int_0^t \sum_{j,k=1}^2 |\tilde{\Phi}_{ijk}(V) n_k| d\tau + \int_0^t \sum_{k=1}^2 |l_i(V) D_x(\tilde{V}, V) r_k(V) m_k| d\tau \\ \leq & |n_{i0}(b_i)| + C\Big(\int_0^t |n(\tau, x(\tau))| d\tau + \int_0^t |m(\tau, x(\tau))| d\tau\Big). \end{aligned} \tag{4.7}$$

Putting (4.6)-(4.7) together, summing up $i = 1,2$ and applying Gronwall's inequality, we have

$$|m(t,x)| + |n(t,x)| \leq (\|m_0\|_{C^0([0,L])} + \|n_0\|_{C^0([0,L])})(1 + CT_1). \tag{4.8}$$

Because of the arbitrariness of $(t,x) \in E_1$ and the boundedness of T_1 in (4.3), we obtain from (4.8) that

$$\max_{(t,x)\in E_1} |m(t,x)| + |n(t,x)| \leq C(\|m_0\|_{C^0([0,L])} + \|n_0\|_{C^0([0,L])}). \tag{4.9}$$

Region 2: the region $E_2 = \{(t,x) | t \geq 0, 0 \leq x \leq L, 0 \leq x \leq x_1^*(t)\}$.

For any point $(t,x) \in E_2$, integrating in (3.27) with respect to x along the i-th characteristic curve, which is assumed to intersect the t-axis at a point $(\tau_i, 0)$, we have from (3.28)-(3.30), (4.1), (4.3) and (4.4) that

$$|\hat{m}_i(t(x), x)| \leq |\hat{m}_{il}(\tau_i)| + C\int_0^x |\hat{m}(t(y), y)| dy. \tag{4.10}$$

For (3.31), applying the same procedures as above, we further use (4.5) to obtain

$$|\hat{n}_i(t(x), x)| \leq |\hat{n}_{il}(\tau_i)| + C\Big(\int_0^x |\hat{n}(t(y), y)| dy + \int_0^x |\hat{m}(t(y), y)| dy\Big). \tag{4.11}$$

Taking the summation of (4.10) and (4.11) and the summation for $i = 1,2$, applying Gronwall's inequality, we have

$$\max_{(t,x)\in E_2} |\hat{m}(t,x)| + |\hat{n}(t,x)| \leq C(\|\hat{m}_l\|_{C^0([0,+\infty))} + \|\hat{n}_l\|_{C^0([0,+\infty))}), \tag{4.12}$$

where we have used the arbitrariness of $(t,x) \in E_2$.

Region 3: in the remaining region

$$E_3 = \{(t,x) | 0 \leq t \leq T_1, 0 \leq x \leq L, x_1^*(t) \leq x \leq x_2^*(t)\}.$$

For any point $(t,x) \in E_3$, integrating the first equation in (3.18) and (3.22) along the first characteristic curve that intersects $x_2^*(t)$ at a point (t_1, x_1), we get from (3.19)-(3.21), (3.23), (3.24), (4.1), (4.3) and (4.4) that

$$|m_1(t,x(t))| \leq |m_1(t_1,x_1)| + C\int_{t_1}^t |m(\tau,x(\tau))| d\tau \leq |m_1(t_1,x_1)| + C\int_0^t |m(\tau,x(\tau))| d\tau, \tag{4.13}$$

$$|n_1(t,x(t))| \leq |n_1(t_1,x_1)| + C(\int_0^t |n(\tau,x(\tau))|d\tau + \int_0^t |m(\tau,x(\tau))|d\tau). \tag{4.14}$$

Similarly, for any point $(t,x) \in E_3$, integrating the second equation in (3.18) and (3.22) along the second characteristic curve that intersects $x_1^*(t)$ at a point (t_2,x_2), we have

$$|m_2(t,x(t))| \leq |m_2(t_2,x_2)| + C\int_0^t |m(\tau,x(\tau))|d\tau, \tag{4.15}$$

$$|n_2(t,x(t))| \leq |n_2(t_2,x_2)| + C(\int_0^t |n(\tau,x(\tau))|d\tau + \int_0^t |m(\tau,x(\tau))|d\tau). \tag{4.16}$$

By applying Gronwall's inequality, the combination of (4.13)-(4.16) gives rise to

$$\begin{aligned}\max_{(t,x)\in E_3}(|m(t,x)|+|n(t,x)|) \leq & C(\|m_0\|_{C^0([0,L])} + \|n_0\|_{C^0([0,L])} \\ & + \|\hat{m}_l\|_{C^0([0,+\infty))} + \|\hat{n}_l\|_{C^0([0,+\infty))}),\end{aligned} \tag{4.17}$$

where we have used (4.9) and (4.12) and the arbitrariness of $(t,x) \in E_3$.

We notice from (4.9), (4.12), (4.17), (3.14) and (3.26) that under the initial and boundary conditions (1.5)-(1.6) for a sufficiently small $\varepsilon > 0$ and the assumption (4.4), we can check the validity of hypothesis (4.1) for some constant $C > 0$. Therefore, we obtain a uniform a-priori estimate for the global C^1 solution. The global existence of solutions to the initial-boundary value problem (3.8) and (3.9)-(3.10) can be checked by the standard continuity method, the details are omitted here.

5. Periodic solution

In this section, we will prove the global solution $V = (\rho(t,x), u(t,x))^\top$ is a time-periodic function with a period $P > 0$.

Using a Riemann invariant of system (1.1)

$$r = \frac{1}{2}(u - \frac{2}{\gamma-1}c), \quad s = \frac{1}{2}(u + \frac{2}{\gamma-1}c), \tag{5.1}$$

(1.1) can be converted into the following form

$$\begin{cases} r_t + \lambda_1(r,s)r_x = \dfrac{\beta(r+s)^{\alpha+1}}{2}, \\ s_t + \lambda_2(r,s)s_x = \dfrac{\beta(r+s)^{\alpha+1}}{2}, \end{cases} \tag{5.2}$$

where

$$\lambda_1 = u - c = \frac{\gamma+1}{2}r - \frac{\gamma-3}{2}s, \quad \lambda_2 = u + c = \frac{3-\gamma}{2}r + \frac{\gamma+1}{2}s.$$

Correspondingly, the initial data and boundary conditions become

$$r(0,x) = r_0(x), \quad s(0,x) = s_0(x), \quad x \in [0,L], \tag{5.3}$$

$$r(t,0) = r_l(t), \quad s(t,0) = s_l(t), \quad t \geq 0, \tag{5.4}$$

where $r_l(t), s_l(t)$ are time-periodic with the period $P > 0$.

For the convenience of later proof, we exchange t and x, then problem (5.2) and (5.3)-(5.4) becomes the following Cauchy problem in the domain E

$$\begin{cases} r_x + \dfrac{1}{\lambda_1} r_t = \dfrac{\beta(r+s)^{\alpha+1}}{2\lambda_1}, \\ s_x + \dfrac{1}{\lambda_2} s_t = \dfrac{\beta(r+s)^{\alpha+1}}{2\lambda_2}, \\ r(t,0) = r_l(t), \\ s(t,0) = s_l(t). \end{cases} \tag{5.5}$$

Furthermore, setting

$$W = (r-\tilde{r}, s-\tilde{s})^\top, \quad \Lambda(t,x) = \begin{pmatrix} \frac{1}{\lambda_1(r(t,x),s(t,x))} & 0 \\ 0 & \frac{1}{\lambda_2(r(t,x),s(t,x))} \end{pmatrix},$$

then (5.5) can be rewritten as

$$W_x + \Lambda(t,x) W_t = \frac{\beta}{2} \Lambda(t,x) \begin{pmatrix} (r+s)^{\alpha+1} \\ (r+s)^{\alpha+1} \end{pmatrix} - \frac{\beta}{2} \begin{pmatrix} \dfrac{(\tilde{r}+\tilde{s})^{\alpha+1}}{\tilde{\lambda}_1} \\ \dfrac{(\tilde{r}+\tilde{s})^{\alpha+1}}{\tilde{\lambda}_2} \end{pmatrix}, \tag{5.6}$$

where

$$\begin{aligned} &\tilde{r} = \frac{1}{2}(\tilde{u} - \frac{2}{\gamma-1}\tilde{c}), \quad \tilde{s} = \frac{1}{2}(\tilde{u} + \frac{2}{\gamma-1}\tilde{c}), \\ &\tilde{\lambda}_1 = \lambda_1(\tilde{r},\tilde{s}) = \frac{\gamma+1}{2}\tilde{r} - \frac{\gamma-3}{2}\tilde{s}, \\ &\tilde{\lambda}_2 = \lambda_2(\tilde{r},\tilde{s}) = \frac{3-\gamma}{2}\tilde{r} + \frac{\gamma+1}{2}\tilde{s}. \end{aligned}$$

By

$$\|\rho - \tilde{\rho}\|_{C^1(E)} + \|u - \tilde{u}\|_{C^1(E)} < C\varepsilon,$$

and (5.1), we can get

$$\|r(t,x) - \tilde{r}(x)\|_{C^1(E)} + \|s(t,x) - \tilde{s}(x)\|_{C^1(E)} < K_1\varepsilon \tag{5.7}$$

with $K_1 > 0$ a constant that depends only on $\tilde{\rho}, \tilde{u}, \gamma$ and L.

Next we will show that the following conclusion holds

$$r(t+P,x) = r(t,x), \quad s(t+P,x) = s(t,x), \quad \forall t > T_1, x \in [0,L], \tag{5.8}$$

where T_1 is defined by (1.7).

Letting

$$U(t,x) = W(t+P,x) - W(t,x),$$

then by (5.6), we can get

$$\begin{cases} U_x + \Lambda(t,x) U_t = G(t,x), \\ U(t,0) = 0, \quad t > 0, \end{cases} \tag{5.9}$$

where

$$
\begin{aligned}
G(t,x) = & \frac{\beta}{2}\Lambda(t+P,x)\begin{pmatrix}(r(t+P,x)+s(t+P,x))^{\alpha+1}\\(r(t+P,x)+s(t+P,x))^{\alpha+1}\end{pmatrix}\\
& -\frac{\beta}{2}\Lambda(t,x)\begin{pmatrix}(r(t,x)+s(t,x))^{\alpha+1}\\(r(t,x)+s(t,x))^{\alpha+1}\end{pmatrix}\\
& -[\Lambda(t+P,x)-\Lambda(t,x)]W_t(t+P,x).
\end{aligned}
$$

Noting that λ_1,λ_2 are continuous functions of (r,s), then by (5.7), we can get the following estimates

$$|W_t(t+P,x)| \le K_1\varepsilon, \tag{5.10}$$
$$|r(t+P,x)+s(t+P,x)| \le K_2, \tag{5.11}$$
$$|\Lambda_t(r(t,x),s(t,x))| \le K_3\varepsilon, \tag{5.12}$$
$$|\Lambda(t+P,x)-\Lambda(t,x)| \le K_4|U(t,x)|, \tag{5.13}$$
$$|\Lambda(t,x)| \le K_5, \tag{5.14}$$

where constants K_2,K_3,K_4,K_5 depend only on $\tilde{\rho},\tilde{u},\gamma$ and L.

It follows from (5.10)-(5.11), (5.13)-(5.14) that

$$
\begin{aligned}
|G(t,x)| \le & \frac{|\beta|}{2}|\Lambda(t,x)|\begin{pmatrix}(\alpha+1)|\eta|^{\alpha}|U(t,x)|\\(\alpha+1)|\eta|^{\alpha}|U(t,x)|\end{pmatrix}\\
& +\frac{|\beta|}{2}|\Lambda(t+P,x)-\Lambda(t,x)|\begin{pmatrix}|r(t+P,x)+s(t+P,x)|^{\alpha+1}\\|r(t+P,x)+s(t+P,x)|^{\alpha+1}\end{pmatrix}\\
& +|\Lambda(t+P,x)-\Lambda(t,x)||W_t(t+P,x)|\\
\le & K_6|U(t,x)|,
\end{aligned} \tag{5.15}
$$

where η lies between $u(t,x)$ and $u(t+P,x)$, the definition of K_6 is the same as above.

For a fixed point (t_0,x_0) with $t_0>T_1, 0<x_0<L$, we can draw two characteristic curves $\Gamma_1: t=t_1^*(x)$ and $\Gamma_2: t=t_2^*(x)$, namely,

$$\frac{dt_1^*}{dx}=\frac{1}{\lambda_1(r(t_1^*,x),s(t_1^*,x))}, t_1^*(x_0)=t_0$$

and

$$\frac{dt_2^*}{dx}=\frac{1}{\lambda_2(r(t_2^*,x),s(t_2^*,x))}, t_2^*(x_0)=t_0$$

for $0<x<x_0$. And we can easily see that Γ_1 lies below Γ_2.

Setting

$$I(x)=\frac{1}{2}\int_{t_1^*(x)}^{t_2^*(x)}|U(t,x)|^2dt, \tag{5.16}$$

where $0\le x<x_0$.

By the definition of T_1 and $t_0>T_1$, we can get that $(t_1^*(0),t_2^*(0))\subset(0,+\infty)$, then by (5.9), we have $U(t,0)\equiv 0$ in this interval.

Therefore,

$$I(0)=0.$$

Taking derivative of $I(x)$ with respect to x, we get

$$\begin{aligned}
I^{'}(x)=&\int_{t_1^*(x)}^{t_2^*(x)} U(t,x)^T U_x(t,x)dt+\frac{1}{2}|U(t_2^*(x),x)|^2\frac{1}{\lambda_2(r(t_2^*(x),x),s(t_2^*(x),x))}\\
&-\frac{1}{2}|U(t_1^*(x),x)|^2\frac{1}{\lambda_1(r(t_1^*(x),x),s(t_1^*(x),x))}\\
\le&-\int_{t_1^*(x)}^{t_2*(x)} U(t,x)^T\Lambda(t,x)U_t(t,x)dt+\int_{t_1^*(x)}^{t_2^*(x)} U(t,x)^T G(t,x)dt\\
&+\frac{1}{2}U(t,x)^T\Lambda(t,x)U(t,x)|_{t=t_1^*(x)}^{t=t_2^*(x)}\\
=&-\frac{1}{2}\int_{t_1^*(x)}^{t_2^*(x)}(U(t,x)^T\Lambda(t,x)U(t,x))_t-U(t,x)^T\Lambda_t(t,x)U(t,x)dt\\
&+\int_{t_1^*(x)}^{t_2^*(x)} U(t,x)^T G(t,x)dt+\frac{1}{2}U(t,x)^T\Lambda(t,x)U(t,x)|_{t=t_1^*(x)}^{t=t_2^*(x)}\\
=&\frac{1}{2}\int_{t_1^*(x)}^{t_2^*(x)} U(t,x)^T\Lambda_t(t,x)U(t,x)dt+\int_{t_1^*(x)}^{t_2^*(x)} U(t,x)^T G(t,x)dt\\
\le&(K_3\varepsilon+2K_6)I(x).
\end{aligned}$$

In the last inequality we have used (5.12) and (5.15).

Hence, by Gronwall's inequality, we can get that $I(x)\equiv 0$. Furthermore, by continuity of $I(x)$, we have $I(x_0)=0$, then $U(t_0,x_0)=0$.

Since (t_0,x_0) is arbitrary, so we have

$$U(t,x)\equiv 0, \quad \forall t>T_1, x\in[0,L],$$

that is, we complete the proof of (5.8). Then, using (5.1) and $c=\sqrt{\gamma}\rho^{\frac{\gamma-1}{2}}$, we can get that $(\rho,u)^\top$ is also a periodic function with a period $P>0$.

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